On "M-functions" closely related to the distribution of L'/L-values

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[Abstract] *For each global field K, we shall construct and study a basic arithmetic function $M_\sigma^{(K)}(z)$ on $\mathbb{C}$ parametrized by $\sigma > 1/2$, together with its Fourier transform $\tilde{M}_\sigma^{(K)}(z)$. This function $M_\sigma^{(K)}(z)$ is closely related to the density measure for the distribution of values on $\mathbb{C}$ of the logarithmic derivatives of L-functions $L(\chi, s)$, where s is fixed, with $Re(s) = \sigma$, and χ runs over a (natural) infinite family of Dirichlet characters on K.*

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1 Introduction

1.1

By a global field, we mean either an algebraic number field of finite degree, or an algebraic function field of one variable over a finite field. For each global field K, we shall construct two basic functions $M_\sigma(z) = M_\sigma^{(K)}(z)$ and $\tilde{M}_\sigma(z) = \tilde{M}_\sigma^{(K)}(z)$ of $z \in \mathbb{C}$, each parametrized by $\sigma > 1/2$, and in some special cases establish explicit relations with the density measure for the distribution of values of $L'(\chi, s)/L(\chi, s)$ on $\mathbb{C}$. Here, s is *fixed*, $\sigma = Re(s)$, and χ runs over a suitable *infinite* family of Dirichlet characters on K. (Unless the L-functions $L(\chi, s)$ contain a local $\wp$-factor for which $\{\chi(\wp)\}_\chi$ is not uniformly distributed on the unit circle $\mathbb{C}^1$, the distribution measure basically depends only on σ).

Symbolical relations among them, under optimal circumstances are,

$$(1.1.1) \qquad M_\sigma(z) = \mathrm{Avg}_\chi \delta_z\left(\frac{L'(\chi, s)}{L(\chi, s)}\right), \quad \tilde{M}_\sigma(z) = \mathrm{Avg}_\chi \psi_z\left(\frac{L'(\chi, s)}{L(\chi, s)}\right).$$

Here, Avg_χ means a certain weighted average, $\psi_z(w) = \exp(i.\mathrm{Re}(\bar{z}w))$ is the additive character $\mathbb{C} \mapsto \mathbb{C}^1$ parametrized by z, and $\delta_z(w)|dw|$ is the Dirac delta measure on $\mathbb{C}$ with support at z, where $|dw|$ denotes the self-dual Haar measure on $\mathbb{C}$ with respect to the self-dual pairing of $\mathbb{C}$ defined by $\psi_z(w) = \psi_w(z)$; namely, $|dw| = (2\pi)^{-1}dxdy$ for $w = x + iy$. In other words, the first formula of (1.1.1) means that

$$(1.1.2) \qquad \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw| = \mathrm{Avg}_\chi \Phi\left(\frac{L'(\chi, s)}{L(\chi, s)}\right)$$

holds for any test function Φ on $\mathbb{C}$, and the second formula is its special case where $\Phi = \psi_z$. The case of polynomial functions $\Phi(w) = \bar{w}^a.w^b$ will also appear as the coefficient of $z^a\bar{z}^b$ in the $(z, \bar{z})$-expansion of $\tilde{M}_\sigma(z)$ at $z = 0$.

1.2

The function $M_\sigma(z)$ to be constructed is real valued, ≥ 0, and belongs to C^∞, while $\tilde{M}_\sigma(z)$ is complex-valued, $|\tilde{M}_\sigma(z)| \leq 1$, and real-analytic. They are the Fourier transforms of each other in the sense that

$$(1.2.1) \qquad \tilde{M}_\sigma(z) = \int_{\mathbb{C}} M_\sigma(w)\psi_z(w)|dw|, \quad M_\sigma(z) = \int_{\mathbb{C}} \tilde{M}_\sigma(w)\psi_{-z}(w)|dw|.$$

Both are continuous also in σ, and $\tilde{M}_\sigma(z)$ is even real-analytic in σ. They have quite interesting arithmetic and analytic properties. $\tilde{M}_\sigma(z)$ has a convergent *Euler product* expansion each of whose $\wp$-factor can be expressed in terms of Bessel functions, and correspondingly, $M_\sigma(z)$ has a *convolution Euler product* expansion, each of whose $\wp$-factor being a certain hyperfunction. Also, $\tilde{M}_\sigma(z)$ has an everywhere convergent power series expansion (in $z, \bar{z}$) whose coefficients are some arithmetic Dirichlet series in σ. Both decay rapidly as $|z| \mapsto \infty$. Thus, even when $1/2 < \sigma < 1$ in the number field case, where we do not know much about the zeros of $L(\chi, s)$ and hence about the poles of $L'(\chi, s)/L(\chi, s)$, and hence about the distribution of $L'(\chi, s)/L(\chi, s)$ near $z = \infty$, still, the corresponding function $M_\sigma(z)$ can be constructed independently and can be proved to be rapidly decreasing with $|z|$. It seems that these functions $M_\sigma(z), \tilde{M}_\sigma(z)$ are interesting in themselves, and also that one can hope for applications to the distribution of L'/L-values after further studies of their analytic properties.

The construction and study of $M_\sigma(z)$ are in §2, and those of $\tilde{M}_\sigma(z)$, in §3 (Theorems $1 \sim 5$).

1.3

The main idea is as follows. Fix $s \in \mathbb{C}$, with $\sigma = \mathrm{Re}(s)$.

[**Local constructions**] Let $\sigma > 0$, and P be a finite set of non-archimedean primes of K. Put

$$T_P = \prod_{\wp \in P} \mathbb{C}^1 \tag{1.3.1}$$

(a torus), and let $g_{\sigma,P} : T_P \longmapsto \mathbb{C}$ be defined by

$$g_{\sigma,P}(t) = \sum_{\wp \in P} g_{\sigma,\wp}(t_\wp) = \sum_{\wp \in P} \frac{t_\wp \log N(\wp)}{t_\wp - N(\wp)^\sigma} \tag{1.3.2}$$

($t = (t_\wp) \in T_P$). For a Dirichlet character χ on K, let

$$L_P(\chi, s) = \prod_{\wp \in P} (1 - \chi(\wp) N(\wp)^{-s})^{-1} \tag{1.3.3}$$

be the partial L-function. If the conductor $\mathbf{f}_\chi$ is coprime with P, then

$$\frac{L'_P(\chi, s)}{L_P(\chi, s)} = g_{\sigma,P}(\chi_P . N(P)^{-i.\tau}), \tag{1.3.4}$$

where $\tau = \mathrm{Im}(s)$ and

$$\chi_P = (\chi(\wp))_\wp, \quad N(P)^{-i.\tau} = (N(\wp)^{-i.\tau})_\wp \tag{1.3.5}$$

are points of T_P. (Through (1.3.4), we are viewing $L'_P(\chi, s)/L_P(\chi, s)$ as a function of χ.) Now, for each family of χ that we shall consider, all but finitely many χ have conductors coprime with P and, moreover, $\{\chi_P\}_\chi$ for such χ can be shown to be uniformly distributed on T_P. Therefore, (1.1.1), with L_P in place of L, must be given by the corresponding integrals

$$(1.3.6) \qquad M_{\sigma,P}(z) = \int_{T_P} \delta_z(g_{\sigma,P}(t))d^\star t, \quad \tilde{M}_{\sigma,P}(z) = \int_{T_P} \psi_z(g_{\sigma,P}(t))d^\star t.$$

($d^\star t$: the normalized Haar measure on T_P. Note that the contribution of $\mathrm{Im}(s)$ is "averaged away".) These already serve as definitions of the local functions $M_{\sigma,P}(z), \tilde{M}_{\sigma,P}(z)$. We thus have

$$(1.3.7) \qquad \int_{\mathbb{C}} M_{\sigma,P}(w)\Phi(w)|dw| = \mathrm{Avg}_\chi \Phi\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)$$

for any continuous function Φ on $\mathbb{C}$. (Each $M_{\sigma,P}(z)$ is compactly supported.) The summation over $\wp \in P$ in (1.3.2) is translated into "the basic product expansions"

$$(1.3.8) \qquad M_{\sigma,P}(z) = *_{\wp\in P} M_{\sigma,\wp}(z), \quad \tilde{M}_{\sigma,P}(z) = \prod_{\wp\in P} \tilde{M}_{\sigma,\wp}(z),$$

where $*$ denotes the convolution product. Using the simple fact that each $g_{\sigma,\wp}$ maps $\mathbb{C}^1$ to another small circle on $\mathbb{C}$ with center $c_{\sigma,\wp}$ and radius $r_{\sigma,\wp}$ given by

$$(1.3.9) \qquad c_{\sigma,\wp} = -\frac{\log N(\wp)}{N(\wp)^{2\sigma}-1}, \quad r_{\sigma,\wp} = \frac{N(\wp)^\sigma \log N(\wp)}{N(\wp)^{2\sigma}-1},$$

we are able to compute each of $M_{\sigma,\wp}(z)$ and $\tilde{M}_{\sigma,\wp}(z)$ explicitly.

Each $M_{\sigma,\wp}(z)$ is a hyperfunction (Schwartz distribution), but when $|P| > 1$, $M_{\sigma,P}(z)$ is a function (with values ≥ 0) with compact support, which gets smoother (and the support larger) as $|P|$ increases. On the other hand, each $\tilde{M}_{\sigma,\wp}(z)$ is already a ($\mathbb{C}$-valued) real-analytic function expressible by Bessel functions, which satisfies $|\tilde{M}_{\sigma,\wp}(z)| \leq 1$, and $= O((1+|z|)^{-1/2})$.

[**Global constructions**] Let $\sigma > 1/2$, and $P = P_y = \{\wp;\ N(\wp) \leq y\}$. Then the key point is that each $M_{\sigma,P}(z)$ (resp. $\tilde{M}_{\sigma,P}(z)$) *converges* uniformly to a (not-everywhere vanishing) function $M_\sigma(z)$ (resp. $\tilde{M}_\sigma(z)$), when $y \mapsto \infty$.

Thus, these are the functions obtained from $\delta_z(L'_P(\chi,s)/L_P(\chi,s))$ (resp. $\psi_z(L'_P(\chi,s)/L_P(\chi,s))$) first by fixing P and averaging over an infinite family of characters χ, and then by letting $y \mapsto \infty$. This way we can enter the region $1/2 < \sigma < 1$ unconditionally! Since we do not know, in the number field case with $\sigma < 1$, whether the convergence

$$(1.3.10) \qquad L'_P(\chi,s)/L_P(\chi,s) \longmapsto L'(\chi,s)/L(\chi,s)$$

holds (this convergence for all $\sigma > 1/2$ would of course imply the Generalized Riemann Hypothesis), the other approach is blocked by the "GRH-barrier".

Then one asks. How can one connect the averages of the global $\delta_z(L'(\chi,s)/L(\chi,s))$ (resp. $\psi_z(L'(\chi,s)/L(\chi,s)))$ with $M_\sigma(z)$ (resp. $\tilde{M}_\sigma(z)$) ? When $\sigma > 1$, the local relation (1.3.7) directly passes over to the global relation (1.1.2), because then (1.3.10) not only converges but moreover the convergence is uniform with respect to the characters χ. Thus, in this case, the only key point is the local uniformity of distribution of $\{\chi_P\}_\chi$ for each P. The main results for this case will be given in §4, Theorem 6, after having made clear what family of χ and what weighted average over χ we shall take.

When $1/2 < \sigma \leq 1$, the same argument does not work, because even in the function field case where (1.3.10) converges, its speed apparently depends on the size of the norm of the conductor $N(\mathbf{f}_\chi)$. The main purpose of §5-6 is to overcome this difficulty, at least partly. We shall prove (§6, Theorem 7) that if $\sigma > 3/4$, similar global relations are indeed valid in the function field case. This will be done by Fourier analysis of the function $\psi_z(g_{\sigma,P}(t))$ on T_P (§5), and a quantitative version of the uniform distribution of $\{\chi_P\}_\chi$ on T_P.

1.4

Our main results may be summarized as follows.

Theorem $\tilde{M}$ *Let K be any global field, and $\zeta_K(s)$ be its Dedekind zeta function.*
(i) *For each non-archimedean prime $\wp$ of K, consider the function of $\sigma > 0$ and $z \in \mathbb{C}$ defined by the convergent series*

$$\tilde{M}_{\sigma,\wp}(z) = 1 + \sum_{n=1}^{\infty} \frac{G_n(-\frac{i}{2}z\log N(\wp))G_n(-\frac{i}{2}\bar{z}\log N(\wp))}{N(\wp)^{2\sigma n}}, \tag{1.4.1}$$

where $i = \sqrt{-1}$ and

$$G_n(w) = \sum_{k=1}^{n} \frac{1}{k!}\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} w^k. \tag{1.4.2}$$

Then

$$\tilde{M}_{\sigma,\wp}(z) = \exp(ic_{\sigma,\wp}\mathrm{Re}(z))H_{\sigma,\wp}(z), \tag{1.4.3}$$

with

$$H_{\sigma,\wp}(z) = J_0(r_{\sigma,\wp}|z|) + 2\sum_{n=1}^{\infty}\left(\frac{i}{N(\wp)^\sigma}\right)^n \cos(n\mathrm{Arg}(z))J_n(r_{\sigma,\wp}|z|), \tag{1.4.4}$$

$J_n(x)$ being the Bessel function of order n.
(ii) *When $\sigma > 1/2$, the Euler product*

$$\tilde{M}_\sigma(z) = \prod_\wp \tilde{M}_{\sigma,\wp}(z) = \exp\left(i.\frac{\zeta_K'(2\sigma)}{\zeta_K(2\sigma)}\mathrm{Re}(z)\right)\prod_\wp H_{\sigma,\wp}(z), \tag{1.4.5}$$

converges in the following sense. For any compact subset Σ of $\mathbb{C}$, there exists a finite set S_Σ of $\wp$ such that $H_{\sigma,\wp}(z)$ and (hence also) $\tilde{M}_{\sigma,\wp}(z)$ have no zeros on Σ for $\wp \notin S_\Sigma$ and that their product over all $\wp \notin S_\Sigma$ converge absolutely to nowhere vanishing functions of $z \in \Sigma$. This function $\tilde{M}_\sigma(z)$ is real analytic in σ, z, and as a function of z, belongs to L^p for all $1 \leq p \leq \infty$. It has an everywhere convergent power series expansion

$$\tilde{M}_\sigma(z) = 1 + \sum_{a,b=1}^{\infty} (-i/2)^{a+b} \mu_\sigma^{(a,b)} \frac{z^a \bar{z}^b}{a!b!}, \tag{1.4.6}$$

and a convergent Dirichlet series expansion on $\sigma > 1/2$

$$\tilde{M}_\sigma(z) = \sum_{D:integral} \frac{\lambda_D(z)\lambda_D(\bar{z})}{N(D)^{2\sigma}}, \tag{1.4.7}$$

with positive real constants $\mu_\sigma^{(a,b)}$ and polynomials $\lambda_D(z)$ defined in §3.7. Here,D runs over all "integral" divisors of K, i.e., the products of non-negative powers of non-archimedean primes.

Theorem M *There exists a unique continuous function $M_\sigma(z)$ of $\sigma > 1/2$ and z such that*

$$\tilde{M}_\sigma(z) = \int_{\mathbb{C}} M_\sigma(w)\psi_z(w)|dw|, \quad M_\sigma(z) = \int_{\mathbb{C}} \tilde{M}_\sigma(w)\psi_{-z}(w)|dw|. \tag{1.4.8}$$

It is non-negative real valued, C_∞ in z, and satisfies

$$\int_{\mathbb{C}} M_\sigma(z)|dz| = 1. \tag{1.4.9}$$

As for the connections with L'/L-values, presently, we shall restrict our attention to the case where K is either the rational number field $\mathbb{Q}$, an imaginary quadratic field, or a function field of one variable over a finite field (see §4.1 for related discussions). In the function field case, we assume that K is given together with an "infinite" prime divisor $\wp_\infty$ of degree 1 which will be considered "archimedean" and excluded from the ζ_K, L and $\tilde{M}$, M Euler factors. Let χ run over all Dirichlet characters on K with *prime* conductors $\mathbf{f}_\chi$ satisfying

$$\chi(\wp_\infty) = 1. \tag{1.4.10}$$

We define the weighted average over such χ by

$$\mathrm{Avg}_\chi \Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \lim_{m\mapsto\infty} \mathrm{Mean}_{N(\mathbf{f})\leq m}\left(\mathrm{Mean}_{\mathbf{f}_\chi=\mathbf{f}} \Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right)\right), \tag{1.4.11}$$

(Φ: any function on $\mathbb{C}$) whenever the limit exists, where Mean means the usual arithmetic mean.

Theorem $L \sim M$ *Let* $s \in \mathbb{C}$ *with* $\sigma = \mathrm{Re}(s) > 1/2$. *At least if* $\sigma > 1$ ($K = \mathbb{Q}$ *or imaginary quadratic*), *or* $\sigma > 3/4$ (K *a function field*), *then*
(i)

$$\mathrm{Avg}_\chi \Phi\left(\frac{L'(\chi, s)}{L(\chi, s)}\right) = \int_{\mathbb{C}} M_\sigma(z)\Phi(z)|dz| \tag{1.4.12}$$

holds for any "mild" test function Φ *on* $\mathbb{C}$ (*see Theorems 6,7 for details*).
(ii)

$$\mathrm{Avg}_\chi \psi_z\left(\frac{L'(\chi, s)}{L(\chi, s)}\right) = \tilde{M}_\sigma(z), \tag{1.4.13}$$

(iii)

$$\mathrm{Avg}_\chi P^{(a,b)}\left(\frac{L'(\chi, s)}{L(\chi, s)}\right) = (-1)^{a+b}\mu_\sigma^{(a,b)}, \tag{1.4.14}$$

for the polynomials $P^{(a,b)}(w) = \bar{w}^a w^b$ $(a, b \geq 0)$.

We expect that Theorem $L \sim M$ should hold for any $\sigma > 1/2$. But even in the function field case where the Weil's Riemann Hypothesis is valid, the above restriction $\sigma > 3/4$ seems to be the limit of our method (see §6).

We have also left untouched various basic questions related to the functions $M_\sigma(z)$, $\tilde{M}_\sigma(z)$; for example, their zeros, their values at some special points (such as $M_\sigma(0)$, $M_\sigma(\zeta_K'(2\sigma)/\zeta_K(2\sigma))$), determination of the value of

$$\int_{\mathbb{C}} M_\sigma(z)^2|dz| = \int_{\mathbb{C}} |\tilde{M}_\sigma(z)|^2|dz|, \tag{1.4.15}$$

etc. We hope to be able to discuss these in the near future, together with more applications.

2 Constructions of $M_{\sigma,P}(z)$ and $M_\sigma(z)$

2.1

We fix a global field K. By $\wp$ we shall denote any non-archimedean prime divisor of K, and by P any non-empty finite set of such $\wp$. For $y > 1$, put

$$P_y = \{\wp;\, N(\wp) \le y\}. \tag{2.1.1}$$

We shall construct, for each P, a function $M_{\sigma,P}(z)$ on $\mathbb{C}$ parametrized by $\sigma > 0$, and then show that $M_{\sigma,P_y}(z)$ converges uniformly, as $y \mapsto \infty$, to a function $M_\sigma(z)$ when $\sigma > 1/2$.

As in §1, $T_P = \prod_{\wp\in P} \mathbb{C}^1$, and $g_{\sigma,P} : T_P \longmapsto \mathbb{C}$ is defined by

$$g_{\sigma,P}(t_P) = \sum_{\wp\in P} g_{\sigma,\wp}(t_\wp), \qquad g_{\sigma,\wp}(t_\wp) = \frac{t_\wp \log N(\wp)}{t_\wp - N(\wp)^\sigma}, \tag{2.1.2}$$

where $t_P = (t_\wp)_{\wp\in P}$.

Theorem 1 *Let $\sigma > 0$. There exists a unique function $M_{\sigma,P}(z)$ of $z \in \mathbb{C}$, which is a hyperfunction (Schwartz distribution) when $|P| = 1$, that satisfies*

$$\int_{\mathbb{C}} M_{\sigma,P}(w)\Phi(w)|dw| = \int_{T_P} \Phi(g_{\sigma,P}(t_P))d^*t_P \tag{2.1.3}$$

for any continuous function $\Phi(w)$ on $\mathbb{C}$, where $|dw| = (2\pi)^{-1}dxdy$ ($w = x+yi$), and d^*t_P is the normalized Haar measure on T_P. It is compactly supported, and satisfies

$$M_{\sigma,P}(z) \ge 0, \qquad \int_{\mathbb{C}} M_{\sigma,P}(w)|dw| = 1. \tag{2.1.4}$$

Before the proof, we note that each linear fractional function $g_{\sigma,\wp}$ maps the unit circle $\mathbb{C}^1$ to another circle, with center $c_{\sigma,\wp}$ and radius $r_{\sigma,\wp}$ given respectively by

$$c_{\sigma,\wp} = \frac{-\log N(\wp)}{N(\wp)^{2\sigma} - 1}, \quad r_{\sigma,\wp} = \frac{N(\wp)^\sigma \log N(\wp)}{N(\wp)^{2\sigma} - 1}. \tag{2.1.5}$$

If we write $g_{\sigma,\wp}(t_\wp) = c_{\sigma,\wp} + r_{\sigma,\wp}.t'_\wp$, then

$$t'_\wp = \frac{N(\wp)^\sigma t_\wp - 1}{t_\wp - N(\wp)^\sigma}, \quad t_\wp = \frac{N(\wp)^\sigma t'_\wp - 1}{t'_\wp - N(\wp)^\sigma} \tag{2.1.6}$$

(involutive), and $t_\wp \in \mathbb{C}^1$ if and only if $t'_\wp \in \mathbb{C}^1$. The image of the normalized Haar measure $d^*t_\wp = (2\pi i t_\wp)^{-1}dt_\wp$ of $\mathbb{C}^1$ on the $t'_\wp$-unit circle is given by

$$d^*t_\wp = \frac{N(\wp)^{2\sigma} - 1}{|N(\wp)^\sigma - t'_\wp|^2}.d^*t'_\wp, \tag{2.1.7}$$

where $d^*t'_\wp = (2\pi i t'_\wp)^{-1} dt'_\wp$.

Proof of Theorem 1 The uniqueness is obvious. The solution is given explicitly as

$$M_{\sigma,\wp}(c_{\sigma,\wp} + r.e^{i\theta}) = \frac{N(\wp)^{2\sigma} - 1}{|N(\wp)^\sigma - e^{i\theta}|^2} . \frac{\delta(r - r_{\sigma,\wp})}{r} \tag{2.1.8}$$

($r \geq 0$, $\theta \in \mathbb{R}$, $\delta(r)$: the usual 1-dimensional Dirac delta function), and

$$M_{\sigma,P}(z) = *_{\wp \in P} M_{\sigma,\wp}(z), \tag{2.1.9}$$

where $*$ denotes the convolution product with respect to $|dz|$. □

Note that

$$M_{\sigma,P}(\bar{z}) = M_{\sigma,P}(z) = \overline{M_{\sigma,P}(z)}. \tag{2.1.10}$$

It is clear from (2.1.3) that

$$\int_U M_{\sigma,P}(w)|dw| = \mathrm{Vol}(g_{\sigma,P}^{-1}(U)) \tag{2.1.11}$$

for any open set U on $\mathbb{C}$, where Vol denotes the volume with respect to d^*t_P. Therefore, the support of $M_{\sigma,P}(z)$ is exactly the image of $g_{\sigma,P}$:

$$\mathrm{Supp}(M_{\sigma,P}(z)) = \{\sum_{\wp \in P} (c_{\sigma,\wp} + r_{\sigma,\wp} e^{i\theta_\wp}), \quad 0 \leq \theta_\wp < 2\pi\}; \tag{2.1.12}$$

hence it is contained in the disk with center $c_{\sigma,P}$ and radius $r_{\sigma,P}$ given by

$$c_{\sigma,P} = \sum_{\wp \in P} c_{\sigma,\wp}, \quad r_{\sigma,P} = \sum_{\wp \in P} r_{\sigma,\wp}. \tag{2.1.13}$$

When $|P| = 1$, this support is a circle, and when $|P| > 1$, this can either be an annulus or a disk, depending on P and σ.

2.2

For any P and $\wp \notin P$, one can express the convolution product $M_{\sigma,P\cup\wp} = M_{\sigma,P} * M_{\sigma,\wp}$ explicitly as

$$M_{\sigma,P\cup\wp}(z) = \frac{N(\wp)^{2\sigma} - 1}{2\pi} \int_0^{2\pi} \frac{M_{\sigma,P}(z - c_{\sigma,\wp} - r_{\sigma,\wp} e^{i\theta})}{|N(\wp)^\sigma - e^{i\theta}|^2} d\theta. \tag{2.2.1}$$

So, $M_{\sigma,P\cup\wp}(z)$ is obtained by averaging $M_{\sigma,P}(z)$ over the circle with center $z-c_{\sigma,\wp}$ and radius $r_{\sigma,\wp}$, with respect to the image of $d^*t_\wp$ on this circle.

When $P=\{\wp,\wp'\}$ with $r_{\sigma,\wp}\geq r_{\sigma,\wp'}$, we see easily that $M_{\sigma,P}(z)$ is a (non-negative real valued) function whose support is

$$r_{\sigma,\wp}-r_{\sigma,\wp'}\leq |z-c_P|\leq r_{\sigma,\wp}+r_{\sigma,\wp'}. \tag{2.2.2}$$

But $M_{\sigma,\wp\cup\wp'}(z)$ is unbounded near the border of support. When $|P|=3$, $M_{\sigma,P}(z)$ is bounded, but still discontinuous at the border. We shall see that $M_{\sigma,P}(z)$ gets smoother and smoother as $|P|$ increases.

In fact, as a reflection of the rapid decaying property of its Fourier dual (Cor 3.3.3), we obtain

Proposition 2.2.3 *$M_{\sigma,P}(z)$ belongs to class C_k if $|P|>2(k+2)$.*

Remark 2.2.4 The actual bound for $|P|$ seems to be a little better. For example, although the author has not checked it in full detail, it seems that $M_{\sigma,P}(z)$ is continuous already for $|P|=4$.

2.3

Now let (a,b) be any pair of non-negative integers, and consider the derivation

$$D^{(a,b)}=\frac{\partial^{a+b}}{\partial z^a\partial\bar{z}^b}. \tag{2.3.1}$$

If $|P|>2(a+b+2)$, then $M_{\sigma,P}(z)$ belongs to C_{a+b}; hence $D^{(a+b)}$ acts on (2.2.1) and commutes with the integration with respect to the parameter θ. Thus,

$$D^{(a,b)}M_{\sigma,P\cup\wp}(z)=\frac{N(\wp)^{2\sigma}-1}{2\pi}\int_0^{2\pi}\frac{(D^{(a,b)}M_{\sigma,P})(z-c_{\sigma,\wp}-r_{\sigma,\wp}e^{i\theta})}{|N(\wp)^\sigma-e^{i\theta}|^2}d\theta \tag{2.3.2}$$

holds whenever $|P|>2(a+b+2)$. In particular,

$$\mathrm{Max}_z|D^{(a,b)}M_{\sigma,P\cup\wp}(z)|\leq \mathrm{Max}_z|D^{(a,b)}M_{\sigma,P}(z)|. \tag{2.3.3}$$

Therefore, for each (a,b), there exists a positive constant $\mathbf{m}_\sigma^{(a,b)}$ such that

$$|D^{(a,b)}M_{\sigma,P}(z)|\leq \mathbf{m}_\sigma^{(a,b)} \tag{2.3.4}$$

holds for any $P=P_y$ with $|P|>2(a+b+2)$. (We restrict ourselves here to those P of the form P_y only to ensure that for any two $P,\ P'$ in consideration, there is an inclusion relation between them in one way or the other.)

2.4

We shall need the following

Lemma 2.4.1 *Fix $\sigma > 0$ and $a, b \geq 0$. Then for any $P = P_y$ with $|P| > 2(a+b+4)$ and $\wp \notin P$,*

$$|D^{(a,b)} M_{\sigma,P\cup\wp}(z) - D^{(a,b)} M_{\sigma,P}(z)| \ll \left(\frac{\log N(\wp)}{N(\wp)^\sigma}\right)^2, \tag{2.4.2}$$

where $\ll$ is independent of $P, \wp, z$.

The proof is based on (2.3.2) for $(a,b), (a+1,b), (a,b+1), (a+1,b+1)$, and on the following well-known formula in harmonic analysis.

Sublemma 2.4.3 *Let $\triangle = 4D^{(1,1)} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ be the Laplacian on $\mathbb{C} = \mathbb{R}^2$ and take any $R > 0$. Then for any complex valued function $u(z)$ belonging to class C_2 on a domain $\subset \mathbb{C}$ containing the disk $|z - z_0| \leq R$,*

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})d\theta - u(z_0) = \frac{1}{2\pi}\int_{|z-z_0|\leq R} \log\left(\frac{R}{|z-z_0|}\right)(\triangle u)(z)dxdy. \tag{2.4.4}$$

Corollary 2.4.5 *If $|\triangle u(z)| \leq U$ on $|z - z_0| \leq R$, then*

$$|\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})d\theta - u(z_0)| \leq \frac{1}{4}UR^2. \tag{2.4.6}$$

Proof of Lemma 2.4.1 Let us suppress σ from the notation, and write

$$q = N(\wp)^\sigma,\ c = c_{\sigma,\wp},\ r = r_{\sigma,\wp},\ z' = z - c. \tag{2.4.7}$$

Decompose

$$D^{(a,b)} M_{\sigma,P\cup\wp}(z) - D^{(a,b)} M_{\sigma,P}(z) = A + B + C, \tag{2.4.8}$$
$$\begin{cases} A = \frac{q^2-1}{2\pi}\int_0^{2\pi}\left(\frac{1}{|q-e^{i\theta}|^2} - \frac{1}{q^2-1}\right) D^{(a,b)} M_P(z' - re^{i\theta})d\theta, \\ B = \frac{1}{2\pi}\int_0^{2\pi} D^{(a,b)} M_P(z' - re^{i\theta})d\theta - D^{(a,b)} M_P(z'), \\ C = D^{(a,b)} M_P(z') - D^{(a,b)} M_P(z). \end{cases}$$

We shall estimate each of A, B, C.

First, it is clear that

$$C \ll |c|(\mathbf{m}^{(a+1,b)} + \mathbf{m}^{(a,b+1)}) \ll |c| \ll \frac{\log N(\wp)}{N(\wp)^{2\sigma}}. \tag{2.4.9}$$

Secondly, it follows directly from Cor 2.4.5 that

$$B \ll r^2 \mathbf{m}^{(a+1,b+1)} \ll r^2 \ll \frac{(\log N(\wp))^2}{N(\wp)^{2\sigma}}. \tag{2.4.10}$$

As for A, decompose it as

$$A = \frac{1}{\pi}\int_0^{2\pi} \frac{q\cos\theta}{|q-e^{i\theta}|^2} D^{(a,b)} M_P(z' - re^{i\theta}) d\theta - \frac{1}{\pi}\int_0^{2\pi} \frac{D^{(a,b)} M_P(z' - re^{i\theta})}{|q-e^{i\theta}|^2} d\theta. \tag{2.4.11}$$

Observe now that the absolute value of the second term on the right hand side is bounded by $(q-1)^{-2}\mathbf{m}^{(a,b)} \ll N(\wp)^{-2\sigma}$. As for the first term, this decomposes as

$$\frac{q}{\pi}\int_0^{2\pi} \frac{\cos\theta}{|q-e^{i\theta}|^2} (D^{(a,b)} M_P(z' - re^{i\theta}) - D^{(a,b)} M_P(z')) d\theta + 2(q^2-1)^{-1} D^{(a,b)} M_P(z'), \tag{2.4.12}$$

because

$$\frac{q}{\pi}\int_0^{2\pi} \frac{\cos\theta d\theta}{|q-e^{i\theta}|^2} = 2(q^2-1)^{-1} \qquad (q>1). \tag{2.4.13}$$

Since the absolute value of the first (resp. the second) term of (2.4.12) is $\ll q^{-1} r(\mathbf{m}^{(a+1,b)} + \mathbf{m}^{(a,b+1)})$ (resp. $q^{-2}\mathbf{m}^{(a,b)}$), we conclude that

$$A \ll q^{-2} + q^{-1} r \ll \frac{\log N(\wp)}{N(\wp)^{2\sigma}}. \tag{2.4.14}$$

Therefore,

$$A + B + C \ll \frac{(\log N(\wp))^2}{N(\wp)^{2\sigma}}. \tag{2.4.15}$$

□

2.5

Since the sum of the right-hand side of (2.4.2) over all $\wp$ converges when $\sigma > 1/2$, we immediately obtain the first two items (i)(ii) of the following theorem.

Theorem 2 *Let $\sigma > 1/2$, $P = P_y$ and let $y \mapsto \infty$. Then*
(i) $M_{\sigma,P}(z)$ converges uniformly to a non-negative real valued C_∞-function $M_\sigma(z)$.
(ii) Each $D^{(a,b)}M_{\sigma,P}(z)$ converges uniformly to $D^{(a,b)}M_\sigma(z)$ (starting with $|P|$ sufficiently large).
(iii) For any $n \geq 1$, $|z|^n M_\sigma(z)$ belongs to L^2.
(iv) The function $M_\sigma(z)$ is not identically zero; in fact,

$$\int_{\mathbb{C}} M_\sigma(z)|dz| = 1. \tag{2.5.1}$$

It satisfies

$$M_\sigma(\bar{z}) = M_\sigma(z) = \overline{M_\sigma(z)}. \tag{2.5.2}$$

Remark 2.5.3 (i) $M_\sigma(z)$ is continuous also in (σ, z) (see Cor 3.11.11).
(ii) When $\sigma > 1, \sum_\wp r_{\sigma,\wp} < \infty$; hence $M_\sigma(z)$ is compactly supported.

For the proofs of (iii) and (iv), we need some results on the limit of the Fourier transform $\tilde{M}_{\sigma,P}(z)$ of $M_{\sigma,P}(z)$. This will be given in the next §3 ((3.11.9), (3.11.10)).

3 Constructions of $\tilde{M}_{\sigma,P}(z)$ and $\tilde{M}_\sigma(z)$

3.1

For each non-archimedean prime $\wp$ of K and $\sigma > 0$, $\tilde{M}_{\sigma,\wp}(z)$ is, by definition, the Fourier transform of $M_{\sigma,\wp}(z)$;

$$\tilde{M}_{\sigma,\wp}(z) = \int_{\mathbb{C}} M_{\sigma,\wp}(w)\psi_z(w)|dw|, \tag{3.1.1}$$

where $\psi_z(w) = \exp(i.\mathrm{Re}(\bar{z}w))$ and $|dw|$ is the self-dual measure w.r.t. ψ_z, i.e., $|dw| = (2\pi)^{-1}dxdy$ for $w = x + yi$. Thus, either from (2.1.3) or (2.1.8), it follows directly that

$$\begin{aligned}\tilde{M}_{\sigma,\wp}(z) &= \int_{\mathbb{C}^1} \psi_z(g_{\sigma,\wp}(t_\wp))d^*t_\wp \\ &= \exp(i.c_{\sigma,\wp}.\mathrm{Re}(z)).H_{\sigma,\wp}(z),\end{aligned} \tag{3.1.2}$$

where

$$H_{\sigma,\wp}(z) = \frac{N(\wp)^{2\sigma} - 1}{2\pi}.\int_0^{2\pi} \frac{\exp(ir_{\sigma,\wp}|z|\cos(\theta - \vartheta))}{|N(\wp)^\sigma - \exp(i\theta)|^2}d\theta, \tag{3.1.3}$$

with $\vartheta = \mathrm{Arg}(z)$. Let

$$J_n(x) = \frac{i^{-n}}{2\pi}\int_0^{2\pi} \exp(ix\cos(\theta))\cos(n\theta)d\theta \tag{3.1.4}$$

be the Bessel function of order n. Then

$$H_{\sigma,\wp}(z) = \sum_{n=0}^{\infty} \epsilon_n(\frac{i}{N(\wp)^\sigma})^n \cos(n\vartheta)J_n(r_{\sigma,\wp}|z|), \tag{3.1.5}$$

where ϵ_n is the Neumann factor $\epsilon_n = 1(n = 0),\ = 2(n \geq 1)$. Indeed,

$$(N(\wp)^{2\sigma} - 1)|N(\wp)^\sigma - \exp(i\theta)|^{-2} = \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta)N(\wp)^{-n\sigma}, \tag{3.1.6}$$

and ($\sin(n\theta)$ being an odd function)

$$\int_0^{2\pi} \exp(ix\cos(\theta - \vartheta))\cos(n\theta)d\theta = \cos(n\vartheta)\int_0^{2\pi} \exp(ix\cos(\theta))\cos(n\theta)d\theta, \tag{3.1.7}$$

from which (3.1.5) follows directly.

Since

$$J_n(x) = (\frac{x}{2})^n j_n\left((\frac{x}{2})^2\right), \qquad j_n(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!(n+k)!}, \tag{3.1.8}$$

with an entire function $j_n(z)$ on $\mathbb{C}$, (3.1.5) may be rewritten as an everywhere convergent power series in $z, \bar{z}$;

$$H_{\sigma,\wp}(z) = j_0\left((\frac{r_{\sigma,\wp}}{2})^2 z\bar{z}\right) + \sum_{n=1}^{\infty} \left(\frac{ir_{\sigma,\wp}}{2N(\wp)^\sigma}\right)^n (z^n + \bar{z}^n) j_n\left((\frac{r_{\sigma,\wp}}{2})^2 z\bar{z}\right). \tag{3.1.9}$$

Clearly, $H_{\sigma,\wp}(z)$, and hence also $\tilde{M}_{\sigma,\wp}(z)$, are real-analytic functions of z. And by their definitions,

$$|\tilde{M}_{\sigma,\wp}(z)| = |H_{\sigma,\wp}(z)| \leq 1. \tag{3.1.10}$$

We also note that $H_{\sigma,\wp}(z)$ is an eigenfunction of the Laplacian $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$;

$$\triangle H_{\sigma,\wp}(z) = -r_{\sigma,\wp}^2 H_{\sigma,\wp}(z). \tag{3.1.11}$$

This is because $\frac{\partial^2}{\partial z \partial \bar{z}}\psi_z(w) = (\frac{iw}{2})(\frac{i\bar{w}}{2})\psi_z(w)$, and $|g_{\sigma,\wp}(t_\wp) - c_{\sigma,\wp}| = r_{\sigma,\wp}$.

3.2

For any finite set P of non-archimedean primes of K, define

$$\tilde{M}_{\sigma,P}(z) = \prod_{\wp \in P} \tilde{M}_{\sigma,\wp}(z), \qquad H_{\sigma,P}(z) = \prod_{\wp \in P} H_{\sigma,\wp}(z), \tag{3.2.1}$$

so that $\tilde{M}_{\sigma,P}(z) = e^{ic_{\sigma,P}\mathrm{Re}(z)} H_{\sigma,P}(z)$. Note that

$$\tilde{M}_{\sigma,P}(z) = \int_{\mathbb{C}} M_{\sigma,P}(w)\psi_z(w)|dw| = \int_{T_P} \psi_z(g_{\sigma,P}(t_P))d^*t_P. \tag{3.2.2}$$

The Fourier inversion formula gives

$$M_{\sigma,P}(z) = \int_{\mathbb{C}} \tilde{M}_{\sigma,P}(w)\psi_{-z}(w)|dw|. \tag{3.2.3}$$

These functions $H_{\sigma,P}(z)$, $\tilde{M}_{\sigma,P}(z)$ are also obviously real analytic, and satisfy $|H_{\sigma,P}(z)| = |\tilde{M}_{\sigma,P}(z)| \leq 1$ and

$$\begin{gathered} \tilde{M}_{\sigma,P}(0) = H_{\sigma,P}(0) = 1 \qquad (all\ P), \\ |\tilde{M}_{\sigma,P'}(z)| \leq |\tilde{M}_{\sigma,P}(z)| \leq 1 \qquad (P \subseteq P'). \end{gathered} \tag{3.2.4}$$

Also, note that

$$\tilde{M}_{\sigma,P}(\bar{z}) = \tilde{M}_{\sigma,P}(z), \quad \tilde{M}_{\sigma,P}(-z) = \overline{\tilde{M}_{\sigma,P}(z)}. \tag{3.2.5}$$

3.3

We shall show now that

Proposition 3.3.1 *Let $\sigma > 0$ and P be fixed. Then*

$$|\tilde{M}_{\sigma,P}(z)| = O\left((1+|z|)^{-\frac{|P|}{2}}\right). \tag{3.3.2}$$

In particular, $|z|^k \tilde{M}_{\sigma,P}(z)$ belongs to L^1 if $|P| > 2(k+2)$.

Thus the Fourier dual satisfies:

Corollary 3.3.3 *$M_{\sigma,P}(z)$ belongs to class C_k when $|P| > 2(k+2)$.*

To prove Prop 3.3.1, we need the following

Lemma 3.3.4 *There exists an absolute positive constant A such that*

$$x^{\frac{1}{2}}|J_n(x)| < A(n+1)^{\frac{1}{2}} \tag{3.3.5}$$

holds for any non-negative integer n and $x \geq 0$.

It is well-known that $x^{1/2}|J_n(x)|$ is bounded for each n, and also that this bound must depend on n. (In fact, by Cauchy, $n^{1/2}|J_n(n)| \sim n^{1/6}$.) Since the author could not find a suitable reference for a simple explicit bound like (3.3.5), we shall give this a full proof.

We first need:

Sublemma 3.3.6 *$x^{\frac{1}{4}}|J_n(x)|$ for $x \geq 0, n = 0, 1, 2, \ldots$ has a universal upper bound.*

Proof The Schläfli-Neumann formula for $J_n(x)^2$ ([Wa §2.6]) gives

$$J_n(x)^2 = \frac{1}{\pi}\int_0^{\pi} J_0(2x\sin\theta)\cos(2n\theta)d\theta. \tag{3.3.7}$$

But since $x^{1/2}|J_0(x)| \ll 1$,

$$J_n(x)^2 \ll \int_0^{\pi} \frac{d\theta}{\sqrt{x\sin\theta}} \ll \frac{1}{\sqrt{x}}. \tag{3.3.8}$$

□

Proof of lemma 3.3.4 As for the constant A, it suffices that (3.3.5) holds for $n = 0, 1$ and that $2^{-1/4}A$ exceeds the universal upper bound for $x^{1/4}|J_n(x)|$. We shall fix such A and $x \geq 0$, and prove (3.3.5) by induction on $n \geq 2$.

[Case $n^2 \leq x/2$] By the recurrence formula

$$J_n(x) = \frac{2(n-1)}{x}J_{n-1}(x) - J_{n-2}(x) \tag{3.3.9}$$

and the assumptions, we obtain

$$x^{\frac{1}{2}}|J_n(x)| \leq \left(\frac{n-1}{n^2}n^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}\right) A \tag{3.3.10}$$
$$< n^{\frac{1}{2}}\left(n^{-1} + (1-n^{-1})^{\frac{1}{2}}\right) A < (n+1)^{\frac{1}{2}} A,$$

as desired. The last inequality follows from $(1+x)^{1/2} - (1-x)^{1/2} > x$ for $0 < x < 1$, in particular for $x = n^{-1}$ $(n \geq 2)$.

[Case $n^2 > x/2$] In this case, by the sublemma and the assumptions, we obtain

$$x^{\frac{1}{2}}|J_n(x)| \leq x^{\frac{1}{4}} 2^{-\frac{1}{4}} A < (2n^2)^{\frac{1}{4}} 2^{-\frac{1}{4}} A < A(n+1)^{\frac{1}{2}}, \tag{3.3.11}$$

as desired. This proves lemma 3.3.4.

$\square$

Proof of Prop 3.3.1 We shall only use a weak version $x^{1/2}|J_n(x)| \ll n+1$ of lemma 3.3.4. By this and (3.1.5), we obtain

$$|H_{\sigma,\wp}(z)| \ll (r_{\sigma,\wp}|z|)^{-1/2} \sum_{n=0}^{\infty} \epsilon_n (n+1) N(\wp)^{-\sigma n} \tag{3.3.12}$$
$$= (r_{\sigma,\wp}|z|)^{-1/2} \left(2(1-N(\wp)^{-\sigma})^{-2} - 1\right).$$

But since $N(\wp)^{\sigma} r_{\sigma,\wp} \geq \log N(\wp) \geq \log 2$, and $(1-N(\wp)^{-\sigma})^{-2} < (1-2^{-1/2})^{-2}$, this gives

$$|H_{\sigma,\wp}(z)| \ll N(\wp)^{\sigma/2} |z|^{-1/2}, \tag{3.3.13}$$

where $\ll$ is absolute. Since $H_{\sigma,P}(z) = \prod_{\wp \in P} H_{\sigma,\wp}(z)$, the proof is completed. $\square$

Remark 3.3.14 The exponent $|P|/2$ in Prop 3.3.1 is the best possible, because (3.1.5) gives, for each $R > 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} H_{\sigma,\wp}(Re^{i\vartheta}) d\vartheta = J_0(r_{\sigma,\wp} R), \tag{3.3.15}$$

and $J_0(x) \sim (2/\pi x)^{1/2} \cos(x - \pi/4)$.

3.4

By Prop 3.3.1, $\tilde{M}_{\sigma,P}(z) \in L^{\infty}$ (continuous, and for any $\epsilon > 0$ there exists $R > 0$ such that $|\tilde{M}_{\sigma,P}(z)| < \epsilon$ for $|z| > R$), and if $|P| > 4$, $\tilde{M}_{\sigma,P}(z) \in L^1 \cap L^{\infty}$; hence $\in L^t$ $(1 \leq t \leq \infty)$. The main goal of §3 is to prove the following

Theorem 3 *Let $\sigma > 1/2$. Then*
(i) *When $P = P_y$ and $y \mapsto \infty$, $\tilde{M}_{\sigma,P}(z)$ converges uniformly on $\sigma \geq 1/2 + \epsilon$ and $z \in \mathbb{C}$, to a continuous function $\tilde{M}_\sigma(z)$ of σ and z.*
(ii) *For each $\sigma > 1/2$, the function $\tilde{M}_\sigma(z)$ of z belongs to L^t for any $1 \leq t \leq \infty$, and the convergence $\tilde{M}_{\sigma,P}(z) \mapsto \tilde{M}_\sigma(z)$ is also L^t-convergence.*
(iii) *$\tilde{M}_\sigma(z)$ is real analytic in σ and z.*
(iv) *$\tilde{M}_\sigma(z) = O((1+|z|)^{-n})$ for any $n \geq 1$.*
(v) *$M_\sigma(z)$ and $\tilde{M}_\sigma(z)$ are Fourier transforms of each other;*

$$\tilde{M}_\sigma(z) = \int_{\mathbb{C}} M_\sigma(w)\psi_z(w)|dw|, \quad M_\sigma(z) = \int_{\mathbb{C}} \tilde{M}_\sigma(w)\psi_{-z}(w)|dw|. \tag{3.4.1}$$

(vi) *$\tilde{M}_\sigma(z)$ has a power series expansion*

$$\tilde{M}_\sigma(z) = \sum_{a,b=0}^{\infty} (-i/2)^{a+b} \mu_\sigma^{(a,b)} \frac{z^a \bar{z}^b}{a!b!} \qquad (z \in \mathbb{C}), \tag{3.4.2}$$

with the Dirichlet series coefficients

$$\mu_\sigma^{(a,b)} = \sum_{D\, integral} \frac{\Lambda_a(D)\Lambda_b(D)}{N(D)^{2\sigma}} \qquad (\sigma > 1/2). \tag{3.4.3}$$

Here, D runs over all integral ideals (effective divisors) of K, and $\Lambda_k(D)$ is as defined later in §3.7. The expansion (3.4.2) can also be regarded as a Dirichlet series expansion

$$\tilde{M}_\sigma(z) = \sum_{D\, integral} \frac{\lambda_D(z)\lambda_D(\bar{z})}{N(D)^{2\sigma}} \qquad (\sigma > 1/2), \tag{3.4.4}$$

with the polynomial coefficients $\lambda_D(z)\lambda_D(\bar{z})$, where

$$\lambda_D(z) = \sum_{k=0}^{\infty} (-i/2)^k \frac{\Lambda_k(D)}{k!} z^k \tag{3.4.5}$$

(which is actually a polynomial in z).

Remark 3.4.6 Clearly, $|\tilde{M}_\sigma(z)| \leq 1$, and

$$\tilde{M}_\sigma(0) = 1. \tag{3.4.7}$$

In particular, $\tilde{M}_\sigma(z)$ does not vanish identically. Finally, note also that

$$\tilde{M}_\sigma(\bar{z}) = \tilde{M}_\sigma(z) = \overline{\tilde{M}_\sigma(-z)}. \tag{3.4.8}$$

The proof of Theorem 3 requires, among other things, a complex analytic treatment (in 3 complex variables s, z_1, z_2). We shall go on to this, and leave the final stage of the proof of Theorem 3 until the end of §3.

3.5

First, for any $s, u_1, u_2 \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, and a real parameter $q > 1$, define the complex analytic function

$$h_q(s; u_1, u_2) = \sum_{a,b=0}^{\infty} i^{a+b} q^{-s|b-a|} \frac{u_1^a u_2^b}{a!b!}, \tag{3.5.1}$$

of 3 variables s, u_1, u_2, where $i = \sqrt{-1}$. (Note the absolute value $|b-a|$ instead of $b-a$ itself, which makes this function not as simple as a product of two exponential series.) Obviously, this series converges absolutely. Rearrange this with respect to $n = |b-a|$ to get

$$h_q(s; u_1, u_2) = \frac{1}{2} \sum_{n=0}^{\infty} \epsilon_n \left(\frac{i}{q^s} \right)^n (u_1^n + u_2^n) j_n(u_1 u_2), \tag{3.5.2}$$

$\epsilon_n, j_n(x)$ being as in §3.1. It has the following integral expression

$$h_q(s; u_1, u_2) = \int_{\mathbb{C}^1} \exp\left(i(\frac{q^s - t}{1 - q^s t} u_1 + \frac{1 - q^s t}{q^s - t} u_2) \right) d^*t, \tag{3.5.3}$$

where $d^*t = dt/(2\pi i t)$. (Note that the integrand is invariant under $(t, u_1, u_2) \mapsto (t^{-1}, u_2, u_1)$.) Indeed, the right hand side is holomorphic in u_1, u_2, and the Taylor coefficient of each $u_1^a u_2^b$, computed by operating $\partial^{a+b}/\partial u_1^a \partial u_2^b$ under the integral sign is given by

$$\frac{i^{a+b}}{a!b!} \int_{\mathbb{C}^1} \left(\frac{1 - q^s t}{q^s - t} \right)^{b-a} d^*t = \frac{i^{a+b}}{a!b!} \int_{\mathbb{C}^1} \left(\frac{1 - q^s t}{q^s - t} \right)^{a-b} d^*t \qquad (\text{by } t \mapsto t^{-1}). \tag{3.5.4}$$

Depending on whether $b \geq a$ or $a \geq b$, use the left (resp. right) expression and compute the residue at $t = 0$. This shows that the value of (3.5.4) is $i^{a+b} q^{-|b-a|s}/a!b!$, as desired.

Now let K and P be as before. Set

$$\begin{aligned} H_{s,\wp}(z_1, z_2) &= h_{N(\wp)}(s; \frac{r_{s,\wp}}{2} z_1, \frac{r_{s,\wp}}{2} z_2), \\ H_{s,P}(z_1, z_2) &= \prod_{\wp \in P} H_{s,\wp}(z_1, z_2), \end{aligned} \tag{3.5.5}$$

where

$$r_{s,\wp} = \frac{N(\wp)^s \log N(\wp)}{N(\wp)^{2s} - 1}. \tag{3.5.6}$$

Note that these are complex analytic functions of s, z_1, z_2 on $\mathrm{Re}(s) > 0$, and

$$H_{\sigma,P}(z) = H_{\sigma,P}(z, \bar{z}) \qquad (\sigma > 0). \tag{3.5.7}$$

3.6

Theorem 4 *Fix any $\epsilon > 0$ and $R > 0$. Then the sum*

$$\sum_{\wp} |H_{s,\wp}(z_1, z_2) - 1|, \tag{3.6.1}$$

where $\wp$ runs over all non-archimedean primes of K, converges uniformly on the region $\mathrm{Re}(s) \geq \frac{1}{2} + \epsilon$, $|z_1|, |z_2| \leq R$. *In particular, there exists $y = y_{\epsilon,R}$ such that the sum*

$$\sum_{N(\wp)>y} \log H_{s,\wp}(z_1, z_2) \tag{3.6.2}$$

converges absolutely and uniformly on this region, and hence the product

$$\prod_{N(\wp)>y} H_{s,\wp}(z_1, z_2) \tag{3.6.3}$$

converges absolutely and uniformly to a nowhere vanishing analytic function on this region.

Proof The key point is to reduce to the fact that the series

$$\sum_{\wp} (\log N(\wp))^2 N(\wp)^{-2\sigma} \tag{3.6.4}$$

converges uniformly on $\sigma \geq 1/2 + \epsilon$. To avoid inessential complication of the notation (to worry about ϵ), we shall fix $\sigma > 1/2$. The uniformity statement for $\sigma \geq 1/2 + \epsilon$ should be clear from the argument.

We first claim that if $|z_1|, |z_2| \leq R$, $\sigma > 1/2$ and if $N(\wp)$ is so large as to satisfy

$$Rr_{\sigma,\wp} \leq 2, \tag{3.6.5}$$

then

$$|H_{s,\wp}(z_1, z_2) - 1| < \frac{5}{2}(Rr_{\sigma,\wp})^2 + 2Rr_{\sigma,\wp}N(\wp)^{-\sigma}. \tag{3.6.6}$$

In fact, by (3.5.1)(3.5.3), (writing $r = r_{\sigma,\wp}$ and $q = N(\wp)^\sigma$ here),

$$\begin{aligned} |H_{s,\wp}(z_1, z_2) - 1| &\leq \sum_{(a,b)\neq(0,0)} q^{-|b-a|} \frac{1}{a!b!}(Rr/2)^{a+b} \\ &= \sum_{k=1}^{\infty} \frac{1}{(k!)^2}(Rr/2)^{2k} + 2\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} q^{-n}\frac{1}{k!(k+n)!}(Rr/2)^{2k+n} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k!}(Rr/2)^{2k} + 2\left(\sum_{n=1}^{\infty}\frac{1}{n!}(Rr/2q)^n\right)\left(\sum_{k=0}^{\infty}\frac{1}{k!}(Rr/2)^{2k}\right) \\ &= \big(\exp((Rr/2)^2) - 1\big) + 2\exp((Rr/2)^2)\left(\exp(Rr/2q) - 1\right). \end{aligned} \tag{3.6.7}$$

But since $e^{x/2}-1<x$ for $0\leq x\leq 2$, and $Rr\leq 2$, we obtain

$$|H_{s,\wp}(z_1,z_2)-1|\leq \frac{1}{2}(Rr)^2+2(1+\frac{1}{2}(Rr)^2)(Rr/q)<\frac{5}{2}(Rr)^2+2Rr/q, \tag{3.6.8}$$

as desired. Since

$$r_{\sigma,\wp}^2\ll\frac{(\log N(\wp))^2}{N(\wp)^{2\sigma}},\qquad r_{\sigma,\wp}N(\wp)^{-\sigma}\ll\frac{\log N(\wp)}{N(\wp)^{2\sigma}}, \tag{3.6.9}$$

the series (3.6.1) converges uniformly on this region.

Now let $N(\wp)$ be even so large that $Rr_{\sigma,\wp}<1/5$. Then (3.6.6) gives

$$|H_{s,\wp}(z_1,z_2)-1|<\frac{1}{2}. \tag{3.6.10}$$

For such s,z_1,z_2, and over such $\wp$ that satisfy $Rr_{\sigma,\wp}<1/5$, consider the infinite sum

$$\sum_{\wp\ as\ above}\log H_{s,\wp}(z_1,z_2), \tag{3.6.11}$$

where log takes the principal values. Then, since $|w|\leq 1/2$ implies $|\log(1+w)|\leq(3/2)|w|$, and hence

$$|\log H_{s,\wp}(z_1,z_2)|\leq\frac{3}{2}|H_{s,\wp}(z_1,z_2)-1|, \tag{3.6.12}$$

(3.6.11) converges uniformly and absolutely. □

3.7

For each $\wp$, we define the analytic function $\tilde{M}_{s,\wp}(z_1,z_2)$ of s,z_1,z_2 (Re$(s)>0$) by

$$\begin{aligned}\tilde{M}_{s,\wp}(z_1,z_2)&=\exp\left(\frac{i}{2}c_{s,\wp}(z_1+z_2)\right)H_{s,\wp}(z_1,z_2)\\&=\int_{\mathbb{C}^1}\exp\left(\frac{i}{2}(z_1g_{s,\wp}(\bar{t}_\wp)+z_2g_{s,\wp}(t_\wp)\right)d^*t_\wp,\end{aligned} \tag{3.7.1}$$

where

$$c_{s,\wp}=\frac{-\log N(\wp)}{N(\wp)^{2s}-1},\qquad g_{s,\wp}(t_\wp)=\frac{t_\wp\log N(\wp)}{t_\wp-N(\wp)^s}. \tag{3.7.2}$$

The second equality in (3.7.1) follows directly from (3.5.3).

For Re$(s)>1/2$, we also define the global functions

$$H_s(z_1,z_2)=\prod_\wp H_{s,\wp}(z_1,z_2), \tag{3.7.3}$$

(3.7.4)
$$\tilde{M}_s(z_1,z_2)=\prod_{\wp}\tilde{M}_{s,\wp}(z_1,z_2)=\exp\left(\frac{i}{2}\cdot\frac{\zeta_K'(2s)}{\zeta_K(2s)}(z_1+z_2)\right)H_s(z_1,z_2),$$

$\zeta_K(s)$ being the Dedekind zeta function of K. Note here that

(3.7.5)
$$\sum_{\wp}c_{s,\wp}=\frac{\zeta_K'(2s)}{\zeta_K(2s)}.$$

In particular,

(3.7.6)
$$\tilde{M}_\sigma(z)=\tilde{M}_\sigma(z,\bar{z})=\exp\left(i\frac{\zeta_K'(2\sigma)}{\zeta_K(2\sigma)}\mathrm{Re}(z)\right)H_\sigma(z),$$

where

(3.7.7)
$$H_\sigma(z)=H_\sigma(z,\bar{z})=\prod_{\wp}H_{\sigma,\wp}(z).$$

Theorem 5 *The analytic function $\tilde{M}_s(z_1,z_2)$ has the following power series and Dirichlet series expansions.* (*The notation for their coefficients will be defined in §3.8.*)

(3.7.8)
$$\tilde{M}_s(z_1,z_2)=\sum_{a,b=0}^{\infty}(-i/2)^{a+b}\mu_s^{(a,b)}\frac{z_1^a z_2^b}{a!b!},$$

(3.7.9)
$$\begin{aligned}\tilde{M}_s(z_1,z_2)&=\sum_{D\ integral}\frac{\lambda_D(z_1)\lambda_D(z_2)}{N(D)^{2s}}\\&=\prod_{\wp}\left(\sum_{n=0}^{\infty}\frac{\lambda_{\wp^n}(z_1)\lambda_{\wp^n}(z_2)}{N(\wp)^{2ns}}\right).\end{aligned}$$

In fact, each $\wp$-facor in (*3.7.9*) *is equal to $\tilde{M}_{s,\wp}(z_1,z_2)$. Here, D runs over all integral ideals, and $\wp$, all non-archimedean prime divisors of K. The series* (*3.7.8*) *converges for all $z_1,z_2\in\mathbb{C}$, and* (*3.7.9*) *for all s with* $\mathrm{Re}(s)>1/2$.

3.8

To define the coefficients in Theorem 5, first, for any integral ideal D of K, set

(3.8.1)
$$\begin{aligned}\Lambda(D)&=\log N(\wp) &&\cdots if\ D=\wp^r,\ r\geq 1,\\&=0 &&\cdots otherwise,\end{aligned}$$

for a prime divisor $\wp$. Then define $\Lambda_k(D)$ ($k \geq 0$, $k \in \mathbb{Z}$) by

$$\Lambda_0(D) = 1 \qquad \cdots if\ D = (1)$$
$$= 0 \qquad \cdots otherwise, \tag{3.8.2}$$

$$\Lambda_k(D) = \sum_{D = D_1 \cdots D_k} \Lambda(D_1) \cdots \Lambda(D_k) \qquad (k \geq 1). \tag{3.8.3}$$

Here, the summation is over all ordered k-ples of integral ideals $(D_1, \cdots D_k)$ whose product is equal to D. (One may assume that each D_i is a prime power, for $\Lambda(D_i) = 0$ otherwise.) Thus, if $D = \prod_\wp \wp^{n_\wp}$ is the prime factorization of D, then $\Lambda_k(D)$ is the coefficient of $\prod_\wp x_\wp^{n_\wp}$ in the polynomial

$$\left(\sum_\wp (\log N(\wp))(x_\wp + \cdots x_\wp^{n_\wp}) \right)^k, \tag{3.8.4}$$

where $x_\wp$ are independent variables. In particular, (put $x_\wp = 1$ for all $\wp$),

$$\Lambda_k(D) \leq (\log N(D))^k. \tag{3.8.5}$$

Also note that

$$\Lambda_k(D) = 0 \qquad if \quad k > \sum_\wp n_\wp. \tag{3.8.6}$$

For each D, by (3.8.6), the following $\lambda_D(z)$ is a polynomial of z.

$$\lambda_D(z) = \sum_{k=0}^{\infty} (-i/2)^k \frac{\Lambda_k(D)}{k!} z^k. \tag{3.8.7}$$

And for each pair (a, b) of non-negative integers and Re(s)> 1/2, define the Dirichlet series

$$\mu_s^{(a,b)} = \sum_D \frac{\Lambda_a(D)\Lambda_b(D)}{N(D)^{2s}}. \tag{3.8.8}$$

By (3.8.5), this Dirichlet series converges absolutely on Re(s)> 1/2.

Remark 3.8.9 Since
$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_D \frac{\Lambda(D)}{N(D)^s},$$

we have

$$(3.8.10) \qquad \left(-\frac{\zeta_K'(s)}{\zeta_K(s)}\right)^k = \sum_D \frac{\Lambda_k(D)}{N(D)^s} \qquad (k \geq 1).$$

Only when $K = \mathbb{Q}$ in which case $N(D)$ determines D uniquely, this can be used as an alternative definition of $\Lambda_k(D)$.

These arithmetic functions $\Lambda_k(D)$ and $\lambda_D(z)$ enjoy the following properties which are direct consequences of their definitions.

Proposition 3.8.11 (i) *When D, D' are integral ideals with $(D, D') = 1$,*

$$(3.8.12) \qquad \frac{\Lambda_k(DD')}{k!} = \sum_{\substack{a+b=k \\ a,b\geq 0}} \frac{\Lambda_a(D)\Lambda_b(D')}{a!b!},$$

$$(3.8.13) \qquad \lambda_{DD'}(z) = \lambda_D(z)\lambda_{D'}(z).$$

(ii) *When $\wp$ is a prime, we have $\Lambda_k(\wp^n) = 0$ $(n < k)$, and*

$$(3.8.14) \qquad \Lambda_k(\wp^n) = \binom{n-1}{k-1} (\log N(\wp))^k \qquad (n \geq k),$$

$$(3.8.15) \qquad \lambda_{\wp^n}(z) = G_n\left(-\frac{i}{2}(\log N(\wp))z\right),$$

where $G_n(w)$ is the polynomial of w defined by

$$(3.8.16) \qquad \exp\left(\frac{wt}{1-t}\right) = \sum_{n=0}^{\infty} G_n(w)t^n \qquad (|t| < 1);$$

namely, $G_0(w) = 1$ and

$$(3.8.17) \qquad G_n(w) = \sum_{k=1}^{n} \frac{1}{k!} \binom{n-1}{k-1} w^k \qquad (n \geq 1).$$

3.9

In this §3.9, we shall reduce the proof of Theorem 5 to some estimations of $|\lambda_D(z)|$. First, by (3.8.16), applied to $w \mapsto (-iz/2)\log N(\wp)$, $t \mapsto tN(\wp)^{-s}$, and by (3.8.15), we obtain

$$\exp\left(\frac{iz}{2}.\frac{t\log N(\wp)}{t-N(\wp)^s}\right) = \sum_{n=0}^{\infty} \lambda_{\wp^n}(z)N(\wp)^{-ns}.t^n \qquad (|t| < N(\wp)^\sigma). \tag{3.9.1}$$

By (3.7.1), $\tilde{M}_{s,\wp}(z_1,z_2)$ is equal to the constant term of the Fourier expansion of

$$\exp\{\frac{i}{2}(z_1 g_{s,\wp}(\bar{t}_\wp) + z_2 g_{s,\wp}(t_\wp))\} \tag{3.9.2}$$

in $t_\wp$ on $\mathbb{C}^1$. But by (3.9.1), this constant term is equal to

$$\sum_{n=0}^{\infty} \lambda_{\wp^n}(z_1)\lambda_{\wp^n}(z_2)N(\wp)^{-2ns}. \tag{3.9.3}$$

Therefore,

$$\tilde{M}_{s,\wp}(z_1,z_2) = \sum_{n=0}^{\infty} \lambda_{\wp^n}(z_1)\lambda_{\wp^n}(z_2)N(\wp)^{-2ns}. \tag{3.9.4}$$

Among the two statements in Theorem 5, we first pay attention to the second equality (3.7.9). Note that (3.9.4) and Prop 3.8.11 give the *formal* Euler product decomposition. But we must also show that the global Dirichlet series converges on Re(s)$>1/2$. We have already established the absolute convergence of the Euler product as analytic function on this domain, but the absolute convergence of the Dirichlet series on this domain is (at least a priori) a separate matter. We shall use the following estimations of $|\lambda_D(z)|$.

Proposition 3.9.5 *(i) For any $n \geq 1$,*

$$|\lambda_{\wp^n}(z)| < \exp\sqrt{2n|z|\log N(\wp)} \qquad (n \geq 1).$$

(ii) For any non-trivial integral divisor $D \neq (1)$,

$$|\lambda_D(z)| < \exp\{(\log N(D))\sqrt{2C_K|z|/(\log\log N(D)+2)}\},$$

where C_K is a positive constant depending only on K.

The proof of Prop 3.9.5 will be postponed until §3.10.

Remark 3.9.6 The inequality (3.8.5) leads only to $|\lambda_k(D)| \leq N(D)^{|z|/2}$, from which follows only that (3.7.9) converges on Re(s)$>(1+|z|)/2$.

Proof of Theorem 5 assuming Prop 3.9.5

First, by Prop 3.9.5 (ii), it is clear that for any given $\epsilon > 0$ and $R > 0$, $|\lambda_D(z)| \ll N(D)^\epsilon$ holds for all $|z| \leq R$ if $N(D)$ is sufficiently large. Therefore, (3.7.9) converges absolutely and uniformly in the wider sense on Re(s)> 1/2.

Secondly, to prove (3.7.8), fix s with Re(s)> 1/2. Since (3.7.9) converges uniformly on $|z_1|$, $|z_2| \leq 1$, we can compute the derivative $(\partial^{a+b}/\partial z_1^a \partial z_2^b)\tilde{M}_s(z_1, z_2)$ at $z_1 = z_2 = 0$ by termwise differentiation. And since

$$\frac{\partial^k}{\partial z^k}\lambda_D(z) \mid_{(0)} = (-i/2)^k \Lambda_k(D), \tag{3.9.7}$$

the Taylor expansion of $\tilde{M}_s(z_1, z_2)$ at 0 is as given by (3.7.8). But $\tilde{M}_s(z_1, z_2)$ being analytic everywhere, this power series must converge everywhere.

Thus, Theorem 5 is reduced to Prop 3.9.5.

3.10

For the proof of Prop 3.9.5, we need two sublemmas.

Sublemma 3.10.1 *Let*

$$L_n(x) = \sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} x^k \qquad (n \geq 0). \tag{3.10.2}$$

($L_n(-x)$ is Laguerre's polynomial.) Then

$$L_n(x) \leq \exp(2\sqrt{nx}) \qquad (x > 0). \tag{3.10.3}$$

Proof Take any $t > 0$. Then

$$\begin{aligned} L_n(x) &= \sum_{k=0}^{n} \binom{n}{k} t^{-k} \frac{1}{k!}(tx)^k \\ &\leq (1 + t^{-1})^n \exp(tx) < \exp(nt^{-1} + tx). \end{aligned} \tag{3.10.4}$$

Take $t = (n/x)^{1/2}$. This gives $L_n(x) \leq \exp(2\sqrt{nx})$, as desired. □

Sublemma 3.10.5 *For each global field, there exists a positive constant C_K such that*

$$|\mathrm{Supp}(D)| \leq C_K \frac{\log N(D)}{\log\log N(D) + 2} \tag{3.10.6}$$

holds for any integral divisor $D \neq (1)$ of K. Here, Supp(D) *denotes the support of the effective divisor D, i.e., the set of prime factors of D.*

This is well-known, together with that one can take $C_K = 1+\epsilon$, for $N(D)$ sufficiently large.

Proof of Prop 3.9.5 (i) Recall that $\lambda_{\wp^n}(z) = G_n(-\frac{i}{2}(\log N(\wp))z)$. Since $\begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \le \begin{pmatrix} n \\ k \end{pmatrix}$, we have $G_n(x) \le L_n(x)$ for $x \ge 0$. Hence

$$|\lambda_{\wp^n}(z)| \le L_n(\frac{1}{2}\log N(\wp)|z|) \le \exp\sqrt{2n|z|\log N(\wp)}, \tag{3.10.7}$$

by Sublemma 3.10.1.

(ii) Let $D = \prod_{\wp \in P} \wp^{n_\wp}$, with $n_\wp \ge 1$, $P = \mathrm{Supp}(D)$. Then

$$\sum_{\wp \in P}(n_\wp \log N(\wp))^{1/2} \le \left(|P|\sum_{\wp\in P} n_\wp \log N(\wp)\right)^{1/2} = (|P|\log N(D))^{1/2}. \tag{3.10.8}$$

This, combined with (i) and Sublemma 3.10.5 gives

$$|\lambda_D(z)| = \prod_{\wp\in P}|\lambda_{\wp^{n_\wp}}(z)| < \exp\{(\log N(D))\,(2C_K|z|/(\log\log N(D)+2))^{1/2}\}, \tag{3.10.9}$$

as desired.

This settles the proof of Prop 3.9.5 and hence also that of Theorem 5.

3.11 Proof of Theorem 3

Proofs of (i)-(iii) As for (i), since we have proved Theorem 4, it remains to show the uniformity of convergence without restriction on the range of $|z|$ (namely, (ii) for $t=\infty$). This and (ii) follow directly by combining the following three properties of $\tilde{M}_{\sigma,P}(z)$. Here, t is *fixed*, with $1 \le t \le \infty$.

(a) If $|P_0| > 4$, then $\tilde{M}_{\sigma,P_0} \in L^t$; in particular, for any $\epsilon > 0$, there exists $R > 0$ such that

$$\begin{cases} \int_{|z|\ge R} |\tilde{M}_{\sigma,P_0}(z)|^t |dz| < \epsilon & \cdots if\, t \ne \infty, \\ \mathrm{Sup}_{|z|\ge R}|\tilde{M}_{\sigma,P_0}(z)| < \epsilon & \cdots if\, t = \infty. \end{cases} \tag{3.11.1}$$

(b) $|\tilde{M}_{\sigma,\wp}(z)| \le 1$ for each $\wp$; hence

$$|\tilde{M}_{\sigma,P}(z)| \le |\tilde{M}_{\sigma,P_0}(z)| \qquad \cdots for\ any\ P \supseteq P_0, \tag{3.11.2}$$

and

$$(3.11.3) \qquad |\tilde{M}_\sigma(z) - \tilde{M}_{\sigma,P}(z)|^t = |\prod_{\wp \notin P} \tilde{M}_{\sigma,\wp}(z) - 1|^t |\tilde{M}_{\sigma,P}(z)|^t \leq 2^t |\tilde{M}_{\sigma,P_0}(z)|^t.$$

(c) $\tilde{M}_{\sigma,P}(z)$ converges to $\tilde{M}_\sigma(z)$ uniformly on $|z| \leq R$ for any given $R > 0$.
(For a given $\epsilon > 0$, first choose R to validate (a); then apply (3.11.3), then choose $P \supseteq P_0$ large enough to make the integral over $|z| \leq R$ also small.)

(iii) is obvious by Theorem 4.

Proof of (iv) Also obvious by Prop 3.3.1, because $|\tilde{M}_\sigma(z)| \leq |\tilde{M}_{\sigma,P}(z)|$.

Proof of (v) In general, use the symbols $\wedge, \vee$ for

$$(3.11.4) \qquad f^\wedge(z) = \int_{\mathbb{C}} f(w)\psi_z(w)|dw|,$$

$$(3.11.5) \qquad g^\vee(z) = \int_{\mathbb{C}} g(w)\psi_{-z}(w)|dw|.$$

Recall that $\tilde{M}_{\sigma,P} = M^\wedge_{\sigma,P}$, $M_{\sigma,P} = \tilde{M}^\vee_{\sigma,P}$ for each P. Recall also that for each t ($1 \leq t \leq \infty$), $\tilde{M}_{\sigma,P}$ (for $|P| > 4$) "L^t-converges" to $\tilde{M}_\sigma$. The case $t = 2$ reflects to that $\tilde{M}^\vee_{\sigma,P}$ L^2-converges to $\tilde{M}^\vee_\sigma$. But $\tilde{M}_\sigma$ belongs to L^1; hence $\tilde{M}^\vee_\sigma$ is continuous. Therefore, $\tilde{M}^\vee_\sigma$ must coincide with the L^∞-limit M_σ of $\tilde{M}^\vee_{\sigma,P} = M_{\sigma,P}$.

$$(3.11.6) \qquad \tilde{M}^\vee_\sigma(z) = M_\sigma(z).$$

Now, since $M_{\sigma,P}(z)$ converges uniformly to $M_\sigma(z)$ (Theorem 2), and each $M_{\sigma,P}(z)$ has total volume 1, we have

$$(3.11.7) \qquad \int_{\mathbb{C}} M_\sigma(z)|dz| \leq 1;$$

hence ($M_\sigma(z)$ being non-negative real valued) $M_\sigma \in L^1$. Therefore, $M^\wedge_\sigma$ is continuous. But $M^\wedge_\sigma = (\tilde{M}^\vee_\sigma)^\wedge$ is equal to $\tilde{M}_\sigma$ in L^2, i.e., $M^\wedge_\sigma = \tilde{M}_\sigma$ almost everywhere. Both being continuous, we conclude

$$(3.11.8) \qquad M^\wedge_\sigma(z) = \tilde{M}_\sigma(z),$$

as desired.

(vi) This is a special case of Theorem 5. □

Since $M_\sigma = \tilde{M}^\vee_\sigma$ and $\tilde{M}_\sigma$ belongs to C_∞ (being real analytic), we also obtain

Corollary 3.11.9 *$|z|^n M_\sigma(z)$ belongs to L^2 for any $n \geq 1$.*

Also, since $\tilde{M}_\sigma = M_\sigma^\wedge$, we obtain the expected equality

$$\int_{\mathbb{C}} M_\sigma(z)|dz| = \tilde{M}_\sigma(0) = 1. \tag{3.11.10}$$

.

Corollary 3.11.11 *$M_\sigma(z)$ is continuous in (σ, z).*

Proof Since $\tilde{M}_\sigma(w)\psi_{-z}(w)$ is continuous in (σ, z, w), the integral

$$\int_{|w|\leq R} \tilde{M}_\sigma(w)\psi_{-z}(w)|dw| \tag{3.11.12}$$

is continuous in (σ, z) for each $R > 0$, and as $R \mapsto \infty$, this converges uniformly in the wider sense to $M_\sigma(z)$, because if we choose any P with $|P| = 5$, then

$$|\tilde{M}_\sigma(w)| \leq |\tilde{M}_{\sigma,P}(w)| \ll \left(\prod_{\wp\in P} N(\wp)\right)^{\sigma/2} |w|^{-5/2}. \tag{3.11.13}$$

by (3.3.13). $\square$

Remark 3.11.14 By (3.7.6), $H_\sigma(z)$ is the Fourier transform of

$$M_\sigma(z + \frac{\zeta_K'(2\sigma)}{\zeta_K(2\sigma)}). \tag{3.11.15}$$

4 Connections with $L'(\chi, s)/L(\chi, s)$; (I) Case $\sigma > 1$

4.1

In general, it is not clear to the author what family of characters χ one should treat, and how one should define the "average" of $\Phi(L'(\chi, s)/L(\chi, s))$ over χ. Eventually, we wish to be able to treat Grössencharacters and archimedean L-factors, too, under as general a setting as possible. But at this stage, we restrict our attention to Dirichlet characters and non-archimedean L-factors. In the function field case, we shall fix an "infinite prime" $\wp_\infty$ with $\deg(\wp_\infty) = 1$ and impose $\chi(\wp_\infty) = 1$, to kill the effect of infinitely many trivial twists. We shall consider $\wp_\infty$ *as archimedean* and exclude it from the L-factors and M-factors.

In order not to worry about repeated occurrence of $\chi(\wp)$ being 0, we shall consider only those χ (the non-archimedean part of) whose conductor is a prime divisor. Also, in order not to worry about the question as to whether there does exist χ with a given conductor, we restrict ourselves to the case where the unit group of K is finite, i.e., either K is $\mathbb{Q}$, or imaginary quadratic, or K is a function field over a finite field $\mathbb{F}_q$. (Note that then, the $\wp_\infty$-unit group will also be finite.) Thus, in §4 (and §6), we impose that

(i) The field K is either $\mathbb{Q}$, or an imaginary quadratic number field, or a function field over $\mathbb{F}_q$, with an assigned prime divisor $\wp_\infty$ with degree 1.

(ii) The set of primes P, the L-functions and the $M, \tilde{M}$-functions shall *not* contain any archimedean factors (including $\wp_\infty$).

(iii) The characters χ runs over all Dirichlet characters on K (the non-archimedean part of) whose conductor is a prime divisor, such that $\chi(\wp_\infty) = 1$. (We may or may not impose χ even when $K = \mathbb{Q}$.)

(iv) The average of any complex valued function $\phi(\chi)$ of χ will be defined as follows. First, for each prime divisor $\mathbf{f}$, we take the usual average of $\phi(\chi)$ over all those χ with the (non-archimedean part of the) conductor $\mathbf{f}$. Then we take the average of this average over all $\mathbf{f}$ with $N(\mathbf{f}) \le m$;

$$\text{(4.1.1)} \qquad \mathrm{Avg}_{N(\mathbf{f})\le m}\phi(\chi) = \frac{\sum_{N(\mathbf{f})\le m}(\sum_{\mathbf{f}_\chi=\mathbf{f}}\phi(\chi))/(\sum_{\mathbf{f}_\chi=\mathbf{f}}1)}{\sum_{N(\mathbf{f})\le m}1},$$

where the summation $\sum_{N(\mathbf{f})\le m}$ is over all non-archimedean prime divisors $\mathbf{f}$ of K with $N(\mathbf{f}) \le m$. Finally, we define

$$\text{(4.1.2)} \qquad \mathrm{Avg}_\chi\phi(\chi) = \lim_{m\mapsto\infty}(\mathrm{Avg}_{N(\mathbf{f})\le m}\phi(\chi)),$$

whenever the limit exists. When we state a formula for $\mathrm{Avg}_\chi\phi(\chi)$, it will first mean that it exists.

4.2

The main purpose of §4 is to prove the following

Theorem 6 *Let $s \in \mathbb{C}$ be fixed, with $\sigma = \mathrm{Re}(s) > 1$. Then*
(i)

$$\mathrm{Avg}_\chi \Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw|$$

holds for any continuous function $\Phi(w)$ on $\mathbb{C}$.
(ii)

$$\mathrm{Avg}_\chi \psi_z\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \tilde{M}_\sigma(z),$$

(iii)

$$\mathrm{Avg}_\chi P^{(a,b)}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = (-1)^{(a+b)}\mu_\sigma^{(a,b)},$$

where $\psi_z(w) = \exp(i\mathrm{Re}(\bar{z}w))$, $P^{(a,b)}(w) = \bar{w}^a w^b \quad (a, b \in \mathbb{Z}, a, b \geq 0)$, and $\mu_\sigma^{(a,b)}$ is as in §3.8.

Corollary 4.2.1 *When $\mathrm{Re}(s) > 1$, and k is an odd positive integer,*

$$\mathrm{Avg}_\chi \left(\mathrm{Re}(\frac{L'(\chi,s)}{L(\chi,s)})\right)^k \leq 0, \tag{4.2.2}$$

with the equality if and only if $k = 1$.

Proof This average is equal to

$$(-2)^{-k} \sum_{a+b=k} \left(\begin{array}{c} k \\ a \end{array}\right) \mu_\sigma^{(a,b)}, \tag{4.2.3}$$

but $\mu_\sigma^{(a,b)} \geq 0$ with the equality if and only if $ab = 0$. □

4.3

The first key to the proof is the uniformity of distribution of $\{\chi_P\}_\chi$ on T_P for each P.

Lemma 4.3.1 *Let P be any finite set of non-archimedean primes of K, and set $T_P = \prod_{\wp \in P} \mathbb{C}^1$. Let χ run over the family of characters on K described in §4.1, but exclude those (finitely many) χ such that $\mathbf{f}_\chi \in P$. For each such χ, put $\chi_P = (\chi(\wp))_{\wp \in P} \in T_P$. Then $(\chi_P)_\chi$ is uniformly distributed on T_P; namely, for any continuous function $\Psi : T_P \mapsto \mathbb{C}$, we have*

$$\mathrm{Avg}_\chi(\Psi(\chi_P)) = \int_{T_P} \Psi(t_P) d^* t_P. \tag{4.3.2}$$

Proof Let $\mathbb{Z}_P = \prod_{\wp \in P} \mathbb{Z}$, and for $n = (n_\wp) \in \mathbb{Z}_P$ and $t = (t_\wp) \in T_P$, write $t^n = \prod_{\wp \in P} t_\wp^{n_\wp} \in \mathbb{C}^1$ (a dual pairing between T_P and $\mathbb{Z}_P$). By Weyl's criterion for uniform distribution, it suffices to prove (4.3.2) when $\Psi(t)$ is any character $\Psi(t) = t^n$, or what amounts to the same, it suffices to prove

$$\text{Avg}_\chi(\chi_P^n) = 0 \qquad (n \in \mathrm{Z}_P \setminus (0)). \tag{4.3.3}$$

To prove (4.3.3), pick any $n = (n_\wp) \in \mathbb{Z}_P \setminus (0)$, and call P^n the divisor defined by $\prod_{\wp \in P} \wp^{n_\wp}$. Note that $P^n \neq (1)$ and that $\chi(P^n) = \chi_P^n$. Now if $\mathbf{f}$ is any non-archimedean prime not contained in P, the orthogonality of characters gives

$$\sum_{\mathbf{f}_\chi | \mathbf{f}} \chi(P^n) / \sum_{\mathbf{f}_\chi | \mathbf{f}} 1 \quad = \begin{cases} 1 \cdots \text{“P}^\text{n} \equiv 1 \ (\text{mod}\ \mathbf{f})\text{”}, \\ 0 \cdots \text{otherwise}, \end{cases} \tag{4.3.4}$$

where for any divisor D of K, “$D \equiv 1 (\text{mod}\ \mathbf{f})$” means that D belongs to the common kernel of all χ with $\mathbf{f}_\chi | \mathbf{f}$. But since $P^n \neq (1)$, and the unit group of our field K is finite, there exist at most finitely many $\mathbf{f}$ such that $P^n \equiv 1 (\text{mod}\ \mathbf{f})$. Just from this follows (4.3.3) by easy estimations. This is omitted here, because a more detailed quantitative estimation will be carried out in §6. □

4.4 Proof of Theorem 6

Write $s = \sigma + \tau i$. First, take any finite set P of non-archimedean primes of K. Recall that

$$\frac{L'_P(\chi, s)}{L_P(\chi, s)} = g_{\sigma,P}(N(P)^{-\tau.i} \chi_P) \tag{4.4.1}$$

if $(\mathbf{f}_\chi, P) = 1$. First, let χ run over all characters described in §4.1 such that $(\mathbf{f}_\chi, P) = 1$. Then since $\{\chi_P\}_\chi$ is uniformly distributed on T_P, so is its translate $\{N(P)^{-\tau.i}\chi_P\}_\chi$. Therefore, by Lemma 4.3.1 applied to $\Psi = \Phi \circ g_{\sigma,P}$, we obtain

$$\begin{aligned} \text{Avg}'_\chi \left(\Phi(\frac{L'_P(\chi, s)}{L_P(\chi, s)}) \right) &= \int_{T_P} \Phi(g_{\sigma,P}(t_P)) d^* t_P \\ &= \int_{\mathbb{C}} M_{\sigma,P}(w) \Phi(w) |dw| \end{aligned} \tag{4.4.2}$$

(cf. Theorem 1). Here, Avg'_χ means that we excluded finitely many χ such that $\mathbf{f}_\chi \in P$. But since this difference does not affect the value of Avg_χ, we obtain

$$\text{Avg}_\chi \left(\Phi(\frac{L'_P(\chi, s)}{L_P(\chi, s)}) \right) = \int_{\mathbb{C}} M_{\sigma,P}(w) \Phi(w) |dw|. \tag{4.4.3}$$

Now since $\mathrm{Re}(s) > 1$ (and s is fixed), $L'_P(\chi, s)/L_P(\chi, s)$ tends *uniformly* to $L'(\chi, s)/L(\chi, s)$. Indeed,

$$(4.4.4) \qquad \left|\frac{L'(\chi, s)}{L(\chi, s)} - \frac{L'_P(\chi, s)}{L_P(\chi, s)}\right| \leq \sum_{\wp \notin P} \frac{\log N(\wp)}{N(\wp)^\sigma - 1},$$

and the right hand side tends to 0 when $P = P_y$ and $y \mapsto \infty$. Moreover, since $|L'(\chi, s)/L(\chi, s)|$ and $|L'_P(\chi, s)/L_P(\chi, s)|$ are uniformly bounded (by $|\zeta'_K(\sigma)/\zeta_K(\sigma)|$), and $M_\sigma(w)$ is compactly supported (because $\sigma > 1$), the effect of Φ is only within these bounds; hence we may assume Φ to be equicontinuous. Therefore, $\Phi(L'_P(\chi, s)/L_P(\chi, s))$ tends uniformly to $\Phi(L'(\chi, s)/L(\chi, s))$. And since $M_{\sigma,P}(w)$ tends uniformly to $M_\sigma(w)$ (Theorem 2), we obtain from (4.4.3) by letting $P = P_y, y \mapsto \infty$, the statement (i) of Theorem 6. The second statement (ii) is a special case of (i). The last formula (iii) is also a special case where $\Phi(w) = P^{(a,b)}(w)$. In fact, since the Fourier transform of $M_\sigma(z)$ is $\tilde{M}_\sigma(z)$ (Theorem 3 (v)), that of $P^{(a,b)}(z)M_\sigma(z)$ is $(2/i)^{a+b}\frac{\partial^{a+b}}{\partial z^a \partial \bar{z}^b}\tilde{M}_\sigma(z)$; hence

$$(4.4.5) \qquad \begin{aligned} \mathrm{Avg}_\chi P^{(a,b)}\left(\frac{L'(\chi, s)}{L(\chi, s)}\right) &= \int_{\mathbb{C}} M_\sigma(w) P^{(a,b)}(w)|dw| \\ &= \left(\frac{2}{i}\right)^{a+b} \frac{\partial^{a+b}}{\partial z^a \partial \bar{z}^b}\tilde{M}_\sigma(z) \,|_{z=0} = (-1)^{a+b}\mu_\sigma^{a+b} \end{aligned}$$

(the last equality by Theorem 3 (vi)). This completes the proof of Theorem 6. □

5 Some Fourier analysis of $\psi_z(g_{\sigma,P}(t))$

5.1

We come back to the general situation where K is any global field, P is any finite set of non-archimedean primes of K, and $T_P = \prod_{\wp\in P}\mathbb{C}^1$, $\mathbb{Z}_P = \prod_{\wp\in P}\mathbb{Z}$, with the dual pairing

$$(5.1.1)\qquad t^n = \prod_{\wp\in P} t_\wp^{n_\wp} \in \mathbb{C}^1 \qquad (t=(t_\wp)\in T_P,\ n=(n_\wp)\in\mathbb{Z}_P).$$

For $\sigma>0$, put, as before,

$$(5.1.2)\qquad g_{\sigma,P}(t) = \sum_{\wp\in P} g_{\sigma,\wp}(t_\wp), \qquad g_{\sigma,\wp}(t_\wp) = \frac{t_\wp \log N(\wp)}{t_\wp - N(\wp)^\sigma}.$$

For $z_1, z_2, w \in \mathbb{C}$, put

$$(5.1.3)\qquad \psi_{z_1,z_2}(w) = \exp(\frac{i}{2}(z_1\bar{w} + z_2 w)).$$

Thus, $\psi_{z_1,z_2} : \mathbb{C} \mapsto \mathbb{C}^\times$ is a quasi-character of the additive group $\mathbb{C}$, which is a character into $\mathbb{C}^1$ when $z_2 = \bar{z}_1$. In our previous notation,

$$(5.1.4)\qquad \psi_{z,\bar{z}}(w) = \psi_z(w).$$

We shall study the Fourier expansion of $\psi_{z_1,z_2}(g_{\sigma,P}(t))$, as a preparation for §6. First, we shall prove the following

Proposition 5.1.5 *For each $\sigma>0$, $z_1,z_2\in\mathbb{C}$ and P, the function $\psi_{z_1,z_2}(g_{\sigma,P}(t))$ of $t\in T_P$ has an absolutely convergent Fourier expansion*

$$(5.1.6)\qquad \psi_{z_1,z_2}(g_{\sigma,P}(t)) = \sum_{n\in\mathbb{Z}_P} A_{\sigma,P}(n;\, z_1,z_2)t^n,$$

with

$$(5.1.7)\qquad \begin{aligned} A_{\sigma,P}(n;\, z_1,z_2) &= \int_{T_P} \psi_{z_1,z_2}(g_{\sigma,P}(t))t^{-n}d^*t \\ &= \sum_{D_2D_1^{-1}=P^n} \lambda_{D_1}(z_1)\lambda_{D_2}(z_2)N(D_1D_2)^{-\sigma}. \end{aligned}$$

Here, the last summation is over all integral ideals D_1, D_2 with supports in P such that $D_2D_1^{-1} = P^n (= \prod_{\wp\in P}\wp^{n_\wp})$.

Proof Each side of these formulas being multiplicative, it suffices to prove them when P consists of a single prime $\wp$. So, write $t = t_\wp$. Since $\exp(\frac{i}{2}z.g_{\sigma,\wp}(t))$ is a holomorphic function of t outside the point $t = N(\wp)^\sigma$, its Taylor expansion (cf. (3.9.1)).

$$\exp(\frac{i}{2}z.g_{\sigma,\wp}(t)) = \sum_{n=0}^{\infty} \lambda_{\wp^n}(z) N(\wp)^{-n\sigma} t^n \tag{5.1.8}$$

at $t = 0$ is absolutely convergent on $|t| < N(\wp)^\sigma$. Therefore, $\psi_{z_1,z_2}(g_{\sigma,\wp}(t))$ is the product of two absolutely convergent series, for $\exp(\frac{i}{2}z_2.g_{\sigma,\wp}(t))$ and for $\exp(\frac{i}{2}z_1.g_{\sigma,\wp}(\bar{t}))$, on the domain $|t| < N(\wp)^\sigma$. By restricting this to $|t| = 1$, replacing $\bar{t}$ by t^{-1} and rearranging the absolutely convergent double series, we obtain the absolutely convergent series (5.1.6) for $P = \{\wp\}$, with

$$A_{\sigma,\wp}(n;\, z_1, z_2) = \sum_{\substack{n_1,n_2 \geq 0 \\ n_2 - n_1 = n}} \lambda_{\wp^{n_1}}(z_1)\lambda_{\wp^{n_2}}(z_2) N(\wp)^{-(n_1+n_2)\sigma}. \tag{5.1.9}$$

□

It is clear that

$$A_{\sigma,P}(n;\, z_1, z_2) = \prod_{\wp \in P} A_{\sigma,\wp}(n_\wp;\, z_1, z_2) \qquad (n = (n_\wp)), \tag{5.1.10}$$

$$A_{\sigma,P}(-n;\, z_1, z_2) = A_{\sigma,P}(n;\, z_2, z_1), \tag{5.1.11}$$

$$A_{\sigma,P}(0;\, z_1, z_2) = \tilde{M}_{\sigma,P}(z_1, z_2). \tag{5.1.12}$$

(cf. (3.7.1)(5.1.7)).

Put

$$A_{\sigma,P}(n, z) = A_{\sigma,P}(n;\, z, \bar{z}) \qquad (n \in \mathbb{Z}_P,\ z \in \mathbb{C}), \tag{5.1.13}$$

so that

$$A_{\sigma,P}(0, z) = \tilde{M}_{\sigma,P}(z). \tag{5.1.14}$$

Then, clearly,

$$|\sum_{n \in \mathbb{Z}_P} A_{\sigma,P}(n, z) t^n| = |\psi_z(g_{\sigma,P}(t))| = 1 \qquad (t \in T_P), \tag{5.1.15}$$

and the Plancherel formula gives also that

$$\sum_{n \in \mathbb{Z}_P} |A_{\sigma,P}(n, z)|^2 = \int_{T_P} |\psi_z(g_{\sigma,P}(t))|^2 d^*t = 1. \tag{5.1.16}$$

On the other hand, the value of the (finite) sum

$$\sum_{n\in\mathbb{Z}_P} |A_{\sigma,P}(n,z)| \tag{5.1.17}$$

grows (unboundedly when $\sigma \le 1$) with P (see Remark 5.2.23). What we shall actually need is Cor 5.2.18 giving an estimation of a sum similar to (5.1.17), for $P = \{\wp\}$; Remark 5.2.23 says that this is essentially the best possible.

5.2

In this subsection, we shall first generalize the formulas given in §3.5 for the function (5.1.12), to the case $n \neq 0$. By (5.1.10)(5.1.11), it suffices to give the formula for $A_{\sigma,\wp}(n_\wp;\, z_1, z_2)$ when $n_\wp > 0$. Then we apply this formula to the estimations mentioned above.

Proposition 5.2.1 *Let $n > 0$, and write $q = N(\wp)^\sigma, \lambda = \log N(\wp)$. Then*

$$\exp(-\frac{i}{2}c_{\sigma,\wp}(z_1+z_2))A_{\sigma,\wp}(n;\, z_1, z_2) = \frac{1}{q^n}\sum_{\nu=1}^{n}\begin{pmatrix} n-1 \\ \nu-1 \end{pmatrix}\left(\frac{-i\lambda z_2}{2}\right)^{\nu} B^{(\nu)}_{\sigma,\wp}(z_1, z_2), \tag{5.2.2}$$

where

$$B^{(\nu)}_{\sigma,\wp}(z_1, z_2) = \sum_{\ell=0}^{\infty}\left(\frac{ir_{\sigma,\wp}z_2}{2q}\right)^{\ell}\begin{pmatrix} \nu+\ell \\ \nu \end{pmatrix} j_{\nu+\ell}\left(\frac{r^2_{\sigma,\wp}z_1z_2}{4}\right). \tag{5.2.3}$$

In particular,

$$\exp(-ic_{\sigma,\wp}\mathrm{Re}(z))A_{\sigma,\wp}(n,z) = \frac{1}{q^n}\sum_{\nu=1}^{n}\begin{pmatrix} n-1 \\ \nu-1 \end{pmatrix}(1-q^2)^{\nu}\left(\frac{i}{q}\right)^{\nu}\left(\frac{\bar{z}}{|z|}\right)^{\nu} C^{(\nu)}_{\sigma,\wp}(z), \tag{5.2.4}$$

with

$$C^{(\nu)}_{\sigma,\wp}(z) = \sum_{\ell=0}^{\infty}\left(\frac{i}{q}\right)^{\ell}\left(\frac{\bar{z}}{|z|}\right)^{\ell}\begin{pmatrix} \nu+\ell \\ \nu \end{pmatrix} J_{\nu+\ell}(r_{\sigma,\wp}|z|). \tag{5.2.5}$$

Proof As in §2.1, we change variables,

$$g_{\sigma,\wp}(t) = c_{\sigma,\wp} + r_{\sigma,\wp}t', \tag{5.2.6}$$

$$t' = \frac{1-qt}{q-t}. \tag{5.2.7}$$

Recall that $|t| = 1$ if and only if $|t'| = 1$. As a function of t, t' is holomorphic outside $t = q$; hence $(t')^a(\bar{t}')^b$ $(a, b \geq 0)$ has an absolutely convergent Fourier expansion

$$(t')^a(\bar{t}')^b = \sum_{n\in\mathbb{Z}} \gamma^{(a,b)}(n)t^n \qquad (|t| = 1), \tag{5.2.8}$$

with

$$\gamma^{(a,b)}(n) = \int_{\mathbb{C}^1} \left(\frac{q-t}{1-qt}\right)^{b-a} t^{-n}d^*t = \int_{\mathbb{C}^1} \left(\frac{q-t}{1-qt}\right)^{a-b} t^n d^*t. \tag{5.2.9}$$

By direct residue calculus at $t = q^{-1}$, we obtain $\gamma^{(a,b)}(n) = 0$ if $a \leq b$, and

$$\gamma^{(a,b)}(n) = \frac{1}{q^{n+a-b}} \sum_{\nu=1}^{\min(n,a-b)} \binom{n-1}{\nu-1}\binom{a-b}{\nu}(1-q^2)^\nu \qquad (a > b). \tag{5.2.10}$$

(In fact, when $a > b$, $\gamma^{(a,b)}(n)$ is the coefficient of y^{a-b-1} in

$$(y + 1 - q^2)^{a-b}(y + 1)^{n-1}$$

(y corresponds to $qt - 1$).)

Now, each $A_{\sigma,\wp}(n;\, z_1, z_2)$ is an analytic function of z_1, z_2, and the coefficients of its Taylor expansion at (z)=(0) can be obtained by operating $\partial^{a+b}/\partial z_1^a \partial z_2^b$ under the integration symbol in (5.1.7) and by putting (z)=(0). But since

$$\psi_{z_1,z_2}(g_{\sigma,\wp}(t)) = \exp(\frac{i}{2}c_{\sigma,\wp}(z_1 + z_2)) \exp(\frac{i}{2}r_{\sigma,\wp}(z_2t' + z_1\bar{t}')), \tag{5.2.11}$$

we obtain

$$A_{\sigma,\wp}(n;\, z_1, z_2) = \exp(\frac{i}{2}c_{\sigma,\wp}(z_1 + z_2)) \sum_{a,b=0}^{\infty} \left(\frac{ir_{\sigma,\wp}}{2}\right)^{a+b} \gamma^{(a,b)}(n)\frac{z_2^a z_1^b}{a!b!}. \tag{5.2.12}$$

By inserting (5.2.10) into (5.2.12), and by rearranging the series (use $\ell = a - b - \nu$, and note that $r(1 - q^2) = -\lambda q$), we obtain the desired formula. □

Corollary 5.2.13 *When* $n \neq 0$,

$$|A_{\sigma,\wp}(n, z)| \leq \frac{1}{N(\wp)^{\sigma|n|}} e^{\frac{1}{2}|c_{\sigma,\wp}z|} \sum_{\nu=1}^{|n|} \frac{1}{\nu!}\binom{|n|-1}{\nu-1}\left(\frac{|z|\log N(\wp)}{2}\right)^\nu. \tag{5.2.14}$$

Proof Since

$$|J_n(w)| \le \frac{1}{n!}\left(\frac{|w|}{2}\right)^n e^{|\mathrm{Im}(w)|} \tag{5.2.15}$$

(cf. e.g. [Wa] §3.31), with the notation of Prop 5.2.1, we obtain

$$\begin{aligned} |C^{(\nu)}_{\sigma,\wp}(z)| \le & \sum_{\ell=0}^{\infty} \frac{1}{q^\ell}\begin{pmatrix} \nu+\ell \\ \nu \end{pmatrix} \frac{1}{(\nu+\ell)!}\left(\frac{r_{\sigma,\wp}|z|}{2}\right)^{\nu+\ell} \\ &= \frac{1}{\nu!}\left(\frac{r_{\sigma,\wp}|z|}{2}\right)^{\nu} \exp\left(\frac{r_{\sigma,\wp}|z|}{2q}\right); \end{aligned} \tag{5.2.16}$$

$$|A_{\sigma,\wp}(n,z)| \le \frac{1}{q^{|n|}}\sum_{\nu=1}^{|n|}\begin{pmatrix} |n|-1 \\ \nu-1 \end{pmatrix}(q-q^{-1})^\nu |C^{(\nu)}_{\sigma,\wp}(z)|. \tag{5.2.17}$$

But since $(q-q^{-1})r_{\sigma,\wp} = \log N(\wp)$ and $q^{-1}r_{\sigma,\wp} = |c_{\sigma,\wp}|$, (5.2.14) follows. □

Corollary 5.2.18 *Put* $q = N(\wp)^\sigma$, $\lambda = \log N(\wp)$. *Then*

$$\sum_{n\in\mathbb{Z}} |A_{\sigma,\wp}(n,z)|(|n|+1) < \exp\{2\lambda|z|\left(\frac{1}{q-1}+\frac{4}{q^2-1}\right)\}. \tag{5.2.19}$$

Proof By (5.2.14) (and by $|A_{\sigma,\wp}(0,z)| \le 1$), the left hand side of (5.2.19) is bounded by

$$1 + 2e^{\frac{1}{2}|c_{\sigma,\wp}z|}\{\sum_{n=1}^{\infty}\frac{n+1}{q^n}\sum_{\nu=1}^{n}\frac{1}{\nu!}\begin{pmatrix} n-1 \\ \nu-1 \end{pmatrix}\left(\frac{\lambda|z|}{2}\right)^\nu\}. \tag{5.2.20}$$

Now since

$$\sum_{k=0}^{\infty}\begin{pmatrix} \nu+k-1 \\ k \end{pmatrix}(\nu+k+1)t^k = t^{-\nu}\frac{d}{dt}\left(t^{\nu+1}(1-t)^{-\nu}\right) = (\nu+1-t)(1-t)^{-\nu-1},$$

for $\nu \ge 1$, the sum in the braces in (5.2.20) may be rewritten (using $k = n-\nu \ge 0$) as

$$\begin{aligned} &\sum_{\nu=1}^{\infty}\frac{1}{\nu!}\left(\frac{\lambda|z|}{2q}\right)^\nu\{\sum_{k=0}^{\infty}\begin{pmatrix} \nu+k-1 \\ k \end{pmatrix}\frac{(\nu+k+1)}{q^k}\} \\ &= \sum_{\nu=1}^{\infty}\frac{1}{\nu!}\left(\frac{\lambda|z|}{2q}\right)^\nu(\nu+1-q^{-1})(1-q^{-1})^{-\nu-1} \\ &= \left(1+\frac{\lambda|z|}{2q(1-q^{-1})^2}\right)\exp\left(\frac{\lambda|z|}{2(q-1)}\right) - 1 \\ &< \exp\{\frac{\lambda|z|}{2}\left(\frac{q}{(q-1)^2}+\frac{1}{q-1}\right)\} - 1. \end{aligned} \tag{5.2.21}$$

Since $|c_{\sigma,\wp}| = \lambda(q^2-1)^{-1}$, we obtain

$$\sum_{n\in\mathbb{Z}} |A_{\sigma,\wp}(n,z)|(|n|+1) < 2\exp\{\lambda|z|\frac{q^2+q-1}{(q-1)(q^2-1)}\} - 1 \tag{5.2.22}$$
$$\le \exp\{2\lambda|z|\frac{q^2+q-1}{(q-1)(q^2-1)}\} < \exp\{2\lambda|z|\left(\frac{1}{q-1}+\frac{4}{q^2-1}\right)\}$$

(because $q \ge \sqrt{2} > 4/3$). □

Remark 5.2.23 We can show that if $N(\wp)$ is sufficienly large compared with $|z|$, then

$$\sum_{n\in\mathbb{Z}} |A_{\sigma,\wp}(n,z)| \ge |\tilde{M}_{\sigma,\wp}(z)| \exp\left(\frac{\lambda B}{q-1}\right), \tag{5.2.24}$$

where $B > 0$ depends on σ, K and z. Thus, the "core" of Cor 5.2.18 that will be used later cannot be expected to be improved.

Remark 5.2.25 This is just to draw a full circle and not for applications in the present paper. Theorem 5 in §3.7 gives a Dirichlet series expansion for $\tilde{M}_s(z_1,z_2)$. By (5.1.7), we meet a more general Dirichlet series

$$\sum_{D_2D_1^{-1}=D_0} \lambda_{D_1}(z_1)\lambda_{D_2}(z_2)N(D_1D_2)^{-s}, \tag{5.2.26}$$

where D_0 is a given fractional divisor of K, and D_1, D_2 run over all integral divisors of K such that $D_2D_1^{-1} = D_0$. (D_0 corresponds to P^n). This series is also absolutely convergent on $\mathrm{Re}(s) > 1/2$.

6 Connections with $L'(\chi, s)/L(\chi, s)$; (II) Case $\sigma > 3/4$

6.1

The main goal of §6 is to prove the following

Theorem 7 *Consider the function field case of §4. Then, among the equalities* (i) $\sim$ (iii) *in Theorem 6, namely*
(i)

$$\mathrm{Avg}_\chi \Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw|,$$

(ii)

$$\mathrm{Avg}_\chi \psi_z\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \tilde{M}_\sigma(z),$$

(iii)

$$\mathrm{Avg}_\chi P^{(a,b)}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = (-1)^{(a+b)}\mu_\sigma^{(a,b)},$$

(ii) *and* (iii) *hold also when* $\sigma > 3/4$. *The equality* (i) *holds if either*
(*a*) $\sigma > 3/4$, $\Phi \in L^1 \cap L^\infty$ *and moreover the Fourier transform of* Φ *has compact support, or*
(*b*) $\sigma > 5/6$ *and* Φ *is a standard function in the sense of [We].*

Conjecture *The theorem holds for any* $\sigma > 1/2$ *and any continuous function* $\Phi(w)$ *with compact support.*

Remarks 6.1.1 1. Remarks alluded to the number field case will be in §6.9.

2. Cor 4.2.1 holds also in the function field case if $\sigma > 3/4$.

6.2

First, we fix some notation. We denote by Cl_K the group of divisor classes of K in degree 0, and $h_K = |Cl_K|$ the class number. For a *prime* divisor $\mathbf{f} \neq \wp_\infty$, denote by $I_\mathbf{f}$ (resp. $I_\mathbf{f}^{(0)}$) the group of divisors of K (resp. those of degree 0) that are coprime with $\mathbf{f}$, and by $G_\mathbf{f}$ the quotient of $I_\mathbf{f}$ by the subgroup generated by $\wp_\infty$ and all the principal divisors of the form (α) with $\alpha \equiv 1 \pmod{\mathbf{f}}$. Since $\deg(\wp_\infty) = 1$, $G_\mathbf{f}$ is canonically isormorphic to the quotient of $I_\mathbf{f}^{(0)}$ by the subgroup generated by those (α). Call

(6.2.1) $$i_\mathbf{f} : I_\mathbf{f} \longmapsto G_\mathbf{f},$$

$$(6.2.2) \qquad j_{\mathbf{f}} : G_{\mathbf{f}} \longmapsto Cl_K,$$

the projections, so that $\mathrm{Ker}(j_{\mathbf{f}}) \cong \kappa_{\mathbf{f}}^{\times}/\mathbb{F}_q^{\times}$, where $\kappa_{\mathbf{f}}$ denotes the residue field of $\mathbf{f}$. Thus,

$$(6.2.3) \qquad |G_{\mathbf{f}}| = h_K(N(\mathbf{f}) - 1)/(q-1).$$

For any finite abelian group G, $\hat{G}$ will denote its character group. Thus, $\hat{j}_{\mathbf{f}}$ embeds $\hat{Cl}_K$ into $\hat{G}_{\mathbf{f}}$. The Dirichlet characters χ of K with conductor $\mathbf{f}$ satisfying $\chi(\wp_\infty) = 1$ are the elements of $\hat{G}_{\mathbf{f}} \setminus \hat{j}_{\mathbf{f}}(\hat{Cl}_K)$; hence

$$(6.2.4) \qquad \#\{\chi;\ \mathbf{f}_\chi = \mathbf{f}\} = |G_{\mathbf{f}}| - |Cl_K| = h_K(N(\mathbf{f}) - q)/(q-1).$$

6.3

Now let P be any finite set of primes $\neq \wp_\infty$ of K. For a prime $\mathbf{f} \neq \wp_\infty$, consider two averages

$$(6.3.1) \qquad S'_{\mathbf{f}} = \frac{\sum_{\mathbf{f}_\chi = \mathbf{f}} \psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)}{\sum_{\mathbf{f}_\chi = \mathbf{f}} 1} = \frac{\sum_{\chi \in \hat{G}_{\mathbf{f}} \setminus \hat{j}_{\mathbf{f}}(\hat{Cl}_K)} \psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)}{|G_{\mathbf{f}}| - |Cl_K|},$$

$$(6.3.2) \qquad S_{\mathbf{f}} = \frac{\sum_{\mathbf{f}_\chi | \mathbf{f}} \psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)}{\sum_{\mathbf{f}_\chi | \mathbf{f}} 1} = \frac{\sum_{\chi \in \hat{G}_{\mathbf{f}}} \psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)}{|G_{\mathbf{f}}|}.$$

Clearly, $|S_{\mathbf{f}}|$, $|S'_{\mathbf{f}}| \leq 1$. It is also easy to see that

$$(6.3.3) \qquad |S'_{\mathbf{f}} - S_{\mathbf{f}}| \leq \frac{2(q-1)}{N(\mathbf{f}) - 1} \ll \frac{1}{N(\mathbf{f})}.$$

When $\mathbf{f} \notin P$, $S_{\mathbf{f}}$ can be expressed in terms of the Fourier series

$$(6.3.4) \qquad \psi_z(g_{\sigma,P}(t_P)) = \sum_{n \in \mathbb{Z}_P} A_{\sigma,P}(n;\ z) t_P^n,$$

(cf. §5; $A_{\sigma,P}(n;\ z) = A_{\sigma,P}(n;\ z, \bar{z})$) as follows. Since

$$(6.3.5) \qquad \frac{L'_P(\chi,s)}{L_P(\chi,s)} = g_{\sigma,P}\left(\chi_P N(P)^{-\tau i}\right)$$

($\tau = \mathrm{Im}(s)$), we have

$$(6.3.6) \qquad \psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right) = \sum_{n \in \mathbb{Z}_P} A_{\sigma,P}(n;\ z)(\chi_P N(P)^{-\tau i})^n.$$

For each $n \in \mathbb{Z}_P$, define the divisor P^n of K by

$$P^n = \prod_{\wp \in P} \wp^{n_\wp}, \tag{6.3.7}$$

where $n = (n_\wp)_{\wp \in P}$. Then $\chi_P^n = \chi(P^n)$, $(N(P)^{-\tau i})^n = N(P^n)^{-\tau i}$; hence the orthogonality relation for characters gives

$$\frac{1}{|G_{\mathbf{f}}|} \sum_{\chi \in \hat{G}_{\mathbf{f}}} \chi_P^n = \begin{cases} 1 \cdots i_{\mathbf{f}}(P^n) = 1, \\ 0 \ldots otherwise. \end{cases} \tag{6.3.8}$$

Therefore,

$$S_{\mathbf{f}} = \sum_{\substack{n \in \mathbb{Z}_P \\ i_{\mathbf{f}}(P^n)=1}} A_{\sigma,P}(n;\, z) N(P^n)^{-\tau i}, \tag{6.3.9}$$

whenever $\mathbf{f} \notin P$. Now let us estimate the quantity

$$|\mathrm{Avg}_{N(\mathbf{f}_\chi) \le m} \psi_z \left(\frac{L'_P(\chi, s)}{L_P(\chi, s)} \right) - \tilde{M}_{\sigma,P}(z)| \tag{6.3.10}$$

when m is large compared with $|P|$. Let $\pi(x) = \pi_K(x)$ denote the number of prime divisors $\mathbf{f} \neq \wp_\infty$ with $N(\mathbf{f}) \le x$. Then $\pi(x) \sim x/\log x$; hence by our definition of Avg (§4.1),

(6.3.11)

$$\begin{aligned} \mathrm{Avg}_{N(\mathbf{f}_\chi) \le m} \psi_z \left(\frac{L'_P(\chi, s)}{L_P(\chi, s)} \right) &= \frac{\sum_{N(\mathbf{f}) \le m} S'_{\mathbf{f}}}{\sum_{N(\mathbf{f}) \le m} 1} \\ &= \frac{\sum_{N(\mathbf{f}) \le m} S_{\mathbf{f}}}{\pi(m)} + O\left(\frac{\sum_{N(\mathbf{f}) \le m} N(\mathbf{f})^{-1}}{\pi(m)} \right) \\ &= \frac{\sum_{N(\mathbf{f}) \le m,\, \mathbf{f} \notin P \cup \{\wp_\infty\}} S_{\mathbf{f}}}{\pi(m)} + O\left(\frac{\log m}{m} |P| + \frac{\log m \log\log m}{m} \right), \end{aligned}$$

and by (6.3.9) the main term on the last line of (6.3.11) is given by

$$\sum_{n \in \mathbb{Z}_P} \epsilon^{(m)}(n) A_{\sigma,P}(n;\, z) N(P^n)^{-\tau i}, \tag{6.3.12}$$

where

$$\epsilon^{(m)}(n) = \frac{1}{\pi(m)} \#\{\mathbf{f} \notin P \cup (\wp_\infty);\, N(\mathbf{f}) \le m,\, i_{\mathbf{f}}(P^n) = 1\}. \tag{6.3.13}$$

Note that $0 \leq \epsilon^{(m)}(n) \leq 1$, and that if $P = P_y$ with $y < m$, then

$$\epsilon^{(m)}(0) = 1 - \frac{\pi(y)}{\pi(m)}. \tag{6.3.14}$$

Recall (5.1.12):

$$A_{\sigma,P}(0;\, z) = \tilde{M}_{\sigma,P}(z). \tag{6.3.15}$$

Now we are going to let both m and y grow, but with m much faster than y. We shall take

$$y \leq (\log m)^b \tag{6.3.16}$$

(and $y \mapsto \infty$), where b is a positive constant to be specified later which will depend only on σ. For such a case,

$$\epsilon^{(m)}(0) = 1 + O\left(\frac{\log^{b+1} m}{m}\right); \tag{6.3.17}$$

hence

$$\epsilon^{(m)}(0) A_{\sigma,P}(0;\, z) = \tilde{M}_{\sigma,P}(z) + O\left(\frac{\log^{b+1} m}{m}\right) \tag{6.3.18}$$

(since $|\tilde{M}_{\sigma,P}(z)| \leq 1$); hence we obtain the first basic formula:

(6.3.19)
$$\mathrm{Avg}_{N(\mathbf{f}_\chi)\leq m}\psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right) - \tilde{M}_{\sigma,P}(z) = \sum_{n\in\mathbb{Z}_P\setminus(0)} \epsilon^{(m)}(n) A_{\sigma,P}(n;\, z) N(P^n)^{-\tau i} + O\left(\frac{\log^{b+1} m}{m}\right),$$

when $P = P_y$ with $y \leq (\log m)^b$.

This basic formula is, at the same time, a branch point for the choice of the next way to proceed. The road we choose here is less challenging, as the goal is "$\sigma > 3/4$ results". This method takes the absolute value of each term of the Fourier expansion in (6.3.19), and estimate the sum of these absolute values. This can be done, as we shall see below, but by this method, it is unlikely that we can get anything better than our "$\sigma > 3/4$ results" (see Remark 6.6.4 (ii)). The other, seemingly more difficult way towards "$\sigma > 1/2$ results" is to hope that one can make use of the fact that

$$|\sum_{n\in\mathbb{Z}_P} A_{\sigma,P}(n;\, z) N(P^n)^{-\tau i}| = |\psi_z(g_{\sigma,P}(N(P)^{-\tau i}))| \leq 1, \tag{6.3.20}$$

and that the "expected average" of $\epsilon^{(m)}(n)$ is so small as

$$\sim \frac{1}{\pi(m)} \sum_{N(\mathbf{f}) \leq m} |G_{\mathbf{f}}|^{-1} \ll \frac{1}{m}(\log m)(\log\log m). \tag{6.3.21}$$

Presently, the author is not able to show (even when $\tau = 0$) that the size of $\epsilon^{(m)}(n)$ and ,say, the sign of $\mathrm{Re}(A_{\sigma,P}(n;\, z))$ are independently distributed, and so we must rely on the evaluation of the sum of absolute values.

6.4

In order to estimate the sum

$$\sum_{n \in \mathbb{Z}_P \setminus (0)} \epsilon^{(m)}(n) |A_{\sigma,P}(n;\, z)|, \tag{6.4.1}$$

we need the following two sublemmas.

Sublemma 6.4.2 *Let D be any divisor of K such that $D \neq (1)$, $\mathrm{Supp}(D) \not\ni \wp_\infty$. Then*

$$\#\{\mathbf{f};\ i_{\mathbf{f}}(D) = 1\} \ll \frac{\log \parallel D \parallel}{\log\log \parallel D \parallel + 2}. \tag{6.4.3}$$

Here, the condition on $\mathbf{f}$ preassumes that it is a prime not contained in $\mathrm{Supp} D \cup \{\wp_\infty\}$, and we put $\parallel D \parallel = \prod_{\wp \in P} N(\wp)^{|n_\wp|}$ for $D = \prod_{\wp \in P} \wp^{n_\wp}$.

Proof In fact, $i_{\mathbf{f}}(D) = 1$ holds only if $D\wp_\infty^{-\deg D} = (\alpha)$ and $\alpha \equiv c \,(\mathrm{mod}\, \mathbf{f})$ with some $c \in \mathbb{F}_q^\times$. So, the left hand side of (6.4.3) is bounded by the sum of $|\mathrm{Supp}(\alpha - c)|$ over all $c \in \mathbb{F}_q^\times$. But since $(\alpha - c)$ has the same denominator as (α), we have $\parallel \alpha - c \parallel = \parallel \alpha \parallel \leq \parallel D \parallel^2$; hence (6.3.4) follows directly from Sublemma 3.10.5. □

Corollary 6.4.4 *For $P = P_y$, $y < (\log m)^b$, and $n = (n_\wp) \in \mathbb{Z}_P \setminus (0)$,*

$$\epsilon^{(m)}(n) \ll \frac{(\log m)(\log\log m)}{m} \prod_{\wp \in P} (|n_\wp| + 1). \tag{6.4.5}$$

Proof Since

$$\log \parallel P^n \parallel = \sum_{\wp \in P} |n_\wp| \log N(\wp) < \left(\prod_{\wp \in P} (|n_\wp| + 1) \right) \log y, \tag{6.4.6}$$

we obtain by Sublemma 6.4.2 that

$$\#\{\mathbf{f};\ i_{\mathbf{f}}(P^n) = 1\} \ll (\log\log m) \prod_{\wp \in P} (|n_\wp| + 1); \tag{6.4.7}$$

hence (6.4.5). □

Sublemma 6.4.8 *Let $\sigma > 1/2$. Then*

$$\sum_{N(\wp)\leq y} \frac{\log N(\wp)}{N(\wp)^\sigma - 1} < q^{1-\sigma}\frac{y^{1-\sigma}-1}{q^{1-\sigma}-1}\log q + C'_{K,\sigma}. \tag{6.4.9}$$

When $\sigma = 1$, the first term on the right hand side means the limit value at $\sigma = 1$; namely, $\log y$.

Proof This is standard and is an easy exercise. But we shall sketch the proof, in order to show how Weil's Riemann hypothesis for curves is used and to make it explicit where $C'_{K,\sigma}$ comes from.

The left hand side of (6.4.9) is equal to

$$\sum_{\substack{N(\wp)\leq y \\ k\geq 1}} \frac{\log N(\wp)}{N(\wp)^{k\sigma}} = \sum_{N(\wp^k)\leq y} + \sum_{\substack{N(\wp)\leq y \\ N(\wp^k)>y}}, \tag{6.4.10}$$

and the first sum can be rewritten as

$$\sum_{dk\leq \log_q y} \frac{dB_d}{q^{dk\sigma}}\log q = \sum_{n\leq \log_q y} \frac{N_n}{q^{n\sigma}}\log q, \tag{6.4.11}$$

where B_d is the number of prime divisors of K with degree d, and $N_n = \sum_{d|n} dB_d$ is the number of $\mathbb{F}_{q^n}$-rational points of the corresponding curve. Use the consequence of the Weil Riemann Hypothesis

$$N_n \leq q^n + 1 + 2gq^{n/2} \tag{6.4.12}$$

to bound (6.4.11), and use

$$\sum_{\substack{N(\wp)\leq y \\ N(\wp^k)>y}} \leq \sum_{k\geq 2,\, \wp} \frac{\log N(\wp)}{N(\wp)^{\sigma k}} = \sum_{\wp} \frac{\log N(\wp)}{N(\wp)^{2\sigma}(1-N(\wp)^{-\sigma})} < \infty \tag{6.4.13}$$

to bound the second sum on the right hand side of (6.4.10). The main term on the right hand side of (6.4.11) of course comes from q^n on the right hand side of (6.4.12). □

Now, by Corollary 6.4.4 and by Cor 5.2.18, we obtain

(6.4.14)

$$\sum_{n\in\mathbb{Z}_P\backslash(0)} \epsilon^{(m)}(n)|A_{\sigma,P}(n;\,z)| \ll \frac{(\log m)(\log\log m)}{m}\prod_{\wp\in P}\left(\sum_{n_\wp\in\mathbb{Z}}|A_{\sigma,\wp}(n_\wp;\,z)|(|n_\wp|+1)\right)$$
$$\ll \frac{(\log m)(\log\log m)}{m}\exp\{2|z|\sum_{\wp\in P}\left(\frac{\log N(\wp)}{N(\wp)^\sigma-1}+\frac{4\log N(\wp)}{N(\wp)^{2\sigma}-1}\right)\}$$
$$\ll \frac{(\log m)(\log\log m)}{m}\exp\{2|z|\left(\sum_{\wp\in P}\frac{\log N(\wp)}{N(\wp)^\sigma-1}+C\text{"}_{K,\sigma}\right)\},$$

with some positive constant $C\text{"}_{K,\sigma}$ depending only on K and σ. Therefore, by (6.3.19) and Sublemma 6.4.8, we obtain

(6.4.15)

$$|\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)-\tilde{M}_\sigma(z)| \ll \frac{(\log m)(\log\log m)}{m}\exp\{2|z|C_{K,\sigma}(y^{1-\sigma}+1)\}$$
$$+|\tilde{M}_{\sigma,P}(z)-\tilde{M}_\sigma(z)|+\frac{\log^{b+1}m}{m}\cdots\sigma\neq 1,$$

and when $\sigma=1$, $y^{1-\sigma}$ is to be replaced by $\log y$. Here, $\ll$ and $C_{K,\sigma}$ depend only on K and σ.

6.5

In order to estimate the difference

$$|\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\psi_z\left(\frac{L'(\chi,s)}{L(\chi,s)}\right)-\tilde{M}_\sigma(z)|, \tag{6.5.1}$$

we need, in addition to (6.4.15), the following lemma for which our assumption that K be a function field is more essential. (Indeed, the validity of the lemma implies the Generalized Riemann Hypothesis for $L(s,\chi)$.)

Lemma 6.5.2 *Let K be any function field over $\mathbb{F}_q$, and $P=P_y$ $(y>1)$ denote the set of prime divisors of K with $N(\wp)\le y$. Let χ be any non-principal Dirichlet character on K and $L(\chi,s)$ (resp. $L_P(\chi,s)$) be the associated L-function (resp. the partial L-function)*

$$L_P(\chi,s)=\prod_{\wp\in P}(1-\chi(\wp)N(\wp)^{-s})^{-1} \qquad (\mathrm{Re}(s)>0). \tag{6.5.3}$$

Let now $\sigma = \mathrm{Re}(s) > 1/2$. Then

$$|\frac{L'(\chi, s)}{L(\chi, s)} - \frac{L'_P(\chi, s)}{L_P(\chi, s)}| \ll (\log N(\mathbf{f}_\chi))y^{\frac{1}{2}-\sigma}. \tag{6.5.4}$$

Here, if $\epsilon > 0$ is fixed and $\sigma \geq \frac{1}{2} + \epsilon$, the implied constant depends only on K and ϵ.

This is an easy exercise, but being also a basic point in our argument, we shall sketch the proof.

Proof First note that (for $\mathrm{Re}(s) > 0$)

$$\begin{aligned} -\frac{L'_P(\chi, s)}{L_P(\chi, s)} &= -\sum_{N(\wp)\leq y} \frac{\chi(\wp)\log N(\wp)}{\chi(\wp) - N(\wp)^s} \\ &= \sum_{\substack{N(\wp)\leq y \\ k\geq 1}} \frac{\chi(\wp)^k \log N(\wp)}{N(\wp)^{ks}}. \end{aligned} \tag{6.5.5}$$

Divide the last double sum over $\wp$ and k into two parts

$$A = \sum_{N(\wp^k)\leq y}, \qquad B = \sum_{\substack{N(\wp)\leq y \\ N(\wp^k)>y}}, \tag{6.5.6}$$

so that

$$|\frac{L'(\chi, s)}{L(\chi, s)} - \frac{L'_P(\chi, s)}{L_P(\chi, s)}| \leq |\frac{L'(\chi, s)}{L(\chi, s)} + A| + |B|. \tag{6.5.7}$$

Estimation of $|\frac{L'(\chi,s)}{L(\chi,s)} + A|$

By Weil, since χ is non-principal, $L(\chi, s)$ is a polynomial of $u = q^{-s}$ of the form

$$L(\chi, s) = \prod_{\nu=1}^{D_\chi}(1 - \pi_\nu u), \tag{6.5.8}$$

where

$$D_\chi = 2g - 2 + \deg \mathbf{f}_\chi \tag{6.5.9}$$

(g: the genus), and $|\pi_\nu| = q^{1/2}$ for all ν. And since $du = -u(\log q)ds$,

$$\frac{L'(\chi, s)}{L(\chi, s)} = \sum_{\nu=1}^{D_\chi} \frac{\pi_\nu u}{1 - \pi_\nu u} \log q. \tag{6.5.10}$$

On the other hand, as a power series of u,

$$(6.5.11)\qquad \frac{L'(\chi,s)}{L(\chi,s)} = -\left(\sum_{\wp,\,k\geq 1} \chi(\wp)^k(\deg\wp)u^{k.\deg\wp}\right)\log q.$$

Therefore, if n denotes the integral part of $\log_q y$, then $L'(\chi,s)/L(\chi,s)+A$ is nothing but the "degree $> n$ -part" of (6.5.11); hence that of (6.5.10); hence is given by

$$(6.5.12)\qquad \sum_{\nu=1}^{D_\chi} \frac{(\pi_\nu u)^{n+1}}{1-\pi_\nu u}\log q.$$

But since $|\pi_\nu u| = q^{1/2-\sigma}$, we obtain

$$(6.5.13)\qquad |\frac{L'(\chi,s)}{L(\chi,s)}+A| \leq D_\chi \frac{y^{\frac{1}{2}-\sigma}}{1-q^{\frac{1}{2}-\sigma}}(\log q).$$

Estimation of $|B|$

With the notation as in (6.4.11), we have

$$(6.5.14)\qquad \begin{aligned} |B| &\leq \sum_{\substack{N(\wp)\leq y\\ N(\wp)^k>y}} \frac{\log N(\wp)}{N(\wp)^{k\sigma}} = \sum_{\substack{d\leq n\\ dk>n}} \frac{dB_d}{q^{dk\sigma}}\log q \\ &\leq \left(\sum_{\substack{d\leq n\\ dk>n}} \frac{q^d+2gq^{d/2}+1}{q^{dk\sigma}}\right)\log q \\ &\ll \sum_{d\leq n} q^d(\log q)\left(\sum_{k>n/d} q^{-dk\sigma}\right). \end{aligned}$$

And the inner sum over k is bounded by the geometric series with the ratio $q^{-d\sigma}$ starting from $q^{-n\sigma}$. But when $d \geq n/2$, we must use a better bound

$$(6.5.15)\qquad \sum_{k>n/d} q^{-dk\sigma} \leq \sum_{k\geq 2} q^{-dk\sigma}$$

to make it effecitve for our purpose. We thus obtain

$$(6.5.16)\qquad |B| \ll \frac{y^{\frac{1}{2}-\sigma}}{1-q^{1-2\sigma}}\log q < \frac{y^{\frac{1}{2}-\sigma}}{1-q^{\frac{1}{2}-\sigma}}\log q.$$

□

Corollary 6.5.17

$$(6.5.18)\qquad |\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\psi_z\left(\frac{L'(\chi,s)}{L(\chi,s)}\right)-\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)|\ll(\log m)|z|y^{\frac{1}{2}-\sigma}.$$

Proof Since the segment of a circle is shorter than the arc,

$$(6.5.19)\qquad \begin{aligned}|\psi_z(w')-\psi_z(w)|=|\exp(i\mathrm{Re}(\bar z(w'-w)))-1|&\le|\mathrm{Re}(\bar z(w'-w))|\\&\le|z||w'-w|.\end{aligned}$$

Therefore, the Corollary follows immediately from Lemma 6.5.2. □

Remark 6.5.20 Note that $(\log m)$ in (6.5.18) comes from $\log N(\mathbf{f}_\chi)$ in (6.5.4). Since, here, we average over χ, it would be possible that the former can be replaced by something smaller. For the effect of such a possible replacement, see Remark 6.6.4 (i).

6.6 Proof of Theorem 7 (ii)

We first prove (ii). By (6.4.15) and Cor 6.5.17, we have, for $y=(\log m)^b$, $b>0$, $\sigma>1/2$,

(6.6.1)

$$\begin{aligned}|\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\psi_z\left(\frac{L'(\chi,s)}{L(\chi,s)}\right)-\tilde M_\sigma(z)|\ll\frac{(\log m)(\log\log m)}{m}&\exp\{2|z|C_{K,\sigma}(y^{1-\sigma}+1)\}\\+|\tilde M_{\sigma,P_y}(z)-\tilde M_\sigma(z)|+&\frac{\log^{b+1}m}{m}\\+&(\log m)|z|y^{\frac{1}{2}-\sigma},\end{aligned}$$

where $y^{1-\sigma}$ inside the exponential braces should be replaced by $\log y$ when $\sigma=1$.

Note that if

$$(6.6.2)\qquad 1<(\sigma-\frac{1}{2})b,$$

the last term on the right hand side of (6.6.1) tends to 0 as $m\mapsto\infty$, while if

$$(6.6.3)\qquad (1-\sigma)b<1,$$

then the first term has this property (including the case $\sigma=1$). Note finally that the middle terms tend to 0 for any b (cf. Theorem 3(i)). The necessary and sufficient condition for σ to have a solution b satisfying both (6.6.2) and (6.6.3) is that either (i) $\sigma\ge1$, or (ii) $\sigma<1$ and $(\sigma-1/2)/(1-\sigma)>1$ holds, i.e., simply that $\sigma>3/4$ holds. Therefore, when $\sigma>3/4$, (6.6.1) tends to 0 as ($y=(\log m)^b$ (b being as above) and) $m\mapsto\infty$. This proves (ii).

Remarks 6.6.4 (i) If $(\log m)$ in (6.5.18) can be replaced by, say, $(\log m)^{1/2+\epsilon}$ (resp. $(\log m)^{\epsilon}$) for any $\epsilon > 0$, then (6.6.2) will be replaced by $(\sigma - 1/2)b > 1/2$ (resp. > 0); hence it would imply that Theorem 7(ii) holds for $\sigma > 2/3$ (resp. $\sigma > 1/2$).
(ii) As regards the estimation of the sum (6.4.1), the partial sum over $\| P^n \| \leq \log m$, as well as that over $\| P^n \| \geq (\log m)^{b+\epsilon}$, tend to 0 as $m \mapsto \infty$. The crucial part is the "middle" sum

$$\sum_{\log m < \| P^n \| < (\log m)^{b+\epsilon}} \varepsilon^{(m)}(n) |A_{\sigma,P}(n,z)|. \tag{6.6.5}$$

For this, even if we replace $\varepsilon^{(m)}(n)$ by $1/m$, it does not tend to 0 unless $b(1-\sigma) < 1$.

6.7 Proof of Theorem 7 (iii)

The point is to show that the analytic function

$$\mathrm{Avg}_{N(\mathbf{f}_\chi) \leq m} \psi_{z_1,z_2} \left(\frac{L'(\chi,s)}{L(\chi,s)} \right) \tag{6.7.1}$$

of $z_1, z_2 \in \mathbb{C}$ tends to $\tilde{M}_\sigma(z_1,z_2)$ *uniformly* on some neighborhood U of $(0,0)$; say, $U = \{(z_1,z_2); |z_1|, |z_2| < 1\}$. This can be done by a slight modification of Cor 5.2.18 and the above arguments. Thus, for each $a, b \geq 0$, if we write $D^{(a,b)} = \partial^{a+b}/\partial z_1^a \partial z_2^b$, then

$$\mathrm{Avg}_{N(\mathbf{f}_\chi) \leq m} D^{(a,b)} \psi_{z_1,z_2} \left(\frac{L'(\chi,s)}{L(\chi,s)} \right) \tag{6.7.2}$$

tends (uniformly) to $D^{(a,b)} \tilde{M}_\sigma(z_1,z_2)$ on U. But since

$$D^{(a,b)} \psi_{z_1,z_2}(w)|_{(z)=(0)} = (i/2)^{a+b} \bar{w}^a w^b, \tag{6.7.3}$$

$$D^{(a,b)} \tilde{M}_\sigma(z_1,z_2)|_{(z)=(0)} = (-i/2)^{a+b} \mu_\sigma^{(a,b)}, \tag{6.7.4}$$

(by Theorem 5 of §3), (iii) follows.

6.8 Proof of Theorem 7 (i)

First, since M_σ and $\tilde{M}_\sigma = M_\sigma^\wedge$ belong to $L^1 \cap L^\infty$, and since $\overline{M_\sigma(w)} = M_\sigma(w)$, $\overline{\tilde{M}_\sigma(w)} = \tilde{M}_\sigma(-w)$, we have

$$\int_{\mathbb{C}} M_\sigma(w) \Phi(w) |dw| = \int_{\mathbb{C}} \tilde{M}_\sigma(-z) \Phi^\wedge(z) |dz|. \tag{6.8.1}$$

And since

$$\Phi(w) = \int_{\mathbb{C}} \Phi^\wedge(z)\psi_{-z}(w)|dz|, \tag{6.8.2}$$

we have

$$\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \int_{\mathbb{C}} \Phi^\wedge(z)\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\psi_{-z}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right)|dz|. \tag{6.8.3}$$

Therefore,

$$\begin{aligned} &\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) - \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw| \\ &= \int_{\mathbb{C}} \Phi^\wedge(z)\left(\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\psi_{-z}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) - \tilde{M}_\sigma(-z)\right)|dz|. \end{aligned} \tag{6.8.4}$$

Since (6.6.1) tends uniformly to 0 on any domain where $|z|$ is bounded, the case (a) is settled.

When $\sigma > 5/6$, so that $2(1-\sigma) < \sigma - \frac{1}{2}$, we can choose $b > 0$ such that $2(1-\sigma) < b^{-1} < \sigma - \frac{1}{2}$. (We may assume $\sigma \le 1$; hence this also implies $1 - \sigma < b^{-1}$.) Then $a := b(1-\sigma) < \frac{1}{2}$.

Now, $\Phi(z)$ is assumed to be a standard function, i.e., the product of a polynomial of x, y and $\exp(-Q(x,y))$, where $z = x + yi$ and $Q(x,y)$ is some positive definite quadratic form (with real coefficients). This implies that $\Phi^\wedge(z)$ is also a standard function; hence

$$|\Phi^\wedge(z)| \ll \exp(-A|z|^2), \tag{6.8.5}$$

with some $A > 0$. But it is easy to see that for any constant $C > 0$,

$$\int_0^\infty \exp(-Ar^2 + 2Cr(\log m)^a)rdr \ll (\log m)^a \exp(\frac{C^2}{A}(\log m)^{2a}). \tag{6.8.6}$$

By (6.6.1), this shows that the absolute value of the right hand side of (6.8.4) is bounded by

$$\frac{(\log m)^{a+2}}{m}\exp(\frac{C_{K,\sigma}^2}{A}(\log m)^{2a}) + o\left(1\right). \tag{6.8.7}$$

But since $2a < 1$, the main term must also tend to 0 as $m \mapsto \infty$.

This completes the proof of Theorem 7. $\square$

6.9 Remarks alluded to the number field case

In §6, we have restricted our attention solely to the function field case. We note here that this restriction was necessary only for the validity of Sublemma 6.4.8 and Lemma 6.5.2.

As for Sublemma 6.4.8, if we assume GRH (the Generalized Riemann Hypothesis) for $\zeta_K(s)$ for a number field K, then the following substitute holds.

Sublemma 6.9.1 (Under GRH) *If $\sigma > 1/2$ and $y > 1$,*

$$\sum_{N(\wp)\le y} \frac{\log N(\wp)}{N(\wp)^\sigma - 1} \ll \frac{y^{1-\sigma}-1}{1-\sigma}, \tag{6.9.2}$$

where the implied constant depends only on K and σ, and the right hand side is to be replaced by $\log y$ *when $\sigma = 1$.*

As for Lemma 6.5.2, its validity would of course imply the holomorphy of $L'(\chi,s)/L(\chi,s)$ on $\mathrm{Re}(s) > 1/2$ and hence the GRH for $L(\chi,s)$ ($\chi \neq \chi_0$). It is not clear to the author whether conversely the GRH implies Lemma 6.5.2 (except when $s = 1$ [IMS]).

At any rate, the validity of Theorem 7 in the number field case (§4.1) depends "only" on that of (6.9.2) and of Lemma 6.5.2 for this case.

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