LIE GROUPS, BURAU REPRESENTATION, AND NON-CONJUGATE BRAIDS WITH THE SAME CLOSURE LINK

This is a preprint. I would be grateful for any comments and corrections.

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Abstract. We use the unitarization of the Burau representation, found by Squier, and some Lie group arguments, to extend the previous construction of infinite sequences of pairwise non-conjugate braids with the same closure link of a non-minimal number of (and at least 4) strands. In particular we show that such a sequence always exists for non-torus alternating links. It also exists for minimal strand braid representations of any composite knot of braid index at least 6.

1. Introduction

The theory of braid groups took its origins in the 1930s from the work of Artin [2] and Alexander [1]. By a classical theorem of Alexander, knots and links embedded in real 3-dimensional space are all realized as closures of braids. Contrarily, Markov's theorem relates braids realizing the same link by two moves. These moves are conjugacy in the braid group, and (de)stabilization. Stabilization adds a new strand and Artin generator at the edge of a braid, and the inverse operation, called destabilization, removes them. Alexander's and Markov's theorems are the basis of the expectation to apply braids (in particular their group structure) in knot theory. However, we gradually understood that the effect of (de)stabilization on general conjugacy classes of braids is extremely difficult. Only in very special situations these conjugacy classes can be well described.

Birman-Menasco proved that up to 3 strands at most 3 conjugacy classes of braids with the same strand number have the same closure link [6]. For minimal braid representations (i.e. representations realizing the braid index) in general braid groups sometimes only finitely many conjugacy classes seem to occur. Birman in fact conjectured that there would always be a single class, but Murasugi and Thomas [30] found some counterexamples[1].

The situation changes even more when (as will be the main focus in this paper) we abandon minimality. Morton [24] discovered an infinite sequence of conjugacy classes of 4-braids with unknotted closure, and also constructed an irreducible one [25]. (An irreducible conjugacy class is one containing no braid which admits a destabilization as in Markov's theorem.) Then Fiedler [11] combined both properties and showed the existence of an infinite sequence of irreducible conjugacy classes. Unfortunately, the argument in the irreducibility proofs remains restricted to 4-braids only, and we do not know of generalizations to higher braid groups.

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[1] Unlike what they state, their examples apply only to 4-braids. Their argument for 4 strands uses the homomorphism $B_4 \to B_3$; it has no generalization for arbitrary braid groups.

For reducible braids, later, Fukunaga [12, 13], using Garside's normal form [14], and alternatively Fiedler's invariant, obtained two different proofs of an extension of Morton's original result, showing infinitely many conjugacy classes of 4-braids associated to $(2,k)$-torus links. Recently, Shinjo [32] obtained the following even more general theorem for *knots*.

Theorem 1 (Shinjo) If there is an $n-1$-strand braid having a knot K as closure ($n \geq 4$), then there exists an infinite sequence of pairwise non-conjugate braids of n strands realizing K (as closure).

Shinjo's argument does not apply immediately to general links of several components, and in a previous paper [35], we made some effort to extend theorem 1, obtaining the following result. A braid is called central if it commutes with any other braid. The property 'pure' is defined by the triviality of the associated strand permutation. A subbraid of β is a braid obtained by choosing a proper subset of the strands of β.

Theorem 2 ([35]) Assume there exists an $n-1$-strand braid β having a link L as closure, $n \geq 4$, and β is either a non-pure braid, or a pure braid that contains a non-central 3-strand braid as a sub-braid. Then there exists an infinite sequence of pairwise non-conjugate braids of n strands realizing L.

The case of pure braids in theorem 2 was handled by some argument based on the Burau representation ψ_n for $n = 3$. This representation associates to a braid β in the n-strand braid group B_n an $(n-1)\times(n-1)$ matrix with entries in $\mathbb{Z}[t^{\pm 1}]$. It remains of fundamental importance to braid and link theory (see §3 or, for example, [3, 15]).

In this paper we use a more detailed, but different, study of the Burau representation, and some more sophisticated machinery of Lie group theory, to obtain the following result, which assumes in general even weaker restrictions on the (sub)braids.

Theorem 3 Assume $L = \hat{\beta}'$, and $\beta' \in B_{n-1}$ for $n > 4$, such that β' or a subbraid of β' of 4 or more strands has a non-scalar Burau matrix. (That is, the Burau matrix is not a multiple, that may depend on t, of the identity.) Then there exists an infinite sequence of pairwise non-conjugate braids of n strands realizing L.

Again the construction of non-conjugate braids is simply the stablization of conjugates by braids α of a given braid representation β' of L. Herein the effort will be to show that letting α vary over the whole $n-1$-strand braid group B_{n-1}, we obtain enough non-conjugate n-braids after stabilization. The proof of this combines several ingredients. First we show a result of independent interest that concerns the image of the Burau representation of B_n in $GL(n-1,\mathbb{C})$ for some particular values of t.

Theorem 4 When $t \in \mathbb{C}$ with $|t| = 1$ and t is close to 1, but not a root of unity, and $n > 3$, then $\overline{\psi_n(B_n)} \simeq U(n-1)$ as a subgroup of $GL(n-1,\mathbb{C})$.

Here bar means closure in the usual topology of $M(n-1,\mathbb{C})$ and isomorphy is meant up to conjugation with a matrix depending on t. The question on the Burau image has some importance. Cooper and Long [8] studied it as an abstract group, but in this form it appears too hard to describe. (It is likely not finitely presented, even with matrix coefficients in $\mathbb{Z}[t^{\pm 1}]$ taken modulo 2.) Theorem 4 says something about its embedding in $GL(n-1,\mathbb{C})$ for the t in question. The unitarity condition is known by a result of Squier [33]. To show density, we apply some arguments on Lie groups, in particular a part of Dynkin's seminal work [10] on the classification of maximal subgroups of the complex groups. The property "close to 1" can be concretified with a bit more effort, if needed.

To complete the proof of theorem 3, finally we show that a non-central conjugacy class in $SU(n-2)$ satisfies no linear conditions except the invariance of the trace (proposition 3), and the Burau trace of the stabilized braid (in B_n) gives such a condition.

Certainly, by incorporating theorem 2, we can replace both occurrences of '4' by '3' in theorem 3. (Note that a central braid has scalar Burau matrix, and the converse is true for 3-braids; see the proof of lemma 5 and remark 2.) The point is that the proof of theorem 3 differs completely from this of theorem 2, and the case $n = 4$ creates a slight

(Lie group theoretic) problem in the new proof. While it seems remediable, we felt that it would be better to leave the case entirely to the merit of the previous results. For $n \neq 4$, theorem 3 clearly extends theorem 1.

As an application, we use the relation between the Burau representation and the Alexander polynomial in §4 to restrict the possible values of the Alexander polynomial of links on which our construction could fail. This restriction is very weak, and allows us to settle the case of non-trivial alternating links (theorem 6).

Of course, if β' is contained in the center of B_{n-1} (in other words, the subgroup of elements commuting with all of B_{n-1}), conjugacy is trivial, and our construction must certainly fail. (This requires to exclude the unlink in theorem 6, for example.) However, the center is very small, being generated by the (pure) full twist braid, and our theorem exhausts many of the non-central cases. On the other hand, a result of Long [21] implies that still there exist non-central braids neither contained in the class specified in theorem 3 (see remark 1). At the end of the paper, in §6, we discuss some related problems.

Although we focus in theorem 3 on reducible representations, we will explain in §5 how to adapt the arguments in the proof to some minimal representations. See theorem 7. It gives a general condition for finding non-conjugate minimal braids. Such constructions relate to Birman-Menasco's exchange move [7], so the theorem gives a tool to decide that this move creates non-conjugate braids.

The paper [35] was written originally in Japanese; for completeness and clarity we reproduce an English proof of theorem 2 in the appendix. This is the more useful, that some work here (theorem 6) depends on it non-trivially.

2. Lie groups (joint with T. Yoshino)

In this section we make some Lie group theoretic preparations. We prove propositions 1, 2 and 3, which are needed for the proof of theorem 3.

2.1. Correspondence between compact and complex Lie groups

Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$. Compactness implies in particular that G is real, finite-dimensional and linear reductive (see example 5.38 of [20]). Linear reductive means for a closed subgroup $G \subset GL(n,\mathbb{C})$ that the number of connected components is finite and G is closed under conjugated matrix transposition $M \mapsto \overline{M^T}$ (see definition 5.36 of [20]).

A *linear representation* of G is understood as a pair $\rho = (V,\pi)$ made of a vector space V and a homomorphism $\pi : G \to \mathrm{Aut}(V)$. We will often omit π and identify ρ with V for simplicity, if unambiguous. A representation is *irreducible* if it has no non-trivial (i.e. proper and non-zero) invariant subspaces. Linear reductiveness of G implies that each invariant subspace of a linear representation of G has a complementary invariant subspace, so that each representation of G is *completely reducible* as direct sum of irreducible representations.

To G there exists a uniquely determined complex connected linear reductive Lie group $G_{\mathbb{C}}$, with

(i) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is the Lie algebra of $G_{\mathbb{C}}$, and
(ii) $G \subset G_{\mathbb{C}}$ as a closed subgroup.

Then $G_{\mathbb{C}}$ is called a *complexification* of G. If G is simply connected, so is $G_{\mathbb{C}}$, and then any other connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is a covering of $G_{\mathbb{C}}$. The real group G is always a maximal compact subgroup of $G_{\mathbb{C}}$; we call it the *compact real form* of $G_{\mathbb{C}}$ (see [20, theorem 12.27]).

Thus we have a one-to-one correspondence between a compact connected real (simply-connected) Lie group and a (simply-connected) connected linear reductive complex Lie group. Under this correspondence to $G = SU(n)$ we have $G_{\mathbb{C}} = SL(n,\mathbb{C})$. The groups are connected and simply-connected.

The correspondence behaves well w.r.t. many properties. The real form G is simple, if and only if $G_{\mathbb{C}}$ is too. (In particular, if G is semisimple, so is $G_{\mathbb{C}}$.) For every complex representation $\rho = (V,\pi)$ of G ('complex' means that V is a complex vector space) we have an 'extension' to a representation $\tilde{\rho} = (V,\pi_{\mathbb{C}})$ of $G_{\mathbb{C}}$, such that $\pi_{\mathbb{C}}$ is an extension of π from G to $G_{\mathbb{C}}$. *If $G_{\mathbb{C}}$ is simply-connected,* it is proved by Weyl's unitary trick (Theorem 12.19 and Remark 12.20 in [20]), that ρ *is irreducible if and only if $\tilde{\rho}$ is so.*

2.2. Symmetric pairs

Let G be a Lie group and σ an involution. Define

$$G^{\sigma} := \{ g \in G : \sigma(g) = g \}$$

to be the σ-invariant subgroup of G and G_0^{σ} the connected component of the identity. Then a pair (G,H) for a closed subgroup H with $G_0^{\sigma} \subset H \subset G^{\sigma}$ is called a *symmetric pair*.

In the case $G = SU(n)$ the symmetric pairs have been classified by Cartan. See [18, Chapter IX.4.A, table p. 354]. In this case H is some of $S(U(m) \times U(n-m))$, $Sp(n/2)$ if n is even, or $SO(n)$.

Let us give the corresponding involutions σ that define the symmetric pairs (see p. 348 of [18]).

Define $M_{i,j}$ to be the matrix with all entries 0 except that at the (i,j)-position, which is 1. Let $\mathrm{diag}(x_1,\dots,x_n) = \sum_{i=1}^{n} x_i M_{i,i}$ be the diagonal matrix with entries $x_1,\dots,x_n$, so that $Id_n = \mathrm{diag}(1,\dots,1)$ (with n entries '1') is the identity matrix.

For $S(U(m) \times U(l))$, with $m+l=n$, the involution σ is of the form $\sigma_{m,l} : M \mapsto I_{m,l} M I_{m,l}$, where

$$I_{m,l} = \mathrm{diag}(1,\dots,\underset{m}{1},\underset{m+1}{-1},\dots,-1)\,.$$

For $n = 2n'$ even, $Sp(n')$ respects the involution $\sigma_J : M \mapsto J^{-1}\bar{M}J$, where

$$J = \left(\begin{array}{c|c} 0 & -Id_{n'} \\ \hline Id_{n'} & 0 \end{array} \right), \tag{1}$$

and $\bar{M}$ is the complex conjugation (of all entries) of M. For $SO(n)$, the involution σ is given by $\sigma(M) = \bar{M}$.

These subgroups can be also defined in the standard representation by the linear transformations that respect a certain (complex) non-degenerate bilinear form, which is Hermitian, skew-symmetric or symmetric resp. (see table 7.2 on p. 315 of [20]). All transformations that respect such a form determine, up to conjugacy, a subgroup of one of the three types.

Analogous three types of subgroups $R(m,n,\mathbb{C})$, $Sp(n/2,\mathbb{C})$ and $SO(n,\mathbb{C})$ can be defined for $SL(n,\mathbb{C})$. Here $R(m,n,\mathbb{C})$ is the group of all (complex-)linear unit determinant transformations of $\mathbb{C}^n$ that leave invariant a subspace of dimension m. (In contrast to the unitary case, there is not necessarily a complementary invariant subspace!)

We call the three types of groups *reducible*, *symplectic* and *orthogonal* resp. We call a representation V of G after one of the types, if it is contained in a conjugate of a group of the same name. Orthogonal and symplectic subgroups/representations will be called also *symmetric*, the others *asymmetric*.

In the real-complex correspondence we explained, we have that *if the representation ρ of G is symmetric, then so is the representation $\tilde{\rho}$ of $G_{\mathbb{C}}$.* This is easily seen by restricting the respected bilinear form to the reals.

2.3. Maximal subgroups of the complex groups

In the 1950s, Dynkin published a series of ground-breaking papers, in which he gave tremendous impetus to the theory of Lie groups. We will use a part of his classification of maximal subgroups of classical Lie groups [10]. (See theorems 1.3, 1.5 and 2.1 in [10].)

Theorem 5 (Dynkin [10]) A maximal proper (compact) subgroup of $SL(n,\mathbb{C})$ is conjugate in $SL(n,\mathbb{C})$ to
(i) some symmetric representation, i.e., $SO(n,\mathbb{C})$ or $Sp(n/2,\mathbb{C})$ when n is even, *or*
(ii) to $SL(m,\mathbb{C}) \otimes SL(m',\mathbb{C})$ with $mm' = n$ and $m,m' \geq 2$ (one which is non-simple irreducible), *or*
(iii) to $R(m,n,\mathbb{C})$ (one which is reducible), *or*
(iv) it is an irreducible representation of a simple Lie group.

For a non-simple group $H = H_1 \times H_2$, one considers $\mathbb{C}^n = \mathbb{C}^{mm'} \simeq \mathbb{C}^m \otimes \mathbb{C}^{m'}$ as a tensor (Kronecker) product, and H_1 resp. H_2 acts on $\mathbb{C}^m$ resp. $\mathbb{C}^{m'}$.

We call the first 3 types of subgroups *orthogonal*, *symplectic*, *product* and *reducible* resp.

We will need in theorem 5 mainly case (iii). Also, case (i) is in fact included in case (iv), but singled out due to the somewhat special treatment it will receive below. It can be handled by elementary means, instead of (although we could do as well by) appealing to a larger extent to the Lie theory described, mainly in the appendix, in [10].

The irreducible representations will be handled with the following lemma.

Lemma 1 An irreducible representation of dimension k of a simple complex Lie group of rank at least $k-1$ is the standard representation of $SL(k,\mathbb{C})$.

Note that the rank of G is the dimension of a maximal commutative subgroup, and equal to the number of simple roots resp. the number of nodes in the Dynkin diagram.

Proof. The irreducible representations of a simple complex Lie group in $SL(k,\mathbb{C})$ can be specified by a labelling of the nodes of the Dynkin diagram, as explained on p. 329 of [10]. Now, the number n of the nodes of the Dynkin diagram is equal to the dimension of the maximal torus (see p. 320–322 of [10]), that is, the rank of G. So by assumption we have $k \le n+1$.

Now it follows from Weyl's formula for the dimension (theorem 0.24 in §31 of the "Supplement" in [10]) and Cartan's description of maximal weights (theorem 0.9 in §10 ibid.) that increasing the label of a node strictly increases the dimension of the representation, and the dimensions of the basic representations (only one node labelled, with a '1') are given in Figure 30 of [10]. This dimension must be less than or equal to $n+1$. Only the standard representation of A_n has such a dimension, and $k-1 = n$. □

Proposition 1 Let V be an n-dimensional irred. faithful representation (for $n > 1$) of a complex group G of rank at least $n-1$. Then it is the standard representation of $SL(n,\mathbb{C})$ (and the rank is $n-1$).

Proof. Any irreducible group G of linear transformations is semisimple by work of Cartan (see Remark B after theorem 1.5 on p. 253 of [10]). If G is not the full $SL(n,\mathbb{C})$, go over to the maximal proper subgroup of $SL(n,\mathbb{C})$ containing G. By abuse of notation we call it again G. If G is simple, we are done by the previous lemma. If G is not simple, then by theorem 1.3 of [10], the inclusion of G in $\mathrm{Aut}(V)$ is contained in a tensor product representation $V_1 \otimes V_2$, and $V_{1,2}$ are themselves the standard representations of $SL(n_{1,2},\mathbb{C})$ (of dimension n_i at least two). The rank $\mathrm{rk}\, G$ is monotonous under inclusion, and additive under cross product, so $\mathrm{rk}\, G \le \mathrm{rk}\, SL(n_1,\mathbb{C}) + \mathrm{rk}\, SL(n_2,\mathbb{C}) = n_1+n_2-2$. Contrarily the dimension of V is multiplicative under Kronecker product, and $\dim V = n_1 n_2 > n_1+n_2-1$, so we see that V cannot be the representation we assumed. □

Proposition 2 Let V be an n-dimensional faithful representation of a semisimple linear reductive complex Lie group G of rank at least $n-1$ in $SL(n,\mathbb{C})$. Then V is irreducible, and so $G = SL(n,\mathbb{C})$.

Proof. Assume V is reducible. By linear reductiveness, V decomposes as a direct sum of irreducible representations V_i of dimensions n_i, with $\sum n_i = n$. Since G is semisimple, all $n_i \ge 2$. But now the rank and dimension are additive under direct sum, and we get a contradiction from proposition 1. □

2.4. Unique conjugacy invariance of the trace

It is well-known that central matrices in $SU(n)$ are scalar and that the trace is a conjugacy invariant. We show that, apart from these trivial cases, there are no linear functions of matrices invariant on a conjugacy class.

Proposition 3 Assume that $f : M(n,\mathbb{C}) \simeq \mathbb{C}^{n^2} \to \mathbb{C}$ is a linear function, which is not a multiple of the trace. Let X be a non-central element in $SU(n)$. Then f is not constant on the conjugacy class of X in $SU(n)$ (considered as a subset of $M(n,\mathbb{C})$).

We found two proofs of this fact. The first proof uses simple matrix algebra. The second one is more contextual. It applies in more generality and is shorter, but appeals again to some Lie group theory.

Proof. We fix a presentation of f by

$$f(X) := \sum_{s=1}^{l} \alpha_{i_s,j_s} x_{i_s,j_s}\,, \tag{2}$$

where $x_{i,j}$ is the (i,j)-entry of X and $\alpha_{a,b} \in \mathbb{C}$ are coefficients. Now we find one by one matrices A such that the identity

$$f(AXA^{-1}) = f(X) \tag{3}$$

restricts gradually the possible values of $\alpha_{a,b}$. Hereby, it is admissible to conjugate with matrices $A \in U(n)$ instead of $SU(n)$, because $U(n)$ is just a central extension.

Let $M_{i,j}$ and $\mathrm{diag}\,(x_1,\dots,x_n)$ be the notation of §2.2. First we use (3) with $A = \mathrm{diag}\,(1,\dots,1,\underset{l}{-1},1,\dots,1)$ for $l = 1,\dots,n$. Then we find

$$\sum_{\text{either } i_s = l \text{ or } j_s = l} \alpha_{i_s,j_s} x_{i_s,j_s} = 0\,. \tag{4}$$

Using $A = \mathrm{diag}\,(1,\dots,1,\underset{i}{-1},1,\dots,1,\underset{k}{-1},1,\dots,1)$, we obtain similarly

$$\sum_{|\{i,k\}\cap\{i_s,j_s\}|=1} \alpha_{i_s,j_s} x_{i_s,j_s} = 0\,. \tag{5}$$

Let us complete the α_{i_s,j_s} to $\alpha_{i,j}$ for $1 \le i,j \le n$, by setting $\alpha_{i,j} = 0$ when (i,j) does not occur as (i_s,j_s). Now for $i \ne k$, subtracting (5) from the sum of the two instances of (4) for $l = i$ and $l = k$, we find

$$x_{i,k}\alpha_{i,k} + x_{k,i}\alpha_{k,i} = 0\,. \tag{6}$$

Still we can use in (6) instead of $x_{i,k}$ also the entries of any other matrix conjugate to X. This gives a family of new conditions on the $x_{i,k}$. Conjugating with

$$A_{i,j} = Id - M_{i,i} - M_{j,j} + M_{j,i} - M_{i,j} \tag{7}$$

shows

$$x_{i,k}\alpha_{k,i} + x_{k,i}\alpha_{i,k} = 0\,. \tag{8}$$

We would like to conclude now that

$$\alpha_{i,k} = 0 \text{ for all } i \ne k\,. \tag{9}$$

The conditions (6) and (8) together imply that if $x_{i,k} \ne \pm x_{k,i}$, then $\alpha_{i,k} = \alpha_{k,i} = 0$. Since one can obtain all pairs (i,k) with $i \ne k$ up to conjugacy, we see that if X is not symmetric or antisymmetric, then (9) is true.

Now assume that X is symmetric or antisymmetric, and not diagonal, i.e. $x_{i,k} \ne 0$ for some $i \ne k$. Then a non-trivial solution $(\alpha_{i,k},\alpha_{k,i})$ to (6) and (8) has $\alpha_{i,k} = -\alpha_{k,i}$ (for X symmetric) or $\alpha_{i,k} = \alpha_{k,i}$ (for X antisymmetric). When conjugating X with $A = \mathrm{diag}\,(1,\dots,1,\underset{i}{\lambda},1,\dots,1)$, with $\lambda \in \mathbb{C}$ of norm 1, then $x_{i,k}$ gets multiplied with λ, and $x_{k,i}$ by $1/\lambda$, so again by choosing λ properly, we conclude $\alpha_{i,k} = \alpha_{k,i} = 0$.

So for (9) it remains to consider the case that X is diagonal. Now, by assumption X is not a multiple of the identity. So there is a vector not mapped to a multiple of itself by X. Using Gram-Schmidt, one easily can (unitarily) conjugate X to a non-diagonal matrix, and so (9) follows.

Now with (9), the form (2) reduces to

$$f(X) := \sum_{i=1}^{n} \alpha_i x_{i,i}\,.$$

Letting $A = A_{i,j}$ be again as in (7), the condition (3) reduces to

$$\alpha_i x_{i,i} + \alpha_j x_{j,j} = \alpha_i x_{j,j} + \alpha_j x_{i,i}\,,$$

which can be rewritten as $(\alpha_i - \alpha_j)(x_{i,i} - x_{j,j}) = 0$. Now by assumption, the second factor does not vanish for at least one pair $i < j$. Then the first factor must vanish for that i, j, and then for all other i, j by conjugacy. So f is a multiple of the trace, as desired. □

Here is the more general approach.

Proposition 4 Let G be a connected and simply connected simple complex Lie group with Lie algebra $\mathfrak{g}$ and U the compact real form of G. Then the adjoint representation of U on $\mathfrak{g}$ is irreducible as a representation over $\mathbb{C}$.

Proof. We have that the representation $ad : \mathfrak{g} \to End(\mathfrak{g})$ is irreducible because $\mathfrak{g}$ is a simple Lie algebra. Thus Weyl's unitary trick implies that the adjoint representation of the compact real form U on $\mathfrak{g}$ is also irreducible. □

Corollary 1 $Ad(SU(n)) \curvearrowright \mathfrak{sl}(n,\mathbb{C})$ is irreducible. □

(Note that $\mathfrak{sl}(n,\mathbb{C})$, the Lie algebra of $SL(n,\mathbb{C})$, are the traceless complex $n \times n$ matrices.)

Proof of proposition 3 (alternative version). We use indirect proof. We assume f is constant on the conjugacy class of some non-central (i.e. non-scalar) matrix $X \in SU(n)$, and want to prove that f is a constant times the trace. By adding a multiple of the identity to X (still X is not scalar), we can assume w.l.o.g. that $\mathrm{tr}(X) \neq 0$. Since f is linear, it is still constant on the conjugacy class of the modified X. Define for $Y \in M(n,\mathbb{C})$ a function

$$F(Y) := f(Y) - \frac{f(X)}{\mathrm{tr}(X)} \cdot \mathrm{tr}(Y)\,.$$

Then again F is a linear function and $F(gXg^{-1}) = F(X) = 0$ for all $g \in SU(n)$.

Let $K = SU(n)$, $V = \mathfrak{gl}(n,\mathbb{C}) = M(n,\mathbb{C})$ and $V_0 = \mathfrak{sl}(n,\mathbb{C})$. Consider

$$T := \mathbb{C}\text{-span}\{\, gXg^{-1} : g \in SU(n)\,\} \subset V.$$

This is an $Ad(K)$ invariant $\mathbb{C}$-subspace of V. It is clearly not contained in V_0 (because $\mathrm{tr}(X) \neq 0$) or in $\mathbb{C} \cdot Id_n$ (because X is not scalar). However, it follows from corollary 1 and the linear reductiveness of K that the only $Ad(K)$ invariant $\mathbb{C}$-subspaces of V are $\{0\}, \mathbb{C} \cdot Id_n, V_0$ and V. So $T = V$ and $F \equiv 0$ on V. Then

$$f(Y) = \frac{f(X)}{\mathrm{tr}(X)} \cdot \mathrm{tr}(Y)$$

for all $Y \in V$, and so f is a multiple of the trace. □

3. Unitarization of the Burau representation

The n-strand *braid group* B_n is considered generated by the Artin *standard generators* σ_i for $i = 1, \dots, n-1$. These are subject to relations of the type $[\sigma_i, \sigma_j] = 1$ for $|i-j| > 1$, which we call *commutativity relations* (the bracket denotes the commutator) and $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$, which we call *Yang-Baxter* (or shortly YB) relations.

The Burau representation $\tilde{\psi}_n$ of B_n, for a parameter $t \in \mathbb{C}$, originally acts on $\mathbb{C}^n$ by

$$\tilde{\psi}_n(\sigma_i) = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1-t & t & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}, \tag{10}$$

with $i = 1,\dots,n-1$ and the entry $1-t$ at position (i,i). This form leaves the subspace generated by $(1,\dots,1)$ invariant. So one takes a complementary basis $(0,\underset{i}{1},-1,0,\dots,0) = e_i$, and on that basis, ψ_n gets the shape known as (reduced) Burau representation

$$\psi_n(\sigma_i) = \left[\begin{array}{cccccccc} 1 & & & & & & & \\ & \ddots & & & & 0 & & \\ & & 1 & & & & & \\ & & & 1 & -t & & & \\ & & & & -t & & & \\ & & & & -1 & 1 & & \\ & & & & & & 1 & \\ & & 0 & & & & & \ddots \\ & & & & & & & 1 \end{array}\right] \quad \text{for } 1 < i < n-1,$$

$$\psi_n(\sigma_1) = \left[\begin{array}{ccccc} -t & 0 & & & \\ -1 & 1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{array}\right], \quad \text{and} \quad \psi_n(\sigma_{n-1}) = \left[\begin{array}{ccccc} 1 & & 0 & & \\ & \ddots & & & \\ & & 1 & & \\ & 0 & & 1 & -t \\ & & & 0 & -t \end{array}\right],$$

where at position (i,i) there is always the entry $-t$.

Albeit $\tilde\psi$ contains an extraneous dimension, it displays some features easier than ψ. We will switch back and forth between both forms when convenient.

The permutation representation of the symmetric group S_n of n elements is given by $\mathbb{C}^n$, with the permutation of coordinates. Since B_n surjects onto S_n, we can view the permutation representation also as a representation of B_n. It is obtained from $\tilde\psi$ for $t = 1$.

Lemma 2 The permutation representation of S_n for $n \ge 3$ has no proper invariant subspaces except the one C of vectors with all entries being equal, and the one D of vectors with all entries adding up to 0.

Proof. Clearly the two spaces C and D are invariant. Now a permutation is unitary (w.r.t. the standard Hermitian scalar product), so it is diagonalizable, and all invariant subspaces are sums of eigenspaces. Moreover, such eigenspaces are mutually orthogonal. So a non-trivial invariant subspace E, different from C and D, must be contained in D. The space D has the obvious basis $e_i = (0,\dots,0,\underset{i}{1},-1,0,\dots,0)$, for $i = 1,\dots,n-1$. Each e_i is an eigenvector (to eigenvalue -1) for the action of a transposition of elements i and $i+1$. So E must be either (i) the 1-dimensional space E_i of some e_i, and contained in the orthogonal complement $E_j^\perp$ of E_j for all $j \ne i$, or (ii) it must lie in the intersection of all $E_i^\perp$. Former alternative is excluded because e_i and $e_{i\pm 1}$ are not orthogonal, for whatever of $e_{i\pm 1}$ makes sense. (In order at least one of both to do so, here the condition $n \ge 3$ becomes necessary.) Latter alternative is excluded because the e_i are a basis, so $\bigcap_i E_i^\perp = \{0\}$. □

Note that, since for $t = 1$ the Burau representation degenerates into a permutation representation, theorem 3 (and likewise theorem 2) would apply on a non-pure braid directly, without need to consider subbraids. So it is enough to consider subbraids only of pure braids in the proof of theorem 3. This bypasses the unpleasant problem to understand conjugacy and closures of subbraids of non-pure braids. We also see that theorem 3 contains theorem 1 for $n \ne 4$, and also most of theorem 2. The exceptions would be pure braids β' (it is not clear if such exist) of $n-1 \ge 4$ strands, such that all the subbraids of β' of 4 or more strands have scalar Burau matrix, but some 3-strand subbraid has not.

We consider the unitarization of the Burau representation, found by Squier [33]. Squier proved that ψ is unitarized by a certain Hermitian form for $|t| = 1$. For $t = 1$ this form is the standard Hermitian form. One can verify that Squier's form for ψ_n degenerates exactly in the n-th roots of unity. Therefore, when $t = e^{i\lambda}$, $\lambda \in \mathbb{R}$ and $|\lambda| < 2\pi/n$, then ψ_n is unitary, so it is conjugate to a scalar times a $SU(n-1)$ representation $\psi_u = \psi_{u,n}$. (Here 'u' is a literal and stands for 'unitary', and the integer n will be omitted when fixed.) This extends the above observation for $t = 1$.

Proof of theorem 3. Write, with the e_i as above,

$$(x_1,\dots,x_{n-1})_* = \sum_{i=1}^{n-1} x_i e_i .$$

We use angle brackets to denote the group generated by some elements. Now let $\beta \in B_{2,n} = \langle \sigma_2,\dots,\sigma_{n-1}\rangle$, which is the subgroup B_{n-1} of braids in B_n with isolated leftmost strand. We consider all braids $\sigma_1\alpha\beta\alpha^{-1}$ with $\alpha \in B_{2,n}$, and want to show that infinitely many are non-conjugate. We assume the contrary and derive a contradiction. In fact, it is enough to assume that the $\sigma_1\alpha\beta\alpha^{-1}$ admit only a finite number of Burau traces.

Now complete $\tilde{e} = \tilde{e}_1 := -(1-n,1\dots,1) = (n-1,n-2,\dots,1)_*$ to a basis of $\mathbb{C}^{n-1}$ by $\tilde{e}_i = (0,\dots,0,\underset{i}{1},0,\dots,0)_*$ for $i = 2,\dots,n-1$. Set

$$(x_1,\dots,x_{n-1})_{**} = \sum_{i=1}^{n-1} x_i \tilde{e}_i .$$

Then in the basis $\{\tilde{e}_i\}$ we have for $\beta \in B_{2,n}$ the shape

$$\psi(\beta) = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \psi'(\beta') & \\ 0 & & & \end{array}\right], \tag{11}$$

where ψ' is the reduced Burau representation of B_{n-1} and $\beta' \in B_{n-1}$ is obtained from β by removing the left (isolated) strand. We can assume (up to replacing β' with some of its subbraids) that $\psi'(\beta')$ is not scalar. Similarly let α' be the image of α under the isomorphism $B_{2,n} \to B_{n-1}$.

Now by [33], for $|t| = 1$ and t close to 1, ψ' is conjugate to a $U(n-2)$ representation. So $\psi'_u = (-t)^{-[\cdot]/(n-2)} \cdot \psi'$ is a $SU(n-2)$ representation.

First note that $B_{2,n}$ is generated by two elements $\gamma = \sigma_2$ and $\delta = \sigma_2 \dots \sigma_{n-1}$, i.e., $B_{2,n} = \langle \gamma, \delta\rangle$.

Lemma 3 $B' := \langle \psi'_u(\gamma'), \psi'_u(\delta')\rangle$ is dense in $SU(n-2)$ for $|t| = 1$ and t close to 1 but not a root of unity, and $n \geq 5$.

Proof. For $t = 1$ one can determine the invariant subspaces of $\psi(\gamma)$ and $\psi(\delta)$ by going back from the ordinary (reduced) to the full Burau representation. Then $\tilde{\psi}$ is just the permutation homomorphism. One sees immediately from (10) that the subspace $\tilde{E}_1$ generated by $-(1-n,1\dots,1) = (n-1,n-2,\dots,1)_* = \tilde{e}_1$ is a common invariant subspace of γ,δ for any t. By lemma 2, it is the only such subspace for $t = 1$ (up to orthogonal complement). Invariant subspaces depend continuously on t, and their coincidence is a closed condition. So when t is close to 1, then $\tilde{E}_1$ remains the only common invariant subspace. This means that B' is not contained in a subgroup $S(U(m)\times U(n-2-m))$, i.e. its (unitary) representation is irreducible. By Cartan's work (see the proof of proposition 1) the closure $\overline{B'}$ of this group B' is therefore semisimple.

Let $B'_{\mathbb{C}}$ be the complex group that corresponds, in the terms of §2.1, to the identity connected component $\overline{B'_0}$ of the closure $\overline{B'}$ of B'. This complex group $B'_{\mathbb{C}}$ is linear reductive, because $\overline{B'}$, and so $\overline{B'_0}$, is compact. Also, $B'_{\mathbb{C}}$ is semisimple, because $\overline{B'}$, and hence $\overline{B'_0}$, is.

Note that the representation of B' is faithful *per sé*, because we consider B' to be the subgroup included in Aut$(\mathbb{C}^{n-2})$. The complexification procedure is compatible with this inclusion, i.e. the representation of the complex group is still faithful, since the real group is compact. If $B'_{\mathbb{C}}$ contains a commutative Lie subgroup of dimension $n-3$, by propositions 1 and 2, we could then conclude that $B'_{\mathbb{C}} = SL(n-2,\mathbb{C})$, and so (by dimension reasons) that $B' = \langle \psi'_u(\gamma'), \psi'_u(\delta')\rangle$ must be dense in $SU(n-2)$. We show equivalently that $\overline{B'}$ contains an $(n-3)$-dimensional torus.

Observe that the full twist braids $(\sigma_1 \dots \sigma_{k-1})^k$ on the leftmost $k = 2,\dots,n-2$ strands commute. It is not hard to see that the eigenvalues of their (reduced, but unnormalized w.r.t. determinant 1 or unitarity) Burau matrices are $k-1$ copies of a certain (rational) power of t, that correspond to eigenspaces which are in the span of the first $k-1$

coordinate vectors, and $n-k-1$ copies of 1. When choosing t not to be a root of unity, we can thus conclude that these $n-3$ elements generate (together) a free commutative group, with the infinite cyclic subgroups generated by the elements (alone) being dense in a circle. Thus the ($n-2$-dimensional) representation contains (also after normalizing) an $(n-3)$-dimensional torus, and we are done with lemma 3. □

So $\{\psi'_u(\alpha\beta\alpha^{-1})' \,:\, \alpha\in B_{2,n}\}$ is dense in a conjugacy class in $SU(n-2)$. Now for $t=1$,

$$\begin{aligned}
\psi(\sigma_1)(\tilde e_1) &= -(1,1-n,1,\dots,1)=(-1,n-2,n-3,\dots,1)_*=\left(-\frac{1}{n-1},\frac{n}{n-1}\cdot(n-2),\dots,\frac{n}{n-1}\cdot 1\right)_{**},\\
\psi(\sigma_1)(\tilde e_2) &= (1,0,-1,0,\dots,0)=(1,1,0,\dots,0)_*=\left(\frac{1}{n-1},\frac{1}{n-1},-\frac{n-3}{n-1},-\frac{n-4}{n-1},\dots,-\frac{1}{n-1}\right)_{**},\\
\psi(\sigma_1)(\tilde e_i) &= \tilde e_i \quad \text{for } i=3,\dots,n-1\,.
\end{aligned}$$

Since for $t=1$ Squier's form is just the standard one, to get ψ conjugated within $SU(n-1)$, we should normalize $\tilde e_i$. Let $\bar e_i := \tilde e_i/||\tilde e_i||$. We have $||\tilde e_i||=\sqrt{2}$ for $i>1$ and $||\tilde e_1||=\sqrt{n(n-1)}$. So with $\eta=\sqrt{\frac{2}{n(n-1)}}$, we have in the basis $\{\bar e_i\}$

$$\psi(\sigma_1)=\begin{bmatrix} -\frac{1}{n-1} & \frac{\eta}{n-1} & 0 & \cdots & 0\\ \frac{n\eta}{n-1}\cdot(n-2) & \frac{1}{n-1} & 0 & \cdots & 0\\ \frac{n\eta}{n-1}\cdot(n-3) & -\frac{n-3}{n-1} & 1 & & 0\\ \vdots & \vdots & & \ddots & \\ \frac{n\eta}{n-1} & -\frac{1}{n-1} & 0 & & 1\end{bmatrix}. \tag{12}$$

(Note that (11) remains unchanged when switching between the bases $\{\tilde e_i\}$ and $\{\bar e_i\}$.)

Now $SU(n-2)$ is connected and $\{\psi'_u(\alpha')\,:\,\alpha\in B_{2,n}\}$ is dense therein. Since $\mathrm{tr}\,(\psi(\sigma_1\alpha\beta\alpha^{-1}))$ is continuous in α and must take only a finite number of values, it is constant. This would mean that the coefficients of the matrices $\{\psi'(\alpha')\psi'(\beta')\psi'(\alpha'^{-1})\}$ satisfy some linear relation.

Since the minor of the matrix in (12) obtained by deleting the first row and column is not a scalar matrix (multiple of Id_{n-2}) one sees that for $t=1$ this linear relation is (non-trivial and) not a trace condition. (A trace condition is meant to be here a linear condition that involves only diagonal entries, and all of them with the same coefficient.) In other words,

$$\mathrm{tr}\left(\left[\begin{array}{c|c} 1 & 0\cdots 0\\ \hline 0 & \\ \vdots & M\\ 0 & \end{array}\right]\cdot\psi(\sigma_1)\right)$$

is not determined by $\mathrm{tr}\,M$. Note that we did not make an effort to find a basis such that (11) becomes a unitary matrix. But a linear condition on M that comes from $\mathrm{tr}\,(AMA^{-1})$ for any $A\in SL(n,\mathbb{C})$ is again just the trace.

So the linear condition resulting from $\mathrm{tr}\,(\psi(\sigma_1\alpha\beta\alpha^{-1}))$ on $\psi'(\alpha'\beta'\alpha'^{-1})$ will not be a trace condition either, for $t=1$. Then the same holds for t close to 1 by continuity. However, by proposition 3, such a condition cannot be satisfied on the conjugacy class of $\psi'(\beta')$. This is a contradiction, and completes the proof of theorem 3. □

Proof of theorem 4. This is essentially contained in lemma 3. It shows that the projection $U(n-1)\to SU(n-1)$ is surjective on $\overline{\psi_n(B_n)}$ for $n>3$. This projection is of codimension one, and by incorporating the full twist braid of all strands into the commuting family in the proof of lemma 3, we see that $\overline{\psi_n(B_n)}$ contains a torus of a higher dimension than (and so cannot have connected component of the identity isomorphic to) $SU(n-1)$. □

4. Alternating links

In many situations, it is more useful to move from (conditions on) braid representations to intrinsic properties of links. In that sense, we will handle the case of alternating links from the point of view of theorems 2 and 3.

Let $b(L)$ be the *braid index* of L, i.e. the smallest n with some $\beta\in B_n$, such that $L=\hat\beta$. With the increasing importance braids gained in link theory, mainly through the work of Jones [15], the study of this invariant gradually expanded. See for example [27]. The main aim of this section is to prove

Theorem 6 Let L be an alternating link, which is not a $(2,k)$-torus link, or a trivial split link (that is, unlink, of any number of components). Then L admits for each $n > b(L)$ infinitely many conjugacy classes of braid representations of n strands.

The exclusion of the $(2,k)$-torus links is obviously necessary. Similarly, it follows from the work of Birman and Menasco [5] (see the proof of Theorem 1, p. 604, therein) that a trivial split link of n components has no non-trivial minimal (i.e. n-strand) braid representation. So at least our construction in theorem 3 can certainly not apply.

In this section, we will use the symbol Δ to denote the (1-variable) Alexander polynomial. It is an invariant of links with values in $\mathbb{Z}[t^{\pm 1/2}]$. We make use of the properties of the Alexander polynomial of alternating links proved by Murasugi [28, 29], and its relation to the Burau matrix (see sections 2 and 7 of [15]; here '$\doteq$' denotes equality up to units in $\mathbb{Z}[t^{\pm 1/2}]$):

$$\frac{1-t^n}{1-t}\,\Delta(\hat{\beta}) \doteq \det(Id_{n-1} - \psi_{n-1}(\beta))\,. \tag{13}$$

We normalize Δ so that $\Delta(t) = \pm\Delta(1/t)$. By $\max\deg\Delta$ we mean the largest number (half-integer) p such that Δ has a non-zero coefficient in t^p. This coefficient is called the leading coefficient. A monic polynomial is one with leading coefficient ± 1.

Let us first record a lemma that has also some independent meaning.

Lemma 4 Let L be a link of n components and braid index $b(L) > 2$. Then L has infinitely many conjugacy classes of braid representations of (a fixed number of) more than $b(L)$ strands, *unless* L satisfies the following 3 properties:

(a) $b(L) = n$, which in particular implies that all components of L are unknotted,

(b) all pairs of components have the same linking number, say k, and

(c) L has the same Alexander polynomial as the (n,kn) torus link $T(n,kn)$,

$$\Delta(T(n,kn)) \doteq \frac{(t^{nk}-1)^{n-1}(1-t)}{1-t^n}\,. \tag{14}$$

Proof. The first two conditions follow directly from theorem 2. So we show the third condition.

Let β be a minimal strand braid representation for L. Excluding cases theorems 2 and 3 apply on, we assume that all components of L are unknotted and all linking numbers are equal to k. Also β is a pure braid, so $b(L) = n$ is the number of components of L, and $\psi(\beta)$ is scalar, as well as $\psi(\beta')$ is for all subbraids β' of β.

Now k determines the exponent sum $[\beta] = n(n-1)k$ of β, and since $\psi(\beta)$ is scalar, also $\psi(\beta)$ up to a root ω of unity. By a continuity argument, ω must be constant in t, and setting $t = 1$ we find that $\omega = 1$. (See also proof of lemma 9.3 in [15].) So k determines $\psi(\beta)$, and hence the Alexander polynomial $\Delta(L)$ by (13).

Since the (n,kn) torus link $T(n,kn)$ has an obvious braid representation

$$\beta_0 = (\sigma_1 \dots \sigma_{n-1})^{nk}\,, \tag{15}$$

with the same n and k, it follows that L must have the same Alexander polynomial as $T(n,kn)$. With the preceding argument about ω one easily verifies that $\psi(\beta_0) = t^{nk} Id_{n-1}$, so (14) follows from (13). □

Proof of theorem 6. We consider k and β as in the previous proof.

If $k = 0$, then by lemma 4, $\Delta = 0$, and so by [29], L is split. We can recur this case to the non-split one, by arguing with some split component L' of L with $b(L') > 2$ instead. By Menasco [23], L' is alternating. It must still have all linking numbers vanishing, and (since β is pure) is the closure of a subbraid β' of β, which has also scalar Burau matrix by assumption. So by the proof of lemma 4 we would obtain a contradiction. Arguing with L' fails if all L' are $(2,l_i)$-torus links. However, then linking number equality in L implies that all even l_i are 0, and triviality of components that all odd l_i are ± 1, so L is a trivial split link, which we excluded.

So we can assume $k \neq 0$, and so w.l.o.g. up to mirroring that $k > 0$. Then, using the fact that $T(n,kn)$ has a positive braid representation (15) of exponent sum $n(n-1)k$ and n strands, we obtain easily that

$$2\max\deg\Delta(L) = 2\max\deg\Delta(T(n,kn)) = n(n-1)k - n + 1\,.$$

Moreover, looking at (14) (or using that a non-split link with a positive braid representation is fibered), we see that $\Delta(T(n,kn)) = \Delta(L)$ is monic, and so by [28], L is also fibered. Now by [27], any reduced alternating diagram D of L has $s(D) = b(L) = n$ Seifert circles, and so by [29], it has crossing number

$$c(D) = 2\max\deg\Delta(L) + s(D) - 1 = n(n-1)k\,.$$

Now we know that all components of L have linking number k, so if $c(D) = n(n-1)k$, all the crossings in D must be positive. So D is a special alternating diagram, and L a fibered special alternating link. Such a link is a connected sum of $(2,l_i)$-torus links (see [36]). Since all components are unknotted, we can assume that all l_i are even (and positive). But then again equality of linking numbers implies that there can be only one $(2,l)$-torus link in this connected sum. So we are done. □

5. Non-conjugate minimal braids

Of course, one could try to apply the proof of theorem 3 to more general situations. It makes, in particular, some sense to look at minimal (strand number) braid representations. Note that the only point where stabilization entered into the proof of theorem 3 was in the matrix (12), and we had to verify that for $\beta \in B_{2,n}$ the map $\psi_{n-1}(\beta') \mapsto tr[\psi_n(\beta)\psi_n(\sigma_1)]$ is not a trace on $\psi_{n-1}(\beta')$. Then one can obtain more general statements. To formulate the following one, chosen still more with regard to simplicity rather than maximality, let $B_{k,l} \subset B_n$ for $1 \le k < l \le n$ be the subgroup (isomorphic to B_{l-k+1}) $\langle \sigma_k, \dots, \sigma_{l-1} \rangle$ of braids that act only on strands $k, \dots, l$.

Theorem 7 Let $\beta \in B_{k,n} \simeq B_{n-k+1}$ for $n - k \ge 3$, and $\gamma \in B_n$. Assume that γ permutes the last $n-k+1$ strands non-trivially (that is, the permutation in S_n associated to γ does not fix all of $k, \dots, n$) and $\psi_{n-k+1}(\beta)$ is not a scalar. Then infinitely many braids in $\{\, \alpha\beta\alpha^{-1}\gamma : \alpha \in B_{k,n} \,\}$ are pairwise non-conjugate.

Proof. Define a basis $\mathcal{B}' = \{e_i\}_{i=0}^{n-1}$ on $\mathbb{C}^n$ by

$$\begin{array}{rcll} e_0 & = & (1,\dots,1), & \\ e_i & = & (0,\dots,\underset{i-1}{0},\underset{i}{i-n},1,\dots,1) & \text{for } i = 1,\dots,k-1 \\ e_j & = & (0,\dots,0,\underset{j}{1},\underset{j+1}{-1},0,\dots,0) & \text{for } j = k,\dots,n-1 \end{array}$$

Let $\mathcal{B} = \mathcal{B}' \setminus \{e_0\}$, $\mathcal{X} = \{e_k,\dots,e_{n-1}\}$ and $X = \mathcal{L}\mathcal{X}$, where $\mathcal{L}$ is the (complex-)linear hull.

It is easy to see that if a coordinate permutation on $\mathbb{C}^n$ acts as a scalar on X, then (for $n-k \ge 2$) the permutation fixes the last $n-k+1$ elements/strands. So by assumption, $\tilde{\psi}(\gamma)$ does not act as a scalar on X for $t = 1$. So, when (ignoring the invariant space $\mathcal{L}e_0$ and) writing $\psi_n(\gamma)$ in the basis $\mathcal{B}$, the minor of the last $n-k+1$ rows and columns is not a scalar matrix. The same is then true when orthonormalizing $\mathcal{X}$ (which is already orthogonal to $\mathcal{B}' \setminus \mathcal{X}$). Then the condition on $M \in SU(n-k)$ coming from

$$\mathrm{tr}\left(\left[\begin{array}{c|c} Id_{k-1} & 0 \\ \hline 0 & M \end{array} \right] \cdot \psi_n(\gamma) \right)$$

is not a trace condition, for $t = 1$. The rest of the proof is as for theorem 3. □

Corollary 2 Let L be a composite link of braid index $n \ge 6$, which factors as $L_1 \# L_2$ in such a way, that both components of $L_{1,2}$ the connected sum is perfomed at are knotted (e.g. any composite *knot* L will do). Then L admits infinitely many non-conjugate minimal representations.

Proof. If $\beta_{1,2} \in B_{n_{1,2}}$ are braids, then we can form a *composite braid* $\beta_1\#\beta_2 \in B_n$ for $n = n_1 + n_2 - 1$ by identifying B_{n_1} with $B_{1,n_1} \subset B_n$ and B_{n_2} with $B_{n_1,n}$, and taking the product $\beta_1\beta_2$ (in B_n). By [7], if $\beta_{1,2}$ are minimal representations of $L_{1,2}$, then $\beta = \beta_1\#\beta_2$ is a minimal representation of L. Now by assumption neither of $\beta_{1,2}$ is pure, and neither fixes the position of the strand n_1 in β. Moreover, $n \geq 6$ implies that one of $n_{1,2}$, say n_1, is at least 4. So we can consider $\alpha\beta_1\alpha^{-1}\#\beta_2$ for all $\alpha \in B_{n_1}$. □

Links of $b = 4,5$ are again a tribute to leaving out stabilized 3-braids in theorem 3, and can be presumably handled by a bit of extra arguments. On the other hand, the statement of the corollary could apply also to further composite links (though it is slightly more technical to formulate).

The construction in theorem 7 also applies to the more general instance of Birman-Menasco's [7] exchange move. It is a move that underlies the switch between conjugacy classes with the same closure. Such move can indeed be described as $\beta\gamma \mapsto \alpha\beta\alpha^{-1}\gamma$ for suitably chosen α, β, γ. So one could apply theorem 7 for many particular exchange moves.

For example let $\beta \in B_{k,n}$ and $\gamma = \delta\gamma'\delta^{-1}$ with $\gamma' \in B_{1,l}$ for some $l > k$, and

$$\delta \in \left\langle \sigma_{l'} \dots \sigma_{k'+1}\sigma_{k'}^2\sigma_{k'+1} \dots \sigma_{l'} \,:\, l' \geq l \geq k' \geq k \right\rangle ,$$

and consider for $\alpha \in B_{l+1,n}$ the braids $\beta\alpha^{-1}\gamma\alpha$. Then an exchange move cancels $\delta^{\pm 1}$ in γ, and $\alpha^{\pm 1}$ cancel then by braid isotopy.

It is not hard to choose β and γ so that $\beta\gamma$ is a minimal representation; for example make $\beta\gamma$ alternating and use Murasugi's result [27], or positive, with all syllables having exponent sum ≥ 2 [31], or make β and γ to be products of conjugates of $\sigma_1^{\pm p}$ for a fixed odd $p \geq 3$ (see [15, corollary 15.5]). We do not know, however, how to prove irreducibility of non-minimal representations, except the special argument of [25] for 4 strands.

Fiedler [11] uses a quantity, called defect (see proposition 5 in op.cit.), related to the conjugacy invariant he defines, to detect when exchange moves alter the conjugacy class. Since Morton shows [26] that Fiedler's invariant is determined by the Burau matrix, one should expect our criterion to be more general. For example, Fiedler's defect vanishes when β is conjugate to its mirror image, but in general such β will still have non-scalar Burau matrix.

6. Related questions and problems

The last section describes some quest for generalizations of the preceding constructions, and points to some occurring problems.

Remark 1 As noted in [16] (see the appendix), a result of Long [21] implies that, if (because) ψ is not faithful for $n = 4$ (resp. $n \geq 5$), its kernel contains Brunnian pure braids, i.e., pure braids all of whose proper subbraids are trivial (see e.g. [34]). So even studying the Burau representation on all subbraids is not powerful enough to allow us to deal with all non-central braids β in the sense of theorems 2 and 3.

On the other hand, we do not know if there are braids which are not covered by theorem 3 but by theorem 2.

Question 1 Are the braids (of 4 or more strands) such that all their subbraids of 4 or more strands have scalar Burau matrix, but some 3-strand subbraid has not?

The 6- and 7-braids of Long-Paton [22] and the 5- and 6-braids of Bigelow (see [3]) with trivial Burau matrix have all a 4-strand subbraid with non-scalar Burau matrix.

Now, using the (faithful) Krammer representation V_n of B_n [17, 4], one could hope to deal with the missing few pure braids in theorem 2 and 3. Let $p = \dim V_n$. So it is interesting also what is known on the above arguments for $SL(p) := SL(p, \mathbb{C})$. For a non-compact Lie group, however, many new problems seem to enter. One of them relates to density in a Lie group, as in lemma 3.

Question 2 When do two matrices A, B in $SL(p, \mathbb{C})$ generate a dense (or Zariski dense) subgroup? To have no common invariant $\mathbb{C}^p$ subspace is obviously a required condition. When combined with what other conditions is it sufficient?

There is a criterion of Chevalley, see proposition 3.1 (b), p. 40, of [9], which can be used to establish denseness. Unfortunately, it requires to check that a set of generators do not leave invariant any elements in arbitrary tensor products of the defining representation of $SL(p)$, which are not left invariant by the whole $SL(p)$. However, since $SL(p)$ is not compact, even this is not enough.

A Borel subgroup of $SL(p)$ fixes the same tensors as the whole $SL(p)$. (See remark 3.2 (a), p. 41, in [9].) So even if these generators fix no more tensors than $SL(p)$, then the best one could prove is that (the V_n image of) their generated group is dense in a Borel subgroup. (The Borel subgroup coincides with the full Lie group for compact Lie groups, but not for $SL(p)$.)

The braids of theorems 2 and 3 can be described alternatively in the following way. Here and below, $\Delta^2 = (\sigma_1 \dots \sigma_{n-1})^n$ is the full twist braid, the generator of the center of B_n. (It has a square root Δ, the half-twist, which however will not occur itself here. This notation is a *deviation from section 4*, where Δ was used for the Alexander polynomial.)

Lemma 5 Let $\beta \in B_n$ be a pure braid. Then $\psi(\beta') = x_{\beta'} \cdot Id$ is a scalar for all subbraids β' of β (incl. $\beta' = \beta$) $\Longleftrightarrow$ $\beta = \Delta^{2l} \cdot \tilde{\beta}$, where $\tilde{\beta}$ and all its subbraids have trivial Burau matrix.

Proof. Since ψ is irreducible, the center of B_n must be mapped to scalars by Schur's lemma (see [15, sections 3 and 9, and note 5.7]). So $\Leftarrow$ follows.

Now to prove is $\Rightarrow$. First look at $n = 3$. Using the argument in the proof of lemma 4 that $\omega = 1$, we find that $\psi(\beta) = t^{[\beta]/2} Id$. Thus there exist $l, m \in \mathbb{Z}$ such that $\psi(\beta)^l = \psi(\Delta^2)^m$. Since ψ_3 is faithful, $\beta^l = \Delta^{2m}$. This means in particular that all strand linking numbers in β are equal, say k. Then $\dot{\beta} = \Delta^{-2k}\beta$ is a braid with scalar $\psi(\dot{\beta}) = x_{\dot{\beta}} Id$, all strand linking numbers 0, so $[\dot{\beta}] = 0$. Now $\det(\psi(\beta)) = 1 = x_{\dot{\beta}}^{n-1}$. So $x_{\dot{\beta}}(t)$ is a continuous function in t, taking values in (a discrete set of) $(n-1)$-st roots of unity. Then we can evaluate it by setting $t = 1$. In that case ψ is essentially a permutation representation, and since $\dot{\beta}$ is pure, $\psi_{\dot{\beta}}(1) = Id$. So $x_{\dot{\beta}} = 1$, and $\psi(\dot{\beta}) = Id$ independently on t. Since ψ_3 is faithful, $\dot{\beta} = Id$, so $\beta = \Delta^{2k}$.

Now let $n > 3$. A look at 3 strand subbraids and the previous argument show, that all linking numbers between strands of β are equal, say m. Then $\dot{\beta} = \Delta^{-2m}\beta$ and all its subbraids have scalar ψ, and all strand linking numbers are 0, so $[\dot{\beta}] = 0$. Then as above we see $\psi(\dot{\beta}) = Id$, and the same holds for all subbraids of $\dot{\beta}$. □

Remark 2 I do not know if $\psi(\beta)$ being scalar (but *without* assuming this condition on its subbraids) implies that

$$\beta = \Delta^{2l} \cdot \tilde{\beta}, \quad \text{where } \psi(\tilde{\beta}) = Id. \tag{16}$$

It is certainly true for $n = 3$.

Since if $\psi(\beta)$ is scalar, we always have $\psi(\beta)^l = \psi(\Delta^2)^m$ for proper l and m, the argument in the proof of lemma 5 will allow us to conclude (16) if $\Delta^{-2m}\beta^l \in \operatorname{Ker}\psi$ has all its strand linking numbers equal (and then equal to 0), because then the same property is enjoyed by β. This leads to the following question:

Question 3 Has every element in $\operatorname{Ker}\psi$ all strand linking numbers equal to 0?

The 6- and 7-braids of Long-Paton and the 5- and 6-braids of Bigelow with trivial Burau matrix have all their strand linking numbers equal to 0 (i.e. all 2-strand subbraids are trivial), though all they have non-trivial 3-strand subbraids.

Note that for every β in the proof of lemma 5, the braid $\tilde{\beta} = \beta^{n(n-1)/2}\Delta^{-[\beta]}$ would have scalar ψ and exponent sum 0. So by the previous argument we have $\tilde{\beta} \in \operatorname{Ker}\psi$, and so property (16) always holds for $\beta^{n(n-1)/2}$. The following related question becomes of some interest:

Question 4 Are there elements β in $\operatorname{Ker}\psi$ which have no (non-trivial) roots, i.e. $\beta \neq \alpha^l$ for any $l > 1$ and $\alpha \in B_n$?

Appendix A. Proof of theorem 2

The aim of this appendix is to prove theorem 2, which we restate:

Theorem 2 Assume there exists an n-strand braid β having a link L as closure, $n \geq 3$, and β is contained in the below classes:

1. a non-pure braid, or

2. a pure braid that contains a non-central 3-strand braid as a sub-braid.

Then there exists an infinite sequence of pairwise non-conjugate braids of $n+1$ strands realizing L.

The proof partly uses Shinjo's [32] method.

Definition 1 For a braid β, we define the *axis (addition) link* $L(\beta)$ of β as the link consisting of the closure of β and its axis.

As conjugate braids have the same axis link, for a proof of non-conjugacy we will study invariants of the axis link. As such an invariant we will employ the Conway polynomial ∇, which takes values in $\mathbb{Z}[z]$ and is given by the relation

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0) \tag{17}$$

for a skein triple $L_{\pm,0}$ and the value 1 on the unknot.

To obtain non-conjugate braids, we conjugate again β by certain braids α_n, and obtain the desired family $\{\beta_n\}$ by stabilization. Herein the difficulty is the proper choice of α_n and the non-conjugacy proof of β_n. We already remarked that if β is a central braid, such a construction cannot apply, but that further problematic braids β occur. In [34, 37] Stanford and the author constructed Brunnian braids. Brunnian means that the braid is itself non-trivial, but removing any single strand gives a trivial braid. With σ_k being the k-th Artin standard generator, the iterated commutator

$$[\sigma_1^{\pm 2}, [\sigma_2^{\pm 2}, [\ldots, [\sigma_{n-1}^{\pm 2}, \sigma_n^{\pm 2}] \ldots]]]$$

has this property. Multiplying such a braid with a central braid, we see that there exist non-central braids neither contained in the classes specified in theorem 2. Therefore, by just studying subbraids, we will likely not be able to arrive at the maximal possible validity scope of the theorem.

Proof of theorem 2. If the closure link L has a component K', which is the closure of a subbraid β' of β of at least 3 strands, we consider its axis link $L(\beta')$. We can move properly the strands of K' within β to the right (see lemma 2.3 of [32]) and perform the construction of [32], and obtain a family of links that contain infinitely many sublinks. Then the family is infinite itself.

From now on, we assume the components of L are obtained all as closures of 1 or 2 strand subbraids of β. As β is not pure, we can choose a component K of L of a 2 strand subbraid γ, and as β has at least 3 strands, we can find also another component K' of a subbraid γ'. Now we ignore components except K, K' and the axis of β. Performing with K the previous construction of [32], we create a sequence of braids $\{\beta_i\}$. That is, we set

$$\beta_i = \sigma_{n-1}^{-i}\sigma_n\sigma_{n-1}^{i}\beta\,, \tag{18}$$

choosing the crossings corresponding to the $\sigma_{n-1}^{\pm 1}$ surrounding the σ_n to contain two strands of K (see lemma 2.3 of [32]). Then β_i contain the subbraids γ_i corresponding to K, and $\beta_i = \gamma_i \cup \gamma'$.

Next we recall the formula, written down by Hoste, expressing the lowest coefficient of the Conway polynomial in terms of component linking numbers.

Theorem 8 (Hoste [19]) Let L be a p-component link and number the components by $1,\dots,p$. Let the linking number between components k and m be denoted by $l_{k,m}$. Then the coefficient $[\nabla]_{p-1}$ of the Conway polynomial in degree $p-1$ is

$$[\nabla]_{p-1}(L) = \sum_T \prod_{(k,m)\in T} l_{k,m}\,. \tag{19}$$

Herein the sum ranges over spanning trees T of the complete graph G on the vertex set $\{1,\dots,p\}$, and (k,m) denotes the edge in G connecting the k-th and m-th vertex.

Next, we fix i. Let L_0 be the axis link of the braid obtained from β_{i+1} by deleting the σ_{n-1}^{-1} occurring in (18) directly before σ_n, resp. L_0' the axis link of the braid obtained from β_i by deleting the σ_{n-1} directly after σ_n. The number of components of $L(\beta_i)$ and $L(\beta_{i+1})$ is 3, that of L_0, L_0' becomes 4. Using the skein relation (17), we express the difference of the degree-4-coefficient of the Conway polynomials of $L(\beta_i)$ and $L(\beta_{i+1})$ by the degree-3-coefficients of L_0, L_0' (see lemma 2.2 of [32]):

$$[\nabla]_4 L(\beta_i) - [\nabla]_4 L(\beta_i) = [\nabla]_3(L_0) - [\nabla]_3(L_0')\,,$$

and calculate the right hand-side by Hoste's formula. In L_0, L_0', the component K breaks apart into two components $K_{1,2}$, $K_{1,2}'$, and the linking numbers with the axes K_3, K_3' satisfy

$$lk(K_1,K_2) = lk(K_1',K_2')\,,\quad lk(K_1,K_3) = lk(K_1',K_3')+1\,,\quad \text{and}\quad lk(K_2,K_3) = lk(K_2',K_3')-1\,. \tag{20}$$

Thus, if n is the number of strands of K, and we write $L_* = K_1\cup K_2\cup K_3$, $L_*' = K_1'\cup K_2'\cup K_3'$, then

$$[\nabla]_2(L_*) - [\nabla]_2(L_*') = lk(K_2,K_3) - lk(K_1,K_3)+1 = n-2\,. \tag{21}$$

In the previous case of [32], assuming $n>2$, the degree-3-coefficients of the Conway polynomials of the axis links of γ_i form a non-trivial linear progression, and we conclude that the axis links are mutually distinct.

Presently, $n=2$, so to arrive at the same situation, we take into account the other component K', and consider β_i instead of γ_i. K' becomes in L_0, L_0' a 4-th component K_0, K_0'. The equalities (20) still hold, in addition for $i=1,2,3$ we have $lk(K_0,K_i) = lk(K_0',K_i')$.

The 4-th component adds 13 terms to the sum in (19), but the calculation of $[\nabla]_3(L_0) - [\nabla]_3(L_0')$ is not too difficult. We break the sum into 4 parts. Into the first partial sum Σ_0 enter the terms corresponding to trees T containing neither $(1,3)$ nor $(2,3)$, into the 4-th sum Σ_3 the trees containing both $(1,3)$ and $(2,3)$, resp. the second and third sums Σ_1 and Σ_2 range over trees containing only $(1,3)$ resp. $(2,3)$.

$$\Sigma_0(L_0) = \Sigma_0(L_0')$$

is trivial, and because $\Sigma_3(L_0)-\Sigma_3(L_0')$ is a multiple of the term in (21) and the number of strands of γ is $n=2$, that difference becomes also 0. Using (20), we can calculate $\Sigma_{1,2}(L_0)-\Sigma_{1,2}(L_0')$ directly. The pairs of edges that complete $(1,3)$ and $(2,3)$ to a spanning tree are largely the same, and their contributions to $\Sigma_{1,2}$ cancel, and we obtain

$$\Sigma_1(L_0)+\Sigma_2(L_0)-\Sigma_1(L_0')-\Sigma_2(L_0') = lk(K_3,K_0)lk(K_0,K_2) - lk(K_3,K_0)lk(K_0,K_1)\,.$$

As $lk(K_3,K_0)\in\{1,2\}$ is not 0, the remaining troublesome case is $lk(K_0,K_2)=lk(K_0,K_1)$. However, in that case, we make the right strand of β turn m times around the other strands, that is, we can conjugate by the m-th power of $\sigma_{n-1}^{-1}\dots\sigma_1^{-1}\sigma_1^{-1}\dots\sigma_{n-1}^{-1}$. When creating L_0 and L_0', the rotations added before the point of stabilization enter into $lk(K_0,K_2)$, those added after it (of opposite sign) enter into $lk(K_0,K_1)$, so we can choose m large enough the way that $lk(K_0,K_2)=lk(K_0,K_1)$ does not hold. The proof of the claim corresponding to the first class in the theorem is completed.

The second claim is a direct consequence of the following lemma.

Lemma 6 If $\beta\in B_3$ is not central, we can choose a braid $\alpha\in B_3$ so that the stabilizations of $\alpha^n\beta\alpha^{-n}$ contain an infinite family of non-conjugate 4-braids.

Proof. The lemma's proof uses the Burau representation. The Burau representation ψ of B_3 is a homomorphism into the algebra of 2×2 matrices with entries in $\mathbb{Z}[t,t^{-1}]$, and can be defined as follows:

$$\psi(\sigma_1) = \left[\begin{array}{cc} -t & 0 \\ -1 & 1 \end{array}\right], \quad \psi(\sigma_2) = \left[\begin{array}{cc} 1 & -t \\ 0 & -t \end{array}\right].$$

It is known that the Burau representation is faithful in the case of B_3.

As β is not central, we find an element α not commuting with β. Assume $\alpha^n\beta\alpha^{-n}\sigma_3^{\pm 1} \in B_4$ decompose only into a finite number of conjugacy classes. We use the homomorphism $B_4 \to B_3$ sending $\sigma_{1,2,3}$ to $\sigma_{1,2,1}$. Then the images $\alpha^n\beta\alpha^{-n}\sigma_1^{\pm 1} \in B_3$ are also contained in finitely many conjugacy classes. Let $\beta_n = \alpha^n\beta\alpha^{-n}$. Then, after a possible index shift, we can assume w.l.o.g. that

$$\operatorname{tr}\psi(\beta_n) = \operatorname{tr}\psi(\beta), \quad \operatorname{tr}\psi(\beta_n\sigma_1) = \operatorname{tr}\psi(\beta\sigma_1), \quad \operatorname{tr}\psi(\beta_n\sigma_1^{-1}) = \operatorname{tr}\psi(\beta\sigma_1^{-1}) \tag{22}$$

hold simultaneously for infinitely many values of the natural number n. (tr denotes the matrix trace.) Let Δ denote $\sigma_1\sigma_2\sigma_1$ (the square root of the center generator). If we replace β by $\Delta\beta\Delta^{-1}$, in (22) we can exchange σ_1 and σ_2, add two more conditions, and together with (22) obtain 5 equations. These equations pose 3 linearly independent conditions on $\psi(\beta_n)$. Thus the difference matrix $\psi(\beta_n) - \psi(\beta)$ is determined by its upper left entry z. Besides,

$$\det(\psi(\beta_n)) = \det(\psi(\beta)),$$

so we obtain a separate condition on z. This is a quadratic equation. One can directly verify that this equation does not degenerate for general $\psi(\beta)$ if $t \neq e^{\pm 2\pi i/3}$, and so it has only two roots. Then we conclude that the set $\{\psi(\beta_n)\}$ must be finite. To obtain a contradiction, and with this to complete the proof, we use the faithfulness of the representation, and show that, for proper choice of α, the set $\{\beta_n\}$ becomes infinite. This fact is well-known, but we include two independent simple arguments.

We use Garside's normal form for braid words (see [14, §2.2,2.3]). Express (uniquely) β in the form $\Delta^k\beta'$. Here $k \in \mathbb{Z}$ and Δ is the previously occurred square root of the center generator. The word β' is characterized by two conditions. First, it is a positive braid word, that is, it contains only positive powers of σ_1 and σ_2. Moreover, β' contains no $\Delta = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ as a subword. (Garside's condition to be prime to Δ reduces to this for a 3-braid.) We call $k = p(\beta)$ the *power* and β' the *base* of β. As β is pure and not central, β' is not trivial. If β' contains both σ_1 and σ_2, we choose α to be the last letter in β'. Now $\beta_i = \beta_{i'}$ is equivalent to $\alpha^{i'-i}\beta = \beta\alpha^{i'-i}$. As σ_i commute with Δ^{2k}, we can easily compare the normal forms of $\alpha^i\beta$ and $\beta\alpha^i$ and see that they do not coincide at least for large i. (It is possible that $p(\alpha^i\beta) > p(\beta)$, while always $p(\beta\alpha^i) = p(\beta)$.) So we conclude the desired property. If β' contains only one letter of σ_1 and σ_2, we choose α to be the other letter, and can argue similarly.

An alternative argument uses Artin's "combed" normal form [2] for pure braids. For $1 \le i < j \le 3$, let $[ij]$ be the "tooth" of the comb, turning strand i clockwise once around strands $i+1,\dots,j$. So $[12] = \sigma_1^2$, $[23] = \sigma_2^2$ and $[13] = \sigma_1\sigma_2^2\sigma_1$. Then there is a unique expression $\beta = \beta'\beta''$, where $\beta' = [23]^k$ and $\beta'' \in <[13],[12]>$, latter being the group freely generated by $[13]$ and $[12]$. If β'' contains $[12]$ (i.e. is not a power of $[13]$), then choose $\alpha = [13]$. Since $[13]$ and $[23]$ commute, we can compare the normal forms of $\alpha^i\beta$ and $\beta\alpha^i$ and immediately see that they do not coincide. As $\beta_i = \beta_{i'}$ is equivalent to $\alpha^{i'-i}\beta = \beta\alpha^{i'-i}$, we conclude the desired property. So let $\beta'' = [13]^l$. Then $\beta' = [23]^k$ with $l \neq k$ (because for $l = k$ we have $\beta = \Delta^{2k}$). Then let $\alpha = [12]$. We have $[12][23] = [23][13][12][13]^{-1}$. So

$$\alpha\beta = [12][23]^k[13]^l = [23]^k[13]^k[12][13]^{l-k},$$

and inductively

$$\alpha^i\beta = [12]^i[23]^k[13]^l = [23]^k[13]^k[12]^i[13]^{l-k}.$$

By comparing the subwords of $\alpha^i\beta$ and $\beta\alpha^i$ in $[13]$ and $[12]$, we see that they differ (because $k \neq l$). □

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