BUILDING BLOCKS OF ÉTALE ENDOMORPHISMS OF COMPLEX PROJECTIVE MANIFOLDS

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Abstract. Étale endomorphisms of complex projective manifolds are constructed from two building blocks up to isomorphism if the good minimal model conjecture is true. They are the endomorphisms of abelian varieties and the nearly étale rational endomorphisms of weak Calabi–Yau varieties.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. In this paper, we shall give a systematic study of étale endomorphisms of nonsingular projective varieties. The étaleness assumption is quite natural because every surjective endomorphism of X is étale provided that X is a nonsingular projective variety and is non-uniruled. In the study of birational classification of algebraic varieties, we usually have the following three reductions, where κ denotes the Kodaira dimension and q denotes the irregularity:

(A) Varieties of $\kappa > 0 \Rightarrow$ varieties of $\kappa = 0$, by the Iitaka fibration.

(B) Varieties of $\kappa = 0 \Rightarrow$ abelian varieties and varieties with $\kappa = q = 0$, by the Albanese map.

(C) Uniruled varieties $\Rightarrow$ non-uniruled varieties, by the maximal rationally connected fibration (cf. [8] and [34]).

We want to show that there are similar reductions in the study of étale endomorphisms of nonsingular projective varieties. Theorems A, B, and C below correspond to the reductions (A), (B), and (C), accordingly. See [50] for automorphisms of algebraic manifolds of dimension ≥ 3.

1.1. **The reduction (A).** This reduction is based on the Iitaka fibration. Let X be a nonsingular projective variety of $\kappa(X) > 0$. Then any surjective endomorphism f of X is étale. From a standard argument of pluricanonical systems, we infer that f induces an automorphism g of the base space Y of the Iitaka fibration $X \dashrightarrow Y$. In Theorem

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A, we shall show that the order of g is finite. This is conjectured in several papers (cf. [1, Proposition 6.4], [18, Proposition 3.7]). Theorem A treats not only holomorphic surjective endomorphisms of projective varieties of $\kappa > 0$ but also dominant meromorphic endomorphisms of compact complex manifolds of $\kappa > 0$ in the class $\mathcal{C}$ in the sense of Fujiki [15]. Note that a compact complex manifold is in the class $\mathcal{C}$ if and only if it is bimeromorphic to a compact Kähler manifold (cf. [46]).

Theorem A. *Let X be a compact complex manifold in the class $\mathcal{C}$ of $\kappa(X) \geq 1$ and let $f\colon X \cdots\to X$ be a dominant meromorphic map. Let W_m be the image of the m-th pluricanonical map*

$$\Phi_m\colon X \cdots\to |mK_X|^\vee = \mathbb{P}(\mathrm{H}^0(X, mK_X))$$

giving rise to the Iitaka fibration of X. Then there is an automorphism g of W_m of finite order such that $\Phi_m \circ f = g \circ \Phi_m$.

Remark.

(1) If f is holomorphic, then, resolving the indeterminacy points of Φ_m, we may assume that both $f\colon X \to X$ and $\Phi_m\colon X \to W_m$ are holomorphic so that $\Phi_m \circ f = g \circ \Phi_m$. This is because f is étale and we can take an equivariant resolution of the graph of Iitaka fibration (cf. Section 1.4 and the proof of Lemma 5.2).

(2) Theorem A is known to be true by Deligne and Nakamura–Ueno when X is Moishezon and f is a bimeromorphic automorphism (cf. [45, Theorem 14.10], [40]).

1.2. **The reduction (B).** For a compact Kähler manifold M with $c_1(M)_{\mathbb{R}} = 0$, we have a finite étale cover $\widetilde{M} \to M$ such that $\widetilde{M} \simeq T \times F$ for a complex torus T and a simply connected manifold F with $c_1(F) = 0$, by Bogomolov's decomposition theorem (cf. [6], [2]). For a normal projective variety V with only canonical singularities and with torsion K_V, we have the following weak decomposition by Kawamata [27, Corollary 8.4]:

There exists a finite étale covering $F \times A \to V$ for a weak Calabi–Yau variety F and for an abelian variety A.

Here, a normal projective variety F is called *weak Calabi–Yau* if F has only canonical singularities, $K_F \sim_{\mathbb{Q}} 0$, and

$$q^{\max}(F) := \max\{q(F') \mid F' \to F \text{ is finite étale}\} = 0$$

(cf. Section 4.1). If F is a nonsingular weak Calabi–Yau variety, then F is simply connected by Bogomolov's decomposition theorem, so F is expressed as a product of holomorphic symplectic manifolds and of Calabi–Yau manifolds.

In order to study the surjective endomorphisms of a nonsingular projective variety X of $\kappa(X) = 0$, we assume the existence of a good minimal model V of X; but we allow the variety V to have canonical singularities as in [27]. Then it has a meaning to consider the reduction of the endomorphisms to those of weak Calabi–Yau varieties F and of abelian varieties A by an étale covering $F \times A \to V$. Unfortunately, a holomorphic surjective endomorphism of X induces only a rational map $V \cdots\to V$, but it satisfies the condition of *nearly étale map*, which is introduced in Section 3. Therefore, Theorem B below is meaningful for the reduction of type (B):

Theorem B. *Let V be a normal projective variety with only canonical singularities such that $K_V \sim_{\mathbb{Q}} 0$. Let $h\colon V \cdots\to V$ be a dominant rational map which is nearly étale. Then there exist an abelian variety A, a weak Calabi–Yau variety F, a finite étale morphism $\tau\colon F \times A \to V$, a nearly étale dominant rational map $\varphi_F\colon F \cdots\to F$, and a finite étale morphism $\varphi_A\colon A \to A$ such that $\tau \circ (\varphi_F \times \varphi_A) = h \circ \tau$, i.e., the diagram below is commutative*:

$$\begin{array}{ccc} F \times A & \overset{\varphi_F \times \varphi_A}{\cdots\to} & F \times A \\ \tau \downarrow & & \downarrow \tau \\ V & \overset{h}{\cdots\to} & V. \end{array}$$

Remark.

(1) We have $\deg h = \deg \varphi_F \deg \varphi_A$. In particular, the commutative diagram above is birationally cartesian.

(2) If F has only terminal singularities, then φ_F is étale in codimension one, i.e., there are closed subsets B_1, $B_2 \subset F$ with $\operatorname{codim} B_1 \geq 2$, $\operatorname{codim} B_2 \geq 2$ such that φ_F induces a finite étale morphism $F \setminus B_1 \to F \setminus B_2$ (cf. Lemma 3.3, Remark 3.8).

(3) If the algebraic fundamental group $\pi_1^{\mathrm{alg}}(F)$ is finite, then φ_F is a birational automorphism (cf. Section 4.4). In particular, if V has only terminal singularities and $q^{\max}(V) = \dim V - 2$, then φ_F is an automorphism.

1.3. **The reduction (C).** Let X be a uniruled nonsingular projective variety. A *maximal rationally connected fibration* of X in the sense of [8] and [34] is obtained by a certain rational map $X \cdots\to \operatorname{Chow}(X)$ into the Chow variety $\operatorname{Chow}(X)$, which assigns a general point $x \in X$ a maximal rationally connected subvariety containing x. Let Y be the normalization of the image of $X \cdots\to \operatorname{Chow}(X)$ and let $\pi\colon X \cdots\to Y$ be the induced rational fibration. Assume that X admits an étale endomorphism $f\colon X \to X$. Then we have a functorial morphism $f_*\colon \operatorname{Chow}(X) \to \operatorname{Chow}(X)$. Thus, it induces an endomorphism $h\colon Y \to Y$ such that $\pi \circ f = h \circ \pi$. Since rationally connected manifolds are simply connected, the endomorphism f is induced from h. In Theorem C below, we shall show that h is nearly étale (cf. Section 3).

Theorem C. *Let X be a projective complex manifold with an étale endomorphism f. Assume that X is uniruled. Then there exist a projective manifold M with an étale endomorphism $f_M\colon M \to M$, a non-uniruled normal projective variety Y with a nearly étale endomorphism $h\colon Y \to Y$, a birational morphism $\mu\colon M \to X$, and a surjective morphism $\pi\colon M \to Y$ such that*

(1) $\mu \circ f_M = f \circ \mu$, $\pi \circ f_M = h \circ \pi$,
(2) $\deg f = \deg f_M = \deg h$,
(3) $\pi \circ \mu^{-1}\colon X \dashrightarrow M \to Y$ *is birational to the maximal rationally connected fibration of X.*

Remark.

(1) To distinguish well, we denote $f_M\colon M_1 = M \to M_2 = M$ and $h\colon Y_1 = Y \to Y_2 = Y$. Then M_1 is isomorphic to the normalization of $M_2 \times_{Y_2} Y_1$ so that $f_M\colon M_1 \to M_2$ can be regarded as the natural projection (cf. Remark 5.5).
(2) Let us denote by $Y_{\mathrm{rat}} \subset Y$ the open subset consisting of the smooth points and the points of rational singularity. Then $h^{-1}(Y_{\mathrm{rat}}) = Y_{\mathrm{rat}}$ and the restriction $Y_{\mathrm{rat}} \to Y_{\mathrm{rat}}$ of h is étale, by Proposition 3.12.
(3) If Y has the relative canonical model Y_{can} for resolutions of singularities of Y, then h lifts to an étale endomorphism of Y_{can} and also lifts to an étale endomorphism of a certain resolution Y' of singularities of Y, by Lemma 3.9. The recent paper [3] has announced a proof of the existence of minimal models of varieties of general type even in a relative setting. The existence of our relative canonical model Y_{can} follows from the result.

1.4. **Equivariant resolutions.** Let V be a normal projective complex variety and let $f\colon V \to V$ be an étale endomorphism. Then there exists a resolution of singularities $\mu\colon X \to V$ such that the induced rational map

$$X \xrightarrow{\mu} V \xrightarrow{f} V \overset{\mu^{-1}}{\dashrightarrow} X$$

is a holomorphic étale endomorphism of X. This is known as the existence theorem of an equivariant resolution when f is an automorphism. However, the proof is also effective for non-isomorphic étale endomorphisms:

A method of resolution of singularities is called to have a functoriality if, for any smooth morphism $X \to Y$, and for the resolutions of singularities $X' \to X$ and $Y' \to Y$ given by the method, X' is isomorphic to the fiber product $X \times_Y Y'$. The recent methods by Bierstone–Milman and by Villamayor using *invariants* have the functoriality (cf. [4], [11], [12], [48], [33]).

Therefore, we call the resolution $X \to V$ above also an *equivariant resolution* even if f is a non-isomorphic endomorphism of V.

1.5. **The meaning of our reduction.** Let X be a nonsingular projective variety with an étale endomorphism f.

First, assume that X is uniruled. In view of Theorem C, f is considered to be built from a nearly étale endomorphism h of a non-uniruled normal variety Y up to isomorphism. Moreover, we can replace Y to be a nonsingular variety and h to be an étale endomorphism, provided that the minimal model conjecture is true for varieties birational to Y.

It is conjectured that a variety X is not uniruled if and only if $\kappa(X) \geq 0$. This is regarded as a weak version of the abundance conjecture, and the three-dimensional case is proved by Miyaoka [37], which is a key to the proof of the three-dimensional abundance conjecture by [38] and [28].

The good minimal model conjecture is the combination of the minimal model conjecture and the abundance conjecture. Thus, under the assumption of good minimal model conjecture, the study of étale endomorphisms is reduced to that of étale endomorphisms of varieties of $\kappa \geq 0$.

Second, assume that $\kappa(X) > 0$. Then we have the Iitaka fibration $\Phi\colon X \dashrightarrow Y$, and f induces an automorphism g of Y of finite order by Theorem A. By replacing X with a birational model, we may assume that Φ is holomorphic as in the remark mentioned just after Theorem A. By iterating f, we may assume f to be a morphism over Y. Then f induces an étale endomorphism of a general fiber F of Φ. This is a kind of reduction of f to an endomorphism of varieties of $\kappa = 0$. In fact, from the viewpoint of complex dynamics, several results are known on the topological entropies and the dynamical degrees of f and $f|_F$. In Appendix A, we shall show that $d_1(f) = d_1(f|_F)$ for the first dynamical degree d_1 (cf. Theorem A.10), and that $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(f|_F)$ for the topological entropies h_{top} (cf. Theorem D).

However, even if we know the endomorphisms of fibers very well, it is rather difficult to determine the structure of f, as in the papers [17] and [18], which have determined the structure of endomorphisms of 3-dimensional projective manifolds of $\kappa \geq 0$.

Third, assume that $\kappa(X) = 0$. As in Section 1.2, an étale endomorphism f of X induces a nearly étale rational endomorphism of a weak Calabi–Yau variety F and an endomorphism of an abelian variety A, provided that the good minimal model conjecture is true. However, it is not clear that the nearly étale endomorphism induces an étale endomorphism of a certain resolution of singularities of F. Further, for the converse direction, it seems to remain the problem on recovering the original endomorphism f from the two endomorphisms of F and A (cf. [17] for the 3-dimensional case).

Therefore, we can conclude, under the assumption of good minimal model conjecture, that the nearly étale endomorphisms of weak Calabi–Yau varieties and the endomorphisms of abelian varieties are the building blocks of all the étale endomorphisms of projective manifolds.

For non-étale surjective endomorphisms (necessarily on uniruled manifolds), on the one hand, we know many examples of rationally connected varieties admitting non-isomorphic surjective endomorphisms, such as the projective space, toric varieties, etc. (cf. [42]). On the other hand, at the moment, we have no good idea to consider the building blocks of all the endomorphisms on them.

Construction of the paper. Section 2 is devoted to proving Theorem A and to the application to the pluricanonical representation of the bimeromorphic automorphism group (cf. Corollary 2.4). In Section 3, we introduce a key notion of nearly étale maps and study some elementary properties. Sections 4 and 5 are devoted to Theorem B and to Theorem C, respectively. In Appendix A, we shall prove the equalities on the first dynamical degrees and on the topological entropies mentioned above.

Notation and terminology. We refer to [29] for the definitions of minimal models, and singularities including terminal, canonical, log terminal, and rational singularities. Also, we refer to [32], [35], [39] for additional information of the birational geometry and the minimal model theory.

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2. The case of positive Kodaira dimension

2.1. **Iitaka fibration.** In the situation of Theorem A, we may assume that X is a compact Kähler manifold without loss of generality. We have a natural isomorphism $f^*\colon \mathrm{H}^0(X, mK_X) \xrightarrow{\simeq} \mathrm{H}^0(X, mK_X)$. In fact, there is a bimeromorphic morphism $\mu\colon Z \to X$ from another compact Kähler manifold Z such that $\varphi := f \circ \mu\colon Z \to X$ is holomorphic; then f^* is defined to be

$$\mathrm{H}^0(X, mK_X) \xrightarrow{\varphi^*} \mathrm{H}^0(Z, mK_Z) \underset{\simeq}{\xleftarrow{\mu^*}} \mathrm{H}^0(X, mK_X),$$

and it does not depend on the choice of $\mu\colon Z \to X$. The homomorphism f^* induces an automorphism g of $|mK_X|^\vee$ preserving W_m. The problem is to show the finiteness of the order of $g \in \mathrm{Aut}(W_m)$.

We begin with the following simple result.

Lemma 2.1. *If* Theorem A *does not hold, then there is a positive-dimensional connected commutative algebraic subgroup $G \subset \mathrm{Aut}(W_m)$ such that $g^k \in G$ for some $k > 0$. In particular, W_m is ruled in this case.*

Proof. Let $\widetilde{G} \subset \mathrm{PGL} = \mathrm{Aut}(|mK_X|^\vee)$ be the Zariski closure of the cyclic group $I = \{g^j \mid j \in \mathbb{Z}\}$. Then $\widetilde{G}$ is abelian, since it is contained in the abelian algebraic subgroup $Z(I) \cap Z(Z(I))$, where $Z(S)$ denotes the algebraic subgroup

$$\{\gamma \in \mathrm{PGL} \mid \gamma s = s\gamma \text{ for any } s \in S\}$$

for a subset $S \subset \mathrm{PGL}$. Let G be the identity connected component of $\widetilde{G}$. Then $\dim G > 0$ and $g^k \in G$ for some $k > 0$. Since the action of G preserves W_m, G acts non-trivially on W_m, which implies that W_m is ruled by a result of Matsumura. □

Remark 2.2. There is another proof of Lemma 2.1 by an argument similar to [45, Proposition 14.7]: In fact, we can show that f^* is expressed as a diagonal matrix. Thus, G is contained in an algebraic torus. We can prove more on f^* by the argument of [45, Proposition 14.1]: If λ be an eigenvalue of $f^*\colon \mathrm{H}^0(X, mK_X) \to \mathrm{H}^0(X, mK_X)$, then λ is an algebraic integer with $|\lambda|^{2/m} = \deg f$.

The following is a key to the proof of Theorem A.

Proposition 2.3. *Let $\pi\colon X \to Y$ be a fiber space from a compact Kähler manifold X into a nonsingular rational curve $Y \simeq \mathbb{P}^1$. Let $f\colon X \dashrightarrow X$ be a dominant meromorphic map and $g\colon Y \xrightarrow{\simeq} Y$ an automorphism such that $\pi \circ f = g \circ \pi$. If $\kappa(X) \geq 0$, then g is of finite order.*

Proof. Step 1. *We first prove in the case where* $p_g(X) = \dim \mathrm{H}^0(X, K_X) > 0$: Since π is smooth over a dense open subset U of Y, we have a variation of Hodge structure $H_U = \mathrm{R}^d\, \pi_* \mathbb{Z}_X|_U$ for $d = \dim X/Y = \dim X - 1$.

Let $\mu\colon Z \to X$ be a bimeromorphic morphism from another compact Kähler manifold Z such that $\varphi := f \circ \mu\colon Z \to X$ is holomorphic. Then

$$f^*_\omega\colon g^*(\pi_*\omega_{X/Y}) \xrightarrow{\varphi^*} g^*(\pi_*\varphi_*\omega_{Z/Y}) \underset{\simeq}{\xleftarrow{\mu^*}} \pi_*\omega_{X/Y}$$

is injective. Note that $\pi_*\omega_{X/Y}$ is just the d-th filter $\mathcal{F}^d({}^u\mathcal{H})$ of the upper canonical extension ${}^u\mathcal{H}$ of $H_U \otimes \mathcal{O}_U$ in the sense of Kollár [30].

We have also the pullback homomorphism

$$f^*_{\mathrm{coh}} \colon g^{-1}(\mathrm{R}^d \pi_* \mathbb{Z}_X) \xrightarrow{\varphi^*} g^{-1} \mathrm{R}^d (\pi \circ \varphi)_* \mathbb{Z}_Z \xrightarrow{\mu_*} \mathrm{R}^d \pi_* \mathbb{Z}_X,$$

where μ_* is induced from the trace map $\mathrm{R}\,\mu_* \mathbb{Z}_Z[2n] \to \mathbb{Z}_X[2n]$ for $n = \dim X$. Note that f^*_{coh} is compatible with f^*_ω, i.e., the diagram

$$\begin{array}{ccc} (g^{-1}\,\mathrm{R}^d \pi_* \mathbb{Z}_X)|_{U'} \otimes \mathcal{O}_{U'} & \xrightarrow{f^*_{\mathrm{coh}}} & (\mathrm{R}^d \pi_* \mathbb{Z}_X)|_{U'} \otimes \mathcal{O}_{U'} \\ \Big\uparrow & & \Big\uparrow \\ (g^* \pi_* \omega_{X/Y})|_{U'} & \xrightarrow{f^*_\omega} & \pi_* \omega_{X/Y}|_{U'} \end{array}$$

is commutative over the open subset $U' = U \cap g^{-1} U$. Let $J_k \subset \mathrm{R}^d \pi_* \mathbb{Z}_X$ be the image of

$$(f^*_{\mathrm{coh}})^k \colon (g^k)^{-1}(\mathrm{R}^d \pi_* \mathbb{Z}_X) \xrightarrow{(g^{k-1})^{-1} f^*_{\mathrm{coh}}} (g^{k-1})^{-1}(\mathrm{R}^d \pi_* \mathbb{Z}_X) \to \cdots$$
$$\cdots \to g^{-1}(\mathrm{R}^d \pi_* \mathbb{Z}_X) \xrightarrow{f^*_{\mathrm{coh}}} \mathrm{R}^d \pi_* \mathbb{Z}_X$$

for $k > 0$. Then $J_k \supset J_{k+1}$ and $f^*_{\mathrm{coh}}(g^{-1} J_k) = J_{k+1}$. Thus $J_k = J_{k+1} = \cdots$ for some $k > 0$, since any stalk of $\mathrm{R}^d \pi_* \mathbb{Z}_X$ is a finitely generated abelian group. We set $J := J_k$ for $k \gg 0$. Then, for a non-empty Zariski open subset $U'' \subset U$, $J|_{U''}$ defines a variation of Hodge substructure of $\mathrm{R}^d \pi_* \mathbb{Z}_X|_{U''}$, and $J|_{U''} \otimes \mathcal{O}_{U''}$ contains $\pi_* \omega_{X/Y}|_{U''}$ as the d-th Hodge filter. Furthermore, there is an isomorphism $g^{-1} J \simeq J$ compatible with f^*_ω. Let $U_{\max}$ be the maximum open subset of Y such that there is a variation of Hodge structure $J_{\max}$ on $U_{\max}$ with $J_{\max}|_{U''} \simeq J|_{U''}$. Then $g^{-1} U_{\max} = U_{\max}$. Thus, we may assume that $Y \setminus U_{\max}$ consists of at most two points; otherwise, g is of finite order. Note that a Kähler form of X defines a real polarization of the variation of Hodge structure $J_{\max}$. If $U_{\max} = Y \simeq \mathbb{P}^1$ or if $U_{\max} \simeq \mathbb{C}$, then $J_{\max}$ is a trivial variation of Hodge structure, and hence $\pi_* \omega_{X/Y}$ is a free $\mathcal{O}_Y$-module. Thus, we have a contradiction: $\mathrm{H}^0(X, K_X) \simeq \mathrm{H}^0(Y, \pi_* \omega_{X/Y} \otimes \omega_Y) = 0$. Hence, we may assume $U_{\max} = \mathbb{C}^\star$. Then, the period map associated with $J_{\max}$ is constant, since the universal covering space of $U_{\max}$ is $\mathbb{C}$. In particular, the image of the monodromy representation $\pi_1(U_{\max}, y) \to \mathrm{Aut}(J_{\max,y})$ is finite. Let $\tau \colon Y' \simeq \mathbb{P}^1 \to Y$ be a cyclic covering étale over $U_{\max}$ such that $\tau^{-1} J_{\max}$ extends to a trivial constant sheaf of Y'. We may assume that g lifts to an automorphism g' of Y'. Let $X' \to X \times_Y Y'$ be a resolution of singularities and let $\pi' \colon X' \to Y'$ be the induced morphism. Then $f \times g'$ induces a meromorphic endomorphism of X' and $p_g(X') > 0$. Since $\tau^{-1} J_{\max}$ has trivial monodromy, a similar variation of Hodge structure $J'_{\max}$ is defined on Y'. Thus $\pi'_* \omega_{X'/Y'}$ is a free $\mathcal{O}_{Y'}$-module, and we have the same contradiction as above.

Step 2. *General case*: Let $s \in \mathrm{H}^0(X, mK_X)$ be an eigenvector of f^*. We shall consider a cyclic covering corresponding to taking the m-th root of s. Let $\mathcal{A} = \bigoplus_{i=0}^{m-1} \mathcal{O}_X(-iK_X)$ be the $\mathcal{O}_X$-algebra determined by $s \colon \mathcal{O}_X(-mK_X) \to \mathcal{O}_X$ and let $\widehat{X} \to \mathit{Spec}_X\, \mathcal{A}$ be a resolution of singularities. Then, for the composite $\tau\colon \widehat{X} \to \mathit{Spec}_X \mathcal{A} \to X$, $\tau^* s \in \mathrm{H}^0(\widehat{X}, mK_{\widehat{X}})$ is expressed as σ^m for a section $\sigma \in \mathrm{H}^0(\widehat{X}, K_{\widehat{X}})$. Let X' be a connected component of $\widehat{X}$. Then $\kappa(X') = \kappa(X)$ and $p_g(X') > 0$. Let $\pi'\colon X' \to Y'$ be the fiber space and let $\lambda\colon Y' \to Y$ be the finite morphism obtained as the Stein factorization of $X' \to X \to Y$.

Since s is an eigenvalue of f, we have a meromorphic map $\hat{f}\colon \widehat{X} \cdots\to \widehat{X}$ such that $\tau \circ \hat{f} = f \circ \tau$. Replacing f with a suitable power f^k, we may assume that $\hat{f}$ preserves X'. Let $f'\colon X' \cdots\to X'$ be the induced rational map. Then there is an automorphism g' of Y' such that

$$\pi' \circ f' = g' \circ \pi' \quad \text{and} \quad \lambda \circ g' = g \circ \lambda.$$

If g' is of finite order, then so is g. If the genus of Y' is greater than one, then g' is of finite order. Even if the genus of Y' is one, g' is of finite order since g' preserves the ample invertible sheaf $\lambda^*\mathcal{O}(1)$. If the genus of Y' is zero, then g' is of finite order by *Step* 1. Thus, we are done. □

Remark. If f is a holomorphic endomorphism of X and if X is projective, then Proposition 2.3 follows from [47], since π has at least three singular fibers preserved by g.

2.2. **Proof of Theorem A.** We may assume that the order of g is infinite and that $g \in G$ for the connected commutative algebraic group $G \subset \mathrm{Aut}(W_m)$ in Lemma 2.1. Let $Y \to W_m$ be an equivariant resolution of singularities so that G acts on Y holomorphically. There is a sequence

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_l = G$$

of algebraic subgroups such that $\dim G_i/G_{i-1} = 1$ for $1 \le i \le l$. In particular, $G_i/G_{i-1} \simeq \mathbb{G}_\mathtt{m}$ or $\mathbb{G}_\mathtt{a}$. Let $Y \cdots\to Y_1 \subset \mathrm{Hilb}(Y)$ be the meromorphic quotient of Y by G_1 (cf. [16, Theorem 4.1]). Then G/G_1 acts on Y_1. Replacing Y and Y_1 by their non-singular models, we may assume that $Y \to Y_1$ is holomorphic and G-equivariant. Let $Y_1 \cdots\to Y_2 \subset \mathrm{Hilb}(Y_1)$ be the meromorphic quotient of Y_1 by G_2/G_1, and replace Y, Y_1, and Y_2 by their non-singular models so that $Y \to Y_1$ and $Y_1 \to Y_2$ are G-equivariant morphisms. Continuing similar constructions, we have a sequence of G-equivariant morphisms

$$Y = Y_0 \to Y_1 \to \cdots \to Y_l$$

such that $Y \to Y_i$ is birational to the meromorphic quotient by G_i. Then, for $0 \le i < l$, a general fiber of $Y_i \to Y_{i+1}$ is a smooth rational curve which is the closure of an orbit of G_{i+1}/G_i. There is a similar and stronger assertion in [36, Theorem 4.6].

Let us consider the composition $\phi_i\colon X \to Y \to Y_i$ for $1 \le i \le l$. For a very general point $y_i \in Y_i$, let $F_i \subset X$ be the fiber $\phi_i^{-1}(y_i)$ and let $C_{i-1} \subset Y_{i-1}$ be the fiber of $Y_{i-1} \to Y_i$

over y_i. Then $\kappa(F_i) > 0$ for $i > 0$ by the easy addition formula

$$\kappa(X) = \dim Y \le \kappa(F_i) + \dim Y_i.$$

We may assume that g acts trivially on Y_l. Then $f\colon X \cdots\to X$ is a meromorphic map over Y_l. Thus a dominant meromorphic map $F_l \cdots\to F_l$ is induced. By Proposition 2.3, the action of g on C_{l-1} is of finite order. Thus g^k acts on Y_{l-1} trivially for some $k > 0$. Hence, $\{g^{kj} \mid j \in \mathbb{Z}\} \subset G_{l-1}$. This contradicts that G is the identity component of the Zariski closure of $\{g^j \mid j \in \mathbb{Z}\}$. This completes the proof of Theorem A. □

Theorem A has an application to the pluricanonical representations of the bimeromorphic automorphism group $\mathrm{Bim}(X)$ of compact complex manifolds X in the class $\mathcal{C}$. The following result is proved in [45, §14] (cf. [40]) when X is a Moishezon manifold.

Corollary 2.4. *Let X be a compact complex manifold in the class $\mathcal{C}$ and let*

$$\rho_m\colon\ \mathrm{Bim}(X) \to \mathrm{Aut}(\mathrm{H}^0(X, mK_X))$$

be the m-th pluricanonical representation (cf. [45, §14]) *for a positive integer m with* $\mathrm{H}^0(X, mK_X) \ne 0$. *Then the image of the induced group homomorphism below is finite*:

$$\rho'_m\colon\ \mathrm{Bim}(X) \to \mathrm{PGL}(\mathrm{H}^0(X, mK_X)) = \mathrm{Aut}(\mathrm{H}^0(X, mK_X))/\mathbb{C}^\star.$$

Proof. Let W_m be the image of m-th pluricanonical map $X \cdots\to |mK_X|^\vee$. Then W_m is not contained in any hyperplane of $|mK_X|^\vee$. Thus, for $\gamma \in \mathrm{Bim}(X)$, the order of $\rho'_m(\gamma)$ equals the order of the action of γ on W_m.

For $l > 0$, we have a natural rational map $\Psi_{m,ml}\colon W_{ml} \cdots\to W_m$ such that $\Phi_m = \Psi_{m,ml} \circ \Phi_{ml}$. In fact, $\Psi_{m,ml}$ is defined by the commutative diagram

$$\begin{array}{ccccc} W_{ml} & \hookrightarrow & |mlK_X| & = & \mathbb{P}(\mathrm{H}^0(X, mlK_X)) \\ {\scriptstyle \Psi_{m,ml}}\ \vdots\downarrow & & & & \vdots\downarrow\ {\scriptstyle \mu} \\ W_m & \hookrightarrow & |mK_X| & \overset{\iota}{\hookrightarrow} & \mathbb{P}(\mathrm{Sym}^l\, \mathrm{H}^0(X, mK_X)), \end{array}$$

where ι is the Veronese embedding and μ is induced from the natural homomorphism $\mathrm{Sym}^l\, \mathrm{H}^0(X, mK_X) \to \mathrm{H}^0(X, mlK_X)$. Since $\Psi_{m,ml}$ is compatible with the actions of $\mathrm{Bim}(X)$ on W_m and W_{ml}, we may replace m with any multiple ml. Therefore, we may assume that $\Phi_m\colon X \cdots\to W_m$ gives rise to the Iitaka fibration of X.

Then, the order of $\rho'_m(\gamma)$ is finite for any $\gamma \in \mathrm{Bim}(X)$, by Theorem A. In particular, $\rho_m(\gamma)$ is expressed as a diagonal matrix and its eigenvalues are $(\alpha\theta_1, \alpha\theta_2, \dots, \alpha\theta_k)$ for $k = \dim \mathrm{H}^0(X, mK_X)$, for a constant $\alpha \in \mathbb{C}^\star$, and for a root θ_i of unity, where $\theta_1 = 1$. By [45, Proposition 14.1] and by the same argument as in [45, 14.10], $\alpha\theta_i$ are algebraic integers, $|\alpha| = 1$, and the degree $[\mathbb{Q}(\alpha\theta_i) : \mathbb{Q}]$ of field extension $\mathbb{Q}(\alpha\theta_i)/\mathbb{Q}$ is bounded above by a suitable constant N which depends neither on i nor on γ. Thus the degree $[\mathbb{Q}(\theta_i) : \mathbb{Q}]$ is bounded above by N^2 for any i. Hence, the order of $\rho'_m(\gamma)$ is uniformly bounded. So the image of ρ'_m is a finite group by a theorem of Burnside (cf. [45, Theorem 14.9]). □

3. Nearly étale maps

In this section, we introduce the notion of nearly étale map and study basic properties.

Definition 3.1 (cf. [25])**.** Let $h\colon V \cdots\to W$ be a rational (resp. meromorphic) map between algebraic (resp. complex analytic) varieties. The map h is called *proper* if the projections $p_1\colon \Gamma_h \to X$ and $p_2\colon \Gamma_h \to Y$ are both proper for the graph $\Gamma_h \subset V \times W$. For algebraic varieties V and W, V is called proper birational to W if there exists a proper birational map $V \cdots\to W$.

Remark. The first projection $p_1\colon \Gamma_h \to X$ is proper for a meromorphic map h. In particular, a bimeromorphic map is always proper.

Definition 3.2. Let $h\colon V \cdots\to W$ be a proper rational (resp. meromorphic) map between algebraic (resp. complex analytic) varieties. The map h is called *nearly étale* if there exist proper birational (resp. bimeromorphic) maps $\mu\colon Y \cdots\to W$, $\nu\colon X \cdots\to V$ and a morphism $f\colon X \to Y$ such that

(1) X and Y are algebraic (resp. complex analytic) varieties,
(2) f is a finite étale morphism, and
(3) $\mu \circ f = h \circ \nu$, i.e.,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\nu"'] & Y \arrow[d, dashed, "\mu"] \\ V \arrow[r, dashed, "h"] & W \end{tikzcd}$$

is commutative.

Remark.

- A nearly étale map is dominant and generically finite.
- A proper birational (resp. bimeromorphic) map is nearly étale.
- A finite étale morphism is nearly étale.
- Any nearly étale rational (resp. meromorphic) map $V \cdots\to W$ is the composition of a proper birational (resp. bimeromorphic) map $V \cdots\to W^\sharp$ and a nearly étale finite morphism $W^\sharp \to W$, where $W^\sharp \to W$ is regarded as the Stein factorization of $V \cdots\to W$; in other words, $W^\sharp$ is the normalization of W in the function field $\mathbb{C}(V) \supset \mathbb{C}(W)$ when V is a normal algebraic variety.

For the sake of simplicity, we shall study basic properties of nearly étale maps only between algebraic varieties.

The following is a close relation between étale morphisms and nearly étale maps.

Lemma 3.3. *Let $h\colon V \cdots\to W$ be a nearly étale rational map over a normal algebraic variety W. Suppose that $\pi_1^{\mathrm{alg}}(W) \simeq \pi_1^{\mathrm{alg}}(Y)$ for a resolution of singularities $Y \to W$. Let $W^\sharp \to W$ be the normalization of W in $\mathbb{C}(V)$. Then $W^\sharp \to W$ is étale.*

Proof. Since $\pi_1^{\mathrm{alg}}(Y)$ is a proper birational invariant, we may replace Y with a nonsingular variety proper birational to W. Thus, by Definition 3.2, we may assume that there exists a finite étale morphism $X \to Y$ from a nonsingular variety X proper birational to V. Then X is just the normalization of Y in $\mathbb{C}(V)$. Let $\widetilde{W} \to W$ be the finite étale covering such that the finite-index subgroup $\pi_1(\widetilde{W})$ of $\pi_1(W)$ is isomorphic to the subgroup $\pi_1(X) \subset \pi_1(Y)$. Then $\widetilde{W}$ is proper birational to X, and thus $W^\sharp = \widetilde{W}$. □

In particular, we can show that the composition of two nearly étale maps are also nearly étale. The next shows that a nearly étale finite morphism is turned to be an étale morphism by a suitable base change.

Corollary 3.4. *Let $V \to W$ be a nearly étale finite morphism between normal varieties. Let $Z \to W$ be a morphism from a normal variety such that $\pi_1^{\mathrm{alg}}(Z) \simeq \pi_1^{\mathrm{alg}}(M)$ for a resolution of singularities $M \to Z$. Let $Z^\sharp$ be the normalization of $V \times_W Z$. Then $Z^\sharp \to Z$ is étale.*

Proof. For any connected component $Z^\sharp_\lambda$ of $Z^\sharp$, the finite morphism $Z^\sharp_\lambda \to Z$ is nearly étale. Thus it is étale by Lemma 3.3. □

Definition 3.5 (cf. [31, (7.1.2)], [41, Section 2])**.** Let (V, P) be a germ of normal singularity. Let $\mu\colon Z \to V$ be a resolution of singularity.

(1) (V, P) is called an *algebraically π_1-rational singularity* if the algebraic fundamental group of $\mu^{-1}(P)$ is trivial.

(2) (V, P) is called a *π_1-rational singularity* if the fundamental group of $\mu^{-1}(P)$ is trivial.

Remark 3.6.

- If (V, P) is an algebraically π_1-rational singularity, then

$$(\mathrm{R}_1\, \mu_* \mathbb{Z}_Z)_P \simeq \mathrm{H}_1(\mu^{-1}(P), \mathbb{Z}) = 0, \quad \text{and} \quad (\mathrm{R}^1\, \mu_* \mathbb{Z}_Z)_P \simeq \mathrm{H}^1(\mu^{-1}(P), \mathbb{Z}) = 0.$$

- If (V, Δ) is log-terminal (klt) at the point P for a $\mathbb{Q}$-divisor Δ with $\llcorner \Delta \lrcorner = 0$, then (V, P) is an algebraically π_1-rational singularity by [31, Theorem 7.5], and is in fact a π_1-rational singularity by [44].

Definition 3.7. Let V be a normal variety.

(1) V_{reg} denotes the nonsingular locus of V.
(2) V_{rat} denotes the set of points $P \in V$ such that (V, P) is nonsingular or is a rational singularity.
(3) V_{apr} denotes the set of points $P \in V$ such that (V, P) is an algebraically π_1-rational singularity.

Remark 3.8.

- V_{reg} and V_{rat} are open subset of V, but V_{apr} is not necessarily open.
- If $V = V_{\text{apr}}$, then $\pi_1^{\text{alg}}(V) \simeq \pi_1^{\text{alg}}(Y)$ for any resolution of singularities $Y \to V$. In particular, this holds if (V, Δ) is log-terminal for a $\mathbb{Q}$-divisor Δ with $\llcorner \Delta \lrcorner = 0$ (cf. Remark 3.6, [31, Theorem 7.8]).

The result below says that a nearly étale endomorphism induces an étale endomorphism of a certain nonsingular model, provided that the minimal model conjecture is true.

Lemma 3.9. *Let $h\colon V \to V$ be a nearly étale finite morphism for a normal variety V. Assume that there exists the relative canonical model V_{can} for resolutions of singularities of V; equivalently, the relative canonical ring*

$$\mathcal{R}_V := \bigoplus\nolimits_{m \geq 0} \mu_* \mathcal{O}_M(mK_M)$$

is a finitely generated $\mathcal{O}_V$-algebra for a resolution of singularities $\mu\colon M \to V$, where $V_{\text{can}} = \mathrm{Proj}_V \mathcal{R}_V$. Then h induces an étale endomorphism of a certain nonsingular variety proper birational to V.

Proof. Since V_{can} has only π_1-rational singularities, by Corollary 3.4, h induces a finite étale morphism $h_{\text{can}}\colon V_{\text{can}}^\sharp \to V_{\text{can}}$ together with a commutative diagram

$$\begin{array}{ccc} V_{\text{can}}^\sharp & \longrightarrow & V \\ {\scriptstyle h_{\text{can}}}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ V_{\text{can}} & \longrightarrow & V, \end{array}$$

which is birationally cartesian. Then $V_{\text{can}}^\sharp$ is also the relative canonical model, since it has only canonical singularities and its canonical divisor is also relatively ample over V. Thus, $V_{\text{can}}^\sharp \simeq V_{\text{can}}$ and h_{can} is regarded as an étale endomorphism of V_{can}. Hence, it is enough to take an equivariant resolution of singularities of V_{can} with respect to h_{can} (cf. Section 1.4). □

The result below gives a kind of descent of finite étale morphisms. It is used in the proof of Proposition 3.11 and Lemma 5.3 below.

Lemma 3.10 (cf. [31, Theorem 5.2]). *For a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\psi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ V & \xrightarrow{h} & W \end{array}$$

of proper surjective morphisms of normal complex analytic varieties, suppose that the following conditions are satisfied:

(1) X *and* Y *are nonsingular.*

(2) ψ *and* π *have connected fibers.*

(3) $X \to V \times_W Y$ *is an isomorphism over a dense open subset of* V.

(4) f *is a finite étale morphism, and* h *is a finite morphism.*

(5) $\mathrm{R}^i\,\pi_*\mathcal{O}_Y = 0$ *and* $\mathrm{R}^i\,\psi_*\mathcal{O}_X = 0$ *for any* $i > 0$.

Then, $h\colon V \to W$ *is étale.*

Proof. We may assume that $\deg f = \deg h > 1$. Let $P \in W$ be an arbitrary point. It is enough to show that the cardinality $\sharp h^{-1}(P)$ of $h^{-1}(P)$ equals $\deg h$. For the proof, we may replace Y by a bimeromorphic morphism $Y' \to Y$ from a nonsingular variety Y'. In fact, the pullback $f'\colon X' \to Y'$ of $f\colon X \to Y$ by $Y' \to Y$ induces a similar commutative diagram satisfying the same conditions. Thus, we may assume that the reduced structure $E = \pi^{-1}(P)_{\mathrm{red}}$ of the fiber $\pi^{-1}(P) = Y \times_W \{P\}$ is a normal crossing divisor. Then $f^{-1}E$ is also a reduced normal crossing divisor of X. Here, $f^{-1}E$ is the disjoint union of the reduced structures $\widetilde{E}_Q$ of the fibers $\psi^{-1}(Q)$ for points $Q \in h^{-1}(P)$.

We shall show $\chi(E, \mathcal{O}_E) = 1$. In the natural commutative diagram

$$\begin{array}{ccc} (\mathrm{R}^i\,\pi_*\mathbb{C}_Y)_P & \xrightarrow{\simeq} & \mathrm{H}^i(E, \mathbb{C}) \\ \big\downarrow & & \big\downarrow \\ (\mathrm{R}^i\,\pi_*\mathcal{O}_Y)_P & \longrightarrow & \mathrm{H}^i(E, \mathcal{O}_E), \end{array}$$

the right vertical arrow is surjective by the theory of mixed Hodge structures on normal crossing varieties. Therefore, $\mathrm{H}^i(E, \mathcal{O}_E) = 0$ for $i > 0$ by the condition on the vanishing of $\mathrm{R}^i\,\pi_*\mathcal{O}_Y$. In particular, $\chi(E, \mathcal{O}_E) = 1$. By the same argument, we also have $\chi(\widetilde{E}_Q, \mathcal{O}) = 1$. On the other hand, $\chi(f^{-1}E, \mathcal{O}) = (\deg f)\chi(E, \mathcal{O}_E) = \deg f$, since f is étale. Therefore,

$$\sharp h^{-1}(P) = \sum \chi(\widetilde{E}_Q, \mathcal{O}) = \chi(f^{-1}E, \mathcal{O}) = \deg f = \deg h. \qquad \square$$

Proposition 3.11. *Let* $h\colon V \to W$ *be a nearly étale finite morphism between normal algebraic varieties. Then*:

(1) $h^{-1}(W_{\mathrm{apr}}) \subset V_{\mathrm{apr}}$ *and* h *is étale along* $h^{-1}(W_{\mathrm{apr}})$.

(2) h *is étale along* V_{rat}.

(3) $h^{-1}(W_{\mathrm{apr}} \cap W_{\mathrm{rat}}) = V_{\mathrm{apr}} \cap V_{\mathrm{rat}}$.
(4) $h^{-1}(W_{\mathrm{reg}}) = V_{\mathrm{reg}}$.

Proof. (1) follows from Lemma 3.3 or Corollary 3.4. Moreover, (3) and (4) are derived from (1) and (2), since $h(V_{\mathrm{rat}}) \subset W_{\mathrm{rat}}$. Therefore, it suffices to show (2).

Let $\mu\colon Y \to W$ be a resolution of singularities, and let $f\colon X := Y^{\sharp} \to Y$ be the étale morphism induced from μ (cf. Corollary 3.4). We have a proper birational morphism $\nu\colon X \to V$ with $\mu \circ f = h \circ \nu$. For a point $Q \in V_{\mathrm{rat}}$ and $P = h(Q) \in W$, we have analytic open neighborhoods $\mathcal{W}$ and $\mathcal{V}$ of P and Q, respectively, such that:

- $\mathcal{W}$ is simply connected,
- $\mathcal{V}$ is a connected component of $h^{-1}(\mathcal{W})$, and
- $\mathcal{V} \cap h^{-1}(P) = \{Q\}$, set-theoretically.

Applying Lemma 3.10 to the commutative diagram

$$\begin{array}{ccc} \nu^{-1}(\mathcal{V}) & \xrightarrow{f} & \mu^{-1}(\mathcal{W}) \\ {\scriptstyle\nu}\downarrow & & {\scriptstyle\mu}\downarrow \\ \mathcal{V} & \xrightarrow{h} & \mathcal{W}, \end{array}$$

we infer that $\mathcal{V} \to \mathcal{W}$ is an isomorphism. Thus h is étale along V_{rat}. □

For a nearly étale endomorphism, we have the following stronger property:

Proposition 3.12. *Let $h\colon V \to V$ be a nearly étale finite surjective endomorphism of a normal algebraic variety V. Then $h^{-1}(V_{\mathrm{rat}}) = V_{\mathrm{rat}}$ and h is étale over V_{rat}.*

Proof. Let B be the complement of V_{rat} in V. Then $h^{-1}(B) \subset B$. Thus, the proof is reduced to Proposition 3.11 above and Lemma 3.13 below. □

Lemma 3.13. *Let $h\colon V \to V$ be a finite surjective endomorphism of an algebraic variety V. Suppose $h^{-1}(B) \subset B$ for a closed subset $B \subset V$. Then $h^{-1}(B) = B$. Further, $\Gamma \mapsto h(\Gamma)$ induces an automorphism of the set I_B of irreducible components Γ of B. The inverse map is given by $\Gamma \mapsto h^{-1}(\Gamma)$.*

Proof. Let $I_B^{(k)}$ be the set of irreducible components of B of dimension k. Then $I_B = \bigcup I_B^{(k)}$. Let $J_B^{(k)}$ be the subset of $I_B^{(k)}$ consisting of $\Gamma \in I_B$ with $h(\Gamma) \subset B$. It is enough to show the following two properties for any k:

$(1)_k$: $I_B^{(k)} = J_B^{(k)}$.
$(2)_k$: $h_*\colon J_B^{(k)} \to I_B^{(k)}$ given by $\Gamma \mapsto h(\Gamma)$ is bijective.

We shall prove by descending induction on k.

We set $d = \dim B$. For $\Gamma \in I_B^{(d)}$, some $\Gamma' \in I_B^{(d)}$ is contained in $h^{-1}(\Gamma)$. Here, $\Gamma' \in J_B^{(d)}$, since $h(\Gamma') = \Gamma$. Hence $h_*\colon J_B^{(d)} \to I_B^{(d)}$ is surjective. Since $I_B^{(d)}$ is a finite set, $J_B^{(d)} = I_B^{(d)}$ and $h_*\colon J_B^{(d)} \to I_B^{(d)}$ is bijective.

Next, assume that $(1)_l$ and $(2)_l$ hold for any integer $l > k$. If $\Gamma \in I_B^{(k)}$, then an irreducible component Δ of $h^{-1}(\Gamma)$ dominates Γ, and $\Delta \subset \Gamma'$ for some $\Gamma' \in I_B$. If $\dim \Gamma' > k$, then $h(\Gamma') \subset B$ by induction, and hence $\Gamma \subset h(\Gamma')$; this is a contradiction. Thus $\dim \Gamma' = k$, $\Gamma' = \Delta$, and $\Gamma' \in J_B^{(k)}$. Hence, $h_*\colon J_B^{(k)} \to I_B^{(k)}$ is surjective. Therefore, $I_B^{(k)} = J_B^{(k)}$ and $h_*\colon J_B^{(k)} \to I_B^{(k)}$ is bijective. Thus, we are done. □

If $f\colon X \to X$ is a surjective endomorphism of nonsingular projective variety X of $\kappa(X) \geq 0$, then f is étale. But if we drop the condition of nonsingularity, then we can expect neither the étaleness nor even the nearly étaleness. Indeed, we have:

Example 3.14. Let A be an abelian surface and let V be the quotient of A by the involution $\iota\colon A \ni a \mapsto -a \in A$ with respect to a group structure of A. The minimal resolution of singularities of V is a K3 surface called the Kummer surface associated with A and is denoted by $\mathrm{Km}(A)$. The variety V has only canonical singularities and $K_V \sim 0$. Let $\mu = \mu_m\colon A \to A$ be the multiplication map $a \mapsto ma$ by an odd integer m. Then it descends to a surjective endomorphism $\mu_V\colon V \to V$. Here, μ_V is not nearly étale since $\mathrm{Km}(A)$ is simply connected.

4. The case of zero Kodaira dimension

4.1. **Albanese closure.**

Definition 4.1. Let V be a normal projective variety. We define $q^{\max}(V)$ to be the supremum of the irregularities $q(V') = \dim \mathrm{H}^1(V', \mathcal{O}_{V'})$ for all the finite étale coverings $V' \to V$.

Let X be a nonsingular projective variety with $\kappa(X) = 0$. Then $q(X) \leq q^{\max}(X) \leq \dim X$ by [26].

Let V be a normal projective variety with only canonical singularities such that $\kappa(V) = 0$. Let X be a nonsingular projective variety birational to V. Then $q(X) = q(V)$ and $\mathrm{Alb}(X) \simeq \mathrm{Alb}(V)$, since V has only rational singularities. Furthermore, $\pi_1^{\mathrm{alg}}(X) \simeq \pi_1^{\mathrm{alg}}(V)$ by Remark 3.8. In particular, the category of finite étale coverings over X is equivalent to that of finite étale coverings over V. Therefore, $q^{\max}(X) = q^{\max}(V)$.

Definition 4.2. Let F be a normal projective variety with only canonical singularities such that $K_F \sim_{\mathbb{Q}} 0$. If $q^{\max}(F) = 0$, then F is called a *weak Calabi–Yau variety.*

Let V be a normal projective variety with only canonical singularities such that $K_V \sim_{\mathbb{Q}} 0$. Then, by [27, Corollary 8.4], there is a finite étale covering $F \times A \to V$ with F a weak Calabi-Yau variety and A an abelian variety. Here, $q^{\max}(V) = \dim A$.

The result below guarantees the uniqueness (up to isomorphism) of minimal étale cover $V^{\sim}$ of V realizing $q^{\max}(V)$ as $q(V^{\sim})$.

Proposition 4.3. *Let V be a normal projective variety with only canonical singularities such that $\kappa(V) = 0$. Then there exists a finite étale Galois covering $V^{\sim} \to V$, unique up to* (non-canonical) *isomorphisms, such that*:

(1) *$q(V^{\sim}) = q^{\max}(V)$, and*
(2) *if $V' \to V$ is a finite étale covering from a variety V' with $q(V') = q^{\max}(V)$, then $V' \to V$ factors through $V^{\sim} \to V$.*

We call the Galois cover $V^{\sim} \to V$ the *Albanese closure.*

Proof. There is a finite étale Galois covering $V_0 \to V$ with $q^{\max}(V) = q(V_0)$. For the Galois group G_0 of $V_0 \to V$, let H_0 be the kernel of the natural homomorphism

$$G_0 \to \mathrm{Aut}(H_1(\mathrm{Alb}(V_0), \mathbb{Z})).$$

We set $V^{\sim}$ to be the quotient space $H_0 \backslash V_0$. Then $H_0 \backslash \mathrm{Alb}(V_0)$ is an abelian variety and $V^{\sim} \to H_0 \backslash \mathrm{Alb}(V_0)$ is the Albanese map of $V^{\sim}$. In particular, $q^{\max}(V^{\sim}) = q(V)$.

Let $V_1 \to V$ be a finite étale Galois covering such that $V_1 \to V$ factors as $V_1 \to V_0 \to V$. For the Galois group G_1 of $V_1 \to V$, let $H_1 \subset G_1$ be the kernel of

$$G_1 \to \mathrm{Aut}(H_1(\mathrm{Alb}(V_1), \mathbb{Z})).$$

Then H_1 is just the pullback of $H_0 \subset G_0$ by $G_1 \to G_0$, since $\mathrm{Alb}(V_1) \to \mathrm{Alb}(V_0)$ is an isogeny. In particular, $H_1 \backslash V_1 \simeq H_0 \backslash V_0 = V^{\sim}$. Therefore, $V^{\sim}$ is independent of the choice of Galois covering $V_0 \to V$.

For an arbitrary finite étale cover $V' \to V$ with $q(V') = q^{\max}(V)$, we have a finite Galois cover $V_0 \to V$ which factors through $V' \to V$. Then the Galois group G_0' of $V_0 \to V'$ acts on $\mathrm{Alb}(V_0)$ and, acts on $H_1(\mathrm{Alb}(V_0), \mathbb{Z})$ trivially, since $\mathrm{Alb}(V_0) \to \mathrm{Alb}(V')$ is an isogeny. Thus $G_0' \subset H_0$ and we have an factorization $V' \to V^{\sim} \to V$. □

4.2. **Splitting endomorphisms.** We shall show that any nearly étale rational endomorphism φ of the product $F \times A$ of a weak Calabi-Yau variety F and an abelian variety A is split as the product $\varphi_F \times \varphi_A$ of a nearly étale rational endomorphism φ_F of F and an étale endomorphism φ_A of A. A slightly general assertion is proved in Proposition 4.8 below. To begin with, we recall the following well known result:

Lemma 4.4. *Let F be a normal projective variety such that $q(F) = 0$. If F is not ruled, then* $\mathrm{Aut}(F)$ *is discrete.*

Proof. Let $\mathcal{H}$ be a very ample invertible sheaf of F. For an automorphism f of F belonging to the identity component $\mathrm{Aut}^0(F)$, the invertible sheaf $f^*\mathcal{H}$ is isomorphic to $\mathcal{H}$, since the tangent space of the Picard scheme of F at $\mathcal{O}_F$ is isomorphic to the zero-dimensional vector space $H^1(F, \mathcal{O}_F)$. Let $\Phi\colon F \hookrightarrow \mathbb{P}^N$ be the embedding defined by the very ample linear system $|\mathcal{H}|$. Then f induces an automorphism $\rho(f)\colon \mathbb{P}^N \to \mathbb{P}^N$ such that the diagram below is commutative:

$$\begin{array}{ccc} F & \xrightarrow{\Phi} & \mathbb{P}^N \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \rho(f)} \\ F & \xrightarrow{\Phi} & \mathbb{P}^N. \end{array}$$

The automorphism $\rho(f)$ is contained in a linear subgroup of $\mathrm{PGL}(N, \mathbb{C})$ preserving $\Phi(F)$. Since F is not ruled, we infer that the linear subgroup acts on F trivially. Therefore, $f = \mathrm{id}_F$. Hence, $\mathrm{Aut}^0(F) = \{\mathrm{id}_F\}$. □

The following is the first splitting criterion for an étale morphism.

Lemma 4.5. *Let F and F' be non-ruled normal projective varieties such that $q(F) = q(F') = 0$ and* $\dim F = \dim F'$. *Let A and A' be abelian varieties with* $\dim A = \dim A'$. *Let $\varphi\colon F \times A \to F' \times A'$ be a surjective étale morphism. Then $\varphi = \varphi_1 \times \varphi_2$ for surjective étale morphisms $\varphi_1\colon F \to F'$ and $\varphi_2\colon A \to A'$.*

Proof. The second projections $p_2\colon F \times A \to A$ and $p_2'\colon F' \times A' \to A'$ are the Albanese maps of $F \times A$ and $F' \times A'$, respectively. Thus an étale morphism $\varphi_2\colon A \to A'$ is induced so that the diagram below is commutative:

$$\begin{array}{ccc} F \times A & \xrightarrow{\varphi} & F' \times A' \\ {\scriptstyle p_2}\big\downarrow & & \big\downarrow{\scriptstyle p_2'} \\ A & \xrightarrow{\varphi_2} & A'. \end{array}$$

So for any $a \in A$, there is an étale morphism $\rho_a\colon F \to F'$ such that

$$\varphi(x, a) = (\rho_a(x), \varphi_2(a)).$$

The collection $\{\rho_a\}$ gives rise to a morphism from A into the scheme $\mathrm{Hom}(F, F')$ of morphisms from F to F'. For a surjective étale morphism $\psi\colon F \to F'$, the tangent space of $\mathrm{Hom}(F, F')$ at the point $[\psi] \in \mathrm{Hom}(F, F')$ corresponding to ψ is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_F}(\psi^*\Omega^1_{F'}, \mathcal{O}_F) \simeq \mathrm{Hom}_{\mathcal{O}_F}(\Omega^1_F, \mathcal{O}_F) \simeq \mathrm{H}^0(F, \Theta_F)$$

for the tangent sheaf Θ_F. In particular, the dimension of the tangent space equals that of $\mathrm{Aut}^0(F)$. So the tangent space is zero by Lemma 4.4. Hence, ρ_a is independent of the choice of $a \in A$. Thus, $\varphi = \varphi_1 \times \varphi_2$ for $\varphi_1 = \rho_a$. □

The following is a partial generalization of Lemma 4.5 in the birational case.

Lemma 4.6. *Let F and F' be non-ruled nonsingular projective varieties such that $q(F) = q(F') = 0$ and $\dim F = \dim F'$. Let A and A' be abelian varieties with $\dim A = \dim A'$. Let $\varphi\colon F \times A \dashrightarrow F' \times A'$ be a birational map. Then $\varphi = \varphi_1 \times \varphi_2$ for a birational map $\varphi_1\colon F \dashrightarrow F'$ and an isomorphism $\varphi_2\colon A \xrightarrow{\simeq} A'$.*

Proof. There is an isomorphism $\varphi_2\colon A \to A'$ such that $p'_2 \circ \varphi = \varphi_2 \circ p_2$ for the second projections $p_2\colon F \times A \to A$ and $p'_2\colon F' \times A' \to A'$. Hence, we may assume that $A = A'$ and $\varphi_2 = \mathrm{id}_A$. Then φ is a birational map $F \times A \dashrightarrow F' \times A$ over A. For a general point $a \in A$, we have a birational map $f_a\colon F \dashrightarrow F'$ as the restriction of φ to $F \times \{a\}$. Thus, we may also replace F' with F by a suitable f_a. Therefore, we may assume from the beginning that φ is a birational map $F \times A \dashrightarrow F \times A$ over A. Then φ induces a rational map from A into the scheme $\mathrm{Bir}(F)$ of birational automorphisms studied in [22]. By [23, Theorem (2.1)], we have $\dim \mathrm{Bir}(F) = 0$, and hence the map $A \dashrightarrow \mathrm{Bir}(F)$ is constant. Therefore, $\varphi = \varphi_F \times \mathrm{id}_A$ for a birational map $\varphi_F\colon F \dashrightarrow F$. □

The next is a sufficient condition to split the variety into a product.

Lemma 4.7. *Let V be a normal projective variety with only canonical singularities such that $K_V \sim_{\mathbb{Q}} 0$. Suppose that V is birational to $F \times A$ for an abelian variety A and a normal variety F with only canonical singularities such that $K_F \sim_{\mathbb{Q}} 0$ and $q(F) = 0$. Then $V \simeq F' \times A$ for a variety F' birational to F.*

Proof. The composition $V \dashrightarrow F \times A \to A$ with the second projection is the Albanese map of V. There is a finite Galois étale covering $A' \to A$ from an abelian variety A' such that $V \times_A A' \simeq F' \times A'$ over A' for a variety F', by [27, Theorem 8.3]. Then F' is normal with only canonical singularities, $K_{F'} \sim_{\mathbb{Q}} 0$, $q(F') = 0$, and we have a birational map

$$\varphi\colon F \times A' = (F \times A) \times_A A' \dashrightarrow V \times_A A' \simeq F' \times A'$$

over A'. By Lemma 4.6, $\varphi = \varphi_F \times \mathrm{id}_{A'}$ for a birational map $\varphi_F\colon F \dashrightarrow F'$, since the irregularities of nonsingular models of F and F' are both zero. The action of the Galois group G of $A' \to A$ on $F' \times A' \simeq V \times_A A'$ is written as a diagonal action by Lemma 4.5. Moreover, it is compatible with the action of G on $F \times A'$ by φ, where G acts trivially on the first factor F. Therefore, G acts trivially on the first factor F', and hence, $V \simeq F' \times A$. □

The following is also a partial generalization of Lemma 4.5.

Proposition 4.8. *Let F and F' be normal projective varieties with only canonical singularities such that $K_F \sim_{\mathbb{Q}} 0$, $K_{F'} \sim_{\mathbb{Q}} 0$, and $q(F) = q(F') = 0$. Let A and A' be abelian varieties with $\dim A = \dim A'$. Let $\varphi\colon F \times A \cdots\to F' \times A'$ be a nearly étale dominant rational map such that $p_2' \circ \varphi = \varphi_A \circ p_2$ for an étale morphism $\varphi_A\colon A \to A'$ and for second projections $p_2\colon F \times A \to A$ and $p_2'\colon F' \times A' \to A'$. Then $\varphi = \varphi_F \times \varphi_A$ for a nearly étale dominant rational map $\varphi_F\colon F \cdots\to F'$.*

Proof. Let $V^\sharp \to F' \times A'$ be the étale covering obtained as the Stein factorization of φ. Since there is a birational map $F \times A \cdots\to V^\sharp$, by Lemma 4.7, $V^\sharp \simeq F^\sharp \times A$ for a normal projective variety $F^\sharp$ birational to F. Further, the étale covering $V^\sharp \to F' \times A'$ is isomorphic to $\psi \times \varphi_A$ for finite étale morphisms $\psi\colon F^\sharp \to F'$ and $\varphi_A\colon A \to A'$ by Lemma 4.5. So we have only to show that the birational map $F \times A \cdots\to V^\sharp \simeq F^\sharp \times A$ is the product of a birational map $F \cdots\to F^\sharp$ and the identity map $A \to A$. This is done by Lemma 4.6, since the irregularities of nonsingular models of F and $F^\sharp$ are both zero. □

4.3. **Proof of Theorem B.** Let $\lambda\colon V^\sharp \to V$ be the étale morphism such that h is the composite of a birational map $V \cdots\to V^\sharp$ and λ. Let $\delta\colon V^\sim \to V$ be the Albanese closure of V and let U be a connected component of the fiber product $V^\sharp \times_V V^\sim$. Then $q(U) = q(V^\sim) = q^{\max}(V)$, and hence $U \to V^\sharp$ factors through the Albanese closure $(V^\sharp)^\sim \to V^\sharp$ of $V^\sharp$. On the other hand, $(V^\sharp)^\sim$ is birational to $V^\sim$ by the birational map $V \cdots\to V^\sharp$. Hence, $U = V^\sharp \times_V V^\sim \simeq (V^\sharp)^\sim$. Therefore, we have a nearly étale dominant rational map $h^\sim\colon V^\sim \cdots\to V^\sim$ such that $\delta \circ h^\sim = h \circ \delta$.

Let $\alpha\colon V^\sim \to A$ be the Albanese map of $V^\sim$. Then $h^\sim$ induces an étale endomorphism $h_A\colon A \to A$ such that $\alpha \circ h^\sim = h_A \circ \alpha$. By [27, Theorem 8.3, Corollary 8.4], there exist a normal projective variety F, an isogeny $A' \to A$ of abelian varieties, and an isomorphism $V^\sim \times_A A' \simeq F \times A'$ over A', such that $q^{\max}(F) = 0$. Since the induced morphism $A' \to A$ is an isogeny, we may assume that $A' \simeq A$ and $A' \to A$ is the multiplication map by $m > 0$. By Lemma 4.9 below, there is an étale endomorphism φ_A of A such that $h_A(ma) = m\varphi_A(a)$ for any $a \in A$. Hence, we have a nearly étale dominant rational map $\varphi\colon F \times A \cdots\to F \times A$ such that $p_2 \circ \varphi = \varphi_A \circ p_2$ and $\theta \circ \varphi = h^\sim \circ \theta$, where $\theta\colon F \times A \to V^\sim$ is the composite of the isomorphism $F \times A' \simeq V^\sim \times_A A'$ and the projection $V^\sim \times_A A' \to V^\sim$.

By Proposition 4.8, $\varphi = \varphi_F \times \varphi_A$ for a nearly étale dominant rational map $\varphi_F\colon F \cdots\to F$. This completes the proof of Theorem B. □

The following is used in the proof above:

Lemma 4.9. *Let $\mu\colon A \to A$ be the multiplication map $a \mapsto ma$ by $m > 0$ for an abelian variety A with a given abelian group structure. For a morphism $h\colon A \to A$, there is a morphism $h'\colon A \to A$ such that $\mu \circ h' = h \circ \mu$.*

Proof. There exist a homomorphism $\varphi\colon A \to A$ of abelian group and a point $c \in A$ such that $h(a) = \varphi(a) + c$ for any $a \in A$. There is a point $c' \in A$ with $mc' = c$, since A is divisible. We define $h'\colon A \to A$ by $h'(a) = \varphi(a) + c'$ for $a \in A$. Then

$$mh'(a) = m\varphi(a) + mc' = \varphi(ma) + c = h(ma). \qquad \square$$

4.4. **Conjectural discussion.** We shall pose the following:

Conjecture 4.10. Let X be a nonsingular projective variety such that $\kappa(X) = q^{\max}(X) = 0$. Then $\pi_1(X)$ is finite.

This is true for X with K_X numerically trivial, by Bogomolov's decomposition theorem. Furthermore, it is true when $\dim X \leq 3$ (cf. [43]).

Lemma 4.11. *Let V be a normal projective variety such that $\pi_1^{\mathrm{alg}}(M)$ is finite for a resolution of singularities $M \to V$. Then any nearly étale rational endomorphism $h\colon V \cdots\to V$ is birational.*

Proof. We may assume that V is smooth. Consider the Stein factorization $V \cdots\to V^{\sharp(l)} \to V$ of the l-th power $h^l = h \circ \cdots \circ h$. Then $V^{\sharp(l)} \to V$ is an étale morphism of degree $(\deg h)^l$ by Lemma 3.3. Now, $\pi_1^{\mathrm{alg}}(V)$ is finite, since $\pi_1(M) \simeq \pi_1(V)$. Hence $\deg h = 1$. $\square$

So, if Conjecture 4.10 is true, then a nearly étale rational endomorphism of a weak Calabi–Yau variety is birational. In particular, the building blocks of the étale endomorphisms of projective varieties with $\kappa = 0$ would then turn out to be the endomorphisms of abelian varieties and the birational automorphisms of weak Calabi–Yau varieties.

5. The uniruled case

5.1. **Maximal rationally connected fibration.** A projective variety X is called *uniruled* if there is a dominant rational map $\mathbb{P}^1 \times Y \cdots\to X$ for a variety Y of dimension $\dim Y = \dim X - 1$.

For a nonsingular projective variety X, the following conditions are all equivalent (cf. [7, §3], [34, §2]):

(1) Any two points of X are connected by a chain of rational curves.
(2) Any two general points are contained in one and the same rational curve.
(3) There is a nonsingular rational curve with ample normal bundle.

If X satisfies one of the conditions above, then X is called *rationally connected*.

Remark 5.1 (cf. [7, §3], [34, Proposition (2.5)]). A nonsingular rationally connected variety X has the following properties:

- X is simply connected.
- $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i > 0$.
- $\mathrm{H}^0(X, (\Omega^1_X)^{\otimes m}) = 0$ for any $m > 0$.

If X is a uniruled nonsingular projective variety, then there exists uniquely up to birational equivalence, a rational fiber space $\pi\colon X \cdots\to Y$ into a nonsingular projective variety Y satisfying the following conditions, by [8], [34], [19]:

(1) π is weakly holomorphic, i.e., there exists open dense subsets $U \subset X$, $V \subset Y$ such that π induces a proper surjective morphism $U \to V$.
(2) For a general point $P \in U$, the fiber over $\pi(P)$ is a maximal rationally connected submanifold of X containing P.
(3) Y is not uniruled (cf. [19, Corollary 1.4]).

The fibration π is called the *maximal rationally connected fibration* of X.

The result below is used in the proof of Theorem C.

Lemma 5.2. *Let $f\colon X \to X$ be an étale endomorphism of a uniruled projective variety X. Then there exist a proper birational morphism $\mu\colon M \to X$, a proper surjective morphism $\pi\colon M \to Y$, an étale endomorphism f_M of M, and an endomorphism h of Y satisfying the following conditions*:

(1) *M is a nonsingular projective variety.*
(2) *Y is a normal and non-uniruled projective variety.*
(3) *π is birational to the maximal rationally connected fibration of M.*
(4) *$\mu \circ f_M = f \circ \mu$ and $\pi \circ f_M = h \circ \pi$.*

Proof. We may assume that X is nonsingular, by replacing it with an equivariant resolution of singularities with respect to f. Let $X \cdots\to \mathrm{Chow}(X)$ be a rational map to the Chow variety of X which defines the maximal rationally connected fibration. By associating a cycle Z of X with the push-forward f_*Z, we have a functorial morphism $h_c\colon \mathrm{Chow}(X) \to \mathrm{Chow}(X)$, which is compatible with f. Let $X \cdots\to Y \to \mathrm{Chow}(X)$ be the Stein factorization and let $h\colon Y \to Y$ be the induced endomorphism from h_c and f. Note that Y is not uniruled. We consider the graph $\Gamma \subset X \times Y$ of the rational map $X \cdots\to Y$. Then $f \times h$ induces an endomorphism of Γ. Let $T \to \Gamma$ be the normalization and f_T the induced endomorphism of T. Let T_1 be the fiber product of $f\colon X \to X$ and the first projection $p_1\colon T \to X$ over X. Then $T_1 \to T$ is étale and we have a finite birational morphism $T \to T_1$ over T by the commutative diagram

$$\begin{tikzcd} T \arrow[r, "p_1"] \arrow[d, "f_T"'] & X \arrow[d, "f"] \\ T \arrow[r, "p_1"] & X. \end{tikzcd}$$

Thus, $T \simeq T_1$ and $f_T \colon T \to T$ is étale. Let $M \to T$ be an equivariant resolution of singularities with respect to f_T, and let f_M be the lift of f_T to M as an étale endomorphism. Thus, we are done. □

The following is proved essentially in [31, Theorem 5.2], where F is assumed to be a rationally connected manifold. We present a slightly different proof.

Lemma 5.3. *Let $g \colon M \to N$ be a proper surjective morphism between nonsingular varieties. For a general fiber F of g, suppose that*

(1) *F is connected,*
(2) *F is simply connected, and*
(3) $\mathrm{H}^i(F, \mathcal{O}_F) = 0$ *for any $i > 0$.*

Then $g_* \colon \pi_1(M) \to \pi_1(N)$ is isomorphic.

Proof. Step 1. If $N^\circ \subset N$ is a Zariski open subset with the codimension of $N \setminus N^\circ$ bigger than one, then $\pi_1(N^\circ) \simeq \pi_1(N)$, and $\pi_1(g^{-1}(N^\circ)) \to \pi_1(M)$ is surjective. Thus, if $\pi_1(g^{-1}(N^\circ)) \to \pi_1(N^\circ)$ is isomorphic, then so is $\pi_1(M) \to \pi_1(N)$. Hence, we may replace N with such an open subset N°. In particular, we may assume that $g \colon M \to N$ is smooth outside a nonsingular divisor $D = \sum D_i$ of N, where D_i is an irreducible component.

Step 2. Let $\mathcal{U}_i$ be an analytic open neighborhood of a point $P_i \in D_i$ of D such that $\mathcal{U}_i$ is biholomorphic to a unit polydisc

$$\{(t_1, t_2, \ldots, t_n) \in \mathbb{C}^n \,;\, |t_j| < 1 \text{ for any } j\}$$

in which P_i is mapped to the origin $0 = (0, 0, \ldots, 0)$ and $\mathcal{U}_i \cap D$ is mapped to the coordinate hypersurface $\{t_1 = 0\}$. Since $\mathcal{U}_i \setminus D = \mathcal{U}_i \setminus D_i$ is homotopic to a circle, we have a generator δ_i of $\pi_1(\mathcal{U}_i \setminus D)$. By van Kampen's theorem, or other topological argument, we infer that the kernel of the surjection $\pi_1(N \setminus D) \to \pi_1(N)$ is generated by the conjugacy classes of the images $\overline{\delta}_i$ of δ_i under the homomorphisms $\pi_1(\mathcal{U}_i \setminus D) \to \pi_1(N \setminus D)$.

Step 3. By the assumptions (1) and (2), and by the homotopy exact sequence, we infer that the natural homomorphism $\pi_1(M \setminus g^{-1}D) \to \pi_1(N \setminus D)$ is an isomorphism. Let $\hat{\delta}_i \in \pi_1(M \setminus g^{-1}D)$ be the element corresponding to $\overline{\delta}_i \in \pi_1(N \setminus D)$. In order to show the homomorphism $\pi_1(M) \to \pi_1(N)$ to be isomorphic, it is enough to show that $\hat{\delta}_i$ is contained in the kernel of $\pi_1(M \setminus g^{-1}D) \to \pi_1(M)$. Let $C_i \subset \mathcal{U}_i$ be a curve corresponding to an axis with respect to the coordinate $(t_1, \dots, t_n)$ of the polydisc such that $C_i \cap D = \{0\}$ and let X_i be the fiber product $M \times_N C_i$. By changing P_i, $\mathcal{U}_i$, and coordinates $(t_1, \dots, t_n)$ slightly, we may assume that X_i is nonsingular. Then $\hat{\delta}_i$ comes from $\pi_1(X_i \setminus g^{-1}(P_i)) \simeq \pi_1(C_i \setminus \{P_i\}) = \mathbb{Z}\delta_i$. Thus, we have only to show that $\pi_1(X_i) = 0$, or equivalently, $\pi_1(g^{-1}(P_i)) = 0$, since $g^{-1}(P_i)$ is a deformation retract of X_i.

Step 4. By *Step* 3, the proof of Lemma 5.3 is reduced to the case where N is a unit disc and g is smooth outside the origin 0. We shall show $\pi_1(g^{-1}(0)) = 0$ in this case. By shrinking N if necessary, we have a holomorphic curve $T \subset M$ such that T is biholomorphic to a unit disc and $T \to N$ is a finite surjective morphism branched only at $0 \in N$. We have:

- $\pi_1(M \setminus g^{-1}(0)) \simeq \pi_1(N \setminus \{0\}) \simeq \mathbb{Z}$,
- $\pi_1(T \setminus g^{-1}(0)) \to \pi_1(N \setminus \{0\})$ is an injection into a finite-index subgroup, and
- $\pi_1(M \setminus g^{-1}(0)) \to \pi_1(M)$ is surjective.

Therefore, $\pi_1(M)$ is a finite cyclic group. Let $\lambda \colon \widetilde{M} \to M$ be the universal covering map and let $\widetilde{M} \to \widetilde{N} \to N$ be the Stein factorization of $\widetilde{M} \to M \to N$. Then $\widetilde{M} \to M \times_N \widetilde{N}$ is isomorphic over $(N \setminus \{0\}) \times_N \widetilde{N}$ by (2). By Lemma 3.10, we infer that $\widetilde{N} \to N$ is étale. In fact, the condition (5) of Lemma 3.10 is true over N (and similarly over $\widetilde{N}$), since the higher direct image sheaves $\mathrm{R}^i\, g_*\mathcal{O}_M$ are torsion free over the nonsingular curve N. Hence, $\widetilde{N} \simeq N$ and $\widetilde{M} \simeq M$. Therefore, M and $g^{-1}(0)$ are simply connected. □

The following gives a sufficient condition for a finite morphism to be nearly étale.

Proposition 5.4. *Let* $h \colon \widetilde{Y} \to Y$ *be a finite surjective morphism between normal varieties with* $\deg h > 1$. *Then* h *is nearly étale if the following conditions are satisfied*:

1. *There exist proper surjective morphisms* $\pi \colon M \to Y$ *and* $\tilde{\pi} \colon \widetilde{M} \to \widetilde{Y}$ *from nonsingular varieties* M *and* $\widetilde{M}$.
2. *There exists a finite étale morphism* $f \colon \widetilde{M} \to M$ *with* $\pi \circ f = h \circ \tilde{\pi}$.
3. *For a general fiber* F *of* π, F *is connected and simply connected, and* $\mathrm{H}^i(F, \mathcal{O}_F) = 0$ *for* $i > 0$.

Remark 5.5. Let M_1 be the fiber product $\widetilde{Y} \times_Y M$. So f induces a finite morphism $\lambda \colon \widetilde{M} \to M_1$ over M:

$$\begin{array}{ccccc} \widetilde{M} & \xrightarrow{\lambda} & M_1 & \longrightarrow & M \\ {\scriptstyle\tilde{\pi}}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \widetilde{Y} & = & \widetilde{Y} & \xrightarrow{h} & Y. \end{array}$$

Then λ is just the normalization of the reduced structure $M_{1,\mathrm{red}}$ of M_1. This is shown as follows: Note that M_1 is irreducible since h is finite. Since a general fiber of π is simply connected, λ is an isomorphism over $U \times_Y M \subset Y \times_Y M = M_1$ for a dense open subset $U \subset Y$. Let $\widetilde{M}_1 \to M_{1,\mathrm{red}}$ be the normalization. Then λ induces a finite and birational morphism $\widetilde{M} \to \widetilde{M}_1$. Hence, $\widetilde{M} \simeq \widetilde{M}_1$.

Proof of Proposition 5.4. Let $\nu\colon N \to Y$ be a resolution of singularities and let $M' \to M$ be a proper birational morphism from a nonsingular variety M' such that $M' \to M \dashrightarrow N$ is a morphism. Then we have an isomorphism $\pi_1(M') \simeq \pi_1(N)$ of fundamental groups by Lemma 5.3. Let $\Pi \subset \pi_1(N)$ be the image of $f_*(\pi_1(\widetilde{M})) \subset \pi_1(M)$ under the isomorphism $\pi_1(M) \simeq \pi_1(M') \simeq \pi_1(N)$. Then Π is a finite-index subgroup corresponding to a finite étale covering $\phi\colon \widetilde{N} \to N$. Let $\widetilde{M'} \to M'$ be the étale covering which is the pullback of f by $M' \to M$. Then we have a commutative diagram:

$$\begin{array}{ccccc} \widetilde{M} & \longleftarrow & \widetilde{M'} & \longrightarrow & \widetilde{N} \\ {\scriptstyle f}\downarrow & & \downarrow & & {\scriptstyle \phi}\downarrow \\ M & \longleftarrow & M' & \longrightarrow & N. \end{array}$$

By considering the Stein factorization of $\nu \circ \phi$, we have a proper birational morphism $\tilde{\nu}\colon \widetilde{N} \to \widetilde{Y}$ such that $h \circ \tilde{\nu} = \nu \circ \phi$. Thus, the diagram

$$\begin{array}{ccccc} \widetilde{M} & \xrightarrow{\tilde{\pi}} & \widetilde{Y} & \xleftarrow{\tilde{\nu}} & \widetilde{N} \\ {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \phi}\downarrow \\ M & \xrightarrow{\pi} & Y & \xleftarrow{\nu} & N \end{array}$$

is also commutative. Therefore, h is nearly étale. $\square$

5.2. **Proof of Theorem C.** We apply Lemma 5.2 to the given étale endomorphism f of X. Let $\mu\colon M \to X$, $\pi\colon M \to Y$, f_M, and h be the same objects as in Lemma 5.2. By Proposition 5.4, h is nearly étale. This completes the proof of Theorem C. $\square$

Appendix A. Topological entropies and fiber spaces

In the appendix, we shall prove the following:

Theorem D. *Let $\pi\colon X \to Y$ be a fiber space from a compact Kähler manifold X to a compact complex analytic variety Y and let $f\colon X \to X$ be an étale endomorphism such that $\pi \circ f = \pi$. Then the equality $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(f|_F)$ holds for the topological entropies h_{top} of $f\colon X \to X$ and the restriction $f|_F\colon F \to F$ to a smooth fiber F of π.*

The proof is based on basic properties of spectral radii, a generalized notion of Kähler cone, a simple calculation of dynamical degrees, and the results of Gromov [20] and Yomdin [49] on topological entropies. The proof is given at the end.

A.1. **Spectral radii and Kähler cones.** We recall some basic properties of spectral radii, especially a generalization of the Perron–Frobenius theorem. Furthermore, we introduce a notion of Kähler (k,k)-forms and the Kähler cones in $\mathrm{H}^{k,k}(M,\mathbb{R})$ for a compact Kähler manifold M.

The spectral radius $\rho(\varphi) = \rho(V,\varphi)$ of an endomorphism $\varphi\colon V \to V$ of a finite-dimensional $\mathbb{C}$-vector space V is defined to be the maximum of the absolute values of the eigenvalues of φ.

Remark. Let $\|\cdot\|$ be any norm of V and let $\|\cdot\|_1$ be the L^1-norm of $\mathrm{End}_{\mathbb{C}}(V)$ defined by

$$\|\varphi\|_1 := \sup\{\|\varphi(v)\|\,;\,\|v\|=1\} = \sup\left\{\frac{\|\varphi(v)\|}{\|v\|}\,;\,v\neq 0\right\}.$$

Then

$$\rho(\varphi) = \lim\nolimits_{m\to\infty}\left(\|\varphi^m\|_1\right)^{1/m}.$$

Suppose that $V = V_{\mathbb{R}}\otimes\mathbb{C}$ and $\varphi = \varphi_{\mathbb{R}}\otimes\mathbb{C}$ for an endomorphism $\varphi_{\mathbb{R}}\colon V_{\mathbb{R}}\to V_{\mathbb{R}}$ of a real vector space $V_{\mathbb{R}}$. Let $\|\cdot\|$ be a norm on $V_{\mathbb{R}}$. Then $\rho(\varphi) = \lim_{m\to\infty}\left(\|\varphi_{\mathbb{R}}^m\|_1\right)^{1/m}$, where

$$\|\varphi_{\mathbb{R}}\|_1 := \sup\{\|\varphi_{\mathbb{R}}(v)\|\,;\,v\in V_{\mathbb{R}},\ \|v\|=1\}.$$

Notation. Let $V_{\mathbb{R}}$ be a finite-dimensional real vector space. A subset $\mathcal{C}\subset V_{\mathbb{R}}$ is called a *convex cone* if $\mathcal{C}+\mathcal{C}\subset\mathcal{C}$ and $\mathbb{R}_+\mathcal{C}\subset\mathcal{C}$, where $\mathbb{R}_+$ denotes the set of positive real numbers. If $\mathcal{C}\cap(-\mathcal{C})\subset\{0\}$ in addition, then $\mathcal{C}$ is called *strictly convex.*

Remark. A convex cone $\mathcal{C}\subset V_{\mathbb{R}}$ is strictly convex if and only if there exists a linear form $\chi\colon V_{\mathbb{R}}\to\mathbb{R}$ such that $\chi>0$ on $\mathcal{C}\setminus\{0\}$.

The following is known as a generalization of the Perron–Frobenius theorem on real $n\times n$ matrices $A=(a_{ij})$ of positive entries a_{ij}:

Theorem A.1 (cf. [5])**.** *Let $\mathcal{C}$ be a strictly convex closed cone of a finite-dimensional real vector space $V_{\mathbb{R}}$ such that $\mathcal{C}$ generates $V_{\mathbb{R}}$ as a vector space. Let $\varphi\colon V_{\mathbb{R}}\to V_{\mathbb{R}}$ be an endomorphism such that $\varphi(\mathcal{C})\subset\mathcal{C}$. Then the spectral radius $\rho(\varphi)$ is an eigenvalue of φ and there is an eigenvector in $\mathcal{C}$ with the eigenvalue $\rho(\varphi)$.*

Let M be a compact Kähler manifold of dimension n. Let ω be a C^∞-(k,k)-form on M for an integer $1\le k\le n$. For a local coordinate $(z_1,z_2,\ldots,z_n)$ of M, ω is locally expressed as

$$\omega = (\sqrt{-1})^{k^2}\sum_{I,J\subset\{1,2,\ldots,n\}} a_{I,J}\,dz_I\wedge d\bar{z}_J$$

for C^∞-functions $a_{I,J}$, where $\sharp I = \sharp J = k$ and

$$dz_I := dz_{i_1}\wedge dz_{i_2}\wedge\cdots\wedge dz_{i_k}$$

when $I = \{i_1, i_2, \dots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$. Assume that ω is d-closed ($d\omega = 0$) and real ($\overline{\omega} = \omega$); The real condition is equivalent to that $(a_{I,J})$ is an Hermitian matrix. The (k,k)-form ω is called *Kähler* if $(a_{I,J})$ is positive-definite everywhere in M. Note that this is just the usual definition of Kähler forms in case $k = 1$.

The following is easily shown:

Lemma A.2.

(1) *If η is a (usual) Kähler form ($(1,1)$-form), then $\eta^k = \eta \wedge \cdots \wedge \eta$ is a Kähler (k,k)-form for $1 \le k \le n$.*

(2) *If ω is a Kähler (k,k)-form and if ω' is a Kähler $(n-k, n-k)$-form, then $\int_M \omega \wedge \omega' > 0$.*

Inside of the real vector space $\mathrm{H}^{k,k}(M, \mathbb{R}) := \mathrm{H}^{k,k}(M) \cap \mathrm{H}^{2k}(M, \mathbb{R})$, the set $P^k(M)$ of the classes $[\omega]$ of Kähler (k,k)-forms ω on M is a strictly convex open cone. It is called the *Kähler cone* of degree k.

Lemma A.3. *Let $\xi \in \overline{P^k(M)}$ be a non-zero element.*

(1) *$\int_M \xi \cup [\omega] > 0$ for any Kähler $(n-k, n-k)$-form ω.*

(2) *If $\theta \in \overline{P^1(M)}$ and if $\theta - [T] \in P^1(M)$ for a d-closed positive $(1,1)$-current T, then $\int_M \xi \cup \theta^{n-k} > 0$.*

Proof. (1): Let (x, y) denote $\int x \cup y$ for $x \in \mathrm{H}^{k,k}(M, \mathbb{R})$ and $y \in \mathrm{H}^{n-k,n-k}(M, \mathbb{R})$. Then $\mathrm{H}^{n-k,n-k}(M, \mathbb{R})$ is dual to $\mathrm{H}^{k,k}(M, \mathbb{R})$ by $(*, *)$. Since $P^{n-k}(M)$ generates $\mathrm{H}^{n-k,n-k}(M, \mathbb{R})$ as the vector space, we can find a Kähler $(n-k, n-k)$-form ω_0 such that $(\xi, [\omega_0]) \ne 0$. By Lemma A.2, we have $(\xi, [\omega_0]) > 0$. There is a positive constant C such that $C\omega - \omega_0$ is also a Kähler (k,k)-form, since M is compact. Thus, by Lemma A.2,

$$(\xi, [\omega]) \ge C^{-1}(\xi, [\omega_0]) > 0.$$

(2): We set $\alpha = \theta - [T] \in P^1(M)$. Then the cup product $\theta^l = \theta \cup \cdots \cup \theta$ for $l \ge 2$ is calculated as $\theta^l = \alpha^l + z_{l-1} \cup [T]$ for

$$z_{l-1} = \sum\nolimits_{i=0}^{l-1} \theta^i \cup \alpha^{l-1-i} \in \overline{P^{l-1}(M)}. \tag{A–1}$$

Hence, by (1), we have $(\xi, \theta^{n-k}) \ge (\xi, \alpha^{n-k}) > 0$. □

The following is a well-known property of Kähler classes.

Lemma A.4. *Let $\psi \colon M \to X$ be a generically finite morphism into another compact Kähler manifold X. If η is a Kähler form on X, then $[\psi^*\eta] - [T] \in P^1(M)$ for a d-closed real positive $(1,1)$-current T on M.*

Proof. Let $M \to V \to X$ be the Stein factorization. By Hironaka's blowing up, we have a projective bimeromorphic morphism $\nu\colon Z \to V$ from another compact Kähler manifold Z such that $\mu\colon Z \to V \cdots\to M$ is holomorphic. Here, $\mathcal{O}_Z(-E)$ is ν-ample for a ν-exceptional effective divisor E. Then $\mathcal{O}_Y(-E)$ is also relatively ample over X, and $[\mu^*\psi^*\eta] - \varepsilon[E]$ is represented by a Kähler form on Z for some $\varepsilon > 0$ (cf. [15, Lemma 4.4], [14, Lemma 2]). Thus

$$[\mu^*\psi^*\eta] - \varepsilon[E] - [\mu^*\xi] \in P^1(Z)$$

for a Kähler form ξ on M. Hence, $\psi^*[\eta] - [\xi]$ is represented by a d-closed real positive $(1,1)$-current on M. □

A.2. **Dynamical degrees.** Let $f\colon M \to M$ be a surjective endomorphism of a compact Kähler manifold M of dimension n. Then f induces natural homomorphisms

$$f^*\colon \mathrm{H}^i(M,\mathbb{Z}) \to \mathrm{H}^i(M,\mathbb{Z}) \quad \text{and} \quad f_*\colon \mathrm{H}_i(M,\mathbb{Z}) \to \mathrm{H}_i(M,\mathbb{Z})$$

for $0 \le i \le 2n$. Here, the composite

$$\mathrm{H}^i(M,\mathbb{Z}) \xrightarrow{f^*} \mathrm{H}^i(M,\mathbb{Z}) \simeq \mathrm{H}_{2n-i}(M,\mathbb{Z}) \xrightarrow{f_*} \mathrm{H}_{2n-i}(M,\mathbb{Z}) \simeq \mathrm{H}^i(M,\mathbb{Z})$$

is just the multiplication map by $\deg f$, where the isomorphism above is induced from the Poincaré duality. Thus $f^*\colon \mathrm{H}^i(M,\mathbb{R}) \to \mathrm{H}^i(M,\mathbb{R})$ is isomorphic. Moreover, f^* preserves the Hodge structure, i.e., $f^*\,\mathrm{H}^{p,q}(M) = \mathrm{H}^{p,q}(M)$. Note that f is a finite morphism, since $f^*\colon \mathrm{H}^{1,1}(M,\mathbb{R}) \to \mathrm{H}^{1,1}(M,\mathbb{R})$ is an isomorphism.

Definition A.5 (Dynamical degree)**.** Let η be a Kähler form on M. For an integer $1 \le l \le n$, we set

$$\delta_l(f,\eta) := \int_M f^*\eta^l \wedge \eta^{n-l},$$

where η^i denotes the i-th power $\eta\wedge\cdots\wedge\eta$ for $1 \le i \le n$, and $\eta^0 := 1$. The l-th *dynamical degree* of f is defined to be

$$d_l(f) := \lim_{m\to\infty} \left(\delta_l(f^m,\eta)\right)^{1/m}.$$

The following properties on dynamical degrees are well known:

Fact A.6.

(1) The k-th dynamical degree $d_k(f)$ equals the spectral radius of
$$f^* = (f^*)^{(k,k)}\colon \mathrm{H}^{k,k}(M,\mathbb{C}) \to \mathrm{H}^{k,k}(M,\mathbb{C}).$$
(2) $d_0(f) = 1$ and $d_n(f) = \deg f$.
(3) $d_{l-1}(f)d_{l+1}(f) \le d_l(f)^2$ for $1 \le l \le n$. In particular, $d_n(f) = \deg f \le d_1(f)^n$ (cf. [21, Proposition 1.2]).

Lemma A.7. *Let x be an element of $\overline{P^1(M)}$ and y an element of $\overline{P^k(M)}$ such that $x-[T_1] \in P^1(M)$ and $y-[T_k] \in P^k(M)$ for certain d-closed real positive currents T_1 of type $(1,1)$ and T_k of type (k,k) for $1 \le k \le n$. Then*

$$d_{n-k}(f) = \lim_{m\to\infty} \left(\int_M (f^m)^*(x^{n-k}) \cup y \right)^{1/m}.$$

Proof. There exist d-closed positive currents S_1 of type $(1,1)$ and S_k of type (k,k), and a constant $a > 0$ such that $ax - [\eta] = [S_1]$ and $ay - [\eta^k] = [S_k]$. We set $l := n-k$. Then $a^l x^l - [\eta]^l = z \cup [S_1]$ for an element $z \in \overline{P^{l-1}(M)}$ by (A–1). Thus $f^*(a^l x^l - [\eta]^l) = f^*z \cup f^*[S_1]$ is represented by a d-closed positive (l,l)-current, since $f^*[S_1] = [f^*S_1]$ for the positive $(1,1)$-current f^*S_1. Therefore,

$$f^*(a^l x^l) \cup (ay) - f^*[\eta]^l \cup [\eta]^k = f^*z \cup f^*[S_1] \cup (ay) + f^*[\eta]^l \cup [S_k]$$

is represented by a positive (n,n)-current. Hence,

$$a^{l+1} \int_M (f^m)^*(x^l) \cup y \ge \int_M (f^m)^* \eta^l \wedge \eta^k = \delta_l(f,\eta).$$

Conversely, there exists a constant $b > 0$ such that $b[\eta] - x \in P^1(M)$ and $b[\eta^k] - y \in P^k(M)$. Then $b^l[\eta]^l - x^l \in \overline{P^l(M)}$ by (A–1) and hence $f^*(b^l[\eta]^l - x^l) \in \overline{P^l(M)}$. Therefore

$$f^*(b^l[\eta]^l) \cup [\eta]^k - f^*(x^l) \cup y \in \overline{P^n(M)}.$$

Thus,

$$b^{l+1}\delta_l(f,\eta) = b^{l+1} \int_M (f^m)^* \eta^l \wedge \eta^k \ge \int_M (f^m)^*(x^l) \cup y.$$

Hence, we have the expected equality. □

Lemma A.8. *Suppose that there exist a generically finite surjective morphism $\mu \colon M \to X$ into another compact Kähler manifold X and an endomorphism $g \colon X \to X$ satisfying $\mu \circ f = g \circ \mu$. Then $d_l(f) = d_l(g)$ for any l.*

Proof. Let ξ be a Kähler form on X. Then, by Lemma A.4, $[a\mu^*\xi - \eta]$ is represented by a d-closed real positive $(1,1)$-current on M for a certain positive constant a. Then $[a^l \mu^* \xi^l - \eta^l]$ is represented by a d-closed real positive (l,l)-current by (A–1). Therefore, by Lemma A.7,

$$\begin{aligned} d_l(f) &= \lim_{m\to\infty} \left(\int_M (f^m)^*(\mu^*\xi^l) \wedge \mu^*\xi^{n-l} \right)^{1/m} \\ &= \lim_{m\to\infty} \left((\deg \mu) \int_X (g^m)^* \xi^l \wedge \xi^{n-k} \right)^{1/m} = d_l(g). \end{aligned}$$

□

For an element $x \in \mathrm{H}^{k,k}(M,\mathbb{R})$, we set

$$\delta_l(f,\eta;x) := \int_M [f^*\eta^l \wedge \eta^{n-k-l}] \cup x$$

for $l \leq n-k$. Note that $\delta_l(f, \eta; [\eta^k]) = \delta_l(f, \eta)$. There is a constant $C > 0$ such that $C[\eta^k] - x \in P^k(M)$. Therefore, $C\delta_l(f, \eta) \geq \delta_l(f, \eta; x)$ for any $f\colon M \to M$. Hence, we have

$$d_l(f) \geq \overline{\lim}_{m\to\infty} \left(\delta_l(f^m, \eta; x)\right)^{1/m}. \tag{A–2}$$

Proposition A.9. *Let F be a compact Kähler manifold of dimension k and let $\phi\colon F \to M$ be a generically finite morphism such that $\phi \circ h = f \circ \phi$ for a surjective endomorphism $h\colon F \to F$. Then*

$$d_l(f) \geq d_l(h) \quad \text{and} \quad d_{l+n-k}(f) \geq \deg(f)\deg(h)^{-1} d_l(h)$$

for any $1 \leq l \leq k$.

Proof. Let $G_i\colon \mathrm{H}^{2i}(F, \mathbb{C}) \to \mathrm{H}^{2(n-k+i)}(X, \mathbb{C})$ be the Gysin homomorphism

$$\mathrm{H}^{2i}(F, \mathbb{C}) \simeq \mathrm{H}_{2k-2i}(F, \mathbb{C}) \xrightarrow{\phi_*} \mathrm{H}_{2k-2i}(X, \mathbb{C}) \simeq \mathrm{H}^{2n-2k+2i}(X, \mathbb{C})$$

for $0 \leq i \leq k$, where the isomorphisms in both sides are induced from the Poincaré duality. Note that $G_i \, \mathrm{H}^{p,q}(F, \mathbb{C}) \subset \mathrm{H}^{p+n-k, q+n-k}(M, \mathbb{C})$. For $x_F := G_0(1) \in \mathrm{H}^{2(n-k)}(M, \mathbb{R})$, we have the projection formula $\phi_*\phi^* y = y \cup x_F$ for $y \in \mathrm{H}^{2k}(M, \mathbb{R})$. Thus,

$$\int_F (h^m)^*(\phi^*\eta^l) \wedge \phi^*\eta^{k-l} = \int_F \phi^*((f^m)^*\eta^l) \wedge \phi^*\eta^{k-l} = \int_M [(f^m)^*\eta^l) \wedge \eta^{k-l}] \cup x_F$$

for any $m \geq 1$. Since ϕ is generically finite, $[\phi^*\eta] - [T] \in P^1(F)$ for a d-closed real positive $(1,1)$-current T by Lemma A.4. Thus, by Lemma A.7 and (A–2),

$$d_l(h) = \lim_{m\to\infty} \left(\int_M [(f^m)^*\eta^l) \wedge \eta^{k-l}] \cup x_F \right)^{1/m} = \lim_{m\to\infty} \left(\delta_l(f^m, \eta; x_F)\right)^{1/m} \leq d_l(f).$$

From $\phi_* \circ h_* = f_* \circ \phi_*$, we have

$$(\deg h)^{-1} G_i \circ (h^*)^{(i,i)} = (\deg f)^{-1} (f^*)^{(n-k+i, n-k+i)} \circ G_i,$$

since $(\deg h)^{-1} h^*$ is the inverse of

$$\mathrm{H}^{2i}(F, \mathbb{C}) \simeq \mathrm{H}^{2k-2i}(F, \mathbb{C}) \xrightarrow{h_*} \mathrm{H}^{2k-2i}(F, \mathbb{C}) \simeq \mathrm{H}^{2i}(F, \mathbb{C})$$

and $(\deg f)^{-1} f^*$ is the inverse of

$$\mathrm{H}^{2(n-k+i)}(M, \mathbb{C}) \simeq \mathrm{H}^{2k-2i}(M, \mathbb{C}) \xrightarrow{f_*} \mathrm{H}^{2k-2i}(M, \mathbb{C}) \simeq \mathrm{H}^{2(n-k+i)}(M, \mathbb{C}).$$

We can find an eigenvector w of $(h^*)^{(l,l)}$ with the eigenvalue $d_l(h)$ from the cone $\overline{P^l(M)}$ by Theorem A.1. Thus, $f^* G_l(w) = \deg(f)\deg(h)^{-1} d_l(h) G_l(w)$. For the Kähler form η above, we have

$$\int_M G_l(w) \cup [\eta]^{k-l} = \int_F w \cup [\phi^*\eta]^{k-l} > 0$$

by Lemma A.3. Thus, $G_l(w) \neq 0$ and it is an eigenvector of $(f^*)^{(n-k+l, n-k+l)}$ with the eigenvalue $\deg(f)\deg(h)^{-1} d_l(h)$. Thus, $d_{n-k+l}(f) \geq \deg(f)\deg(h)^{-1} d_l(h)$. $\square$

By Proposition A.9, and by a similar argument to [50, Remark 2.1 (11)], we have:

Theorem A.10. *Let $\pi\colon X \to Y$ be a surjective morphism from a compact Kähler manifold into a compact complex analytic variety Y such that a general fiber of π is connected. Let $f\colon X \to X$ be a surjective endomorphism such that $\pi \circ f = \pi$. Then*

$$d_1(f) = d_1(f|_F)$$

for the restriction $f|_F\colon F \to F$ to a smooth fiber F of π.

Proof. We set $d = \dim Y > 0$. We have $d_1(f) \geq d_1(f|_F)$ by Proposition A.9. If $d_1(f) \leq 1$, then $\deg f = d_1(f) = d_1(f|_F) = 1$ by Fact A.6. Thus, we may assume that $d_1(f) > 1$. Let $v \in \overline{P^1(X)} \subset \mathrm{H}^{1,1}(X, \mathbb{R})$ be an eigenvector of f^* with the eigenvalue $d_1(f)$. If $v|_F \in \mathrm{H}^{1,1}(F, \mathbb{R})$ is not zero, then $d_1(f) = d_1(f|_F)$ by Proposition A.9.

We shall show $v|_F \neq 0$. There is a bimeromorphic morphism $\nu\colon S \to Y$ from a compact Kähler manifold S by [46]. Let $M \to X \times_Y S$ be a proper surjective morphism giving a bimeromorphic morphism to the main component of $X \times_Y S$, i.e., the unique component dominating S. Let $\mu\colon M \to X$ be the induced bimeromorphic morphism and let $\varpi\colon M \to S$ be the induced fiber space. We may replace F with a general fiber of π, since $d_1(f|_F)$ depends only on the class $[F] \in \mathrm{H}^{1,1}(X, \mathbb{R})$. Hence we may assume that ν is an isomorphism over a neighborhood of $\pi(F)$ and that μ is an isomorphism along $\mu^{-1}(F)$. Let $f_M = \mu^{-1} \circ f \circ \mu\colon M \dashrightarrow M$ be the meromorphic endomorphism and let $\varphi\colon Z \to M$ be a bimeromorphic morphism from another compact Kähler manifold Z such that $g := f_M \circ \varphi\colon Z \to M$ is holomorphic. Let $(f_M^*)^{(i,i)}$ be an endomorphism of $\mathrm{H}^{i,i}(M, \mathbb{R})$ for $0 \leq i \leq n = \dim X$ defined as

$$(f_M^*)^{(i,i)}\colon \mathrm{H}^{i,i}(M, \mathbb{R}) \xrightarrow{g^*} \mathrm{H}^{i,i}(Z, \mathbb{R}) \xrightarrow{\varphi_*} \mathrm{H}^{i,i}(M, \mathbb{R}).$$

Since $\varphi_* \circ \varphi^* = \mathrm{id}$, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^{i,i}(X, \mathbb{R}) & \xrightarrow{f^*} & \mathrm{H}^{i,i}(X, \mathbb{R}) \\ {\scriptstyle \mu^*}\big\downarrow & & \big\downarrow{\scriptstyle \mu^*} \\ \mathrm{H}^{i,i}(M, \mathbb{R}) & \xrightarrow{(f_M^*)^{(i,i)}} & \mathrm{H}^{i,i}(M, \mathbb{R}). \end{array}$$

Thus, $\mu^* v \in \overline{P^1(M)}$ is also an eigenvector of $(f_M^*)^{(1,1)}$ with the eigenvalue $d_1(f)$. We have also $f_M^* \circ \varpi^* = \varpi^*$ from $\varpi \circ f_M = \varpi$. For Kähler classes $\alpha \in P^1(X)$ and $\beta \in P^1(S)$, and for any $\xi \in \mathrm{H}^{i,i}(M, \mathbb{R})$, by the projection formula, we have

$$(f_M^*)^{(i+j,i+j)}(\xi \cup \mu^*\alpha) = \varphi_*(g^*(\xi \cup \mu^*\alpha)) = \varphi_*(g^*\xi \cup \varphi^*\mu^* f^*\alpha) = (f_M^*)^{(i,i)}(\xi) \cup \mu^* f^*\alpha,$$

$$(f_M^*)^{(i+j,i+j)}(\xi \cup \varpi^*\beta) = \varphi_*(g^*(\xi \cup \varpi^*\beta)) = \varphi_*(g^*\xi \cup \varphi^*\varpi^*\beta) = (f_M^*)^{(i,i)}\xi \cup \varpi^*\beta.$$

We set $x := \mu^*\alpha$ and $y := \varpi^*\beta \in \mathrm{H}^{1,1}(M, \mathbb{R})$. Then

$$y^d = c[\mu^{-1}F] \in \mathrm{H}^{d,d}(M, \mathbb{R})$$

for some $c > 0$. Assuming $v|_F = 0$, we shall derive a contradiction. Then $\mu^*v \cup y^d = 0$. Let $1 \leq s \leq d$ be the minimum number such that $\mu^*v \cup y^s = 0$. Note that $\mu^*v \cup y^{s-1} = 0$ if $\mu^*v \cup y^{s-1} \cup x^{n-s} = 0$ by Lemmas A.3 and A.4, where $n = \dim X$. Then, by [10, Corollaire 3.5], there is a constant b such that

$$(\mu^*v + by) \cup y^{s-1} \cup \mu^*(\alpha_1 \cup \cdots \cup \alpha_{n-s-1}) = 0 \in \mathrm{H}^{n-1,n-1}(M, \mathbb{R})$$

for any $\alpha_i \in \mathrm{H}^{1,1}(X, \mathbb{R})$. Taking f_M^*, we have

$$(d_1(f)\mu^*v + by) \cup y^{s-1} \cup \mu^*(f^*\alpha_1 \cup \cdots \cup f^*\alpha_{n-s-1}) = 0.$$

In particular,

$$(\mu^*v + by) \cup y^{s-1} \cup x^{n-s} = (d_1(f)\mu^*v + by) \cup y^{s-1} \cup x^{n-s} = 0,$$

which implies $\mu^*v \cup y^{s-1} \cup x^{n-s} = 0$. This is a contradiction. □

A.3. **Topological entropies.** Let $f\colon M \to M$ be a surjective endomorphism of a compact Kähler manifold M. We consider the properties of topological entropy $h_{\mathrm{top}}(f)$ of f. Instead of giving the definition of h_{top}, we use the following:

Fact A.11 (Gromov [20], Yomdin [49], (cf. [13])). For the topological entropy $h_{\mathrm{top}}(f)$ of f, one has

$$h_{\mathrm{top}}(f) = \max\nolimits_{1 \leq i \leq n} \log d_i(f) = \log \rho(\mathrm{H}^*(M, \mathbb{C}), f^*).$$

As a corollary of Proposition A.9, we have:

Corollary A.12. *In the situation of* Proposition A.9*,*

$$h_{\mathrm{top}}(f) \geq \log\Big(\deg(f)\deg(h)^{-1}\Big) + h_{\mathrm{top}}(h) \geq h_{\mathrm{top}}(h).$$

Notation A.13. Let $\varphi\colon V \to V$ be an endomorphism of a finite-dimensional $\mathbb{C}$-vector space V.

(1) $\Lambda(V, \varphi)$ denotes the set of eigenvalues of φ.
(2) For $\lambda \in \mathbb{C}$, we set

$$V_\lambda = V_{\lambda,\varphi} := \bigcup\nolimits_{l \geq 1} \mathrm{Ker}(\lambda \,\mathrm{id}_V - \varphi)^l.$$

Remark. If $\lambda \in \Lambda(V, \varphi)$, then V_λ is the *generalized eigenvector subspace* with the eigenvalue λ. We have the decomposition

$$V = \bigoplus\nolimits_{\lambda \in \Lambda(V,\varphi)} V_{\lambda,\varphi}.$$

Moreover, the decomposition is functorial; Let $h\colon V_1 \to V_2$ be a $\mathbb{C}$-linear map of finite-dimensional $\mathbb{C}$-vector spaces. Let $\varphi_i\colon V_i \to V_i$ be an endomorphism for $i = 1, 2$ such that $h \circ \varphi_1 = \varphi_2 \circ h$. Then $h = \bigoplus h_\lambda$ for

$$h_\lambda\colon (V_1)_{\lambda,\varphi_1} \to (V_2)_{\lambda,\varphi_2}.$$

Lemma A.14. *Let Y be a reduced compact complex analytic space and let $\mu\colon X \to Y$ be a proper surjective morphism from another reduced compact complex analytic space X such that $\mu\colon \mu^{-1}(U) \to U$ is a smooth Kähler morphism for a dense Zariski-open subset $U = Y \setminus B \subset Y$. Let $g\colon Y \to Y$ and $f\colon X \to X$ be surjective endomorphisms such that $\mu \circ f = g \circ \mu$ and $g^{-1}(U) = U$. Let $g_B\colon B \to B$ and $f_A\colon A \to A$ for $A = \mu^{-1}(B)$ be the induced endomorphisms. Then, for any p,*

$$\Lambda(\mathrm{H}^p(Y, \mathbb{C}), g^*) \subset \Lambda(\mathrm{H}^p(X, \mathbb{C}), f^*) \cup \Lambda(\mathrm{H}^p(B, \mathbb{C}), g_B^*) \cup \Lambda(\mathrm{H}^{p-1}(A, \mathbb{C}), f_A^*).$$

Proof. For the injection $j\colon U \hookrightarrow Y$, we have a natural exact sequence

$$0 \to j_!\mathbb{Z}_U \to \mathbb{Z}_Y \to \mathbb{Z}_B \to 0$$

inducing a long exact sequence

$$\cdots \to \mathrm{H}^i_c(U, \mathbb{Z}) \to \mathrm{H}^i(Y, \mathbb{Z}) \to \mathrm{H}^i(B, \mathbb{Z}) \to \cdots .$$

Considering a similar exact sequence on X, we have a commutative diagram

$$\begin{CD}
\cdots @>>> \mathrm{H}^i_c(U,\mathbb{Z}) @>>> \mathrm{H}^i(Y,\mathbb{Z}) @>>> \mathrm{H}^i(B,\mathbb{Z}) @>>> \cdots \\
@. @V{\mu^*}VV @V{\mu^*}VV @V{\mu|_A^*}VV \\
\cdots @>>> \mathrm{H}^i_c(\mu^{-1}(U),\mathbb{Z}) @>>> \mathrm{H}^i(X,\mathbb{Z}) @>>> \mathrm{H}^i(A,\mathbb{Z}) @>>> \cdots
\end{CD}$$

of exact sequences, which are compatible with the actions of g^* and f^*. Since $\mu\colon \mu^{-1}(U) \to U$ is a smooth Kähler morphism, we have a quasi-isomorphism

$$\mathbb{R}\mu_*\mathbb{C}_X|_U \sim_{\mathrm{qis}} \bigoplus \mathrm{R}^i\,\mu_*\mathbb{C}_X|_U[-i]$$

in the derived category on U, by the hard Lefschetz theorem on fibers and by [9]. In particular, the homomorphism $\mu^*\colon \mathrm{H}^i_c(U, \mathbb{C}) \to \mathrm{H}^i_c(\mu^{-1}(U), \mathbb{C})$ is injective for any i. For a complex number λ, if $\mathrm{H}^p(Y, \mathbb{C})_{\lambda,g^*} \neq 0$ and if $\mathrm{H}^p(B, \mathbb{C})_{\lambda,g_B^*} = \mathrm{H}^p(X, \mathbb{C})_{\lambda,f^*} = 0$, then $\mathrm{H}^{p-1}(A, \mathbb{C})_{\lambda,f_A^*} \neq 0$ by the commutative diagram above. Thus, we have the assertion. □

Corollary A.15. *Let Y be a reduced compact complex analytic space with a surjective endomorphism $g\colon Y \to Y$. Then there exist a finite set $\{Z_i\}_{1=1}^N$ of closed subvarieties and a positive integer k such that $Y = \bigcup Z_i$, $(g^k)^{-1}(Z_i) = Z_i$ for any $1 \leq i \leq N$, and*

$$\Lambda(\mathrm{H}^p(Y, \mathbb{C}), (g^k)^*) \subset \bigcup\nolimits_{i=1}^N \bigcup\nolimits_{q=0}^p \Lambda(\mathrm{H}^q(Z_i, \mathbb{C}), (g^k|_{Z_i})^*).$$

Proof. Let $\{Y_i\}$ be the set of the irreducible components of Y and let $\tau\colon X := \bigsqcup Y_i \to Y$ be the natural morphism. Let $B \subset Y$ be the minimum closed subset such that $X \setminus \tau^{-1}B \to Y \setminus B$ is an isomorphism. By replacing g with a power g^k, we may assume that $g^{-1}Y_i = Y_i$ for any i. Then

$$\Lambda(\mathrm{H}^p(Y, \mathbb{C}), g^*) \subset \Lambda(\mathrm{H}^p(B, \mathbb{C}), g^*) \cup \bigcup\nolimits_{i=1}^{N} \Big(\Lambda(\mathrm{H}^p(Y_i, \mathbb{C}), g^*) \cup \Lambda(\mathrm{H}^{p-1}(Y_i \cap B, \mathbb{C}), g^*)\Big)$$

for any p by Lemma A.14. Continuing the same argument to B and $Y_i \cap B$, we have the assertion. □

Proposition A.16. *Let Y be a reduced compact complex analytic space and let $\phi\colon M \to Y$ be a proper surjective morphism from a finite disjoint union M of compact Kähler manifolds M_α such that $\phi(M_\alpha)$ is an irreducible component of Y. Let $g\colon Y \to Y$ and $f\colon M \to M$ be étale surjective endomorphisms such that $\phi \circ f = g \circ \phi$. If k is a positive integer such that $(f^k)^{-1}(M_\alpha) = M_\alpha$ for any α, then*

$$\rho(\mathrm{H}^*(Y, \mathbb{C}), (g^k)^*) \le \max\nolimits_\alpha \rho(\mathrm{H}^*(M_\alpha, \mathbb{C}), (f^k|_{M_\alpha})^*) = \max\nolimits_{\alpha,\ell} d_l(f^k|_{M_\alpha})$$

for any p, for the induced endomorphisms $f^k|_{M_\alpha}\colon M_\alpha \to M_\alpha$.

Proof. We shall prove by induction on $\dim M = \max\{\dim M_\alpha\}$. If $\dim M = 0$, then the assertion holds by a trivial reason. Assume that the assertion holds if the dimension is less than $\dim M$. If the estimate of the spectral radius holds for a power g^{kl}, then it holds also for k; in fact, $\rho(\mathrm{H}^p(Y, \mathbb{C}), (g^{kl})^*) = \rho(\mathrm{H}^p(Y, \mathbb{C}), (g^k)^*)^l$. Thus, we may replace k with any power kl. We can find the maximum Zariski-open subset $U \subset Y$ such that $\phi^{-1}(U) \to U$ is smooth. For the complement $B = Y \setminus U$ and $A = \phi^{-1}(B)$, we have $g^{-1}(B) = B$ and $f^{-1}(A) = A$ by the maximum property of U and by the étaleness of g and f. Thus, we can apply Lemma A.14, and hence

$$\rho(\mathrm{H}^*(Y, \mathbb{C}), g^*) \le \max\{\rho(\mathrm{H}^*(M, \mathbb{C}), f^*), \rho(\mathrm{H}^*(B, \mathbb{C}), (f|_B)^*), \rho(\mathrm{H}^*(A, \mathbb{C}), (f|_A)^*)\}.$$

Let $A = \bigcup A_\beta$ be the irreducible decomposition. By replacing g with a power g^l, we may assume that $f^{-1}(M_\alpha) = M_\alpha$ and $f^{-1}(A_\beta) = A_\beta$ for any α and β. Let $Z_\beta \to A_\beta$ be an equivariant resolution of singularities of A_β with respect to the étale endomorphism $f|_{A_\beta}$. Let Z be the disjoint union $\bigsqcup Z_\beta$ and let $h\colon Z \to Z$ be the induced étale endomorphism, i.e., $\psi \circ h = f|_A \circ \psi$ for the induced morphism $\psi\colon Z \to A$. Then $(Z \to A \to B, h, g|_B)$ and $(Z \to A, h, f|_A)$ satisfy the same condition of Proposition A.16 as $(M \to Y, f, g)$. By induction, we have

$$\max\{\rho(\mathrm{H}^*(B, \mathbb{C}), (g|_B)^*), \rho(\mathrm{H}^*(A, \mathbb{C}), (f|_A)^*)\} \le \max\nolimits_\beta \rho(\mathrm{H}^*(Z_\beta, \mathbb{C}), (h|_{Z_\beta})^*)$$

after replacing g with a power g^l. On the other hand,

$$\rho(\mathrm{H}^*(Z_\beta, \mathbb{C}), (h|Z_\beta)^*) = \exp(h_{\mathrm{top}}(h|_{Z_\beta})) \le \exp(h_{\mathrm{top}}(f|_{M_\alpha})) = \rho(\mathrm{H}^*(M_\alpha, \mathbb{C}), (f|M_\alpha)^*)$$

for $A_\beta \subset M_\alpha$, by Proposition A.9. Hence,

$$\rho(\mathrm{H}^*(Y, \mathbb{C}), g^*) \leq \max_\alpha \rho(\mathrm{H}^*(M_\alpha, \mathbb{C}), (f|M_\alpha)^*). \qquad \square$$

Let $\pi\colon X \to Y$ be a surjective morphism from a compact complex manifold X into a compact complex analytic variety Y such that a general fiber of π is connected and $\pi \circ f = \pi$. Let $f\colon X \to X$ be a surjective endomorphism such that $\pi \circ f = \pi$. Then f induces a surjective endomorphism $f|_F\colon F \to F$ of the fiber $F = \pi^{-1}(y)$ over any point y.

Proposition A.17. *There exist a finite set $\{Z_i\}_{i=1}^N$ of closed subvarieties of X and a positive number k such that $(f^k)^{-1}(Z_i) = Z_i$ and $\pi(Z_i)$ is a point for any i, and*

$$\Lambda(\mathrm{H}^p(X, \mathbb{C}), (f^k)^*) \subset \bigcup\nolimits_{q=0}^{p} \bigcup\nolimits_{i=1}^{N} \Lambda(\mathrm{H}^q(Z_i, \mathbb{C}), (f^k|_{Z_i})^*).$$

Proof. We have an endomorphism f^* of the complex $\mathrm{R}\,\pi_*\mathbb{C}_X$ in the derived category of sheaves of abelian groups on Y such that the induced endomorphism of $\mathrm{R}\,\Gamma(Y, \mathrm{R}\,\pi_*\mathbb{C}_X)$ coincides with that of $\mathrm{R}\,\Gamma(X, \mathbb{C})$. In particular, the Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(Y, \mathrm{R}^q\,\pi_*\mathbb{C}_X) \Rightarrow E^{p+q} = \mathrm{H}^{p+q}(X, \mathbb{C})$$

admits the endomorphisms $f^*\colon E_r^{p,q} \to E_r^{p,q}$ compatible with f^* on E^{p+q}.

We can consider the subsheaf $(\mathrm{R}^i\,\pi_*\mathbb{C}_X)_\lambda \subset \mathrm{R}^i\,\pi_*\mathbb{C}_X$ of the generalized eigenvectors with respect to f^* and a complex number λ (cf. Notation A.13). Then

$$(E_2^{p,q})_\lambda \simeq \mathrm{H}^p(Y, (\mathrm{R}^q\,\pi_*\mathbb{C}_X)_\lambda)$$

for any p, q, and λ. Now $E_2^{p,q} \Rightarrow E^{p+q}$ is decomposed into $(E_2^{p,q})_\lambda \Rightarrow (E^{p+q})_\lambda$. If $\mathrm{H}^p(X, \mathbb{C})_{\lambda, f^*} \neq 0$ for a complex number λ, then $\mathrm{H}^{p'}(Y, (\mathrm{R}^{q'}\,\pi_*\mathbb{C}_X)_\lambda) \neq 0$ for some non-negative integers p' and q' with $p' + q' = p$; thus, λ is an eigenvalue of the induced endomorphism on $\mathrm{H}^q(F_y, \mathbb{C})$ for the fiber $F_y = \pi^{-1}(y)$ over certain point $y \in Y$ for some $q \leq p$. Since the sheaves $\mathrm{R}^q\,\pi_*\mathbb{C}_X$ are constructible, we can find finitely many points $\{y_\alpha\}$ such that

$$\Lambda(\mathrm{H}^p(X, \mathbb{C}), f^*) \subset \bigcup\nolimits_\alpha \Lambda\left(\mathrm{H}^q(F_{y_\alpha}, \mathbb{C}), (f|_{F_{y_\alpha}})^*\right).$$

Applying Corollary A.15 to $f|_{F_{y_\alpha}}\colon F_{y_\alpha} \to F_{y_\alpha}$, we can show the assertion. $\square$

We are ready to prove Theorem D.

Proof of Theorem D. Step 1. Let $\nu\colon Y' \to Y$ be a bimeromorphic morphism from a complex analytic variety Y'. Then f induces an étale endomorphism $f \times_Y Y'\colon X \times_Y Y' \to X \times_Y Y'$. Let X' be the main component of $X \times_Y Y'$. Thus, there exists an equivariant resolution of singularities $\widetilde{X} \to X'$ with respect to the étale endomorphism. Let $\tilde{f}$ be the induced étale endomorphism of $\widetilde{X}$. Then $d_l(\tilde{f}) = d_l(f)$ by Lemma A.8. Therefore, we may replace Y by a bimeromorphic morphism, freely. Thus, we may assume that Y is nonsingular and $\pi\colon X \to Y$ is smooth over the complement $Y \setminus D$ for a divisor $D \subset Y$.

Step 2. We set $E = \pi^{-1}(D)_{\mathrm{red}}$. For the prime decomposition $E = \sum E_i$, there is a positive integer k such that $(f^k)^{-1}E_i = E_i$. Let $\nu_i\colon Z_i \to E_i$ be a bimeromorphic morphism from a compact Kähler manifold Z_i such that $h_i := \nu_i^{-1} \circ f^k|_{E_i} \circ \nu_i$ is an étale endomorphism of Z_i. For example, we can take such ν_i as an equivariant resolution of singularities of E_i with respect to $f^k|_{E_i}$. Then $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(f|_F)$ for a smooth fiber F of π or $h_{\mathrm{top}}(f) = \max\{k^{-1}h_{\mathrm{top}}(h_i)\}$ by Proposition A.16 and Proposition A.17. Therefore, it is enough to show that $d_l(f^k|_F) \geq d_l(h_i)$ for any i.

Step 3. Let $\sigma\colon S \to Y$ be a flattening of π, i.e., the main component of $X \times_Y S$ is flat over S; the existence of the flattening is proved by Hironaka [24]. Here, we may assume that $\sigma^{-1}D$ is also a normal crossing divisor. Let $M \to X \times_Y S$ be a bimeromorphic morphism from a compact Kähler manifold M which is given as an equivariant resolution of singularities of the main component with respect to the induced étale endomorphism. Let $\mu\colon M \to X$ be the induced bimeromorphic morphism, $\varpi\colon M \to S$ the induced fiber space, and let $g\colon M \to M$ be the induced étale endomorphism, i.e., $\mu \circ g = f \circ \mu$ and $\varpi \circ g = \varpi$. Let G_i be the proper transform of E_i in M for any i. By taking an equivariant embedded resolution of singularities, we may assume that all G_i are nonsingular. Then $\varpi(G_i)$ is a prime component of $\sigma^{-1}D$ for any i by the flattening. Let P be a general point of $\varpi(G_i)$, and let $C \subset S$ be a holomorphic curve isomorphic to a unit disc such that C intersects $\sigma^{-1}D$ with one point P, transversely. Then $\varpi^{-1}(P) \cap G_i$ is a disjoint union $\Gamma = \bigsqcup \Gamma_\alpha$ of submanifolds Γ_α of G_i, and the scheme-theoretic fiber $\varpi^{-1}(P)$ is flat deformation equivalent to a smooth fiber $\varpi^{-1}(s)$, where $s \in C \setminus P$. For a Kähler form η on M and for $d = \dim Y$, we have

$$\begin{aligned}&\int_\Gamma (g^{mk})^*(\eta^l|_\Gamma) \wedge \eta^{n-d-l} \leq \int_M [(g^{mk})^*\eta^l \wedge \eta^{n-d-l}] \cup [\varpi^{-1}(P)] \\ &= \int_M [(g^{mk})^*\eta^l \wedge \eta^{n-d-l}] \cup [\varpi^{-1}(s)] = \int_F (f^{mk})^*(\eta|_F)^l \wedge (\eta|_F)^{n-d-l} = \delta_l(f^{mk}, \eta|_F)\end{aligned}$$

for a general fiber F of π, for any $1 \leq l \leq n-d$ and any $m \geq 1$.

Step 4. For the proof, we use the induction on $\dim Y$. The assertion holds if $\dim Y = 0$. Assume that $\dim Y > 0$. By *Step* 3, we have a morphism $G_i \to \varpi(G_i)$. Let $G_i \to S_i \to \varpi(G_i)$ be the Stein factorization. Then $g^k|_{G_i}$ is an étale endomorphism defined over S_i and $\dim S_i = \dim Y - 1$. Thus, by induction,

$$h_{\mathrm{top}}(g^{k'}|_{G_i}) = h_{\mathrm{top}}(g^{k'}|_{\Gamma_\alpha})$$

for a suitable k' divisible by k. Hence, $h_{\mathrm{top}}(g^{k'}|_{G_i}) \leq h_{\mathrm{top}}(f^{k'}|_F)$ by *Step* 3. Since we may replace $(Z_i \to E_i, h_i)$ in *Step* 2 with $(G_i \to E_i, g^k|_{G_i})$, we have the expected equality. $\square$

References

[1] E. Amerik and F. Campana, Fibrations meromorphes sur certaines varietes de classe canonique triviale, preprint 2005 (math.AG/0510299).

[2] A. Beauville, Some remarks on Kähler manifolds with $c_1 = 0$, *Classification of Algebraic and Analytic Manifolds* (Katata, 1982, ed. K. Ueno), pp. 1–26, Progr. Math., **39** Birkhäuser 1983.

[3] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, Existence of minimal models for varieties of log general type, preprint 2006 (math.AG/0610203).

[4] E. Bierstone and P. D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math., **128** (1977), 207302.

[5] G. Birkhoff, Linear transformations with invariant cones, Amer. Math. Monthly, **74** (1967), 274–276.

[6] F. Bogomolov, On the decomposition of Kähler manifolds with trivial canonical class, Math. USSR Sbornick, **22** (1974), 580–583.

[7] F. Campana, On twister spaces of the class $\mathcal{C}$, J. Diff. Geom. **33** (1991), 541–549.

[8] F. Campana, Connexité rationnelle des variétés de Fano, Ann. Sci. École Norm. Sup. (4) **25** (1992), 539–545.

[9] P. Deligne, Théorème de Lefschetz et critères de dégénérescence de suites spectrales, Publ. Math. IHES., **35** (1968), 107–126.

[10] T. -C. Dinh and N. Sibony, Groupes commutatifs d'automorphismes d'une variété kählérienne compacte, Duke Math. J. **123** (2004), 311328.

[11] S. Encinas and H. Hauser, Strong resolution of singularities in characteristic zero, Comment. Math. Helv., **77** (2002), 821845.

[12] S. Encinas and O. E. Villamayor, A course on constructive desingularization and equivariance, *Resolution of singularities* (*Obergurgl, 1997*), Progr. Math. vol. **181**, pp. 147227, Birkhäuser, 2000.

[13] S. Friedland, Entropy of polynomial and rational maps, Ann. of Math. **133** (1991), 359368.

[14] A. Fujiki, On the blowing down of analytic spaces, Publ. Res. Inst. Math. Sci. Kyoto Univ., **10** (1975), 473–507.

[15] A. Fujiki, Closedness of the Douady spaces of compact Kähler spaces, Publ. Res. Inst. Math. Sci. Kyoto Univ., **14** (1978), 1–52.

[16] A. Fujiki, On automorphism groups of compact Kähler manifolds, Invent. Math., **44** (1978), 225–258.

[17] Y. Fujimoto, Endomorphisms of smooth projective 3-folds with nonnegative Kodaira dimension, Publ. Res. Inst. Math. Sci. Kyoto Univ., **38** (2002), 33–92.

[18] Y. Fujimoto and N. Nakayama, Endomorphisms of smooth projective 3-folds with non-negative Kodaira dimension, II, preprint RIMS-1566, Res. Inst. Math. Sci. Kyoto Univ. 2006.

[19] T. Graber, J. Harris and J. Starr, Families of rationally connected varieties, J. Amer. Math. Soc. **16** (2003), 57–67.

[20] M. Gromov, On the entropy of holomorphic maps, Enseign. Math. **49** (2003), 217235.

[21] V. Guedj, Ergodic properties of rational mappings with large topological degree, Ann. of Math. **161** (2005), 15891607.

[22] M. Hanamura, On the birational automorphism groups of algebraic varieties, Compos. Math. **63** (1987), 123–142.

[23] M. Hanamura, Structure of birational automorphism groups, I: non-uniruled varieties, Invent. Math., **93** (1988), 383–403.

[24] H. Hironaka, Flattening theorem in complex-analytic geometry, Amer. J. Math., **97** (1975), 503547.

[25] S. Iitaka, *Algebraic Geometry – an Introduction to Birational Geometry of Algebraic Varieties*, Graduate Texts in Mathematics, No. **76**, Springer 1981.

[26] Y. Kawamata, Characterization of abelian varieties, Compositio Math. **43** (1981), 253–276.

[27] Y. Kawamata, Minimal models and the Kodaira dimension of algebraic fiber spaces, J. Reine Angew. Math. **363** (1985), 1–46.

[28] Y. Kawamata, Abundance theorem for minimal threefolds, Invent. Math., **108** (1992), 229–246.

[29] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem, *Algebraic geometry, Sendai, 1985* (T. Oda ed.), pp. 283–360, Adv. Stud. Pure Math., **10**, Kinokuniya and North-Holland, 1987.

[30] J. Kollár, Higher direct images of dualizing sheaves, II, Ann. of Math., **124** (1986), 171–202.

[31] J. Kollár, Shararevich maps and plurigenera of algebraic varieties, Invent. Math., **113** (1993), 177–215.

[32] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. **3** Folge. Springer-Verlag, 1996.

[33] J. Kollár, Resolution of singularities – seattle lecture, *notes for series of lectures given in the AMS Summer Institute*, 2005, preprint (math.AG/0508332).

[34] J. Kollár, Y. Miyaoka and S. Mori, Rational connected varieties, J. Alg. Geom. **1** (1992), 429–448.

[35] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, **134**, Cambridge University Press, 1998.

[36] D. I. Lieberman, Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds, Lecture Notes in Math., **670**, pp. 140–186, Springer, 1978.

[37] Y. Miyaoka, On the Kodaira dimension of minimal threefolds, Math. Ann., **281** (1988), 325–332.

[38] Y. Miyaoka, Abundance conjecture for 3-folds: case $\nu = 1$, Comp. Math., **68** (1988), 203–220.

[39] S. Mori, Classification of higher-dimensional varieties, *Algebraic geometry, Bowdoin, 1985* pp. 269–331, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., 1987.

[40] I. Nakamura and K. Ueno, An addition formula for Kodaira dimension of analytic fibre bundles whose fibre are Moisezon manifolds, J. Math. Soc. Japan **25** (1973), 363–371.

[41] N. Nakayama, Compact Kähler manifolds whose universal covering spaces are biholomorphic to $\mathbb{C}^n$, (a modified version, but in preparation); the original is RIMS preprint **1230**, Res. Inst. Math. Sci. Kyoto Univ. 1999.

[42] N. Nakayama, Ruled surfaces with non-trivial surjective endomorphisms, Kyushu J. Math., **56** (2002), 433–446.

[43] Y. Namikawa and J. Steenbrink, Global smoothing of Calabi-Yau threefolds, Invent. Math., **122** (1995), 403–419.

[44] S. Takayama, Local simple connectedness of resolutions of log-terminal singularities, Internat. J. Math. **14** (2003), no. 8, 825–836.

[45] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Lecture Notes in Mathematics, Vol. **439**, Springer-Verlag, 1975.

[46] J. Varouchas, Stabilité de la classe des variétés kähleriennes pour les certains morphisms propres, Invent. Math., **77** (1984), 117–128.

[47] E. Viehweg and K. Zuo, On the isotriviality of families of projective manifolds over curves, J. Algebraic Geom. **10** (2001), 781–799.

[48] J. Włodarczyk, Simple Hironaka resolution in characteristic zero, J. Amer. Math. Soc., **18** (2005), 779–822.

[49] Y. Yomdin, Volume growth and entropy, Israel J. Math. **57** (1987), 285–300.

[50] D. -Q. Zhang, Dynamics of automorphisms on projective complex manifolds, preprint 2006.

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