HIGHER LEVEL YOUNG WALLS FOR CLASSICAL QUANTUM AFFINE ALGEBRAS - A UNIFIED APPROACH

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ABSTRACT. We provide a unified approach to the construction of crystal bases for classical quantum affine algebras. The higher level irreducible highest weigh crystals are realized as the affine crystals consisting of higher level reduced Young walls.

1. INTRODUCTION

It is one of the major problems in crystal basis theory to construct explicit realization of crystal bases for irreducible highest weight modules over quantum groups. In [4], Kang introduced the combinatorics of *Young walls*, and gave a new realization of level-1 irreducible highest weight crystals for classical quantum affine algebras. More precisely, for the quantum affine algebras of type $A_n^{(1)}$, $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$, the level-1 highest weight crystals were realized as the affine crystals consisting of level-1 *reduced Young walls*.

However, in this work, the case of $C_n^{(1)}$ was missing, because the level-1 perfect crystals for this case are intrinsically of level-2. This difficulty was resolved in [2] by introducing the notion of *splitting blocks* and *slices*, and Hong, Kang and Lee obtained a realization of level-1 irreducible highest weight crystals for the quantum affine algebras of type $C_n^{(1)}$ in terms of reduced Young walls.

Motivated by this work, in [9], Kang and Lee went further to develop the combinatorics of higher level Young walls, which led to a realization of higher level irreducible highest weight crystals for *most* of classical quantum affine algebras. However, this time, the case of $D_n^{(1)}$ was missing, because we took the view point that level-l Young walls should be made up of l layers of level-1 Young walls and that only *whole blocks* should be used in building Young walls.

In [11], Lee filled up this gap by proposing a new idea that level-l Young walls may be constructed by concatenating (the equivalence classes of) level-l slices which are *split forms* of pre-slices. In particular, in her construction, even the broken halves of whole blocks may be used in building Young walls.

In this work, we extend her idea to *all* classical quantum affine algebras. As in the previous works, the notion of splitting blocks and slices will play an important role in our construction, but they are different from the ones given in [2] and [9]. In fact, even the patterns for building Young walls are slightly different. The main difference lies in the fact that we define the slices to be the split forms of pre-slices, which enables us to provide a unified approach to the construction of higher level irreducible highest weight crystals for all classical quantum affine algebras.

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We first show that level-l *perfect crystals* are realized as the equivalence classes of level-l slices as before, but level-l Young walls are viewed as the concatenation of level-l slices rather than l layers of level-1 Young walls. We then proceed to define the notion of proper Young walls, reduced Young walls, ground-state walls, etc. As the main theorem, we prove that higher level irreducible highest weigh crystals are realized as affine crystals consisting of higher level reduced Young walls.

The higher level irreducible highest weight crystals for the quantum affine algebras $U_q(A_n^{(1)})$ have been constructed in [3, 12], and their construction can be regarded as a special case of our Young wall construction (cf. [9]) which dose not use the notion of splitting. Thus, in this paper, we will focus on the rest of classical quantum affine algebras.

2. Perfect crystals

In this section, we briefly review some of the basic properties of perfect crystals. We will follow the general notations given in [1].

Let $(A, P^\vee, P, \Pi^\vee, \Pi)$ be an affine Cartan datum of type $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$, $A_{2n-1}^{(2)}$, $D_n^{(1)}$, $B_n^{(1)}$, or $C_n^{(1)}$. We denote by $I = \{0, 1, \ldots, n\}$ the index set for the simple roots, $A = (a_{ij})_{i,j \in I}$ the affine generalized Cartan matrix, $P^\vee = \left(\bigoplus_{i\in I} \mathbf{Z} h_i\right) \oplus \mathbf{Z} d$ the dual weight lattice, $\mathfrak{h} = \mathbf{C} \otimes_{\mathbf{Z}} P^\vee$ the Cartan subalgebra, $P = \left(\bigoplus_{i \in I} \mathbf{Z}\Lambda_i\right) \oplus \mathbf{Z}\delta$ (and $\bar{P} = \bigoplus_{i\in I} \mathbf{Z}\Lambda_i$) the weight lattice (resp. classical weight lattice), $\Pi^\vee = \{h_i | i \in I\}$ the set of simple coroots and $\Pi = \{\alpha_i | i \in I\}$ the set of simple roots. We also denote by δ the null root, $\Lambda_i (i \in I)$ the fundamental weights and P^+ the set of affine dominant integral weights.

Let $U_q(\mathfrak{g})$ be the quantum affine algebra associated with $(A, P^\vee, P, \Pi^\vee, \Pi)$ and let $e_i, f_i, K_i^{\pm 1} (i \in I), q^d$ be the generators of $U_q(\mathfrak{g})$. The subalgebra $U_q'(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1} (i \in I)$ is also called the quantum affine algebra.

We will not repeat the basic facts such as tensor product rule on general crystal basis theory. We will just recall that for a dominant integral weight $\lambda \in P^+$, the crystal graph $\mathcal{B}(\lambda)$ of the irreducible highest weight module $V(\lambda)$ over $U_q(\mathfrak{g})$ will be referred to as the ***irreducible highest weight crystal***. As is indicated in the introduction, the explicit construction of irreducible highest weight crystal $\mathcal{B}(\lambda)$ is one of the central problems in crystal basis theory.

Let V be a finite dimensional $U_q'(\mathfrak{g})$-module with crystal graph $\mathcal{B}$. For $b \in \mathcal{B}$, we write

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i, \quad \varphi(b) = \sum_{i\in I} \varphi_i(b)\Lambda_i. \tag{2.1}$$

The crystal $\mathcal{B}$ is called a ***perfect crystal*** of ***level-l*** if

(1) there is a finite dimensional $U_q'(\mathfrak{g})$-module with a crystal basis whose crystal graph is isomorphic to $\mathcal{B}$,
(2) $\mathcal{B} \otimes \mathcal{B}$ is connected,
(3) there exists some $\lambda_0 \in \bar{P}$ such that

$$\mathrm{wt}(\mathcal{B}) \subset \lambda_0 + \frac{1}{d_0} \sum_{i \neq 0} \mathbf{Z}_{\leq 0} \alpha_i, \qquad \#(\mathcal{B}_{\lambda_0}) = 1,$$

(4) for any $b \in \mathcal{B}$, we have $\langle c, \varepsilon(b) \rangle \geq l$,

(5) for each $\lambda \in \bar{P}^+$ with $\langle c, \lambda \rangle = l$, there exist unique vectors $b^\lambda \in \mathcal{B}$ and $b_\lambda \in \mathcal{B}$ such that

$$\varepsilon(b^\lambda) = \lambda, \qquad \varphi(b_\lambda) = \lambda.$$

Here, d_0 is the coefficient of α_0 in the null root δ.

Let $\mathcal{B}$ be a perfect crystal of level-$l(>0)$. For any dominant integral weight $\lambda \in \bar{P}^+$ of level-l, it was proved in [6] that there exists a crystal isomorphism

$$\Psi : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathcal{B}(\varepsilon(b_\lambda)) \otimes \mathcal{B} \qquad \text{given by} \qquad u_\lambda \longmapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda,$$

where u_λ (resp. $u_{\varepsilon(b_\lambda)}$) denotes the highest weight vector of $\mathcal{B}(\lambda)$ (resp. $\mathcal{B}(\varepsilon(b_\lambda))$) and b_λ is the unique element of $\mathcal{B}$ with $\varphi(b_\lambda) = \lambda$.

For $k \geq 0$, set

$$\lambda_0 = \lambda,\ \lambda_{k+1} = \varepsilon(b_{\lambda_k}), \qquad \text{and} \qquad b_0 = b_\lambda,\ b_{k+1} = b_{\lambda_{k+1}}.$$

The sequence

$$\mathbf{p}_\lambda = (b_k)_{k=0}^\infty = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0$$

is called the ***ground-state path*** of weight λ. A λ-***path*** in $\mathcal{B}$ is a sequence

$$\mathbf{p} = (\mathbf{p}(k))_{k=0}^\infty = \cdots \otimes \mathbf{p}(k+1) \otimes \mathbf{p}(k) \otimes \cdots \mathbf{p}(1) \otimes \mathbf{p}(0)$$

such that $\mathbf{p}(k) = b_k$ for all $k \gg 0$. The set $\mathcal{P}(\lambda) = \mathcal{P}(\lambda, \mathcal{B})$ of all λ-paths in $\mathcal{B}$ is given a crystal structure by the tensor product rule, which gives the path realization of the irreducible highest weight crystal $\mathcal{B}(\lambda)$ ([6]) :

$$\mathcal{B}(\lambda) \xrightarrow{\sim} \mathcal{P}(\lambda), \qquad u_\lambda \longmapsto \mathbf{p}_\lambda.$$

Hence the realization problem of the irreducible highest weight crystal $\mathcal{B}(\lambda)$ is reduced to the one of finding perfect crystals.

In [5] and [6], a coherent family of perfect crystals $\mathcal{B}^{(l)}$ was constructed for each of the classical quantum affine algebras. For our use in Section 4, we will give an explicit description of those perfect crystals :

(1) $A_{2n}^{(2)}$ $(n \geq 1)$

$$\mathcal{B}^{(l)} = \Big\{ (x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1) \Big| x_i, \bar{x}_i \in \mathbf{Z}_{\geq 0},\ \textstyle\sum_{i=1}^n (x_i + \bar{x}_i) \leq l \Big\}.$$

(2) $D_{n+1}^{(2)}$ $(n \geq 2)$

$$\mathcal{B}^{(l)} = \left\{ (x_1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_1) \left| \begin{array}{l} x_0 = 0 \text{ or } 1,\ x_i, \bar{x}_i \in \mathbf{Z}_{\geq 0}, \\ x_0 + \sum_{i=1}^n (x_i + \bar{x}_i) \leq l \end{array} \right. \right\}.$$

(3) $A_{2n-1}^{(2)}$ $(n \geq 3)$

$$\mathcal{B}^{(l)} = \Big\{ (x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1) \Big| x_i, \bar{x}_i \in \mathbf{Z}_{\geq 0},\ \textstyle\sum_{i=1}^n (x_i + \bar{x}_i) = l \Big\}.$$

(4) $D_n^{(1)}$ $(n \geq 3)$

$$\mathcal{B}^{(l)} = \left\{ (x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1) \left| \begin{array}{c} x_n = 0 \text{ or } \bar{x}_n = 0,\ x_i, \bar{x}_i \in \mathbf{Z}_{\geq 0} \\ \sum_{i=1}^n (x_i + \bar{x}_i) = l \end{array} \right. \right\}.$$

(5) $C_n^{(1)}$ $(n \geq 2)$

$$\mathcal{B}^{(l)} = \Big\{ (x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1) \Big| x_i, \bar{x}_i \in \mathbf{Z}_{\geq 0},\ 2l \geq \textstyle\sum_{i=1}^n (x_i + \bar{x}_i) \in 2\mathbf{Z} \Big\}.$$

(6) $B_n^{(1)}$ $(n \geq 3)$

$$\mathcal{B}^{(l)} = \left\{ (x_1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_1) \;\middle|\; \begin{array}{l} x_0 = 0 \text{ or } 1,\ x_i, \bar{x}_i \in \mathbf{Z}_{\geq 0}, \\ x_0 + \sum_{i=1}^n (x_i + \bar{x}_i) = l \end{array} \right\}.$$

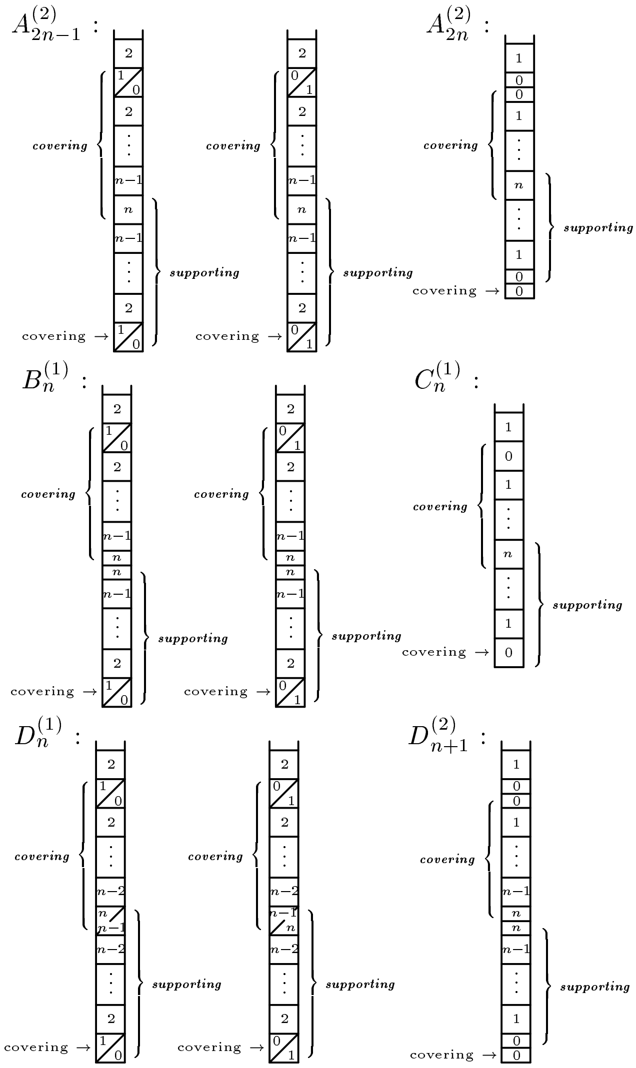
3. Slices and perfect crystals

In this section, we will introduce the notions of splitting blocks and slices, which are different from the ones given in [9]. We will define a classical crystal structure on the set $\mathcal{C}^{(l)}$ of equivalence classes of level-l slices and show that it is isomorphic to the level-l perfect crystal $\mathcal{B}^{(l)}$.

Definition 3.1.

(1) If $\mathfrak{g} \neq C_n^{(1)}$, a ***level-*1 *slice*** is defined to be a set of finitely many blocks stacked in one column of unit depth following the patterns given below.

(2) If $\mathfrak{g} = C_n^{(1)}$, such a set of blocks will be called a ***level-$\frac{1}{2}$ slice***.



In stacking the blocks, no block can be placed on top of a block of half-unit depth. Note that the patterns for stacking blocks are slightly different from the ones given in [9].

As we can see in the figure, the blocks are stacked in a repeating pattern, which is symmetric with respect to the n-block. We say that an i-block is a ***supporting*** (resp. ***covering***) ***i-block*** if it lies in the bottom half (resp. upper half) of one cycle. An i-block that appears only once in one cycle is regarded as both a supporting and a covering block.

An ***i-slot*** is the top of a level-1 (or level-$\frac{1}{2}$) slice where we may add an i-block. The notions of ***supporting i-slots*** and ***covering i-slots*** are defined in a similar manner.

A ***δ-column*** is a set of blocks (and its cyclic variations) that form one cycle of the stacking patterns. For a level-1 (or level-$\frac{1}{2}$) slice c, we define $c+\delta$ (resp. $c-\delta$) to be a level-1 (resp. level-$\frac{1}{2}$) slice obtained from c by adding (resp. removing) a δ-column. For level-1 (or level-$\frac{1}{2}$) slices c and c', we write $c \subset c'$ if c is part of c'. For example, we have $c-\delta \subset c \subset c+\delta$.

Definition 3.2.

(1) If $\mathfrak{g} \neq C_n^{(1)}$, ***a level-$l$ pre-slice*** is defined to be an ordered l-tuple $C = (c_1, \dots, c_l)$ of level-1 slices such that $c_1 \subset c_2 \subset \cdots c_l \subset c_1 + \delta$.

(2) If $\mathfrak{g} = C_n^{(1)}$, ***a level-l pre-slice*** is defined to be an ordered $2l$-tuple $C = (c_1, \dots, c_{2l})$ of level-$\frac{1}{2}$ slices such that $c_1 \subset c_2 \subset \cdots c_{2l} \subset c_1 + \delta$.

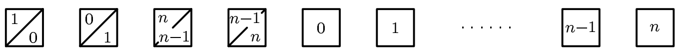
(3) The level-1 (or level-$\frac{1}{2}$) slice c_i in c is called the ***i-th layer*** of C.

Remark 3.3.

(1) For those quantum affine algebras that allow two stacking patterns, we choose only one pattern in building a level-l pre-slice. Still, two different level-l pre-slices can be made from two different stacking patterns.

(2) A level-l pre-slice can be visualized as the set of l columns (or $2l$ columns) with the ith layer placed in front of $(i+1)$-th layer.

Next, we explain the notion of splitting (whole) blocks. By a whole block, we mean a unit cube or a gluing of two half-unit depth blocks as is shown in the following picture :



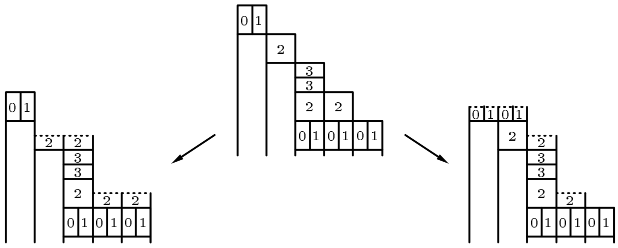
The first two (and the next two) whole blocks all be referred to as $0|1$-blocks (resp. $(n\text{-}1)|n$-blocks). Thus, when we deal with whole blocks, we may choose their colors among $I \cup \{0|1, (n\text{-}1)|n\}$. Similarly, we may consider whole i-slots for $I \cup \{0|1, (n\text{-}1)|n\}$.

Note that in a given pre-slice, there can be at most two heights in which a covering (or supporting) i-block may appear as the top of one layer. Similarly, there may be at most two heights in which a covering (or a supporting) i-slot may appear.

Definition 3.4. Fix $i \in I \cup \{0|1, (n\text{-}1)|n\}$. Suppose that there is a layer whose top is a covering (or supporting) whole i-block and another layer whose top is a supporting (resp. covering) whole i-slot. Choose the covering (resp. supporting) i-block lying in the fore-most layer among the ones with the higher height and the supporting (resp. covering) i-slot lying in the rear-most layer among the ones with the lower hight.

To ***split a whole i-block*** means to break off the top half of the chosen covering (resp. supporting) i-block and to place it in the chosen supporting (resp. covering) i-slot. A ***split form*** of a pre-slice is a result obtained by splitting all the whole blocks that can be split. Note that a pre-slice may have several different split forms.

Example 3.5. This example shows two different split forms of a single pre-slice for $B_3^{(1)}$-type. Neither of the split results allows further splitting.



The lower left figure shows the splitting of a covering 2-block and a supporting 2-block. The lower right figure shows the splitting of a supporting 2-block and a 0|1-block from the same pre-slice.

Definition 3.6. Fix a pattern for building pre-slices. A ***level-l slice*** is a split form of a level-l pre-slice. We denote by $\mathcal{S}^{(l)}$ the set of all level-l slices built on a fixed pattern.

Remark 3.7.

(1) The notions of *i-slot*, *δ-column*, and *layer*, defined for pre-slices, naturally carries over to those of slices. Some care must be exercised, however. For example, δ-columns should allow for halves of blocks to add up to a δ, and we should now consider halves of i-slots.

(2) We would like to remind the readers that in a level-l (pre-)slice, the top of a layer is regarded as an i-slot if it is an i-slot when the layer is viewed a level-1 slice.

We now define the action of Kashiwara operators on the set $\mathcal{S}^{(l)}$ of level-l slices. Let C be a level-l slice and fix an index $i \in I$.

Case 1. Suppose that $i \neq 0, n$ and that the i-block is a unit cube. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by (E1)-(E4) and (F1)-(F4), respectively.

(E1) If C contains both a covering whole i-block and a supporting whole i-block, then remove the upper half i-block from the fore-most covering whole i-block among the ones with the higher hight and another upper half i-block from the fore-most supporting whole i-block among the ones with the higher hight.

(E2) If C contains some whole i-blocks and all of them are of the same type, then remove the upper half i-block from the fore-front whole i-block among the ones with the higher height. This would create a lower half i-block. Consider the (lower) half i-blocks having the same type as this new one. We then remove the fore-most half i-block among the ones with the higher height.

(E3) If C contains no whole i-blocks, but does contain some half i-blocks, then the number of covering half i-blocks and that of supporting half i-blocks must be the same. We remove the fore-most covering half i-block and the fore-most supporting half i-block among the ones with the higher hight.

(E4) If C contains no i-blocks, we define $\tilde{e}_i C = 0$

(F1) If C contains both a covering whole i-slot and a supporting whole i-slot, then we place a half i-block in the rear-most covering whole i-slot among the ones with the lower hight and another half i-block in the rear-most supporting whole i-slot among the ones with the lower height.

(F2) If C contains some whole i-slots and all of them are at the same type, then we place a half i-block in the rear-most whole i-slot among the ones with the lower height. This would create an (upper) half i-slot. Consider the (upper) half i-slots having the same type as this new one. We then place another half i-block in the rear-most (upper) half i-slots among the ones with the lower height.

(F3) If C contains no whole i-slots, but dose contain some half i-slots, then the number of covering half i-slots and that of supporting half i-slots must be the same. We place a half i-block in the rear-most covering half i-slots and another half i-block in the rear-most supporting half i-slots among the ones with the lower height.

(F4) If C contains no i-slots, we define $\tilde{f}_i C = 0$

Case 2. Suppose that $i = 0, n$ and that the i-block is a unit cube. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by (E1)-(E3) and (F1)-(F3), respectively.

(E1) If C contains at least two whole i-blocks, first remove a upper half i-block from the fore-most whole i-block among the ones with the higher hight. Then, do the same once more on the resulting set of blocks. If C contains one whole i-block, then remove the upper half i-block from the whole i-block and remove the fore-most (lower) half i-block among the ones with the higher hight.

(E2) If C contains no whole i-blocks, but does contain some half i-blocks, then the number of half i-blocks must be even. We first remove the fore-most half i-block among the ones with the higher hight. Then, do the same once more on the resulting set of blocks.

(E3) If C contains no i-blocks, we define $\tilde{e}_i C = 0$

(F1) If C contains at least two whole i-slots, we first place a half i-block in the rear-most whole i-slot among the ones with the lower hight. Then, do the same once more on the resulting set of blocks. If C contains one whole i-slot, then we place a half i-block in the whole i-slot and another half i-block in the rear-most (upper) half i-slot among the ones with the lower height.

(F2) If C contains no whole i-slots, but does contain some half i-slots, then the number of half i-slots must be even. We first place a half i-block in the rear-most half i-slot among the ones with the lower height. Then, do the same once more on the resulting set of blocks.

(F3) If C contains no i-slots, we define $\tilde{f}_i C = 0$

Case 3. Suppose that $i = 0$ (resp. $i = n$) and that the i-block is of half-height. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by (E1)-(E3) and (F1)-(F3), respectively.

(E1) If C contains some supporting i-blocks (resp. covering i-blocks), then we remove the fore-most supporting (resp. covering) i-block among the ones with the higher height.

(E2) If C contains some i-blocks and all of them are covering blocks (resp. supporting blocks), then remove the fore-most covering (resp. supporting) i-block among the ones with the higher height.
(E3) If C contains no i-blocks, then $\tilde{e}_i C = 0$.
(F1) If C contains some covering i-slots (resp. supporting i-slots), then we place an i-block in the rear-most covering (resp. supporting) i-slot among the ones with the lower height.
(F2) If C contains some i-slots and all of them are supporting slots (resp. covering slots), then we place an i-block in the rear-most supporting (resp. covering) i-slot among the ones with the lower height.
(F3) If C contains no i-slot, the we define $\tilde{f}_i C = 0$.

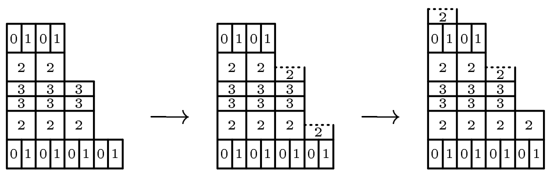
Case 4. Suppose that $i = 0, 1$ (resp. $n-1$, n) and that the i-block is of half-depth. ***Un-split*** all half $0|1$-blocks (resp. $(n\text{-}1)|n$-blocks) in C to obtain C'.

(E1) If C' contains an i-block, remove the fore-most i-block among the ones with the higher height to obtain C''. We split all $0|1$-blocks (resp. $(n\text{-}1)|n$-blocks) in C'' to obtain $\tilde{e}_i C$.
(E2) If C' contains no i-block, we define $\tilde{e}_i C = 0$.
(F1) If C' contains an i-slot, place an i-block in the rear-most i-slot among the ones with the lower height to obtain C''. We split all $0|1$-blocks (resp. $(n\text{-}1)|n$-blocks) in C'' to obtain $\tilde{f}_i C$.
(F2) If C' contains no i-slot, we define $\tilde{f}_i C = 0$.

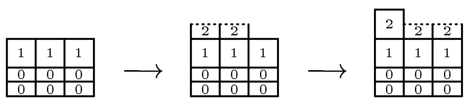
Remark 3.8. In a slice, unlike the case for other whole blocks, the result of un-splitting all $0|1$ and $(n\text{-}1)|n$-blocks, mentioned in the above actions of Kashiwara operators for $i = 0, 1, n-1, n$, is unique.

Example 3.9. We illustrate $\tilde{f}_i$ actions on slices.

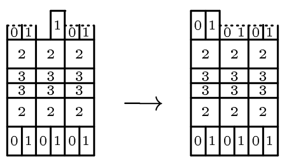
(1) $B_3^{(1)}$-type $\tilde{f}_2$ action; (F1) and (F2) of **Case 1**



(2) $A_4^{(2)}$-type $\tilde{f}_2$ action; (F1) of **Case 2**



(3) $B_3^{(1)}$-type $\tilde{f}_0$ action; (F1) of **Case 4**



4. Equivalence classes of slices

Let $C = (c_1, \ldots, c_l)$ be a level-l slice. We define the slices $C \pm \delta$ to be :

$$\begin{aligned} C + \delta &= (c_2, \ldots, c_l, c_1 + \delta), \\ C - \delta &= (c_l - \delta, c_1, \cdots, c_{l-1}). \end{aligned} \tag{4.1}$$

We say that two slices C and C' are ***related***, denoted by $C \sim C'$, if one of the two may be obtained from the other by adding finitely many δ's. Let

$$\mathcal{C}^{(l)} = \mathcal{S}^{(l)}/\sim \tag{4.2}$$

be the set of equivalence classes of level-l slices under this relation. For the equivalence class containing a level-l slice C, we will use the same symbol C. Since the map $C(\in \mathcal{S}^{(l)}) \mapsto C + \delta$ commutes with the action of Kashiwara operators, we may define the induced Kashiwara operators on $\mathcal{C}^{(l)}$. We also define

$$\begin{aligned} \varphi_i(C) &= \max\{k \mid \tilde{f}_i^k C \in \mathcal{C}^{(l)}\}, \\ \varepsilon_i(C) &= \max\{k \mid \tilde{e}_i^k C \in \mathcal{C}^{(l)}\}, \\ \overline{\mathrm{wt}}(C) &= \sum_i \big(\varphi_i(C) - \varepsilon_i(C)\big)\Lambda_i. \end{aligned} \tag{4.3}$$

Then one can verify in a straightforward manner that the set $\mathcal{C}^{(l)}$, together with the induced Kashiwara operators and the maps φ_i, ε_i $(i \in I)$, $\overline{\mathrm{wt}}$ becomes a $U_q'(\mathfrak{g})$-crystal.

Recall that a typical level-l slice is a split form of a level-l pre-slice. Hence the top of each layer of a level-l slice may be a usual block or a broken half of a block. We will classify the layers of level-l slices into several types depending on the shape of their top parts, and give an explicit description of the set $\mathcal{C}^{(l)}$ in terms of the numbers of the layers of each type.

We then construct a *canonical bijection* $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ and its inverse $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$, which will turn out to be a crystal isomorphism.

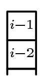
In the following, we will use the symbols s_i, $\bar{s}_i$, t_i, $\bar{t}_i$, for the types of layers, and will write their top parts to their right.

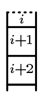
- $A_{2n}^{(2)}$ case

  We first list the types of layers and their top parts :

$$\begin{aligned} s_0 &: \text{supporting 0-slot (=covering 0-block)}, \\ s_i &: \text{supporting } i\text{-slot } (i = 1, \ldots, n), \\ \bar{s}_i &: \text{covering } i\text{-block } (i = 1, \ldots, n), \\ t_i &: \text{half of a supporting } i\text{-block } (i = 1, \ldots, n), \\ \bar{t}_i &: \text{half of a covering } i\text{-block } (i = 1, \ldots, n). \end{aligned}$$

For example, the layers of type s_i and $\bar{t}_i$ have the following form.

s_i = 

$\bar{t}_i$ = 

Let C be a level-l slice in $\mathcal{C}^{(l)}$. Note that the number of layers of type t_i must be the same as that of layers of type $\bar{t}_i$. If we write

$$\begin{aligned} u_0 &= \text{the number of layers of type } s_0 \text{ in } C, \\ y_i &= \text{the number of layers of type } s_i \text{ in } C \ (i = 1, \dots, n), \\ \bar{y}_i &= \text{the number of layers of type } \bar{s}_i \text{ in } C \ (i = 1, \dots, n), \\ z_i &= \text{the number of layers of type } t_i \text{ in } C, \\ &= \text{the number of layers of type } \bar{t}_i \text{ in } C \ (i = 1, \dots, n), \end{aligned}$$

then z_n must be even, $y_i\bar{y}_i = 0$ for all $i = 1, \dots, n$, and $u_0 + \sum_{i=1}^n (y_i + \bar{y}_i) + 2\sum_{i=1}^{n-1} z_i + z_n = l$. Hence the set $\mathcal{C}^{(l)}$ can be characterized as

$$\mathcal{C}^{(l)} = \left\{ (u_0|y_1, \dots, y_n|\bar{y}_1, \dots, \bar{y}_n|z_1, \dots, z_n) \;\middle|\; \begin{array}{l} u_0, y_i, \bar{y}_i, z_i \in \mathbf{Z}_{\geq 0},\ z_n \text{ is even},\ y_i\bar{y}_i = 0, \\ u_0 + \sum_{i=1}^n (y_i + \bar{y}_i) + 2\sum_{i=1}^{n-1} z_i + z_n = l \end{array} \right\}.$$

We define the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by

$$(x_1, \dots, x_n|\bar{x}_n, \dots, \bar{x}_1) \mapsto (u_0|y_1, \dots, y_n|\bar{y}_1, \dots, \bar{y}_n|z_1, \dots, z_n), \text{ where}$$

$$\begin{aligned} u_0 &= l - \textstyle\sum_{i=1}^n (x_i + \bar{x}_i), \\ y_i &= \max\{0, x_i - \bar{x}_i\} \ (i = 1, \dots, n), \\ \bar{y}_i &= \max\{0, \bar{x}_i - x_i\} \ (i = 1, \dots, n), \\ z_i &= \min\{x_i, \bar{x}_i\} \ (i = 1, \dots, n-1), \\ z_n &= 2\min\{x_n, \bar{x}_n\}, \end{aligned}$$

and the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by

$$(u_0|y_1, \dots, y_n|\bar{y}_1, \dots, \bar{y}_n|z_1, \dots, z_n) \mapsto (x_1, \dots, x_n|\bar{x}_n, \dots, \bar{x}_1), \text{ where}$$

$$\begin{aligned} x_i &= y_i + z_i,\ \bar{x}_i = \bar{y}_i + z_i \ (i = 1, \dots, n-1), \\ x_n &= y_n + \frac{1}{2}z_n,\ \bar{x}_n = \bar{y}_n + \frac{1}{2}z_n. \end{aligned}$$

- $D_{n+1}^{(2)}$ case
The types of layers and their top parts are given below :

$$\begin{aligned} s_0 &: \text{covering 0-block (=supporting 0-slot)}, \\ s_i &: \text{supproting } i\text{-slot } (i = 1, \dots, n), \\ \bar{s}_i &: \text{covering } i\text{-block } (i = 1, \dots, n), \\ t_i &: \text{half of a supporting } i\text{-block } (i = 1, \dots, n-1), \\ \bar{t}_i &: \text{half of a covering } i\text{-block } (i = 1, \dots, n-1), \\ t_n &: \text{supproting } n\text{-block (=covering } n\text{-slot)}. \end{aligned}$$

For a level-l slice C in $\mathcal{C}^{(l)}$, we write

$$\begin{aligned}
u_0 &= \text{the number of layers of type } s_0,\\
y_i &= \text{the number of layers of type } s_i \ (i=1,\dots,n),\\
\bar{y}_i &= \text{the number of layers of type } \bar{s}_i \ (i=1,\dots,n),\\
z_i &= \text{the number of layers of type } t_i,\\
&= \text{the number of layers of type } \bar{t}_i \ (i=1,\dots,n-1),\\
z_n &= \text{the number of layers of type } t_n.
\end{aligned}$$

Then we have $y_i\bar{y}_i = 0$ for all $i = 1,\dots,n$ and $u_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=1}^{n-1} z_i + z_n = l$. Hence the set $\mathcal{C}^{(l)}$ can be characterized as

$$\mathcal{C}^{(l)} = \left\{ (u_0|y_1,\dots,y_n|\bar{y}_1,\dots,\bar{y}_n|z_1,\dots,z_n) \;\middle|\; \begin{array}{l} u_0, y_i, \bar{y}_i, z_i \in \mathbf{Z}_{\geq 0},\ y_i\bar{y}_i = 0, \\ u_0 + \sum_{i=1}^{n}(y_i+\bar{y}_i) + 2\sum_{i=1}^{n-1} z_i + z_n = l \end{array} \right\}.$$

We define the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by

$$(x_1,\dots,x_n|x_0|\bar{x}_n,\dots,\bar{x}_1) \mapsto (u_0|y_1,\dots,y_n|\bar{y}_1,\dots,\bar{y}_n|z_1,\dots,z_n), \text{ where}$$

$$\begin{aligned}
u_0 &= l - x_0 - \textstyle\sum_{i=1}^{n}(x_i + \bar{x}_i),\\
y_i &= \max\{0, x_i - \bar{x}_i\} \ (i=1,\dots,n),\\
\bar{y}_i &= \max\{0, \bar{x}_i - x_i\} \ (i=1,\dots,n),\\
z_i &= \min\{x_i, \bar{x}_i\} \ (i=1,\dots,n-1),\\
z_n &= 2\min\{x_n, \bar{x}_n\} + x_0,
\end{aligned}$$

and the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by

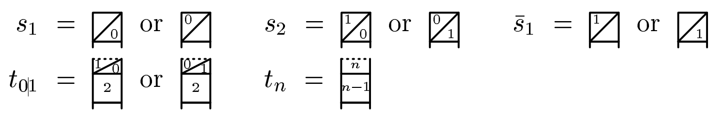
$$(u_0|y_1,\dots,y_n|\bar{y}_1,\dots,\bar{y}_n|z_1,\dots,z_n) \mapsto (x_1,\dots,x_n|x_0|\bar{x}_n,\dots,\bar{x}_1), \text{ where}$$

$$\begin{aligned}
x_0 &= \begin{cases} 0 & \text{if } z_n \text{ is even},\\ 1 & \text{if } z_n \text{ is odd},\end{cases}\\
x_i &= y_i + z_i,\ \bar{x}_i = \bar{y}_i + z_i \ (i=1,\dots,n-1),\\
x_n &= \begin{cases} y_n + z_n & \text{if } z_n \text{ is even},\\ y_n + z_n - 1 & \text{if } z_n \text{ is odd},\end{cases}\\
\bar{x}_n &= \begin{cases} \bar{y}_n + z_n & \text{if } z_n \text{ is even},\\ \bar{y}_n + z_n - 1 & \text{if } z_n \text{ is odd}.\end{cases}
\end{aligned}$$

- $A^{(2)}_{2n-1}$ case
  The types of layers and their top parts are given below :

$$
\begin{aligned}
s_1 &: \text{1-slot with a 0-block},\\
\bar{s}_1 &: \text{0-slot with a 1-block},\\
s_i &: \text{supproting } i\text{-slot } (i = 2, \ldots, n),\\
\bar{s}_i &: \text{covering } i\text{-block } (i = 2, \ldots, n),\\
t_i &: \text{half of a supporting } i\text{-block } (i = 2, \ldots, n-1),\\
\bar{t}_i &: \text{half of a covering } i\text{-block } (i = 2, \ldots, n-1),\\
t_n &: \text{half of an } n\text{-block},\\
t_{0|1} &: \text{half of a } 0|1\text{-block}.
\end{aligned}
$$

For example, we have



For a level-l slice C in $\mathcal{C}^{(l)}$, we write

$$
\begin{aligned}
u_0 &= \text{the number of layers of type } t_{0|1},\\
y_i &= \text{the number of layers of type } s_i \ (i = 1, \ldots, n),\\
\bar{y}_i &= \text{the number of layers of type } \bar{s}_i \ (i = 1, \ldots, n),\\
z_i &= \text{the number of layers of type } t_i,\\
&= \text{the number of layers of type } \bar{t}_i \ (i = 2, \ldots, n-1),\\
z_n &= \text{the number of layers of type } t_n.
\end{aligned}
$$

Then u_0 and z_n must be even, $y_i\bar{y}_i = 0$ for all $i = 1, \ldots, n$, and $u_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=2}^{n-1} z_i + z_n = l$. Hence the set $\mathcal{C}^{(l)}$ can be characterized as

$$
\mathcal{C}^{(l)} = \left\{ (u_0|y_1, \ldots, y_n|\bar{y}_1, \ldots, \bar{y}_n|z_2, \ldots, z_n) \;\middle|\; \begin{array}{l} u_0, y_i, \bar{y}_i, z_i \in \mathbf{Z}_{\geq 0},\ u_0, z_n \text{ are even},\ y_i\bar{y}_i = 0,\\ w_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=2}^{n-1} z_i + z_n = l \end{array} \right\}.
$$

We define the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by

$$(x_1, \ldots, x_n|\bar{x}_n, \ldots, \bar{x}_1) \mapsto (u_0|y_1, \ldots, y_n|\bar{y}_1, \ldots, \bar{y}_n|z_2, \ldots, z_n), \text{ where}$$

$$
\begin{aligned}
u_0 &= 2\min\{x_1, \bar{x}_1\},\\
y_i &= \max\{0, x_i - \bar{x}_i\} \ (i = 1, \ldots, n),\\
\bar{y}_i &= \max\{0, \bar{x}_i - x_i\} \ (i = 1, \ldots, n),\\
z_i &= \min\{x_i, \bar{x}_i\} \ (i = 2, \ldots, n-1),\\
z_n &= 2\min\{x_n, \bar{x}_n\},
\end{aligned}
$$

and the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by

$$(u_0|y_1, \ldots, y_n|\bar{y}_1, \ldots, \bar{y}_n|z_2, \ldots, z_n) \mapsto (x_1, \ldots, x_n|\bar{x}_n, \ldots, \bar{x}_1), \text{ where}$$

$$
\begin{aligned}
x_1 &= y_1 + \frac{1}{2}u_0,\ \bar{x}_1 = \bar{y}_1 + \frac{1}{2}u_0, \\
x_i &= y_i + z_i,\ \bar{x}_i = \bar{y}_i + z_i\ (i = 2, \ldots, n-1), \\
x_n &= y_n + \frac{1}{2}z_n,\ \bar{x}_n = \bar{y}_n + \frac{1}{2}z_n.
\end{aligned}
$$

- $D_n^{(1)}$ case

  The types of layers and their top parts are given below :

$$
\begin{aligned}
s_1 &: \text{1-slot with a 0-block}, \\
\bar{s}_1 &: \text{0-slot with a 1-block}, \\
s_i &: \text{supproting } i\text{-slot } (i = 2, \ldots, n-2), \\
\bar{s}_i &: \text{covering } i\text{-block } (i = 2, \ldots, n-2), \\
s_{n-1} &: (n-1) \text{ and } n\text{-slots}, \\
\bar{s}_{n-1} &: (n-1) \text{ and } n\text{-blocks}, \\
s_n &: n\text{-slot with a } (n-1)\text{-block}, \\
\bar{s}_n &: (n-1)\text{-slot with a } n\text{-block}, \\
t_i &: \text{half of a supporting } i\text{-block } (i = 2, \ldots, n-2), \\
\bar{t}_i &: \text{half of a covering } i\text{-block } (i = 2, \ldots, n-2), \\
t_{0|1} &: \text{half of a } 0|1\text{-block}, \\
t_{(n\text{-}1)|n} &: \text{half of a } (n\text{-}1)|n\text{-block}.
\end{aligned}
$$

For a level-l slice C in $\mathcal{C}^{(l)}$, we write

$$
\begin{aligned}
u_0 &= \text{the number of layers of type } t_{0|1}, \\
w_0 &= \text{the number of layers of type } t_{(n\text{-}1)|n}, \\
y_i &= \text{the number of layers of type } s_i\ (i = 1, \ldots, n), \\
\bar{y}_i &= \text{the number of layers of type } \bar{s}_i\ (i = 1, \ldots, n), \\
z_i &= \text{the number of layers of type } t_i, \\
&= \text{the number of layers of type } \bar{t}_i\ (i = 2, \ldots, n-2).
\end{aligned}
$$

Then u_0 and w_0 must be even, $y_i\bar{y}_i = 0$ for all $i = 2, \ldots, n-1$, and $u_0 + w_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=2}^{n-2} z_i = l$. Hence the set $\mathcal{C}^{(l)}$ can be characterized as

$$
\mathcal{C}^{(l)} = \left\{ (u_0, w_0 | y_1, \ldots, y_n | \bar{y}_1, \ldots, \bar{y}_n | z_2, \ldots, z_{n-2}) \;\middle|\; \begin{array}{l} u_0, w_0 \in 2\mathbf{Z}_{\geq 0},\ y_i, \bar{y}_i, z_i \in \mathbf{Z}_{\geq 0},\ y_i\bar{y}_i = 0, \\ u_0 + w_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=2}^{n-2} z_i = l \end{array} \right\}.
$$

We define the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by

$$
(x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1) \mapsto (u_0, w_0 | y_1, \ldots, y_n | \bar{y}_1, \ldots, \bar{y}_n | z_2, \ldots, z_{n-2}), \text{ where}
$$

$$
\begin{aligned}
u_0 &= 2\min\{x_1, \bar{x}_1\},\ w_0 = 2\min\{x_{n-1}, \bar{x}_{n-1}\}, \\
y_i &= \max\{0, x_i - \bar{x}_i\},\ \bar{y}_i = \max\{0, \bar{x}_i - x_i\}\ (i = 1, \ldots, n-1), \\
y_n &= x_n,\ \bar{y}_n = \bar{x}_n, \\
z_i &= \min\{x_i, \bar{x}_i\}\ (i = 2, \ldots, n-2),
\end{aligned}
$$

and the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by

$$(u_0, w_0|y_1, \ldots, y_n|\bar{y}_1, \ldots, \bar{y}_n|z_2, \ldots, z_{n-2}) \mapsto (x_1, \ldots, x_n|\bar{x}_n, \ldots, \bar{x}_1), \text{ where}$$

$$\begin{aligned}
x_1 &= y_1 + \frac{1}{2}u_0, \quad \bar{x}_1 = \bar{y}_1 + \frac{1}{2}u_0, \\
x_i &= y_i + z_i, \quad \bar{x}_i = \bar{y}_i + z_i \quad (i = 2, \ldots, n-2), \\
x_{n-1} &= y_{n-1} + \frac{1}{2}w_0, \quad \bar{x}_{n-1} = \bar{y}_{n-1} + \frac{1}{2}w_0, \\
x_n &= y_n, \quad \bar{x}_n = \bar{y}_n.
\end{aligned}$$

- $B_n^{(1)}$ case

The types of layers and their top parts are given below :

$$\begin{aligned}
s_1 &: \text{1-slot with a 0-block}, \\
\bar{s}_1 &: \text{0-slot with a 1-block}, \\
s_i &: \text{supproting } i\text{-slot } (i = 2, \ldots, n), \\
\bar{s}_i &: \text{covering } i\text{-block } (i = 2, \ldots, n), \\
t_i &: \text{half of a supporting } i\text{-block } (i = 2, \ldots, n-1), \\
\bar{t}_i &: \text{half of a covering } i\text{-block } (i = 2, \ldots, n-1), \\
t_n &: \text{supproting } n\text{-block (=covering } n\text{-slot)}, \\
t_{0|1} &: \text{half of a } 0|1\text{-block}.
\end{aligned}$$

For a level-l slice C in $\mathcal{C}^{(l)}$, we write

$$\begin{aligned}
u_0 &= \text{the number of layers of type } t_{0|1}, \\
y_i &= \text{the number of layers of type } s_i (i = 1, \ldots, n), \\
\bar{y}_i &= \text{the number of layers of type } \bar{s}_i (i = 1, \ldots, n), \\
z_i &= \text{the number of layers of type } t_i, \\
&= \text{the number of layers of type } \bar{t}_i \ (i = 2, \ldots, n-1), \\
z_n &= \text{the number of layers of type } t_n.
\end{aligned}$$

Then u_0 must be even, $y_i\bar{y}_i = 0$ for all $i = 1, \ldots, n$, and, $u_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=2}^{n-1} z_i + z_n = l$. Hence the set $\mathcal{C}^{(l)}$ can be characterized as

$$\mathcal{C}^{(l)} = \left\{ (u_0|y_1, \ldots, y_n|\bar{y}_1, \ldots, \bar{y}_n|z_2, \ldots, z_n) \;\middle|\; \begin{array}{l} u_0 \in 2\mathbf{Z}_{\geq 0}, \quad y_i, \bar{y}_i, z_i \in \mathbf{Z}_{\geq 0}, \\ u_0 + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=2}^{n-1} z_i + z_n = l \end{array} \right\}.$$

We define the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by

$$(u_0|y_1, \ldots, y_n|\bar{y}_1, \ldots, \bar{y}_n|z_2, \ldots, z_n) \mapsto (x_1, \ldots, x_n|x_0|\bar{x}_n, \ldots, \bar{x}_1), \text{ where}$$

$$\begin{aligned}
u_0 &= 2\min\{x_1, \bar{x}_1\}, \\
y_i &= \max\{0, x_i - \bar{x}_i\} \ (i = 1, \ldots, n), \\
\bar{y}_i &= \max\{0, \bar{x}_i - x_i\} \ (i = 1, \ldots, n), \\
z_i &= \min\{x_i, \bar{x}_i\} \ (i = 2, \ldots, n-1), \\
z_n &= 2\min\{x_n, \bar{x}_n\} + x_0,
\end{aligned}$$

and the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by

$$(x_1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_1) \mapsto (u_0 | y_1, \ldots, y_n | \bar{y}_1, \ldots, \bar{y}_n | z_2, \ldots, z_n), \text{ where}$$

$$\begin{aligned}
x_1 &= y_1 + \frac{1}{2}u_0, \ \bar{x}_1 = \bar{y}_1 + \frac{1}{2}u_0, \\
x_i &= y_i + z_i, \ \bar{x}_i = \bar{y}_i + z_i \ (i = 2, \ldots, n-1), \\
x_n &= y_n + 2[\frac{z_n}{2}], \ \bar{x}_n = \bar{y}_n + 2[\frac{z_n}{2}], \\
x_0 &= 0 \text{ if } z_n \text{ is even}, \ x_0 = 1 \text{ if } z_n \text{ is odd}.
\end{aligned}$$

- $C_n^{(1)}$ case

  The types of layers and their top parts are given below :

$$\begin{aligned}
s_i &: \text{supproting } i\text{-slot } (i = 1, \ldots, n), \\
\bar{s}_i &: \text{covering } i\text{-block } (i = 1, \ldots, n), \\
t_i &: \text{half of a supporting } i\text{-block } (i = 1, \ldots, n-1), \\
\bar{t}_i &: \text{half of a covering } i\text{-block } (i = 1, \ldots, n-1), \\
t_0 &: \text{half of a 0-block}, \\
t_n &: \text{half of a } n\text{-block}.
\end{aligned}$$

For a level-l slice C in $\mathcal{C}^{(l)}$, we write

$$\begin{aligned}
z_0 &= \text{the number of layers of type } t_0, \\
z_n &= \text{the number of layers of type } t_n, \\
y_i &= \text{the number of layers of type } s_i \ (i = 1, \ldots, n), \\
\bar{y}_i &= \text{the number of layers of type } \bar{s}_i \ (i = 1, \ldots, n), \\
z_i &= \text{the number of layers of type } t_i, \\
&= \text{the number of layers of type } \bar{t}_i \ (i = 1, \ldots, n-1).
\end{aligned}$$

Then z_0 and z_n must be even, $y_i\bar{y}_i = 0$ for all $i = 1, \ldots, n$, and $z_0 + z_n + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=1}^{n-1} z_i = 2l$. Hence the set $\mathcal{C}^{(l)}$ can be characterized as

$$\mathcal{C}^{(l)} = \left\{ (y_1, \ldots, y_n | \bar{y}_1, \ldots, \bar{y}_n | z_0, \ldots, z_n) \;\middle|\; \begin{array}{l} z_0, z_n \in 2\mathbf{Z}_{\geq 0}, \quad y_i, \bar{y}_i, z_i \in \mathbf{Z}_{\geq 0}, \\ z_0 + z_n + \sum_{i=1}^{n}(y_i + \bar{y}_i) + 2\sum_{i=1}^{n-1} z_i = 2l \end{array} \right\}.$$

We define the map $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ by

$$(x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1) \mapsto (y_1, \ldots, y_n | \bar{y}_1, \ldots, \bar{y}_n | z_0, \ldots, z_n), \text{ where}$$

$$\begin{aligned}
z_0 &= 2l - \textstyle\sum_{i=1}^{n}(x_1 + \bar{x}_i), \\
y_i &= \max\{0, x_i - \bar{x}_i\}, \ \bar{y}_i = \max\{0, \bar{x}_i - x_i\} \ (i = 1, \ldots, n), \\
z_i &= \min\{x_i, \bar{x}_i\} \ (i = 1, \ldots, n-1), \\
z_n &= 2\min\{x_n, \bar{x}_n\},
\end{aligned}$$

and the map $\phi : \mathcal{C}^{(l)} \to \mathcal{B}^{(l)}$ by

$$(y_1, \ldots, y_n | \bar{y}_1, \ldots, \bar{y}_n | z_0, \ldots, z_n) \mapsto (x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1), \text{ where}$$

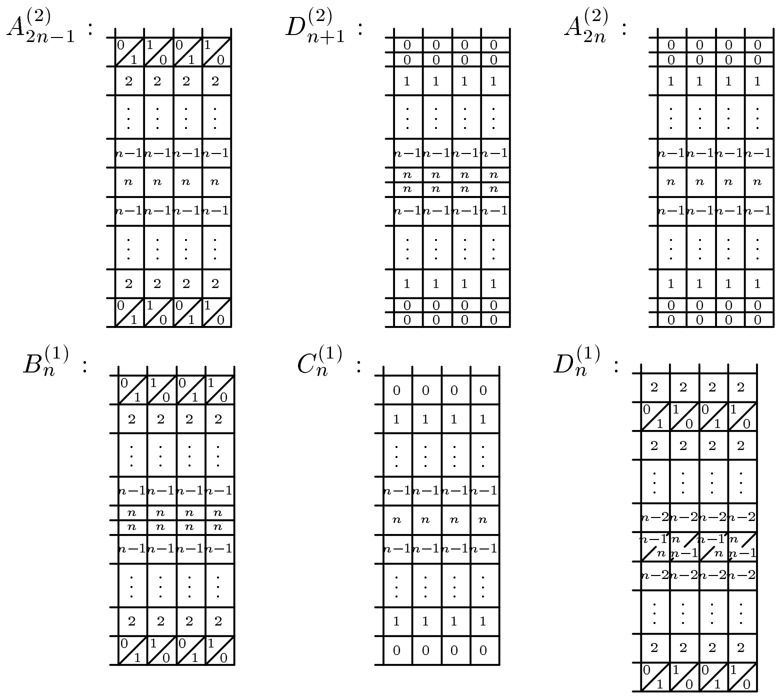
$$x_i = y_i + z_i,\ \bar{x}_i = \bar{y}_i + z_i\ (i = 1, \ldots, n-1),$$
$$x_n = y_n + \frac{1}{2}z_n,\ \bar{x}_n = \bar{y}_n + \frac{1}{2}z_n.$$

It is straightforward to verify that ψ and ϕ are inverse to each other. Also it is easy to see that the maps $\overline{\mathrm{wt}}$, φ_i, ε_i, and the action of the Kashiwara operators commute with ψ. To summarize, we obtain a new realization of level-l perfect crystals as the set of equivalence classes of level-l slices.

Theorem 4.1. *For all classical quantum affine algebras, there is a crystal isomorphism $\psi : \mathcal{B}^{(l)} \to \mathcal{C}^{(l)}$ given by above formulas.*

5. Higher level Young walls

In this section, we will define the notion of (arbitrary level) proper Young walls, reduced Young walls, ground-state walls, etc., and give a realization of arbitrary level irreducible highest weight crystals in terms of reduced Young walls. The patterns for building Young walls are given below :



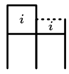
Definition 5.1. A ***level-l Young wall*** is a concatenation of level-l slices, extending infinitely to the left, satisfying the following conditions.

(1) It is concatenated following the pattern given above.
(2) At each layer, there is no free space to the right of any block (or broken half-block).

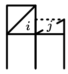
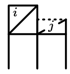
Remark 5.2.

(1) In most cases, it is easy to judge whether there is a free space to the right of a given block (or broken half-block). In addition, the following nontrivial cases will be considered as having a free space to the right of a given block (or broken half-block).

- The left is a whole block and the right is a broken half of a whole i-block.

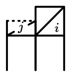
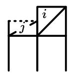


- The left is a single block of half-unit depth and the right is the *upper* broken half of a whole j-block.

 or 

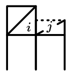
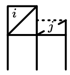
Here, $i = 0, 1$ and $j = 0|1$ or $i = n{-}1, n$ and $j = (n\text{-}1)|n$. Note that the color of the broken half on the right will depend on i.

- The right is a single block of half-unit depth and the left is the *lower* broken half of a whole j-block.

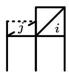
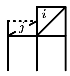
 or 

Here, $i = 0, 1$ and $j = 0|1$ or $i = n{-}1, n$ and $j = (n\text{-}1)|n$.

(2) However, the following cases will be considered as having no free space to the right of a given block (or broken half-block).

 and 

with $i = 0, 1$ and $j = 0|1$ or $i = n{-}1, n$ and $j = (n\text{-}1)|n$ and where the right is a lower half.

 and 

with $i = 0, 1$ and $j = 0|1$ or $i = n{-}1, n$ and $j = (n\text{-}1)|n$ and where the left is an upper half.

Definition 5.3.

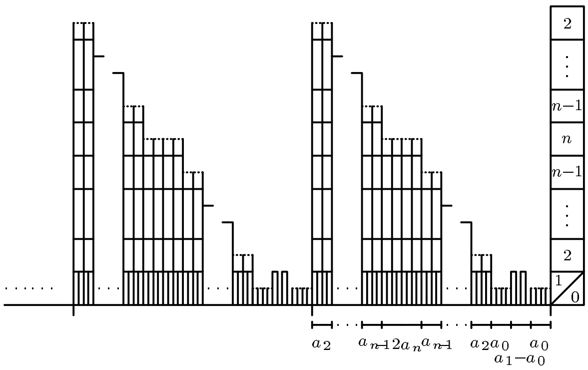
(1) A ***full column*** is a layer of a level-l slice whose height is an integer and whose top is of unit depth.
(2) A level-l Young wall $\mathbf{Y}$ is said to be ***proper*** if for each layer of $\mathbf{Y}$, none of the full columns have the same height.
(3) A column in a level-l proper Young wall is said to contain a ***removable*** δ if one may remove a δ from that column and still obtain a proper Young wall.
(4) A level-l proper Young wall is said to be ***reduced*** if none of its columns contain a removable δ.

Let $\lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_n\Lambda_n$ be a dominant integral weight of level-l so that

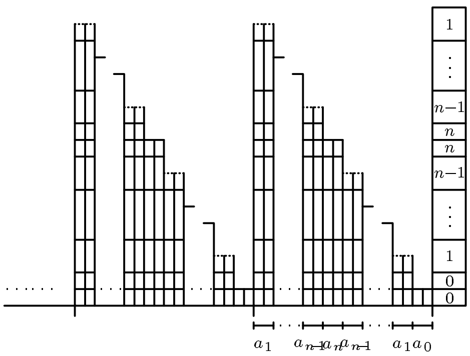
$$
\begin{cases}
l = a_0 + a_1 + 2(a_2 + \cdots + a_n) & A^{(2)}_{2n-1} \text{ case,} \\
l = a_0 + 2(a_1 + \cdots + a_{n-1}) + a_n & D^{(2)}_{n+1} \text{ case,} \\
l = a_0 + 2(a_1 + \cdots + a_n) & A^{(2)}_{2n} \text{ case,} \\
l = a_0 + a_1 + 2(a_2 + \cdots + a_{n-1}) + a_n & B^{(1)}_{n} \text{ case,} \\
l = a_0 + a_1 + \cdots + a_n & C^{(1)}_{n} \text{ case,} \\
l = a_0 + a_1 + 2(a_2 + \cdots + a_{n-2}) + a_{n-1} + a_n & D^{(1)}_{n} \text{ case.}
\end{cases}
\tag{5.1}
$$

We would like to define the ***ground-state wall*** $\mathbf{Y}_\lambda$ ***of weight*** λ. It will be constructed as a level-l reduced Young wall. In the following, we will draw part of $\mathbf{Y}_\lambda$. In the $A_{2n-1}^{(2)}$ and $B_n^{(1)}$ case (resp. $D_n^{(1)}$ case) depending on whether $a_0 \geq a_1$ or $a_0 \leq a_1$ (resp. $a_0 \geq a_1$ or $a_0 \leq a_1$ and $a_{n-1} \geq a_n$ or $a_{n-1} \leq a_n$), there are two (resp. four) different forms of the ground-state wall. We will just draw the case when $a_0 \leq a_1$ (resp. $a_0 \leq a_1$ and $a_{n-1} \geq a_n$).

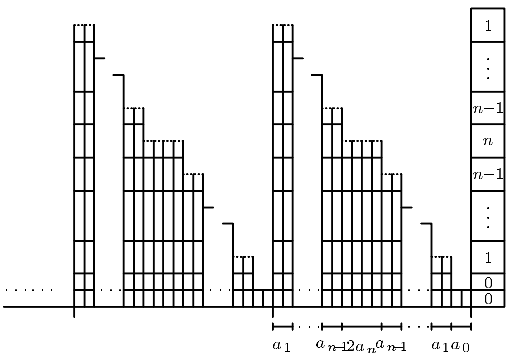
$A_{2n-1}^{(2)}$:



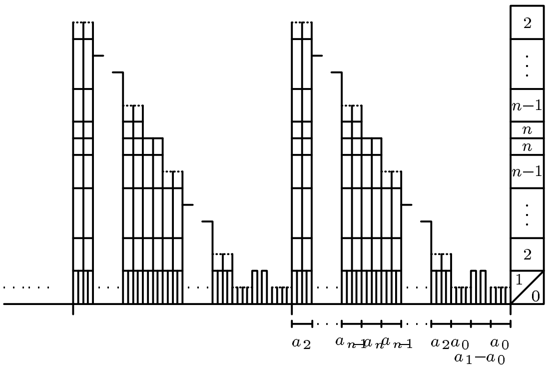
$D_{n+1}^{(2)}$:



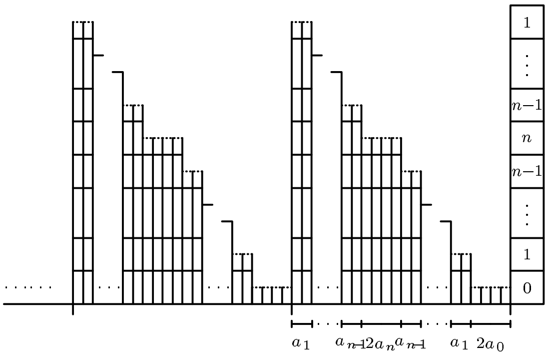
$A_{2n}^{(2)}$:



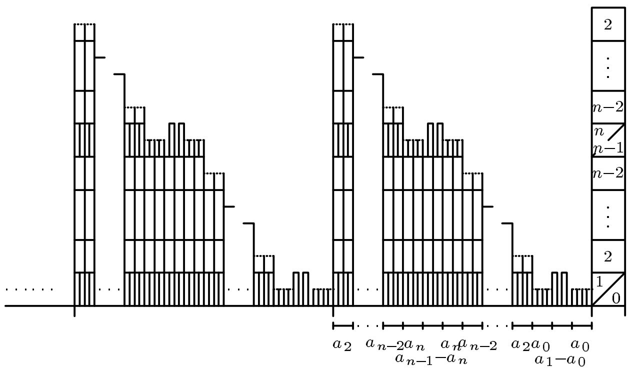
$B_n^{(1)}$:



$C_n^{(1)}$:



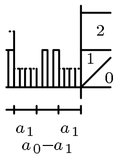
$D_n^{(1)}$:



Recall that a Young wall is a concatenation of slices. We have drawn the left-side-views of the first two slices that make up the Young wall. The actual ground-state wall should extend infinitely to the left, repeating the same pattern. At the right end, we have drawn the pattern for stacking the blocks, so as to show the color of the blocks placed at each height.

For $A_{2n-1}^{(2)}$ and $B_n^{(1)}$ (or $D_n^{(1)}$) cases, the pattern on the right is just for the even ith columns. As given by the figures at the beginning of this section, the odd columns will be stacked in a pattern with $0, 1$ (resp. 0, 1 and $n-1$, n) exchanged. Since the outline of the first two slices given above (actually, all slices) are exactly the same, this means that the even and odd columns are identical except for the exchange of 0 with 1 (resp. 0 with 1 and $n-1$ with n). Below the 0th slice, we have written down how many layers of each shape should be used.

When $a_0 \geq a_1$,



should appear in the first few layers of the 0th column. Notice that in addition to the exchange of a_0 with a_1 from the full diagram above, the position of half-unit depth blocks appearing in the middle has shifted so that they are now 1-blocks instead of the 0-blocks used in the full diagram.

A level-l proper Young wall obtained by adding finitely many blocks to the ground-state wall $\mathbf{Y}_\lambda$ is said to have been ***built on*** $\mathbf{Y}_\lambda$. We denote by $\mathcal{Z}(\lambda)$ (and $\mathcal{Y}(\lambda)$) the set of all proper (resp. reduced) Young walls built on $\mathbf{Y}_\lambda$.

Let $\mathbf{Y}$ be a level-l proper Young wall built on $\mathbf{Y}_\lambda$ and let C be a column of $\mathbf{Y}$. Recall that a column C is a level-l slice and that for each $i \in I$, $\varphi_i(C)$ (resp. $\varepsilon_i(C)$) is the largest integer $k \geq 0$ such that $\tilde{f}_i^k(C) \neq 0$ (resp. $\tilde{e}_i^k(C) \neq 0$).

We now define the action of Kashiwara operators $\tilde{f}_i$, $\tilde{e}_i$ ($i \in I$) on $\mathbf{Y}$ as follows.

(1) For each column C of $\mathbf{Y}$, write $\varepsilon_i(C)$-many 1's followed by $\varphi_i(C)$-many 0's under C. This sequence is called the ***i-signature of*** C.
(2) From this sequence of 1's and 0's, cancel out each (0,1)-pair to obtain a sequence of 1's followed by 0's (reading from left to right). This sequence is called the ***i-signature of*** $\mathbf{Y}$.
(3) We define $\tilde{f}_i\mathbf{Y}$ to be the proper Young wall obtained from $\mathbf{Y}$ by replacing the column C corresponding the leftmost 0 in the i-signature of $\mathbf{Y}$ with the column $\tilde{f}_iC$.
(4) We define $\tilde{e}_i\mathbf{Y}$ to be the proper Young wall obtained from $\mathbf{Y}$ by replacing the column C corresponding the rightmost 1 in the i-signature of $\mathbf{Y}$ with the column $\tilde{e}_iC$.
(5) If there is no 0 (or 1) in the i-signature of $\mathbf{Y}$, we define $\tilde{f}_i\mathbf{Y} = 0$ (resp. $\tilde{e}_i\mathbf{Y} = 0$).

We need to show that the action of Kashiwara operators on $\mathcal{Z}(\lambda)$ is well-defined. We will just deal with the $\tilde{f}_i$ operator.

Since a Young wall extends infinitely to the left, it is not immediately clear as to whether there exists a *leftmost* zero. That is, it is not clear whether the number of zeros in the i-signature of $\mathbf{Y}$ is finite. Let us briefly comment on this here.

Since only finitely many blocks were added to the ground-state wall in building $\mathbf{Y}$, the wall will eventually become identical to the ground-state wall at some point, as it proceeds to the left. Thus it suffices to check if the ground-state walls give finite signatures. This one may do easily with each of the explicit ground-state walls.

Now, we fix some notations. Denote by C the column corresponding to the leftmost 0 in the i-signature of a proper Young wall $\mathbf{Y}$. The column sitting to the right of column C will be denoted by C'.

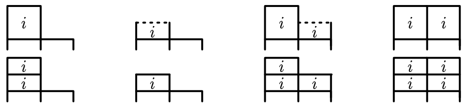
Suppose that $\tilde{f}_i\mathbf{Y}$ is not a proper Young wall. In fact, it could be that $\tilde{f}_i\mathbf{Y}$ is not even a Young wall. In such a case, the following statement would be true.

- There is a free space to the right of some block or half-block in some layer of the column of $\tilde{f}_i\mathbf{Y}$, that corresponds to the column C of $\mathbf{Y}$.

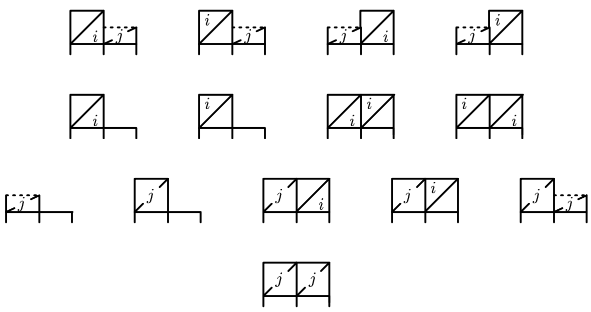
If the result is a Young wall, but just not proper, then the following statement would be true.

- The columns of $\tilde{f}_i\mathbf{Y}$ corresponding to the columns C and C' of $\mathbf{Y}$ contain a layer in which the tops are of unit depth and of the same integer height.

For the index i corresponding to a unit cube or a half-hight block, the following lists all possible non-trivial forms for $\tilde{f}_i\mathbf{Y}$ that satisfy one of the above two statements.



For the indices corresponding to blocks of half-unit depth; that is, for $i = 0, 1$ or $n-1, n$, the following lists (almost) all possible forms for $\tilde{f}_i\mathbf{Y}$ that satisfy one of the above two statements.



Here, $j = 0|1$ or $(n\text{-}1)|n$. As mentioned in Remark 5.2, the right (or left) of the first (resp. last) two diagrams in the first row is the upper (resp. lower) broken half of a whole j-block.

In each of these cases, it is possible to obtain one of the following three conclusions.

(1) There is a free space to the right of some block or half-block in some layer of the column C of $\mathbf{Y}$.
(2) The columns C and C' of the proper Young wall $\mathbf{Y}$ contain a layer in which the tops are of unit depth and of the same integer height.
(3) $\varphi_i(C) \le \varepsilon_i(C')$.

The first of these conclusions violates the assumption that we started out with a Young wall $\mathbf{Y}$. The second conclusion is in violation of the properness of $\mathbf{Y}$. As for the third, since C is the column corresponding to the leftmost 0 in the i-signature of $\mathbf{Y}$ we must have $\varphi_i(C) > \varepsilon_i(C')$. Each of these conclusions brings us to a contradiction, and hence the resulting $\tilde{f}_i\mathbf{Y}$ must have been a proper Young wall.

We define the maps $\mathrm{wt} : \mathcal{Z}(\lambda) \to P$, $\varphi_i, \varepsilon_i : \mathcal{Z}(\lambda) \to \mathbf{Z}$ by setting

$$\begin{aligned} \mathrm{wt}(\mathbf{Y}) &= \lambda - \sum_{i=0}^{n} k_i \alpha_i, \\ \varphi_i(\mathbf{Y}) &= \text{the number of 0's in the } i\text{-signature of } \mathbf{Y}, \\ \varepsilon_i(\mathbf{Y}) &= \text{the number of 1's in the } i\text{-signature of } \mathbf{Y}, \end{aligned} \tag{5.2}$$

where k_i is the number of i-blocks that have been added to $\mathbf{Y}_\lambda$.

Remark 5.4. We have seen that the i-signatures are always finite. Hence it makes sense to count the number of 0's and 1's in the signature.

Now it is straightforward to verify that the following theorem holds.

Theorem 5.5. *The set $\mathcal{Z}(\lambda)$ of all level-l proper Young walls built on $\mathbf{Y}_\lambda$, together with the maps $\tilde{e}_i$, $\tilde{f}_i$, ε_i, φ_i $(i \in I)$, and* wt, *forms a $U_q(\mathfrak{g})$-crystal.*

Finally, we give a new realization of arbitrary level irreducible highest weight crystals in terms of reduced Young walls. Since the irreducible highest weight crystal $\mathcal{B}(\lambda)$ is isomorphic to the crystal $\mathcal{P}(\lambda)$ consisting of λ-paths, it suffices to show that there is a crystal isomorphism $\Phi : \mathcal{Y}(\lambda) \xrightarrow{\sim} \mathcal{P}(\lambda)$.

We define the map $\Phi : \mathcal{Y}(\lambda) \longrightarrow \mathcal{P}(\lambda)$ as follows. Given a reduced Young wall $\mathbf{Y} = (\mathbf{Y}(k))_{k=0}^{\infty}$ in $\mathcal{Y}(\lambda)$, consider the crystal isomorphism $\psi : \mathcal{B}^{(l)} \xrightarrow{\sim} \mathcal{C}^{(l)}$ given in Theorem 4.1, and define $\Phi(\mathbf{Y})$ to be

$$\Phi(\mathbf{Y}) = (\psi^{-1}(\mathbf{Y}(k)))_{k=0}^{\infty}. \tag{5.3}$$

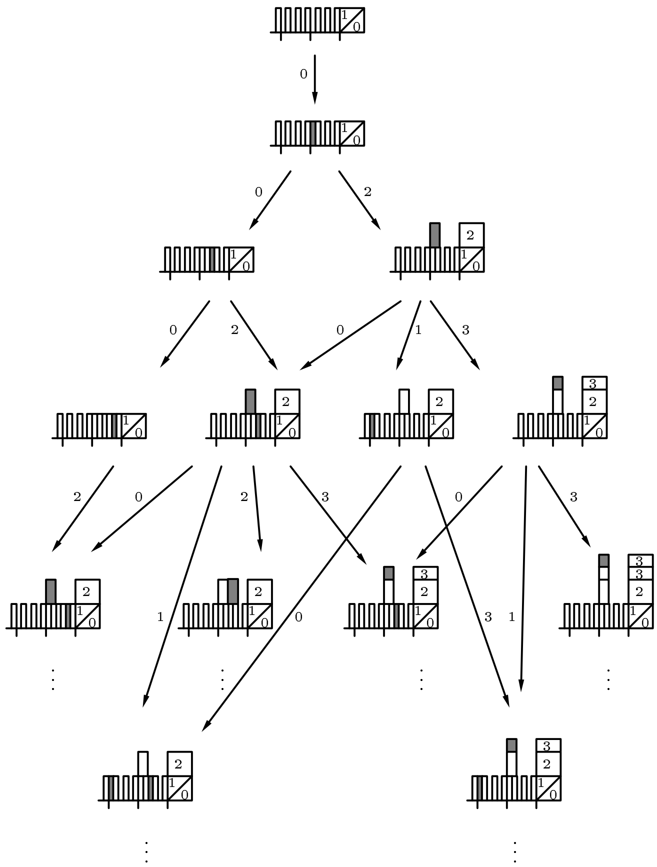
Conversely, to each λ-path $\mathbf{p} = (\mathbf{p}(k))_{k=0}^{\infty}$, by removing an appropriate number of δ's, one can easily see that there exists a unique reduced Young wall $\mathbf{Y} = (\mathbf{Y}(k))_{k=0}^{\infty}$ such that $\psi(\mathbf{p}(k)) = \mathbf{Y}(k)$ for all $k \geq 0$. Hence Φ is a bijection.

Moreover, by the same argument used in the proof of Theorem 6.2 in [9], we can show that $\mathcal{Y}(\lambda)$ is a subcrystal of $\mathcal{Z}(\lambda)$ and the map Φ commutes with the Kashiwara operators. Therefore we obtain our main result.

Theorem 5.6. *We have a $U_q(\mathfrak{g})$-crystal isomorphism*

$$\mathcal{Y}(\lambda) \xrightarrow{\sim} \mathcal{P}(\lambda) \xrightarrow{\sim} \mathcal{B}(\lambda). \tag{5.4}$$

In the following figure, we illustrate the top part of the affine crystal $\mathcal{Y}(3\Lambda_0)$ for $B_3^{(1)}$. The shaded part denotes the change through the action of $\tilde{f}_i$ $(i \in I)$.



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