# Weighted Competition Graphs

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**Abstract**

We introduce a generalization of competition graphs, called weighted competition graphs. The *weighted competition graph* of a digraph $D = (V, A)$, denoted by $C_w(D)$, is an edge-weighted graph $(G, w)$ such that $G = (V, E)$ is the competition graph of $D$, and the *weight* $w(e)$ of an edge $e = xy \in E$ is the number of the common preys of $x$ and $y$ in $D$. We investigate properties of weighted competition graphs.

**Keywords: competition graph, competition number, edge clique cover**

## 1. Introduction

Joel E. Cohen [13] introduced the notion of a competition graph in connection with a problem in ecology in 1968 (also see [14]). Let $D = (V, A)$ be a digraph, which corresponds to a food web. A vertex $x \in V$ in $D$ stands for a species in the food web, and an arc $(x, a) \in A$ in $D$ means that the species $x$ preys on the species $a$. If two species $x$ and $y$ have a common prey $a$, they will compete for the prey $a$. J. E. Cohen defined a graph which represents the relations of competition among the species in the food web. The *competition graph* $C(D)$ of a digraph $D = (V, A)$ is an undirected graph $G = (V, E)$ which has the same vertex set $V$ and has an edge between distinct two vertices $x, y \in V$ if there exist a vertex $a \in V$ and arcs $(x, a), (y, a) \in A$ in $D$. We say that a graph $G$ is a *competition graph* if there exists a digraph $D$ such that $C(D) = G$. This notion is applicable not only in ecology but also in channel assignments, coding, and modeling of complex economic and energy systems (see [36]).

F. S. Roberts [37] observed that, for any graph, the graph with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The minimum such number of isolated vertices was called the *competition number* of the graph $G$ and was denoted by $k(G)$. R. J. Opsut [35] showed that the computation of the competition number of a graph is an NP-hard problem. So it seems to be difficult to compute the competition numbers of graphs, in general. But, for a graph in some special classes, it is easy to get the competition number of the graph. The following are some of famous results for competition numbers, which we use in later.

**Theorem 1.1.** *If $G$ is a chordal graph which has no isolated vertices, then $k(G) = 1$.*

**Theorem 1.2.** *If $G$ is a triangle-free connected graph, then $k(G) = |E(G)| - |V(G)| + 2$.*

Recent studies about competition numbers are found in [6, 8, 26, 30].

Competition graphs and the competition numbers of graph are closely related to edge clique covers and of the edge clique cover numbers of the graphs. A *clique* of a graph $G$ is an empty set or a subset of $V(G)$ such that its induced subgraph of $G$ is a complete graph. A clique consisting of 3 vertices is called a *triangle*. An *edge clique cover* (or an *ECC* for short) of a graph $G$ is a family of cliques of $G$ such that each edge of $G$ is contained in some clique in the family. The minimum size of a edge clique cover of $G$ is called the *edge clique cover number* (or the *ECC number* for short) of the graph $G$, and is denoted by $\theta_e(G)$.

Opsut [35] showed that, for any graph $G$, the competition number satisfies an inequality $\theta_e(G) - |V(G)| + 2 \leq k(G) \leq \theta_e(G)$. R. D. Dutton and R. C. Brigham [15] showed that a graph $G$ is a competition graph if and only if $\theta_e(G) \leq |V(G)|$, and also characterized the competition graphs of acyclic digraphs by using ECCs. F. S. Roberts and J. E. Steif [43] characterized the competition graphs of digraphs which have no loops by using ECCs and ECC numbers. J. R. Lundgren and J. S. Maybee [31] characterized graphs whose competition numbers are less than or equal to a number $m$ by using ECCs. For other applications of ECCs, see [38].

Competition graphs, competition numbers, and their related objects have been studied by many researchers since its appearance. There are various notions closely related to the notion of a competition graph. R. J. Lundgren, and J. S. Maybee [32] introduced the *common enemy graph* of a digraph. D. D. Scott [44] introduced the *competition-common enemy graph* of a digraph and the *double competition number* of a graph (also see [45, 18]). C. Cable, K. F. Jones, J. R. Lundgren, and S. Seager [5] introduced the *niche graph* of a digraph and the *niche number* of a graph (also see [2, 1]). F. S. Roberts, and L. Sheng [40, 41] introduced the *phylogeny graph* of a digraph and the *phylogeny number* of a graph (also see [42, 39]).

There are also various generalizations of competition graphs. S. R. Kim, T. A. McKee, F. R. McMorris, and F. S. Roberts [28, 27] introduced the *$p$-competition graph* of a digraph and the *$p$-competition number* of a graph (also see [22, 23]). R. C. Brigham, F. R. McMorris, and R. P. Vitray [3] introduced a *tolerance competition graph* (also see [4]). H. H. Cho, S. R. Kim, and Y. Nam [10] introduced the *$m$-step competition graph* of a digraph and the *$m$-step competition number* of a graph (also see [11, 12, 20, 21]). M. Sonntag, and H. M. Teichert [46] introduced the *competition hypergraph* of a digraph.

Surveys of the large literature around competition graphs can be found in [24, 25, 39]. For other topics related to competition graphs, see [7, 9, 16, 17, 19, 33, 34].

In this paper, we introduce another new generalization of competition graphs, called *weighted competition graphs*. The weighted competition graph of a digraph has much more information derived from the digraph than the competition graph of the digraph has. The weights of a weighted competition graph represent degree of competition between two species.

This paper is organized as follows. In section 2, we define the weighted competition graph of a digraph and a weighted edge clique cover of a weighted graph. And we give characterizations of weighted competition graphs by using weighted edge clique covers. In section 3, we define the weighted competition number of a weighted graph and investigate it. In section 4, we consider an application of weighted competition graphs and weighted edge clique covers to analysis of $p$-competition graphs. In section 5, we mention some remarks.

**Notation.** In this paper, we use the following notations. For a graph $G$, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. We denote an edge between vertices $x$ and $y$ by $xy$. For a digraph $D$, we denote its vertex set by $V(D)$ and its arc set by $A(D)$. We denote an arc from a vertex $u$ to a vertex $v$ by $(u, v)$. We call an arc $(u, v)$ an *incoming arc* of $v$, and also call it an *outgoing arc* of $u$. We denote the graph of $k$ isolated vertices with no edges by $I_k$.

## 2. Weighted competition graphs

In this section, we define the weighted competition graph of a digraph, and give its characterizations by using a weighted edge clique cover.

## 2.1. The weighted competition graph of a digraph

**Definition.** The *weighted competition graph* of a digraph $D = (V, A)$ is an edge-weighted graph $(G, w)$ such that $G = (V, E)$ is the competition graph of $D$, and the *weight* $w(e)$ of an edge $e = xy \in E$ is the number of the common preys of $x$ and $y$ in $D$. We denote the weighted competition graph of a digraph $D$ by $C_w(D)$.

And we call a weighted graph $(G, w)$ a *weighted competition graph* if there exists a digraph $D$ such that $C_w(D) = (G, w)$. □

**Example.** Let $D$ be a digraph on the left in Figure 1. Then the weighted competition graph of the digraph $D$ is the graph on the center in Figure 1. Note that we can consider the weighted competition graph as a graph which has multiple edges like the graph on the right in Figure 1. □

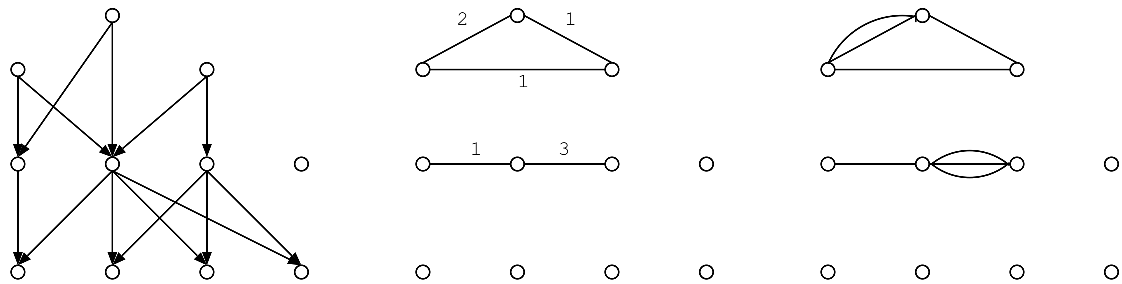


Figure 1:

**Remark 2.1.** It should be noted the relation between weighted competition graphs and $p$-competition graphs (see [28]). Let $p$ be a positive integer. The *$p$-competition graph* $C_p(D)$ of a digraph $D = (V, A)$ is a graph which has same vertex set $V$ and has an edge between distinct vertices $x, y \in V$ if, for some distinct $p$ vertices $a_1, ..., a_p \in V$, there exist arcs $(x, a_i), (y, a_i) \in A$ in the digraph $D$ for each $i = 1, ..., p$. And we call a graph $G$ a *$p$-competition graph* if there exists a digraph $D$ such that $C_p(D) = G$.

Let $C_p(D) = (V, E(C_p(D))$ $(p = 1, 2, 3, ...)$ be the $p$-competition graph of a digraph $D = (V, A)$. Then we have

$$E(C_1(D)) \supseteq E(C_2(D)) \supseteq ... \supseteq E(C_p(D)) \supseteq .... \tag{2.1}$$

We define a graph $G_p = (V, E_p)$ $(p = 1, 2, 3, ...)$ by

$$E_p := E(C_p(D)) - E(C_{p+1}(D)) \quad (p = 1, 2, 3, ...). \tag{2.2}$$

And we define a weighted graph $(G, w)$ as follows; $V(G) := V$,

$$E(G) := E_1 \oplus E_2 \oplus ... \oplus E_p \oplus ... = E(C(D)), \tag{2.3}$$

and $w(e) := p$ if $e \in E_p$. Then this weighted graph $(G, w)$ coincides with the weighted competition graph of the digraph $D$, i.e., $(G, w) = C_w(D)$.

Conversely, let $C_w(D)$ be the weighted competition graph of a digraph $D = (V, A)$. For a weighted graph $(G, w)$, the $p_{\leq}$*-subgraph* $(G, w)_{p\leq}$ of $(G, w)$ is a subgraph $(V, E_{p\leq})$ of $G$ defined by

$$E_{p\leq} := \{e \in E(G) \mid p \leq w(e)\}. \tag{2.4}$$

Then the $p_{\leq}$-subgraph $(C_w(D))_{p\leq}$ of the weighted competition graph $C_w(D)$ coincides with the $p$-competition graph of the digraph $D$, i.e., $(C_w(D))_{p\leq} = C_p(D)$.

So put it shortly, the weighted competition graph $C_w(D)$ of a digraph $D$ has the information about the $p$-competition graphs $C_p(D)$ of $D$ for all $p$. The relation between weighted competition graphs and $p$-competition graphs is considered further in section 4. □

**Definition.** Let $(G, w)$ be a weighted graph. A family $\mathcal{F} = \{S_1, ..., S_r\}$ of cliques of $G$ is called a *weighted edge clique cover* (or a *w-ECC* for short) of the weighted graph $(G, w)$ if each edge $e = xy \in E(G)$ is contained in exactly $w(e)$ cliques $S_i$ in $\mathcal{F}$, i.e., both $x$ and $y$ are contained in exactly $w(e)$ same cliques $S_i$ in $\mathcal{F}$.

For a weighted graph $(G, w)$, the minimum size of weighted edge clique covers of $(G, w)$ is called the *weighted edge clique cover number* (or the *w-ECC number for short* of $(G, w)$ and is denoted by $\theta_e^w(G, w)$. □

The following relation between w-ECC numbers and ECC numbers holds.

**Proposition 2.2.** *Let $G$ be a graph. Then, for any weight $w$ on $E(G)$, we have*

$$\theta_e(G) \leq \theta_e^w(G, w). \tag{2.5}$$

*Proof.* Since a w-ECC of $(G, w)$ is an ECC of $G$, the proposition holds. □

**Theorem 2.3.** *Let $(G, w)$ be a weighted graph with $|V(G)| = n$. Then, $(G, w)$ is a weighted competition graph if and only if $\theta_e^w(G, w) \leq n$.*

*Proof.* Let $V = V(G) = \{v_1, ..., v_n\}$ and $E = E(G)$.

Suppose that $(G, w)$ is a weighted competition graph. Then there exists a digraph $D = (V, A)$ such that $C_w(D) = (G, w)$. Put $S_j := \{v_i \in V \mid (v_i, v_j) \in A\}$ $(j = 1, ..., n)$. For any $x$ and $y$ in $S_j$, since $v_j$ is a common prey of $x$ and $y$ in $D$, $xy$ is an edge in $G$, and thus $S_j$ is a clique of $G$. If $xy \in E$ is an edge which has weight $p$, then there exist exactly $p$ common preys $v_{i_1}, ..., v_{i_p} \in V$ of $x$ and $y$ in $D$. Then exactly $p$ cliques $S_{i_1}, ..., S_{i_p}$ contain both $x$ and $y$. Hence the family $\{S_1, ..., S_n\}$ is a w-ECC of $(G, w)$, and thus we conclude $\theta_e^w(G, w) \leq n$.

Next, suppose that $\theta_e^w(G, w) \leq n$. Then there exists a w-ECC $\mathcal{F} = \{S_1, ..., S_r\}$ of $(G, w)$, where $r \leq n$. We define a digraph $D$ as follows; $V(D) = V$, and $A(D) =$

$\{(v_i, v_j) \mid v_i \in S_j\}$. Then the competition graph of this digraph $D$ is the graph $G$. And if $xy \in E$ is an edge which has weight $p$, then there exist exactly $p$ cliques $S_{i_1}, ..., S_{i_p} \in \mathcal{F}$ such that each clique contains both $x$ and $y$. So $x$ and $y$ have exactly $p$ common preys $v_{i_1}, ..., v_{i_p} \in V$ in the digraph $D$. Thus the weight of the edge $xy$ in $C_w(D)$ is $p$. Hence we have $C_w(D) = (G, w)$, and thus we conclude $(G, w)$ is a weighted competition graph. $\square$

**Example.** Consider weighted graphs shown in Figure 2. A family $\{\{v_1, v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_2, v_3, v_4\}\}$ is a w-ECC of the weighted graph (a), which has size 4. A family $\{\{v_1, v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_2, v_4\}, \{v_3, v_4\}\}$ is a w-ECC of the weighted graph (b) of minimum size 5. A family $\{\{v_1, v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}\}$ is a w-ECC of the weighted graph (c) of minimum size 5. A family $\{\{v_1, v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_3, v_4\}\}$ is a w-ECC of the weighted graph (d), which has size 4.

Since the number of vertices is 4, from Theorem 2.3, we have that the weighted graphs (a), (d) are weighed competition graphs but the weighted graphs (b), (c) are not weighted competition graphs.

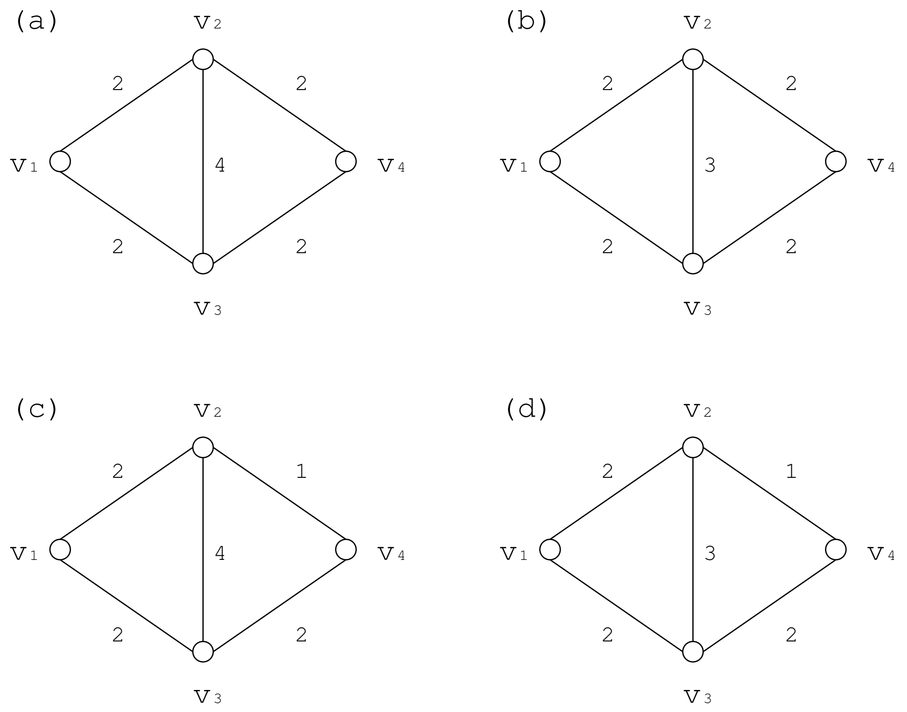


Figure 2:

## 2.2. The weighted competition graph of a loopless digraph

In ordinary situation, it is natural to assume that there are no species that prey themselves in a food web. This assumption corresponds to that a digraph $D$ has no loops.

Let $V$ be a finite set, and $D_i$ be a subset of $V$ and $v_i \in V$ for each $i = 1, ..., r$. Then, $(v_1, ..., v_r)$ is called a *system of distinct representatives* for $\{D_1, ..., D_r\}$ if $v_1, ..., v_r$ are distinct and $v_i \in D_i$ $(i = 1, ..., r)$.

**Theorem 2.4.** *Let $(G, w)$ be a weighted graph. Then the following statements are equivalent.*
*(a) $(G, w)$ is the weighted competition graph of a loopless digraph.*
*(b) There exist an ordering $v_1, ..., v_n$ of the vertices of $G$ and a weighted edge clique cover $\{S_1, ..., S_r\}$ of $(G, w)$ such that $r \leq n$ and $v_j \notin S_j$ $(j = 1, ..., r)$.*
*(c) There exists a weighted edge clique cover $\{S_1, ..., S_r\}$ of $(G, w)$ such that $r \leq n$ and $\{D_1, ..., D_r\}$ has a system of distinct representatives, where $D_j := V(G) - S_j$ $(j = 1, ..., r)$.*

*Proof.* Let $V = V(G) = \{v_1, ..., v_n\}$ and $E = E(G)$.
(a)⇒(b): Let $D = (V, A)$ be a loopless digraph such that $C_w(D) = (G, w)$. Put $S_j := \{v_i \in V \mid (v_i, v_j) \in A\}$ $(j = 1, ..., n)$. Then $\{S_1, ..., S_n\}$ is a w-ECC of $(G, w)$. Since $D$ is loopless, we have $v_j \notin S_j$ $(j = 1, ..., n)$.
(b)⇒(c): Let $v_1, ..., v_n$ be an ordering of vertices of $G$ and $\{S_1, ..., S_r\}$ be a w-ECC of $(G, w)$ such that $r \leq n$ and $v_j \notin S_j$ $(j = 1, ..., r)$. Then $(v_1, ..., v_r)$ is a system of distinct representatives for $\{D_1, ..., D_r\}$.
(c)⇒(a): Let $\{S_1, ..., S_r\}$ be a w-ECC of $(G, w)$ such that $r \leq n$ and that $\{D_1, ..., D_r\}$ has a system of distinct representatives $(v_1, ..., v_r)$. Then we have $v_j \notin S_j$ $(j = 1, ..., r)$. We define a digraph $D$ as follows; $V(D) = V$, and $A(D) = \{(v_i, v_j) \mid v_i \in S_j\}$. Then we have $C_w(D) = (G, w)$, and that $D$ has no loops since $v_j \notin S_j$. □

## 2.3. The weighted competition graph of an acyclic digraph

A digraph $D$ is called *acyclic* if there is no directed cycle in $D$. It is well-known that a digraph $D$ is acyclic if and only if the vertices of $D$ can be labeled so that $(v_i, v_j) \in A(D)$ $\Rightarrow i < j$. We call such a labeling an *acyclic labeling* of $D$.

**Theorem 2.5.** *Let $(G, w)$ be a weighted graph. Then the following statements are equivalent.*
*(a) $(G, w)$ is the weighted competition graph of an acyclic digraph.*
*(b) There exist an ordering $v_1, ..., v_n$ of the vertices of $G$ and a weighted edge clique cover $\{S_1, ..., S_n\}$ of $(G, w)$ such that $v_i \in S_j \Rightarrow i < j$.*
*(c) There exists a weighted edge clique cover $\{S'_1, ..., S'_{n-2}\}$ of $(G, w)$ such that $|S'_1 \cup ... \cup S'_j| \leq j + 1$ for $j = 1, ..., n - 2$.*

*Proof.* Let $V = V(G)$ and $E = E(G)$, and put $n = |V(G)|$.
(a)⇒(b): Let $D = (V, A)$ be an acyclic digraph such that $C_w(D) = (G, w)$. Then there exists an acyclic labeling $v_1, ..., v_n$ of $D$. Put $S_j := \{v_i \in V \mid (v_i, v_j) \in A\}$ $(j = 1, ..., n)$. Then $\{S_1, ..., S_n\}$ is a w-ECC of $(G, w)$ such that $v_i \in S_j \Rightarrow i < j$.
(b)⇒(c): Let $v_1, ..., v_n$ be an ordering of vertices of $G$ and $\{S_1, ..., S_n\}$ be a w-ECC of $(G, w)$ such that $v_i \in S_j \Rightarrow i < j$. Then $S_1 = \emptyset$ and $S_2 = \emptyset$ or $\{v_1\}$. We define $S'_j := S_{j+2}$ $(j = 1, ..., n-2)$. Since $S_1$ and $S_2$ has no edges, $\{S'_1, ..., S'_{n-2}\}$ is also a w-ECC of $(G, w)$. For any $1 \leq j \leq n-2$, if $v_i \in S'_1 \cup ... \cup S'_j$ then $i < j+2$. Hence we have $|S'_1 \cup ... \cup S'_j| \leq j+1$ for $j = 1, ..., n-2$.
(c)⇒(a): Let $\{S'_1, ..., S'_{n-2}\}$ be a w-ECC of $(G, w)$ such that $|S'_1 \cup ... \cup S'_j| \leq j+1$ for $j = 1, ..., n-2$. We label the vertices of $G$ as follows; Let $v_n$ be a vertex in $V \setminus (S'_1 \cup ... \cup S'_{n-2})$ (Since $|S'_1 \cup ... \cup S'_{n-2}| \leq n-1$, $V \setminus (S'_1 \cup ... \cup S'_{n-2}) \neq \emptyset$.), and let $v_{n-1} (\neq v_n)$ be a vertex in $V \setminus (S'_1 \cup ... \cup S'_{n-3})$, ..., and let $v_i (\neq v_j$ for all $j > i)$ be a vertex in $V \setminus (S'_1 \cup ... \cup S'_{i-2})$, ... Finally, for the remaining two vertices, we label arbitrarily as $v_2$ and $v_1$. Let $D$ be a digraph defined as follows; $V(D) = V$, and $A(D) = \{(v_i, v_j) \mid v_i \in S'_{j-2}\}$. Then we have $C_w(D) = (G, w)$. Furthermore, $(v_i, v_j) \in A(D)$ implies $i \leq j-1$. So $v_1, ..., v_n$ is an acyclic labeling of $D$, and thus $D$ is acyclic. □

**Theorem 2.6.** *Let $(G, w)$ be a weighted graph. Then there exists an nonnegative integer $k$ such that $(G \cup I_k, w)$ is the weighted competition graph of some 'acyclic' digraph.*

*Proof.* Put $M$ =max$\{w(e) \mid e \in E(G)\}$. We define a digraph $D$ as follows;

$$V(D) = V(G) \cup \bigcup_{p=1}^{M} \bigcup_{e \in E, w(e)=p} \{a_{e,1}, ..., a_{e,p}\}, \tag{2.6}$$

$$A(D) = \bigcup_{p=1}^{M} \bigcup_{e \in E, w(e)=p} \{(x, a_{e,i}), (y, a_{e,i}) \mid e = xy, 1 \leq i \leq p\}. \tag{2.7}$$

Then the digraph $D$ is acyclic and we have $C_w(D) = (G \cup I_k, w)$ where

$$k = |\bigcup_{p=1}^{M} \bigcup_{e \in E, w(e)=p} \{a_{e,1}, ..., a_{e,p}\}|.$$

Thus the theorem holds. □

# 3. Weighted competition numbers

In this section, we define the weighted competition number of a weighted graph and investigate it.

## 3.1. The weighted competition number of a weighted graph

From Theorem 2.6, we can define the following.

**Definition.** The *weighted competition number* of a weighted graph $(G, w)$ is the smallest nonnegative integer $k$ such that $(G \cup I_k, w)$ is the weighted competition graph of some 'acyclic' digraph $D$. We denote the weighted competition number of a weighted graph $(G, w)$ by $k_w(G, w)$. $\square$

First we see the relation between competition numbers and weighted competition numbers.

**Proposition 3.1.** *Let $G$ be a graph. Then, for any weight $w$ on $E(G)$, we have*

$$k(G) \leq k_w(G, w). \tag{3.1}$$

*Proof.* Take any weight $w$ on $E(G)$. Let $D$ be an acyclic digraph such that $C_w(D) = (G \cup I_k, w)$, where $k = k_w(G, w)$. Then the competition graph of this digraph $D$ is $G \cup I_k$. Thus we have $k(G) \leq k = k_w(G, w)$. $\square$

A weighted graph whose weighted competition number is at most one is characterized as follows.

**Theorem 3.2.** *Let $(G, w)$ be a weighted graph with $|V(G)| = n$. Then, $k_w(G, w) \leq 1$ if and only if there exist an ordering $v_1, ..., v_n$ of vertices of $G$ and a weighted edge clique cover $\{S_1, ..., S_n\}$ of $(G, w)$ such that $v_i \in S_j \Rightarrow i \leq j$.*

*Proof.* Suppose that $k_w(G, w) \leq 1$. If $k_w(G, w) = 0$, then the result follows from Theorem 2.5. So suppose that $k_w(G, w) = 1$. Let $G_1 = G \cup \{a\}$, where $a$ is an extra isolated vertex. Then, from Theorem 2.5, there exist an ordering $v_1, ..., v_n, v_{n+1}$ of vertices of $G_1$ and a w-ECC $\{S'_1, ..., S'_{n+1}\}$ of $(G_1, w)$ such that $v_i \in S'_j \Rightarrow i < j$. Here, $S'_1 = \emptyset$, and $v_{n+1}$ is an isolated vertex in $G_1$. Hence $G \cong G_1 - \{v_{n+1}\}$, and $\{S'_2, ..., S'_{n+1}\}$ is a w-ECC of $(G_1 - \{v_{n+1}\}, w)$. Put $S_j := S'_{j+1}$ $(j = 1, ..., n)$. Then the ordering $v_1, ..., v_n$ and the w-ECC $\{S_1, ..., S_n\}$ satisfy the condition $v_i \in S_j \Rightarrow i \leq j$.

Conversely, suppose that there exist an ordering $v_1, ..., v_n$ of vertices of $G$ and a w-ECC $\{S_1, ..., S_n\}$ of $(G, w)$ such that $v_i \in S_j \Rightarrow i \leq j$. Put $G_1 := G \cup \{v_{n+1}\}$, $S'_1 = \emptyset$, and $S'_j := S_{j-1}$ $(j = 2, ..., n+1)$, where $v_{n+1}$ is an extra isolated vertex. Then the ordering $v_1, ..., v_n, v_{n+1}$ of vertices of $G_1$ and the w-ECC $\{S'_1, ..., S'_{n+1}\}$ of $(G_1, w)$ satisfy the condition $v_i \in S_j \Rightarrow i < j$. Thus, from Theorem 2.5, $(G_1, w)$ is the weighted competition graph of an acyclic digraph. Hence we have $k_w(G, w) \leq 1$. $\square$

More generally, a weighted graph whose weighted competition number is at most $m$ is characterized as follows.

**Theorem 3.3.** *Let $(G, w)$ be a weighted graph with $|V(G)| = n$, and $m$ be a positive integer such that $m \leq n$. Then, $k_w(G, w) \leq m$ if and only if there exist an ordering $v_1, ..., v_n$ of vertices of $G$ and a weighted edge clique cover $\{S_1, ..., S_n\}$ of $(G, w)$ such that $v_i \in S_j \Rightarrow i \leq j + m - 1$.*

*Proof.* We prove the theorem by induction on the number $m$. When $m = 1$, the theorem follows from Theorem 3.2.

Assume that the theorem holds for $m - 1$, i.e., $k_w(G, w) \leq m - 1$ if and only if there exist an ordering $v_1, ..., v_n$ of vertices of $G$ and a w-ECC $\{S_1, ..., S_n\}$ of $(G, w)$ such that $v_i \in S_j \Rightarrow i \leq j + m - 2$.

Suppose that $k_w(G, w) \leq m$. If $k_w(G, w) \leq m - 1$, then the theorem follows from the induction hypothesis. So suppose that $k_w(G, w) = m$. Let $G_1 = G \cup \{a\}$, where $a$ is an extra isolated vertex. Then we have $k_w(G_1, w) = m - 1$. Then, from the induction hypothesis, there exist an ordering $v_1, ..., v_n, v_{n+1}$ of vertices of $G_1$ and a w-ECC $\{S'_1, ..., S'_{n+1}\}$ of $(G_1, w)$ such that $v_i \in S'_j \Rightarrow i \leq j + m - 2$. Here, $S'_1 = \emptyset$, and $v_{n+1}$ is an isolated vertex in $G_1$. Hence $G \cong G_1 - \{v_{n+1}\}$, and $\{S'_2, ..., S'_{n+1}\}$ is a w-ECC of $(G_1 - \{v_{n+1}\}, w)$. Put $S_j := S'_{j+1}$ $(j = 1, ..., n)$. Then the ordering $v_1, ..., v_n$ and the w-ECC $\{S_1, ..., S_n\}$ satisfy the condition $v_i \in S_j \Rightarrow i \leq j + m - 1$.

Conversely, suppose that there exist an ordering $v_1, ..., v_n$ of vertices of $G$ and a w-ECC $\{S_1, ..., S_n\}$ of $(G, w)$ such that $v_i \in S_j \Rightarrow i \leq j + m - 1$. Put $G_1 := G \cup \{v_{n+1}\}$, $S'_1 = \emptyset$, and $S'_j := S_{j-1}$ $(j = 2, ..., n + 1)$, where $v_{n+1}$ is an extra isolated vertex. Then the ordering $v_1, ..., v_n, v_{n+1}$ of vertices of $G_1$ and the w-ECC $\{S'_1, ..., S'_{n+1}\}$ of $(G_1, w)$ satisfy the condition $v_i \in S_j \Rightarrow i \leq j + m - 2$. Thus, from the induction hypothesis, we have $k_w(G_1, w) \leq m - 1$. Hence we have $k_w(G, w) \leq m$. □

## 3.2. Bounds for weighted competition numbers

**Notation.** (1) Let $w : E \to \mathbb{N}$ be a weight function. We denote the sum of weights of all the elements in $E$ by $w(E)$, i.e., $w(E) := \Sigma_{e \in E} w(e)$.
(2) Let $p$ be a positive integer. We denote a weight function $w : E \to \mathbb{N}$ such that $w(e) = p$ for all $e \in E$ by $p\mathbf{1}$. If $p = 1$, then we denote it by $\mathbf{1}$ instead of $1\mathbf{1}$. □

If a graph $G$ has no isolated vertices, then we have the following lower bound for the weighted competition number.

**Proposition 3.4.** *Let $(G, w)$ be a weighted graph. If $G$ has no isolated vertices, then*

$$\min\{w(e) \mid e \in E(G)\} \leq k_w(G, w). \tag{3.2}$$

*Proof.* Let $D$ be an acyclic digraph such that $C_w(D) = (G \cup I_k, w)$, where $k = k_w(G, w)$ and $I_k = \{a_1, ..., a_k\}$. Consider the digraph $D' := D - \{a_1, ..., a_k\}$. Since $D'$ is also acyclic, there exists a vertex $v$ such that $v$ has no outgoing arcs in $D'$. Since the graph $G$ has no isolated vertices, $v$ is an endpoint of an edge $e \in E(G)$. Let $u$ be another endpoint of the edge $e$. Since $C_w(D) = (G \cup \{a_1, ..., a_k\}, w)$, $v$ and $u$ have exactly $w(e)$ common preys in $D$. Here, since $v$ has no outgoing arcs in $D'$, all the preys of $v$ in $D$ are in $\{a_1, ..., a_k\}$. So we have $w(e) \leq k$. Thus we have

$$\min\{w(e) \mid e \in E(G)\} \leq w(e) \leq k = k_w(G, w).$$

Hence the proposition holds. □

**Corollary 3.5.** *Let $K_n$ be a complete graph with $n$ vertices and $p$ be a positive integer. Then,*

$$k_w(K_n, p\mathbf{1}) = p. \tag{3.3}$$

*Proof.* Since a complete graph $K_n$ has no isolated vertices, from Proposition 3.4, we have $k_w(K_n, p\mathbf{1}) \geq p$. Next we define a digraph $D$ as follows; $V(D) = V(K_n) \cup \{a_1, ..., a_p\}$, and $A(D) = \{(v, a_i) \mid v \in V(K_n), 1 \leq i \leq p\}$. Then the digraph $D$ is acyclic and we have $C_w(D) = (K_n \cup \{a_1, ..., a_p\}, p\mathbf{1})$. Thus $k_w(K_n, p\mathbf{1}) \leq p$ holds. Hence we have $k_w(K_n, p\mathbf{1}) = p$. □

If a graph $G$ has no isolated vertices, then we also have the following bounds.

**Theorem 3.6.** *Let $(G, w)$ be a weighted graph. If $G$ has no isolated vertices, then*

$$\theta_e^w(G, w) - |V(G)| + 2 \leq k_w(G, w) \leq \theta_e^w(G, w). \tag{3.4}$$

*Proof.* Let $\{S_1, ..., S_r\}$ be a w-ECC of $(G, w)$, where $r = \theta_e^w(G, w)$.

First, we will show $k_w(G, w) \leq \theta_e^w(G, w)$. We define a digraph $D$ as follows; $V(D) := V(G) \cup \{a_1, ..., a_r\}$, and $A(D) := \{(v, a_i) \mid v \in S_i, 1 \leq r \leq r\}$. Then the digraph $D$ is acyclic and we have $C_w(D) = (G \cup I_r, w)$. Hence $k_w(G, w) \leq r = \theta_e^w(G, w)$.

Second, we will show $\theta_e^w(G, w) - |V(G)| + 2 \leq k_w(G, w)$. Let $k = k_w(G, w)$ and $n = |V(G)|$. Let $D$ be an acyclic digraph such that $C_w(D) = (G \cup I_k, w)$. Consider an acyclic labeling $v_1, ..., v_{n+k}$ of the digraph $D$. Then $v_1$ has no incoming arcs and $v_2$ has at most one incoming arc. Put $S_j := \{v \in V(D) \mid (v, v_j) \in A(D)\}$ $(j = 3, 4, ..., n + k)$. Then, $\{S_3, ..., S_{n+k}\}$ is a w-ECC of $(G, w)$. Hence we have $\theta_e^w(G, w) \leq n + k - 2 = |V(G)| + k_w(G, w) - 2$, i.e., $\theta_e^w(G, w) - |V(G)| + 2 \leq k_w(G, w)$ holds. □

If a graph $G$ is triangle-free and connected, then we have the following explicit formula for the weighted competition number.

**Theorem 3.7.** *Let $(G, w)$ be a weighted graph. If $G$ is triangle-free and connected, then*

$$k_w(G, w) = w(E(G)) - |V(G)| + 2. \tag{3.5}$$

*Proof.* Let $V = V(G)$ and $E = E(G)$.

Since $G$ is triangle-free and connected, $\theta_e^w(G, w) = w(E)$ holds. Thus, from Theorem 3.6, we have $k_w(G, w) \geq w(E) - |V| + 2$.

Next we will show $k_w(G, w) \leq w(E) - |V| + 2$. Since $G$ is triangle-free and connected, from Theorem 1.2, we have $k(G) = |E| - |V| + 2$. So it is enough to show that $k_w(G, w) \leq w(E) - |E| + k(G)$. From the definition of competition numbers, there is an acyclic digraph $D$ such that $C(D) = G \cup I_k$, where $k = k(G)$. Since $G$ is triangle-free, we can take such an acyclic digraph $D$ which satisfies $C_w(D) = (G \cup I_k, \mathbf{1})$. Now we define a digraph $D'$ as follows;

$$V(D') = V(D) \cup \bigcup_{e \in E} \{a_{e,1}, ..., a_{e,w(e)-1}\} \tag{3.6}$$

$$A(D') = A(D) \cup \bigcup_{e \in E} \{(x, a_{e,i}), (y, a_{e,i}) \mid e = xy, 1 \leq i \leq w(e) - 1\} \tag{3.7}$$

Here we note that $l := | \cup_{e \in E} \{a_{e,1}, ..., a_{e,w(e)-1}\}| = \Sigma_{e \in E}(w(e) - 1) = w(E) - |E|$. Then the digraph $D'$ is acyclic, and we have $C_w(D') = (G \cup I_{k+l}, w)$. Thus we have $k_w(G, w) \leq k + l \leq w(E) - |E| + k(G)$. Hence the theorem holds. □

**Corollary 3.8.** *Let $(G, w)$ be a triangle-free connected weighted graph. Then, $k_w(G, w) = 1$ if and only if $G$ is a tree and $w = \mathbf{1}$.*

*Proof.* If $G$ is a tree and $w = \mathbf{1}$, then we have $k_w(G, w) = 1$.

Suppose that $k_w(G, w) = 1$. From Theorem 3.7, we have $k_w(G, w) = w(E(G)) - |V(G)| + 2$. $w(E(G)) - |V(G)| + 2 = 1$ implies $|V(G)| - 1 = w(E(G)) \geq |E(G)|$. Since $G$ is connected, we have $|E(G)| \geq |V(G)| - 1$. Thus we have $|V(G)| - 1 = w(E(G)) = |E(G)|$. Hence $G$ is a tree and $w = \mathbf{1}$. □

**Corollary 3.9.** *Let $P_n$ be a path with $n$ vertices, $C_n$ be a cycle with $n$ vertices, and $p$ be a positive integer. Then,*

$$k_w(P_n, p\mathbf{1}) = (p - 1)(n - 1) + 1. \tag{3.8}$$

$$k_w(C_n, p\mathbf{1}) = (p - 1)n + 2. \tag{3.9}$$

*Proof.* Since paths and cycles are triangle-free and connected, the corollary immediately follow from Theorem 3.7. □

At the end of this section, we show that a weighted competition number $k_w(G, w)$ can be very large even though the competition number $k(G)$ is small and $w = \mathbf{1}$.

**Proposition 3.10.** *For any positive integer $t$, there exists a weighted graph $(G, \mathbf{1})$ with a weight function $\mathbf{1}$ such that $k(G) + t \leq k_w(G, \mathbf{1})$*

*Proof.* Take any positive integer $t$. Let $G$ be a graph with $V(G) = \{a, b, v_1, ..., v_{t+2}\}$ and $E(G) = \{ab, v_1a, ..., v_{t+2}a, v_1b, ..., v_{t+2}b\}$ (see Figure 3). Then $G$ is a connected chordal graph. Thus, from Theorem 1.1, we have $k(G) = 1$. Here we can see that $\theta_e^w(G, \mathbf{1}) = 2t + 3$. Since $G$ has no isolated vertices, from Theorem 3.6, we have $k_w(G, \mathbf{1}) \geq (2t + 3) - (t + 4) + 2 = t + 1 = k(G) + t$. $\square$

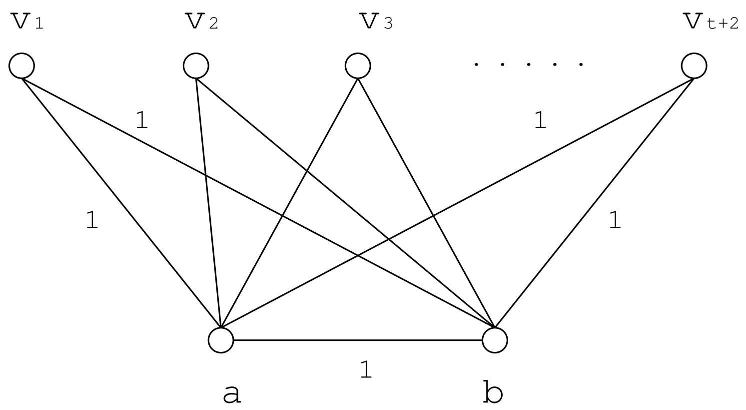


Figure 3:

# 4. Application to $p$-competition graphs

In this section, we will consider an application of weighted competition graphs and weighted edge clique covers to analysis of $p$-competition graphs. The notion of a w-ECC is very useful to determine whether a graph is a $p$-competition graph or not. See Remark 2.1 for the definitions of $p$-competition graphs and $p_{\leq}$-subgraphs.

First, we introduce a graph and a weighted graph which are related to ECC and w-ECC, respectively.

**Definition.** Let $V$ be a finite set and $S_i$ be a subset of $V$ $(i = 1, ..., r)$. The *ECC graph* $G_{\mathcal{F}}^{ecc}$ associated with the family $\mathcal{F} = \{S_1, ..., S_r\}$ is a graph $(V, E)$ which has $V$ as the vertex set and has an edge between distinct vertices $x, y \in V$ if there exists $S_i \in \mathcal{F}$ which contains both $x$ and $y$.

The *w-ECC graph* associated with a family $\mathcal{F} = \{S_1, ..., S_r\}$ is a weighted graph $(G_{\mathcal{F}}^{ecc}, w_{\mathcal{F}})$, where $G_{\mathcal{F}}^{ecc}$ is the ECC graph associated with $\mathcal{F}$, and the weight $w_{\mathcal{F}}(e)$ of an edge $e = xy \in E$ is the number of sets $S_i \in \mathcal{F}$ which contains both $x$ and $y$. $\square$

Note that the subsets $S_j \in \mathcal{F}$ become cliques of $G_{\mathcal{F}}^{ecc}$. And also note that the family $\mathcal{F}$ is a w-ECC of $(G_{\mathcal{F}}^{ecc}, w_{\mathcal{F}})$.

**Proposition 4.1.** *Let $G = (V, E)$ be a graph with $n$ vertices and $p$ be a positive integer. Then, $G$ is a $p$-competition graph if and only if there exists a family $\mathcal{F} = \{S_1, ..., S_r\}$ of subsets of $V$ such that $r \leq n$ and the $p_{\leq}$-subgraph $(G_{\mathcal{F}}^{ecc}, w_{\mathcal{F}})_{p\leq}$ of the w-ECC graph associated with $\mathcal{F}$ coincides with the graph $G$.*

*Proof.* Let $G = (V, E)$ be a graph with $V = \{v_1, ..., v_n\}$. Suppose that $G$ is a $p$-competition graph. Then there exists a digraph $D = (V, A)$ such that $C_p(D) = G$. Put $S_j = \{v_i \in V \mid (v_i, v_j) \in A\}$ $(j = 1, ..., n)$ and $\mathcal{F} = \{S_1, ..., S_n\}$. Then we can see that the $p_{\leq}$-subgraph of w-ECC graph $(G_{\mathcal{F}}^{ecc}, w_{\mathcal{F}})$ associated with the family $\mathcal{F}$ coincides with $G = (V, E)$.

Conversely, suppose that there exists a family $\mathcal{F} = \{S_1, ..., S_r\}$ of subsets of $V$ such that $r \leq n$ and the $p_{\leq}$-subgraph $(G_{\mathcal{F}}^{ecc}, w_{\mathcal{F}})_{p\leq}$ of the w-ECC graph associated with $\mathcal{F}$ coincides with the graph $G = (V, E)$. We define a digraph $D$ as follows; $V(D) = V$, and $A(D) = \{(v_i, v_j) \mid v_i \in S_j\}$. Then we can see that the $p$-competition graph $C_p(D)$ of this digraph $D$ coincides with the graph $G$. Thus the graph $G$ is a $p$-competition graph. $\square$

**Theorem 4.2.** *Let $C_n$ be a cycle with $n$ vertices and $p$ be a positive integer. If $n > 2p$, then $C_n$ is a $p$-competition graph.*

*Proof.* Let $v_0 ... v_{n-1} v_0$ be a cycle $C_n$, and $p$ be a positive integer such that $2p < n$. Put $S_i = \{v_i, v_{i+1}, ..., v_{i+p}\}$ $(i = 0, ..., n-1)$, where the indices are considered in modulo $n$. Let $(G_{\mathcal{F}}^{ecc}, w_{\mathcal{F}})$ be the w-ECC graph associated with the family $\mathcal{F} = \{S_0, ..., S_{n-1}\}$. Then we have $w_{\mathcal{F}}(v_i v_{i+1}) = p$ and $w_{\mathcal{F}}(v_i v_j) < p$ $(j \neq i \pm 1)$ since $n > 2p$. Thus we have $(G_{\mathcal{F}}^{ecc}, w_{\mathcal{F}})_{p\leq} = C_n$. Hence, from Proposition 4.1, a cycle $C_n$ is a $p$-competition graph. $\square$

**Theorem 4.3.** *Let $P_n$ be a path with $n$ vertices and $p$ be a positive integer. If $n > 2p$, then $P_n$ is a $p$-competition graph.*

*Proof.* Let $v_0 ... v_{n-1}$ be a path $P_n$, and $p$ be a positive integer such that $2p < n$. Put $S_i = \{v_i, v_{i+1}, ..., v_{i+p}\}$ $(i = 0, ..., n-2)$ and $S_{n-1} = \{v_0, v_1, ..., v_{p-1}\}$, where the indices are considered in modulo $n$. Let $(G_{\mathcal{F}}^{ecc}, w_{\mathcal{F}})$ be the w-ECC graph associated with the family $\mathcal{F} = \{S_0, ..., S_{n-1}\}$. Then we have $w_{\mathcal{F}}(v_i v_{i+1}) = p$ $(i = 0, ..., n-2)$, $w_{\mathcal{F}}(v_0 v_{n-1}) < p$, and $w_{\mathcal{F}}(v_i v_j) < p$ $(j \neq i \pm 1)$ since $n > 2p$. Thus we have $(G_{\mathcal{F}}^{ecc}, w_{\mathcal{F}})_{p\leq} = P_n$. Hence, from Proposition 4.1, a path $P_n$ is a $p$-competition graph. $\square$

A *star* $K_{1,n}$ with $n+1$ vertices is the graph which consists of a single vertex with $n$ neighbors.

**Theorem 4.4.** *Let $K_{1,n}$ be a star with $n+1$ vertices and $p$ be a positive integer. If $n > p$, then $K_{1,n}$ is a $p$-competition graph.*

*Proof.* Let $u, v_0, ..., v_{n-1}$ be the vertices of a star $K_{1,n}$, where $u$ is the center vertex of $K_{1,n}$, and $p$ be a positive integer such that $p < n$. Put $S_i = \{u, v_i, v_{i+1}, ..., v_{i+p-1}\}$ $(i = 0, ..., n-1)$, where the indices are considered in modulo $n$. Let $(G^{ecc}_{\mathcal{F}}, w_{\mathcal{F}})$ be the w-ECC graph associated with the family $\mathcal{F} = \{S_0, ..., S_{n-1}\}$. Then we have $w_{\mathcal{F}}(uv_i) = p$ $(i = 0, ..., n-1)$ and $w_{\mathcal{F}}(v_i v_j) < p$. Thus we have $(G^{ecc}_{\mathcal{F}}, w_{\mathcal{F}})_{p\leq} = K_{1,n}$. Hence, from Proposition 4.1, a star $K_{1,n}$ is a $p$-competition graph. □

# 5. Concluding remarks

In this paper, we have introduced the concepts of a weighted competition graph, a weighted edge clique cover, and a weighted competition number, and gave fundamental theorems, characterizations of weighted competition graphs, and several bounds for weighted competition numbers. But there still remain a number of problems to be considered.

Finally, we left a conjecture for weighted competition numbers. (This is an analogy to Opsut's Conjecture [35], also see [29, 47, 48].) For a vertex $v \in V(G)$, its *closed neighborhood* is

$$N_G[v] = \{u \in V(G) \mid uv \in E(G)\} \cup \{v\}. \tag{5.1}$$

We denote the subgraph of $G$ induced by $N_G[v]$ by the same symbol $N_G[v]$. And, for a vertex $v \in V(G)$, we denote the restriction of a weight $w$ on $E(G)$ to on the edges of $N_G[v]$ by simply $w|_v$.

**Conjecture 5.1.** *Let $(G, w)$ be a weighted graph. If $\theta^w_e(N_G[v], w|_v) \leq 2$ holds for all $v \in V(G)$, then $k_w(G, w) \leq 2$, and $k_w(G, w) = 2$ holds if and only if $\theta^w_e(N_G[v], w|_v) = 2$ holds for all $v \in V(G)$.* □

Note that, if a weighted graph $(G, w)$ satisfies the assumption of Conjecture 5.1, then the weight $w(e)$ is 1 or 2 for any edge $e \in E(G)$. To challenge this conjecture, it would be a good way to start with considering about the cases that the graph $G$ is a chordal graph, a line graph, or a proper circular arc graph. Note that the example $(G\mathbf{1})$ in the proof of Proposition 3.10 has a chordal graph $G$, and each edge of $G$ has weight 1. But, since $\theta^w_e(N_G[a], \mathbf{1}|_a) = \theta^w_e(G, \mathbf{1}) = 2t + 3$, this weighted graph $(G, \mathbf{1})$ does not satisfy the assumption of Conjecture 5.1.


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