Obstructions to deforming curves on a 3-fold, II: Deformations of degenerate curves on a del Pezzo 3-fold

Hirokazu NASU*

Abstract

We study the Hilbert scheme $\mathrm{Hilb}^{sc} V$ of smooth connected curves on a smooth del Pezzo 3-fold V. We prove that every degenerate curve C, i.e. every curve contained in a smooth hyperplane section S of V, does not deform to a non-degenerate curve if the following two conditions are satisfied: (i) $\chi(V, \mathcal{I}_C(S)) \geq 1$ and (ii) for every line ℓ on S such that $\ell \cap C = \emptyset$, the normal bundle $N_{\ell/V}$ is trivial (i.e. $N_{\ell/V} \simeq \mathcal{O}_{\mathbb{P}^1}{}^{\oplus 2}$). As a consequence, we prove an analogue (for $\mathrm{Hilb}^{sc} V$) of a conjecture of J.O. Kleppe which is concerned with non-reduced components of the Hilbert scheme $\mathrm{Hilb}^{sc} \mathbb{P}^3$ of curves in the 3-dimensional projective space $\mathbb{P}^3$.

1 Introduction

This paper is a sequel to a joint work [7] with Shigeru Mukai. In [7] the embedded deformations of smooth curves C on a smooth projective 3-fold V have been studied under the presence of a smooth surface S such that $C \subset S \subset V$, especially when V is a uniruled 3-fold. In this paper, the same subject is studied in detail especially when V is a del Pezzo 3-fold.

It is known that even if the deformations of $C \subset S$ and $S \subset V$ behave well, those of $C \subset V$ behave badly in general. For example, even if $\mathrm{Hilb}\, V$ and $\mathrm{Hilb}\, S$ are nonsingular of expected dimension $\chi(N_{S/V})$ and $\chi(N_{C/S})$ at $[S]$ and $[C]$ respectively, $\mathrm{Hilb}\, V$ can be generically non-reduced along some component passing through $[C]$ (cf. Mumford's example in [8]). Such a non-reduced component of the Hilbert scheme $\mathrm{Hilb}^{sc} V$ of smooth connected curves on V has been constructed for many uniruled 3-folds V in [7]. The non-reducedness is originated from the non-surjectivity of the restriction map

$$H^0(S, N_{S/V}) \xrightarrow{|_C} H^0(C, N_{S/V}\big|_C). \tag{1.1}$$

*Supported in part by the 21-st century COE program "Formation of an International Center of Excellence in the Frontier of Mathematics and Fostering of Researchers in Future Generations".

If (1.1) is surjective, then C is *stably degenerate*, i.e. every (small) deformation of C in V is contained in a divisor S' of V which is algebraically equivalent to S. If moreover $\operatorname{Hilb} V$ and $\operatorname{Hilb} S$ are respectively nonsingular at $[S]$ and $[C]$, then so is $\operatorname{Hilb} V$ at $[C]$. In this paper, we study the behavior of the deformation of C in V to answer the following problem raised by Mukai:

Problem 1.1. *Suppose that* (1.1) *is not surjective and* $\chi(V, \mathcal{I}_C(S)) > 0$. *Then (1) Is C stably degenerate? (2) Is* $\operatorname{Hilb}^{sc} V$ *singular at* $[C]$*?*

J. O. Kleppe [6] and Ph. Ellia [1] considered this problem for the case where V is the 3-dimensional projective space $\mathbb{P}^3$, $S \subset \mathbb{P}^3$ is a smooth cubic surface and C is a smooth connected curve on S. Kleppe gave a conjecture (cf. Conjectures 5.1), which can be reformulated as follows:

Conjecture 1.2. *Let $C \subset S \subset \mathbb{P}^3$ be as above and assume that $\chi(\mathbb{P}^3, \mathcal{I}_C(3)) \geq 1$. If C is linearly normal, then every (small) deformation $C' \subset \mathbb{P}^3$ of C is contained in a cubic surface $S' \subset \mathbb{P}^3$, i.e. C is stably degenerate.*

As a testing ground of his conjecture, we consider Problem 1.1 for the case where V is a smooth del Pezzo 3-fold of degree n (cf. §2.2) and S is a smooth polarization of V (i.e. a smooth hyperplane section of V when $n \geq 3$) and C is a smooth connected curve on S. The following theorem is an analogue of Kleppe's conjecture.

Theorem 1.3 (Main). *Let $C \subset S \subset V$ be as above and assume that $\chi(V, \mathcal{I}_C(S)) \geq 1$. If every line ℓ on S such that $C \cap \ell = \emptyset$ is a* good line *on V (i.e. the normal bundle $N_{\ell/V}$ of ℓ in V is trivial), then:*

(1) *C is stably degenerate, and*

(2) $\operatorname{Hilb}^{sc} V$ *is nonsingular at $[C]$ if and only if $H^1(V, \mathcal{I}_C(S)) = 0$.*

Let d and g be the degree ($= (C \cdot S)_V$) and genus of C, respectively. Then the condition $\chi(V, \mathcal{I}_C(S)) \geq 1$ is equivalent to $g \geq d - n$. If $g < d - n$, then it follows from a dimension count that C is not stably degenerate (Remark 4.3). If some ℓ is a *bad line* on V (i.e. $N_{\ell/V} \not\simeq \mathcal{O}_{\mathbb{P}^1}{}^{\oplus 2}$) then C is not necessarily stably degenerate (Proposition 5.4). If $g \geq 2$, then the dimension of $\operatorname{Hilb}^{sc} V$ at $[C]$ is equal to $d + g + n$, while its tangential dimension at $[C]$ is equal to $d + g + n + h^1(V, \mathcal{I}_C(S))$. As a corollary to Theorem 1.3, we give a sufficient condition for a maximal family W of degenerate curves on V to be an irreducible component of the Hilbert scheme $\operatorname{Hilb}^{sc} V$ and decide whether $\operatorname{Hilb}^{sc} V$ is generically non-reduced along W or not (Theorem 4.8).

One of the main tools used in this paper is the infinitesimal analysis of the Hilbert scheme developed in [7]. As is well known, every first order infinitesimal deformation $\tilde{C}$

of $C \subset V$ determines a global section $\alpha \in H^0(N_{C/V})$ and a cohomology class $\mathrm{ob}(\alpha) \in H^1(N_{C/V})$ (called the *obstruction*) such that $\tilde{C}$ lifts to a deformation over $\operatorname{Spec} k[t]/(t^3)$ if and only if $\mathrm{ob}(\alpha) = 0$ (cf. §2.3). Let $\pi_S : N_{C/V} \to N_{S/V}\big|_C$ be a natural projection of normal bundles. In [7] Mukai and Nasu studied the *exterior component* of α and $\mathrm{ob}(\alpha)$, i.e. the images of α and $\mathrm{ob}(\alpha)$ by the induced maps $H^i(\pi_S) : H^i(N_{C/V}) \to H^i(N_{S/V}\big|_C)$ ($i = 0, 1$), respectively. They showed that if the exterior component of α lifts to a global section $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$ for some curve E on S, then that of $\mathrm{ob}(\alpha)$ is nonzero provided that certain additional conditions on E, C and v hold (cf. [7, Theorem 1.6]). Such a rational section v of $N_{S/V}$ admitting a pole along E is called an *infinitesimal deformation with a pole*. In §3 we see that an infinitesimal deformation with a pole induces an obstructed infinitesimal deformation of $S^\circ := S \setminus E$ in $V^\circ := V \setminus E$ (Theorem 3.1). By virtue of this obstruction, we prove the main theorem in §4. In §5 we give some examples of generically non-reduced components of the Hilbert scheme of curves on a del Pezzo 3-fold as an application.

Acknowledgements I should like to express my sincere gratitude to Professor Shigeru Mukai. He showed me the example of non-reduced components of the Hilbert scheme of canonical curves given in §5.2 as a simplification of Mumford's example of a non-reduced component of $\operatorname{Hilb}^{sc}\mathbb{P}^3$. This led me to research the topic of this paper. Throughout the research, he made many suggestions which are useful for obtaining and improving the proofs. In particular, according to his suggestion, the deformation theory of an open surface in an open 3-fold is organized in §3 to improve the crucial part of the proof of Proposition 4.6. I am grateful to Professor Jan Oddvar Kleppe for giving me useful comments about the Hilbert-flag scheme, one of which helped me to correct a mistake in the proof of the main theorem.

Notation and Conventions For a given closed subscheme $X \subset V$ of a scheme V, we denote by $\mathcal{I}_X$ the ideal sheaf of X in V and denote by $N_{X/V}$ the normal bundle $(\mathcal{I}_X/{\mathcal{I}_X}^2)^\vee$ of X in V. For a sheaf $\mathcal{F}$ on V, we denote the restriction map $H^i(V, \mathcal{F}) \to H^i(X, \mathcal{F}\big|_X)$ by $\big|_X$. We denote the Euler-Poincaré characteristic of $\mathcal{F}$ by $\chi(V, \mathcal{F})$ or $\chi(\mathcal{F})$. $\operatorname{Hilb}^{sc} V$ denotes the subscheme of the Hilbert scheme $\operatorname{Hilb} V$ whose point corresponds to a smooth connected curve on V. We work over an algebraically closed field k of characteristic 0.

2 Preliminaries

2.1 Del Pezzo surfaces

A *del Pezzo surface* is a smooth surface S with ample anti-canonical divisor $h := -K_S$. Except for $\mathbb{P}^1 \times \mathbb{P}^1$, every del Pezzo surface can be realized as a blow-up of $\mathbb{P}^2$ at fewer than 9 points. We denote the blow-up of $\mathbb{P}^2$ at $(9-n)$-points by S_n. A (-1)-$\mathbb{P}^1$ on S_n is called a *line*. We obtain a del Pezzo surface S_n by blowing down a line from S_{n-1}.

Lemma 2.1. *Let D be a divisor on a del Pezzo surface S. If D is nef and $\chi(-D) \geq 0$, then $H^1(S, -D) = 0$.*

Proof. If D is big (i.e. $D^2 > 0$), then it follows from the Kawamata-Viehweg vanishing theorem. Hence we assume that D is not big. If $S = S_n$, then D is a multiple mq ($m \geq 0$) of conic q (i.e. $q \sim (1;1,0,\dots,0)$ under a suitable blow-up $S_n \to \mathbb{P}^2$). By the Riemann-Roch theorem, we have

$$\begin{aligned}\chi(-D) &= \frac{1}{2}(-mq)\cdot(-mq+h) + \chi(\mathcal{O}_S)\\ &= -m+1,\end{aligned}$$

since $q^2 = 0$ and $q \cdot h = 2$. Thus we have $m = 0$ or 1 by assumption. This implies $H^1(-mq) = 0$. If $S = \mathbb{P}^1 \times \mathbb{P}^1$, then D is of bidegree $(m, 0)$ or $(0, m)$ with $m \geq 0$. Again by the Riemann-Roch theorem, we have $\chi(-D) = -m+1 \geq 0$. Thus $H^1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-D)) = 0$. □

Lemma 2.2. *Let D be an effective divisor on a del Pezzo surface S. Then the lines ℓ such that $D \cdot \ell < 0$ are mutually disjoint. Moreover the fixed part* $\operatorname{Bs}|D|$ *of $|D|$ is equal to*

$$-\sum_{D\cdot\ell<0}(D\cdot\ell)\ell.$$

Proof. It is clear that a line ℓ satisfying $D \cdot \ell < 0$ is contained in $\operatorname{Bs}|D|$. On the other hand, every irreducible curve C on S except for a line can move on S since $\chi(C) \geq 2$ and $H^2(C) = 0$. Therefore the linear system $|D|$ is decomposed into the sum

$$|D| = |D'| + \sum_{i=1}^{k} m_i\ell_i, \qquad (m_i > 0)$$

of a linear system $|D'|$ without base components and union of lines $\ell_1, \dots, \ell_k$. If $\ell_i \cap \ell_j \neq \emptyset$, then $|\ell_i + \ell_j|$ can move by $\chi(\ell_i + \ell_j) = 2$. Thus ℓ_i's are mutually disjoint. Since $m_i = (D' - D) \cdot \ell_i$, it suffices to show that $D' \cdot \ell_i = 0$ for any i. Since D' is effective and nef, we have $(D')^2 \geq 0$. Moreover, since $-K_S$ is ample, $D' - K_S$ is nef and big, Thus we have $H^1(D') = H^1((D' - K_S) + K_S) = 0$. If $D' \cdot \ell_i \geq 1$, then it follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_S(D') \longrightarrow \mathcal{O}_S(D' + \ell_i) \longrightarrow \mathcal{O}_S(D' + \ell_i)\big|_{\ell_i} \longrightarrow 0$$

that $h^0(D' + \ell_i) > h^0(D')$. □

Lemma 2.3. *Let S be a del Pezzo surface and let $\varepsilon : S \to F$ be the blow-down of m lines (i.e. (-1)-$\mathbb{P}^1$'s) ℓ_i $(1 \leq i \leq m)$ from S. If a divisor D on F satisfies $h^0(F, D) \geq m$, then we have*

$$h^0(S, \varepsilon^* D - \sum_{i=1}^{m} \ell_i) = h^0(F, D) - m.$$

Proof. Put $D_j := \varepsilon^* D - \sum_{1 \leq i \leq j} \ell_i$. Since the images of ℓ_i on F are points, we have $h^0(D_j) \geq h^0(D) - j$ for every $1 \leq j \leq m$. Moreover since $D_{j-1} \cdot \ell_j = 0$, Lemma 2.2 shows that ℓ_j is not contained in $\mathrm{Bs}\,|D_{j-1}|$. Hence $\dim |D_j|$ decreases one by one and we have $h^0(D_m) = h^0(D) - m$. □

Let C be a smooth curve of degree d and genus $g > 0$ on S. We consider the restriction of the anti-canonical linear system $|-K_S|$ on S to C. The restriction map $H^0(-K_S) \to H^0(-K_S\big|_C)$ is not surjective in general. We define an effective divisor E on S to be the sum of all lines E_i $(1 \leq i \leq m)$ on S such that $E_i \cap C = \emptyset$. We put $E = 0$ if there exists no such line E_i.

Proposition 2.4. *If $g \geq d - n$, then the restriction map*

$$H^0(S, -K_S + E)) \xrightarrow{|_C} H^0(C, -K_S\big|_C) \tag{2.1}$$

is surjective.

Proof. It suffices to show that $H^1(-K_S + E - C) = 0$ by the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-K_S + E - C) \longrightarrow \mathcal{O}_S(-K_S + E) \longrightarrow \mathcal{O}_S(-K_S)\big|_C \longrightarrow 0.$$

Claim. $D := C + K_S - E$ is nef.

Since S is regular (i.e. $H^1(K_S) = 0$), the restriction map $\big|_C : H^0(C + K_S) \to H^0(K_C)$ is surjective. Since $g > 0$ by assumption, the linear system $|C + K_S|$ on S is non-empty. Let l be a line on S. Since C is not a line, C is nef. Therefore we have $C \cdot \ell \geq 0$ and hence $(C + K_S) \cdot \ell \geq -1$. If ℓ is contained in $\mathrm{Bs}\,|C + K_S|$, then by Lemma 2.2, ℓ is disjoint to C and hence coincides with some E_i. Thus we have $E = \mathrm{Bs}\,|C + K_S|$ and $|D|$ does not have base components. In particular, D is nef.

By the Riemann-Roch theorem, we have $\chi(-D) = g - d + n + m \geq 0$, where $m \geq 0$ is the number of lines E_i. By Lemma 2.1, we have $H^1(-D) = 0$. □

Remark 2.5. If C is not elliptic, then the map (2.1) is an isomorphism. Indeed, since $K_C \not\sim 0$, we have $C + K_S \not\sim E$ by adjunction. Hence $D \not\sim 0$ and $H^0(-D) = 0$.

Lemma 2.6. *If C is not elliptic and $g \geq d - n$, then the restriction map*

$$H^0(S, C + 2K_S - 2E) \xrightarrow{|_E} H^0(E, (C + 2K_S - 2E)\big|_E)$$

is surjective.

Proof. Let $\varepsilon : S \to F$ be the blow-down of E from S. Then $C + 2K_S - 2E$ is a pull back $\varepsilon^* D$ of a divisor D on F. By the Riemann-Roch theorem, we have $\chi(S, \varepsilon^* D) = g - d + n + m \geq m$. Since $(K_S - \varepsilon^* D) \cdot C = (-K_S - C) \cdot C = 2 - 2g < 0$, we have $H^2(S, \varepsilon^* D) \simeq H^0(S, K_S - \varepsilon^* D)^\vee = 0$. Hence $h^0(F, D) = h^0(S, \varepsilon^* D) \geq \chi(S, \varepsilon^* D) \geq m$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(\varepsilon^* D - E) \longrightarrow \mathcal{O}_S(\varepsilon^* D) \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

Since $h^0(\varepsilon^* D) - h^0(\varepsilon^* D - E) = h^0(\mathcal{O}_E)$ by Lemma 2.3, the restriction map is surjective. $\square$

Let S be a smooth projective surface and let L be a line bundle on S.

Lemma 2.7. *Let E be a disjoint union of irreducible curves E_i ($i = 1, \dots, m$) on S with $E_i^2 < 0$ and let $\iota : S^\circ := S \setminus E \hookrightarrow S$ be the open immersion. If $\deg(L\big|_{E_i}) \leq 0$ for every i, then the map*

$$H^1(S, L) \to H^1(S^\circ, L\big|_{S^\circ})$$

induced by the sheaf inclusion $L \hookrightarrow L \otimes \iota_* \mathcal{O}_{S^\circ}$ is injective.

The proof is similar to that of [7, Lemma 2.3] and we omit it here. Lemma 2.7 allows us to identify $H^1(S, L(nE))$ $(n \geq 0)$ with their images in $H^1(S^\circ, L\big|_{S^\circ})$. As a result, under the identification we obtain a natural filtration

$$H^1(S, L) \subset H^1(S, L(E)) \subset H^1(S, L(2E)) \subset \cdots \subset H^1(S^\circ, L\big|_{S^\circ})$$

on $H^1(S^\circ, L\big|_{S^\circ})$.

2.2 Del Pezzo threefolds

A *del Pezzo threefold* is a pair (V, H) consisting of a (smooth) irreducible projective variety V of dimension 3 and an ample Cartier divisor H on V such that $-K_V = 2H$. Here H is called the *polarization* of V and sometimes omitted. The self-intersection number $n := H^3$ is called the *degree* of V. It is known that the linear system $|H|$ on V determines a double cover $\varphi_{|H|} : V \to \mathbb{P}^3$ if $n = 2$, and an embedding $\varphi_{|H|} : V \hookrightarrow \mathbb{P}^{n+1}$ if $n \geq 3$. If S is a

Table 1: Del Pezzo 3-folds

del Pezzo 3-folds	n	ρ	
$V_1 = (6) \subset \mathbb{P}(3,2,1,1,1)$	1	1	a weighted hypersurface of degree 6
$V_2 = (4) \subset \mathbb{P}(2,1,1,1,1)$	2	1	a weighted hypersurface of degree 4 [a]
$V_3 = (3) \subset \mathbb{P}^4$	3	1	a cubic hypersurface
$V_4 = (2) \cap (2) \subset \mathbb{P}^5$	4	1	a complete intersection of two quadrics
$V_5 = [\mathrm{Gr}(2,5) \overset{\text{Plücker}}{\hookrightarrow} \mathbb{P}^9] \cap \mathbb{L}^{(6)}$	5	1	a linear section of Grassmannian
$V_6 = [\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \overset{\text{Segre}}{\hookrightarrow} \mathbb{P}^7]$	6	3	
$V_6' = [\mathbb{P}^2 \times \mathbb{P}^2 \overset{\text{Segre}}{\hookrightarrow} \mathbb{P}^8] \cap \mathbb{L}^{(7)}$	6	2	
$V_7 = \mathrm{Bl}_{\mathrm{pt}}\, \mathbb{P}^3 \subset \mathbb{P}^8$	7	2	the blow-up of $\mathbb{P}^3$ at a point [b]
$V_8 = \mathbb{P}^3 \overset{\text{Veronese}}{\hookrightarrow} \mathbb{P}^9$	8	1	the Veronese image of $\mathbb{P}^3$

[a] Another realization of V_2 is a double cover of $\mathbb{P}^3$ branched along a quartic surface.

[b] V_7 is realized as the projection of $V_8 \subset \mathbb{P}^9$ from one of its point.

smooth member of $|H|$, then the pair $(S, H|_S)$ is a del Pezzo surface of degree n. Every smooth del Pezzo 3-fold is one of V_n $(1 \leq n \leq 8)$ or V_6' in Table 1, in which $\mathbb{L}^{(i)}$ denotes a linear subspace of dimension i, and n and ρ respectively denote the degree and the Picard number of V_n (and of V_6') (cf. [2],[3],[4]). It is known that a smooth 3-fold $V \subset \mathbb{P}^{n+1}$ $(n \geq 3)$ is a del Pezzo 3-fold of degree n if a linear section $[V \subset \mathbb{P}^{n+1}] \cap H_1 \cap H_2$ with two general hyperplanes $H_1, H_2 \subset \mathbb{P}^{n+1}$ is an elliptic normal curve in $\mathbb{P}^{n-1}$.

We briefly review the basics of the Hilbert scheme of lines on a del Pezzo 3-fold. We refer to Iskovskih ([4],[5]) for the detail. Let (V, H) be a smooth del Pezzo 3-fold of degree n. By a *line* on (V, H), we mean a reduced irreducible curve ℓ on V such that $(\ell \cdot H)_V = 1$ and $\ell \simeq \mathbb{P}^1$. If $n \leq 7$ then V contains a line ℓ. Then there are only the following possibilities for the normal bundle of ℓ in V:

$$\begin{array}{lll} (0,0): & N_{\ell/V} \simeq \mathcal{O}_{\mathbb{P}^1}{}^{\oplus 2}, & \cdots \ (\textit{good line}) \\ (1,\text{-}1): & N_{\ell/V} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1), & \\ (2,\text{-}2): & N_{\ell/V} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \quad (\text{only if } n = 1 \text{ or } 2), & \Bigg\} \ (\textit{bad line}) \\ (3,\text{-}3): & N_{\ell/V} \simeq \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \quad (\text{only if } n = 1). & \end{array}$$

The Hilbert scheme Γ of lines on V is called the *Fano surface* of V, and in fact every irreducible (non-embedded) component of Γ is of dimension two. Let $\Gamma_i \subset \Gamma$ be an irreducible component and let S_i be the universal family of lines on V over Γ_i. Then we have a natural diagram.

$$\begin{array}{ccc} S_i & \xrightarrow{p} & V \\ {\scriptstyle\pi}\downarrow & & \\ \Gamma_i. & & \end{array}$$

By [5, Chap.III, Proposition 1.3], if $n \geq 3$ then either (a) or (b) holds:

(a) p is surjective; in this case a general line in Γ_i is a good line;

(b) $p(S_i) \simeq \mathbb{P}^2$ is a plane on $V \subset \mathbb{P}^{n+1}$; in this case every line in Γ_i is a bad line.

The proof works fine for the case $n \leq 2$ as well*. If $n \neq 7$ then every irreducible component of Γ is of type (a). If $n = 7$ then Γ consists of two irreducible components $\Gamma_i \simeq \mathbb{P}^2 (i = 0, 1)$, one of which satisfies (a) and the other satisfies (b). Therefore there exists a good line on V if $n \neq 8$.

Lemma 2.8. *Let (V, H) be a smooth del Pezzo 3-fold of degree n and let S be a general member of $|H|$. If $n \neq 7$ then S does not contain a bad line. If $n = 7$ then S contains three lines, one of which is bad, while the others are good.*

Proof. There exists no line on V_8. If $n \neq 7$, then the locus $\mathfrak{B}$ of bad lines in the Fano surface Γ is of dimension one. Let p_i denote the projection of

$$\{(\ell, S) \mid \ell \subset S\} \subset \Gamma \times |H|$$

to the i-th factor. Since the fiber of p_1 is of dimension $n-1$, $p_2(p_1^{-1}(\mathfrak{B}))$ is a proper closed subset of $|H| \simeq \mathbb{P}^{n+1}$. Hence every general member $S \in |H|$ contains no bad line.

Suppose that $V = V_7$. Then S is a del Pezzo surface S_7 and isomorphic to a blow-up of $\mathbb{P}^2$ at two distinct points. Hence there are three lines (i.e. three (-1)-$\mathbb{P}^1$'s) ℓ_0, ℓ_1, ℓ_2 on S forming the configuration in Figure 1:

Here ℓ_0 is distinguished by the fact that it intersects both of the other lines. Recall that V_7 is isomorphic to the blown-up of $\mathbb{P}^3$ at a point. Then the exceptional divisor $P \simeq \mathbb{P}^2$ is a unique plane on V_7 and ℓ_0 is exactly the intersection of S with P (cf. [5, Chap II, §1.4]). Since $N_{\ell_0/P} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$, ℓ_0 is a bad line on V_7. On the other hand, ℓ_1 and ℓ_2 are good lines on V_7 since S is general. □

*Indeed, they assume $n \geq 3$ only for showing the existence of smooth hyperplane section S of V which contains l. Then there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow N_{\ell/V} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

In particular, ℓ is either of type $(0, 0)$ or $(1, -1)$ if $n \geq 3$.

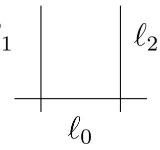

Figure 1: (-1)-$\mathbb{P}^1$'s on S_7

2.3 Infinitesimal deformations and obstructions

Let V be a smooth variety and $X \subset V$ a smooth closed subvariety. An *(embedded) first order infinitesimal deformation* of $X \subset V$ is a closed subscheme $\tilde{X} \subset V \times \operatorname{Spec} k[t]/(t^2)$ which is flat over $\operatorname{Spec} k[t]/(t^2)$ and whose central fiber is $X \subset V$. It is well known that there exists a one to one correspondence between the group of homomorphisms $\alpha : \mathcal{I}_X \to \mathcal{O}_X$ and the first order infinitesimal deformations $\tilde{X}$ of $X \subset V$. In what follows, we identify $\tilde{X}$ with α and abuse the notation. The standard exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{2.2}$$

induces $\delta : \operatorname{Hom}(\mathcal{I}_X, \mathcal{O}_X) \to \operatorname{Ext}^1(\mathcal{I}_X, \mathcal{I}_X)$ as a coboundary map. Then $\alpha \in \operatorname{Hom}(\mathcal{I}_X, \mathcal{O}_X)$ (i.e. $\tilde{X}$) lifts to a deformation over $\operatorname{Spec} k[t]/(t^3)$ if and only if

$$\operatorname{ob}(\alpha) := \delta(\alpha) \cup \alpha \in \operatorname{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$$

is zero, where $\cup$ is the cup product map

$$\operatorname{Ext}^1(\mathcal{I}_X, \mathcal{I}_X) \times \operatorname{Hom}(\mathcal{I}_X, \mathcal{O}_X) \xrightarrow{\cup} \operatorname{Ext}^1(\mathcal{I}_X, \mathcal{O}_X).$$

Then $\operatorname{ob}(\alpha)$ is called the *obstruction* of α (i.e. $\tilde{X}$). Since both X and V are smooth, $\operatorname{ob}(\alpha)$ is contained in $H^1(N_{X/V}) \subset \operatorname{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$.

Since $\operatorname{Hom}(\mathcal{I}_X, \mathcal{O}_X) \simeq H^0(N_{X/V})$, we regard α as a global section of $N_{X/V}$ from now on. If X is a hypersurface of V, then $\operatorname{ob}(\alpha)$ becomes a simpler cup product.

Lemma 2.9. *Let X be a smooth hypersurface of V. Let* †

$$d_X : H^0(X, N_{X/V}) \longrightarrow H^1(X, \mathcal{O}_X) \tag{2.3}$$

be the composite map of the coboundary map $\delta : H^0(N_{X/V}) \to H^1(\mathcal{O}_V)$ of the exact sequence (2.2)$\otimes \mathcal{O}_V(X)$ and the restriction map $\big|_X : H^1(\mathcal{O}_V) \to H^1(\mathcal{O}_X)$. Then $\operatorname{ob}(\alpha)$ is equal to the cup product $d_X(\alpha) \cup \alpha$, where $\cup$ is the cup product map

$$H^1(X, \mathcal{O}_X) \times H^0(X, N_{X/V}) \xrightarrow{\cup} H^1(X, N_{X/V}).$$

†The map d_X is equal to the map $d_{X,\mathcal{O}_V(X)}$ defined in [7, §2.1].

Proof. Since $\mathcal{I}_X \simeq \mathcal{O}_V(-X)$ is a line bundle on V, we have $\mathrm{Ext}^i(\mathcal{I}_X, \mathcal{O}_X) \simeq H^i(N_{X/V})$ $(i = 0, 1)$ and $\mathrm{Ext}^1(\mathcal{I}_X, \mathcal{I}_X) \simeq H^1(\mathcal{O}_V)$. Hence the coboundary map δ appearing in the definition of $\mathrm{ob}(\alpha)$ is a map from $H^0(N_{X/V})$ to $H^1(\mathcal{O}_V)$. Since α is a cohomology class on X, the cup product map $H^1(\mathcal{O}_V) \to H^1(N_{X/V})$ with α factors through the restriction map $\big|_X$. $\square$

We introduce the exterior component defined in [7]. Let Y be a smooth hypersurface of V containing X. Then the natural projection $\pi_Y : N_{X/V} \to N_{Y/V}\big|_X \simeq \mathcal{O}_X(Y)$ of normal bundles induces the maps $H^i(\pi_Y)$ of their cohomology groups for $i = 0, 1$.

Definition 2.10. We denote the images $H^0(\pi_Y)(\alpha)$ and $H^1(\pi_Y)(\mathrm{ob}(\alpha))$ by $\pi_Y(\alpha)$ and $\mathrm{ob}_Y(\alpha)$, respectively and call them the *exterior component* of α and $\mathrm{ob}(\alpha)$ (with respect to Y).

3 Infinitesimal deformations with a pole

Let V be a smooth projective 3-fold, S a smooth surface in V, E a smooth curve on S. We put $V^\circ := V \setminus E$ and $S^\circ := S \setminus E$, the complemental open subvarieties. We study the first order infinitesimal deformations of $S^\circ \subset V^\circ$ when the self-intersection number of E on S is negative. We are interested in a rational section v of $N_{S/V}$ having a pole only along E and of order one, that is, $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$. Let $\iota : S^\circ \hookrightarrow S$ be the open immersion. Then $\iota_*\mathcal{O}_{S^\circ}$ contains $\mathcal{O}_S(nE)$ as a subsheaf for any $n \geq 0$. Hence the natural sheaf injection $N_{S/V}(nE) \hookrightarrow \iota_* N_{S^\circ/V^\circ}$ induces $H^0(S, N_{S/V}(nE)) \hookrightarrow H^0(S^\circ, N_{S^\circ/V^\circ})$ for each n. In particular, v determines a first order infinitesimal deformation of $S^\circ \subset V^\circ$. The main theorem of this section is the following.

Theorem 3.1. *Assume that* $E^2 < 0$ *and* $\det N_{E/V} := \bigwedge^2 N_{E/V}$ *is trivial. If the exact sequence*

$$0 \longrightarrow N_{E/S} \longrightarrow N_{E/V} \longrightarrow N_{S/V}\big|_E \longrightarrow 0 \tag{3.1}$$

does not split, then the first order infinitesimal deformation of $S^\circ \subset V^\circ$ *determined by* v *does not lift to a deformation over* $\operatorname{Spec} k[t]/(t^3)$.

We identify $H^0(N_{S/V}(nE))$ with its image in $H^0(N_{S^\circ/V^\circ})$ from now on. We shall show that the obstruction $\mathrm{ob}(v) \in H^1(N_{S^\circ/V^\circ})$ of $v \in H^0(N_{S^\circ/V^\circ})$ is nonzero. By Lemma 2.9, $\mathrm{ob}(v)$ is equal to the cup product of $d_{S^\circ}(v) \in H^1(\mathcal{O}_{S^\circ})$ with v. A natural injection $\mathcal{O}_S(2E) \hookrightarrow \iota_*\mathcal{O}_{S^\circ}$ of sheaves on S induces a map $H^1(S, \mathcal{O}_S(2E)) \to H^1(S^\circ, \mathcal{O}_{S^\circ})$ of cohomology groups. Since $\deg \mathcal{O}_E(2E) < 0$, this is injective by Lemma 2.7. Similarly there exists a natural injection from $H^1(S, N_{S/V}(3E))$ to $H^1(S^\circ, N_{S^\circ/V^\circ})$, since $\deg N_{S/V}(3E)\big|_E = \deg(\det N_{E/V}) + 2E^2 = 2E^2 < 0$. From now on we identify $H^1(\mathcal{O}_S(2E))$

and $H^1(N_{S/V}(3E))$ with their images in $H^1(\mathcal{O}_{S^\circ})$ and $H^1(N_{S^\circ/V^\circ})$, respectively. Then by [7, Proposition 2.4 (1)], we have

$$d_{S^\circ}(H^0(N_{S/V}(E))) \subset H^1(\mathcal{O}_S(2E)).$$

Hence by the commutative diagram,

$$\begin{array}{ccccc} H^1(\mathcal{O}_{S^\circ}) & \times & H^0(N_{S^\circ/V^\circ}) & \xrightarrow{\cup} & H^1(N_{S^\circ/V^\circ}) \\ \bigcup & & \bigcup & & \bigcup \\ H^1(\mathcal{O}_S(2E)) & \times & H^0(N_{S/V}(E)) & \xrightarrow{\cup} & H^1(N_{S/V}(3E)), \end{array}$$

the image of $H^0(N_{S/V}(E))$ by ob is contained in $H^1(N_{S/V}(3E))$.

The following lemma is essential to the proof of Theorem 3.1.

Lemma 3.2 ([7, Proposition 2.4 (2)]). *Let $v \in H^0(N_{S/V}(E))$ and let $d_{S^\circ}(v)\big|_E \in H^1(\mathcal{O}_E(2E))$ be the restriction of $d_{S^\circ}(v) \in H^1(\mathcal{O}_S(2E))$ to E. Then we have $d_{S^\circ}(v)\big|_E = \partial(v\big|_E)$ in $H^1(\mathcal{O}_E(2E))$, where*

$$\partial : H^0(N_{S/V}(E)\big|_E) \longrightarrow H^1(\mathcal{O}_E(2E)) \simeq H^1(N_{E/S}(E))$$

is the coboundary map of the exact sequence (3.1)$\otimes\mathcal{O}_S(E)$.

Proof of Theorem 3.1. It suffices to show that the restriction $\mathrm{ob}(v)\big|_E \in H^1(N_{S/V}(3E)\big|_E)$ of $\mathrm{ob}(v) \in H^1(N_{S/V}(3E))$ to E is nonzero. By the definition of v, we have $v\big|_E \neq 0$ in $H^0(N_{S/V}(E)\big|_E)$. Here $N_{S/V}(E)\big|_E \simeq \det N_{E/V}$ is trivial. Since (3.1) does not split by assumption, we have $\partial(v\big|_E) \neq 0$. Hence by Lemma 3.2, we conclude that

$$\mathrm{ob}(v)\big|_E = d_{S^\circ}(v)\big|_E \cup v\big|_E = \partial(v\big|_E) \cup v\big|_E \neq 0. \quad \square$$

If E is a (-1)-$\mathbb{P}^1$ on S with $\det N_{E/V} \simeq \mathcal{O}_{\mathbb{P}^1}$, then the exact sequence (3.1) does not split if and only if $N_{E/V}$ is trivial.

Example 3.3. Let E be a good line on a smooth cubic 3-fold V_3 (i.e. N_{E/V_3} is trivial. cf. §2.2). Let $S_3 \supset E$ be a smooth hyperplane section of V_3 and let $\varepsilon : S_3 \to S_4$ be the blow-down of E from S_3. Since $N_{S_3/V_3} \simeq -K_{S_3}$ and $N_{S_3/V_3}(E) \simeq \varepsilon^*(-K_{S_4})$, $N_{S_3/V_3}(E)$ has one more global section than N_{S_3/V_3}. Thus there exists an obstructed infinitesimal deformation of $S_3^\circ \subset V_3^\circ$ by Theorem 3.1.

In the rest of this section, we discuss a generalization of Theorem 3.1, which will be used for the proof of the main theorem. Let E be a disjoint union of smooth irreducible curves E_i $(i = 1, \ldots, m)$ on S such that $E_i^2 < 0$ and $\det N_{E_i/V}$ is trivial. By the same symbol E we also denote the divisor $\sum_{i=1}^m E_i$ on S. We define V° and S° as before and compute the obstruction map $\mathrm{ob}: H^0(N_{S^\circ/V^\circ}) \to H^1(N_{S^\circ/V^\circ})$. Then Lemma 2.7 allows us to regard $H^1(\mathcal{O}_S(2E))$ and $H^1(N_{S/V}(3E))$ as subgroups of $H^1(\mathcal{O}_{S^\circ})$ and $H^1(N_{S^\circ/V^\circ})$, respectively. Then an argument similar to [7, Proposition 2.4 (1)] shows that the image of $H^0(N_{S/V}(E))$ by d_{S° is contained in $H^1(\mathcal{O}_S(2E))$ and hence its image by ob is contained in $H^1(N_{S/V}(3E))$. Moreover, we have

$$\mathrm{ob}(v+v')\big|_E = \mathrm{ob}(v)\big|_E$$

in $H^1(N_{S/V}(3E)\big|_E)$ for any $v \in H^0(N_{S/V}(E))$ and $v' \in H^0(N_{S/V})$. Indeed it follows from the definition of d_{S° that $d_{S^\circ}(v') \in H^1(\mathcal{O}_S)$ and hence

$$\begin{aligned} \mathrm{ob}(v+v') &= (d_{S^\circ}(v) + d_{S^\circ}(v')) \cup (v+v') \\ &= \mathrm{ob}(v) + \underbrace{d_{S^\circ}(v) \cup v' + d_{S^\circ}(v') \cup v + d_{S^\circ}(v') \cup v'}_{\text{contained in } H^1(N_{S/V}(2E))}. \end{aligned}$$

Therefore the obstruction map ob induces a map

$$\overline{\mathrm{ob}}: H^0(N_{S/V}(E))\big/H^0(N_{S/V}) \longrightarrow H^1(N_{S/V}(3E)\big|_E).$$

Proposition 3.4. *If $H^1(N_{S/V}) = 0$ and the exact sequence*

$$0 \longrightarrow N_{E_i/S} \longrightarrow N_{E_i/V} \longrightarrow N_{S/V}\big|_{E_i} \longrightarrow 0 \tag{3.2}$$

does not split for every i, then $\overline{\mathrm{ob}}$ is injective.

This is an immediate consequence of the next lemma.

Lemma 3.5. *Under the assumption of Proposition 3.4, $\overline{\mathrm{ob}}$ is equivalent to the quadratic map*

$$k^m \longrightarrow k^n, \qquad (a_1, \dots, a_m) \longmapsto (a_1^2, \dots, a_m^2, 0, \dots, 0)$$

of diagonal type, where $n = \dim H^1(N_{S/V}(3E)\big|_E)$.

Proof. Since $H^1(N_{S/V}) = 0$, the source of $\overline{\mathrm{ob}}$ is isomorphic to $H^0(N_{S/V}(E)\big|_E)$. Moreover there exist global sections v_i of $N_{S/V}(E_i)$ such that $v_i\big|_E \neq 0$ in $H^0(N_{S/V}(E_i)\big|_{E_i})$ for all i. Since E_i's are mutually disjoint, we have $N_{S/V}(E)\big|_E \simeq \bigoplus_{i=1}^m N_{S/V}(E_i)\big|_{E_i} \simeq \bigoplus_{i=1}^m \mathcal{O}_{E_i}$. Then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H^0(N_{S/V}) & \to & H^0(N_{S/V}(E)) & \to & H^0(N_{S/V}(E)\big|_E) & \to & 0 \\ & & \uparrow{\scriptstyle a_1} & & \uparrow{\scriptstyle a_2} & & \uparrow{\scriptstyle a_3} & & \\ 0 & \to & \bigoplus_i H^0(N_{S/V}) & \to & \bigoplus_i H^0(N_{S/V}(E_i)) & \to & \bigoplus_i H^0(N_{S/V}(E_i)\big|_{E_i}) & \to & 0, \end{array}$$

where a_i $(1 \leq i \leq 3)$ are defined by addition. Since a_1 and a_3 are surjective, so is a_2. Hence every element $v \in H^0(N_{S/V}(E))$ is written as a k-linear combination $\sum_{i=1}^m c_i v_i$ of

$v_i \in H^0(N_{S/V}(E_i))$ and the expression is unique modulo $H^0(N_{S/V})$. By the commutative diagram

$$\begin{array}{ccccc} H^1(\mathcal{O}_S(2E)) & \times & H^0(N_{S/V}(E)) & \xrightarrow{\cup} & H^1(N_{S/V}(3E)) \\ {\scriptstyle |_E}\downarrow & & {\scriptstyle |_E}\downarrow & & {\scriptstyle |_E}\downarrow \\ \bigoplus_i H^1(\mathcal{O}_{E_i}(2E_i)) & \times & \bigoplus_i H^0(N_{S/V}(E_i)\big|_{E_i}) & \xrightarrow{\cup} & \bigoplus_i H^1(N_{S/V}(3E_i)\big|_{E_i}), \end{array}$$

we have

$$\mathrm{ob}(v)\big|_E = (d_{S^\circ}(v) \cup v)\big|_E = d_{S^\circ}(v)\big|_E \cup v\big|_E = \sum_i c_i^2 d_{S^\circ}(v_i)\big|_{E_i} \cup v_i\big|_{E_i}.$$

By Lemma 3.2, $d_{S^\circ}(v_i)\big|_{E_i}$ is equal to $\partial_i(v\big|_{E_i})$ in $H^1(\mathcal{O}_{E_i}(2E_i))$, where ∂_i is the coboundary map of (3.2). Since (3.2) does not split by assumption, we have $\partial_i(v\big|_{E_i}) \neq 0$ and hence $d_{S^\circ}(v_i)\big|_{E_i} \neq 0$ for any i. As a result $\{d_{S^\circ}(v_i)\big|_{E_i} \cup v_i\big|_{E_i}, 1 \leq i \leq m\}$ is a sub-basis of $H^1(N_{S/V}(3E)\big|_E)$. □

We get the following corollary to Proposition 3.4.

Corollary 3.6. *Let E_i $(i = 1, \ldots, m)$ be mutually disjoint (-1)-$\mathbb{P}^1$'s on S such that $N_{E_i/V} \simeq \mathcal{O}_{\mathbb{P}^1}{}^{\oplus 2}$. If $H^1(N_{S/V}) = 0$, then $\overline{\mathrm{ob}}$ is injective.*

4 Obstructions to deforming curves

The purpose of this section is to prove the main theorem. Let $C \subset S \subset V$ be a sequence of a smooth projective 3-fold V, a smooth surface S, and a smooth curve C.

4.1 S-maximal family and S-normality

In this subsection, we recall the definition of the S-maximal family introduced in [7, §3.2]. Let U_S be an irreducible component of $\mathrm{Hilb}\, V$ passing through $[S]$ and let

$$V \times U_S \supset \mathcal{S} \xrightarrow{p_2} U_S$$

be the universal family of U_S. Assume that $H^1(N_{S/V}) = H^1(N_{C/S}) = 0$, that is, $\mathrm{Hilb}\, V$ and $\mathrm{Hilb}^{sc} S$ are nonsingular of expected dimensions $\chi(N_{S/V})$ and $\chi(N_{C/S})$ at $[S]$ and $[C]$, respectively. Then the Hilbert scheme $\mathrm{Hilb}^{sc} \mathcal{S}$, which is the same as the relative Hilbert scheme of $\mathcal{S}/U_S$ or the Hilbert-flag scheme introduced in [6, §2], is nonsingular at $[C]$. Let $\mathcal{W}_{S,C}$ be the irreducible component of $\mathrm{Hilb}^{sc} \mathcal{S}$ passing through $[C]$. The projection $p_1 : \mathcal{S} \to V$ induces a natural morphism $\mathrm{Hilb}^{sc} \mathcal{S} \to \mathrm{Hilb}^{sc} V$. We call the image of $\mathcal{W}_{S,C}$ in $\mathrm{Hilb}^{sc} V$ the *S-maximal family of curves* containing C and denote it by $W_{S,C}$. If the natural morphism $pr_1 : \mathcal{W}_{S,C} \to \mathrm{Hilb}^{sc} V$ is surjective in a neighborhood of $[C] \in \mathrm{Hilb}^{sc} V$, then the answer to the first question of Problem 1.1 is affirmative. By [7, Lemma 3.3] the cokernel (resp. kernel) of the tangential map

$$\kappa_{[C]} : t_{\mathcal{W}_{S,C}} = H^0(N_{C/\mathcal{S}}) \longrightarrow H^0(N_{C/V}) \tag{4.1}$$

of the morphism pr_1 at $[C]$ is isomorphic to that of the restriction map (1.1). In what follows, we use the following convention.

Definition 4.1. Let $C \subset S \subset V$ be as above.

(1) C is said to be *stably degenerate* if every (small) deformation of C in V is contained in a divisor $S' \overset{alg.}{\sim} S$ of V

(2) C is said to be *S-normal* if the restriction map (1.1) is surjective.

If C is S-normal, then pr_1 is surjective in a neighborhood of $[C]$ and hence C is stably degenerate. Then $\operatorname{Hilb}^{sc} V$ is nonsingular at $[C]$ as well.

4.2 Deformation of curves on a del Pezzo 3-fold

In what follows, we assume that V is a smooth del Pezzo 3-fold of degree n with polarization H, S is a smooth member of $|H|$ and C is a smooth connected curve on S of degree d and genus g. Since $-K_V \sim 2S$, by adjunction we have $N_{S/V} = \mathcal{O}_S(S) \simeq -K_S$ and $N_{C/S} \simeq -K_S\big|_C + K_C$. Since $-K_S$ is ample, we have $H^1(N_{S/V}) = H^1(N_{C/S}) = 0$ and hence $\operatorname{Hilb} V$ and $\operatorname{Hilb} S$ are nonsingular of expected dimension $\chi(N_{S/V})$ and $\chi(N_{C/S})$ at $[S]$ and $[C]$, respectively. A natural exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/V} \xrightarrow{\pi_S} N_{S/V}\big|_C \longrightarrow 0 \tag{4.2}$$

of normal bundles induces an isomorphism

$$H^1(N_{C/V}) \simeq H^1(N_{S/V}\big|_C). \tag{4.3}$$

Since $\mathcal{I}_S(S) \simeq \mathcal{O}_V$ and V is del Pezzo, we have $H^i(\mathcal{I}_S(S)) = 0$ for $i = 1, 2$. Hence the exact sequence

$$[0 \longrightarrow \mathcal{I}_S \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{O}_S(-C) \longrightarrow 0] \otimes \mathcal{O}_V(S) \tag{4.4}$$

on V shows that

$$H^1(V, \mathcal{I}_C(S)) \simeq H^1(S, N_{S/V}(-C)), \tag{4.5}$$

where the right hand side is isomorphic to the cokernel of the restriction map (1.1) since $H^1(N_{S/V}) = 0$. Therefore C is S-normal if and only if $H^1(V, \mathcal{I}_C(S)) = 0$.

The dimension of the S-maximal family $W_{S,C}$ of curves containing C can be computed from d and g.

Lemma 4.2. *If $g \geq 2$ or $d \geq n+1$, then we have the following:*

(a) *the natural morphism $pr_1 : \mathcal{W}_{S,C} \to \mathrm{Hilb}^{sc} V$ is a closed embedding;*

(b) $\dim W_{S,C} = d + g + n$.

Proof. (a) The proof is similar to that of [6, Remark 9]. By assumption, we have $(-K_S - C) \cdot C = 2 - 2g < 0$ or $(-K_S - C) \cdot (-K_S) = n - d < 0$. Since both C and $-K_S$ are nef, we have $H^0(N_{S/V}(-C)) = H^0(-K_S - C) = 0$. Similarly we have $H^0(N_{S'/V}(-C')) = 0$ for any member (C', S') of $\mathcal{W}_{S,C}$. This implies that S' is the unique member of $|H|$ containing C'. Hence the map pr_1 is injective. Moreover, since the restriction map $\big|_{C'} : H^0(N_{S'/V}) \to H^0(N_{S'/V}\big|_{C'})$ is injective, so is the tangential map $\kappa_{[C']}$ of pr_1 at $[(C', S')]$ in (4.1).

(b) By the Riemann-Roch theorem, we have $\dim |\mathcal{O}_{S'}(C')| = d + g - 1$ for general member (C', S') of $\mathcal{W}_{S,C}$. Since $\mathcal{W}_{S,C}$ is birationally equivalent to $\mathbb{P}^{d+g-1}$-bundle over an open subset of $|H| \simeq \mathbb{P}^{n+1}$, the dimension of $\mathcal{W}_{S,C}$ is equal to $d+g+n$. Hence we obtain $\dim W_{S,C} = d + g + n$ by (a). □

We denote by $\mathrm{Hilb}^{sc}_{d,g} V$ the open and closed subscheme of $\mathrm{Hilb}^{sc} V$ of curves of degree d and genus g. It is well known that the dimension of every irreducible component of $\mathrm{Hilb}^{sc}_{d,g} V$ is greater than or equal to the expected dimension $\chi(N_{C/V}) = (-K_V \cdot C)_V = 2d$.

Remark 4.3. If $g < d - n$ then C is not stably degenerate. In other words, there exists a deformation $C' \subset V$ of C not contained in any hyperplane section of V. Indeed we have $\dim W_{S,C} \leq \dim \mathcal{W}_{S,C} = d + g + n < 2d$. Hence there exists an irreducible component $W' \supset W_{S,C}$ of $\mathrm{Hilb}^{sc} V$ such that $\dim W' > \dim W_{S,C}$. By the definition of $W_{S,C}$, this implies the existence of such C'.

4.3 Stably degenerate curves

We devote this section to the proof of Theorem 1.3. Throughout this section, we assume that $\chi(V, \mathcal{I}_C(S)) \geq 1$, which is equivalent to $g \geq d - n$.

Lemma 4.4. *If $H^1(N_{S/V}\big|_C) = 0$ then C is S-normal.*

Proof. It suffices to show that $H^1(N_{S/V}(-C)) = 0$ by the exact sequence

$$0 \longrightarrow N_{S/V}(-C) \longrightarrow N_{S/V} \longrightarrow N_{S/V}\big|_C \longrightarrow 0.$$

Since $H^2(N_{S/V}) \simeq H^2(-K_S) = 0$ and $H^1(N_{S/V}\big|_C) = 0$, we obtain $H^2(N_{S/V}(-C)) = 0$. Then by (4.4), we have an inequality

$$\begin{aligned} 0 \leq \chi(V, \mathcal{I}_C(S)) - 1 &= \chi(N_{S/V}(-C)) \\ &= h^0(N_{S/V}(-C)) - h^1(N_{S/V}(-C)). \end{aligned}$$

Therefore if $H^0(N_{S/V}(-C)) = 0$, then $H^1(N_{S/V}(-C)) = 0$. Thus we may assume that $H^0(N_{S/V}(-C)) \neq 0$. Then there exists an effective divisor D on S such that $N_{S/V}(-C) \simeq \mathcal{O}_S(D)$. If $D = 0$, then $H^1(N_{S/V}(-C)) = 0$. Suppose that $D \neq 0$. Let h be a general member of $|-K_S|$. Then h is a smooth elliptic curve on S. Since $-K_S$ is ample, we have $\deg \mathcal{O}_S(D)\big|_h = D \cdot (-K_S) > 0$ and hence $H^1(\mathcal{O}_S(D)\big|_h) = 0$. Since C is connected, we obtain $H^1(D-h) \simeq H^1(-C) = 0$ from the exact sequence $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$. Therefore it follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_S(D-h) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D)\big|_h \longrightarrow 0$$

that $H^1(N_{S/V}(-C)) \simeq H^1(D) = 0$. □

In particular if C is rational ($g = 0$) or elliptic ($g = 1$) then C is S-normal, because we have $H^1(N_{S/V}\big|_C) \simeq H^1(-K_S\big|_C) = 0$ in these two cases.

Let $E_1, \dots, E_m$ be lines on S disjoint to C and define a divisor E on S as in Proposition 2.4. If C is not S-normal, then E is responsible for the abnormality.

Proposition 4.5. (i) *The restriction map*

$$H^0(N_{S/V}(E)) \xrightarrow{|_C} H^0(N_{S/V}\big|_C) \tag{4.6}$$

is surjective.

(ii) *Assume that $g \geq 2$. Then C is S-normal if and only if there exists no line ℓ such that $C \cap \ell = \emptyset$ (i.e. $E = 0$).*

Proof. (i) If $H^1(N_{S/V}\big|_C) = 0$ then we have the assertion by Lemma 4.4. If $H^1(N_{S/V}\big|_C) \neq 0$ then we have $g \geq 2$ and obtain the assertion from Proposition 2.4.

(ii) The 'if' part follows from (i). We prove the 'only if' part. Suppose that there exist such lines on S. Let $\varepsilon : S \to F$ be the blow-down of E from S. Then F is also a del Pezzo surface and $\varepsilon^*(-K_F) = -K_S + E$. Since $\deg F > \deg S$, we have $h^0(-K_F) > h^0(-K_S)$. Hence it follows from $N_{S/V} \simeq -K_S$ that $N_{S/V}(E)$ has more global sections than $N_{S/V}$. Since $g \geq 2$, the map (4.6) is an isomorphism by Remark 2.5 and hence we have $h^0(N_{S/V}\big|_C) = h^0(N_{S/V}(E)) > H^0(N_{S/V})$. Therefore C is not S-normal. □

Let $W_{S,C}$ be the S-maximal family of curves containing C and let $\kappa_{[C]} : t_{\mathcal{W}_{S,C}} \to H^0(N_{C/V})$ be the map (4.1).

Proposition 4.6. *Suppose that C is not S-normal. If every E_i is a good line on V, then the obstruction* $\mathrm{ob}(\alpha)$ *is nonzero for any $\alpha \in H^0(N_{C/V}) \setminus \operatorname{im} \kappa_{[C]}$.*

Proof. We prove that the exterior component $\mathrm{ob}_S(\alpha)$ of $\mathrm{ob}(\alpha)$ is nonzero (cf. Definition 2.10). Our proof is parallel to that of [7, Theorem 1.6], which is carried out by 3 steps. Here we only sketch the proof of each step and refer to [7] for the further detail, especially for the proof of the two equalities (4.7) and (4.9) of cup products. The first step shows that $\mathrm{ob}_S(\alpha)$ is computed from the exterior component $\pi_S(\alpha)$ of α.

Step 1 By Proposition 4.5 (i), there exists a global section v of $N_{S/V}(E)$ whose restriction $v\big|_C \in H^0(N_{S/V}\big|_C)$ coincides with $\pi_S(\alpha)$. Let $d_{S^\circ} : H^0(N_{S^\circ/V^\circ}) \to H^1(\mathcal{O}_{S^\circ})$ be the map (2.3) for $S^\circ \subset V^\circ$. As we saw in §3, the image $d_S(v) := d_{S^\circ}(v)$ for $v \in H^0(N_{S/V}(E))$ is contained in $H^1(\mathcal{O}_S(2E)) \subset H^1(\mathcal{O}_{S^\circ})$. Then we have

$$\mathrm{ob}_S(\alpha) = d_S(v)\big|_C \cup \pi_S(\alpha), \tag{4.7}$$

where $d_S(v)\big|_C \in H^1(\mathcal{O}_C)$ is the restriction of $d_S(v) \in H^1(\mathcal{O}_S(2E))$ to C and $\cup$ is the cup product map

$$H^1(\mathcal{O}_C) \otimes H^0(N_{S/V}\big|_C) \xrightarrow{\cup} H^1(N_{S/V}\big|_C).$$

The second step relates $\mathrm{ob}_S(\alpha)$ to $\mathrm{ob}(v) = d_S(v) \cup v \in H^1(N_{S/V}(3E)) \subset H^1(N_{S^\circ/V^\circ})$ for $v \in H^0(N_{S/V}(E))$, which has been computed in the latter half of §3.

Step 2 Let $\mathbf{k}_C$ and $\mathbf{k}_E$ be the extension classes of the two exact sequences

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_S(-E) \to \mathcal{O}_S \to \mathcal{O}_E \to 0 \tag{4.8}$$

on S, respectively. Then we obtain two cup product maps

$$H^1(N_{S/V}\big|_C) \xrightarrow{\cup \mathbf{k}_C} H^2(N_{S/V}(2E-C)) \quad \text{and} \quad H^1(N_{S/V}(3E)\big|_E) \xrightarrow{\cup \mathbf{k}_E} H^2(N_{S/V}(2E-C)),$$

which are the coboundary maps of (4.8) tensored with some suitable line bundles on S. Moreover, the following equation of cup products in $H^1(N_{S/V}(2E-C))$ holds:

$$\mathrm{ob}_S(\alpha) \cup \mathbf{k}_C = d_S(v)\big|_E \cup v\big|_E \cup \mathbf{k}_E. \tag{4.9}$$

Here the cup product $d_S(v)\big|_E \cup v\big|_E \in H^1(N_{S/V}(3E)\big|_E)$ is clearly equal to the restriction $\mathrm{ob}(v)\big|_E \in H^1(N_{S/V}(3E)\big|_E)$ of $\mathrm{ob}(v)$ to E.

The final step shows that $\mathrm{ob}_S(\alpha) \neq 0$.

Step 3 Since $\alpha \notin \operatorname{im} \kappa_{[C]}$, $v \in H^0(N_{S/V}(E))$ does not belong to $H^0(N_{S/V})$. Since every $N_{E_i/V}$ is trivial bundle on $E_i \simeq \mathbb{P}^1$ by assumption, by virtue of Corollary 3.6, we have $\mathrm{ob}(v)\big|_E = \overline{\mathrm{ob}}(v\big|_E) \neq 0$ in $H^1(N_{S/V}(3E)\big|_E)$. Note that the cup product map

$$H^1(E, N_{S/V}(3E)\big|_E) \xrightarrow{\cup \mathbf{k}_E} H^2(S, N_{S/V}(2E-C))$$

is injective. Indeed since $N_{S/V} \simeq -K_S$ this map is exactly the Serre dual of the restriction map

$$H^0(S, C + 2K_S - 2E) \xrightarrow{|_E} H^0(E, (C + 2K_S - 2E)\big|_E),$$

which is surjective by Lemma 2.6. Hence the right hand side of (4.9) is nonzero and we obtain that $\mathrm{ob}_S(\alpha) \neq 0$ in $H^1(N_{S/V}\big|_C)$. □

Proof of Theorem 1.3. If C is S-normal then C is clearly stably degenerate and $\mathrm{Hilb}^{sc} V$ is nonsingular at $[C]$. Suppose that C is not S-normal. Then by Proposition 4.6, every first order infinitesimal deformation of $C \subset V$ does not lift to a deformation over $\mathrm{Spec}\, k[t]/(t^3)$ except for the ones realized as a member of $W_{S,C}$. This implies that $\mathrm{Hilb}^{sc} V$ is singular at $[C]$ and moreover every small deformation of C in V is contained $W_{S,C}$. Therefore C is stably degenerate. Since C is S-normal if and only if $H^1(V, \mathcal{I}_C(S)) = 0$ by (4.5), the proof of Theorem 1.3 is completed. □

Remark of Theorem 1.3

(1) If $n \neq 7$ and S is a general member of $|H|$, then by Lemma 2.8, every line on S is a good line on V. Hence every curve C on S is stably degenerate by the theorem. If $n = 7$ then there exists a non-stably degenerate curve C on V_7 which is contained in a general member S of $|H|$ (cf. Proposition 5.4).

(2) There exists no line on V_8. Hence if $n = 8$, then the assumption of the theorem concerning lines ℓ on S such that $C \cap \ell = \emptyset$ is empty. In fact, every curve C on $V_8 \simeq \mathbb{P}^3$ is S-normal, provided that $g \geq d - 8$ and hence stably degenerate. This coincides with the result obtained in [9, Appendix] for curves on a smooth quadric surface $Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$.

The following proposition is more practical than Proportion 4.6 in showing the singularity of $\mathrm{Hilb}^{sc} V$ at $[C]$.

Proposition 4.7. *Suppose that $g \geq 2$. If there exists a good line ℓ on V such that $\ell \subset S$ and $C \cap \ell = \emptyset$, then $\mathrm{Hilb}^{sc} V$ is singular at $[C]$.*

The proofs of Proposition 4.6 and Proposition 4.7 are very similar. Take a global section $v \in H^0(N_{S/V}(\ell)) \setminus H^0(N_{S/V})$ and put $\alpha \in H^0(N_{C/V})$ as a lift of $v\big|_C \in H^0(N_{S/V})$ by the surjective map $\pi_S : H^0(N_{C/V}) \twoheadrightarrow H^0(N_{S/V}\big|_C)$. Then it is enough to show that $\mathrm{ob}_S(\alpha) \neq 0$ in $H^1(N_{S/V}\big|_C)$ by reducing it to $\mathrm{ob}(v)\big|_\ell \neq 0$ as in the proof of Proposition 4.6. We omit the detail.

The following is an analogue of Conjecture 5.1 due to Kleppe and Ellia.

Theorem 4.8. *Let C be the curve in Theorem 1.3. Then the S-maximal family $W_{S,C} \subset \mathrm{Hilb}^{sc} V$ containing $[C]$ is an irreducible component of $(\mathrm{Hilb}^{sc} V)_{red}$. Then $\mathrm{Hilb}^{sc} V$ is generically smooth along $W_{S,C}$ if $H^1(V, \mathcal{I}_C(S)) = 0$, and generically non-reduced along $W_{S,C}$ otherwise.*

Proof. The first part follows from Theorem 1.3. (In fact, we proved that every small deformation of C in V is contained in $W_{S,C}$. This implies that $W_{S,C}$ is maximal as an irreducible closed subset of $\mathrm{Hilb}^{sc} V$.) Now we prove the second part. Let C' be a general member of $W_{S,C}$. Then C' is contained in a smooth surface $S' \sim S$ in V. Since C' is general, S' is a general member of $|H|$. Suppose that C is S-normal. Then since (C', S') is a generalization of (C, S), we have $H^1(\mathcal{I}_{C'}(S')) = H^1(\mathcal{I}_C(S)) = 0$ by the upper semicontinuity. Therefore C' is S'-normal. Hence $\mathrm{Hilb}^{sc} V$ is nonsingular at $[C']$ and hence generically smooth along $W_{S,C}$.

Suppose that C is not S-normal. Then Lemma 4.4 shows that $H^1(N_{S/V}\big|_C) \neq 0$ and hence $g \geq 2$. By Proposition 4.5 (ii), there exists a line ℓ on S such that $C \cap \ell = \emptyset$. Since $H^1(\mathcal{O}_S) = 0$, the Picard group of S does not change under smooth deformation and hence $\operatorname{Pic} S \simeq \operatorname{Pic} S'$. Since $H^1(\mathcal{O}_S(\ell)) = 0$, the line ℓ is deformed to a line ℓ' on S'. We have $C' \cap \ell' = \emptyset$. Moreover since ℓ is a good line, so is ℓ'. Hence $\mathrm{Hilb}^{sc} V$ is singular at $[C']$ by Proposition 4.7. Since C' is a general member of $W_{S,C}$, $\mathrm{Hilb}^{sc} V$ is everywhere singular along $W_{S,C}$ and hence generically non-reduced along $W_{S,C}$. □

5 Original motivation and examples

5.1 Kleppe's conjecture

The original motivation of the present work was to show the following conjecture due to Kleppe. We denote by $\mathrm{Hilb}^{sc}_{d,g} \mathbb{P}^3$ the open and closed subscheme of $\mathrm{Hilb}^{sc} \mathbb{P}^3$ consisting of curves of degree d and genus g.

Conjecture 5.1 (Kleppe, Ellia)**.** *Let W be a maximal irreducible closed subset of* $\mathrm{Hilb}^{sc}_{d,g} \mathbb{P}^3$ *whose general member C is contained in a smooth cubic surface. If*

$$d \geq 14, \quad g \geq 3d - 18, \quad H^1(\mathbb{P}^3, \mathcal{I}_C(3)) \neq 0 \quad \textit{and} \quad H^1(\mathbb{P}^3, \mathcal{I}_C(1)) = 0,$$

then W is a component of $(\mathrm{Hilb}^{sc} \mathbb{P}^3)_{\mathrm{red}}$ *and* $\mathrm{Hilb}^{sc} \mathbb{P}^3$ *is generically non-reduced along W.*

In the original conjecture [6, Conjecture 4] of Kleppe, the assumption of the linearly normality of C (i.e. $H^1(\mathbb{P}^3, \mathcal{I}_C(1)) = 0$) was missing. However Ellia [1] pointed out that the conjecture does not hold for linearly non-normal curves C by a counterexample, and suggested restricting the conjecture to linearly normal ones. The most crucial part to prove this conjecture is the proof of the maximality of W in $(\mathrm{Hilb}^{sc} \mathbb{P}^3)_{\mathrm{red}}$. Once we prove that W is a component of $(\mathrm{Hilb}^{sc} \mathbb{P}^3)_{\mathrm{red}}$, then the non-reducedness of $\mathrm{Hilb}^{sc} \mathbb{P}^3$ along W naturally follows. Therefore Conjecture 5.1 follows from Conjecture 1.2, where the condition $\chi(\mathbb{P}^3, \mathcal{I}_C(3)) \geq 1$ is equivalent to $g \geq 3d - 18$. Recently it has been proved in [9] that Conjecture 5.1 is true when $h^1(\mathbb{P}^3, \mathcal{I}_C(3)) = 1$. Kleppe and Ellia gave a proof for the conjecture under some other conditions, however the whole conjecture is still open.

5.2 Hilbert scheme of canonical curves

In this subsection we prove the following:

Theorem 5.2. *The Hilbert scheme* $\mathrm{Hilb}^{sc} V$ *of smooth connected curves on a smooth del Pezzo* 3*-fold* V *has a generically non-reduced component* W*.*

Let n and H be the degree and the polarization of V. The theorem for the cases $n = 8$ (i.e. $V = V_8 \simeq \mathbb{P}^3$) and $n = 3$ (i.e. V is a smooth cubic 3-fold V_3) were already obtained in [8] and [7], respectively. For the proof, we consider a canonical curve C on a smooth surface $S \in |H|$ which is not S-normal. Here we say that a curve $C \subset V$ is *canonical* if $f^*H = K_C$, where $f : C \hookrightarrow V$ is the embedding. Equivalently C is embedded into V by a linear subsystem of $|K_C|$. Theorem 4.8 gives us the non-reduced component W such that $W_{\mathrm{red}} = W_{S,C}$.

Proof of Theorem 5.2. Since $V_8 \simeq \mathbb{P}^3$, we may assume that $n \le 7$. Then as we saw in §2.2, there exists a good line ℓ on V. Let $S_n \in |H|$ be a smooth del Pezzo surface containing ℓ. We consider the complete linear system $\Lambda := |-2K_{S_n} + 2\ell|$ on S_n. Let S_{n+1} be the the blow-down of ℓ from S_n, which is a del Pezzo surface of degree $n+1$. Then Λ is the pull-back of $|-2K_{S_{n+1}}| \simeq \mathbb{P}^{3n+3}$ on S_{n+1}. Since Λ is base point free, a general member C of Λ is a smooth connected curve of degree $d = 2n + 2$ and genus $g = n + 2$. Therefore we have $g = d - n$. Then ℓ does not intersect C by $(-2K_{S_n} + 2\ell) \cdot \ell = 2 - 2 = 0$. Moreover ℓ is the only such line on S_n. By Theorem 4.8, $W_{S_n,C}$ is an irreducible component of $(\mathrm{Hilb}^{sc} V)_{\mathrm{red}}$. Since $C \cap \ell = \emptyset$, C is not S_n-normal by Proposition 4.5 (ii). Therefore $\mathrm{Hilb}^{sc} V$ is generically non-reduced along $W_{S_n,C}$. □

Remark 5.3. (1) By construction, C is the image of a canonical curve $C' \sim -2K_{S_{n+1}}$ on S_{n+1} by the projection $S_{n+1} \cdots \to S_n$ from a point $p \in S_{n+1}$ outside C'.

(2) The dimension of the irreducible component $W_{S_n,C}$ is equal to $d + g + n = 4n + 4$ by Lemma 4.2.

(3) The tangential dimension of $\mathrm{Hilb}^{sc} V$ at a general point $[C]$ of $W_{S_n,C}$ is equal to $h^0(N_{C/V}) = 4n + 5$. Indeed the exact sequence (4.2) is

$$0 \longrightarrow \mathcal{O}_C(2K_C) \longrightarrow N_{C/V} \longrightarrow \mathcal{O}_C(K_C) \longrightarrow 0,$$

since $N_{S/V}\big|_C \simeq -K_S\big|_C \simeq K_C$. Hence we have

$$h^0(N_{C/V}) = h^0(2K_C) + h^0(K_C) = (3n + 3) + (n + 2) = 4n + 5.$$

The next example shows that the curve C in Theorem 1.3 is not necessarily stably degenerate if there exists a bad line ℓ on S such that $C \cap \ell = \emptyset$.

Let $V_7 \subset \mathbb{P}^8$ be a smooth del Pezzo 3-fold of degree 7 and let $S_7 \subset V_7$ be a smooth hyperplane section. Then there exist three lines ℓ_0, ℓ_1, ℓ_2 on S_7 forming the configuration of Figure 1. Consider a general member C of $\Lambda := |-2K_{S_7} + 2\ell_0|$. Then C is a smooth connected curve of degree 16 and genus $9 = 16 - 7$ and not S_7-normal by $C \cap \ell_0 = \emptyset$.

Proposition 5.4. *Let C be as above. Then there exists a smooth deformation $C' \subset V_7$ of C not contained in any hyperplane section. In other words, C is not stably degenerate.*

Proof. Recall that V_7 is isomorphic to the blow-up of $\mathbb{P}^3$ at a point p. It is realized as the projection of the Veronese image $V_8 \subset \mathbb{P}^9$ of $\mathbb{P}^3$ from $p \in V_8$ (cf. §2.2). Then S_7 is the image by the projection of a hyperplane section $Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ of V_8 containing p. Hence we have a diagram

$$\begin{array}{ccccc} S_7 \simeq \mathrm{Bl}_{2\mathrm{pts}}\,\mathbb{P}^2 & \subset & V_7 \simeq \mathrm{Bl}_p\,\mathbb{P}^3 & \subset & \mathbb{P}^8 \\ \downarrow\uparrow & & {\scriptstyle \pi_p}\downarrow\uparrow{\scriptstyle \Pi_p} & & \uparrow \\ Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 & \subset & V_8 \simeq \mathbb{P}^3 & \subset & \mathbb{P}^9, \end{array} \tag{5.1}$$

where the down arrows (resp. the up arrows) are the blow-up morphisms at (resp. the projections from) $p \in Q_2 \subset V_8 \subset \mathbb{P}^9$. Let $P \simeq \mathbb{P}^2$ denote the exceptional divisor of π_p. Then its intersection with S_7 is equal to the bad line ℓ_0.

Since $C \cap \ell_0 = \emptyset$ and $C \cdot \ell_i = 4$ for each $i = 1, 2$, π_p maps C isomorphically onto a curve of bidegree $(4, 4)$ on Q_2. Let Q_2' be a general hyperplane section of V_8. Then $Q_2' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is mapped isomorphically onto a surface Q_2'' on V_7 by Π_p. Here Q_2'' is linearly equivalent to $S_7 + P$ as a divisor of V_7 and contains a smooth deformation C' of C. Then there exists no hyperplane section of V_7 containing C'. Suppose that there exists such a hyperplane section S_7'. Then the image $\pi_p(C')$ is contained in the intersection of two hyperplane sections $\pi_p(S_7')$ and Q_2' of V_8. Hence the inverse image of $\pi_p(C')$ in $\mathbb{P}^3$ by the Veronese embedding is contained in a complete intersection of two quadrics. This is impossible since the degree of the inverse image is equal to $8 > 4$. □

5.3 Hilbert scheme of curves on a cubic 3-fold

Let V_3 be a smooth cubic 3-fold. Every smooth hyperplane section S of V_3 is isomorphic to a blown-up of $\mathbb{P}^2$ at 6 points. Let $\mathcal{O}_S(a; b_1, \dots, b_6)$ denote the line bundle on S associated to a divisor $a\ell - \sum_{i=1}^{6} b_i e_i$ on S, where ℓ is the pullback of a line on $\mathbb{P}^2$ and e_i $(1 \le i \le 6)$ are the six exceptional curves on S. We have an isomorphism $\mathrm{Pic}\, S \simeq \mathbb{Z}^7$ which sends the class of $\mathcal{O}_S(a; b_1, \dots, b_6)$ to a 7-tuple $(a; b_1, \dots, b_6)$ of integers. When the linear system $|\mathcal{O}_S(a; b_1, \dots, b_6)|$ on S contains a smooth member C, we denote the S-maximal family $W_{S,C}$ of curves containing C by $W_{(a;b_1,\dots,b_6)}$.

Example 5.5. Suppose that S is a general hyperplane section of V_3 and let W be one of the S-maximal families

$$W_{(\lambda+6;\lambda+1,1,1,1,1,0)} \subset \mathrm{Hilb}^{sc}_{d,2d-16} V_3 \quad (d = 2\lambda + 13) \quad \text{and}$$
$$W_{(\lambda+6;\lambda+2,1,1,1,1,0)} \subset \mathrm{Hilb}^{sc}_{d,\frac{3}{2}d-9} V_3 \quad (d = 2\lambda + 12),$$

where $\lambda \in \mathbb{Z}_{\geq 0}$. It is clear that $g \geq d-3$ and e_6 is the only line on S such that $C \cap S = \emptyset$. Since S is general, e_6 is a good line on V_3 by Lemma 2.8. By Theorem 4.8, W is an irreducible component of $(\mathrm{Hilb}^{sc} V_3)_{\mathrm{red}}$ and $\mathrm{Hilb}^{sc} V_3$ is generically non-reduced along W. Thus $\mathrm{Hilb}^{sc} V_3$ has infinitely many non-reduced components.

References

[1] P. Ellia: D'autres composantes non réduites de Hilb $\mathbb{P}^3$, *Math. Ann.* **277**(1987), 433–446.

[2] T. Fujita: On the structure of polarized manifolds with total deficiency one. I, *J. Math. Soc. Japan* **32**(1980), 709–725.

[3] T. Fujita: On the structure of polarized manifolds with total deficiency one. II, *J. Math. Soc. Japan* **33**(1981), 415–434.

[4] V.A. Iskovskih: Fano 3-folds. I, *Math. USSR-Izvstija* **11**(1977), no. 3, 485–527 (English translation).

[5] V.A. Iskovskih: Anticanonical models of three-dimensional algebraic varieties, Current problems in mathematics, *J. Soviet Math.* **13**(1980), 745–814 (English translation).

[6] J. O. Kleppe: Non-reduced components of the Hilbert scheme of smooth space curves in "*Space curves*" (eds. F. Ghione, C. Peskine and E. Sernesi), Lecture Notes in Math. **1266**, Springer-Verlag, 1987, pp.181–207.

[7] S. Mukai and H. Nasu: Obstruction to deforming curves on a 3-fold, I: A generalization of Mumford's example and an application to Hom schemes, preprint math.AG/0609284 (2006).

[8] D. Mumford: Further pathologies in algebraic geometry, *Amer. J. Math.* **84**(1962), 642–648.

[9] H. Nasu: Obstructions to deforming space curves and non-reduced components of the Hilbert scheme, *Publ. Res. Inst. Math. Sci.* **42**(2006), 117–141 (see also math.AG/0505413).

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: nasu@kurims.kyoto-u.ac.jp