ON THE FIRST AND SECOND K-GROUPS OF AN ELLIPTIC CURVE OVER GLOBAL FIELDS OF POSITIVE CHARACTERISTIC

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Abstract. Let E be an elliptic curve over a global field of positive characteristic. Let r be the order of zero at $s = 0$ of the Hasse-Weil L-function with bad factors removed. Parshin conjecture on the vanishing of higher rational K-theory of projective smooth schemes over finite fields implies $\dim_{\mathbb{Q}} K_2(E) \otimes_{\mathbb{Z}} \mathbb{Q} = r$. It is shown that $\dim_{\mathbb{Q}} K_2(E) \otimes_{\mathbb{Z}} \mathbb{Q} \geq r$.

As applications, some information on the structure of the torsion of the first and second K-groups of the elliptic curve, as well as the motivic cohomology groups of open elliptic surfaces over finite fields are obtained.

1. Introduction

Let E be an elliptic curve over a global field k of positive characteristic. Let C be the proper smooth curve over a finite field whose function field is k. We take the flat proper regular minimal model $\mathcal{E} \to C$ of E.

Let us identify the K-theory and the G-theory of regular noetherian schemes. There is the localization sequence of G-theory:

$$K_2(\mathcal{E}) \to K_2(E) \xrightarrow{\oplus \partial_\wp} \bigoplus_{\wp} G_1(\mathcal{E}_\wp) \to K_1(\mathcal{E})$$

where $\wp$ runs over all primes of k, $\mathcal{E}_\wp = \mathcal{E} \times_C \operatorname{Spec} \kappa(\wp)$, and $\kappa(\wp)$ is the residue field at $\wp$.

We use the subscript $-_{\mathbb{Q}}$ to mean $- \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 1.1. *Let the notations be as above. The homomorphism*

$$K_2(E)_{\mathbb{Q}} \xrightarrow{\oplus \partial_s} \bigoplus_{s \in S_0} G_1(\mathcal{E}_s)_{\mathbb{Q}}$$

is surjective.

Parshin conjecture says that, in particular, $K_1(\mathcal{E})_{\mathbb{Q}} = K_2(\mathcal{E})_{\mathbb{Q}} = 0$. Hence the validity of the conjecture implies that the homomorphism in Theorem 1.1 is an isomorphism.

Our principal motivation was to prove the following corollary.

Corollary 1.2. *Let S_0 be the set of primes of k (or, equivalently, closed points of C) at which E has split multiplicative reduction. Let r be the order of pole at $s = 0$ of the Hasse-Weil L-function $L^{S_0}(h^1(E), s)$ with bad factors removed. Then $\dim_{\mathbb{Q}} K_2(E) \otimes_{\mathbb{Z}} \mathbb{Q} \geq r$.*

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Theorem 1.1 implies Corollary 1.2. The reduction of an elliptic curve is well understood; one can verify directly (cf. [Bl-Gr]) that

$$\dim_{\mathbb{Q}} G_1(\mathcal{E}_\wp)_{\mathbb{Q}} = \begin{cases} 1 & \text{if } \wp \in S_0, \\ 0 & \text{if } \wp \notin S_0. \end{cases}$$

Since $|S_0| = r$, the claim follows. □

The analogue of Corollary 1.2 over number fields is that the rational rank of the K-group is greater than or equal to $|S_0| + [k : \mathbb{Q}]$ (see [Ro-Sc, Section 1.2]). It is a consequence of (a strong form of) Beilinson's conjectures, and is not yet proved. Bloch and Grayson gives a method ([Bl-Gr]) for constructing elements in K_2 such that the boundary map is not trivial, but one needs that the image of $\mathrm{Gal}(\overline{k}/k) \to \prod_l \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell(E(\overline{k}))$ (where $\overline{k}$ is the separable closure of k and ℓ runs over all primes) is small, and is not applicable in general.

We have other applications. Using Theorem 1.1, we obtain some information on the motivic cohomology and K-groups of lower degrees of an elliptic curve and an elliptic surface over a finite field associated to it. The following theorem is on K-groups of an elliptic curve. For other main results, we refer to Theorems 12.1, 12.2, 12.3, 13.1, 13.2. In the following theorem, $T'_{(1)}$ is the twisted Mordell-Weil group and S_2 is the set of bad primes. For the precise definitions, we refer to Section 2 and the beginning of Section 12.

Theorem 1.3. (1) *The dimension of the $\mathbb{Q}$-vector space $(K_2(E)^{\mathrm{red}})_{\mathbb{Q}}$ is r.*

(2) *The cokernel of the boundary map $\partial_2 : K_2(E) \to \bigoplus_{\wp \in C_0} G_1(\mathcal{E}_\wp)$ is a finite group of order*
$$\frac{(q-1)^2 |L(h^0(\mathrm{Irr}(\mathcal{E}_{S_2})), -1)|}{|T'_{(1)}| \cdot |L(h^0(C \setminus S_2), -1)|}.$$
(3) *The group $K_1(E)_{\mathrm{div}}$ is uniquely divisible.*

(4) *The kernel of the boundary map $\partial_1 : K_1(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_0(\mathcal{E}_\wp)$ is a finite group of order $(q-1)^2 |T'_{(1)}| \cdot |L(E, 0)|$, where $L(E, s)$ is the L-function of E. The cokernel of ∂_1 is a finitely generated abelian group of rank $2 + |\mathrm{Irr}(\mathcal{E}_{S_2})| - |S_2|$ whose torsion subgroup is isomorphic to $\mathrm{Jac}(C)(\mathbb{F}_q)^{\oplus 2}$, where $\mathrm{Jac}(C)$ denotes the Jacobian of C.*

(5) *Suppose that the Bloch-Kato conjecture holds. Then the group $K_2(E)_{\mathrm{div}}$ is uniquely divisible, and the kernel of the boundary map $\partial : K_2(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_1(\mathcal{E}_\wp)$ is a finite group of order $|L(h^2(\mathcal{E}), 0) L(h^1(C), -1)|$.*

To prove Theorem 1.1, we use the analogue of Beilinson elements, constructed in [Ko-Ya], for Drinfeld modular curves. The key idea is that, over function fields, one has an analogue of Beilinson elements for every place at which the elliptic curve has split multiplicative reduction. The linear independence of the elements amounts to the integrality of the elements.

The sections are organized as follows. The paper is divided in two parts: Sections 3–8 and Sections 9–13. In the first half, we prove Theorem 1.1. In the second half, we compute motivic cohomology and K-groups of elliptic surfaces over finite fields and elliptic curves over function fields.

In Section 3, we consider curves over local fields. We compare the triviality of Chern class map and the triviality of the boundary map. We use this to avoid the construction of morphisms between integral models. It may be possible to actually construct them, however. In Section 4, we define Weil pairing morphism. This has already been done by van den Heiden [vdHe] using the theory of A-motives. Here we take a different approach using the theory of elliptic sheaves. In Section 5, we study bad reduction of Drinfeld modular curves. The essential part of the results is due to Drinfeld; there are also papers [Ge4], [Ge5], [Ge3] by Gekeler. The results in the case of elliptic modular curves are found in the book of Katz-Mazur [Ka-Ma]. In Section 6, we prove the integrality of certain elements in K_2 of Drinfeld modular curves. This section is the function field analogue of [Sc-Sc, Section 7], and we do follow the same line. In Section 7, we construct, starting from the elements of [Ko-Ya], a subspace in the rational K_2 of (the limit of) compactified Drinfeld modular curves which does not vanish under the boundary map at the infinity prime. In Section 8, we give the proof of Theorem 1.1.

In Section 9, we compute the motivic cohomology groups of smooth surfaces X over finite fields. The difficult case is that of $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$ and is treated in Section 9.2. In Section 10, we define Chern characters for singular curves over finite fields. The treatment is quite ad hoc. In Section 11, we give the relation via the Chern class map between K_1 and K_2 of curves over function fields and the motivic cohomology groups. In Section 12, we restrict ourselves to the case of elliptic surfaces. Applying the results in the previous three sections, and using the special features of elliptic surfaces, including Theorem 1.1, we compute the explicit orders of certain torsion groups. We treat the p-part separately in Appendix A. See its introduction for more technical details. In Section 13, we assume that the Bloch-Kato conjecture holds and generalize the results in Section 12. Appendix B is a digression; we determine the structure of the higher Chow groups $\mathrm{CH}^{d+i}(X, i)$ for $i = 1, 2$ where X is a scheme of dimension less than or equal to d, separated, and of finite type over a finite field. This is a generalization of the result of Akhtar [Ak].

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2. Notations

For a finite set M, we let $|M|$ denote the cardinality of M. For a prime number ℓ, we let $|\ |_\ell : \mathbb{Q}_\ell \to \mathbb{Q}$ denote the ℓ-adic absolute value normalized so that $|\ell|_\ell = \ell^{-1}$. For an abelian group M, let M_{tors} (resp. M_{div}) denote the torsion subgroup (resp. the maximal divisible subgroup) of M. We also put $M^{\mathrm{red}} = M/M_{\mathrm{div}}$. For a prime number ℓ, we put $T_\ell M = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, M)$. For a scheme X, let X_0 (resp. $\mathrm{Irr}(X)$) denote the set of the closed points (resp. the irreducible components) of X.

3. From special fiber to generic fiber

Let S be the spectrum of a henselian discrete valuation ring whose residue field is a finite field of characteristic $p > 0$. We denote by s (resp. η) the closed (resp. generic) point in S. Let $X \to S$ be a proper, flat, and surjective morphism from a regular scheme X to S such that the generic fiber X_η is a smooth curve over η. We let Y denote the complement $X \setminus X_\eta$ with the reduced scheme structure. We consider the boundary map

$$\partial_{\mathbb{Q}} : K_2(X_\eta)_{\mathbb{Q}} \to G_1(Y)_{\mathbb{Q}}.$$

We fix a prime number ℓ different from p. We consider the etale Chern class map

$$c_{2,2} : K_2(X_\eta) \to H^2_{\mathrm{et}}(X_\eta, \mathbb{Q}_\ell(2)),$$

(or more precisely, the limit of etale Chern class maps with finite coefficients) introduced in [Gi, Definition 2.22].

The aim of this section is to prove the following

Proposition 3.1. *For an element $x \in K_2(X_\eta)$, $\partial_{\mathbb{Q}}(x) = 0$ if and only if $c_{2,2}(x) = 0$.*

Lemma 3.2. *The diagram*

$$\begin{array}{ccc} K_2(X_\eta)_{\mathbb{Q}} & \xrightarrow{\partial_{\mathbb{Q}}} & G_1(Y)_{\mathbb{Q}} \\ {\scriptstyle c_{2,2}}\downarrow & & \downarrow{\scriptstyle c^Y_{2,1}} \\ H^2_{\mathrm{et}}(X_\eta, \mathbb{Q}_\ell(2)) & \xrightarrow{\partial_{\mathrm{et}}} & H^3_{Y,\mathrm{et}}(X, \mathbb{Q}_\ell(2)) \end{array}$$

is commutative. Here where $c^Y_{2,1}$ is the Chern class map and ∂_{et} is a part of the long exact sequence

$$\cdots \to H^2_{\mathrm{et}}(X, \mathbb{Q}_\ell(2)) \to H^2_{\mathrm{et}}(X_\eta, \mathbb{Q}_\ell(2)) \xrightarrow{\partial_{\mathrm{et}}} H^3_{Y,\mathrm{et}}(X, \mathbb{Q}_\ell(2)) \to \cdots \tag{3.1}$$

of cohomology with support.

Proof. By definition of the Chern class maps in [Gi], the map $c_{2,2}$ (resp. $c^Y_{2,1}$) is described as the composition

$$K_2(X_\eta) \to H^{-2}_{\mathrm{Zar}}(X_\eta, \mathbb{Z}_\infty \mathcal{B}_\bullet \mathcal{GL}(\mathcal{O}_X)) \to H^{-2}_{\mathrm{Zar}}(X_\eta, \mathcal{K}(4, \Gamma(2))) \to H^2_{\mathrm{et}}(X_\eta, \mathbb{Q}_\ell(2))$$

(resp.

$$G_1(Y) \cong K^Y_1(X) \to H^{-1}_{Y,\mathrm{Zar}}(X, \mathbb{Z}_\infty \mathcal{B}_\bullet \mathcal{GL}(\mathcal{O}_X)) \to H^{-1}_{Y,\mathrm{Zar}}(X, \mathcal{K}(4, \Gamma(2))) \to H^3_{Y,\mathrm{et}}(X, \mathbb{Q}_\ell(2))).$$

Here $\mathbb{Z}_\infty \mathcal{B}_\bullet \mathcal{GL}(\mathcal{O}_X)$ and $\mathcal{K}(4, \Gamma(2))$ are as in [Gi] (as the cohomology theory $\Gamma(*)$, we take the ℓ-adic etale cohomology theory on the category of schemes which is separated and is of finite type over $\overline{S}$ (cf. [Gi, Example 1.4 (iii)])). Then the claim follows from the commutativity of the diagram

$$\begin{array}{ccc} K_2(X_\eta) & \longrightarrow & K^Y_1(X) \\ \downarrow & & \downarrow \\ H^{-2}_{\mathrm{Zar}}(X_\eta, \mathbb{Z}_\infty \mathcal{B}_\bullet \mathcal{GL}(\mathcal{O}_X)) & \longrightarrow & H^{-1}_{Y,\mathrm{Zar}}(X, \mathbb{Z}_\infty \mathcal{B}_\bullet \mathcal{GL}(\mathcal{O}_X)) \\ \downarrow & & \downarrow \\ H^{-2}_{\mathrm{Zar}}(X_\eta, \mathcal{K}(4, \Gamma(2))) & \longrightarrow & H^{-1}_{Y,\mathrm{Zar}}(X, \mathcal{K}(4, \Gamma(2))) \\ \downarrow & & \downarrow \\ H^2_{\mathrm{et}}(X_\eta, \mathbb{Q}_\ell(2)) & \xrightarrow{\partial_{\mathrm{et}}} & H^3_{Y,\mathrm{et}}(X, \mathbb{Q}_\ell(2)). \end{array}$$

□

Lemma 3.3. *The Chern class map*

$$c^Y_{2,1} : G_1(Y)_{\mathbb{Q}} \to H^3_{Y,\mathrm{et}}(X, \mathbb{Q}_\ell(2))$$

is injective.

Proof. Let Y_{sing} denote the singular locus of Y. We set $Y_{\mathrm{sm}} = Y \setminus Y_{\mathrm{sing}}$. By [Gi, Lemma 2.23], the diagram

$$\begin{array}{ccc} G_1(Y)_{\mathbb{Q}} & \longrightarrow & G_1(Y_{\mathrm{sm}})_{\mathbb{Q}} \\ {\scriptstyle c^Y_{2,1}}\big\downarrow & & \big\downarrow{\scriptstyle c^{Y_{\mathrm{sm}}}_{2,1}} \\ H^3_{Y,\mathrm{et}}(X, \mathbb{Q}_\ell(2)) & \longrightarrow & H^3_{Y_{\mathrm{sm}},\mathrm{et}}(X - Y_{\mathrm{sing}}, \mathbb{Q}_\ell(2)) \end{array}$$

is commutative. Since $G_1(Y_{\mathrm{sing}})_{\mathbb{Q}} = 0$, the upper horizontal arrow is injective. Hence it suffices to show that $c^{Y_{\mathrm{sm}}}_{2,1}$ is injective. The diagram

$$\begin{array}{ccc} G_1(Y_{\mathrm{sm}})_{\mathbb{Q}} & \xrightarrow{=} & G_1(Y_{\mathrm{sm}})_{\mathbb{Q}} \\ {\scriptstyle -c_{1,1}}\big\downarrow & & \big\downarrow{\scriptstyle c^{Y_{\mathrm{sm}}}_{2,1}} \\ H^1(Y_{\mathrm{sm}}, \mathbb{Q}_\ell(1)) & \xrightarrow{\cong} & H^3_{Y_{\mathrm{sm}},\mathrm{et}}(X - Y_{\mathrm{sing}}, \mathbb{Q}_\ell(2)) \end{array}$$

is commutative by Riemann-Roch theorem ([Gi, Theorem 3.1]. See also [Gi, Corollary 3.7]). Here $c_{1,1}$ is the Chern class map. Thus it suffices to show that $c_{1,1}$ is injective. It is known that the map $-c_{1,1}$ equals the composition

$$G_1(Y_{\mathrm{sm}})_{\mathbb{Q}} \to H^0(Y_{\mathrm{sm}}, \mathbb{G}_m)_{\mathbb{Q}} \to H^1(Y_{\mathrm{sm}}, \mathbb{Q}_\ell(1))$$

where the last map is given by Kummer sequence. From the localization sequence

$$\cdots \to \bigoplus_{x \in Y_{\mathrm{sm},0}} G_1(x) \to G_1(Y_{\mathrm{sm}})_{\mathbb{Q}} \to \bigoplus_{x \in Y_{\mathrm{sm},1}} G_1(x) \to \cdots,$$

we see that $G_1(Y_{\mathrm{sm}})_{\mathbb{Q}} \to H^0(Y_{\mathrm{sm}}, \mathbb{G}_m)_{\mathbb{Q}}$ is an isomorphism. Hence by Kummer theory the map $c_{1,1}$ is injective. $\square$

Proof of Proposition 3.1. By the lemma above, for an element $x \in K_2(X_\eta)_{\mathbb{Q}}$, $\partial_{\mathbb{Q}}(x) = 0$ if and only if $\partial_{\mathrm{et}}(c_{2,2}(x)) = 0$. Therefore, to prove Proposition 3.1, it suffices to show that ∂_{et} is injective.

Let us consider the exact sequence,

$$H^2_{\mathrm{et}}(X, \mathbb{Q}_\ell(2)) \to H^2_{\mathrm{et}}(X_\eta, \mathbb{Q}_\ell(2)) \xrightarrow{\partial} H^1_{Y,\mathrm{et}}(X, \mathbb{Q}_\ell(2)).$$

By [SGA4-3, XII, Corollaire 5.5], we have $H^2_{\mathrm{et}}(X, \mathbb{Q}_\ell(2)) \cong H^2_{\mathrm{et}}(X \times_S s, \mathbb{Q}_\ell(2)) \cong H^2_{\mathrm{et}}(Y, \mathbb{Q}_\ell(2))$. The weight argument shows that the group $H^2_{\mathrm{et}}(Y, \mathbb{Q}_\ell(2))$ is zero. This proves Proposition 3.1. $\square$

4. Drinfeld modules, elliptic sheaves, and Weil pairing

To compute the bad reduction as in Katz-Mazur's book [Ka-Ma], we need Weil pairing. Instead of actually defining the pairing, we construct a morphism, which we call Weil pairing morphism, from the moduli of rank d Drinfeld modules to the moduli of rank 1 Drinfeld modules. This has already been done by van den Heiden [vdHe] using the theory of A-motives. Here we take a different approach using the notion of elliptic sheaves, which is equivalent to that of Drinfeld modules (see [Dr3], [Bl-St, Theorem 3.2.1]). The result will be used in Section 5 when we study the bad reduction of Drinfeld modular curves.

4.1. **Setting.** Let C be a projective smooth geometrically irreducible curve over a finite field $\mathbb{F}_q$ of q elements of characteristic p. We let $k = \mathbb{F}_q(C)$ denote the function field of C. We fix a closed point $\infty \in C$. Let $A = H^0(C \setminus \{\infty\}, \mathcal{O}_C)$ denote the coordinate ring of $C \setminus \{\infty\}$.

4.2. **Drinfeld modules.**

4.2.1. We recall the definition of a Drinfeld module. Let S be an A-scheme. Let d be a positive integer. A Drinfeld module of rank d over S is an A-module scheme E over S satisfying the following three conditions.

(1) Zariski locally on S, E is isomorphic to $\mathbb{G}_a$ as a commutative group scheme.
(2) If we denote the A-action on E by $\varphi : A \to \mathrm{End}_{S\text{-group}}(E)$, then, for every $a \in A \setminus \{0\}$, the a-action $\varphi(a) : E \to E$ on E is finite, locally free of constant degree $|a|_\infty^d$.
(3) The A-action on $\mathrm{Lie}\, E$ induced by φ coincide with the A-action on $\mathrm{Lie}\, E$ which comes from the structure homomorphism $A \to \Gamma(S, \mathcal{O}_S)$.

4.2.2. *Drinfeld level structure and modular variety.* Let d be a positive integer. Let E be a Drinfeld module of rank d over an A-scheme S. Let $E(S)$ be the A-module of the sections of $E \to S$. We regard an element in $E(S)$ as an effective Cartier divisor in E/S (in the sense of [Ka-Ma, 1.1.1]). Let $I \subset A$ be a non-zero ideal. Let $E[I]$ denote the I-torsion part of E. If we take generators $a_1, \ldots, a_m \in I$ of I, then $E[I]$ is identified with the fiber product of the morphism $(a_1, \ldots, a_m) : E^m \to E^m$ and the diagonal embedding $E \to E^m$ (here E^m denotes the m-fold fiber product of E over S). A Drinfeld level I structure on E is a homomorphism $\phi : (I^{-1}/A)^{\oplus d} \to E(S)$ of A-modules such that $\sum_{a \in (I^{-1}/A)^{\oplus d}} \phi(a)$ equals $E[I]$ as an effective Cartier divisor in E/S.

Suppose that $I \neq A$. Then Drinfeld [Dr1, Proposition 5.3] shows that the functor which associates to an A-scheme S the set of isomorphism classes of Drinfeld modules of rank d over S with a Drinfeld level I structure is representable by an affine A-scheme M_I^d. Moreover the A-scheme M_I^d has the following properties.

Lemma 4.1. (1) *M_I^d is a regular equidimensional scheme of Krull dimension d.*
(2) *The structure morphism $M_I^d \to \mathrm{Spec}\, A$ is of finite type, flat, and surjective, and is smooth when restricted to the open $U_I = \mathrm{Spec}\, A \setminus \mathrm{Spec}(A/I) \subset \mathrm{Spec}\, A$.*
(3) *For two non-zero ideals I, J of A with $J \subset I \subsetneqq A$, the canonical "level-lowering" morphism $M_J^d \to M_I^d$ is finite flat.*

Proof. All assertions, except the flatness and the surjectivity in (2), are immediate consequences of [Dr1, Proposition 5.3] and its corollary. The flatness follows from the local description of M_I^d in [Dr1, Proposition 5.4]. Surjectivity follows from [Dr1, §8, Corollary]. □

4.3. **Elliptic sheaves.**

4.3.1. We recall the definition of an elliptic sheaf. Let S be a scheme over X. Let d be a positive integer. An elliptic sheaf of rank d over S is a sequence $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$, where $\mathcal{E}_i$ are locally free $\mathcal{O}_{C\times_{\mathbb{F}_q}S}$-modules of rank d and where $j_i : \mathcal{E}_i \to \mathcal{E}_{i+1}$, $t_i : {}^\tau\mathcal{E}_i \to \mathcal{E}_{i+1}$ are injective $\mathcal{O}_{C\times_{\mathbb{F}_q}S}$-linear homomorphisms. We put ${}^\tau\mathcal{E}_i := (\mathrm{id}_C \times \mathrm{Frob}_S)^*\mathcal{E}_i$, where Frob_S denotes the q-power absolute Frobenius endomorphism of S. The following conditions should hold.

(1) The diagrams are commutative:

$$\begin{array}{ccc} \mathcal{E}_i & \xrightarrow{j_i} & \mathcal{E}_{i+1} \\ {\scriptstyle t_{i-1}}\big\uparrow & & {\scriptstyle t_i}\big\uparrow \\ {}^\tau\mathcal{E}_{i-1} & \xrightarrow{{}^\tau j_{i-1}} & {}^\tau\mathcal{E}_i. \end{array}$$

(2) For each i there exists an isomorphism $\mathcal{E}_{i+d\cdot\deg(\infty)} \cong \mathcal{E}_i(\infty) := \mathcal{E}_i \otimes_{\mathcal{O}_{C\times_{\mathbb{F}_q}S}} (\mathcal{O}_C(\infty)\boxtimes\mathcal{O}_S)$ where $\mathcal{O}_C(\infty) \boxtimes \mathcal{O}_S = \mathrm{pr}_1^*(\mathcal{O}_C(\infty)) \otimes \mathrm{pr}_2^*(\mathcal{O}_S)$ is a sheaf on $C \times_{\mathbb{F}_q} S$ such that the composite
$$\mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1} \hookrightarrow \cdots \hookrightarrow \mathcal{E}_{i+d\cdot\deg(\infty)} \cong \mathcal{E}_i(\infty)$$
is the canonical embedding.

(3) The direct image of $\mathcal{E}_{i+1}/j_i(\mathcal{E}_i)$ under the projection $\mathrm{pr}_S : C \times_{\mathbb{F}_q} S \to S$ is a locally free O_S-module of rank one.

(4) The cokernel $\mathrm{Coker}\, t_j$ is supported by the graph of the structure morphism $\iota_j : S \to C$, and is the direct image of a locally free module on S of rank one by this graph morphism $S \xrightarrow{(\iota_j,\mathrm{id}_S)} C \times_{\mathbb{F}_q} S$.

(5) For any closed point $s \in S$, one has $\deg(\mathcal{E}_0|_{C\times_{\mathbb{F}_q}\{s\}}) = d(g-1)+1$ where g is the genus of the curve $C/\mathbb{F}_q$.

4.3.2. *Level structure.* Let $I \subset A$ be an ideal. Let S be a scheme over $C \setminus \mathrm{Spec}(A/I)$. Let $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ be an elliptic sheaf of rank d over S. The morphisms j_i for $i \in \mathbb{Z}$ identify the restrictions $\mathcal{E}_i|_{\mathrm{Spec}(A/I)\times_{\mathbb{F}_q}S}$. The homomorphism t_0 induces $t_I : {}^\tau\mathcal{E}_0|_{\mathrm{Spec}(A/I)\times_{\mathbb{F}_q}S} \to \mathcal{E}_1|_{\mathrm{Spec}(A/I)\times_{\mathbb{F}_q}S} \cong \mathcal{E}_0|_{\mathrm{Spec}(A/I)\times_{\mathbb{F}_q}S}$. A level I structure on $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ is an isomorphism $\iota : \mathcal{E}_0|_{\mathrm{Spec}(A/I)\times_{\mathbb{F}_q}S} \xrightarrow{\cong} \mathcal{O}^{\oplus d}_{\mathrm{Spec}(A/I)\times_{\mathbb{F}_q}S}$ such that the isomorphism ${}^\tau\iota : {}^\tau\mathcal{E}_0|_{\mathrm{Spec}(A/I)\times_{\mathbb{F}_q}S} \xrightarrow{\cong} {}^\tau(\mathcal{O}_{\mathrm{Spec}(A/I)\times_{\mathbb{F}_q}S})^{\oplus d} = \mathcal{O}^{\oplus d}_{\mathrm{Spec}(A/I)\times_{\mathbb{F}_q}S}$ equals $\iota \circ t_I$. If S is a scheme over $\mathrm{Spec}\, A$, there is a canonical one-to-one correspondence between the isomorphism classes of Drinfeld modules of rank d over S with a Drinfeld level I structure and those of elliptic sheaves of rank d over S with a level I structure ([Dr3], [Bl-St]).

4.3.3. Let $I \subsetneqq A$ be a non-zero ideal. Let us recall the fundamental properties of the moduli scheme M^1_I.

Lemma 4.2. *The k-scheme $M^1_I \otimes_A k$ is isomorphic to the spectrum of a finite abelian extension of k which is completely split at ∞ and unramified outside the primes dividing I. The A-scheme M^1_I is identified with the normalization of A in $M^1_I \otimes_A k$.*

Proof. This follows from [Dr1, §8, Theorem 1] and its proof. $\square$

4.4. **Weil pairing morphism.** Let $I \subsetneq A$ be a non-zero ideal. Let S be a scheme over $C \setminus \operatorname{Spec}(A/I)$. Given an elliptic sheaf $(\mathcal{E}_i, t_i, j_i)_{i\in\mathbb{Z}}$ over S of rank d with a level I structure $\iota^*\mathcal{E}_0 \cong \mathcal{O}^{\oplus d}_{\operatorname{Spec}(A/I)\times_{\mathbb{F}_q} S}$, we define its determinant elliptic sheaf to be the triple $(\mathcal{F}_i, t''_i, j''_i)$, where $\mathcal{F}_i = \det(\mathcal{E}_{i-(d-1)(g-1)})$, $t''_i = \wedge^d t_{i-(d-1)(g-1)}$, and $j''_i = \wedge^d j_{i-(d-1)(g-1)}$.

Lemma 4.3. *The triple $(\mathcal{F}_i, t''_i, j''_i)_{i\in\mathbb{Z}}$ is an elliptic sheaf of rank one.*

Proof. We prove $\mathcal{F}_{i+\deg(\infty)} \cong \mathcal{F}_i(\infty)$. Other conditions in the definition of elliptic sheaves are easily checked. We may assume that S is connected and is of finite type over A. By the conditions (ii) and (iii) in the definition of elliptic sheaves, the scheme $\{\infty\}\times S$ decomposes into a disjoint union $\{\infty\}\times S = \coprod_{j=1}^{\deg(\infty)} S_j$ of $\deg(\infty)$ connected components, and for each i the $\mathcal{O}_{C\times_{\mathbb{F}_q} S}$-module $\mathcal{F}_{i+1}/\mathcal{F}_i$ is the direct image of an invertible module on S_{j_i} for some j_i. By the conditions (iii) and (iv) in the definition of elliptic sheaves, t_i induces an isomorphism ${}^{\tau}(\mathcal{F}_i/\mathcal{F}_{i-1}) \cong \mathcal{F}_{i+1}/\mathcal{F}_i$ for each i. This implies that the components $S_{j_i}, \cdots, S_{j_{i+\deg(\infty)-1}}$ are pairwise distinct. Hence $\mathcal{F}_{i+\deg(\infty)}/\mathcal{F}_i$ is the direct image of an invertible module on $\{\infty\}\times S$. Since $\mathcal{F}_{i+\deg(\infty)}/\mathcal{F}_i$ is a submodule of $\mathcal{F}_{i+d\cdot\deg(\infty)}/\mathcal{F}_i \cong \mathcal{F}_i(d\infty)/\mathcal{F}_i$, it is identified with $\mathcal{F}_i(-\infty)/\mathcal{F}_i$. □

Since $\mathcal{F}_0 = \det(\mathcal{E}_0)$, the level I structure $\iota^*\mathcal{E}_0 \cong \mathcal{O}^{\oplus d}_{\operatorname{Spec}(A/I)\times_{\mathbb{F}_q} S}$ on $(\mathcal{E}_i, t_i, j_i)_{i\in\mathbb{Z}}$ induces a level I structure $\iota^*\mathcal{F}_0 \cong \mathcal{O}_{\operatorname{Spec}(A/I)\times_{\mathbb{F}_q} S}$ on $(\mathcal{F}_i, t''_i, j''_i)_{i\in\mathbb{Z}}$.

We set $U_I = \operatorname{Spec} A \setminus \operatorname{Spec}(A/I)$. Passing to the moduli schemes, we obtain a canonical morphism $M^d_I \times_{\operatorname{Spec} A} U_I \to M^1_I \times_{\operatorname{Spec} A} U_I$. By Lemma 4.2, M^1_I is identified with the normalization of A in $M^1_I \times_{\operatorname{Spec} A} U_I$. Since M^d_I is regular (in particular normal), the morphism $M^d_I \times_{\operatorname{Spec} A} U_I \to M^1_I \times_{\operatorname{Spec} A} U_I$ is uniquely extended to a morphism $w : M^d_I \to M^1_I$, which we call the Weil pairing morphism.

4.5. **Adelic description of the Weil pairing morphism.** We denote by $\mathbb{A}$ the ring of adeles of k, by $\mathbb{A}^\infty$ the ring of finite adeles of k, and by $\widehat{A}$ the projective limit $\varprojlim_I A/I$ where I runs over all non-zero ideals of A. Let d be a positive integer. Let k_∞ denote the completion of k at ∞. Let $\mathfrak{X}_d$ be the Drinfeld symmetric space for GL_d. (When $d = 1$, this is just one point.) Then $M^d_I \otimes_A k_\infty$ has the following rigid analytic description (cf. [Dr1], [Bl-St, 4.3]):

$$(M^d_I \otimes_A k_\infty)^{\mathrm{an}} \cong \mathrm{GL}_d(k)\backslash(\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}_I \times \mathfrak{X}_d)$$

where $\mathbb{K}_I = \mathbb{K}_{d,I} = \operatorname{Ker}[\mathrm{GL}_d(\widehat{A}) \to \mathrm{GL}_d(A/I)]$. Since $\mathfrak{X}_1$ is a point, there exists a canonical morphism

$$\begin{aligned} &\mathrm{GL}_d(k)\backslash(\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}_{d,I} \times \mathfrak{X}_d) \to \mathrm{GL}_d(k)\backslash(\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}_{d,I} \times \mathfrak{X}_1) \\ &\xrightarrow{r} k^\times\backslash((\mathbb{A}^\infty)^\times/\mathbb{K}_{1,I} \times \mathfrak{X}_1) \end{aligned} \tag{4.1}$$

where the last map r is induced by the determinant homomorphism $\det : \mathrm{GL}_d(\mathbb{A}^\infty) \to (\mathbb{A}^\infty)^\times$.

Lemma 4.4. *The morphism r in (4.1) is bijective. The composite morphism in (4.1) is compatible with the Weil pairing morphism $w : M^d_I \to M^1_I$, that is, the diagram*

$$\begin{array}{ccc} (M^d_I \otimes_A k_\infty)^{\mathrm{an}} & \xrightarrow{\cong} & \mathrm{GL}_d(k)\backslash(\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}_{d,I} \times \mathfrak{X}_d) \\ {\scriptstyle w}\downarrow & & \downarrow \\ (M^1_I \otimes_A k_\infty)^{\mathrm{an}} & \xrightarrow{\cong} & k^\times\backslash((\mathbb{A}^\infty)^\times/\mathbb{K}_{1,I} \times \mathfrak{X}_1) \end{array}$$

is commutative.

Proof. The surjectivity of r is clear. We prove the injectivity. We write $\mathfrak{X}_1 = \{x\}$. Suppose that $r(\mathrm{GL}_d(k)g_1\mathbb{K}_{d,I}, x) = r(\mathrm{GL}_d(k)g_2\mathbb{K}_{d,I}, x)$. Replacing g_2 by an element in $\mathrm{GL}_d(k)g_2\mathbb{K}_{d,I}$, we may assume that $\det(g_1) = \det(g_2)$. Since $\mathrm{SL}_d(k)$ is dense in $\mathrm{SL}_d(\mathbb{A}^\infty)$ by the strong approximation theorem, the intersection $\mathrm{SL}_d(k) \cap g_1\mathbb{K}_{d,I}g_2^{-1}$ is non-empty. This implies that $\mathrm{GL}_d(k)g_1\mathbb{K}_{d,I} = \mathrm{GL}_d(k)g_2\mathbb{K}_{d,I}$. Hence r is injective.

Let $\Omega_\infty^{(d)}$ be the $\mathbb{F}_q$-scheme in [Bl-St, Definition 4.1.5]. There is a canonical specialization map $\mathrm{sp} : \mathfrak{X}_d \to \Omega_\infty^{(d)}$ (which is a continuous map of topological spaces). Let $M_{I,\infty}^d$ be the moduli stack of elliptic sheaves of rank d of "infinite characteristic", which is the fiber at ∞ of the moduli stack $M_{I,C}^d$ over C of the elliptic sheaves of rank d (cf. [Bl-St]). The definition of Weil pairing morphism is canonically extended, and gives rise to a morphism $w_C : M_{I,C}^d \to M_{I,C}^1$. Let $|M_{I,\infty}^d|$ denote the set of points of the stack $M_{I,\infty}^d$ with Zariski topology (cf. [La-Mo]). By [Bl-St, 4.1], $M_{I,\infty}^d$ is canonically isomorphic to $\mathrm{GL}_d(k)\backslash(\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}_{d,I} \times \Omega_\infty^{(d)})$. Hence the map sp induces a map $\mathrm{sp} : (M_I^d \otimes_A k_\infty)^{\mathrm{an}} \cong \mathrm{GL}_d(k)\backslash(\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}_{d,I} \times \mathfrak{X}_d) \to |M_{I,\infty}^d|$ which makes the diagram

$$\begin{array}{ccc} (M_I^d \otimes_A k_\infty)^{\mathrm{an}} & \xrightarrow{\mathrm{sp}} & |M_{I,\infty}^d| \\ {\scriptstyle w}\downarrow & & {\scriptstyle w_C}\downarrow \\ (M_I^1 \otimes_A k_\infty)^{\mathrm{an}} & \xrightarrow{\mathrm{sp}} & |M_{I,\infty}^1| \end{array}$$

commutative. Since the composite

$$(M_I^d \otimes_A k_\infty)^{\mathrm{an}} \to \mathrm{GL}_d(k)\backslash(\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}_{d,I} \times \mathfrak{X}_d) \to \mathrm{GL}_d(k)\backslash\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}_{d,I}$$

factors as

$$(M_I^d \otimes_A k_\infty)^{\mathrm{an}} \xrightarrow{\mathrm{sp}} |M_{I,\infty}^d| \to \mathrm{GL}_d(k)\backslash\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}_{d,I},$$

it suffices to show the commutativity of the diagram

$$\begin{array}{ccc} |M_{I,\infty}^d| & \longrightarrow & \mathrm{GL}_d(k)\backslash\mathrm{GL}_d(\mathbb{A}^\infty)/\mathbb{K}_{d,I} \\ {\scriptstyle w_C}\downarrow & & \downarrow \\ |M_{I,\infty}^1| & \longrightarrow & k^\times\backslash(\mathbb{A}^\infty)^\times/\mathbb{K}_{1,I}. \end{array}$$

This follows from the definition of w_C and the construction of the horizontal arrows. $\square$

5. Bad reduction of Drinfeld modular curves

We study bad reduction of Drinfeld modular curves in this section. The local study using the Serre-Tate theory is due to Drinfeld [Dr1]. The modular curves considered in Section 5.4 are the analogue of exotic Igusa curve in the elliptic modular case [Ka-Ma, p.385]. The description of deformation spaces in the case of elliptic $\mathcal{D}$-sheaves is found in [Bo]. He treats the higher dimensional cases as well.

5.1. Throughout this section we fix a prime ideal $\wp \subset A$. We denote by $A_\wp$ the $\wp$-adic completion of A. Let $\kappa(\wp)$ denote the residue field at $\wp$ of A. We fix an algebraic closure $\overline{\kappa(\wp)}$ of $\kappa(\wp)$.

Let $I \subsetneq A$ be a non-zero ideal. We write $M^d_{I/\kappa(\wp)}$ (resp. $M^d_{I/\overline{\kappa(\wp)}}$) for $M^d_I \times_{\operatorname{Spec} A} \operatorname{Spec}\kappa(\wp)$ (resp. $M^d_I \times_{\operatorname{Spec} A} \operatorname{Spec} \overline{\kappa(\wp)}$). Let $w : M^2_I \to M^1_I$ be the Weil pairing morphism on M^2_I. Let us consider the following cartesian diagram:

$$\begin{array}{ccc} M^2_I \times_{M^1_I} M^1_{I/\kappa(\wp)} & \longrightarrow & M^2_I \\ \Big\downarrow f & & \Big\downarrow w \\ M^1_{I/\kappa(\wp)} & \longrightarrow & M^1_I \end{array}$$

where the morphism at the bottom is the canonical closed immersion.

Lemma 5.1. *Suppose that I is prime to $\wp$. Then*
1. *f is smooth.*
2. *The fibers of f are geometrically connected.*

Proof. The claim (1) follows from the smoothness of the morphism $M^2_I \times_{M^1_I} M^1_{I/\kappa(\wp)} \to \operatorname{Spec}\kappa(\wp)$ and the etaleness of the morphism $M^1_{I/\kappa(\wp)} \to \operatorname{Spec}\kappa(\wp)$.

The compactification $\overline{M}^2_I$ of M^2_I, constructed by Drinfeld [Dr1, §9], is a regular scheme which is proper over A. Since the generic fiber $\overline{M}^2_I \otimes_A k$ is the smooth compactification of the curve $M^2_I \otimes_A k$, the Weil pairing morphism $M^2_I \to M^1_I$ gives a morphism $\overline{M}^2_I \otimes_A k \to M^1_I \otimes_A k$. This morphism is uniquely extended to the morphism $\overline{M}^2_I \to M^1_I$ since $\overline{M}^2_I$ is regular and M^1_I equals the normalization of A in $M^1_I \otimes_A k$ (by Lemma 4.2). The morphism $\overline{M}^2_I \to M^1_I$ is proper, since $\overline{M}^2_I \to \operatorname{Spec} A$ is proper. We apply the theory of the Stein factorization to the morphism $\overline{M}^2_I \to M^1_I$. To prove the claim (2), it suffices to prove that the generic fiber of $\overline{M}^2_I \to M^1_I$ is geometrically connected. It can be checked using the rigid analytic description of $M^2_I \otimes_A F_\infty$ in Lemma 4.4. □

5.2. **Supersingular Drinfeld modules of rank 2.** A Drinfeld module of rank 2 over $\overline{\kappa(\wp)}$ is called *ordinary* (resp. *supersingular*) if the corresponding formal $A_\wp$-module is of height 1 (resp. 2) (see [Dr1, §1] for the definition of formal $A_\wp$-modules). We refer to [Ge3, Proposition 4.1] for other equivalent definitions. For a non-zero ideal $I \subsetneq A$, a closed point in $M^2_{I/\overline{\kappa(\wp)}}$ is called ordinary (resp. supersingular) if the corresponding Drinfeld module is ordinary (resp. supersingular).

5.2.1. *Adelic description.* A supersingular Drinfeld module E of rank 2 over $\overline{\kappa(\wp)}$ is known to exist. We fix one such E and put $\mathcal{O} = \operatorname{End}(E)$. Then $\mathcal{O}$ is an A-algebra. Moreover, $B = \mathcal{O} \otimes_A k$ is a quaternion algebra over k which ramifies exactly at $\wp$ and at ∞, and $\mathcal{O}$ is a maximal A-order of B (this is due to Drinfeld ([Dr2, §2])). We say that a left $\mathcal{O}$-module M is invertible if $M \otimes_A A_{\wp'}$ is a free $\mathcal{O} \otimes_A A_{\wp'}$-module of rank one for each finite prime $\wp'$. Let M_1, M_2 be two invertible left $\mathcal{O}$-modules. An isogeny from M_1 to M_2 is an injective homomorphism $M_1 \to M_2$ of left $\mathcal{O}$-modules. We say that an isogeny $M_1 \to M_2$ is prime to $\wp$ if its cokernel has no $\wp$-torsion.

Proposition 5.2. *There exists a canonical equivalence between the following two categories.*

(1) *The category whose objects are the invertible left $\mathcal{O}$-modules, and whose morphisms are the isogenies between them.*
(2) *The category whose objects are the supersingular Drinfeld modules over $\overline{\kappa(\wp)}$, and whose morphisms are the isogenies between them.*

Moreover the notions of prime-to-$\wp$ isogeny in both categories coincide. If a prime-to-$\wp$ isogeny $M_1 \to M_2$ corresponds to a prime-to-$\wp$ isogeny $E_1 \to E_2$, then there is a canonical isomorphism between the cokernel of $M_1 \to M_2$ and the kernel of $E_1(\overline{\kappa(\wp)}) \to E_2(\overline{\kappa(\wp)})$.

Proof. The essential part of this proposition is a consequence of the results in §2 of [Dr2]. Let us remark that the description using left ideals is an analogue of Deuring's result for elliptic curves and is adapted from [Ge3]. $\square$

We fix an isomorphism $\mathcal{O} \otimes_A A_{\wp'} \cong \mathrm{Mat}_2(A_{\wp'})$ for each finite prime $\wp'$ of k different from $\wp$. For a non-zero ideal $I \subset A$ prime to $\wp$, it induces an isomorphism $\mathcal{O}/I\mathcal{O} \cong \mathrm{Mat}_2(A/I)$. Let E be a supersingular Drinfeld module of rank 2 over $\overline{\kappa(\wp)}$ and let M be the invertible left $\mathcal{O}$-module corresponding to E. Then the set of Drinfeld level I structures on E is canonically identified, as a $\mathrm{GL}_2(A/I)$-set, with the set of surjective homomorphisms $M \to \mathcal{O}/I\mathcal{O} \cong \mathrm{Mat}_2(A/I)$. We set $\widehat{A}^{\wp} = \varprojlim_J A/J$, where J runs over the non-zero ideals of A which is prime to $\wp$. We let $\mathbb{A}^{\infty,\wp}$ denote the prime-to-$\wp$-part of $\mathbb{A}^{\infty}$, which is identified with $\widehat{A}^{\wp} \otimes_A k$. We set $\widehat{\mathcal{O}}^{\wp} = \mathcal{O} \otimes_A \widehat{A}^{\wp}$. It is identified with $\mathrm{Mat}_2(\widehat{A}^{\wp})$. Let $\mathbb{K}'_I$ denote the kernel of the homomorphism $(\widehat{\mathcal{O}}^{\wp})^\times \to (\mathcal{O}/I\mathcal{O})^\times$. For $x \in B \otimes_k \mathbb{A}^{\infty,\wp}$, let M_x denote the intersection $B \cap \widehat{\mathcal{O}}^{\wp} \cdot x^{-1}$ in $B \otimes_k \mathbb{A}^{\infty,\wp}$. Then M_x is an invertible left $\mathcal{O}$-module and the composite

$$M_x \hookrightarrow \widehat{\mathcal{O}}^{\wp} \cdot x^{-1} \cong \widehat{\mathcal{O}}^{\wp} \twoheadrightarrow \mathcal{O}/I\mathcal{O}$$

gives a canonical surjection $M_x \to \mathcal{O}/I\mathcal{O}$. Hence it corresponds to a supersingular Drinfeld module of rank 2 over $\overline{\kappa(\wp)}$ with a Drinfeld level I structure. This gives an $(\mathcal{O}/I\mathcal{O})^\times \cong \mathrm{GL}_2(A/I)$-equivariant bijection between the set $\Sigma_{I,\wp}$ of the isomorphism classes of supersingular Drinfeld modules of rank 2 over $\overline{\kappa(\wp)}$ with a Drinfeld level I structure and the double coset

$$B^\times \mathbb{A}^{\infty,\wp\times} \backslash (B \otimes_k \mathbb{A}^{\infty,\wp})^\times / \mathbb{K}'_I.$$

5.3. **Local description of $M^2_{I/\overline{\kappa(\wp)}}$.** For a non-zero ideal $I \subset A$, we set $M^2_{I/\overline{\kappa(\wp)}} = M^2_{I/\kappa(\wp)} \otimes_{\kappa(\wp)} \overline{\kappa(\wp)}$.

Let $q_\wp$ denote the cardinality of $\kappa(\wp)$. We fix a prime element $\pi \in A_\wp$. The canonical projection $A_\wp \to \kappa(\wp)$ has a unique left inverse $\kappa(\wp) \to A_\wp$, which identifies $A_\wp$ with the ring $\kappa(\wp)[[\pi]]$ of formal power series.

5.3.1. *Universal deformation space of formal $A_\wp$-modules over $\overline{\kappa(\wp)}$ with a Drinfeld level structure: height one case.* We define a formal $A_\wp$-module $\widehat{F}_1$ over $A_\wp$ as follows. As a formal group, $\widehat{F}_1 = \widehat{\mathbb{G}_a}$. The action of $a \in \kappa(\wp) \subset A_\wp$ on F is given by the power series aX, and the action of π is given by the power series $f_1(X) = \pi X + X^{q_\wp}$. We put $F_1 = \widehat{F}_1 \otimes_{A_\wp} \kappa(\wp)$. Then F_1 is a formal $A_\wp$-module of height 1 over $\kappa(\wp)$. By [Dr1, Proposition 1.6], any formal $A_\wp$-module over $\overline{\kappa(\wp)}$ of height 1 is isomorphic to $F_1 \widehat{\otimes}_{\kappa(\wp)} \overline{\kappa(\wp)}$. Let $\widehat{A}^{\mathrm{ur}}_\wp$ denote the completion of the strict henselization of $A_\wp$. Then it is easily checked that the formal $A_\wp$-module $\widehat{F}_1 \widehat{\otimes}_{A_\wp} \widehat{A}_\wp^{\mathrm{ur}}$ is identified with the universal deformation of $F_1 \widehat{\otimes}_{\kappa(\wp)} \overline{\kappa(\wp)}$.

The following description of the universal deformation ring $D_{1,n}$ of $F_1 \widehat{\otimes}_{\kappa(\wp)} \overline{\kappa(\wp)}$ of level n is due to Drinfeld [Dr1, §4]. For $r \geq 0$ let $g_{1,r} = f_1 \circ \cdots \circ f_1$ be the r-th iteration of f_1. Then for $n \geq 1$ the ring $D_{1,n}$ is isomorphic to $\widehat{A}_\wp^{\mathrm{ur}}[[x]]/(g_{1,n}(x)/g_{1,n-1}(x))$. We note that the reduction modulo π of $D_{1,n}$ is isomorphic to $\overline{\kappa(\wp)}[[x]]/(x^{q_\wp^n - q_\wp^{n-1}})$.

Let $I \subsetneqq A$ be a non-zero ideal which is prime to $\wp$. We set $I_n = I\wp^n$. Let $\mathcal{P}_n$ denote the set of A-submodules of $(\wp^{-n}/A)^{\oplus 2}$ which is free of rank one over $A/\wp^n$.

Let $\overline{x} \in M^2_{I/\overline{\kappa(\wp)}}$ be an ordinary point, and let $E_{\overline{x}}$ be the corresponding Drinfeld module of rank 2 over $\overline{\kappa(\wp)}$. The following description of the formal completion $\widehat{M}^2_{I_n/\overline{\kappa(\wp)},\overline{x}}$ of $M^2_{I_n/\overline{\kappa(\wp)}}$ along the fiber $M^2_{I_n/\overline{\kappa(\wp)},\overline{x}}$ of $\overline{x}$ is due to Drinfeld. If we fix an isomorphism of the formal $A_\wp$-module associated to $E_{\overline{x}}$ to F_1, then $\widehat{M}^2_{I_n/\overline{\kappa(\wp)},\overline{x}}$ is canonically isomorphic to the disjoint union $\coprod_{Q \in \mathcal{P}_n} \mathrm{Spf}\,(D_{1,n}/(\pi)[[y]])$ of copies of $\mathrm{Spf}\,(D_{1,n}/(\pi)[[y]]) \cong \mathrm{Spf}\,\overline{\kappa(\wp)}[[x,y]]/(x^{q_\wp^n - q_\wp^{n-1}})$.

5.3.2. *Universal deformation space of formal $A_\wp$-modules over $\overline{\kappa(\wp)}$ with a Drinfeld level structure: height two case.* We define a formal $A_\wp$-module $\widehat{F}_2$ over the ring $A_\wp[[t]]$ of formal power series as follows. As a formal group, $\widehat{F}_2 = \widehat{\mathbb{G}_a}$. The action of $a \in \kappa(\wp) \subset A_\wp$ on F is given by the power series aX, and the action of π is given by the power series $f_2(X) = \pi X + tX^{q_\wp} + X^{q_\wp^2}$. We put $F_2 = \widehat{F}_2 \otimes_{A_\wp[[t]]} \kappa(\wp)$. Then F_2 is a formal $A_\wp$-module of height 2 over $\kappa(\wp)$. By [Dr1, Proposition 1.6], any formal $A_\wp$-module over $\overline{\kappa(\wp)}$ of height 2 is isomorphic to $F_2 \widehat{\otimes}_{\kappa(\wp)} \overline{\kappa(\wp)}$.

Lemma 5.3. *The formal $A_\wp$-module $\widehat{F}_2 \widehat{\otimes}_{A_\wp[[t]]} \widehat{A}_\wp^{\mathrm{ur}}[[t]]$ is identified with the universal deformation of $F_2 \widehat{\otimes}_{\kappa(\wp)} \overline{\kappa(\wp)}$.*

Proof. By [Dr1, Proposition 4.2], the universal deformation ring of the formal $A_\wp$-module F_2 equals $\widehat{A}_\wp^{\mathrm{ur}}[[t_1]]$. The universality gives the homomorphism $\varphi : \widehat{A}_\wp^{\mathrm{ur}}[[t_1]] \to \widehat{A}_\wp^{\mathrm{ur}}[[t]]$. Since the coefficient of X in f_2 is π, φ is a homomorphism of $\widehat{A}_\wp^{\mathrm{ur}}$-algebra. It suffices to prove that the image $\varphi(t_1)$ is a topological generator of $\widehat{A}_\wp^{\mathrm{ur}}[[t]]$. Let $\Lambda_{A_\wp}$ (resp. $\widetilde{\Lambda}_{A_\wp}$) be the graded ring Λ_O (resp. $\widetilde{\Lambda}_O$) defined in [Dr1, §1] for $O = A_\wp$. In [Dr1, Proposition 1.4], Drinfeld shows that $\Lambda_{A_\wp}$ equals the polynomial ring with generators $g_1, g_2, \cdots \in \Lambda_{A_\wp}$, $\deg g_i = i$. In view of his construction of the g_j's, we may choose $g_{q_\wp - 1}$ in such a way that the image of $g_{q_\wp-1}$ in $\widetilde{\Lambda}_{A_\wp}$ equals the element u in [Dr1, Proposition 1.3 (2)] for $n = q_\wp$. By the construction of $\widehat{A}_\wp^{\mathrm{ur}}[[t_1]]$, there exists a canonical ring homomorphism $\Lambda_{A_\wp} \to \widehat{A}_\wp^{\mathrm{ur}}[[t_1]]$ such that the image of $g_{q_\wp-1}$ equals t_1. This implies that $\varphi(t_1) = -t/(1-\pi^{q_\wp-1}) + O(t^2)$. $\square$

The following description of the universal deformation ring $D_{2,n}$ of $F_2 \widehat{\otimes}_{\kappa(\wp)} \overline{\kappa(\wp)}$ of level n is due to Drinfeld [Dr1, §4]:

$$D_{2,1} \cong \widehat{A}_\wp^{\mathrm{ur}}[[t, \theta_1, \theta_2]]/(\pi + t\theta_1^{q_\wp - 1} + \theta_1^{q_\wp^2 - 1}, t + \frac{\theta_2^{q_\wp^2-1} - \theta_1^{q_\wp^2-1}}{\theta_2^{q_\wp-1} - \theta_1^{q_\wp-1}}).$$

For $r \geq 0$ let $g_{2,r} = f_2 \circ \cdots \circ f_2$ be the r-th iteration of f_n. Then for $n \geq 1$ the ring $D_{2,n}$ is isomorphic to

$$D_{2,1}[[y_1, y_2]]/(g_{2,n-1}(y_1) - \theta_1, g_{2,n-1}(y_2) - \theta_2).$$

Let $\overline{x} \in M^2_{I/\overline{\kappa(\wp)}}$ be a supersingular point, $E_{\overline{x}}$ be the corresponding Drinfeld module of rank 2 over $\overline{\kappa(\wp)}$. The following description of the formal completion $\widehat{M}^2_{I_n/\overline{\kappa(\wp)},\overline{x}}$ of $M^2_{I_n/\overline{\kappa(\wp)}}$ along the fiber $M^2_{I_n/\overline{\kappa(\wp)},\overline{x}}$ of $\overline{x}$ is due to Drinfeld. If we fix an isomorphism of the formal $A_\wp$-module associated to $E_{\overline{x}}$ to F_2, then $\widehat{M}^2_{I_n/\overline{\kappa(\wp)},\overline{x}}$ is canonically isomorphic to $\mathrm{Spf}\,(D_{2,n}/(\pi))$.

5.3.3. Let $I \subsetneq A$ be a non-zero ideal which is prime to $\wp$. As in Section 5.2.1, we denote by $\Sigma_{I,\wp}$ the set of supersingular points of $M^2_{I/\overline{\kappa(\wp)}}$. Let $S_{I,\wp}$ denote the set of connected components of $M^2_{I/\overline{\kappa(\wp)}}$. There is a canonical map $\Sigma_{I,\wp} \to S_{I,\wp}$.

We fix an algebraic closure $\overline{k}_\infty$ of k_∞. For an integer $n \geq 0$, let $\mathrm{Cusp}_{I,\wp^n}$ denote the set of cusps of $M^2_{I\wp^n} \otimes_A \overline{k}_\infty$, that is, the set of the complement of $M^2_{I\wp^n} \otimes_A \overline{k}_\infty$ in its smooth compactification.

Let $\Sigma_{\lim,\wp}$ (resp. $S_{\lim,\wp}$, resp. $\mathrm{Cusp}_{\lim,\wp^n}$) denote the profinite set $\varprojlim_I \Sigma_{I,\wp}$ (resp. $\varprojlim_I S_{I,\wp}$, resp. $\varprojlim_I \mathrm{Cusp}_{I,\wp^n}$) where I runs over the non-zero ideals $I \subsetneq A$. There is a canonical morphism $\Sigma_{\lim,\wp} \to S_{\lim,\wp}$ of profinite sets. The group $\mathrm{GL}_2(\mathbb{A}^{\infty,\wp})$ acts continuously on the sets $\Sigma_{\lim,\wp}$, $S_{\lim,\wp}$, and the above canonical morphism is $\mathrm{GL}_2(\mathbb{A}^{\infty,\wp})$-equivariant.

By the description of $\Sigma_{I,\wp}$ given in Section 5.2.1, we have an isomorphism

$$\Sigma_{\lim,\wp} \cong B^\times \mathbb{A}^{\infty,\wp\times} \backslash (B \otimes_A \mathbb{A}^{\infty,\wp})^\times$$

of $(B \otimes_A \mathbb{A}^{\infty,\wp})^\times \cong \mathrm{GL}_d(\mathbb{A}^{\infty,\wp})$-sets.

Lemma 5.4. *Let $T \subset \mathrm{GL}_2$ (resp. $N \subset \mathrm{GL}_2$) denote the diagonal torus (resp. the subgroup of upper triangular unipotent matrices) of GL_2.*

(1) *There are canonical isomorphisms*

$$\begin{aligned} S_{I,\wp} &\cong k^\times \backslash \mathbb{A}^\infty / \mathrm{Ker}(\widehat{A}^\times \to (A/I)^\times), \\ \mathrm{Cusp}_{I,\wp^n} &\cong T(k)N(\mathbb{A}^\infty) \backslash \mathrm{GL}_2(\mathbb{A}^\infty)/\mathbb{K}_{I\wp^n}. \end{aligned}$$

(2) *There are canonical isomorphisms*

$$\begin{aligned} S_{\lim,\wp} &\cong k^\times \backslash \mathbb{A}^\infty / A_\wp^\times, \\ \mathrm{Cusp}_{\lim,\wp^n} &\cong T(k)N(\mathbb{A}^\infty) \backslash \mathrm{GL}_2(A^\infty)/(1 + \wp^n \mathrm{Mat}_2(A_\wp)) \end{aligned}$$

of $\mathrm{GL}_2(\mathbb{A}^{\infty,\wp})$-sets. Here $\mathrm{GL}_2(\mathbb{A}^{\infty,\wp})$ acts on $\mathbb{A}^\infty/A_\wp^\times$ via $\det : \mathrm{GL}_2(\mathbb{A}^{\infty,\wp}) \to (\mathbb{A}^{\infty,\wp})^\times$.

Proof. The set of cusps $\mathrm{Cusp}_{I,\wp^n}$ is isomorphic to the double coset $\mathrm{GL}_2(k)\backslash(\mathrm{GL}_s(\mathbb{A}^\infty)/\mathbb{K}_{I\wp^n} \times \mathbb{P}^1(k)) \cong T(k)N(k)\backslash \mathrm{GL}_2(\mathbb{A}^\infty)/\mathbb{K}_{I\wp^n}$ (see, for example, [Ge2]). The claim follows since $N(k)$ is dense in $N(\mathbb{A}^\infty)$.

We prove the claim for $S_{I,\wp}$. By Lemma 5.1, $S_{I,\wp}$ is identified with the set of connected components of $M^1_{I/\overline{\kappa(\wp)}}$. Since $M^1_I \times_{\mathrm{Spec}\,A} U_I$ is finite etale, it is also isomorphic to the set of connected components of $M^1_I \otimes_A \overline{k}_\infty$ By the adelic description of $(M^1_I \otimes_A k_\wp)^{\mathrm{an}}$, it is also isomorphic to the double coset

$$k^\times \backslash \mathbb{A}^\infty / \mathrm{Ker}(\widehat{A}^\times \to (A/I)^\times).$$

This proves (1). Passing to the projective limit, we have (2). □

5.4. Fix an integer $n \geq 1$. For a non-zero ideal $I \subsetneq A$ which is prime to $\wp$, we put $I_n = I\wp^n$. Let us describe the reduction at $\wp$ of the moduli scheme $M^2_{I_n}$.

For $Q \in \mathcal{P}_n$, we let $M^2_{I_n/\kappa(\wp),Q} \subset M^2_{I_n/\kappa(\wp)}$ denote the closed subscheme which classifies the Drinfeld modules of rank 2 with a Drinfeld level I_n structure ϕ such that $Q \subset \operatorname{Ker}\phi$. We set $M^2_{I_n/\overline{\kappa(\wp)},Q} = M^2_{I_n/\kappa(\wp),Q} \times_{\kappa(\wp)} \overline{\kappa(\wp)}$.

Lemma 5.5. (1) *As a topological space $M^2_{I_n/\kappa(\wp)}$ equals the union of $M^2_{I_n/\kappa(\wp),Q}$'s.*

(2) *For $Q, Q' \in \mathcal{P}_n$ with $Q \neq Q'$, $M^2_{I_n/\kappa(\wp),Q}$ intersects $M^2_{I_n/\kappa(\wp),Q'}$ only at supersingular points.*

(3) *$M^2_{I_n/\kappa(\wp),Q}$ is a smooth curve over $\kappa(\wp)$.*

(4) *The morphism $M^2_{I_n/\kappa(\wp),Q} \to M^2_{I/\kappa(\wp)}$ is finite, flat, surjective, of constant degree and is totally ramified at every supersingular point on $M^2_{I/\kappa(\wp)}$. This also induces a bijection between the set of connected components of $M^2_{I_n/\overline{\kappa(\wp)},Q}$ and that of $M^2_{I/\overline{\kappa(\wp)}}$.*

Proof. We will prove the following four statements for every closed point $\overline{x} \in M^2_{I/\overline{\kappa(\wp)}}$:

(1)′ The fiber $M^2_{I_n/\overline{\kappa(\wp)},\overline{x}}$ of $M^2_{I_n/\overline{\kappa(\wp)}}$ at $\overline{x}$ is non-empty and is the union of $M^2_{I_n/\overline{\kappa(\wp)},Q,\overline{x}}$'s.

(2)′ If $\overline{x}$ is an ordinary point, then $M^2_{I_n/\overline{\kappa(\wp)},\overline{x}}$ is the disjoint union of $M^2_{I_n/\overline{\kappa(\wp)},Q,\overline{x}}$'s.

(3)′ The scheme $M^2_{I_n/\overline{\kappa(\wp)},Q}$ is smooth over $\overline{\kappa(\wp)}$ in a neighborhood of $M^2_{I_n/\overline{\kappa(\wp)},Q,\overline{x}}$.

(4)′ If $\overline{x}$ is a supersingular point, then the morphism $M^2_{I_n/\overline{\kappa(\wp)},Q} \to M^2_{I/\overline{\kappa(\wp)},Q}$ is totally ramified at $\overline{x}$, whose ramification index does not depend on the choice of $\overline{x}$.

It is clear that the assertions (1), (2) and (3) follow from (1)′, (2)′, and (3)′. Let us denote by f the morphism $M^2_{I_n/\kappa(\wp),Q} \to M^2_{I/\kappa(\wp)}$ in the statement of (4). Then f is finite flat since $M^2_{I\wp^n} \to M^2_I$ is finite flat. By (1), f is surjective. The remaining assertions in (4) follow from (4)′ and the existence of a supersingular point on each connected component of $M^2_{I/\overline{\kappa(\wp)}}$.

Let $E \to \overline{x}$ denote the Drinfeld module of rank 2 over $\overline{x}$ corresponding to the geometric point $\overline{x}$. Then the $\wp^k$-torsion subgroup of the A-module $E(\overline{x})$ is a free $A/\wp^n$-module of rank ≤ 1. Hence for any A-module homomorphism $(J^{-1}/A)^{\oplus 2} \to E(\overline{x})$, its kernel contains an element Q in $\mathbb{P}^1(A/\wp^n)$. This implies (1)′.

The group $\mathrm{GL}_2(A/\wp)$ acts both on $\mathcal{P}_n$ and on $M^2_{I_n/\wp}$. The action of $g \in \mathrm{GL}_2(A/\wp)$ on $M^2_{I_n/\wp}$ maps $M^2_{I_n/\wp,Q}$ isomorphically onto $M^2_{I_n/\wp,gQ}$. Since the group $\mathrm{GL}_2(A/\wp^n)$ acts transitively on $\mathcal{P}_N \cong \mathbb{P}^1(A/\wp^n)$, all $M^2_{I_n/\wp,Q}$'s are isomorphic as $M^2_{I/\wp}$-schemes. Hence it suffices to show that (3) and (4) hold for $Q = Q_0$, where Q_0 is the second direct summand $\wp^{-n}/A$ of $(\wp^{-n}/A)^{\oplus 2}$.

It follows from the description in Section 5.3.1 that the completion of $M^2_{I_n/\overline{\kappa(\wp)},Q_0}$ at the ordinary point equals

$$(D_{1,n} \otimes_{\widehat{A}^{\mathrm{ur}}_\wp} \overline{\kappa(\wp)})[[y]]/(x) \cong \overline{\kappa(\wp)}[[y]].$$

This proves (2)′.

The claim (3)′ for ordinary $\overline{x}$ immediately follows from the argument in the proof of (2)′. Suppose that $\overline{x}$ is a supersingular point. It follows from the description in Section 5.3.2 that the completion of $M^2_{I_n/\overline{\kappa(\wp)},Q_0}$ along the fiber of $\overline{x}$ equals

$$(D_{2,n} \otimes_{\widehat{A}^{\mathrm{ur}}_\wp} \overline{\kappa(\wp)})/(y_2) = \overline{\kappa(\wp)}[[t,\theta_1,y_1]]/(t+\theta_1^{q_\wp^2-q_\wp}, g_{2,n}(y_1)-\theta_1).$$

From this description, we easily see that the canonical homomorphism $\overline{\kappa(\wp)}[[y_1]] \to (D_{2,n} \otimes_{\widehat{A}^{\mathrm{ur}}_\wp} \overline{\kappa(\wp)})/(y_2)$ is an isomorphism. Hence $M^2_{I_n/\wp,Q_0}$ is smooth over $\overline{\kappa(\wp)}$ in an neighborhood of $M^2_{I_n/\wp,Q_0,\overline{x}}$, and $M^2_{I_n/\wp,Q_0} \to M^2_{I/\wp}$ is totally ramified at $\overline{x}$. This proves $(3)'$.

Let $\overline{x}$ be a supersingular point. Then, using the notations in the proof of $(3)'$, the ring homomorphism from the completion of $M^2_{I/\overline{\kappa(\wp)}}$ at $\overline{x}$ to the completion of $M^2_{I_n/\overline{\kappa(\wp)},Q_0}$ along the fiber of $\overline{x}$ is identified with the homomorphism

$$\overline{\kappa(\wp)}[[t]] \to (D_{2,n} \otimes_{\widehat{A}^{\mathrm{ur}}_\wp} \overline{\kappa(\wp)})/(y_2) \cong \overline{\kappa(\wp)}[[y_1]].$$

This homomorphism is totally ramified and its ramification index is independent of the choice of $\overline{x}$. This proves $(4)'$. □

5.5. Fix a nonzero ideal $I \subsetneqq A$ which is prime to $\wp$. Let $n \geq 0$ be an integer. We write $I_n = I\wp^n$. Let us consider the compactification $\overline{M}^2_{I_n}$ of $M^2_{I_n}$ constructed by Drinfeld [Dr1, §9].

Lemma 5.6. *For $Q \in \mathcal{P}_n$, let $\overline{M}^2_{I_n/\kappa(\wp),Q} \subset \overline{M}^2_{I_n}$ denote the closure of $M^2_{I_n/\kappa(\wp),Q}$. Then*

(1) *$\overline{M}^2_{I_n/\kappa(\wp),Q}$ is a projective smooth curve over $\kappa(\wp)$.*
(2) *For $Q' \in \mathcal{P}_n$ with $Q \neq Q'$, $\overline{M}^2_{I_n/\kappa(\wp),Q}$ and $\overline{M}^2_{I_n/\kappa(\wp),Q'}$ do not intersect at a point in the boundary $\overline{M}^2_{I_n} \setminus M^2_{I_n}$.*

Proof. First we note that, to prove (1), it suffices to prove that $\overline{M}^2_{I_n/\kappa(\wp),Q}$ is nonsingular at the boundary $\overline{M}^2_{I_n/\kappa(\wp),Q} \setminus M^2_{I_n/\kappa(\wp),Q}$.

Since $M^2_{I_n/\kappa(\wp),Q}$ is a smooth curve and $\overline{M}^2_{I_n}$ is regular, $\overline{M}^2_{I_n/\kappa(\wp),Q}$ is reduced. Hence both (1) and (2) follow if we prove that $(\overline{M}^2_{I_n} \otimes_A \kappa(\wp))_{\mathrm{red}}$ is regular at the boundary.

We know that the boundary $\overline{M}^2_{I_n} \setminus M^2_{I_n}$ is a disjoint union of finite number of copies of M^1_I. Let us write $M^1_I = \operatorname{Spec} R$. Then for each component of $\overline{M}^2_{I_n} \setminus M^2_{I_n}$, the completion of $\overline{M}^2_{I_n}$ along it is isomorphic to $R[[t]]$ (cf. [Leh, Chapter 5]). The claim follows since $(R \otimes_A \kappa(\wp)[[t]])_{\mathrm{red}}$ is regular. □

6. Integrality

We give the Drinfeld modular analogue of the result of Beilinson on integrality in the form presented by Schappacher and Scholl [Sc-Sc, Section 7].

6.1. Let M be an abelian group. We write $M[\Sigma_{I,\wp}]$, $M[S_{I,\wp}]$, for the groups of M-valued functions on the sets $\Sigma_{I,\wp}$, $S_{I,\wp}$, respectively. The canonical map $\Sigma_{I,\wp} \to S_{I,\wp}$ induces the homomorphism $\gamma_I : M[S_{I,\wp}] \to M[\Sigma_{I,\wp}]$. We set $\mathcal{C}(\Sigma_{\lim,\wp}, M) = \varinjlim_I M[\Sigma_{I,\wp}]$ and similarly define $\mathcal{C}(S_{\lim,\wp}, M)$, $\mathcal{C}(\mathrm{Cusp}_{\lim,\wp}, M)$, $\gamma_{\lim}$.

Lemma 6.1. *The cokernel of* $\gamma_{\lim} : \mathcal{C}(S_{\lim,\wp}, \mathbb{C}) \to \mathcal{C}(\Sigma_{\lim,\wp}, \mathbb{C})$ *is a direct sum of irreducible admissible representations of* $\mathrm{GL}_d(\mathbb{A}^{\infty,\wp})$*. Each direct summand is cuspidal automorphic, that is, isomorphic to the* $\mathbb{A}^{\infty,\wp}$*-component of a cuspidal irreducible automorphic representations of* $\mathrm{GL}_d(\mathbb{A})$

Proof. This follows from the adelic description of $\Sigma_{\lim,\wp}$ given in Section 5.2.1 and the argument in [Ja-La, Section 14] related to Jacquet-Langlands correspondence. ☐

6.2. For $Q \in \mathcal{P}_n$, let $M^{2,Q}_{I_n}$ denote the complement

$$M^{2,Q}_{I_n} = M^2_{I_n} \otimes_A A_\wp \setminus \bigcup_{Q' \in \mathcal{P}_n,\, Q' \neq Q} M^2_{I_n,\kappa(\wp),Q'}.$$

Let $M^{2,\mathrm{ord}}_{I_n/\kappa(\wp),Q}$ denote the ordinary locus of $M^2_{I_n/\kappa(\wp),Q}$. It is a regular closed subscheme of $M^{2,Q}_{I_n}$ and its open complement equals $M^2_{I_n} \otimes_A k_\wp$. The localization sequence induces the homomorphism

$$\partial_{I_n,Q} : K_2(M^2_{I_n} \otimes_A k_\wp) \to K_1(M^{2,\mathrm{ord}}_{I_n/\kappa(\wp),Q}) \cong \mathcal{O}(M^{2,\mathrm{ord}}_{I_n/\kappa(\wp),Q})^\times.$$

Here the last isomorphism follows from [Ba-Mi-Se, Corollary 4.3].

6.2.1. Take a prime element $\pi \in A_\wp$. Let us consider the composite

$$\begin{aligned}\widetilde{\mu}_{I_n,Q} :\ & \mathcal{O}(M^2_{I_n} \otimes_A k)^\times \hookrightarrow \mathcal{O}(M^2_{I_n} \otimes_A k_\wp)^\times \xrightarrow{\{\ ,\pi\}} K_2(M^2_{I_n} \otimes_A k_\wp) \\ & \xrightarrow{\partial_{I_n,Q}} \mathcal{O}(M^{2,\mathrm{ord}}_{I_n/\kappa(\wp),Q})^\times.\end{aligned}$$

Lemma 6.2. *For* $u \in \mathcal{O}(M^2_{I_n} \otimes_A k)^\times$*,* $\widetilde{\mu}_{I_n,Q}(u)$ *has the same order of pole or zero at each supersingular point of* $M^2_{I_n/\kappa(\wp),Q}$*.*

Proof. Let us consider the composite

$$\mathcal{O}(M^2_{I_n} \otimes_A k)^\times \xrightarrow{\widetilde{\mu}_{I_n,Q}} \mathcal{O}(M^{2,\mathrm{ord}}_{I_n,\wp,Q})^\times \xrightarrow{\mathrm{div}} \mathbb{Z}[\Sigma_{I,\wp}].$$

We prove that its image lies in the image of $\gamma_I : \mathbb{Z}[S_{I,\wp}] \to \mathbb{Z}[\Sigma_{I,\wp}]$.

Consider the composite with the quotient map, and extend the scalars:

$$\mathcal{O}(M^2_{I_n} \otimes_A k)^\times \to \mathbb{C}[\Sigma_{I,\wp}]/\mathbb{C}[S_{I,\wp}].$$

We prove that this map is zero.

We write the above map as the composite

$$\mathcal{O}(M^2_{I_n} \otimes_A k)^\times \to \mathcal{O}(M^{2,\mathrm{ord}}_{I_n/\kappa(\wp),Q})^\times \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{C}[\Sigma_{I,\wp}]/\mathbb{C}[S_{I,\wp}]. \tag{6.1}$$

By Lemmas 5.1 and 5.5, $M^2_{I_n} \otimes_A k$ is a connected smooth curve over k. Let $k(I_n)$ denote its field of constants. Then it is easily checked that the first map of (6.1) factors through the quotient $\mathcal{O}(M^2_{I_n} \otimes_A k)^\times / k(I_n)^\times$. Let $J, J' \subset A$ be two non-zero ideals such that J, J' are prime to $\wp$ and J divides J'. Let us consider a diagram

$$(J^{-1}/A)^{\oplus 2} \xleftarrow{\varpi} N \xrightarrow{\iota} (J'^{-1}/A)^{\oplus 2}$$

in the category of A-modules such that ι is injective and ϖ is surjective. To a Drinfeld module E of rank 2 over an $A_\wp$-scheme S with a Drinfeld level J' structure $\phi : (J'^{-1}/A)^{\oplus 2} \to E(S)$, we associate the Drinfeld module $E' = E/\phi \circ \iota(\mathrm{Ker}\,\varpi)$ and endow E' with the canonical level J

structure $(J^{-1}/A)^{\oplus 2} \xrightarrow{\bar\iota} (J'^{-1}/A)^{\oplus 2}/\iota(\operatorname{Ker}\varpi) \to E'(S)$ where $\bar\iota$ is the homomorphism induced by ι. It defines the morphism $M^2_{J'} \otimes_A A_\wp \to M^2_J \otimes_A A_\wp$, inducing a finite flat morphism $M^{2,Q}_{J'_n} \to M^{2,Q}_{J_n}$. Since $M^{2,\mathrm{ord}}_{J'_n/\kappa(\wp),Q}$ is canonically isomorphic to the fiber product $M^{2,\mathrm{ord}}_{J_n/\kappa(\wp),Q} \times_{M^2_J} M^2_{J'}$, the diagram

$$\begin{array}{ccccccc}
\mathcal{O}(M^2_J \otimes_A k)^\times & \xrightarrow{\{\ ,\pi\}} & K_2(M^2_{J_n} \otimes_A k_\wp) & \xrightarrow{\partial_{J_n,Q}} & \mathcal{O}(M^{2,\mathrm{ord}}_{J_n/\kappa(\wp),Q})^\times & \to & \mathbb{C}[\Sigma_{J,\wp}]/\mathbb{C}[S_{J,\wp}] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}(M^2_{J'} \otimes_A k)^\times & \xrightarrow{\{\ ,\pi\}} & K_2(M^2_{J'_n} \otimes_A k_\wp) & \xrightarrow{\partial_{J'_n,Q}} & \mathcal{O}(M^{2,\mathrm{ord}}_{J'_n/\kappa(\wp),Q})^\times & \to & \mathbb{C}[\Sigma_{J',\wp}]/\mathbb{C}[S_{J',\wp}]
\end{array}$$

is commutative. Passing to the inductive limit with respect to the inclusion $(J^{-1}/A)^{\oplus 2} = (J^{-1}/A)^{\oplus 2} \hookrightarrow (J'^{-1}/A)^{\oplus 2}$ we have a homomorphism

$$\mu_{\lim,Q} : \varinjlim_{J,\,\wp\nmid J} \mathcal{O}(M^2_{J_n} \otimes_A k)^\times/k^\times_{J_n} \to \varinjlim_{J,\,\wp\nmid J} \mathcal{O}(M^{2,\mathrm{ord}}_{J_n/\kappa(\wp),Q})^\times \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathcal{C}(\Sigma_{\lim,\wp}, \mathbb{C})/\mathcal{C}(S_{\lim,\wp}, \mathbb{C}),$$

of $\mathrm{GL}_2(\mathbb{A}^{\infty,\wp})$-modules. It suffices to prove that the image of $\mu_{\lim,Q}$ is zero. The group $\mathcal{O}(M^2_{J_n} \otimes_A k)^\times/k^\times_{J_n}$ is canonically regarded as a subgroup of $\mathbb{Z}[\mathrm{Cusp}_{J,\wp^n}]$. Passing to the inductive limit, we have an injective homomorphism

$$\mathcal{O}(M^2_{J_n} \otimes_A k)^\times/k^\times_{J_n} \to \mathcal{C}(\mathrm{Cusp}_{\lim,\wp^n}, \mathbb{Z}),$$

from which we see that Image $\mu_{\lim,Q} \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to a subquotient of $\mathcal{C}(\mathrm{Cusp}_{\lim,\wp^n}, \mathbb{C})$. By the adelic description of $\mathrm{Cusp}_{\lim,\wp^n}$ in Lemma 5.4, no irreducible subquotient of Image $\mu_{\lim,Q} \otimes_{\mathbb{Z}} \mathbb{C}$ is cuspidal automorphic. By Lemma 6.1, any irreducible subquotient of $\mathcal{C}(\Sigma_{\lim,\wp}, \mathbb{C})/\mathcal{C}(S_{\lim,\wp}, \mathbb{C})$ of $\mu_{\lim,Q}$ is cuspidal automorphic. Hence $\mu_{\lim,Q} \otimes_{\mathbb{Z}} \mathbb{C}$ is zero. □

Lemma 5.6 together with Lemma 6.2 implies the following corollary (cf. [Sc-Sc, Section 7]):

Corollary 6.3. *Let $x \in K_2(M^2_{I_n} \otimes_A k)$ be an element which lies in both the image of the symbol map $\mathcal{O}(M^2_{I_n} \otimes_A k)^\times \otimes \mathcal{O}(M^2_{I_n} \otimes_A k)^\times \to K_2(M^2_{I_n} \otimes_A k)$ and the kernel of the boundary map $K_2(M^2_{I_n} \otimes_A k) \to K_1(\overline{M}^2_{I_n} \setminus M^2_{I_n}) \otimes \mathbb{Q}$. Then for any $Q \in \mathcal{P}_n$, the element $\partial_{I_n,Q}(x) \in \mathcal{O}(M^{2,\mathrm{ord}}_{I_n/\kappa(\wp),Q})^\times$ is of finite order.* □

7. A non-vanishing result

The aim of this section is to show that there exist certain elements in the K_2 of the compactification of Drinfeld modular curves such that the boundary at the infinity prime is nontrivial. We start with elements in the (open) Drinfeld modular curves such that the image under the regulator is nontrivial. The tasks are then to apply Bloch's method, and to compare the regulator map and the Chern class map.

7.1. Let k, E, C be as in Section 1. We fix a prime ∞ of k at which E has split multiplicative reduction. Let $A = H^0(C \setminus \{\infty\}, \mathcal{O}_C)$. We use the notations in Section 4 and Section 5. We also fix a separable closure $\overline{k}_\infty$ of k_∞.

Let $M^2_{\lim,k}$ (resp. $\overline{M}^2_{\lim,k}$) denote the projective limit $\varprojlim_I (M^2_I \otimes_A k)$ (resp. $\varprojlim_I (\overline{M}^2_I \otimes_A k)$) of schemes. The function field analogue of Shimura-Taniyama conjecture is proved by Drinfeld, and is worked out in detail by Gekeler and Reversat ([Ge-Re]). Thus there exists a non-constant morphism $\varphi_E : \overline{M}^2_{\lim,k} \to E$ of k-schemes, which is called "Weil uniformization" or "modular parametrization". The morphism φ_E factors through the canonical projection $\overline{M}^2_{\lim,k} \to \overline{M}^2_I \otimes_A k$ for sufficiently small I. For $J \subset I$, let $\varphi_{E,J}$ denote the composite $\overline{M}^2_J \otimes_A k \to \overline{M}^2_I \otimes k \to E$.

We fix a prime number ℓ different from p. For each prime $\wp$ of k, we put

$$H^2_{\mathrm{et}}(M^2_{\lim,k_\wp}, \mathbb{Q}_\ell(2)) = \varinjlim_I H^2_{\mathrm{et}}(M^2_I \otimes_A k_\wp, \mathbb{Q}_\ell(2)),$$
$$H^2_{\mathrm{et}}(\overline{M}^2_{\lim,k_\wp}, \mathbb{Q}_\ell(2)) = \varinjlim_I H^2_{\mathrm{et}}(\overline{M}^2_I \otimes_A k_\wp, \mathbb{Q}_\ell(2)).$$

The homomorphism

$$\frac{1}{\deg(\varphi_{E,J})} \cdot \varphi_{E,J_*} : H^2_{\mathrm{et}}(\overline{M}^2_J \otimes_A k_\wp, \mathbb{Q}_\ell(2)) \to H^2_{\mathrm{et}}(E \otimes_k k_\wp, \mathbb{Q}_\ell(2))$$

gives rise to the homomorphism $H^2_{\mathrm{et}}(\overline{M}^2_{\lim,k_\wp}, \mathbb{Q}_\ell(2)) \to H^2_{\mathrm{et}}(E \otimes_k k_\wp, \mathbb{Q}_\ell(2))$ which we denote by $\varphi_{*,\wp,\mathrm{et}}$.

Lemma 7.1. *The homomorphism $\varphi_{*,\wp,\mathrm{et}}$ is surjective.*

Proof. Let us consider the morphism $\varphi_{E,I}$ for a sufficiently small I. Take a separable closure $\overline{k}_\wp$ of $k_\wp$. We have a commutative diagram

$$\begin{array}{ccc} H^2_{\mathrm{et}}(\overline{M}^2_I \otimes_A k_\wp, \mathbb{Q}_\ell(2)) & \longrightarrow & H^1(k_\wp, H^1_{\mathrm{et}}(\overline{M}^2_I \otimes_A \overline{k}_\wp, \mathbb{Q}_\ell(2))) \\ {\scriptstyle \varphi_{E,I_*}}\downarrow & & \downarrow \\ H^2_{\mathrm{et}}(E \otimes_k k_\wp, \mathbb{Q}_\ell(2)) & \longrightarrow & H^1(k_\wp, H^1_{\mathrm{et}}(E \otimes_k \overline{k}_\wp, \mathbb{Q}_\ell(2))). \end{array}$$

Since $\varphi_{E,I}$ is non-constant, the homomorphism

$$H^1_{\mathrm{et}}(\overline{M}^2_I \otimes_A \overline{k}_\wp, \mathbb{Q}_\ell(2)) \to H^1_{\mathrm{et}}(E \otimes_k \overline{k}_\wp, \mathbb{Q}_\ell(2))$$

of $\mathrm{Gal}(\overline{k}_\wp/k_\wp)$-modules is split surjective. Hence φ_{E,I_*} is surjective. □

For each prime $\wp$ of k, the homomorphism

$$K_2(\overline{M}^2_I \otimes_A k) \to K_2(\overline{M}^2_I \otimes_A k_\wp) \xrightarrow{c_{2,2}} H^2_{\mathrm{et}}(\overline{M}^2_I \otimes_A k_\wp, \mathbb{Q}_\ell(2))$$

gives rise to the homomorphism $K_2(\overline{M}^2_{\lim,k}) \to H^2_{\mathrm{et}}(\overline{M}^2_{\lim,k_\wp}, \mathbb{Q}_\ell(2))$ which we denote by $c_{2,2,\wp}$. For each non-zero ideal $I \subsetneq A$, let $W_I = \mathrm{Image}[\mathcal{O}(M^2_I \otimes_A k)^\times \otimes \mathcal{O}(M^2_I \otimes_A k)^\times \to K_2(M^2_I \otimes_A k)]$. We let $W_{0,I}$ denote the inverse image of W_I by the homomorphism $K_2(\overline{M}^2_I \otimes_A k) \to K_2(M^2_I \otimes_A k)$. We put $W = \varinjlim_I W_I$, $W_0 = \varinjlim_I W_{0,I}$. By Corollary 6.3, it follows that $c_{2,2,\wp}(W_0) = 0$ for $\wp \neq \infty$. The aim of this section is to prove the following proposition.

Proposition 7.2. *The composite*

$$W_0 \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to K_2(\overline{M}^2_{\lim,k}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{c_{2,2,\infty}} H^2_{\mathrm{et}}(\overline{M}^2_{\lim,k_\infty}, \mathbb{Q}_\ell(2))$$

is surjective.

Proof. Let $\mathcal{BT}$ denote the Bruhat-Tits building for $\mathrm{PGL}_2(k_\infty)$. Let $I \subsetneq A$ be a non-zero ideal. In [Ko-Ya], the authors defined the regulator map

$$\mathrm{reg}_I : K_2(M^2_I \otimes_A k) \to H_I$$

to the module H_I of $\mathrm{GL}_2(k)$-invariant harmonic 1-cochains on the simplicial complex

$$\mathrm{GL}_2(\mathbb{A}^\infty)/\mathbb{K}_I \times \mathcal{BT} = \coprod_{\gamma \in \mathrm{GL}_2(\mathbb{A}^\infty)/\mathbb{K}_I} \mathcal{BT}.$$

Let $H_{0,I} \subset H_I$ denote the submodule of $\mathrm{GL}_2(k)$-invariant harmonic 1-cochains whose support is finite modulo $\mathrm{GL}_2(k)$. We put $H = \varinjlim_I H_I$, $H_0 = \varinjlim_I H_{0,I}$. Passing to the inductive limit we obtain a $\mathrm{GL}_2(\mathbb{A})$-equivariant homomorphism

$$\mathrm{reg} : K_2(M^2_{\mathrm{lim},k}) \to H.$$

Let $\mathcal{I} \subset \mathrm{GL}_2(k_\infty)$ denote the Iwahori subgroup. Since the set of pointed edges in $\mathcal{BT}$ is canonically isomorphic to $\mathrm{GL}_2(k_\infty)/k_\infty^\times \mathcal{I}_\infty$, we can regard H (resp. H_0) as a module of locally constant (resp. locally constant, compactly supported) $\mathbb{Z}$-valued functions on the double coset $\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A})/k_\infty^\times \mathcal{I}_\infty$. There is the analogue of Petersson inner product $\langle\ ,\ \rangle : H_{0,\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$ defined as $\langle f_1, f_2 \rangle = \int_{\mathrm{GL}_2(k)k_\infty^\times \backslash \mathrm{GL}_2(\mathbb{A})} f_1(g) f_2(g) dg$, where dg denotes the Haar measure of $\mathrm{GL}_2(\mathbb{A})$ with $\mathrm{vol}(\prod_\wp \mathrm{GL}_2(A_\wp)) = 1$. Since the restriction of $\langle\ ,\ \rangle$ to $H_{0,I,\mathbb{Q}} \times H_{0,I,\mathbb{Q}}$ is non-degenerate, there exists a unique homomorphism $P_I : H_{I,\mathbb{Q}} \to H_{0,I,\mathbb{Q}}$ such that $\langle f_1, f_2 \rangle = \langle f_1, P_I f_2 \rangle$ for all $f_1 \in H_{0,I,\mathbb{Q}}$, $f_2 \in H_{I,\mathbb{Q}}$. Passing to the inductive limit we obtain a $\mathrm{GL}_2(\mathbb{A})$-equivariant homomorphism $P : H_{\mathbb{Q}} \to H_{0,\mathbb{Q}}$.

In [Ko-Ya], it is shown that the composite

$$\mathrm{reg}'_{\mathbb{Q}} : W_{\mathbb{Q}} \xrightarrow{\mathrm{reg}} H_{\mathbb{Q}} \xrightarrow{P} H_{0,\mathbb{Q}}$$

is surjective.

We apply Bloch's method ([De-Wi, Lemma 5.2]) in the case of elliptic curves to our Drinfeld modular context. By the Weil pairing morphism, we may regard elements of $\mathcal{O}(M_I^1 \times_A k)$ as elements of $\mathcal{O}(M_I^2 \times_A k)$. We note that every cusp of $\overline{M}_I^2 \otimes_A k$ is $M_I^1 \otimes_A k$-rational and that the Drinfeld modular analogue of Drinfeld-Manin theorem is proved by Gekeler ([Ge2]). Thus for each $\kappa \in K_2(M_I^2 \otimes_A k)$, one can find an integer $N \geq 1$ and an element $\overline{\kappa}$ in Image$[K_2(\overline{M_I^2} \otimes_A k) \to \varinjlim_I K_2(M_I^2 \otimes_A k)]$ such that $\overline{\kappa} - N\kappa \in \mathrm{Im}[\mathcal{O}(M_I^2 \otimes_A k)^\times \otimes \mathcal{O}(M_I^1 \otimes_A k)^\times \to K_2(\overline{M_I^2} \otimes_A k)]$.

Lemma 7.3. *Let $u_{12} \in \mathcal{O}(M_I^1 \times_A k)^\times \otimes \mathcal{O}(M_I^2 \times_A k)^\times$. Then $\mathrm{reg}_I(u_{12})$ equals zero.*

Proof. Using the formula [Ko-Ya, Lemma 6.3] for the regulator on symbols it is easy to see that the harmonic 1-cochain $\mathrm{reg}_I(u_{12})$ is a 1-coboundary. This implies that $\langle f, \mathrm{reg}_I(u_{12}) \rangle = 0$ for all $f \in H_{0,I}$. Hence $\mathrm{reg}_I(u_{12}) = 0$. $\square$

The above lemma shows that the composite

$$\mathrm{reg}''_{\mathbb{Q}} : W_{0,\mathbb{Q}} \to K_2(M^2_{\mathrm{lim},k})_{\mathbb{Q}} \xrightarrow{\mathrm{reg}} H_{\mathbb{Q}} \xrightarrow{P} H_{0,\mathbb{Q}}$$

is surjective.

We now use the following lemma, whose proof will be given in Section 7.2.

Lemma 7.4. *The kernel of reg_I contains the kernel of the composite*

$$K_2(M_I^2 \otimes_A k) \to K_2(M_I^2 \otimes_A k_\infty) \xrightarrow{c_{2,2}} H^2_{\mathrm{et}}(M_I^2 \otimes_A k_\infty, \mathbb{Q}_\ell(2)).$$

We use the subscript $-_{\mathbb{Q}_\ell}$ to mean $- \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$. This lemma implies that $H_{0,\mathbb{Q}_\ell}$ is a quotient of the image $\overline{W}_0$ of

$$W_{0,\mathbb{Q}_\ell} \to K_2(\overline{M}^2_{\mathrm{lim},k})_{\mathbb{Q}_\ell} \xrightarrow{c_{2,2,\infty}} H^2_{\mathrm{et}}(\overline{M}^2_{\mathrm{lim},k_\infty}, \mathbb{Q}_\ell(2)).$$

We have a canonical isomorphism $H_{0,\mathbb{Q}_\ell} \cong \mathrm{Hom}_{\mathrm{GL}_2(k_\infty)}(\mathrm{St}, \mathcal{A}_{\mathrm{cusp}})$ of $\mathrm{GL}_2(\mathbb{A}^\infty)$-modules, where $\mathcal{A}_{\mathrm{cusp}}$ denote the space of $\mathbb{Q}_\ell$-valued cusp forms on $\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A})$, and St denotes the Steinberg representation of $\mathrm{GL}_2(k_\infty)$. Let V_ℓ be a two dimensional $\mathbb{Q}_\ell$-representation of $\mathrm{Gal}(\overline{k}_\infty/k_\infty)$ which admits a non-split exact sequence

$$0 \to \mathbb{Q}_\ell \to V_\ell \to \mathbb{Q}_\ell(-1) \to 0.$$

Such V_ℓ is unique up to an isomorphism. By the fundamental theorem in [Dr1, §11], we have an isomorphism

$$\begin{aligned} H^2_{\mathrm{et}}(\overline{M}^2_{\mathrm{lim},k_\infty}, \mathbb{Q}_\ell(2)) &\cong \varinjlim_I H^1(k_\infty, H^1_{\mathrm{et}}(\overline{M}^2_I \otimes_A \overline{k}_\infty, \mathbb{Q}_\ell(2)) \\ &\cong H^1(k_\infty, V_\ell) \otimes_{\mathbb{Q}_\ell} \mathrm{Hom}_{\mathrm{GL}_2(k_\infty)}(\mathrm{St}, \mathcal{A}_{\mathrm{cusp}}) \end{aligned}$$

of $\mathrm{GL}_2(\mathbb{A}^\infty)$-modules. Since $H^1(k_\infty, V_\ell)$ is one-dimensional, the module $H^2_{\mathrm{et}}(\overline{M}^2_{\mathrm{lim},k_\infty}, \mathbb{Q}_\ell(2))$ is isomorphic to $H_{0,\mathbb{Q}_\ell}$ as a $\mathrm{GL}_2(\mathbb{A}^\infty)$-module. On the other hand, $H_{0,\mathbb{Q}_\ell}$ is a quotient of the $\mathrm{GL}_2(\mathbb{A}^\infty)$-submodule $\overline{W}_0$ of $\varinjlim_I H^2_{\mathrm{et}}(\overline{M}^2_I \otimes_A k_\infty, \mathbb{Q}_\ell(2))$. Since $H_{0,\mathbb{Q}_\ell}$ is an admissible $\mathrm{GL}_2(\mathbb{A}^\infty)$-module, we conclude that $\overline{W}_0$ equals $H^2_{\mathrm{et}}(\overline{M}^2_{\mathrm{lim},k_\infty}, \mathbb{Q}_\ell(2))$. This completes the proof of Proposition 7.2. □

7.2. **Proof of Lemma 7.4.** Let $\mathfrak{X}$ be the Drinfeld upper half plane over k_∞. The rigid analytic uniformization of M^2_I gives a canonical morphism

$$\coprod_{\gamma \in \mathrm{GL}_2(\mathbb{A}^\infty)/\mathbb{K}_I} \mathfrak{X} \to (M^2_I \otimes_A k_\infty)^{\mathrm{an}}$$

of rigid analytic spaces. For $\gamma \in \mathrm{GL}_2(\mathbb{A}^\infty)$, let $v_\gamma : \mathfrak{X} \to (M^2_I \otimes_A k_\infty)^{\mathrm{an}}$ denote its $\gamma\mathbb{K}_I$-component.

There is the specialization map: $\mathrm{sp} : \mathfrak{X} \to \mathcal{BT}$. Let e be an oriented edge (including the endpoints) in the Bruhat-Tits building $\mathcal{BT}$. Then $\mathrm{sp}^{-1}(e)$ is isomorphic to $\mathrm{Spm}\, B_e$ for some affinoid algebra B_e. Let Y_e denote the special fiber of the formal model of $\mathrm{Spm}\, B_e$. Let $v_{\gamma,e} : \mathrm{Spm}\, B_e \to (M^2_I \otimes_A k_\infty)^{\mathrm{an}}$ denote the restriction of v_γ to $\mathrm{sp}^{-1}(e)$.

Lemma 7.5. *Associated to $v_{\gamma,e}$, there exists a canonical ring homomorphism $\mathcal{O}(M^2_I \otimes_A k_\infty) \to \mathrm{Spm}\, B_e$.*

Proof of Lemma 7.5. We take an embedding $M^2_I \otimes_A k_\infty \hookrightarrow V$ of M^2_I into an affine space V over k_∞. Let $B(r)$ denote the open ball in V centered at the origin of radius r. Then $V_i = M^2_I \cap B(i)$ is an affinoid, say $\mathrm{Spm}\, A_i$. There is a canonical ring homomorphism $\mathcal{O}(M^2_I) \to A_i$. Now since $\mathrm{Spm}\, B_e$ is quasi-compact, $\mathrm{Spm}\, B_e \to (M^2_I \otimes_A k_\infty)^{\mathrm{an}}$ factors through V_n for some n. This means that there is a ring homomorphism $A_n \to B_e$. The composition $A \to A_n \to B_e$ does not depends on the choice of an embedding $M^2_I \otimes_A k_\infty \hookrightarrow V$ and gives the desired ring homomorphism. □

By the definition of reg_I given in [Ko-Ya], the kernel of reg_I contains the kernel of the composite

$$K_2(M_I^2 \otimes_A k) \to \prod_{\gamma,e} K_2(B_e) \xrightarrow{\partial} \prod_{\gamma,e} G_1(Y_e) \to \prod_{\gamma,e} G_1(Y_e)/G_1(Y_e)_{\mathrm{tors}}.$$

Let K, L_e denote the total quotient ring of $\mathcal{O}(M_I^2 \otimes_A k_\infty)$, $\mathcal{O}(Y_e)$, respectively. Consider the following commutative diagram:

$$\begin{tikzcd}
K_2(M_I^2 \otimes_A k_\infty) \arrow[r] \arrow[d] & K_2(B_e) \arrow[r, "\partial"] \arrow[d] & G_1(Y_e)/G_1(Y_e)_{\mathrm{tors}} \arrow[d] \\
K_2(K) \arrow[r] & K_2(\mathrm{Frac}(B_e)) \arrow[r, "\partial"] & K_1(L_e)/K_1(L_e)_{\mathrm{tors}}
\end{tikzcd}$$

where $\mathrm{Frac}(B_e)$ is the field of fractions of B_e. We note that the right vertical homomorphism $G_1(Y_e)/G_1(Y_e)_{\mathrm{tors}} \to K_1(L_e)/K_1(L_e)_{\mathrm{tors}}$ is injective since Y_e is a normal crossing curve over a finite field. Hence the kernel of reg_I contains the kernel of the composite

$$K_2(M_I^2 \otimes_A k) \to K_2(K) \to \prod_{\gamma,e} K_2(\mathrm{Frac}(B_e)) \xrightarrow{\partial} \prod_{\gamma,e} K_1(L_e)/K_1(L_e)_{\mathrm{tors}}.$$

For an abelian group M, let $M^\wedge = \varprojlim_n M/l^n \otimes_{\mathbb{Z}} \mathbb{Q}$. The following diagram is commutative:

$$\begin{tikzcd}
K_2(K) \arrow[r] \arrow[d] & \prod_{\gamma,e} G_1(L_e)/G_1(L_e)_{\mathrm{tors}} \arrow[d, "(1)"'] \\
K_2(K)^\wedge \arrow[r] & (\prod_{\gamma,e} G_1(L_e)/G_1(L_e)_{\mathrm{tors}})^\wedge.
\end{tikzcd}$$

Since L_e is the product of two copies of the field of rational functions over a finite field, $G_1(L_e)/G_1(L_e)_{\mathrm{tors}}$ is a free abelian group. Hence the map (1) is injective. Therefore, the kernel of reg_I contains the kernel of the composite

$$K_2(M_I^2 \otimes_A k) \to K_2(K) \to K_2(K)^\wedge. \tag{7.1}$$

We note that K is the product of finite number of fields. By the theorem of Merkurjev-Suslin ([Me-Su1]), the symbol map $K_2(K) \to H^2_{\mathrm{et}}(K, \mathbb{Q}_\ell(2))$ gives an isomorphism $K_2(K)^\wedge \cong H^2_{\mathrm{et}}(K, \mathbb{Q}_\ell(2))$. Since the diagram

$$\begin{tikzcd}
K_2(M_I^2 \otimes_A k) \arrow[r] \arrow[d] & K_2(K) \arrow[r] & K_2(K)^\wedge \arrow[d, "\cong"] \\
K_2(M_I^2 \otimes_A k_\infty) \arrow[r, "c_{2,2}"'] & H^2(M_I^2 \otimes_A k_\infty, \mathbb{Q}_\ell(2)) \arrow[r] & H^2(K, \mathbb{Q}_\ell(2))
\end{tikzcd}$$

is commutative, the kernel of (7.1) contains the kernel of $K_2(M_I^2 \otimes_A k) \to K_2(M_I^2 \otimes_A k_\infty) \to H^2(M_I^2 \otimes_A k_\infty, \mathbb{Q}_\ell(2))$. This completes the proof of Lemma 7.4. $\square$

8. Proof of Theorem 1.1

Assume $S_0 \neq \emptyset$ and take $s \in S_0$. We use the notations in Section 7 for $\infty = s$. Combining Proposition 7.2 with Lemma 7.1, we see that there exists an element $\kappa \in W_0$ whose image by the composite

$$W_0 \hookrightarrow K_2(\overline{M}^2_{\mathrm{lim},k}) \to H^2_{\mathrm{et}}(\overline{M}^2_{\mathrm{lim},k_\infty}, \mathbb{Q}_\ell(2)) \xrightarrow{\varphi_{*,\infty,\mathrm{et}}} H^2_{\mathrm{et}}(E \otimes_k k_\infty, \mathbb{Q}_\ell(2))$$

is non-zero. Let us consider the morphism $\varphi_{E,I}$ for a sufficiently small I. The homomorphism

$$\frac{1}{\deg(\varphi_{E,I})} \cdot \varphi_{E,I_*} : K_2(\overline{M}^2_I \otimes_A k) \to K_2(E)_{\mathbb{Q}}$$

gives rise to the homomorphism $\varphi_{*,K} : K_2(\overline{M}^2_{\mathrm{lim},k}) \to K_2(E)_{\mathbb{Q}}$. We put $\kappa_s = \varphi_{*,K}(\kappa) \in K_2(E)_{\mathbb{Q}}$.

Lemma 8.1. *For $s, s' \in S_0$, $\partial_{s'}(\kappa_s) = 0$ if and only if $s \neq s'$.*

Proof. For each $s' \in S_0$, we have a commutative diagram,

$$\begin{array}{ccc}
K_2(\overline{M}^2_{\mathrm{lim},k}) & \xrightarrow{c_{2,2,s'}} & H^2_{\mathrm{et}}(\overline{M}^2_{\mathrm{lim},k_{s'}}, \mathbb{Q}_\ell(2)) \\
\downarrow{\scriptstyle \varphi_{*,K}} & & \downarrow{\scriptstyle \varphi_{*,s',\mathrm{et}}} \\
K_2(E) & \xrightarrow{c_{2,2,s'}} & H^2_{\mathrm{et}}(E \otimes_k k_{s'}, \mathbb{Q}_\ell(2)).
\end{array}$$

Hence $c_{2,2,s'}(\kappa_\infty) = 0$ if $s' \neq \infty$ and $c_{2,2,\infty}\kappa_\infty \neq 0$. Now Proposition 3.1 gives the claim. □

Proof of Theorem 1.1. Consider the subspace of $K_2(E)_{\mathbb{Q}}$ generated by $\{\kappa_s | s \in S_0\}$. Using Lemma 8.1 one sees that it maps surjectively onto the right hand side. This completes the proof of Theorem 1.1. □ □

9. Motivic cohomology groups of smooth surfaces

Aside from the uniquely divisible part, we understand the motivic cohomology groups of smooth surfaces over finite fields fairly well. The divisible part is conjecturally zero.

9.1. **Motivic cohomology of surfaces over a finite field.** Let $\mathbb{F}_q$ be a finite field of characteristic p.

For a separated scheme X which is essentially of finite type over $\mathbb{F}_q$, we define the motivic cohomology group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ as the homology group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) = H_{2j-i}(z^j(X, \bullet))$ of Bloch's cycle complex $z^j(X, \bullet)$ ([Bl2] see also [Ge-Le2, 2.5] to remove the condition that X is quasi-projective). When X is essentially smooth over $\mathbb{F}_q$, it coincides with the motivic cohomology group defined in [Lev1] or [Vo-Su-Fr] (cf. [Lev2], [Vo2]). For a discrete abelian group M, we put $H^i_{\mathcal{M}}(X, M(j)) = H_{2j-i}(z^j(X, \bullet) \otimes_{\mathbb{Z}} M)$.

We will compute the motivic cohomology group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ modulo uniquely divisible subgroup when X is a smooth surface over $\mathbb{F}_q$. The group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ for $j \leq 1$ has been computed. By definition, $H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) = 0$ for $j \leq 0$ and $(i, j) \neq (0, 0)$, and $H^0_{\mathcal{M}}(X, \mathbb{Z}(0)) = H^0_{\mathrm{Zar}}(X, \mathbb{Z})$. We also have $H^i_{\mathcal{M}}(X, \mathbb{Z}(1)) = 0$ for $i \neq 1, 2$, $H^1_{\mathcal{M}}(X, \mathbb{Z}(1)) = H^0_{\mathrm{Zar}}(X, \mathbb{G}_m)$, and $H^1_{\mathcal{M}}(X, \mathbb{Z}(1)) = \mathrm{Pic}(X)$ ([Bl2, Theorem 6.3]).

The following conjecture is a part of the Bloch-Kato conjecture ([Ka, §1, Conjecture 1]).

Conjecture 9.1. *Let $j \geq 1$ be an integer. Then for any finitely generated field K over $\mathbb{F}_q$ and for any $\ell \neq p$, the symbol map $K_j^M(K) \to H^j_{\mathrm{et}}(\operatorname{Spec} K, \mathbb{Z}/\ell(j))$ is surjective.*

Conjecture 9.1 is known to hold when $j \leq 2$ or $\ell = 2$ (cf. [Me-Su1], [Vo1]). We note that Conjecture 9.1 for j implies Conjecture 9.1 for any $j' \leq j$.

Definition 9.2. Let M be an abelian group. We say that M is *finitely generated modulo uniquely divisible subgroup* (resp. *finite modulo uniquely divisible subgroup*) if M_{div} is uniquely divisible and M^{red} is finitely generated (resp. M_{div} is uniquely divisible and M^{red} is finite).

We note that, if M is finite modulo uniquely divisible subgroup, then M_{tors} is a finite group and $M = M_{\mathrm{div}} \oplus M_{\mathrm{tors}}$.

The aim of Section 9.1 is to prove the following theorem.

Theorem 9.3. *Let X be a smooth surface over $\mathbb{F}_q$. Let R denote the number of connected components of X which is projective over $\mathbb{F}_q$.*

(1) *The group $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))$ is finitely generated modulo uniquely divisible subgroup if $i \neq 3$ or if X is projective. More precisely,*
  (a) *The group $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))$ is zero for $i \geq 5$.*
  (b) *The group $H^4_{\mathcal{M}}(X, \mathbb{Z}(2))$ is a finitely generated abelian group of rank R.*
  (c) *If $i \leq 1$ or if X is projective and $i \leq 3$, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))$ is finite modulo uniquely divisible subgroup.*
  (d) *The group $H^2_{\mathcal{M}}(X, \mathbb{Z}(2))$ is finitely generated modulo uniquely divisible subgroup.*
  (e) *For $i \leq 2$, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{tors}}$ is canonically isomorphic to the direct sum $\bigoplus_{\ell \neq p} H^{i-1}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$. In particular, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))$ is uniquely divisible for $i \leq 0$.*
  (f) *If X is projective, then the group $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{tors}}$ is isomorphic to the direct sum of the group $\bigoplus_{\ell \neq p} H^2_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ and a finite p-group of order $|\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)| \cdot |L(h^2(X), 0)|_p^{-1}$. Here $\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)$ denotes the set of morphisms $\mathrm{Pic}^o_{X/\mathbb{F}_q} \to \mathbb{G}_m$ of $\mathbb{F}_q$-group schemes.*

(2) *Let $j \geq 3$ be an integer and suppose that Conjecture 9.1 is true for j. Then for any integer i, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is finite modulo uniquely divisible subgroup. More precisely,*
  (a) *The group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is zero for $i \geq \max(6, j+1)$, is isomorphic to $(\mathbb{Z}/(q^{j-2} - 1))^{\oplus R}$ for $(i,j) = (5,3)$, $(5,4)$, and is finite for $(i,j) = (4,3)$.*
  (b) *The group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ is canonically isomorphic to $\bigoplus_{\ell \neq p} H^{i-1}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$. In particular, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is uniquely divisible for $i \leq 0$ or $5 \leq i \leq j$, and the group $H^1_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ is cyclic of order $q^j - 1$.*

In the following table, we summarize the description of the groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ stated in Theorem 9.3. For $j \geq 3$, we assume that Conjecture 9.1 for j holds. Here we write u.d, f./u.d., f.g./u.d., f., f.g. for uniquely divisible, finite modulo uniquely divisible, finite generated modulo uniquely divisible, finite, finitely generated respectively.

<table>
<tr><th>$j \setminus i$</th><th>< 0</th><th>0</th><th>$0 < i < j$</th><th>j</th><th>$j+1$</th><th>$j+2$</th><th>$\geq j+3$</th></tr>
<tr><td>0</td><td>0</td><td>$H^0(\mathbb{Z})$</td><td>-</td><td colspan="4">0</td></tr>
<tr><td>1</td><td colspan="2">0</td><td>-</td><td>$H^0(\mathbb{G}_m)$</td><td>$\mathrm{Pic}(X)$</td><td colspan="2">0</td></tr>
<tr><td rowspan="2">2</td><td colspan="2" rowspan="2">u. d.</td><td rowspan="2">f./u. d.</td><td>f. g./u. d.</td><td>?</td><td rowspan="2">f. g.</td><td rowspan="2">0</td></tr>
<tr><td colspan="2">f./u. d. if projective</td></tr>
<tr><td>3</td><td colspan="2">u. d.</td><td colspan="2">f./u. d.</td><td colspan="2">f.</td><td>0</td></tr>
<tr><td>4</td><td colspan="2">u. d.</td><td colspan="2">f./u. d.</td><td>f.</td><td colspan="2">0</td></tr>
<tr><td rowspan="2">≥ 5</td><td colspan="2" rowspan="2">u. d.</td><td colspan="2">f./u. d.</td><td colspan="3" rowspan="2">0</td></tr>
<tr><td colspan="2">u. d. if $6 \leq i \leq j$</td></tr>
</table>

Lemma 9.4. *Let X be a separated scheme which is essentially of finite type over $\mathbb{F}_q$. Let i, j be integers. If both $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$ and $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$ are finite, then $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is finite modulo uniquely divisible subgroup and its torsion subgroup is isomorphic to $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$.*

Proof. Let us consider the exact sequence

$$0 \to H^{i-1}_{\mathcal{M}}(X, \mathbb{Z}(j)) \otimes \mathbb{Q}/\mathbb{Z} \to H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}} \to 0. \tag{9.1}$$

Since $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$ is a finite group, all the groups in the above exact sequence are finite groups. Then the group $H^{i-1}_{\mathcal{M}}(X, \mathbb{Z}(j)) \otimes \mathbb{Q}/\mathbb{Z}$ must be zero since it is finite and divisible. Hence we have a canonical isomorphism $H^{i-1}_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j)) \to H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$. The finiteness of $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ implies that the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{div}}$ is uniquely divisible and the canonical homomorphism

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(j))^{\mathrm{red}} \to \varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}(j))/m$$

is injective. The latter group $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}(j))/m$ is canonically embedded in the finite group $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$. Hence we conclude that $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))^{\mathrm{red}}$ is finite. This proves the claim. □

Lemma 9.5. *Let X be a smooth projective surface over $\mathbb{F}_q$. Let j be an integer and suppose that Conjecture 9.1 is true for j. Then the group $H^i_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$ and the group $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$ are finite if $i \neq 2j$ or $j \geq 3$.*

Proof. The claim for $j \leq 1$ is clear. Suppose that $j = 2$. Then the claim for $i \geq 5$ is clear. If $p \nmid m$, by [Ge-Le2, Corollary 1.2. See also Corollary 1.4] and Merkurjev-Suslin theorem, the cycle class map $H^i_{\mathcal{M}}(X, \mathbb{Z}/m(2)) \to H^i_{\mathrm{et}}(X, \mathbb{Z}/m(2))$ is an isomorphism for $i \leq 2$ and is injective for $i = 3$. By [Co-Sa-So, Théorème 2] and the exact sequence [Co-Sa-So, 2.1 (29) p.781], the group $\varinjlim_{m, p \nmid m} H^i_{\mathrm{et}}(X, \mathbb{Z}/m(2))$ and the group $\varprojlim_{m, p \nmid m} H^i_{\mathrm{et}}(X, \mathbb{Z}/m(2))$ are finite for $i \leq 3$. Let $W_n\Omega^{\bullet}_{X,\log}$ denote the logarithmic de Rham-Witt sheaf (cf. [Il1, I, 5.7]). This was introduced by Milne in [Mi1]. There is an isomorphism $H^i_{\mathcal{M}}(X, \mathbb{Z}/p^n(2)) \cong H^{i-2}_{\mathrm{Zar}}(X, W_n\Omega^2_{\mathcal{E},\log})$ (cf. [Ge-Le1, Theorem 8.4]). In particular, we have $H^i_{\mathcal{M}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) = 0$ for $i \leq 1$. By [Co-Sa-So, §2, Théorème 3], $\varinjlim_n H^i_{\mathrm{et}}(X, W_n\Omega^2_{X,\log})$ is a finite group for $i = 0, 1$. Using the argument in [Co-Sa-So, 2.2], we see that $\varprojlim_n H^i_{\mathrm{et}}(X, W_n\Omega^2_{X,\log})$ is also finite for $i = 0, 1$ and is isomorphic to $\varinjlim_n H^{i-1}_{\mathrm{et}}(X, W_n\Omega^2_{X,\log})$. Since the homomorphism

$$H^i_{\mathrm{Zar}}(X, W_n\Omega^2_{X,\log}) \to H^i_{\mathrm{et}}(X, W_n\Omega^2_{X,\log}),$$

induced by the change of topology $\varepsilon : X_{\mathrm{et}} \to X_{\mathrm{Zar}}$, is an isomorphism for $i = 0$ and is injective for $i = 1$, we see that $\varprojlim_n H^2_{\mathcal{M}}(X, \mathbb{Z}/p^n(2))$ is zero, and that both $H^2_{\mathcal{M}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ and $\varprojlim_n H^3_{\mathcal{M}}(X, \mathbb{Z}/p^n(2))$ are finite groups. This proves the claim for $j = 2$.

Suppose $j \geq 3$. The claim for $i \geq 2j$ is clear. Since $j \geq 3$, we have $H^i_{\mathcal{M}}(X, \mathbb{Z}/p^n(j)) \cong H^{i-2}_{\mathrm{Zar}}(X, W_n\Omega^j_{\mathcal{E},\log}) = 0$. Assume Conjecture 9.1 for j. Then by [Ge-Le2, Theorem 1.1], the group $H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$ is isomorphic to the group $H^i_{\mathrm{Zar}}(X, \tau_{\leq j} R\varepsilon_* \mathbb{Z}/m(j))$ if $p \nmid m$. Since any affine surface over $\mathbb{F}_q$ has ℓ-cohomological dimension 3 for any $\ell \neq p$, we have $H^i_{\mathrm{Zar}}(X, \tau_{\leq j} R\varepsilon_* \mathbb{Z}/m(j)) \cong H^i_{\mathrm{et}}(X, \mathbb{Z}/m(j))$ for all i. Hence by [Co-Sa-So, Théorème 2] and the exact sequence [Co-Sa-So, 2.1 (29) p.781], the group $H^i_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(j))$ and the group $\varprojlim_m H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j))$ are finite for $i \leq 2j - 1$. This proves the claim for $j \geq 3$. □

Lemma 9.6. *Let Y be a scheme of dimension $d \leq 1$ which is of finite type over $\mathbb{F}_q$. Then $H^i_{\mathcal{M}}(Y, \mathbb{Z}(j))$ is a torsion group unless $0 \leq j \leq d$ and $j \leq i \leq 2j$.*

Proof. By taking a smooth affine open of Y_{red} whose complement is of dimension zero, and using the localization sequence of motivic cohomology, we are reduced to the case where Y is connected, affine, and smooth over $\mathbb{F}_q$. When $d = 0$ (resp. $d = 1$), the claim follows from the result of Quillen [Qu] (resp. Harder [Hard, Korollar 3.2.3] (see [Gr, Theorem 0.5] for the correct interpretation of his result)) on the structure of the K-groups of Y, combined with the Riemann-Roch theorem for higher Chow groups [Bl2, Theorem 9.1]. □

We use the following lemma, whose proof is easy and is left to the reader.

Lemma 9.7. *Let $\varphi : M \to M'$ be a homomorphism of abelian groups such that* $\mathrm{Ker}\,\varphi$ *is finite and* $(\mathrm{Coker}\,\varphi)_{\mathrm{div}} = 0$. *If M_{div} or M'_{div} is uniquely divisible, then φ induces an isomorphism $M_{\mathrm{div}} \xrightarrow{\cong} M'_{\mathrm{div}}$.*

Proof of Theorem 9.3 (1). Without loss of generality, we may assume that X is connected. We first prove the claims assuming X is projective. It is clear that the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is zero for $i \geq \min(j + 3, 2j + 1)$. It is known that the degree map $H^4_{\mathcal{M}}(X, \mathbb{Z}(2)) = \mathrm{CH}_0(X) \to \mathbb{Z}$ has finite kernel and cokernel ([Bl1, p.232 (5)], [Mi2], see also [Co-Sa-So]). This proves the claim for $i \geq \min(j + 3, 2j)$. Fix $j \geq 2$ and assume Conjecture 9.1 for j. For $i \leq 2j - 1$, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is finite modulo uniquely divisible subgroup by Lemmas 9.4 and 9.5. The claim on the identification of $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ with the etale cohomology follows immediately from the argument in the proof of Lemma 9.5 except for the p-primary part of $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$, which follows from Proposition A.1.

To finish the proof, it remains to prove that $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{div}}$ is zero for $j \geq 3$ and $i = j+1, j+2$. It suffices to prove that $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is a torsion group for $j \geq 3$ and $i \geq j + 1$. Consider the limit

$$\varinjlim_Y H^{i-2}_{\mathcal{M}}(Y, \mathbb{Z}(j-1)) \to H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \to \varinjlim_Y H^i_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j))$$

of the localization sequence where Y runs over the reduced closed subschemes of X of pure codimension one. The group $H^{i-2}_{\mathcal{M}}(Y, \mathbb{Z}(j-1))$ is torsion by Lemma 9.6 and we have $\varinjlim_Y H^{i-2}_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j-1)) = 0$ for dimension reasons. Hence the claim follows. This completes the proof in the case where X is projective.

For general connected X, take an embedding $X \hookrightarrow X'$ of X into a smooth projective surface X' over $\mathbb{F}_q$ such that $Y = X' \setminus X$ is of pure codimension one in X'. We can show that such an

X' exists by using [Na] and a resolution of singularities ([Ab], [Lip]). Then the claims, except for that on the identification of $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ with the etale cohomology, easily follow from Lemma 9.7 and by using the localization sequence

$$\cdots \to H^{i-2}_{\mathcal{M}}(Y, \mathbb{Z}(j-1)) \to H^i_{\mathcal{M}}(X', \mathbb{Z}(j)) \to H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \to \cdots .$$

The claim on the identification of $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))_{\mathrm{tors}}$ with the etale cohomology can be obtained in a way similar to that in the proof of Lemma 9.5. This completes the proof. $\square$

9.2. **A criterion for the finiteness of $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{tors}}$.**

Proposition 9.8. *Let X be a smooth surface over $\mathbb{F}_q$. Let $X \hookrightarrow X'$ be an open immersion such that X' is smooth projective and $Y = X' \setminus X$ is of pure codimension one in X'. Then the following conditions are equivalent.*

1. *The group $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$ is finitely generated modulo uniquely divisible subgroup.*
2. *The group $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{tors}}$ is finite.*
3. *The pull-back $H^3_{\mathcal{M}}(X', \mathbb{Z}(2)) \to H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$ induces an isomorphism*
$$H^3_{\mathcal{M}}(X', \mathbb{Z}(2))_{\mathrm{div}} \xrightarrow{\cong} H^3_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{div}}.$$
4. *The kernel of the pull-back map $H^3_{\mathcal{M}}(X', \mathbb{Z}(2)) \to H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$ is finite.*
5. *The cokernel of the boundary homomorphism $\partial : H^2_{\mathcal{M}}(X, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(Y, \mathbb{Z}(1))$ is finite.*

Moreover, if the above equivalent conditions are satisfied, then the group $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{tors}}$ is isomorphic to the direct sum of the group $\bigoplus_{\ell \neq p} H^2_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$ and a finite group of p-power order, and the localization sequence induces the long exact sequence

$$\cdots \to H^{i-2}_{\mathcal{M}}(Y, \mathbb{Z}(1)) \to H^i_{\mathcal{M}}(X', \mathbb{Z}(2))^{\mathrm{red}} \to H^i_{\mathcal{M}}(X, \mathbb{Z}(2))^{\mathrm{red}} \to \cdots \tag{9.2}$$

of finitely generated abelian groups.

Proof. The condition (1) clearly implies the condition (2). The localization sequence shows that the conditions (4) and (5) are equivalent and the condition (3) implies the condition (1). By the localization sequence and Lemma 9.7, the condition (4) implies the condition (3).

We claim that the condition (2) implies the condition (4). Assume the condition (2) and suppose that the condition (4) is not satisfied. We put $M = \mathrm{Ker}[H^3_{\mathcal{M}}(X', \mathbb{Z}(2)) \to H^3_{\mathcal{M}}(X, \mathbb{Z}(2))]$. The localization sequence shows M is finitely generated. By assumption, M is not torsion. Since $H^3_{\mathcal{M}}(X', \mathbb{Z}(2))$ is finite modulo uniquely divisible subgroup, the intersection $H^3_{\mathcal{M}}(X', \mathbb{Z}(2))_{\mathrm{div}} \cap M$ is a non-trivial free abelian group of finite rank. Hence the group $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$ contains a group isomorphic to $H^3_{\mathcal{M}}(X', \mathbb{Z}(2))_{\mathrm{div}}/(H^3_{\mathcal{M}}(X', \mathbb{Z}(2))_{\mathrm{div}} \cap M)$, which contradicts the condition (2). Hence the condition (2) implies the condition (4). This completes the proof the equivalence of the conditions (1)-(5).

Suppose that the conditions (1)-(5) are satisfied. The localization sequence shows that the kernel (resp. the cokernel) of the pull-back $H^i_{\mathcal{M}}(X, \mathbb{Z}(2)) \to H^i_{\mathcal{M}}(X', \mathbb{Z}(2))$ is a torsion group (resp. has no non-trivial divisible subgroup) for any $i \in \mathbb{Z}$. Hence, by Lemma 9.7, $H^i_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{div}}$ is uniquely divisible and the sequence (9.2) is exact. The condition (2) and the exact sequence (9.1) for $(i, j) = (3, 2)$ give the isomorphism $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{tors}} \cong H^2_{\mathcal{M}}(X, \mathbb{Q}/\mathbb{Z}(2))^{\mathrm{red}}$. Then the claim on the structure of $H^3_{\mathcal{M}}(X, \mathbb{Z}(2))_{\mathrm{tors}}$ follows from [Ge-Le2, Corollary 1.2. See also Corollary 1.4] and Merkurjev-Suslin theorem. This completes the proof. $\square$

Let X be a smooth projective surface over $\mathbb{F}_q$. Suppose that X admits a flat, surjective and generically smooth morphism $f : X \to C$ to a connected, smooth projective curve C over $\mathbb{F}_q$. For each point $\wp \in C$, let $X_\wp$ denote the fiber of f at $\wp$.

Corollary 9.9. *Let the notations be as above. Let $\eta \in C$ denote the generic point. Suppose that the cokernel of the homomorphism $\partial : H^2_{\mathcal{M}}(X_\eta, \mathbb{Z}(2)) \to \bigoplus_{\wp \in C_0} H^1_{\mathcal{M}}(X_\wp, \mathbb{Z}(1))$, which is the inductive limit of the boundary maps of the localization sequences, is a torsion group. Then the group $H^i_{\mathcal{M}}(X_\eta, \mathbb{Z}(2))_{\mathrm{div}}$ is uniquely divisible for all $i \in \mathbb{Z}$ and the inductive limit of localization sequences induces the long exact sequence*

$$\cdots \to \bigoplus_{\wp \in C_0} H^{i-2}_{\mathcal{M}}(X_\wp, \mathbb{Z}(1)) \to H^i_{\mathcal{M}}(X, \mathbb{Z}(2))^{\mathrm{red}} \to H^i_{\mathcal{M}}(X_\eta, \mathbb{Z}(2))^{\mathrm{red}} \to \cdots .$$

Proof. Since the group $\bigoplus_{\wp \in C_0} H^i_{\mathcal{M}}(X_\wp, \mathbb{Z}(1))$ has no non-trivial divisible subgroup for all $i \in \mathbb{Z}$, and is torsion for $i \neq 1$ by Lemma 9.6, the claim follows from Lemma 9.7. $\square$

10. Motivic Chern characters for singular curves over finite fields

We construct Chern characters for singular curves over finite fields in an ad hoc manner. We apply them to the bad reductions of an elliptic curve.

10.1. Given an essentially smooth scheme X over $\mathbb{F}_q$ and integers $i, j \geq 0$, we let $c_{i,j} : K_i(X) \to H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j))$ denote the Chern class map, which is constructed and is denoted by $c_X^{j,2j-i}$ in [Lev1]. The map $c_{i,j}$ is a group homomorphism if $i \geq 1$ or $(i,j) = (0,1)$. We let $\mathrm{ch}_{0,0} : K_0(X) \to H^0_{\mathcal{M}}(X, \mathbb{Z}(0)) \cong H^0_{\mathrm{Zar}}(X, \mathbb{Z})$ denote the homomorphism which sends the class of locally free $\mathcal{O}_X$-module $\mathcal{F}$ to the rank of $\mathcal{F}$. For $i \geq 1$ and $a \in K_i(X)$, we put formally $\mathrm{ch}_{i,0}(a) = 0$.

Lemma 10.1. *Let X be an $\mathbb{F}_q$-scheme which is a localization of a smooth quasi-projective $\mathbb{F}_q$-scheme. Let $Y \subset X$ be a closed subscheme of pure codimension d which is essentially smooth over $\mathbb{F}_q$. Then for $i, j \geq 1$ or $(i,j) = (0,1)$, the diagram*

$$\begin{array}{ccc} K_i(Y) & \xrightarrow{\alpha_{i,j}} & H^{2j-i-2d}_{\mathcal{M}}(Y, \mathbb{Z}(j-d)) \\ \downarrow & & \downarrow \\ K_i(X) & \xrightarrow{c_{i,j}} & H^{2j-i}_{\mathcal{M}}(X, \mathbb{Z}(j)) \\ \downarrow & & \downarrow \\ K_i(X \setminus Y) & \xrightarrow{c_{i,j}} & H^{2j-i}_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(j)) \\ \downarrow & & \downarrow \\ K_{i-1}(Y) & \xrightarrow{\alpha_{i-1,j}} & H^{2j-i-2d+1}_{\mathcal{M}}(Y, \mathbb{Z}(j-d)) \end{array} \tag{10.1}$$

is commutative. Here the homomorphism $\alpha_{i,j}$ is defined as follows: for $a \in K_i(Y)$, $\alpha_{i,j}(a)$ equals

$$G_{d,j-d}(\mathrm{ch}_{i,0}(a), c_{i,1}(a), \ldots, c_{i,j-d}(a); c_{0,1}(\mathcal{N}), \ldots, c_{0,j-d}(\mathcal{N})),$$

where $G_{d,j-d}$ is the universal polynomial in [SGA6, Exposé 0, Appendice, Proposition 1.5] *and $\mathcal{N}$ is the conormal sheaf of Y in X), and the left (resp. the right) vertical sequence is the localization sequence of K-theory (resp. of higher Chow groups established in* [Bl3]*).*

Proof. We may assume that X is quasi-projective and smooth over $\mathbb{F}_q$. It follows from [Lev1, Part I, Chapter III, 1.5.2] and the Riemann-Roch theorem without denominators [Gi, Theorem 3.1] that the diagram (10.1) is commutative if we replace the right vertical sequence by the Gysin sequence in [Lev1, Part I, Chapter III, 2.1]. It suffices to show that the Gysin sequence is identified with the localization sequence of higher Chow groups. We use the notations in [Lev1, Part I, Chapter I, II]. Suppose that $S = \operatorname{Spec} \mathbb{F}_q$ and $\mathcal{V}$ is the category of essentially smooth $\mathbb{F}_q$-schemes. We put $\Gamma = \mathbb{Z}_{X,Y}(j)[2j-i]$ which we regard as an object in $\mathbf{C}^b_{\mathrm{mot}}(\mathcal{V})^*$. We have canonical homomorphisms

$$H^0(\mathcal{Z}_{\mathrm{mot}}(\Gamma,*)) \xrightarrow{\mathbb{H}^0} \mathcal{CH}(\Gamma) \xrightarrow{\mathrm{cl}(\Gamma)} \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1,\Gamma).$$

These homomorphisms give a canonical homomorphism

$$\begin{array}{rl} & \mathrm{CH}^{j-d}(Y,i) \cong H_i(\mathrm{Cone}(z^j(X,\bullet) \to z^j(X \setminus Y,\bullet)[-1])) \\ \to & \mathrm{Hom}_{\mathbf{D}^b_{\mathrm{mot}}(\mathcal{V})}(1,\mathbb{Z}_{X,Y}(j)[2j-i]) = H^{2j-i}_Y(X,\mathbb{Z}(j)) \cong H^{2j-i-2d}_{\mathcal{M}}(Y,\mathbb{Z}(j-d)). \end{array}$$

Using the deformation diagram [Lev1, Part I, Chapter III, (2.1.2.1)], we see that this homomorphism equals the isomorphism $\mathrm{cl}_Y^{j-d,2j-i-2d} \circ i_Y^{j-d,i} : \mathrm{CH}^{j-d}(Y,i) \cong H^{2j-i-2d}_{\mathcal{M}}(Y,\mathbb{Z}(j-d))$ in [Lev1, Part I, Chapter II, Theorem 3.6.6]. This proves the claim. □

Remark 10.2. For $j = d$, we have $\alpha_{i,d} = (-1)^{d-1}(d-1)! \cdot \mathrm{ch}_{i,0}$. For $i \geq 1$ and $j = d+1$, we have $\alpha_{i,d+1} = (-1)^d d! \cdot c_{i,1}$.

Suppose that $d = 1$ and $\mathcal{N} \cong \mathcal{O}_Y$. Then we have $\alpha_{i,1} = \mathrm{ch}_{i,0}$ and

$$\alpha_{i,j}(a) = (-1)^{j-1} Q_{j-1}(c_{i,1}(a), \ldots, c_{i,j-1}(a))$$

for $i \geq 0$, $j \geq 2$, where Q_{j-1} denotes the $(j-1)$-st Newton polynomial which expresses the $(j-1)$-st power sum polynomial in terms of the elementary symmetric polynomials. In particular, $\alpha_{i,2} = -c_{i,1}$ for $i \geq 0$, and $\alpha_{i,j} = -(j-1)c_{i,j-1}$ and for $i \geq 1$, $j \geq 2$.

10.2. Let Z be an $\mathbb{F}_q$-scheme of pure dimension one which is separated of finite type over $\mathbb{F}_q$. We construct a canonical homomorphism $\mathrm{ch}'_{i,j} : G_i(Z) \to H^{2j-i}_{\mathcal{M}}(Z,\mathbb{Z}(j))$ for $(i,j) = (0,0)$, $(0,1)$, $(1,1)$, and $(1,2)$. Then we will show (Proposition 10.4) that the homomorphism

$$(\mathrm{ch}'_{i,i}, \mathrm{ch}'_{i,i+1}) : G_i(Z) \to H^i_{\mathcal{M}}(Z,\mathbb{Z}(i)) \oplus H^{i+2}_{\mathcal{M}}(Z,\mathbb{Z}(i+1)) \tag{10.2}$$

is an isomorphism for $i = 0, 1$. Since G-theory of Z and Z_{red} are isomorphic, and the same holds for motivic cohomology, it suffices to treat the case where Z is reduced.

Take a dense affine open smooth subscheme $Z_{(0)} \subset Z$, and let $Z_{(1)} = Z \setminus Z_{(0)}$ be the complement of $Z_{(0)}$ with the reduced scheme structure. We define $\mathrm{ch}'_{0,0}$ to be the composite

$$G_0(Z) \to K_0(Z_{(0)}) \xrightarrow{\mathrm{ch}_{0,0}} H^0_{\mathcal{M}}(Z_{(0)},\mathbb{Z}(0)) \cong H^0_{\mathcal{M}}(Z,\mathbb{Z}(0)).$$

We use the following lemma.

Lemma 10.3. *For $i = 0$ (resp. $i = 1$), the diagram*

$$\begin{array}{ccc} K_{i+1}(Z_{(0)}) & \longrightarrow & K_i(Z_{(1)}) \\ {\scriptstyle c_{i+1,i+1}}\downarrow & & \downarrow{\scriptstyle c_{i,i}\ (\textit{resp. } \mathrm{ch}_{0,0})} \\ H^{i+1}_{\mathcal{M}}(Z_{(0)},\mathbb{Z}(i+1)) & \longrightarrow & H^i_{\mathcal{M}}(Z_{(1)},\mathbb{Z}(i)) \end{array}$$

where each horizontal arrow is a part of localization sequence, is commutative.

Proof. Let $\widetilde{Z}$ denote the normalization of Z. We write $\widetilde{Z}_{(0)} = Z_{(0)} \times_Z \widetilde{Z}(\cong Z_{(0)})$ and $\widetilde{Z}_{(1)} = (Z_{(1)} \times_Z \widetilde{Z})_{\mathrm{red}}$. Comparing the diagrams

$$\begin{array}{ccc} K_{i+1}(\widetilde{Z}_{(0)}) & \to & K_i(\widetilde{Z}_{(1)}) \\ \downarrow & & \downarrow \\ K_{i+1}(Z_{(0)}) & \to & K_i(Z_{(1)}) \end{array} \quad \text{and} \quad \begin{array}{ccc} H^{i+1}_{\mathcal{M}}(\widetilde{Z}_{(0)}, \mathbb{Z}(i+1)) & \to & H^{i}_{\mathcal{M}}(\widetilde{Z}_{(1)}, \mathbb{Z}(i)) \\ \downarrow & & \downarrow \\ H^{i+1}_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(i+1)) & \to & H^{i}_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(i)) \end{array}$$

reduces us to proving the same claim for $\widetilde{Z}_{(0)}$ and $\widetilde{Z}_{(1)}$, which follows from Lemma 10.1. □

We define $\mathrm{ch}'_{1,1}$ to be the composite

$$\begin{aligned} &G_1(Z) \to \mathrm{Ker}[K_1(Z_{(0)}) \to K_0(Z_{(1)})] \\ &\xrightarrow{c_{1,1}} \mathrm{Ker}[H^1_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1)) \to H^0_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(0))] \cong H^1_{\mathcal{M}}(Z, \mathbb{Z}(1)). \end{aligned}$$

Next we define $\mathrm{ch}'_{1,2}$ when Z is connected. If Z is not proper, then $H^3_{\mathcal{M}}(Z, \mathbb{Z}(2))$ is zero by Proposition B.1. We put $\mathrm{ch}'_{1,2} = 0$ in this case. If Z is proper, then the push-forward map $H^3_{\mathcal{M}}(Z, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\mathrm{Spec}\, H^0(Z, \mathcal{O}_Z), \mathbb{Z}(1)) \cong K_1(H^0(Z, \mathcal{O}_Z))$ is an isomorphism. We define $\mathrm{ch}'_{1,2}$ to be (-1)-times the composite

$$G_1(Z) \to K_1(\mathrm{Spec}\, H^0(Z, \mathcal{O}_Z)) \cong H^3_{\mathcal{M}}(Z, \mathbb{Z}(2)).$$

We define $\mathrm{ch}'_{1,2}$ for non-connected Z to be the direct sum of $\mathrm{ch}'_{1,2}$ for each connected component of Z.

Observe that the group $G_0(Z)$ is generated by the two subgroups $M_1 = \mathrm{Image}[K_0(Z_{(1)}) \to G_0(Z)]$ and $M_2 = \mathrm{Image}[K_0(\widetilde{Z}) \to G_0(Z)]$. One can see by using Lemma 10.3 and the localization sequences that the isomorphism $\mathrm{ch}_{0,0} : K_0(Z_{(1)}) \xrightarrow{\cong} H^0_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(0))$ induces a homomorphism $\mathrm{ch}'_{0,1} : M_1 \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$. The kernel of $K_0(\widetilde{Z}) \to G_0(Z)$ is contained in the image of $K_0(\widetilde{Z}_{(1)}) \to K_0(\widetilde{Z})$. It is easily checked that the composite

$$K_0(\widetilde{Z}_{(1)}) \to K_0(\widetilde{Z}) \xrightarrow{c_{0,1}} H^2_{\mathcal{M}}(\widetilde{Z}, \mathbb{Z}(1)) \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$$

equals the composite

$$K_0(\widetilde{Z}_{(1)}) \to K_0(Z_{(1)}) \twoheadrightarrow M_1 \xrightarrow{\mathrm{ch}'_{0,1}} H^2_{\mathcal{M}}(Z, \mathbb{Z}(1)).$$

Hence the homomorphism $c_{0,1} : K_0(\widetilde{Z}) \to H^2_{\mathcal{M}}(\widetilde{Z}, \mathbb{Z}(1))$ induces a homomorphism $\mathrm{ch}'_{0,1} : M_2 \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$ such that the two homomorphisms $\mathrm{ch}'_{0,1} : M_i \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$, $i = 1, 2$, coincide on $M_1 \cap M_2$. Thus we obtain a homomorphism $\mathrm{ch}'_{0,1} : G_0(Z) \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$.

It is easily seen that the four homomorphisms $\mathrm{ch}'_{0,0}$, $\mathrm{ch}'_{0,1}$, $\mathrm{ch}'_{1,1}$, and $\mathrm{ch}'_{1,2}$ do not depend on the choice of $Z_{(0)}$.

Proposition 10.4. *The homomorphism (10.2) for $i = 0, 1$ is an isomorphism.*

Proof. It follows from [Ba-Mi-Se, Corollary 4.3] that the map $c_{1,1} : K_1(Z_{(0)}) \to H^1_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1))$ is an isomorphism. Hence, by construction, $\mathrm{ch}'_{1,1}$ is surjective and its kernel equals the image of $K_1(Z_{(1)}) \to G_1(Z)$. It follows from the vanishing of K_2 groups of finite fields that the homomorphism $c_{2,2} : K_2(Z_{(0)}) \to H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(2))$ is an isomorphism. We then have isomorphisms

$$\mathrm{Image}[K_1(Z_{(1)}) \to G_1(Z)] \cong \mathrm{Image}[H^1_{\mathcal{M}}(Z_{(1)}, \mathbb{Z}(1)) \to H^3_{\mathcal{M}}(Z, \mathbb{Z}(2))] \cong H^3_{\mathcal{M}}(Z, \mathbb{Z}(2)),$$

the first of which is by Lemma 10.3, and the second is by [Ba-Mi-Se, Corollary 4.3]. Therefore the composite $\mathrm{Ker}\,\mathrm{ch}'_{1,1} \hookrightarrow G_1(Z) \xrightarrow{\mathrm{ch}'_{1,2}} H^3_{\mathcal{M}}(Z, \mathbb{Z}(2))$ is an isomorphism. This proves the claim for $G_1(Z)$.

By the construction of $\mathrm{ch}'_{0,1}$, the image of $\mathrm{ch}'_{0,1}$ contains the image of $H^0_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(0)) \to H^2_{\mathcal{M}}(Z, \mathbb{Z}(1))$, and the composite $K_0(\widetilde{Z}) \to G_0(Z) \xrightarrow{\mathrm{ch}'_{0,1}} H^2_{\mathcal{M}}(Z, \mathbb{Z}(1)) \to H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1))$ equals the composite $K_0(\widetilde{Z}) \to K_0(\widetilde{Z}_{(0)}) \cong K_0(Z_{(0)}) \xrightarrow{c_{0,1}} H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1))$. This implies that $\mathrm{ch}'_{0,1}$ is surjective and the homomorphism

$$\mathrm{Ker}\,\mathrm{ch}'_{0,1} \to \mathrm{Ker}[K_0(Z_{(0)}) \xrightarrow{c_{0,1}} H^2_{\mathcal{M}}(Z_{(0)}, \mathbb{Z}(1))]$$

is an isomorphism. This proves the claim for $G_0(Z)$. □

11. K-groups and motivic cohomology of curves over a function field

From the computations of motivic cohomology of a surface with a fibration, we deduce some results concerning the K-groups of low degrees of the generic fiber. We relate the two using Chern class maps and by taking the limit.

Let C be a smooth projective, geometrically connected curve over a finite field $\mathbb{F}_q$. Let k denote the function field of C. Let X be a smooth projective geometrically connected curve over k. Let $\mathcal{X}$ be a regular model of X which is proper and flat over C.

Lemma 11.1. *The map*

$$K_1(X) \xrightarrow{(c_{1,1}, c_{1,2})} k^\times \oplus H^3_{\mathcal{M}}(X, \mathbb{Z}(2))$$

is an isomorphism. The group $H^4_{\mathcal{M}}(X, \mathbb{Z}(3))$ is a torsion group and there exists a canonical short exact sequence

$$0 \to H^4_{\mathcal{M}}(X, \mathbb{Z}(3)) \xrightarrow{\beta} K_2(X) \xrightarrow{c_{2,2}} H^2_{\mathcal{M}}(X, \mathbb{Z}(2)) \to 0$$

such that the composite $c_{2,3} \circ \beta$ equals the multiplication-by-2 map.

Proof. Let X_0 denote the set of closed points of X. Construct a commutative diagram by connecting the localization sequence

$$\begin{aligned} &\textstyle\bigoplus_{x \in X_0} K_2(\kappa(x)) \to K_2(X) \to K_2(k(X)) \\ \to\ &\textstyle\bigoplus_{x \in X_0} K_1(\kappa(x)) \to K_1(X) \to K_1(k(X)) \to \bigoplus_{x \in X_0} K_0(x) \end{aligned}$$

with the localization sequence

$$\begin{aligned} &\textstyle\bigoplus_{x \in X_0} H^0_{\mathcal{M}}(\mathrm{Spec}\,\kappa(x), \mathbb{Z}(1)) \to H^2_{\mathcal{M}}(X, \mathbb{Z}(2)) \to H^2_{\mathcal{M}}(\mathrm{Spec}\,k(X), \mathbb{Z}(2)) \\ \to\ &\textstyle\bigoplus_{x \in X_0} H^1_{\mathcal{M}}(\mathrm{Spec}\,\kappa(x), \mathbb{Z}(1)) \to H^3_{\mathcal{M}}(X, \mathbb{Z}(2)) \to H^3_{\mathcal{M}}(\mathrm{Spec}\,k(X), \mathbb{Z}(2)), \end{aligned}$$

using the Chern class maps. Since $H^0(\mathrm{Spec}\,\kappa(x), \mathbb{Z}(1)) = 0$ and the K-groups and motivic cohomology groups of fields agree in low degrees, the claim for $K_1(X)$ follows from diagram chasing.

It also follows from diagram chasing that

$$K_3(k(X)) \to \bigoplus_{x \in X_0} K_2(\kappa(x)) \to K_2(X) \xrightarrow{c_{2,2}} H^2_{\mathcal{M}}(X, \mathbb{Z}(2)) \to 0$$

is exact. By [Ne-Su] and [To], the group $H^3_{\mathcal{M}}(k(X), \mathbb{Z}(3))$ and the group $H^2_{\mathcal{M}}(\operatorname{Spec}\kappa(x), \mathbb{Z}(2))$ for each $x \in X_0$ are isomorphic to the Milnor K-groups $K_3^M(k(X))$ and $K_2^M(\kappa(x))$ respectively. We easily see from the definition of these isomorphisms in [To] that the boundary map $H^3_{\mathcal{M}}(k(X), \mathbb{Z}(3)) \to H^2_{\mathcal{M}}(\operatorname{Spec}\kappa(x), \mathbb{Z}(2))$ is identified with the boundary map $K_3^M(k(X)) \to K_2^M(\kappa(x))$ under these isomorphisms. Hence by [Me-Su2, Proposition 11.11], we have isomorphisms

$$\begin{aligned} & \operatorname{Coker}[K_3(k(X)) \to \textstyle\bigoplus_{x \in X_0} K_2(\kappa(x))] \\ \xrightarrow{\cong}\ & \operatorname{Coker}[H^3_{\mathcal{M}}(k(X), \mathbb{Z}(3)) \to \textstyle\bigoplus_{x \in X_0} H^2_{\mathcal{M}}(\operatorname{Spec}\kappa(x), \mathbb{Z}(2))] \\ \xrightarrow{\cong}\ & H^4_{\mathcal{M}}(X, \mathbb{Z}(3)). \end{aligned}$$

This gives the desired short exact sequence. The identity $c_{2,3} \circ \beta = 2$ follows from Remark 10.2. Since $H^2_{\mathcal{M}}(\operatorname{Spec}\kappa(x), \mathbb{Z}(2))$ is a torsion group for each $x \in X_0$, the group $H^4_{\mathcal{M}}(X, \mathbb{Z}(3))$ is a torsion group. This completes the proof. $\square$

Lemma 11.2. *Let $U \subset C$ be a non-empty open. We denote by $\mathcal{X}^U$ the complement $\mathcal{X} \setminus \mathcal{X} \times_C U$ with the reduced scheme structure. Then For $(i,j) = (0,0)$, $(0,1)$ or $(1,1)$, the diagram*

$$\begin{array}{ccc} K_{i+1}(X) & \longrightarrow & G_i(\mathcal{X}^U) \\ {\scriptstyle c_{i+1,j+1}}\downarrow & & \downarrow{\scriptstyle (-1)^j \mathrm{ch}'_{i,j}} \\ H^{2j-i+1}_{\mathcal{M}}(X, \mathbb{Z}(j+1)) & \longrightarrow & H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(j)) \end{array} \tag{11.1}$$

where each horizontal arrow is a part of localization sequence, is commutative.

Proof. Let $\mathcal{X}^U_{\mathrm{sm}} \subset \mathcal{X}^U$ denote the smooth locus. The commutativity of the diagram (11.1) for $(i,j) = (1,1)$ (resp. for $(i,j) = (0,0)$) follows from the commutativity of the diagram

$$\begin{array}{ccc} K_{i+1}(X) & \longrightarrow & K_i(\mathcal{X}^U_{\mathrm{sm}}) \\ {\scriptstyle c_{i+1,j+1}}\downarrow & & \downarrow{\scriptstyle -c_{1,1}\ (\text{resp. } \mathrm{ch}_{0,0})} \\ H^{2j-i+1}_{\mathcal{M}}(X, \mathbb{Z}(j+1)) & \longrightarrow & H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U_{\mathrm{sm}}, \mathbb{Z}(j)) \end{array}$$

and the injectivity of $H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(j)) \to H^{2j-i}_{\mathcal{M}}(\mathcal{X}^U_{\mathrm{sm}}, \mathbb{Z}(j))$.

By Lemma 11.1, the group $K_1(X)$ is generated by the image of the push-forward homomorphism $\bigoplus_{x \in X_0} K_1(\kappa(x)) \to K_1(X)$ and the image of the pull-back $K_1(k) \to K_1(X)$. Then the commutativity of the diagram (11.1) for $(i,j) = (0,1)$ follows from the commutativity of the diagram

$$\begin{array}{ccc} K_0(Y) & \longrightarrow & G_0(\mathcal{X}^U) \\ {\scriptstyle \mathrm{ch}_{0,0}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}'_{0,1}} \\ H^0_{\mathcal{M}}(Y, \mathbb{Z}(0)) & \longrightarrow & H^2_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(1)) \end{array}$$

for any finite reduced closed subscheme $Y \subset \mathcal{X}^U$, and the fact that the composite $K_0(C\setminus U) \xrightarrow{f^{U*}} G_0(\mathcal{X}^U) \xrightarrow{\mathrm{ch}'_{0,1}} H^2_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(1))$ is zero. Here $f^U : \mathcal{X}^U \to C \setminus U$ denotes the morphism induced by the morphism $\mathcal{X} \to C$. $\square$

Lemma 11.3. *The diagram*

$$\begin{array}{ccccccccc} 0 \to & H^4_{\mathcal{M}}(X, \mathbb{Z}(3)) & \to & K_2(X) & \xrightarrow{c_{2,2}} & H^2_{\mathcal{M}}(X, \mathbb{Z}(2)) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \bigoplus_{\wp \in C_0} H^3_{\mathcal{M}}(\mathcal{X}_\wp, \mathbb{Z}(2)) & \to & \bigoplus_{\wp \in C_0} G_1(\mathcal{X}_\wp) & \xrightarrow{-\mathrm{ch}'_{1,1}} & \bigoplus_{\wp \in C_0} H^1_{\mathcal{M}}(\mathcal{X}_\wp, \mathbb{Z}(1)) & \to 0 \end{array} \tag{11.2}$$

with exact rows, is commutative.

Proof. The commutativity of the right square follows from Lemma 11.2. For each closed point $x \in X_0$, let D_x denotes the closure of x in $\mathcal{X}$ and write $D_{x,\wp} = D_x \times_C \operatorname{Spec} \kappa(\wp)$. Then the commutativity of the left square in (11.2) follows from the commutativity of the diagram

$$\begin{array}{ccccccc} H^4_{\mathcal{M}}(X, \mathbb{Z}(3)) & \leftarrow & H^2_{\mathcal{M}}(\operatorname{Spec} \kappa(x), \mathbb{Z}(2)) & \cong & K_2(\kappa(x)) & \to & K_2(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{\wp \in C_0} H^3_{\mathcal{M}}(\mathcal{X}_\wp, \mathbb{Z}(2)) & \leftarrow & \bigoplus_{\wp \in C_0} H^1_{\mathcal{M}}(D_{x,\wp}, \mathbb{Z}(1)) & \cong & \bigoplus_{\wp \in C_0} G_1(D_{x,\wp}) & \to & \bigoplus_{\wp \in C_0} G_1(\mathcal{X}_\wp). \end{array}$$

$\square$

Lemma 11.4. *Let $\partial : H^4_{\mathcal{M}}(\mathcal{X} \times_C U, \mathbb{Z}(3)) \to H^3_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(2))$ denote the boundary map of the localization sequence. Then the composite*

$$\alpha : \operatorname{Coker} \partial \hookrightarrow H^5_{\mathcal{M}}(\mathcal{X}, \mathbb{Z}(3)) \to H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}(1)) \cong \mathbb{F}_q^\times$$

is an isomorphism.

Proof. For each closed point $x \in X_0$, let D_x denote the closure of x in $\mathcal{X}$. We put $D_{x,U} = D_x \times_C U$. Let D_x^U denote the complement $D_x \setminus D_{x,U}$ with the reduced scheme structure. Let $\iota_x : D_x \hookrightarrow \mathcal{X}$, $\iota_{x,U} : D_{x,U} \hookrightarrow \mathcal{X}_U$, $\iota_x^U : D_x^U \hookrightarrow \mathcal{X}^U$ denote the canonical inclusions. Let us consider the commutative diagram

$$\begin{array}{ccccc} H^2_{\mathcal{M}}(D_{x,U}, \mathbb{Z}(2)) & \longrightarrow & H^1_{\mathcal{M}}(D_x^U, \mathbb{Z}(1)) & \xrightarrow{\beta} & H^3_{\mathcal{M}}(D_x, \mathbb{Z}(2)) \\ \downarrow & & {\scriptstyle \iota^U_{x*}}\downarrow{\scriptstyle\cong} & & \\ H^4_{\mathcal{M}}(\mathcal{X} \times_C U, \mathbb{Z}(3)) & \xrightarrow{\partial} & H^3_{\mathcal{M}}(\mathcal{X}^U, \mathbb{Z}(2)) & \longrightarrow & \operatorname{Coker} \partial \to 0 \end{array}$$

with exact rows. Since X is geometrically connected, the Stein factorization shows that every fiber of $\mathcal{X} \to C$ is connected. In particular D_x intersects every connected component of $\mathcal{X}^U$. This implies that the homomorphism ι^U_{x*} in the above diagram is surjective. Hence we have a surjective homomorphism $\operatorname{Image} \beta \twoheadrightarrow \operatorname{Coker} \partial$. Let $\mathbb{F}(x)$ denote the finite field $H^0(D_x, \mathcal{O}_{D_x})$. The push-forward $H^3_{\mathcal{M}}(D_x, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{F}(x), \mathbb{Z}(1)) \cong \mathbb{F}(x)^\times$ gives an isomorphism $\operatorname{Image} \beta \cong \mathbb{F}(x)^\times$. Hence $|\operatorname{Coker} \partial|$ divides $\gcd_{x \in X_0}(|\mathbb{F}(x)^\times|) = q - 1$, where the equality follows from Lemma B.8. It is easily checked that the composite

$$\mathbb{F}(x)^\times \cong \operatorname{Image} \beta \twoheadrightarrow \operatorname{Coker} \partial \xrightarrow{\alpha} \mathbb{F}_q^\times$$

equals the norm map $\mathbb{F}(x)^\times \to \mathbb{F}_q^\times$, which implies $|\mathrm{Coker}\,\partial| \geq q-1$. Hence $|\mathrm{Coker}\,\partial| = q-1$ and the homomorphism α is an isomorphism. The claim is proved. $\square$

12. Applications

We give some applications of Theorem 1.1. In this section, we do not use the Bloch-Kato conjecture. The objective is to prove Theorems 12.1, 12.2, 12.3. The statements give some information on the structures of K-groups and motivic cohomology groups of elliptic curves over global fields and of the (open) complement of some fibers of elliptic surfaces over finite fields. Milne [Mi2] expresses the special values of zeta functions in terms of the order of arithmetic etale cohomology groups. We compute the orders of some torsion groups, in terms of the special values of L-functions, the torsion subgroup of (twisted) Mordell-Weil group, and some invariants of the base curve.

Let us list the ingredients of the proof. Using Theorem 1.1, we deduce that the torsion subgroups we are interested in are actually finite. Then the theorems of Geisser-Levine and Merkurjev-Suslin relate the motivic cohomology groups modulo uniquely divisible part and the etale cohomology and the cohomology of de-Rham Witt complexes. We use the arguments which appear in [Mi1], [Co-Sa-So], [Gro-Su] to compute such cohomology groups. The computation of the exact orders of the torsion may be new. One geometric property of elliptic surfaces which makes this explicit calculation possible is that the (abelian) fundamental group is isomorphic to that of the base curve. This follows from a theorem in [Sh] for the prime-to-p part. The use of the class field theory of Kato-Saito for surfaces over finite fields ([Ka-Sa]) is somewhat indirect but we then know that the groups of zero-cycles on the elliptic surface and on the base curve are isomorphic.

12.1. **Notations.** Let k, E, S_0, r, C, and $\mathcal{E}$ be as in Section 1. We also let S_1 (resp. S_2) denote the set of primes of k at which E has multiplicative (resp. bad) reduction. Thus we have $S_0 \subset S_1 \subset S_2$. Let p denote the characteristic of k. The closure of the origin of E in $\mathcal{E}$ gives a section to $\mathcal{E} \to C$, which we denote by $\iota : C \to \mathcal{E}$. Throughout this section, we assume that the structure morphism $f : \mathcal{E} \to C$ is not smooth. For any scheme X over C, let $\mathcal{E}_X$ denote the base change $\mathcal{E} \times_C X$. For any non-empty open $U \subset C$, we denote by $\mathcal{E}^U$ the complement $\mathcal{E} \setminus \mathcal{E}_U$ with the reduced scheme structure.

Let $\mathbb{F}_q$ denote the field of constants of C. We take an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Let $\mathrm{Frob} \in G_{\mathbb{F}_q} = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ denote the geometric Frobenius. For a scheme X over $\mathbb{F}_q$, we denote by $\overline{X}$ its base change $\overline{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ to $\overline{\mathbb{F}}_q$. We often regard the set $\mathrm{Irr}(X)$ as a finite etale $\mathbb{F}_q$-scheme corresponding to the $G_{\mathbb{F}_q}$-set $\mathrm{Irr}(\overline{X})$.

12.2. **Results.** We put $T = E(k \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)_{\mathrm{tors}}$ and, for each integer $j \in \mathbb{Z}$, $T'_{(j)} = \bigoplus_{\ell \neq p} (T \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(j))^{G_{\mathbb{F}_q}}$.

Theorem 12.1. *Let the notations and the assumptions be as above. Let $L(E,s)$ denote the L-function of the elliptic curve E over the global field k.*

1. *The $\mathbb{Q}$-vector space $(K_2(E)^{\mathrm{red}})_{\mathbb{Q}}$ is of dimension r.*

(2) *The cokernel of the boundary map* $\partial_2 : K_2(E) \to \bigoplus_{\wp \in C_0} G_1(\mathcal{E}_\wp)$ *is a finite group of order*
$$\frac{(q-1)^2 |L(h^0(\mathrm{Irr}(\mathcal{E}_{S_2})),-1)|}{|T'_{(1)}| \cdot |L(h^0(C \setminus S_2),-1)|}.$$
(3) *The group* $K_1(E)_{\mathrm{div}}$ *is uniquely divisible.*
(4) *The kernel of the boundary map* $\partial_1 : K_1(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_0(\mathcal{E}_\wp)$ *is a finite group of order* $(q-1)^2 |T'_{(1)}| \cdot |L(E,0)|$. *The cokernel of* ∂_1 *is a finitely generated abelian group of rank* $2 + |\mathrm{Irr}(\mathcal{E}_{S_2})| - |S_2|$ *whose torsion subgroup is isomorphic to* $\mathrm{Jac}(C)(\mathbb{F}_q)^{\oplus 2}$, *where* $\mathrm{Jac}(C)$ *denotes the Jacobian of* C.

For an $\mathbb{F}_q$-scheme X of finite type and for $i \in \mathbb{Z}$, choose a prime number $\ell \neq p$ and put $L(h^i(X),s) = \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell))$. In all the cases considered in this paper, the function $L(h^i(X),s)$ does not depend on the choice of ℓ.

For each non-empty open $U \subset C$, let T_U denote the torsion subgroup of the group $\mathrm{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}$ of divisors on $\overline{\mathcal{E}}_U$ modulo algebraic equivalence. For each integer $j \in \mathbb{Z}$, we put $T'_{U,(j)} = \bigoplus_{\ell \neq p} (T_U \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(j))^{G_{\mathbb{F}_q}}$. It is easily checked that the canonical homomorphism $T_U \to \varinjlim_{U'} T_{U'}$ is injective and is an isomorphism if $\mathcal{E}_U \to U$ is smooth. By [Sh], there is a canonical isomorphism $\varinjlim_{U'} T_{U'} \cong T$. In particular, we have an injection $T'_{U,(j)} \hookrightarrow T'_{(j)}$ which is an isomorphism if $\mathcal{E}_U \to U$ is smooth.

In Section 12.7, we deduce Theorem 12.1 from the following two theorems.

Theorem 12.2. *Let the notations and the assumptions be as above. Let* $\partial^i_{\mathcal{M},j} : H^i_{\mathcal{M}}(E,\mathbb{Z}(j))^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} H^{i-1}_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(j-1))$ *denote the homomorphism induced by the boundary map of the localization sequence established in* [Bl3].

(1) *For any* $i \in \mathbb{Z}$, *the group* $H^i_{\mathcal{M}}(E,\mathbb{Z}(2))_{\mathrm{div}}$ *is uniquely divisible.*
(2) *For* $i \leq 0$, *the group* $H^i_{\mathcal{M}}(E,\mathbb{Z}(2))$ *is uniquely divisible. The group* $H^1_{\mathcal{M}}(E,\mathbb{Z}(2))$ *is finite modulo uniquely divisible subgroup and the group* $H^1_{\mathcal{M}}(E,\mathbb{Z}(2))_{\mathrm{tors}}$ *is cyclic of order* q^2-1.
(3) *The kernel (resp. cokernel) of* $\partial^2_{\mathcal{M},2}$ *is a finite group of order* $|L(h^1(C),-1)|$ *(resp. of order*
$$\frac{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}_{S_2})),-1)|}{|T'_{(1)}| \cdot |L(h^0(C \setminus S_2),-1)|}).$$
(4) *The kernel (resp. cokernel) of the homomorphism* $\partial^3_{\mathcal{M},2}$ *is a finite group of order* $(q-1)|T'_{(1)}| \cdot |L(E,0)|$ *(resp. is isomorphic to* $\mathrm{Pic}(C)$*).*
(5) *For* $i \geq 4$, *the group* $H^i_{\mathcal{M}}(E,\mathbb{Z}(2))$ *is zero.*
(6) *The group* $H^4_{\mathcal{M}}(E,\mathbb{Z}(3))$ *is a torsion group, and the cokernel of the homomorphism* $\partial^4_{\mathcal{M},3}$ *is a finite cyclic group of order* $q-1$.

Theorem 12.3. *Let* $U \subset C$ *be a non-empty open. Then*

(1) *For any* $i \in \mathbb{Z}$, *the group* $H^i_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(2))$ *is finitely generated modulo uniquely divisible subgroup.*
(2) *For* $i \leq 0$, *the group* $H^i_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(2))$ *is uniquely divisible. The group* $H^1_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(2))$ *is finite modulo uniquely divisible subgroup and the group* $H^1_{\mathcal{M}}(\mathcal{E}_U,\mathbb{Z}(2))_{\mathrm{tors}}$ *is cyclic of order* q^2-1.

(3) *The rank of* $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ *is* $|S_0 \backslash U|$. *If* $U = C$ *(resp.* $U \neq C$*), the torsion subgroup of* $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ *is of order* $|L(h^1(C), -1)|$ *(resp. of order* $|T'_{U,(1)}| \cdot |L(h^1(C), -1) L(h^0(C \setminus U), -1)|/(q-1)$*).*

(4) *If* $U = C$ *(resp.* $U \neq C$*), the cokernel of* $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ *is zero (resp. is finite of order*
$$\frac{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}{|T'_{U,(1)}| \cdot |L(h^0(C \setminus U), -1)|}).$$

(5) *The rank of* $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ *is* $\max(|C \setminus U| - 1, 0)$. *If* $U = C$ *(resp.* $U \neq C$*), the torsion subgroup* $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}$ *is finite of order* $|L(h^2(\mathcal{E}), 0)|$ *(resp. of order*
$$\frac{|T'_{U,(1)}| \cdot |L(h^2(\mathcal{E}), 0) L^*(h^1(\mathcal{E}^U), 0) L(h^0(C \setminus U), -1)|}{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}).$$
Here
$$L^*(h^1(\mathcal{E}^U), 0) = \lim_{s \to 0} (s \log q)^{-|S_0 \backslash U|} L(h^1(\mathcal{E}^U), s)$$
is the leading coefficient of $L(h^1(\mathcal{E}^U), s)$.

(6) *The group* $H^4_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ *is canonically isomorphic to* $\mathrm{Pic}(U)$. *For* $i \geq 5$*, the group* $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ *is zero.*

12.3. **Relation between $L(E, s)$ and the congruence zeta function of $\mathcal{E}$.** Let $\ell \neq p$ be a prime number. By the Grothendieck-Lefschetz trace formula, we have
$$L(E, s) = \prod_{i=0}^{2} \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell))^{(-1)^{i-1}}.$$

Lemma 12.4. *Let* D *be a proper* $\mathbb{F}_q$*-scheme of dimension* ≤ 1. *Let* $\ell \neq p$ *be an integer. Then the group* $H^i_{\mathrm{et}}(\overline{D}, \mathbb{Z}_\ell)$ *is torsion free for any* $i \in \mathbb{Z}$*, and is zero for* $i \neq 0, 1, 2$. *The group* $H^i_{\mathrm{et}}(\overline{D}, \mathbb{Q}_\ell)$ *is pure of weight* i *for* $i \neq 1$*, and is mixed of weight* $\{0, 1\}$ *for* $i = 1$. *The group* $H^1_{\mathrm{et}}(\overline{D}, \mathbb{Q}_\ell)$ *is pure of weight one (resp. pure of weight zero) if* D *is smooth (resp. every irreducible component of* $\overline{D}$ *is rational).*

Proof. We may assume that D is reduced. Let D' be the normalization of D. Let $\pi : D' \to D$ denote the canonical morphism. Let $\mathcal{F}_n$ denote the cokernel of the homomorphism $\mathbb{Z}/\ell^n \to \pi_*(\mathbb{Z}/\ell^n)$ of etale sheaves. The sheaf $\mathcal{F}_n$ is supported on the singular locus D_{sing} of D and is isomorphic to $i_*(\mathrm{Coker}[\mathbb{Z}/\ell^n \to \pi_{\mathrm{sing}*}(\mathbb{Z}/\ell^n)])$, where $i : D_{\mathrm{sing}} \hookrightarrow D$ is the canonical inclusion and $\pi_{\mathrm{sing}} : D' \times_D D_{\mathrm{sing}} \to D_{\mathrm{sing}}$ is the base change of π. Then the claim follows from the long exact sequence
$$\cdots \to H^i_{\mathrm{et}}(\overline{D}, \mathbb{Z}/\ell^n) \to H^i_{\mathrm{et}}(\overline{D}', \mathbb{Z}/\ell^n) \to H^i_{\mathrm{et}}(\overline{D}, \mathcal{F}_n) \to \cdots.$$
□

Lemma 12.5. *For* $i \neq 1$, $H^i_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) = 0$.

Proof. For any point $x \in \overline{C}(\overline{\mathbb{F}}_q)$ lying over a closed point $\wp \in C$, the canonical homomorphism $H^0_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) \to H^1_{\mathrm{et}}(\mathcal{E}_x, \mathbb{Q}_\ell)$ is injective since $H^0_{\mathrm{et},c}(\overline{C} \setminus \{x\}, R^1 f_* \mathbb{Q}_\ell) = 0$. By Lemma 12.4, the module $H^1_{\mathrm{et}}(\mathcal{E}_x, \mathbb{Q}_\ell)$ is pure of weight 1 (resp. of weight 0) if $\mathcal{E}_x$ is smooth (resp. is not smooth). Since we have assumed that $\mathcal{E} \to C$ is not smooth, $H^0_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) = 0$.

Take a non-empty open $U \subset C$ such that $f_U : \mathcal{E}_U \to U$ is smooth. By Poincare duality, the group $H^2_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell) \cong H^2_{\mathrm{et},c}(\overline{U}, R^1 f_{U*}\mathbb{Q}_\ell)$ is the dual of $H^0_{\mathrm{et}}(\overline{U}, R^1 f_{U*}\mathbb{Q}_\ell(1))$. Assume $H^0_{\mathrm{et}}(\overline{U}, R^1 f_{U*}\mathbb{Q}_\ell(1)) \neq 0$. Let $T_\ell(E)$ denote the ℓ-adic Tate module of E. The etale fundamental group $\pi_1(U)$ acts on $T_\ell(E)$. By the assumption, the $\pi_1(\overline{U})$-invariant part $V = (T_\ell(E) \otimes \mathbb{Q}_\ell)^{\pi_1(\overline{U})}$ is non-zero. Since $\mathcal{E} \to C$ is not smooth, V is one dimensional. Hence we have a non-zero homomorphism $\pi_1(\overline{U})^{\mathrm{ab}} \to \mathrm{Hom}(T_\ell(E) \otimes \mathbb{Q}_\ell/V, V)$ of $G_{\mathbb{F}_q}$-module. By the weight argument, we see that this is impossible. Hence $H^2_{\mathrm{et}}(\overline{C}, R^1 f_* \mathbb{Q}_\ell(1)) = 0$. □

Corollary 12.6. *The spectral sequence*

$$E_2^{i,j} = H^i_{\mathrm{et}}(\overline{C}, R^j f_* \mathbb{Q}_\ell) \Rightarrow H^{i+j}_{\mathrm{et}}(\overline{\mathcal{E}}, R^j f_* \mathbb{Q}_\ell)$$

is E_2-degenerate.

Lemma 12.7. *Let $U \subset C$ be a non-empty open such that $f_U : \mathcal{E}_U \to U$ is smooth. Let $\mathrm{Irr}^0(\mathcal{E}^U) \subset \mathrm{Irr}(\mathcal{E}^U)$ denote the subscheme of the irreducible components of $\mathcal{E}^U$ which does not intersect $\iota(C)$. Then*

$$L(h^i(\mathcal{E}), s) = \begin{cases} (1 - q^{-s}), & \textit{if } i = 0, \\ L(h^1(C), s), & \textit{if } i = 1, \\ (1 - q^{1-s})^2 L(E, s) L(h^0(\mathrm{Irr}_0(\mathcal{E}^U)), s-1), & \textit{if } i = 2, \\ L(h^1(C), s-1), & \textit{if } i = 3, \\ (1 - q^{2-s}), & \textit{if } i = 4. \end{cases}$$

Proof. We prove the lemma for $i = 2$; other cases are easy. Since $R^2 f_{U*}\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(-1)$, there exists an exact sequence

$$0 \to H^0_{\mathrm{et}}(\overline{C}, R^2 f_* \mathbb{Q}_\ell) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \to H^1_{\mathrm{et},c}(\overline{U}, \mathbb{Q}_\ell(-1)).$$

The map $H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \to H^1_{\mathrm{et},c}(\overline{U}, \mathbb{Q}_\ell(-1))$ decomposes as

$$H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \to H^0_{\mathrm{et}}(\overline{C} \setminus \overline{U}, \mathbb{Q}_\ell(-1)) \to H^1_{\mathrm{et},c}(\overline{U}, \mathbb{Q}_\ell(-1)).$$

Hence $H^0_{\mathrm{et}}(\overline{C}, R^2 f_* \mathbb{Q}_\ell)$ is isomorphic to the inverse image of the image of $H^0_{\mathrm{et}}(\overline{C}, \mathbb{Q}_\ell(-1)) \to H^0_{\mathrm{et}}(\overline{C} \setminus \overline{U}, \mathbb{Q}_\ell(-1))$ by the surjective homomorphism $H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell) \to H^0_{\mathrm{et}}(\overline{C} \setminus \overline{U}, \mathbb{Q}_\ell(-1))$. This proves the claim. □

12.4. **The fundamental group of $\mathcal{E}$.**

Lemma 12.8. *For $i = 0, 1$, the pull-back $H^i(C, \mathcal{O}_C) \to H^i(\mathcal{E}, \mathcal{O}_\mathcal{E})$ is an isomorphism.*

Proof. The claim for $i = 0$ is clear. We prove the claim for $i = 1$. Let us write $\mathcal{L} = R^1 f_* \mathcal{O}_\mathcal{E}$. It suffices to prove $H^0(C, \mathcal{L}) = 0$. We note that $\mathcal{L}$ is an invertible $\mathcal{O}_C$-module since $\mathcal{E} \to C$ has no multiple fiber. The Leray spectral sequence $E_2^{i,j} = H^i(C, R^j f_* \mathcal{O}_\mathcal{E}) \Rightarrow H^{i+j}(\mathcal{E}, \mathcal{O}_\mathcal{E})$ shows that the Euler-Poincare characteristic $\chi(\mathcal{O}_\mathcal{E})$ equals $\chi(\mathcal{O}_C) - \chi(\mathcal{L}) = -\deg \mathcal{L}$. By the well-known inequality $\chi(\mathcal{O}_\mathcal{E}) > 0$ (cf. [Og], [Do], or [Ogu, Theorem 2]), we have $\deg \mathcal{L} < 0$. This proves $H^0(C, \mathcal{L}) = 0$. □

Lemma 12.9. (1) *The canonical homomorphism $\pi_1^{\mathrm{ab}}(\mathcal{E}) \to \pi_1^{\mathrm{ab}}(C)$ between the abelian (etale) fundamental groups is an isomorphism.*

(2) *The canonical morphism $\mathrm{Pic}^o_{C/\mathbb{F}_q} \to \mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q}$ between the identity components of the Picard schemes is an isomorphism.*

Proof. The homomorphism $\mathrm{Pic}^o_{C/\mathbb{F}_q} \to \mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q,\mathrm{red}}$ is an isomorphism by [Sh, Theorem 4.1]. This, combined with the cohomology long exact sequence of the Kummer sequence, shows that $H^1_{\mathrm{et}}(C,\mathbb{Z}/m) \to H^1_{\mathrm{et}}(\mathcal{E},\mathbb{Z}/m)$ is an isomorphism if $p \nmid m$. Hence, to prove (1), we are reduced to showing that $H^1_{\mathrm{et}}(C,\mathbb{Z}/p^n) \to H^1_{\mathrm{et}}(\mathcal{E},\mathbb{Z}/p^n)$ is an isomorphism for all $n \geq 1$. For any smooth $\mathbb{F}_q$-scheme X, there exists an exact sequence

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \to W_n\mathcal{O}_X \xrightarrow{1-\sigma} W_n\mathcal{O}_X \to 0$$

of etale sheaves, where $W_n\mathcal{O}_X$ is the sheaf of Witt vectors and $\sigma : W_n\mathcal{O}_X \to W_n\mathcal{O}_X$ is the Frobenius endomorphism. This gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\cdots \xrightarrow{1-\sigma} & H^0(C,W_n\mathcal{O}_C) & \to & H^1_{\mathrm{et}}(C,\mathbb{Z}/p^n) & \to & H^1(C,W_n\mathcal{O}_C) & \xrightarrow{1-\sigma} \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \xrightarrow{1-\sigma} & H^0(\mathcal{E},W_n\mathcal{O}_\mathcal{E}) & \to & H^1_{\mathrm{et}}(\mathcal{E},\mathbb{Z}/p^n) & \to & H^1(\mathcal{E},W_n\mathcal{O}_\mathcal{E}) & \xrightarrow{1-\sigma} \cdots .
\end{array}$$

By Lemma 12.8 and induction on n, we see that $H^i(C,W_n\mathcal{O}_C) \to H^i(\mathcal{E},W_n\mathcal{O}_\mathcal{E})$ is an isomorphism for $i = 0, 1$. Thus the homomorphism $H^1_{\mathrm{et}}(C,\mathbb{Z}/p^n) \to H^1_{\mathrm{et}}(\mathcal{E},\mathbb{Z}/p^n)$ is an isomorphism. This proves the claim (1).

For (2), it suffices to prove that the homomorphism $\mathrm{Lie}\,\mathrm{Pic}_{C/\mathbb{F}_q} \to \mathrm{Lie}\,\mathrm{Pic}_{\mathcal{E}/\mathbb{F}_q}$ between the tangent spaces is an isomorphism. Since this homomorphism is identified with the homomorphism $H^1(C,\mathcal{O}_C) \to H^1(\mathcal{E},\mathcal{O}_\mathcal{E})$, the claim (2) follows from Lemma 12.8. $\square$

Remark 12.10. Using Lemma 12.9 (1), we can prove that the homomorphism $\pi_1(\mathcal{E}) \to \pi_1(C)$ is an isomorphism. Since it is not used in this paper, let us only sketch the proof.

Let $x \to C$ be a geometric point. Since the morphism $f : \mathcal{E} \to C$ has a section, the fiber $\mathcal{E}_x$ of f at x has a reduced irreducible component. Hence, by the same argument as in the proof of [SGA1, X, Proposition 1.2, Théorème 1.3], we have an exact sequence

$$\pi_1(\mathcal{E}_x) \to \pi_1(\mathcal{E}) \to \pi_1(C) \to 1.$$

In particular, the kernel of $\pi_1(\mathcal{E}) \to \pi_1(C)$ is abelian. Applying Lemma 12.9(1) to $\mathcal{E} \times_C C' \to C'$ for each finite connected etale cover $C' \to C$, we obtain the bijectivity of $\pi_1(\mathcal{E}) \to \pi_1(C)$.

The statements in Lemma 12.9 and the statement above that the fundamental groups are isomorphic are also valid for $\mathcal{E}$ a regular, proper, non-smooth, minimal elliptic fibration with a section over C a proper smooth curve over an arbitrary perfect base field.

Corollary 12.11. *For any prime number $\ell \neq p$ and for any $i \in \mathbb{Z}$, the group $H^i_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible.*

Proof. The claim for $i \neq 1, 2$ is obvious. By Lemma 12.9, we have $H^1_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong H^1_{\mathrm{et}}(\overline{C},\mathbb{Q}_\ell/\mathbb{Z}_\ell)$. Hence $H^1_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible. The group $H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible since $H^2_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\mathrm{red}}$ is isomorphic to the Pontryagin dual of $H^1_{\mathrm{et}}(\overline{\mathcal{E}},\mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$. $\square$

Corollary 12.12. *For $i \in \mathbb{Z}$, we put $M^i_j = \bigoplus_{\ell \neq p} H^i_{\mathrm{et}}(\mathcal{E},\mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$. For a rational number a, we write $|a|^{(p')} = |a| \cdot |a|_p$.*

(1) *For $i \leq -1$ or $i \geq 6$, the group M^i_j is zero.*

(2) *For $j \neq 2$ (resp. $j = 2$), the group M^5_j is zero (resp. is isomorphic to $\bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$).*

(3) *For* $j \neq 0$, *the group* M_j^0 *is cyclic of order* $q^{|j|} - 1$. *The group* M_0^0 *is isomorphic to* $\bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$.
(4) *For* $j \neq 0$, *the group* M_j^1 *is finite of order* $|L(h^1(C), 1-j)|^{(p')}$.
(5) *For* $j \neq 1$, *the group* M_j^2 *is finite of order* $|L(h^2(\mathcal{E}), 2-j)|^{(p')}$.
(6) *For* $j \neq 1$, *the group* M_j^3 *is finite of order* $|L(h^1(C), 2-j)|^{(p')}$.
(7) *For* $j \neq 2$, *the group* M_j^4 *is cyclic of order* $q^{|2-j|} - 1$. *The group* M_2^4 *is isomorphic to* $\bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$.

Proof. By Corollary 12.11, if $i \neq 2j+1$ and $\ell \neq p$, $H^i_{\mathrm{et}}(\mathcal{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is isomorphic to the $G_{\mathbb{F}_q}$-invariant part of $H^i_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$. Hence by Poincare duality,

$$|H^i_{\mathrm{et}}(\mathcal{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))| = |L(h^{4-i}(\mathcal{E}), 2-j)|_\ell^{-1}$$

for $i \neq 2j, 2j+1$. Hence the claim follows from Lemma 12.7. $\square$

12.5. **Torsion in the etale cohomology of open elliptic surfaces.** We fix a non-empty open subscheme $U \subset C$.

Lemma 12.13. *Let* $\ell \neq p$ *be a prime number. For* $i \in \mathbb{Z}$, *let* γ_i *denote the pull-back* $\gamma_i : H^i_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell) \to H^i_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell)$.
(1) *For* $i \neq 0, 2$, *the homomorphism* γ_i *is zero.*
(2) $(\mathrm{Coker}\,\gamma_2)_{\mathbb{Q}_\ell}$ *is isomorphic to the kernel of* $H^0_{\mathrm{et}}(\overline{C} \setminus \overline{U}, \mathbb{Q}_\ell(-1)) \to H^0_{\mathrm{et}}(\mathrm{Spec}\,\overline{\mathbb{F}}_q, \mathbb{Q}_\ell(-1))$.
(3) *There is a canonical isomorphism*

$$\mathrm{Hom}_{\mathbb{Z}}(T_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) \cong (\mathrm{Coker}\,\gamma_2)_{\mathrm{tors}}.$$

Proof. By Lemma 12.9, the pull-back $H^1_{\mathrm{et}}(\overline{C}, \mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell)$ is an isomorphism. Hence the homomorphism $H^1_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell)$ is zero. The claim (1) follows.

Let $\mathrm{NS}(\overline{\mathcal{E}}) = \mathrm{Pic}_{\mathcal{E}/\mathbb{F}_q}(\overline{\mathbb{F}}_q)/\mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q}(\overline{\mathbb{F}}_q)$ denote the Neron-Severi group of $\overline{\mathcal{E}}$.

We have an exact sequence

$$0 \to \mathrm{NS}(\overline{\mathcal{E}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\mathrm{cl}_\ell} H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1)) \to T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m) \to 0.$$

We note that $T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m)$ is torsion free. For $D \in \mathrm{Irr}(\overline{\mathcal{E}^U})$, let $[D] \in \mathrm{NS}(\overline{\mathcal{E}})$ denote the class of the Weil divisor D_{red} on $\overline{\mathcal{E}}$. By [SGA4$\frac{1}{2}$, Cycle, 2.3], the D-component of the homomorphism $\gamma_2 : H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell) \cong \mathrm{Map}(\mathrm{Irr}(\overline{\mathcal{E}^U}), \mathbb{Z}_\ell(-1))$ is identified with the homomorphism

$$H^2_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell) \xrightarrow{\cup \mathrm{cl}_\ell([D])} H^4_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell(1)) \cong \mathbb{Z}_\ell(-1).$$

Let $M \subset \mathrm{NS}(\overline{\mathcal{E}})$ denote the subgroup generated by $\{[D] \mid D \in \mathrm{Irr}(\overline{\mathcal{E}^U})\}$. This is a free abelian group with basis $\mathrm{Irr}^0(\overline{\mathcal{E}^U}) \cup \{D'\}$, where D' is an arbitrary element in $\mathrm{Irr}(\overline{\mathcal{E}^U}) \setminus \mathrm{Irr}^0(\overline{\mathcal{E}^U})$. This proves (2).

By Corollary 12.11, the cup-product

$$H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1)) \times H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1)) \to H^4_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)) \cong \mathbb{Z}_\ell$$

is a perfect pairing. Since $\mathrm{Coker}\,\mathrm{cl}_\ell$ is torsion free as we noted, this pairing gives a duality between the torsion part of $\mathrm{Coker}\,\gamma_2$ and the torsion part of $(\mathrm{NS}(\overline{\mathcal{E}})/M) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong (\mathrm{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. Thus we have the claim (3). $\square$

Corollary 12.14. *For $i \neq 3$, $H^i_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell)$ is torsion free, and $H^3_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell)_{\mathrm{tors}}$ is canonically isomorphic to* $\mathrm{Hom}_{\mathbb{Z}}(T_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))$*. We put*

$$L(h^i_{c,\ell}(\mathcal{E}_U), s) = \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Q}_\ell)).$$

Then if $U \neq C$, we have

$$L(h^i_{c,\ell}(\mathcal{E}_U), s) = \begin{cases} 1, & \text{if } i \leq 0 \text{ or } i \geq 5, \\ \frac{L(h^1(C),s)L(h^0(C\setminus U),s)}{1-q^{-s}} & \text{if } i = 1, \\ \frac{L(h^2(\mathcal{E}),s)L(h^1(\mathcal{E}^U),s)L(h^0(C\setminus U),s-1)}{(1-q^{1-s})L(h^2(\mathcal{E}^U),s)} & \text{if } i = 2, \\ \frac{L(h^1(C),s-1)L(h^0(C\setminus U),s-1)}{1-q^{1-s}} & \text{if } i = 3, \\ 1 - q^{2-s}, & \text{if } i = 4. \end{cases}$$

Proof. This follows from Lemmas 12.7 and 12.13, and the long exact sequence

$$\cdots \to H^i_{\mathrm{et},c}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell) \to H^i_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell) \to H^i_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Z}_\ell) \to \cdots .$$

□

Remark 12.15. Corollary 12.14 in particular shows that the function $L(h^i_{c,\ell}(\mathcal{E}_U), s)$ is independent of $\ell \neq p$. We can show the ℓ-independence of $L(h^i_{c,\ell}(X), s) = \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{c,\mathrm{et}}(X, \mathbb{Q}_\ell))$ for any normal surface X over $\mathbb{F}_q$ which is not necessarily proper. Since we will not need it, let us only give a sketch. There is a proper smooth surface X' and a closed subset $D \subset X'$ of pure codimension one such that $X = X' \setminus D$. One can express the cokernel and kernel of the restriction map $H^1_{\mathrm{et}}(\overline{X}', \mathbb{Q}_l) \to H^1_{\mathrm{et}}(\overline{D}, \mathbb{Q}_l)$ in terms of $\mathrm{Pic}_{X'/\mathbb{F}_q}$ and the Jacobian of the normalization of each irreducible component of D. Then we apply the same method as above to obtain the result.

Corollary 12.16. *Suppose that $U \neq C$. Then*

(1) *The group $H^i_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is zero for $i \leq -1$ or $i \geq 5$.*

(2) *For $j \neq 0$, the group $H^0_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is isomorphic to $\mathbb{Z}_\ell/(q^j - 1)$, and we have $H^0_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) = \mathbb{Q}_\ell/\mathbb{Z}_\ell$.*

(3) *For $j \neq 0, 1$, the group $H^1_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is finite of order*

$$\frac{|T'_{U,(j-1)}|_\ell^{-1} \cdot |L(h^1(C), 1-j)L(h^0(C \setminus U), 1-j)|_\ell^{-1}}{|q^{j-1} - 1|_\ell^{-1}}.$$

The group $H^1_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))$ is isomorphic to the direct sum of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ and a finite group of order

$$\frac{|T'_{U,(-1)}|_\ell^{-1} \cdot |L(h^1(C), 1)L(h^0(C \setminus U), 1)|_\ell^{-1}}{|q - 1|_\ell^{-1}}.$$

(4) *For $j \neq 1, 2$, the group $H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is finite of order*

$$\frac{|T'_{U,(j-1)}|_\ell^{-1} \cdot |L(h^2(\mathcal{E}), 2-j)L(h^1(\mathcal{E}^U), 2-j)L(h^0(C \setminus U), 1-j)|_\ell^{-1}}{|(q^{j-1} - 1)L(h^2(\mathcal{E}^U), 2-j)|_\ell^{-1}}.$$

(5) *For $j \neq 1, 2$, the group $H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is finite of order*

$$\frac{|L(h^1(C), 2-j)L(h^0(C \setminus U), 2-j)|_\ell^{-1}}{|q^{j-2} - 1|_\ell^{-1}}.$$

The group $H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$ *is isomorphic to the direct sum of* $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus|C\setminus U|-1}$ *and a finite group of order*

$$\frac{|L(h^1(C),1)L(h^0(C\setminus U),1)|_\ell^{-1}}{|q-1|_\ell^{-1}}.$$

(6) *For* $j \neq 2$ *(resp.* $j=2$*), the group* $H^4_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ *is zero (resp. is isomorphic to* $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus|C\setminus U|-1}$*).*

Proof. The group $H^i_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is the Pontryagin dual of the group $H^{5-i}_{\mathrm{et},c}(\mathcal{E}_U, \mathbb{Z}_\ell(2-j))$. The claim follows from Corollary 12.14 and the short exact sequence

$$0 \to H^{4-i}_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2-j))_{G_{\mathbb{F}_q}} \to H^{5-i}_{c,\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2-j)) \to H^{5-i}_{c,\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2-j))^{G_{\mathbb{F}_q}} \to 0.$$

□

Lemma 12.17. *Suppose that* $U \neq C$*. Then* $H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$ *is finite of order*

$$\frac{|T'_{U,(1)}|_\ell^{-1} \cdot |L(h^2(\mathcal{E}),0)L^*(h^1(\mathcal{E}^U),0)L(h^0(C\setminus U),-1)|_\ell^{-1}}{|(q-1)L(h^0(\mathrm{Irr}(\mathcal{E}^U)),-1)|_\ell^{-1}}.$$

Proof. We note that the group $H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^{\mathrm{red}}$ is canonically isomorphic to the group $H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2))_{\mathrm{tors}}$. Let us consider the long exact sequence

$$\cdots \to H^i_{\mathcal{E}^U,\mathrm{et}}(\mathcal{E}, \mathbb{Z}_\ell(2)) \xrightarrow{\mu_i} H^i_{\mathrm{et}}(\mathcal{E}, \mathbb{Z}_\ell(2)) \to H^i_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2)) \to \cdots$$

The group $\operatorname{Ker}\mu_4$ is isomorphic to the Pontryagin dual of the cokernel of $H^1_{\mathrm{et}}(\mathcal{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\mathcal{E}^U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. By Lemma 12.9, this homomorphism factors through $H^1_{\mathrm{et}}(C\setminus U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1_{\mathrm{et}}(\mathcal{E}^U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. In particular, $(\operatorname{Ker}\mu_4)_{\mathrm{tors}}$ is isomorphic to the Pontryagin dual of the group $(H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_{\mathbb{F}_q}})^{\mathrm{red}}$. By the weight argument we see that $\operatorname{Coker}\mu_3$ is a finite group. It follows that

$$|H^3_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2))_{\mathrm{tors}}| = |L^*(h^1(\mathcal{E}^U),0)| \cdot |\operatorname{Coker}\mu_3|.$$

Let μ' denote the homomorphism $H^2_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))$. We have the exact sequence

$$\operatorname{Ker}\mu_3 \to H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}} \to (\operatorname{Coker}\mu')_{G_{\mathbb{F}_q}} \to \operatorname{Coker}\mu_3 \to 0. \tag{12.1}$$

Since $\operatorname{Ker}\mu_3 \cong \operatorname{Coker}[H^2_{\mathrm{et}}(\mathcal{E}, \mathbb{Z}_\ell(2)) \to H^2_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Z}_\ell(2))]$, the cokernel of

$$\operatorname{Ker}\mu_3 \to H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}}$$

is isomorphic to the cokernel of the homomorphism

$$\nu' : H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}} \to H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}}.$$

Let us consider the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Coker}\mu' \arrow[r] \arrow[d, "1-\mathrm{Frob}"'] & H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2)) \arrow[r, "\nu"] \arrow[d, "1-\mathrm{Frob}"'] & H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)) \arrow[d, "1-\mathrm{Frob}"'] \\
0 \arrow[r] & \operatorname{Coker}\mu' \arrow[r] & H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2)) \arrow[r, "\nu"] & H^3_{\overline{\mathcal{E}^U},\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)).
\end{tikzcd}$$

Since $(\operatorname{Coker}\nu)^{G_{\mathbb{F}_q}} \subset H^3_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))^{G_{\mathbb{F}_q}} = 0$, $\operatorname{Coker}\nu'$ is isomorphic to the kernel of $(\operatorname{Coker}\mu')_{G_{\mathbb{F}_q}} \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}}$. Hence by (12.1), $|\operatorname{Coker}\mu_3|$ equals the order of

$$\begin{aligned} M'' &= \operatorname{Image}[(\operatorname{Coker}\mu')_{G_{\mathbb{F}_q}} \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}}] \\ &= \operatorname{Image}[H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}} \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))_{G_{\mathbb{F}_q}}]. \end{aligned}$$

We put $M' = \operatorname{Image}[H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(2)) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2))]$. From the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & \operatorname{NS}(\overline{\mathcal{E}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \to & H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1)) & \to & T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & (\operatorname{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \to & H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(1)) & \to & T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{G}_m) & \to 0 \end{array} \tag{12.2}$$

and the exact sequence

$$0 \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m) \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{G}_m) \to H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}/\mathbb{Z}),$$

we obtain an exact sequence

$$0 \to M' \to H^2_{\mathrm{et}}(\overline{\mathcal{E}}_U, \mathbb{Z}_\ell(2)) \to T_\ell H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)).$$

By the weight argument, we have $(T_\ell H^1_{\mathrm{et}}(\overline{\mathcal{E}^U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)))^{G_{\mathbb{F}_q}} = \{0\}$. Hence the canonical surjection $M'_{G_{\mathbb{F}_q}} \to M''$ is an isomorphism. From (12.2) we have an exact sequence

$$0 \to (\operatorname{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1) \to M' \to T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m)(1) \to 0.$$

By the weight argument, we have $(T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m)(1))^{G_{\mathbb{F}_q}} = 0$. Hence

$$\begin{aligned} 0 &\to ((\operatorname{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1))_{G_{\mathbb{F}_q}} \to M'_{G_{\mathbb{F}_q}} \\ &\to (T_\ell H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{G}_m)(1))_{G_{\mathbb{F}_q}} \to 0 \end{aligned}$$

is exact. Therefore $|\operatorname{Coker}\mu_3| = |M'_{G_{\mathbb{F}_q}}|$ equals

$$\frac{|(T_U \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1))_{G_{\mathbb{F}_q}}| \cdot |\det(1 - \mathrm{Frob};\, H^2_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell(2)))|_\ell^{-1}}{|\det(1 - \mathrm{Frob};\, \operatorname{Ker}[\operatorname{NS}(\overline{\mathcal{E}}) \to \operatorname{Div}(\overline{\mathcal{E}}_U)/\sim_{\mathrm{alg}}] \otimes_{\mathbb{Z}} \mathbb{Q}_\ell(1))|_\ell^{-1}}.$$

This proves the claim. □

12.6. Fix a non-empty open $U \subset C$. Let $f^U : \mathcal{E}^U \to C \setminus U$ denote the structure morphism and let $\iota^U : C \setminus U \to \mathcal{E}^U$ denote the morphism induced from $\iota : C \to \mathcal{E}$.

Lemma 12.18. *The homomorphism*

$$(\mathrm{ch}'_{1,1}, f^U_*) : G_1(\mathcal{E}^U) \to H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \oplus K_1(C \setminus U)$$

is an isomorphism.

Proof. The morphism $f^U : \mathcal{E}^U \to C \setminus U$ has connected fibers. Hence the claim follows from Proposition 10.4 and the construction of $\mathrm{ch}'_{1,2}$. □

Lemma 12.19. *The group $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is finitely generated of rank $|C \setminus U|$. Moreover, the group $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}$ is of order $|L^*(h^1(\mathcal{E}^U), 0)|$.*

Proof. It suffices to prove the following claim: if E has good reduction (resp. non-split multiplicative reduction, resp. split multiplicative or additive reduction) at $\wp \in C_0$, then $H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))$ is a finitely generated abelian group of rank one, and $|H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))_{\mathrm{tors}}|$ equals $|\mathcal{E}_\wp(\kappa(\wp))|$ (resp. 2, resp. 1). We put $\mathcal{E}_{\wp,(0)} = (\mathcal{E}_{\wp,\mathrm{red}})_{\mathrm{sm}} \setminus \iota(\wp)$ and $\mathcal{E}_{\wp,(1)} = \mathcal{E}^U \setminus \mathcal{E}_{\wp,(0)}$. We have an exact sequence

$$H^1_{\mathcal{M}}(\mathcal{E}_{\wp,(0)}, \mathbb{Z}(1)) \to H^0_{\mathcal{M}}(\mathcal{E}_{\wp,(1)}, \mathbb{Z}(0)) \to H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1)) \to \mathrm{Pic}(\mathcal{E}_{\wp,(0)}) \to 0.$$

First suppose that E does not have non-split multiplicative reduction at $\wp$, or E has non-split multiplicative reduction at $\wp$ and $\mathcal{E}_\wp \otimes_{\kappa(\wp)} \overline{\mathbb{F}}_q$ has an even number of irreducible components. Then, using the classification due to Kodaira, Neron and Tate (cf. [Liu, 10.2]) of singular fibers of $\mathcal{E} \to C$, we can verify the equality

$$\begin{aligned} & \mathrm{Image}[H^0_{\mathcal{M}}(\mathcal{E}_{\wp,(1)}, \mathbb{Z}(0)) \to H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))] \\ = \ & \mathrm{Image}[\iota_* : H^0_{\mathcal{M}}(\operatorname{Spec} \kappa(\wp), \mathbb{Z}(0)) \to H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))]. \end{aligned}$$

This shows that $H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))$ is isomorphic to the direct sum of $H^0_{\mathcal{M}}(\operatorname{Spec} \kappa(\wp), \mathbb{Z}(0)) \cong \mathbb{Z}$ and $\mathrm{Pic}(\mathcal{E}_{\wp,(0)})$. In particular, we have $H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))_{\mathrm{tors}} \cong \mathrm{Pic}(\mathcal{E}_{\wp,(0)})$, from which we easily deduce the claim.

Now suppose that E has non-split multiplicative reduction at $\wp$ and $\mathcal{E}_\wp \otimes_{\kappa(\wp)} \overline{\mathbb{F}}_q$ has an odd number of irreducible components. In this case, we can directly verify that the image of $H^0_{\mathcal{M}}(\mathcal{E}_{\wp,(1)}, \mathbb{Z}(0)) \to H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$ and $\mathrm{Pic}(\mathcal{E}_{\wp,(0)}) = 0$. The claim in this case follows. □

Lemma 12.20. *The diagram*

$$\begin{array}{ccc} K_1(E) & \longrightarrow & G_0(\mathcal{E}^U) \\ {\scriptstyle \iota^*}\Big\downarrow & & {\scriptstyle \iota^{U*}}\Big\downarrow \\ K_1(k) & \longrightarrow & K_0(C \setminus U) \end{array}$$

is commutative.

Proof. The group $K_1(E)$ is generated by the image of $f^* : K_1(k) \to K_1(E)$ and the image of $\bigoplus_{x \in E_0} K_1(\kappa(x)) \to K_1(E)$. The claim follows from the fact that the localization sequence in G-theory commutes with flat pull-backs and finite push-forwards. □

12.7. **Proofs of Theorems 12.1, 12.2, and 12.3.**

Lemma 12.21. *For any non-empty open $U \subset C$, the cokernel of the boundary map ∂_U : $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2)) \to H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is finite.*

Proof. If suffices to prove the claim for sufficiently small U. Hence we may assume that $\mathcal{E}_U \to U$ is smooth. Since $K_2(\mathcal{E}_U)_{\mathbb{Q}} \to K_2(E)_{\mathbb{Q}}$ is an isomorphism in this case, the claim follows from Theorem 1.1 and Lemma 11.2. □

Proof of Theorem 12.3. The claims (1) and (2) follow from Theorem 9.3, Proposition 9.8 and Lemma 12.21. Proposition 9.8 gives the exact sequence

$$\begin{aligned} & 0 \to H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \to H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}} \\ \xrightarrow{\partial^2_U} \ & H^1_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \to H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \to H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}} \\ \xrightarrow{\partial^3_U} \ & H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \to \mathrm{CH}_0(\mathcal{E}) \to \mathrm{CH}_0(\mathcal{E}_U) \to 0. \end{aligned}$$

By Lemma 12.21, Coker ∂_U^2 is a finite group, which implies that the group $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ is of rank $|S_0 \setminus U|$. By Theorem 9.3, $|H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}|$ equals

$$\prod_{\ell \neq p} |H^1_{\mathrm{et}}(\mathcal{E}_U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))|.$$

By Corollaries 12.12 and 12.16, it equals

$$|T'_{U,(1)}| \cdot |L(h^1(C), -1)L(h^0(C \setminus U, -1)/(q-1)|.$$

This proves the claim (3).

As we have noted in the proof of Theorem 9.3 (1), the group $\mathrm{CH}_0(\mathcal{E})$ is a finitely generated abelian group of rank one and $\mathrm{CH}_0(\mathcal{E}_U)$ is finite if $U \neq C$. By Lemma 12.19, $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is a finitely generated abelian group of rank $|C \setminus U|$. Hence the rank of $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))^{\mathrm{red}}$ equals $\max(|C \setminus U| - 1, 0)$.

From the class field theory of varieties over finite fields ([Ka-Sa, Theorem 1], see also the introduction in [Co-Ra]) and Lemma 12.9, it follows that the push-forward map $\mathrm{CH}_0(\mathcal{E}) \to \mathrm{Pic}(C)$ is an isomorphism. Hence the homomorphism $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \to \mathrm{CH}_0(\mathcal{E}) \cong \mathrm{Pic}(C)$ factors though the push-forward map $f^U_* : H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1)) \to H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(0))$. By the surjectivity of f^U_*, we have isomorphisms

$$\mathrm{CH}_0(\mathcal{E}^U) \cong \mathrm{Coker}[H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(0)) \to \mathrm{Pic}(C)] \cong \mathrm{Pic}(U),$$

which proves the claim (6). Since the group $H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(0))$ is torsion free, the image of $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}$ in $\mathrm{CH}_0(\mathcal{E})$ is zero. Thus we have the exact sequence

$$0 \to \mathrm{Coker}\, \partial_U^2 \to H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \to H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}} \to H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}} \to 0.$$

By Proposition 9.8 and Lemma 12.17, the group $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}$ is finite of order

$$\frac{p^m |T'_{U,(1)}| \cdot |L(h^2(\mathcal{E}), 0) L^*(h^1(\mathcal{E}^U), 0) L(h^0(C \setminus U), -1)|}{(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}$$

for some $m \in \mathbb{Z}$. By Lemma 12.19, the group $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}$ is finite of order $|L^*(h^1(\mathcal{E}^U), 0)|$. By Lemma 12.9, $\mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q}$ is an abelian variety and in particular $\mathrm{Hom}(\mathrm{Pic}^o_{\mathcal{E}/\mathbb{F}_q}, \mathbb{G}_m) = \{0\}$. Hence by Theorem 9.3 and Corollary 12.12, the group $H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}$ is of order $|L(h^2(\mathcal{E}), 0)|$. Therefore,

$$|\mathrm{Coker}\, \partial_U^2| = \frac{|H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}| \cdot |H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))_{\mathrm{tors}}|}{|H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))_{\mathrm{tors}}|} = \frac{p^{-m}(q-1)|L(h^0(\mathrm{Irr}(\mathcal{E}^U)), -1)|}{|T'_{U,(1)}| \cdot |L(h^0(C \setminus U), -1)|}.$$

Since $|\mathrm{Coker}\, \partial_U^2|$ is prime to p, we have $m = 0$. This proves the claims (4) and (5). This completes the proof. $\square$

Proof of Theorem 12.2. The claim (5) is clear. The claim (1) follows from Corollary 9.9 and Theorem 1.1. It is easily checked that $H^i_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(1))$ is zero for $i \leq 0$. By the localization sequence of higher Chow groups (cf. [Bl3]), we have $H^i_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2)) \cong H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(2))$ for $i \leq 1$. Taking the inductive limit with respect to U, we obtain the claim (2).

By Corollary 9.9, we have an exact sequence

$$
\begin{aligned}
&0 \to H^2_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}} \xrightarrow{\alpha} H^2_{\mathcal{M}}(E, \mathbb{Z}(2))^{\mathrm{red}} \\
&\xrightarrow{\partial^2_{\mathcal{M},2}} \bigoplus_{\wp \in C_0} H^1_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1)) \to H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}. \to H^3_{\mathcal{M}}(E, \mathbb{Z}(2))^{\mathrm{red}} \\
&\xrightarrow{\partial^3_{\mathcal{M},2}} \bigoplus_{\wp \in C_0} H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1)) \to \mathrm{Pic}(C) \to 0.
\end{aligned}
\tag{12.3}
$$

Hence by Theorem 9.3 and Corollary 12.12, the group $\mathrm{Ker}\,\partial^2_{\mathcal{M},2}$ is finite of order $|L(h^1(C), -1)|$. For a non-empty open subsets $U \subset C$, let us consider the group $\mathrm{Coker}\,\partial^2_U$ in the proof of Theorem 12.3. For two non-empty open subsets $U', U \subset C$ with $U' \subset U$, the homomorphism $\mathrm{Coker}\,\partial^2_U \to \mathrm{Coker}\,\partial^2_{U'}$ is injective since both $\mathrm{Coker}\,\partial^2_U$ and $\mathrm{Coker}\,\partial^2_{U'}$ canonically inject into $H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}$. The claim (3) follows from the claim (4) of Theorem 12.3 by passing to the inductive limit. The claim (4) follows from the exact sequence (12.3) and Lemma 12.7.

By the localization sequence, we see that the push-forward $\bigoplus_{x \in E_0} H^2_{\mathcal{M}}(\mathrm{Spec}\,\kappa(x), \mathbb{Z}(2)) \to H^4_{\mathcal{M}}(E, \mathbb{Z}(3))$ is surjective. Hence $H^4_{\mathcal{M}}(E, \mathbb{Z}(3))$ is a torsion group and the claim (6) follows from Lemma 11.4. This completes the proof. □

Proof of Theorem 12.1. Let us consider the restriction $\gamma : \mathrm{Ker}\,c_{2,3} \to H^2_{\mathcal{M}}(E, \mathbb{Z}(2))$ of $c_{2,2}$ to $\mathrm{Ker}\,c_{2,3}$. By Lemma 11.1, both $\mathrm{Ker}\,\gamma$ and $\mathrm{Coker}\,\gamma$ are killed by 2. This implies that the image of γ contains $H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$ and that the group $\mathrm{Ext}^1_{\mathbb{Z}}(H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}, \mathrm{Ker}\,\gamma)$ is zero. Hence the map γ induces an isomorphism $(\mathrm{Ker}\,c_{2,3})_{\mathrm{div}} \xrightarrow{\cong} H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$. This shows that the homomorphism $K_2(E)^{\mathrm{red}} \to H^2_{\mathcal{M}}(E, \mathbb{Z}(2))^{\mathrm{red}}$ induced by $c_{2,2}$ is surjective with torsion kernel. Thus the claim (1) follows from Theorem 12.2 (3).

The claim (3) follows from Theorem 12.2 (1) and Lemma 11.1.

For $\wp \in C_0$, let $\iota_\wp : \mathrm{Spec}\,\kappa(\wp) \to \mathcal{E}_\wp$ denote the fiber of the morphism $\iota : C \to \mathcal{E}$. The diagram (11.2) gives an exact sequence

$$\mathrm{Coker}\,\partial^4_{\mathcal{M},3} \to \mathrm{Coker}\,\partial_2 \to \mathrm{Coker}\,\partial^2_{\mathcal{M},2} \to 0.$$

By Lemma 11.4, we have an isomorphism $\mathrm{Coker}\,\partial^4_{\mathcal{M},3} \cong \mathbb{F}_q^\times$. By the construction of this isomorphism, we see that the composite

$$\mathbb{F}_q^\times \cong \mathrm{Coker}\,\partial^4_{\mathcal{M},3} \to \mathrm{Coker}\,\partial_2 \hookrightarrow K_1(\mathcal{E}) \to K_1(\mathrm{Spec}\,\mathbb{F}_q) \cong \mathbb{F}_q^\times$$

equals the identity. Hence the map $\mathrm{Coker}\,\partial^4_{\mathcal{M},3} \to \mathrm{Coker}\,\partial_2$ is injective. Then the claim (2) follows from Theorem 12.2 (3).

From Proposition 10.4 and Lemmas 11.1, 11.2, and 12.20, it follows that the homomorphism $\partial_1 : K_1(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_0(\mathcal{E}_\wp)$ is identified with the direct sum of

$$\partial'_1 : k^\times \to \bigoplus_{\wp \in C_0} H^0_{\mathcal{M}}(\mathrm{Spec}\,\kappa(\wp), \mathbb{Z}(0)) \to \bigoplus_{\wp} H^0_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(0))$$

and $\partial^3_{\mathcal{M},2} : H^3_{\mathcal{M}}(E, \mathbb{Z}(2))^{\mathrm{red}} \to \bigoplus_\wp H^2_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(1))$. We have isomorphisms

$$
\begin{aligned}
&\mathrm{Ker}\,\partial'_1 \cong \mathbb{F}_q^\times,\ \mathrm{Coker}\,\partial'_1 \cong \mathrm{Pic}(C) \oplus \textstyle\bigoplus_\wp \mathbb{Z}^{|\mathrm{Irr}(\mathcal{E}_\wp)|-1}, \\
&\mathrm{Ker}\,\partial^3_{\mathcal{M},2} \cong H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(2))_{\mathrm{tors}}/\mathrm{Coker}\,\partial^2_{\mathcal{M},2},\ \mathrm{Coker}\,\partial^3_{\mathcal{M},2} \cong \mathrm{Pic}(C).
\end{aligned}
$$

The claim (4) follows. This completes the proof of Theorem 12.1. □

13. Some consequences of the Bloch-Kato conjecture

In this section we assume that the Bloch-Kato conjecture holds. We obtain results generalizing the theorems in Section 12, but the proofs of the results use neither class field theory nor Drinfeld modular curves.

For integers i, j, let us consider the boundary map

$$\partial^i_{\mathcal{M},j} : H^i_{\mathcal{M}}(E, \mathbb{Z}(j))^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} H^{i-1}_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(j-1)).$$

Theorem 13.1. *Let $j \geq 3$ be an integer. Suppose that Conjecture 9.1 is true for j.*

(1) *For any $i \in \mathbb{Z}$, both $\operatorname{Ker} \partial^i_{\mathcal{M},j}$ and $\operatorname{Coker} \partial^i_{\mathcal{M},j}$ are finite groups.*

(2) *We have*

$$|\mathrm{Ker}\partial^i_{\mathcal{M},j}| = \begin{cases} 0, & \text{if } i \leq 0 \text{ or } i \geq 5, \\ q^j - 1, & \text{if } i = 1, \\ |L(h^1(C), 1-j)|, & \text{if } i = 2, \\ \frac{|T'_{U,(j-1)}| \cdot |L(h^2(\mathcal{E}), 2-j)|}{q^{j-1}-1}, & \text{if } i = 3, \\ |L(h^1(C), 2-j)|, & \text{if } i = 4. \end{cases}$$

Moreover, the group $\mathrm{Ker}\partial^1_{\mathcal{M},j}$ is cyclic of order $q^j - 1$.

(3) *We have*

$$|\operatorname{Coker} \partial^i_{\mathcal{M},j}| = \begin{cases} 0, & \text{if } i \leq 1,\ i = 3,\ \text{or } i \geq 5, \\ \frac{q^{j-1}-1}{|T'_{U,(j-1)}|}, & \text{if } i = 2, \\ |L(h^1(C), 2-j)|, & \text{if } i = 4. \end{cases}$$

(4) *Let $U \subset C$ be a non-empty open. Then the group $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))$ is finite modulo uniquely divisible subgroup for any $i \in \mathbb{Z}$. The group $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))$ is zero if $i \geq \max(6, j)$, and is finite for $(i, j) = (4, 3), (5, 3), (4, 4), (5, 4)$, or $(5, 5)$.*

(5) *The group $H^i_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))$ is uniquely divisible for $i \leq 0$ or $6 \leq i \leq j$, and the group $H^1_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))_{\mathrm{tors}}$ is cyclic of order $q^j - 1$.*

(6) *Suppose that $U = C$ (resp. $U \neq C$). Then the group $H^2_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))_{\mathrm{tors}}$ is of order $|L(h^1(C), 1-j)|$ (resp. of order*

$$\frac{|T'_{U,(j-1)}| \cdot |L(h^1(C), 1-j) L(h^0(C \setminus U), 1-j)|}{q^{j-1} - 1}),$$

the group $H^3_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))_{\mathrm{tors}}$ is of order $|L(h^2(\mathcal{E}), 2-j)|$ (resp. of order

$$\frac{|T'_{U,(j-1)}| \cdot |L(h^2(\mathcal{E}), 2-j) L(h^1(\mathcal{E}^U), 2-j) L(h^0(C \setminus U), 1-j)|}{(q^{j-1} - 1)|L(h^2(\mathcal{E}^U), 2-j)|}),$$

the group $H^4_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))_{\mathrm{tors}}$ is of order $|L(h^1(C), 2-j)|$ (resp. of order

$$\frac{|L(h^1(C), 2-j) L(h^0(C \setminus U), 2-j)|}{q^{j-2} - 1}),$$

and the group $H^5_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(j))_{\mathrm{tors}}$ is cyclic of order $q^{j-2} - 1$ (resp. is zero).

Theorem 13.2. *Suppose that Conjecture 9.1 is true for $j = 3$. Then*

(1) *The group* $K_2(E)_{\mathrm{div}}$ *is uniquely divisible and the map* $c_{2,2}$ *induces an isomorphism* $K_2(E)_{\mathrm{div}} \cong H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$.
(2) *The kernel of the boundary map* $\partial : K_2(E)^{\mathrm{red}} \to \bigoplus_{\wp \in C_0} G_1(\mathcal{E}_\wp)$ *is a finite group of order* $|L(h^2(\mathcal{E}), 0)L(h^1(C), -1)|$.

Lemma 13.3. *Let* X *be a smooth projective geometrically connected curve over a global field* k'. *Let* $k'(X)$ *denote the function field of* X. *Then the Milnor* K*-group* $K^M_n(k'(X))$ *is torsion for* $n \geq 2 + \mathrm{gon}(X)$, *and is of exponent* 2 *(resp. is zero) for* $n \geq 3 + \mathrm{gon}(X)$ *if* $\mathrm{char}(k') = 0$ *(resp.* $\mathrm{char}(k') > 0$*). Here* $\mathrm{gon}(X)$ *denotes the gonality of* X*, namely, the minimal degree of morphisms from* X *to* $\mathbb{P}^1_{k'}$.

Proof. The field $k'(X)$ is an extension of degree $\mathrm{gon}(X)$ of a subfield K of the form $K = k'(t)$. Looking at the split exact sequence

$$0 \to K^M_n(k') \to K^M_n(K) \to \bigoplus_P K^M_{n-1}(k'[t]/P) \to 0$$

in [Milno, Theorem 2.3] (where P runs over the irreducible monic polynomials in $k'[t]$), and using [Ba-Ta, Chapter II, (2.1)], we see that $K^M_n(K)$ is torsion for $n \geq 3$, and is of exponent 2 (resp. is zero) for $n \geq 4$ if $\mathrm{char}(k') = 0$ (resp. $\mathrm{char}(k') > 0$). Take a flag $K = V_1 \subset V_2 \subset \cdots \subset V_{\mathrm{gon}(X)} = k'(X)$ of K-subspaces of $k'(X)$ with $\dim_K V_i = i$. For each i we put $V_i^* = V_i \setminus \{0\}$. Suppose $i \geq 2$ and $\alpha, \beta \in V_i \setminus V_{i-1}$. Then there exists $a, b \in K^\times$ such that $\gamma = a\alpha + b\beta \in V_{i-1}$. If $\gamma = 0$ (resp. $\gamma \neq 0$), then $\{a\alpha, b\beta\} = 0$ (resp. $\{a\alpha/\gamma, b\beta/\gamma\} = 0$) in $K^M_2(k'(X))$. Expanding this equality, we see that $\{\beta, \gamma\}$ belongs to the subgroup of $K^M_2(k'(X))$ generated by $\{V_i^*, V_{i-1}^*\}$. Hence for $n \geq \mathrm{gon}(X) - 1$, the group $K^M_n(k'(X))$ generated by the image of $\{V^*_{\mathrm{gon}(X)}, \ldots, V_2^*\} \times K^M_{n-\mathrm{gon}(X)+1}(K)$, which proves the claim. □

Lemma 13.4. *Suppose that Conjecture 9.1 is true for* j. *Then the push-forward homomorphism* $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(j-1)) \to H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))$ *is zero.*

Proof. Let us consider the composite

$$H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(j-1)) \to H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j)) \xrightarrow{f_*} H^2_{\mathcal{M}}(C, \mathbb{Z}(j-1))$$

of push-forwards. This is the zero map since this factors through the group $H^0_{\mathcal{M}}(C \setminus U, \mathbb{Z}(j-2))$ which is zero by [Ge-Le2, Corollary 1.2]. By Lemma 9.6, the group $H^2_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(j-1))$ is torsion. Hence it suffices to show that the homomorphism $f_{*,\mathrm{tors}} : H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))_{\mathrm{tors}} \to H^2_{\mathcal{M}}(C, \mathbb{Z}(j-1))_{\mathrm{tors}}$ induced by f_* is an isomorphism.

Let us consider the commutative diagram

$$\begin{array}{ccc}
H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))_{\mathrm{tors}} & \xrightarrow{f_{*,\mathrm{tors}}} & H^2_{\mathcal{M}}(C, \mathbb{Z}(j-1))_{\mathrm{tors}} \\
\cong \big\uparrow & & \cong \big\uparrow \\
H^3_{\mathcal{M}}(\mathcal{E}, \mathbb{Q}/\mathbb{Z}(j)) & \longrightarrow & H^1_{\mathcal{M}}(C, \mathbb{Q}/\mathbb{Z}(j-1)) \\
\cong \big\downarrow & & \cong \big\downarrow \\
\bigoplus_{\ell \neq p} H^3_{\mathrm{et}}(\overline{\mathcal{E}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))^{G_{\mathbb{F}_q}} & \longrightarrow & \bigoplus_{\ell \neq p} H^1_{\mathrm{et}}(\overline{C}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j-1))^{G_{\mathbb{F}_q}}.
\end{array}$$

The homomorphism at the bottom is an isomorphism by Lemma 12.9. Hence $f_{*,\mathrm{tors}}$ is an isomorphism, as desired. □

Proof of Theorem 13.1. Let $j \geq 3$ and assume Conjecture 9.1 for j. The claims (4) and (5) follow from Theorem 9.3 and Lemma 13.3. The claim (6) follows from Theorem 9.3 and Corollary 12.16. In a manner similar to that in the proof of Corollary 9.9, we can show that the pull-back induces an isomorphism $H^i_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))_{\mathrm{div}} \cong H^i_{\mathcal{M}}(E, \mathbb{Z}(j))_{\mathrm{div}}$ for all $i \in \mathbb{Z}$, and the localization sequence induces the long exact sequence

$$\cdots \to \bigoplus_{\wp \in C_0} H^{i-2}_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(j-1)) \to H^i_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(j))_{\mathrm{tors}} \to H^i_{\mathcal{M}}(E, \mathbb{Z}(j))_{\mathrm{tors}} \to \cdots . \tag{13.1}$$

By assumption, Conjecture 9.1 holds for $j-1$. From this we easily see that for any $\wp \in C_0$, the group $H^i_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(j-1))$ is finite for all i, is zero for $i \leq 0$ or $i \geq 4$, and is cyclic of order $q^{j-1}-1$ for $i = 1$. By looking at the exact sequence (13.1) and using Lemma 13.4, we can deduce the claims (1), (2) and (3) from the claims (4), (5) and (6). This completes the proof. □

Proof of Theorem 13.2. Let $U \neq C$ and suppose that Conjecture 9.1 is true for $j = 3$. Then by Lemmas 11.4 and 13.4, the sequence

$$0 \to H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(3)) \to H^4_{\mathcal{M}}(\mathcal{E}_U, \mathbb{Z}(3)) \xrightarrow{\partial} H^3_{\mathcal{M}}(\mathcal{E}^U, \mathbb{Z}(2)) \xrightarrow{\alpha} \mathbb{F}_q^\times \to 1$$

is exact. By taking the inductive limit, we obtain the exact sequence

$$0 \to H^4_{\mathcal{M}}(\mathcal{E}, \mathbb{Z}(3)) \to H^4_{\mathcal{M}}(E, \mathbb{Z}(3)) \xrightarrow{\partial^4_{\mathcal{M},3}} \bigoplus_{\wp \in C_0} H^3_{\mathcal{M}}(\mathcal{E}_\wp, \mathbb{Z}(2)) \to \mathbb{F}_q^\times \to 1. \tag{13.2}$$

By Theorem 9.3 and Corollary 9.9, the group $H^4_{\mathcal{M}}(E, \mathbb{Z}(3))_{\mathrm{div}}$ is zero. Hence we have $K_2(E)_{\mathrm{div}} \subset \operatorname{Ker} c_{2,3}$. In the proof of Theorem 12.1, we saw that the map $c_{2,2}$ induces an isomorphism $(\operatorname{Ker} c_{2,3})_{\mathrm{div}} \xrightarrow{\cong} H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$. Hence the map $c_{2,2}$ induces an isomorphism $K_2(E)_{\mathrm{div}} \cong H^2_{\mathcal{M}}(E, \mathbb{Z}(2))_{\mathrm{div}}$, which proves the claim (1). The claim (2) follows from Theorems 12.1 and 12.2, the commutative diagram (11.2), and the exact sequence (13.2). This completes the proof. □

Appendix A. A proposition on the p-part

The aim of this Appendix is to give a proof of Proposition A.1 below. It is used in the proof of Theorem 9.3. Nothing in this Appendix is new except possibly the definition of the Frobenius map on the inductive limit (not on the inverse limit) given in Section A.3. A similar situation has already appeared in the work of Milne ([Mi1]) and Nygaard ([Ny]).

Proposition A.1. *Let X be a smooth projective geometrically connected surface over a finite field $\mathbb{F}_q$ of characteristic p. Let $W_n\Omega^i_{X,\log}$ denote the logarithmic de Rham-Witt sheaf (cf.* [Il1, I, 5.7]*). Then the inductive limit $\varinjlim_n H^0_{\mathrm{et}}(X, W_n\Omega^2_{X,\log})$ with respect to the multiplication-by-p is finite of order $|\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)|_p^{-1} \cdot |L(h^2(X), 0)|_p^{-1}$. Here $\mathrm{Hom}(\mathrm{Pic}^o_{X/\mathbb{F}_q}, \mathbb{G}_m)$ denotes the set of homomorphisms $\mathrm{Pic}^o_{X/\mathbb{F}_q} \to \mathbb{G}_m$ of $\mathbb{F}_q$-group schemes, and $L(h^2(X), s)$ is the (Hasse-Weil) L-function of $h^2(X)$.*

A.1. **The de Rham-Witt complex.** In this Appendix, let k be a perfect field of characteristic p. Let X be a smooth k-scheme of dimension δ. For $i, n \in \mathbb{Z}$, let $W_n\Omega_X^\bullet$ denote the de Rham-Witt complex (cf. [Il1]) of the ringed topos of schemes over X with Zariski topology. We let $R : W_n\Omega_X^i \to W_{n-1}\Omega_X^i$, $F : W_n\Omega_X^i \to W_{n-1}\Omega_X^i$, and $V : W_n\Omega_X^i \to W_{n+1}\Omega_X^i$ denote the restriction, the Frobenius, and the Verschiebung, respectively. For each $i \in \mathbb{Z}$, the sheaf $W_n\Omega_X^i$ has a canonical structure of coherent $W_n\mathcal{O}_X$-modules, which enables us to regard $W_n\Omega_X^i$ as an etale sheaf. From now on until the end of section, we work on the category of etale sheaves on schemes over X.

A.2. **Logarithmic de Rham-Witt sheaves.** For $n \in \mathbb{Z}$, let $W_n\Omega_{X,\log}^i \subset W_n\Omega_X^i$ denote the logarithmic de Rham-Witt sheaf (cf. [Il1, I, 5.7]).

Lemma A.2. *The homomorphism $V : W_n\Omega_X^i \to W_{n+1}\Omega_X^i$ sends $W_n\Omega_{X,\log}^i$ into $W_{n+1}\Omega_{X,\log}^i$.*

Proof. Let $x \in W_n\Omega_{X,\log}^i$ be an etale local section. By the definition of $W_n\Omega_{X,\log}^i$, there exists an etale local section $y \in W_{n+1}\Omega_{X,\log}^i$ such that $x = Ry$. We easily see that $Ry = Fy$. Hence $Vx = VRy = VFy = py \in W_{n+1}\Omega_{X,\log}^i$. □

Let $CW\Omega_X^i$ denote the inductive limit $CW\Omega_X^i = \varinjlim_{n,V} W_n\Omega_X^i$ with respect to V. The above lemma enables us to define the inductive limit $CW\Omega_{X,\log}^i = \varinjlim_{n,V} W_n\Omega_{X,\log}^i$.

A.3. **Modified Frobenius operator.** In this subsection we define an operator $F' : CW\Omega_X^i \to CW\Omega_X^i$ such that the sequence

$$0 \to CW\Omega_{X,\log}^i \to CW\Omega_X^i \xrightarrow{1-F'} CW\Omega_X^i \to 0 \tag{A.1}$$

is exact.

For $n \geq 0$, let $\widetilde{W}_n\Omega_X^i$ denote the cokernel of the homomorphism $V^n : \Omega_X^i = W_1\Omega_X^i \to W_{n+1}\Omega_X^i$. The homomorphisms R, F and V on $W_{n+1}\Omega_X^i$ induce homomorphisms on $\widetilde{W}_n\Omega_X^i$ which we denote by the same notations. If $n \geq 1$, the homomorphisms $R, F : W_{n+1}\Omega_X^i \to W_n\Omega_X^i$ factor through the canonical surjection $W_{n+1}\Omega_X^i \to \widetilde{W}_n\Omega_X^i$. We let $\widetilde{R}, \widetilde{F} : \widetilde{W}_n\Omega_X^i \to W_n\Omega_X^i$ denote the induced homomorphisms. Then both $\widetilde{R}$ and $\widetilde{F}$ commute with R, F and V. For $n \geq 0$, we let $\widetilde{W}_n\Omega_{X,\log}^i$ denote the image of $W_{n+1}\Omega_{X,\log}^i$ by the canonical surjection $W_{n+1}\Omega_X^i \to \widetilde{W}_n\Omega_X^i$. The restriction of $\widetilde{R} : \widetilde{W}_n\Omega_X^i \to W_n\Omega_X^i$ to $\widetilde{W}_n\Omega_{X,\log}^i$ gives a surjective homomorphism $\widetilde{R}_{\log} : \widetilde{W}_n\Omega_{X,\log}^i \to W_n\Omega_{X,\log}^i$.

Lemma A.3. *The homomorphisms $\widetilde{R}$, $\widetilde{R}_{\log}$ induce isomorphisms*

$$\varinjlim_{n\geq 0, V} \widetilde{W}_n\Omega_X^i \cong CW\Omega_X^i, \quad \varinjlim_{n\geq 0, V} \widetilde{W}_n\Omega_{X,\log}^i \cong CW\Omega_{X,\log}^i.$$

Proof. The surjectivity is clear. By [Il1, I, Proposition 3.2], the kernel of $\widetilde{R}$ equals the image of the composite $W_1\Omega_X^i \xrightarrow{dV^n} W_{n+1}\Omega_X^i \twoheadrightarrow \widetilde{W}_n\Omega_X^i$. Since $Vd = pdV$, we have $V(\operatorname{Ker}\widetilde{R}) = 0$. This proves the injectivity. □

We easily see that $\widetilde{W}_n\Omega^i_{X,\log}$ is contained in the kernel of $\widetilde{R}-\widetilde{F} : \widetilde{W}_n\Omega^i_X \to W_n\Omega^i_X$. Hence

$$0 \to \widetilde{W}_n\Omega^i_{X,\log} \to \widetilde{W}_n\Omega^i_X \xrightarrow{\widetilde{R}-\widetilde{F}} W_n\Omega^i_X \to 0 \tag{A.2}$$

is a complex.

Lemma A.4. *The inductive limit*

$$0 \to \varinjlim_{n\geq 0,\, V} \widetilde{W}_n\Omega^i_{X,\log} \to \varinjlim_{n\geq 0,\, V} \widetilde{W}_n\Omega^i_X \to CW\Omega^i_X \to 0$$

of (A.2) with respect to V is exact.

Proof. The argument in the proof of [Il1, I, Théorème 5.7.2] shows that the kernel of $R - F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$ is contained in $W_{n+1}\Omega^i_{X,\log} + \operatorname{Ker} R$. Hence the claim follows from Lemma A.3. □

The inductive limit of $\widetilde{F} : \widetilde{W}_n\Omega^i_X \to \widetilde{W}_{n+1}\Omega^i_X$ gives the endomorphism $F' : CW\Omega^i_X \cong \varprojlim_{n\geq 1,\, V} \widetilde{W}_n\Omega^i_X \to CW\Omega^i_X$. By Lemma A.3 and Lemma A.4, we have a canonical exact sequence (A.1).

A.4. **The duality.** Suppose further that X is proper. Let $H^*(X, W_n\Omega^i_X)$ denote the cohomology groups of $W_n\Omega^i_X$ with respect to the Zariski topology.

The trace map $\mathrm{Tr} : H^\delta(X, W_n\Omega^\delta_X) \cong W_n(\mathbb{F}_q)$ is defined in [Il2]. This commutes with the homomorphisms R, F and V. For $0 \leq i, j \leq \delta$, the product $m : W_n\Omega^i_X \times W_n\Omega^{\delta-i}_X \to W_n\Omega^\delta_X$ gives a $W_n(k)$-bilinear paring

$$(\ ,\ ) : H^j(X, W_n\Omega^i_X) \times H^{\delta-j}(X, W_n\Omega^{\delta-i}_X) \to H^\delta(X, W_n\Omega^\delta_X) \xrightarrow{\mathrm{Tr}} W_n(k).$$

By [Il2], this pairing is perfect.

Since $m \circ (\mathrm{id} \otimes V) = V \circ m \circ (F \otimes \mathrm{id})$, the diagram

$$\begin{array}{ccccc} W_{n+1}\Omega^i_X & \times & W_{n+1}\Omega^{\delta-i}_X & \longrightarrow & W_{n+1}(k) \\ {\scriptstyle F}\big\downarrow & & {\scriptstyle V}\big\uparrow & & {\scriptstyle V}\big\uparrow \\ W_n\Omega^i_X & \times & W_n\Omega^{\delta-i}_X & \longrightarrow & W_n(k) \end{array}$$

is commutative. Hence this induces an isomorphism

$$H^{\delta-j}(X, CW\Omega^{\delta-i}_X) \cong \varinjlim_n \mathrm{Hom}_{W_n(k)}(H^j(X, W_n\Omega^i_X), W_n(k)) \tag{A.3}$$

where the transition map in the inductive limit of the right hand side is given by $f \mapsto V \circ f \circ F$. We endow each $H^j(X, W_n\Omega^i_X)$ with the discrete topology. We put $H^j(X, W'\Omega^i_X) = \varprojlim_{n,\, F} H^j(X, W_n\Omega^i_X)$ and endow it with the induced topology. We turn $H^j(X, W'\Omega^i_X)$ into a $W(k)$-module by letting $a \cdot (b_n) = (\sigma^{-n}(a)b_n)$ for $a \in W(k)$, $b_n \in H^j(X, W_n\Omega^i_X)$. We put $D = \varinjlim_{n,\, V} W_n(k)$ and endow it with the discrete topology. We make D into a $W(k)$-module by letting $a \cdot c_n = \sigma^{-n}(a)c_n$ for $a \in W(k)$, $c_n \in W_n(k)$. Then the right hand side of (A.3) equals $\mathrm{Hom}_{W(k),\mathrm{cont}}(H^j(X, W'\Omega^i_X), D)$. The homomorphism $R : H^j(X, W_n\Omega^i_X) \to H^j(X, W_{n-1}\Omega^i_X)$ induces the endomorphism $R' : H^j(X, W'\Omega^i_X) \to H^j(X, W'\Omega^i_X)$. The Frobenius endomorphism $\sigma : W_n(k) \to W_n(k)$ induces the endomorphism $\sigma : D \to D$.

Lemma A.5. *Under the isomorphism (A.3), the endomorphism* $F' : H^{\delta-j}(X, CW\Omega_X^{\delta-i}) \to H^{\delta-j}(X, CW\Omega_X^{\delta-i})$ *is identified with the endomorphism of* $\mathrm{Hom}_{W(k),\mathrm{cont}}(H^j(X, W'\Omega_X^i), D)$ *which send a homomorphism* $f : H^j(X, W'\Omega_X^i) \to D$ *to the homomorphism* $\sigma \circ f \circ R'$.

Proof. Immediate from the definition of the isomorphism (A.3) and the module D. □

A.5. We are mainly concerned with the case where $i = 0$. We denote $H^j(X, W'\Omega_X^0)$ by $H^j(X, W'\mathcal{O}_X)$. Recall that $F : W_n\Omega_X^0 \to W_{n-1}\Omega_X^0$ equals the composite $W_n\mathcal{O}_X \xrightarrow{\sigma} W_n\mathcal{O}_X \xrightarrow{R} W_{n-1}\mathcal{O}_X$. By [Il1, II, Proposition 2.1], $H^j(X, W\Omega_X^i) \to \varprojlim_{n,R} H^j(X, W_n\Omega_X^i)$ is an isomorphism. Hence $H^j(X, W'\mathcal{O}_X)$ is isomorphic to the projective limit

$$\widetilde{H}^j(X, W\mathcal{O}_X) = \varprojlim[\cdots \xrightarrow{\sigma} H^j(X, W\mathcal{O}_X) \xrightarrow{\sigma} H^j(X, W\mathcal{O}_X)].$$

The endomorphism $\sigma : H^j(X, W\mathcal{O}_X) \to H^j(X, W\mathcal{O}_X)$ induces an automorphism

$$\sigma : \widetilde{H}^j(X, W\mathcal{O}_X) \xrightarrow{\cong} \widetilde{H}^j(X, W\mathcal{O}_X).$$

We easily see that the endomorphism R' on $H^j(X, W'\mathcal{O}_X)$ corresponds to the endomorphism σ^{-1} on $\widetilde{H}^j(X, W\mathcal{O}_X)$.

Let $K = \mathrm{Frac}W(k)$ denote the field of fractions of $W(k)$. The homomorphism $\sigma^n/p^n : W_n(k) \to K/W(k)$ for each $n \geq 1$ induces a canonical isomorphism $D \cong K/W(k)$ of $W(k)$-modules which commutes with the action of σ.

A.6. **Proof of Proposition A.1.** Suppose that $k = \mathbb{F}_q$. Then by Lemma A.5, $H^0(X, CW\Omega_X^d)$ is isomorphic to the Pontryagin dual of $\widetilde{H}^\delta(X, W\mathcal{O}_X)$. Hence the group

$$H^0(X, CW\Omega_{X,\log}^\delta) \cong \mathrm{Ker}[H^0(X, CW\Omega_X^\delta) \xrightarrow{1-F'} H^0(X, CW\Omega_X^\delta)]$$

is isomorphic to the Pontryagin dual of the cokernel of $1 - \sigma^{-1}$ on $\widetilde{H}^\delta(X, W\mathcal{O}_X)$.

Proposition A.6. *Let* $k = \mathbb{F}_q$ *be a finite field and* X *be a projective smooth* k*-scheme of dimension* δ. *Suppose that the* V*-torsion part* T *of* $H^\delta(X, W\mathcal{O}_X)$ *is finite. Then* $H^0(X, CW\Omega_X^\delta)$ *is a finite group of order* $|T^\sigma| \cdot |L(h^\delta(X), 0)|_p^{-1}$. *Here* T^σ *denotes the* σ*-invariant part of* T.

Proof. By the argument above, the order of $H^0(X, CW\Omega_X^\delta)$ equals the order of the cokernel of $1 - \sigma$ on $\widetilde{H}^\delta(X, W\mathcal{O}_X)$ if it is finite. The torsion subgroup of $\widetilde{H}^\delta(X, W\mathcal{O}_X)$ is finite since it injects into T. By [Il1, II, Corollaire 3.5], $\widetilde{H}^\delta(X, W\mathcal{O}_X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is isomorphic to the slope zero part of $H^\delta_{\mathrm{crys}}(X/W(k)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Hence the claim follows. □

Proof of Proposition A.1. Let the notations be as above, and suppose that $\delta = 2$. Then by [Il1, II, Remarque 6.4], the module T in the above proposition is canonically isomorphic to the group

$$\mathrm{Hom}_{W(F_q)}(M(\mathrm{Pic}^o_{X/\mathbb{F}_q}/\mathrm{Pic}^o_{X/\mathbb{F}_q,\mathrm{red}}), K/W(\mathbb{F}_q))$$

where $M(\ )$ denotes the contravariant Dieudonne module functor. In particular, T is a finite group. Let T_σ denote the σ-coinvariant of T. Then by Dieudonne theory (cf. [Dem]), $\mathrm{Hom}_{W(F_q)}(T_\sigma, K/W(\mathbb{F}_q))$ is canonically isomorphic to $\mathrm{Hom}(\mathrm{Pic}_{X/\mathbb{F}_q}, \mathbb{G}_m)$. Hence the claim follows from Proposition A.6. □

Appendix B. On some motivic cohomology groups of schemes over finite fields

Let $\mathbb{F}_q$ be a finite field of characteristic p. For a separated scheme X of finite type over $\mathbb{F}_q$, and a discrete abelian group M, let $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ be as in Section 9.1. In particular, the group $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ equals Bloch's higher Chow group $\mathrm{CH}^j(X, 2j-i)$. This coincides with the standard notation when X is essentially smooth over $\mathbb{F}_q$. The aim of Appendix B is to prove the following proposition.

Proposition B.1. *Let X be a connected scheme of pure dimension d which is separated and of finite type over $\mathbb{F}_q$. Then for $i = 1, 2$, the push-forward map*

$$H^{2d+1}_{\mathcal{M}}(X, \mathbb{Z}(d+i)) \to H^1_{\mathcal{M}}(\operatorname{Spec} H^0(X, \mathcal{O}_X), \mathbb{Z}(i))$$

is an isomorphism if X is proper, and the group $H^{2d+1}_{\mathcal{M}}(X, \mathbb{Z}(d+i))$ is zero if X is not proper.

Remark B.2. We note that, for a proper $\mathbb{F}_q$-scheme X, the cycle class map gives an isomorphism

$$H^1_{\mathcal{M}}(\operatorname{Spec} H^0(X, \mathcal{O}_X), \mathbb{Z}(i)) \cong H^0_{\mathcal{M}}(\operatorname{Spec} H^0(X, \mathcal{O}_X), \mathbb{Q}/\mathbb{Z}(i)) \cong \bigoplus_{\ell \neq p} H^0_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

for $i = 1, 2$, by [Ge-Le2, Corollary 1.2] and Merkurjev-Suslin theorem. In particular, if X is a connected scheme of pure dimension d which is proper over $\mathbb{F}_q$, Proposition B.1 shows that the group $H^{2d+1}_{\mathcal{M}}(\operatorname{Spec} H^0(X, \mathcal{O}_X), \mathbb{Z}(d+i))$ is cyclic of order $|H^0(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})|^i - 1$ for $i = 1, 2$.

Remark B.3. Let X be a scheme of dimension less than or equal to d which is separated of finite type over $\mathbb{F}_q$. If we let $X' \subset X$ denote the union of the irreducible components of X of dimension d, then it is immediate from definition that the push-forward map $H^{2d+1}_{\mathcal{M}}(X', \mathbb{Z}(d+i)) \to H^{2d+1}_{\mathcal{M}}(X, \mathbb{Z}(d+i))$ is an isomorphism for any $i \in \mathbb{Z}$. Hence Proposition B.1 determines the structure of $H^{2d+1}_{\mathcal{M}}(X, \mathbb{Z}(d+i))$ for $i = 1, 2$.

Remark B.4. Proposition B.1 generalizes [Ak, Theorem 3.1] where the claim is proved for $i = 1$ and X smooth projective over $\mathbb{F}_q$. Our proof of Proposition B.1 is independent of [Ak], and we do not require a Bertini-type theorem.

Lemma B.5. *Let $\mathbb{F}'_1$, $\mathbb{F}'_2$ be two finite extensions of $\mathbb{F}_q$ with $\mathbb{F}'_1 \subset \mathbb{F}'_2$. Then for $i = 1, 2$, the push-forward map $H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{F}'_2, \mathbb{Z}(i)) \to H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{F}'_1, \mathbb{Z}(i))$ is surjective.*

Proof. By Remark B.2, the cycle class map gives an isomorphism $\alpha : H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{F}'_j, \mathbb{Z}(i)) \cong \bigoplus_{\ell \neq p} H^0_{\mathrm{et}}(\operatorname{Spec} \mathbb{F}'_j, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ for $j = 1, 2$. Thus the claim follows from the compatibility of the cycle class maps with finite push-forwards. □

Lemma B.6. *For $i = 1, 2$, the group $H^3_{\mathcal{M}}(U, \mathbb{Z}(1+i))$ is zero for any smooth affine curve U over $\mathbb{F}_q$.*

Remark B.7. The claim for $i = 1$ follows from the fact that $SK_1(U) = 0$ proved in [Ba-Mi-Se, Corollary 4.3], or from [Ge-Le2, Corollary 1.2] and Merkurjev-Suslin theorem. Our proof of Lemma B.6 below, specialized to the case $i = 1$, gives another proof of the fact that $SK_1(U) = 0$. We also note that the claim for $i = 2$ follows from [Ge-Le2, Corollary 1.2] if we assume Conjecture 9.1 for $j = 3$.

Proof. We may assume that U is geometrically connected. Let K denote the function field of U. Let $i \in \{1,2\}$. We know that the push-forward map $M = \bigoplus_{x \in U_0} H^1_{\mathcal{M}}(\operatorname{Spec} \kappa(x), \mathbb{Z}(i)) \to H^3_{\mathcal{M}}(U, \mathbb{Z}(1+i))$ is surjective. This is clear for $i = 1$, and is a consequence of the fact that $K_3^M(K) = 0$ proved in [Ba-Ta, Chapter II, (2.1)] for $i = 2$. Hence the group $H^2_{\mathcal{M}}(U, \mathbb{Z}(1+i))$ is isomorphic to the cokernel of the boundary map $\partial = (\partial_x)_{x \in U_0} : H^2_{\mathcal{M}}(\operatorname{Spec} K, \mathbb{Z}(1+i)) \to M$. In particular, $H^2_{\mathcal{M}}(U, \mathbb{Z}(1+i))$ is a torsion group whose p-primary part is zero.

We claim that $H^2_{\mathcal{M}}(U, \mathbb{Z}(1+i))_{\mathrm{div}} = 0$. By Lemma 10.1, we have the commutative diagram

$$\begin{array}{ccccc} \bigoplus_{x \in U_0} K_{2i-1}(x) & \longrightarrow & K_{2i-1}(U) & & \\ {\scriptstyle -i \cdot c_{2i-1,i}} \downarrow & & {\scriptstyle c_{2i-1,1+i}} \downarrow & & \\ M & \longrightarrow & H^2_{\mathcal{M}}(U, \mathbb{Z}(1+i)) & \longrightarrow & 0. \end{array}$$

By [So, IV.2], the cokernel of the left vertical homomorphism is killed by i^2. Hence the same holds for the right vertical homomorphism. The group $K_{2i-1}(U)$ is finitely generated by [Gr, Theorem 0.4] (for $i = 1$, it also follows from the argument in [Se, Chapitre II] and the stability [Ba, Chapter V, (4.2)]). Hence the group $H^2_{\mathcal{M}}(U, \mathbb{Z}(1+i))_{\mathrm{div}}$ is zero, as we claimed.

It suffices to prove, for any fixed integer $m \geq 1$ with $p \nmid m$, that the group $(\operatorname{Coker} \partial)/m$ is zero. We fix an integer $n \geq 1$ satsifying $m | q^{in} - 1$. Take a smooth compactification $U \hookrightarrow C$, and a closed point $\infty \in \overline{C} \setminus \overline{U}$. Let $x \in U_0$ be a closed point. Take a closed point $x' \in \overline{U}_0$ lying over x. Then the divisor $[x'] - [\infty]$ of $\overline{C}$ gives a $\overline{\mathbb{F}}_q$-rational point of $\operatorname{Jac}(C)$. The homomorphism $1 - \mathrm{Frob}^n : \operatorname{Jac}(C)(\overline{\mathbb{F}}_q) \to \operatorname{Jac}(C)(\overline{\mathbb{F}}_q)$ is surjective since $\operatorname{Jac}(C)(\overline{\mathbb{F}}_q)$ is a divisible torsion group whose Pontryagin dual is topologically finitely generated and $\operatorname{Ker}(1 - \mathrm{Frob}^n)$ is finite. Hence there is a divisor D on $\overline{C}$ and an element $f \in (K \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)^\times$ such that $\operatorname{div}(f) = [x'] - [\infty] + (\mathrm{Frob}^n - 1)D$. Let $\mathbb{F}'$ be a finite extension of $\mathbb{F}_q$ contained in $\overline{\mathbb{F}}_q$ such that f, x', ∞ and D are defined over $\mathbb{F}'$. For $a \in H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{F}', \mathbb{Z}(i))$, let $c_{x,a} \in H^2_{\mathcal{M}}(\operatorname{Spec} K, \mathbb{Z}(1+i))$ be the image under the push-forward map of the restriction of the product $f \cup a \in H^2_{\mathcal{M}}(\operatorname{Spec}(K \otimes_{\mathbb{F}_q} \mathbb{F}') \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F}', \mathbb{Z}(1+i))$ to $\operatorname{Spec}(K \otimes_{\mathbb{F}_q} \mathbb{F}') = \operatorname{Spec}(K \otimes_{\mathbb{F}_q} \mathbb{F}') \times_{\operatorname{Spec} \mathbb{F}'} \operatorname{Spec} \mathbb{F}'$. Then it is easily checked that, modulo $(q^{in} - 1)M$, $\partial_y(c_{x,a})$ is congruent to zero for $y \in U_0$, $y \neq x$, and $\partial_x(c_{x,a})$ is congruent to the image of a under the push-forward map $H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{F}', \mathbb{Z}(i)) \to H^1_{\mathcal{M}}(\operatorname{Spec} \kappa(x), \mathbb{Z}(i))$. Hence, by Lemma B.5, we have $M = \operatorname{Image} \partial + (q^{in} - 1)M$. This proves the claim. $\square$

Lemma B.8. *Let X be an integral scheme which is of finite type over $\mathbb{F}_q$. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_q$ in $H^0(X, \mathcal{O}_X)$. Then we have $[\mathbb{F} : \mathbb{F}_q] | [\kappa(x) : \mathbb{F}_q]$ for all closed points $x \in X_0$. Moreover, if X is normal, we have the equality $[\mathbb{F} : \mathbb{F}_q] = \gcd_{x \in X_0} [\kappa(x) : \mathbb{F}_q]$.*

Proof. For each $x \in X_0$, the composite $\mathbb{F} \hookrightarrow H^0(X, \mathcal{O}_X) \to \kappa(x)$ is injective since $\mathbb{F}$ is a field. Hence $[\mathbb{F} : \mathbb{F}_q]$ divides $[\kappa(x) : \mathbb{F}_q]$. Suppose that X is normal of dimension d. Let us consider the zeta function $Z(X, s) = \prod_{x \in X_0} (1 - |\kappa(x)|^{-s})^{-1}$. Take a prime $\ell \neq p$. Then $Z(X, s) = \prod_{i=0}^{2d} \det(1 - \mathrm{Frob} \cdot q^{-s}; H^i_{c,\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell))^{(-1)^{i-1}}$. By assumption, we have $H^{2d}_{c,\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell)) \cong H^0(\operatorname{Spec}(\mathbb{F} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q), \mathbb{Q}_\ell(-d))$ and $H^i_{c,\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell))$ is mixed of weight $\leq 2d - 1$ for $i \leq 2d - 1$. This implies that $\gcd_{x \in X_0} [\kappa(x) : \mathbb{F}_q]$ divides $[\mathbb{F} : \mathbb{F}_q]$. This proves the claim. $\square$

Lemma B.9. *Let $d \geq 0$ be an integer. Suppose that Proposition B.1 holds for all connected normal non-proper $\mathbb{F}_q$-schemes of pure dimension d. Then Proposition B.1 holds for all connected normal proper $\mathbb{F}_q$-schemes of pure dimension d.*

Proof. Let X be a connected normal proper $\mathbb{F}_q$-scheme of pure dimension d. Let $i \in \{1,2\}$. For a closed point $x \in X_0$, the group $H^{2d+1}_{\mathcal{M}}(X \setminus \{x\}, \mathbb{Z}(d+i))$ is zero by assumption. Hence the localization sequence shows that the push-forward map $H^1_{\mathcal{M}}(\operatorname{Spec} \kappa(x), \mathbb{Z}(i)) \to H^{2d+1}_{\mathcal{M}}(X, \mathbb{Z}(d+i))$ is surjective. This implies that the group $H^{2d+1}_{\mathcal{M}}(X, \mathbb{Z}(d+i))$ is of order dividing $\gcd_{x \in X_0} |\kappa(x)^\times|$ and the push-forward map $\alpha_X : H^{2d+1}_{\mathcal{M}}(X, \mathbb{Z}(d+i)) \to H^1_{\mathcal{M}}(\operatorname{Spec} H^0(X, \mathcal{O}_X), \mathbb{Z}(i))$ is surjective. Hence the bijectivity of α_X follows from Lemma B.8. □

Lemma B.10. *Let $d \geq 0$ be an integer. Suppose that Proposition B.1 holds for all connected proper $\mathbb{F}_q$-schemes of pure dimension $< d$ and for all connected normal $\mathbb{F}_q$-schemes of pure dimension d. Then Proposition B.1 holds for all connected proper $\mathbb{F}_q$-schemes of pure dimension d.*

Proof. Let X be a connected proper $\mathbb{F}_q$-scheme of pure dimension d. Without loss of generality we may assume that X is reduced. Suppose that X is not normal. Let $\pi : X' \to X$ denote the normalization of X. The $\mathbb{F}_q$-scheme X' is proper since π is finite by [EGAII, Remarque 6.3.10]. Take a reduced closed subscheme $Y \subset X$ of pure codimension one such that $X \setminus Y$ is normal and put $Y' = (Y \times_X X')_{\mathrm{red}}$. By assumption, Proposition B.1 holds for each connected component of $X \setminus Y$, X', Y and Y'.

Let $i \in \{1,2\}$. Let us consider the commutative diagram

$$\begin{array}{ccc} H^{2d-1}_{\mathcal{M}}(Y, \mathbb{Z}(d-1+i)) & \xrightarrow{\beta} & H^{2d+1}_{\mathcal{M}}(X, \mathbb{Z}(d+i)) \\ {\scriptstyle \alpha_Y}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \alpha_X}\big\downarrow \\ H^1_{\mathcal{M}}(\operatorname{Spec} H^0(Y, \mathcal{O}_Y), \mathbb{Z}(i)) & \xrightarrow{\gamma} & H^1_{\mathcal{M}}(\operatorname{Spec} H^0(X, \mathcal{O}_X), \mathbb{Z}(i)) \end{array}$$

where all the morphisms are push-forwards. Since α_Y is an isomorphism and γ is surjective, the homomorphism α_X is surjective. Since $H^{2d+1}_{\mathcal{M}}(X \setminus Y, \mathbb{Z}(d+i))$ is zero, the localization sequence shows that the map β is surjective. Since the diagram

$$\begin{array}{ccc} H^0_{\mathrm{et}}(X, \mathbb{Z}/m(-i)) & \longrightarrow & H^0_{\mathrm{et}}(X', \mathbb{Z}/m(-i)) \\ \big\downarrow & & \big\downarrow \\ H^0_{\mathrm{et}}(Y, \mathbb{Z}/m(-i)) & \longrightarrow & H^0_{\mathrm{et}}(Y', \mathbb{Z}/m(-i)) \end{array}$$

is cartesian for all integers $m \geq 1$ with $p \nmid m$, the diagram

$$\begin{array}{ccc} H^1_{\mathcal{M}}(\operatorname{Spec} H^0(Y', \mathcal{O}_{Y'}), \mathbb{Z}(i)) & \longrightarrow & H^1_{\mathcal{M}}(\operatorname{Spec} H^0(X', \mathcal{O}_{X'}), \mathbb{Z}(i)) \\ \big\downarrow & & \big\downarrow \\ H^1_{\mathcal{M}}(\operatorname{Spec} H^0(Y, \mathcal{O}_Y), \mathbb{Z}(i)) & \longrightarrow & H^1_{\mathcal{M}}(\operatorname{Spec} H^0(X, \mathcal{O}_X), \mathbb{Z}(i)) \end{array}$$

is cocartesian. Hence the surjective homomorphism β factors through the homomorphism

$$H^{2d-1}_{\mathcal{M}}(Y, \mathbb{Z}(d-1+i)) \cong H^1_{\mathcal{M}}(\operatorname{Spec} H^0(Y, \mathcal{O}_Y), \mathbb{Z}(i)) \to H^1_{\mathcal{M}}(\operatorname{Spec} H^0(X, \mathcal{O}_X), \mathbb{Z}(i)).$$

This proves that $|H^{2d+1}_{\mathcal{M}}(X, \mathbb{Z}(d+i))|$ divides $|H^1_{\mathcal{M}}(\operatorname{Spec} H^0(X, \mathcal{O}_X), \mathbb{Z}(i))|$. Hence α_X is an isomorphism. This completes the proof. □

Proof of Proposition B.1. First suppose $d = 1$. The claim for X normal and non-proper follows from Lemma B.6. Then the claim for X proper follows from Lemmas B.9 and B.10.

To prove the claim for non-proper X, we are easily reduced, by induction on the number of irreducible components of X, to the case where X is integral. Take an open immersion from X to a connected proper $\mathbb{F}_q$-scheme X' of dimension one such that the complement $X' \setminus X$ is zero dimensional. Let $i \in \{1,2\}$. We have proved that the push-forward map $H^3_{\mathcal{M}}(X,\mathbb{Z}(1+i)) \to H^1_{\mathcal{M}}(\operatorname{Spec} H^0(X,\mathcal{O}_X),\mathbb{Z}(i))$ is an isomorphism. This implies that the push-forward map $H^1_{\mathcal{M}}(X' \setminus X,\mathbb{Z}(i)) \to H^3_{\mathcal{M}}(X',\mathbb{Z}(1+i))$ is surjective. Hence, by the localization sequence, we have $H^3_{\mathcal{M}}(X,\mathbb{Z}(1+i)) = 0$.

Next suppose that $d \geq 2$ and X is affine. Let $i \in \{1,2\}$. The localization sequence gives an exact sequence

$$\varinjlim_{Y} H^{2d-1}_{\mathcal{M}}(Y,\mathbb{Z}(d-1+i)) \to H^{2d+1}_{\mathcal{M}}(X,\mathbb{Z}(d+i)) \to \varinjlim_{Y} H^{2d+1}_{\mathcal{M}}(X \setminus Y,\mathbb{Z}(d+i)),$$

where Y runs over the reduced closed subschemes of X of pure codimension one. For dimension reasons, we have $\varinjlim_Y H^{2d+1}_{\mathcal{M}}(X \setminus Y,\mathbb{Z}(d+i)) = 0$. Hence by induction on d, we have $H^{2d+1}_{\mathcal{M}}(X,\mathbb{Z}(d+i)) = 0$.

Next suppose that $d \geq 2$ and X is not proper. We are easily reduced, by induction on the number of irreducible components of X, to the case where X is integral. Take an open immersion from X to a connected proper $\mathbb{F}_q$-scheme X' of pure dimension d such that X is dense in X'. Take a non-empty affine open $U \subset X$ and put $Y = X' \setminus U$. Let us take an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. By [Go] and [Hart, Chapter II, §3, §6], each irreducible component X'' of $X' \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, $X'' \setminus U \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ is connected and is of pure codimension one in X'. This shows that Y is connected and is of pure codimension one in X'. In particular, every connected component of $Y \cap X$ is not proper. Let $i \in \{1,2\}$. Since U is affine, the localization sequence

$$H^{2d}_{\mathcal{M}}(Y \cap X,\mathbb{Z}(d-1+i)) \to H^{2d+1}_{\mathcal{M}}(X,\mathbb{Z}(d+i)) \to H^{2d+1}_{\mathcal{M}}(U,\mathbb{Z}(d+i))$$

shows by induction on d that $H^{2d+1}_{\mathcal{M}}(X,\mathbb{Z}(d+i))$ is zero (to remove the hypothesis that the schemes in the localization sequence are quasi-projective, we refer to [Lev2] and [Ge-Le2, 2.6]). This proves the claim for X not proper.

The claim for X proper follows from Lemmas B.9 and B.10. This completes the proof. □

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