The greedy algorithm for strict cg-matroids

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Abstract

A matroid-like structure defined on a convex geometry, called a cg-matroid, is defined by S. Fujishige, G. A. Koshevoy, and Y. Sano in [6]. Strict cg-matroids are the special subclass of cg-matroids. In this paper, we show that the greedy algorithm works for strict cg-matroids with natural weightings, and also show that the greedy algorithm works for a hereditary system on a convex geometry with any natural weighting if and only if the hereditary system is a strict cg-matroid.

Keywords: matroid, convex geometry, hereditary system, greedy algorithm

1. Introduction

A matroid which was introduced by H. Whitney [12] in 1935 is one of the most important structures in combinatorial optimization. Many researchers have studied and extended the matroid theory (see [11] and [8]). One of the reasons that matroids are important is that matroids are closely related to the greedy algorithm, which solves the maximum base problem efficiently. U. Faigle [5] considered the greedy algorithm for a hereditary system on the lattice formed by all ideals of a poset in 1979. The greedy algorithm has been studied in more general framework. A *greedoid* is a system for which the greedy algorithm works (see [7] for detail).

F. D. J. Dunstan, A. W. Ingleton, and D. J. A. Welsh [3] introduced the concept of a *supermatroid* defined on a poset in 1972 as a generalization of a matroid. A supermatroid on a distributive lattice is also called a *poset matroid*. In 1993 and 1998, M. Barnabei, G.

Nicoletti, and L. Pezzoli [1] [2] studied poset matroids in terms of the poset structure of the ground set.

S. Fujishige, G. A. Koshevoy, and Y. Sano [6] generalized poset matroids by considering convex geometries, instead of posets, as underlying combinatorial structures on which they define matroid-like structures, called *cg-matroids*. They also considered a special class of cg-matroids, called *strict cg-matroids*, for which rank functions are naturally defined. And they show the equivalence of the concept of a strict cg-matroid and that of a supermatroid defined on the lattice of closed sets of a convex geometry.

In this paper, we will consider about the greedy algorithm for strict cg-matroids, which contains the case of poset matroids. It should be emphasized that *strict cg-matroids are not greedoids*. We show that the greedy algorithm works for strict cg-matroids with natural weightings, and give a characterization of strict cg-matroids by using the greedy algorithm. This paper is organized as follows. In Section 2, we give definitions and some preliminaries on convex geometries, strict cg-matroids, and the greedy algorithm. In Section 3, we show that the greedy algorithm works for strict cg-matroids with natural weightings, and also show that the greedy algorithm works for a hereditary system on a convex geometry with any natural weighting if and only if the hereditary system is a strict cg-matroid.

2. Definitions and preliminaries

In this section, we give the definitions of convex geometries, strict cg-matroids, and the greedy algorithm, and we show some lemmas.

2.1. Convex geometries

A convex geometry is a fundamental combinatorial structure defined on a finite set (see [4]).

Definition. Let E be a nonempty finite set and $\mathcal{F}$ be a family of subsets of E. The pair $(E, \mathcal{F})$ is called a *convex geometry* on E if $\mathcal{F}$ satisfies the following three conditions:

(F0) $\emptyset,\ E \in \mathcal{F}$.

(F1) $X, Y \in \mathcal{F} \Longrightarrow X \cap Y \in \mathcal{F}$.

(F2) $\forall X \in \mathcal{F} \setminus \{E\}$, $\exists e \in E \setminus X$: $X \cup \{e\} \in \mathcal{F}$.

The set E is called the *ground set* of the convex geometry $(E, \mathcal{F})$, and each member of $\mathcal{F}$ is called a *closed set*. It should be noted that the condition (F2) is equivalent to the following condition:

(F2)′ Every maximal chain $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n = E$ in $\mathcal{F}$ has length $n = |E|$. □

Example 2.1. (a) Let E be a finite set of points in a Euclidean space $\mathbb{R}^d$. Define $\mathcal{F} = \{X \in 2^E \mid X = \mathrm{Conv}(X) \cap E\}$, where $\mathrm{Conv}(X)$ denotes the convex hull of X in $\mathbb{R}^d$. Then $(E, \mathcal{F})$ is a convex geometry, called a *convex shelling*.
(b) Let E be the vertex set of a tree T. Define $\mathcal{F} = \{X \in 2^E \mid X$ is the vertex set of a subtree of $T\}$. Then $(E, \mathcal{F})$ is a convex geometry, called a *tree shelling*.
(c) Let E be a partially ordered set (poset). Define $\mathcal{F} = \{X \in 2^E \mid X$ is an (order) ideal of $E\}$. Then $(E, \mathcal{F})$ is a convex geometry, called a *poset shelling*. It is well-known that a convex geometry $(E, \mathcal{F})$ is a poset shelling if and only if $\mathcal{F}$ is closed with respect to set union. □

Next, we define operators associated with a convex geometry.

Definition. Let $(E, \mathcal{F})$ be a convex geometry.
The *closure operator* of $(E, \mathcal{F})$ is an operator $\tau : 2^E \to \mathcal{F}$ defined by

$$\tau(X) = \bigcap \{Y \in \mathcal{F} \mid X \subseteq Y\} \quad (X \in 2^E). \tag{2.1}$$

That is, $\tau(X)$ is the unique minimal closed set containing X.
The *extreme-point operator* of $(E, \mathcal{F})$ is an operator $\mathrm{ex} : \mathcal{F} \to 2^E$ defined by

$$\mathrm{ex}(X) = \{e \in X \mid X \setminus \{e\} \in \mathcal{F}\} \quad (X \in \mathcal{F}). \tag{2.2}$$

An element in $\mathrm{ex}(X)$ is called an *extreme point* of $X \in \mathcal{F}$.
The *co-extreme-point operator* of $(E, \mathcal{F})$ is an operator $\mathrm{ex}^* : \mathcal{F} \to 2^E$ defined by

$$\mathrm{ex}^*(X) = \{e \in E \setminus X \mid X \cup \{e\} \in \mathcal{F}\} \quad (X \in \mathcal{F}). \tag{2.3}$$

An element in $\mathrm{ex}^*(X)$ is called a *co-extreme point* of $X \in \mathcal{F}$. □

Lemma 2.2. *For any $X \in 2^E$, we have $\mathrm{ex}(\tau(X)) \subseteq X$.*

Proof. Take $e \in \mathrm{ex}(\tau(X))$. Then we have $\tau(X) \setminus \{e\} \in \mathcal{F}$. From the definition of a closure operator, we have $X \setminus \{e\} \subseteq \tau(X) \setminus \{e\}$. From a property of a closure operator, we have $\tau(X \setminus \{e\}) \subseteq \tau(X) \setminus \{e\}$. This implies $\tau(X \setminus \{e\}) \neq \tau(X)$, and thus $e \in X$. □

2.2. Strict cg-matroids

Definition. Let $(E, \mathcal{F})$ be a convex geometry and $\mathcal{I} \subseteq \mathcal{F}$ be a subfamily. $(E, \mathcal{F}; \mathcal{I})$ is called a *hereditary system* on the convex geometry $(E, \mathcal{F})$ if $\mathcal{I}$ satisfies the following properties.

(I0) $\emptyset \in \mathcal{I}$.

(I1) $I_1 \in \mathcal{F},\ I_2 \in \mathcal{I},\ I_1 \subseteq I_2 \implies I_1 \in \mathcal{I}$. $\square$

Definition. Let $(E, \mathcal{F}; \mathcal{I})$ be a hereditary system on a convex geometry. Then $(E, \mathcal{F}; \mathcal{I})$ is called a *strict cg-matroid* if $\mathcal{I}$ satisfies the following equivalent properties.

(IsA) (Strict Augmentation Property)
For any $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$,
there exists $e \in \tau(I_1 \cup I_2) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

(ILA) (Local Augmentation Property)
For any $I_1, I_2 \in \mathcal{I}$ with $|I_1| + 1 = |I_2|$,
there exists $e \in \tau(I_1 \cup I_2) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

(IS) For each $X \in \mathcal{F}$, all the maximal elements of $\mathcal{I}^{(X)} := \{X \cap I \mid I \in \mathcal{I}\}$ have the same cardinality (as subsets of E). $\square$

See [6] for the proof of the equivalence of above three properties. Now we see some examples of strict cg-matroids.

Example 2.3. Let $(E, \mathcal{F})$ be any convex geometry and k be an integer such that $0 \leq k \leq |E|$. Define

$$\mathcal{I} = \{X \in \mathcal{F} \mid |X| \leq k\}. \tag{2.4}$$

Then $(E, \mathcal{F}; \mathcal{I})$ is a strict cg-matroid, called a *k-uniform cg-matroid*. $\square$

Example 2.4. Let $(E, \mathcal{F})$ be a convex shelling in $\mathbb{R}^d$. We call a finite set X of points in $\mathbb{R}^d$ a *simplex* if $\dim(\mathrm{Conv}(X)) = |X| + 1$. Let

$$\mathcal{I} = \{X \in \mathcal{F} \mid \dim(\mathrm{Conv}(X)) = |X| + 1\} \tag{2.5}$$

be the family of closed sets which are simplices in $\mathbb{R}^d$. Then $(E, \mathcal{F}; \mathcal{I})$ is a strict cg-matroid.

Proof. Since $\mathrm{Conv}(\emptyset) = \emptyset$ and $\dim(\emptyset) = -1$ by convention, the empty set $\emptyset$ is a simplex in $\mathbb{R}^d$. So we have $\emptyset \in \mathcal{I}$ and thus (I0) holds.

Suppose that $I_1 \in \mathcal{F}$, I_2 is a simplex in $\mathbb{R}^d$, and $I_1 \subseteq I_2$. Since any subset of a simplex is also a simplex, we have $I_1 \in \mathcal{I}$ and thus (I1) holds.

Take any simplices $I_1, I_2 \in \mathcal{I}$ such that $|I_1| < |I_2|$. Then, since $\dim(\mathrm{Conv}(I_1)) < \dim(\mathrm{Conv}(I_2)) \leq \dim(\mathrm{Conv}(\tau(I_1 \cup I_2)))$, there exists a point e in $\tau(I_1 \cup I_2)$ such that the point e is not contained in the affine hull of I_1 and that $I_1 \cup \{e\}$ is a closed set. Then $I_1 \cup \{e\}$ is a simplex in $\mathbb{R}^d$ since $\dim(\mathrm{Conv}(I_1 \cup \{e\})) = \dim(\mathrm{Conv}(I_1)) + 1 = |I_1| + 2 = |I_1 \cup \{e\}| + 1$. So we have $I_1 \cup \{e\} \in \mathcal{I}$ with $e \in \tau(I_1 \cup I_2) \setminus I_1$, and thus (IsA) holds. $\square$

Remark. Note that strict cg-matroids are the special subclass of cg-matroids, whose definition is as follows. A hereditary system $(E, \mathcal{F}; \mathcal{I})$ on a convex geometry is called a *cg-matroid* if $\mathcal{I}$ satisfies the following property.

(IA) (Augmentation Property)
For any $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$ and I_2 being maximal in $\mathcal{I}$,
there exists $e \in \tau(I_1 \cup I_2) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

See [6] for detail about cg-matroids. $\square$

Definition. Let $(E, \mathcal{F})$ be a convex geometry and $X = \{e_1, ..., e_k\} \in \mathcal{F}$ be a closed set, where $1 \leq k \leq |E|$. An ordering $(e_1, ..., e_k)$ of the elements of X is called an *$\mathcal{F}$-feasible ordering* of X if $X_i := \{e_1, ..., e_i\} \in \mathcal{F}$ holds for all $1 \leq i \leq k$. $\square$

Definition. Let $(E, \mathcal{F})$ be a convex geometry and $w : E \to \mathbb{R}_{\geq 0}$ be a nonnegative weight function on E. w is called a *natural weighting* on $(E, \mathcal{F})$ if there exists an $\mathcal{F}$-feasible ordering $(e_1, ..., e_n)$ of E such that $w(e_1) \geq ... \geq w(e_n)$. $\square$

Lemma 2.5. *Let $(E, \mathcal{F})$ be a convex geometry and $w : E \to \mathbb{R}_{\geq 0}$ be a natural weighting on $(E, \mathcal{F})$. Then, for any closed set $X \in \mathcal{F}$, there exists an $\mathcal{F}$-feasible ordering $(e_1, ..., e_k)$ of X such that $w(e_1) \geq ... \geq w(e_k)$.*

Proof. Since $w : E \to \mathbb{R}_{\geq 0}$ is a natural weighting on $(E, \mathcal{F})$, there exists an $\mathcal{F}$-feasible ordering $(e_1, ..., e_n)$ of E such that $w(e_1) \geq ... \geq w(e_n)$. Put $Y_i = \{e_1, ..., e_i\} \in \mathcal{F}$ $(1 \leq i \leq n)$ and $Y_0 = \emptyset$. Also put $Z_i = X \cap Y_i$ $(0 \leq i \leq n)$. Then we have $Z_i \in \mathcal{F}$ and

$$\emptyset = Z_0 \subseteq Z_1 \subseteq ... \subseteq Z_n = X.$$

Here we can take the strictly increasing maximal subchain of this chain.

$$\emptyset = Z_{i_0} \subsetneq Z_{i_1} \subsetneq ... \subsetneq Z_{i_k} = X,$$

where $k = |X|$. Take $\hat{e}_t \in Z_{i_t} \setminus Z_{i_{t-1}}$ $(1 \leq t \leq k)$. Then $(\hat{e}_1, ..., \hat{e}_k)$ is an $\mathcal{F}$-feasible ordering of X such that $w(\hat{e}_1) \geq ... \geq w(\hat{e}_k)$. Thus the lemma holds. $\square$

Lemma 2.6. *Let $(E, \mathcal{F})$ be a convex geometry and $w : E \to \mathbb{R}_{\geq 0}$ be a natural weighting of $(E, \mathcal{F})$. Then, for any closed set $X \in \mathcal{F}$, there exists $\hat{e} \in \mathrm{ex}(X)$ such that $w(\hat{e}) =\min\{w(e) \mid e \in X\}$.*

Proof. Take a closed set $X \in \mathcal{F}$. Then, from Lemma 2.5, there exists an $\mathcal{F}$-feasible ordering $(e_1, ..., e_k)$ of X such that $w(e_1) \geq ... \geq w(e_k)$, where $k = |X|$. Since $\{e_1, ..., e_{k-1}\} \in \mathcal{F}$, we have $\hat{e} := e_k \in \mathrm{ex}(X)$ and $w(\hat{e}) =\min\{w(e) \mid e \in X\}$. $\square$

2.3. Greedy algorithm

Let $(E, \mathcal{F}; \mathcal{I})$ be a hereditary system and $w : E \to \mathbb{R}_{\geq 0}$ be a nonnegative weight function on E. We denote $\Sigma_{e \in X} w(e)$ by $w(X)$. We consider the following problem.

$$(P_w) \qquad \text{maximize} \qquad w(I) \tag{2.6}$$

$$\text{subject to} \qquad I \in \mathcal{I} \tag{2.7}$$

The *greedy algorithm* is the following algorithm.

Greedy Algorithm.

Set $I^{(0)} \leftarrow \emptyset$. For $i = 0$ to $n - 1$, do

step i: If there exists $e \in E \setminus I^{(i)}$ such that $I^{(i)} \cup \{e\} \in \mathcal{I}$, then choose such an element e_{i+1} of maximum weight, i.e.,

$$w(e_{i+1}) = \max\{w(e) \mid e \in E \setminus I^{(i)}, I^{(i)} \cup \{e\} \in \mathcal{I}\}. \tag{2.8}$$

Let $I^{(i+1)} \leftarrow I^{(i)} \cup \{e_{i+1}\}$ and go to step $i + 1$.
Otherwise, let $I_G \leftarrow I^{(i)}$ and stop. □

Definition. Let $(E, \mathcal{F}; \mathcal{I})$ be a hereditary system on a convex geometry and $w : E \to \mathbb{R}_{\geq 0}$ be a weight function on E. If the greedy algorithm produces an optimal solution of (P_w), then we say the greedy algorithm *works* for $(E, \mathcal{F}; \mathcal{I})$ with the weighting w. □

3. Main results

In this section, we show our main result that the greedy algorithm works for a hereditary system on a convex geometry with any natural weighting if and only if the hereditary system is a strict cg-matroid.

First, we show that the greedy algorithm works for any strict cg-matroids with any natural weightings.

Theorem 3.1. *Let $(E, \mathcal{F}; \mathcal{I})$ be a strict cg-matroid. Then the greedy algorithm works for $(E, \mathcal{F}; \mathcal{I})$ with any natural weighting on $(E, \mathcal{F})$.*

Proof. Fix any natural weighting $w : E \to \mathbb{R}_{\geq 0}$ on $(E, \mathcal{F})$. Let $I_G = \{e_1, ..., e_r\} \in \mathcal{I}$ be a solution obtained by the greedy algorithm. Note that $(e_1, ..., e_r)$ is an $\mathcal{F}$-feasible ordering such that $w(e_1) \geq ... \geq w(e_r)$. Since w is nonnegative, if $X \subseteq Y$ then $w(X) \leq w(Y)$. Take any $I' \in \mathcal{I}$ which is maximal in $\mathcal{I}$. Then, from (IS), I' has also r elements. From Lemma 2.5, there exists an $\mathcal{F}$-feasible ordering $(e'_1, ..., e'_r)$ of I' such that $w(e'_1) \geq ... \geq w(e'_r)$. Then it follows from the following Lemma 3.2 that $w(e_i) \geq w(e'_i)$ for all $1 \leq i \leq r$. Thus we have $w(I_G) = \Sigma_{i=1}^{r} w(e_i) \geq \Sigma_{i=1}^{r} w(e'_i) = w(I')$. Hence I_G is an optimal solution of the problem (P_w), and the theorem holds. □

Lemma 3.2. *The settings are the same as in the proof of Theorem 3.1. Then* $w(e_i) \geq w(e'_i)$ *holds for all* $1 \leq i \leq r$.

Proof. Suppose that the lemma does not hold. Let k be the minimum number such that $w(e_k) < w(e'_k)$. Put $I_1 = \{e_1, ..., e_{k-1}\}$ and $I_2 = \{e'_1, ..., e'_k\}$. Then we have $I_1 \in \mathcal{F}$ and $I_2 \in \mathcal{F}$ since $(e_1, ..., e_r)$ and $(e'_1, ..., e'_r)$ are $\mathcal{F}$-feasible orderings. Thus it follows from (I1) that $I_1 \in \mathcal{I}$ and $I_2 \in \mathcal{I}$. Since $|I_1| < |I_2|$, from (IsA), there exists $e \in \tau(I_1 \cup I_2) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$. Here there are the following two cases.
(Case 1) $e \in I_2 \setminus I_1$.

Since e'_k has the minimum weight in I_2, we have $w(e) \geq w(e'_k) > w(e_k)$. This is a contradiction to the choice of e_k in the greedy algorithm step $k-1$.
(Case 2) $e \in \tau(I_1 \cup I_2) \setminus (I_1 \cup I_2)$.

From Lemma 2.6, there exists $\hat{e} \in \mathrm{ex}(\tau(I_1 \cup I_2))$ such that $w(\hat{e})$ =min$\{w(e) \mid e \in \tau(I_1 \cup I_2)\}$. Here, from Lemma 2.2, we have $\mathrm{ex}(\tau(I_1 \cup I_2)) \subseteq I_1 \cup I_2$. So we have $\hat{e} \in I_1 \cup I_2$, and thus $e \neq \hat{e}$. Since e'_k has the minimum weight in $I_1 \cup I_2$ and $\hat{e} \in I_1 \cup I_2$, we have $w(\hat{e}) \geq w(e'_k)$. Therefore we have

$$w(e) \geq \min\{w(e) \mid e \in \tau(I_1 \cup I_2)\} = w(\hat{e}) \geq w(e'_k) > w(e_k).$$

This is a contradiction to the choice of e_k in the greedy algorithm step $k-1$.

Hence the lemma holds. □

Next, we show that a hereditary system on a convex geometry such that the greedy algorithm works for it with any natural weighting is a strict cg-matroid.

Theorem 3.3. *Let* $(E, \mathcal{F}; \mathcal{I})$ *be a hereditary system on a convex geometry. Suppose that*

(G) *The greedy algorithm works for* $(E, \mathcal{F}; \mathcal{I})$ *with any natural weighting on* $(E, \mathcal{F})$.

Then $(E, \mathcal{F}; \mathcal{I})$ *is a strict cg-matrioid.*

Proof. We will show that (IsA) holds. Take any $I_1, I_2 \in \mathcal{I}$ such that $|I_1| < |I_2|$. If $I_1 \subset I_2$ then it is easy to see that (IsA) holds. So we suppose that $I_1 \not\subset I_2$, and suppose that (IsA) does not hold, i.e., there is no element $e \in \tau(I_1 \cup I_2) \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Then we have $0 < |I_1 - I_2| = |I_1| - |I_1 \cap I_2| < |I_2| - |I_1 \cap I_2| = |I_2 - I_1|$. Take a positive number $\epsilon > 0$ which satisfies $0 < (1+\epsilon)|I_1 - I_2| < |I_2 - I_1|$. Define a weight function $w : E \to \mathbb{R}_{\geq 0}$ as follows.

$$w(e) = \begin{cases} 2 & (\ e \in I_1 \cap I_2\ ) \\ 1/|I_1 - I_2| & (\ e \in I_1 \setminus I_2\ ) \\ (1+\epsilon)/|I_2 - I_1| & (\ e \in \tau(I_1 \cup I_2) \setminus I_1\ ) \\ 0 & (\ e \in E \setminus \tau(\ I_1 \cup I_2)\ ) \end{cases} \tag{3.1}$$

Then w is a natural weighting on $(E, \mathcal{F})$. Because any maximal chain of $\mathcal{F}$ that contains $I_1 \cap I_2$, I_1, and $\tau(I_1 \cup I_2)$ naturally defines an $\mathcal{F}$-feasible ordering $(e_1, ..., e_n)$ of E such that $w(e_1) \geq ... \geq w(e_n)$.

Put $k = |I_1|$ and consider the greedy algorithm. In step $k-1$, we have $I^{(k)} = I_1$. From the assumption, we can not take an element $e \in \tau(I_1 \cup I_2) \setminus I_1$ in step k. Let $I_G \in \mathcal{I}$ be the solution obtained by the greedy algorithm. We claim that I_G does not contain the elements in $\tau(I_1 \cup I_2) \setminus I_1$, i.e., $I_G \cap \tau(I_1 \cup I_2) = I_1$. If there exist such elements $e_{i_1}, ..., e_{i_t}$, then consider a maximal chain in $\mathcal{F}$ which contains I_1 and the subset $I_1 \cup \{e_{i_1}, ..., e_{i_t}\} = I_G \cap \tau(I_1 \cup I_2) \in \mathcal{F}$. Then $I_1 \cup \{e_i\} \in \mathcal{F}$ for some $e_i \in \{e_{i_1}, ..., e_{i_t}\}$. Since $I_1 \cup \{e_i\} \subseteq I_G \in \mathcal{I}$, from (I1), we have $I_1 \cup \{e_i\} \in \mathcal{I}$, but this is a contradiction to the assumption.

Now we have the following.

$$w(I_G) = w(I_1) = 2|I_1 \cap I_2| + 1, \tag{3.2}$$

$$w(I_2) = 2|I_1 \cap I_2| + 1 + \epsilon. \tag{3.3}$$

Thus we have $w(I_G) < w(I_2)$, i.e., I_G is not an optimal solution of (P_w). This is a contradiction to (G).

Hence (IsA) holds, and thus $(E, \mathcal{F}; \mathcal{I})$ is a strict cg-matroid. $\square$

Combining Theorem 3.1 and Theorem 3.3, we get the following our main theorem.

Theorem 3.4. *Let $(E, \mathcal{F}; \mathcal{I})$ be a hereditary system on a convex geometry. Then, $(E, \mathcal{F}; \mathcal{I})$ is a strict cg-matroid if and only if the greedy algorithm works for $(E, \mathcal{F}; \mathcal{I})$ with any natural weighting on $(E, \mathcal{F})$.* $\square$

At the end of this section, we see some examples, which show that the greedy algorithm fails for a strict cg-matroid with a "not natural" weighting and also fails for a "not strict" cg-matroid with a natural weighting.

Example 3.5. Let $(E, \mathcal{F})$ be the tree shelling of a path with five vertices, i.e., $E = \{1, 2, 3, 4, 5\}$ and $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}$. Consider the 3-uniform cg-matroid on this convex geometry $(E, \mathcal{F})$, i.e., $\mathcal{I} = \{X \in \mathcal{F} \mid |X| \leq 3\}$ (see Figure 1). Then $(E, \mathcal{F}; \mathcal{I})$ is a strict cg-matroid.

Let $w : E \to \mathbb{R}_{\geq 0}$ be a weight function on E defined by $w(1) = 10$, $w(2) = 1$, $w(3) = 2$, $w(4) = 8$, $w(5) = 9$. This is not a natural weighting on $(E, \mathcal{F})$ because the ordering $(1, 5, 4, 3, 2)$ is not an $\mathcal{F}$-feasible ordering.

Now the greedy algorithm produces a solution $I_G = \{1, 2, 3\}$ with $w(I_G) = 13$. But this is not an optimal solution of (P_w). The optimal solution of (P_w) is $I = \{3, 4, 5\}$ with $w(I) = 19$. $\square$

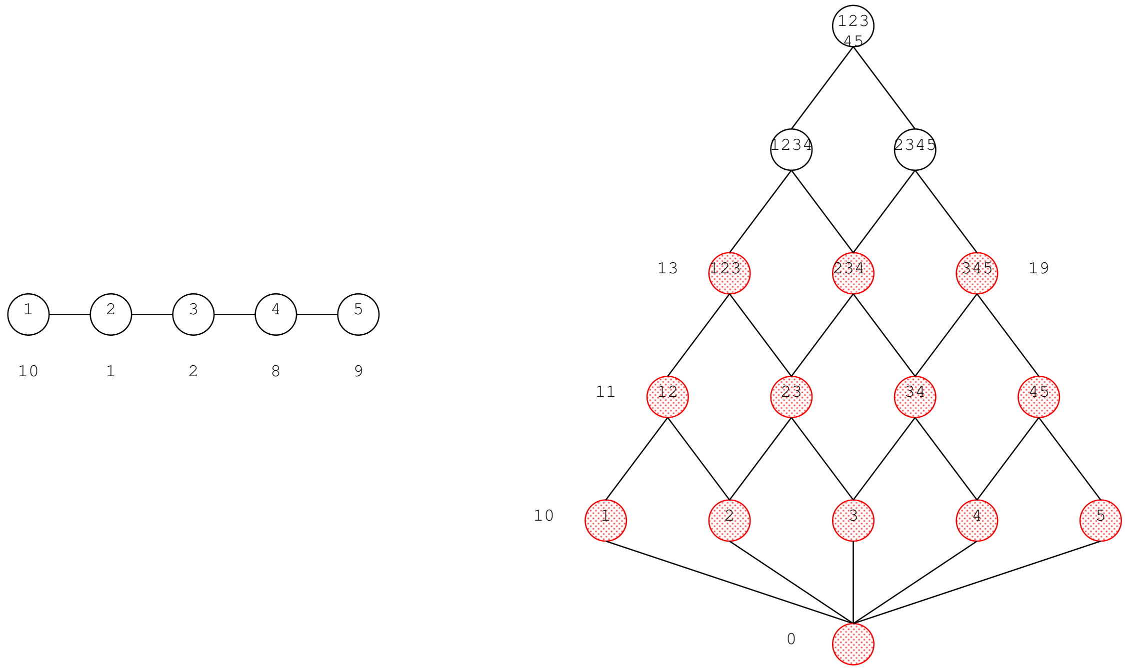

Figure 1: A tree shelling of a path with five vertices

Example 3.6. Let $(E, \mathcal{F})$ be the convex shelling of the five points in the plane given in the left of Figure 2, i.e., $E = \{1, 2, 3, 4, 5\}$ and $\mathcal{F} = 2^E \setminus \{\{1, 2, 4, 5\}, \{1, 2, 4\}, \{1, 2, 5\}\}$. Define $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{1, 2, 3\}, \{2, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}$ (see the right of Figure 2). Then $(E, \mathcal{F}; \mathcal{I})$ is a cg-matroid but is not a strict cg-matroid.

Let $w : E \to \mathbb{R}_{\geq 0}$ be a weight function on E defined by $w(1) = 10$, $w(2) = 1$, $w(3) = 2$, $w(4) = 8$, $w(5) = 9$. This is a natural weighting on $(E, \mathcal{F})$ because there is an $\mathcal{F}$-feasible ordering $(1, 5, 4, 3, 2)$, which satisfies $w(1) \geq w(5) \geq w(4) \geq w(3) \geq w(2)$.

Now the greedy algorithm produces a solution $I_G = \{1, 3, 2\}$ with $w(I_G) = 13$. But this is not an optimal solution of (P_w). The optimal solution of (P_w) is $I = \{2, 4, 5\}$ with $w(I) = 18$. $\square$

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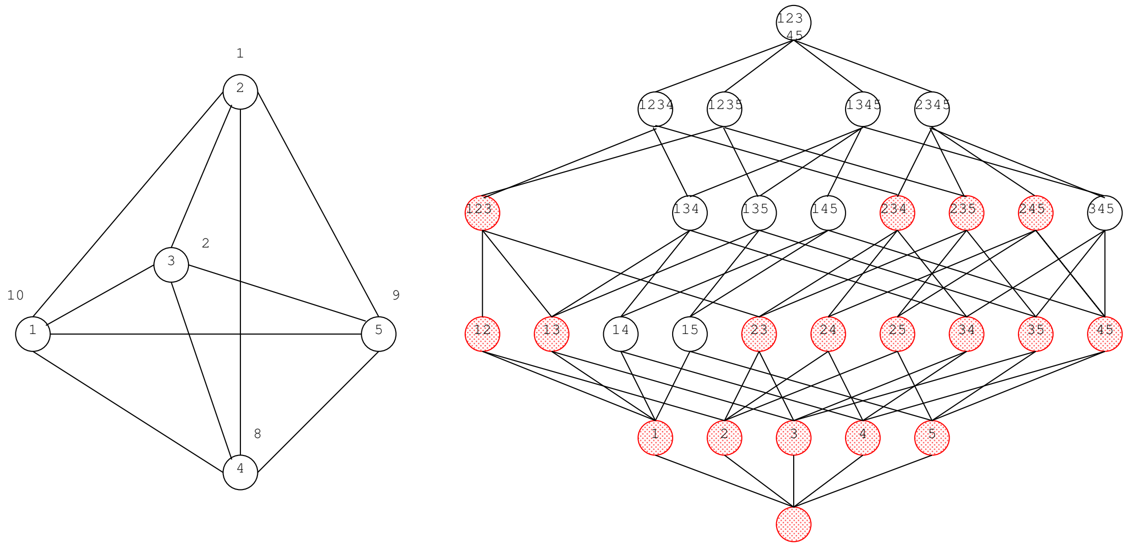

Figure 2: A convex shelling of five points in the plane.

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