The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p, q)$

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Abstract

The indefinite orthogonal group $G = O(p, q)$ has a distinguished infinite dimensional irreducible unitary representation π for $p+q$ even and greater than 4, which is the "smallest" in the sense that the Gelfand–Kirillov dimension of π attains its (positive) minimum value $p+q-3$ among the unitary dual of G. Moreover, π is the *minimal representation* if $p+q > 6$.

The *Schrödinger model* realizes π on the Hilbert space $L^2(C)$ consisting of square integrable functions on a Lagrangean submanifold C of the minimal nilpotent coadjoint orbit. Among various concrete models of π, the Hilbert structure (e.g. inner product) of the Schrödinger model is so simple, whereas the G-action on $L^2(C)$ has not been well-understood except for a specific maximal parabolic subgroup.

The subject of this paper is the analysis of the Schrodinger model of the minimal representation. We establish the "global formula" for the Schrödinger model with an explicit description of the action of the whole group G. For this, we describe the unitary operator $\pi(w_0)$

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on $L^2(C)$ for the "conformal inversion" w_0 as a singular integral, and find its kernel by means of a distribution which we call a *Bessel distribution*. Our results generalize the well-established case, namely, the original Schrödinger model $L^2(\mathbb{R}^n)$ for the Weil representation of the metaplectic group, where the "conformal inversion" gives rise to the Fourier transform. However, a new mysterious phenomenon arises in our case $G = O(p,q)$, namely, the kernel distribution is not always locally integrable. In fact, this happens if and only if π is a non-highest weight minimal representation, equivalently, $p,q > 2$ and $p+q > 6$. We analyze the kernel distributions by using singular Radon transforms and Mellin–Barnes type integral formulas.

Large group symmetries in the minimal representations bring us naturally to functional equations of various special functions, which we also emphasize in this paper. For example, we find explicit K-finite vectors on $L^2(C)$ by means of Bessel functions for every K-type, and give a representation theoretic proof of the inversion formula and the Plancherel formula for Meijer's G-transforms.

Contents

1 Introduction and statement of main results

The subject of this paper is the analysis of the L^2-model (*Schrödinger model*) for the minimal representation π of the indefinite orthogonal group $G = O(p, q)$ for even $p + q$. Motivated by recent developments of algebraic representation theory, we try to shed new light on geometric analysis including a group theoretic approach to special functions.

1.1 Minimal representation and L^2-model

For a reductive Lie group a particularly interesting irreducible unitary representation, sometimes referred to as the *minimal representation*, is the one corresponding via "geometric quantization" to the minimal nilpotent coadjoint orbit $\mathcal{O}$. Minimal representations are one of the most fundamental irreducible unitary representations in the sense that they cannot be built up from any smaller groups by existing methods of (ordinary or cohomological) induced representations.

One of the important algebraic properties of minimal representations is that the Gelfand–Kirillov dimension attains its minimum at minimal representations among all irreducible unitary representations of the same group. Analytically, this in turn implies that there are more "symmetries" on the representation space of the minimal representation than other irreducible unitary representations. Then, realizing the minimal representation in the space of certain functions, we could expect an abundant and fruitful theory of concrete global analysis for the minimal representations. Thus, we initiate a new line of investigation on various special functions (e.g. Bessel functions, Appell's hypergeometric functions, Meijer's G-functions, etc.) arising from the minimal representation by group theoretic approaches.

By the *Schrödinger model*, we mean a realization of π on the Hilbert space $L^2(C)$ consisting of L^2-functions on a Lagrangean variety C of $\mathcal{O}$. Although the variety C is so small that the whole group G cannot act on C, a maximal parabolic subgroup $\overline{P^{\max}}$ acts on C, and correspondingly we can define naturally a unitary representation of $\overline{P^{\max}}$ on $L^2(C)$. We proved in [35] that this $\overline{P^{\max}}$ action on $L^2(C)$ extends to an irreducible unitary representation of G, leading us to construction of the Schrödinger model of the minimal representation. In a series of papers [33, 34, 35], we also found some basic properties of this model including the infinitesimal character of the center of the enveloping algebra $U(\mathfrak{g})$ and an explicit vector that belongs to the minimal K-type in terms of the K-Bessel function. The missing piece of [35] is an explicit formula for the action of the whole group G on

$L^2(C)$ other than the action of $\overline{P^{\max}}$. In light of the Bruhat decomposition $G = \overline{P^{\max}} \coprod \overline{P^{\max}} w_0 \overline{P^{\max}}$, the crux of the theory is to find an explicit formula for $\pi(w_0)$. In this paper, we shall find the integro-differential kernel for the unitary "inversion operator" $\pi(w_0)$ on $L^2(C)$.

1.2 Schrödinger model for the Weil representation of $Mp(n, \mathbb{R})$

Our group $G = O(p, q)$ ($p + q$ even) is a reductive group of type D. In order to clarify our motivation, we begin with the best understood minimal representation of a reductive group of type C, that is, the (Segal–Shale–)Weil representation ϖ, or sometimes referred to as the oscillator representation, of the metaplectic group $G' = Mp(n, \mathbb{R})$, the twofold cover of the real symplectic group $Sp(n, \mathbb{R})$. Let ξ_0 denote the (unique) non-trivial element in the kernel of the homomorphism $G' \to Sp(n, \mathbb{R})$.

The Schrödinger model is originally the term concerning with a realization of the Weil representation ϖ on the Hilbert space $L^2(\mathbb{R}^n)$. Since our model $(\pi, L^2(C))$ of the minimal representation of G has a strong resemblance to $(\varpi, L^2(\mathbb{R}^n))$ of G', we list some important features of the Schrödinger model of ϖ (see e.g. [11], [25]):

C1 The representation is realized on a very explicit Hilbert space, that is, $L^2(\mathbb{R}^n)$.

C2 The restriction of ϖ to the Siegel parabolic subgroup P_{Siegel} is still irreducible, and the restriction to P_{Siegel} has a relatively simple form (translations and multiplications).

C3 The infinitesimal action $d\varpi$ of the Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ is given by differential operators of at most second order.

C4 There is a distinguished element w_0' of G' that sends P_{Siegel} to the opposite parabolic subgroup. The corresponding unitary operator $\varpi(w_0')$ on $L^2(\mathbb{R}^n)$ is proportional to the Fourier transform $\mathcal{F}$. Correspondingly to the fact that $(w_0')^4 = \xi_0$ and $\varpi(\xi_0) = -\operatorname{id}$, the Fourier transform $\mathcal{F}$ is of order four ($\mathcal{F}^4 = \operatorname{id}$).

Since G' is generated by P_{Siegel} and w_0', C2 and C4 determine the action of G' on $L^2(\mathbb{R}^n)$ (see [42] for an explicit formula for the action of the whole group G' on $L^2(\mathbb{R}^n)$). C3 asserts in particular that the action $d\varpi$ is not given by vector fields. This reflects the fact that G' acts only on $L^2(\mathbb{R}^n)$, not on $\mathbb{R}^n$.

1.3 Schrödinger model for the minimal representation of $O(p,q)$

Now, let us consider our representation π. To fix notation, we set

$$I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

and define the indefinite orthogonal group $G = O(p,q)$ of signature (p,q) as the following matrix group:

$$O(p,q) := \{g \in GL(p+q,\mathbb{R}) : {}^t g I_{p,q} g = I_{p,q}\}.$$

The unitary representation π of $O(p,q)$ was constructed by Kostant in [36] for $p = q = 4$, and by Binegar and Zierau [5], Huang and Zhu [24], and Kobayashi and Ørsted [33, 35] for general $p, q \geq 2$ such that $p+q$ is an even integer and greater than 4. Yet another realization was studied in Brylinski and Kostant [6], and Torasso [51]. Among various different realizations of π, we have proved in [35] that π can be realized on the Hilbert space $L^2(C)$ consisting of square integrable functions on the conical subvariety

$$C := \{\zeta := (\zeta_1, \cdots, \zeta_{p+q-2}) \in \mathbb{R}^{p+q-2} \setminus \{0\} : Q(\zeta) = 0\},$$

where

$$Q(\zeta) := \zeta_1^2 + \cdots + \zeta_{p-1}^2 - \zeta_p^2 - \cdots - \zeta_{p+q-2}^2. \tag{1.3.1}$$

We remark that C is defined in $\mathbb{R}^{p+q-2}$, and $G = O(p,q)$ cannot act (non-trivially) on C. (In fact, any (non-trivial) G-space is of dimension at least greater than $p+q-2 = \dim C + 1$.)

In our model $(\pi, L^2(C))$ of the indefinite orthogonal group G, the maximal parabolic subgroup $\overline{P^{\max}}$ (see Subsection 2.2 for definition) plays a similar role of the Siegel parabolic subgroup P_{Siegel}, and it is proved in [35] that analogous results to the properties C1, C2 and C3 also hold. If we set

$$w_0 := I_{p,q},$$

then w_0 sends $\overline{P^{\max}}$ to the opposite parabolic subgroup, and G is generated by w_0 and $\overline{P^{\max}}$.

Our main concern of this paper is to establish an analogous result to C4 for $G = O(p,q)$, namely, to find the unitary operator $\pi(w_0)$ on $L^2(C)$ for the "conformal inversion" w_0 (see Subsection 2.4 for a geometric meaning). We shall give an explicit kernel distribution $K(\zeta, \zeta')$ of the unitary operator $\pi(w_0)$.

1.4 Bessel distributions

For a locally integrable function $f(t)$ on $\mathbb{R}$, we write

$$f(t_+) := \begin{cases} f(t) & (t > 0) \\ 0 & (t \leq 0), \end{cases} \qquad f(t_-) := \begin{cases} 0 & (t \geq 0) \\ f(|t|) & (t < 0). \end{cases}$$

If $f_\lambda(t)$ is a locally integrable function on $\mathbb{R}$ with parameter λ in a certain domain in $\mathbb{C}$, and if $f_\lambda(t_+)$ (or $f_\lambda(t_-)$) extends meromorphically as a distribution, we shall use the same notation $f_\lambda(t_+)$ (or $f_\lambda(t_-)$).

In order to state our main theorem, we introduce the following tempered distributions (which we call *Bessel distributions*) on $\mathbb{R}$ by

$$\Phi_m^+(t) := (2t)_+^{-\frac{m}{2}} J_m(2\sqrt{2t_+}), \tag{1.4.1}$$

$$\Psi_m^+(t) := (2t)_+^{-\frac{m}{2}} J_m(2\sqrt{2t_+}) - \sum_{k=1}^{m} \frac{(-1)^{k-1}}{2^k (m-k)!} \delta^{(k-1)}(t), \tag{1.4.2}$$

$$\Psi_m(t) := (2t)_+^{-\frac{m}{2}} Y_m(2\sqrt{2t_+}) + \frac{2(-1)^{m+1}}{\pi} (2t)_-^{-\frac{m}{2}} K_m(2\sqrt{2t_-}). \tag{1.4.3}$$

Here, $J_\nu(x)$, $Y_\nu(x)$ and $K_\nu(z)$ are the (modified) Bessel functions (see Appendix 7.2), and $\delta^{(l)}(t)$ denotes the l-th differential of the Dirac delta function $\delta(t)$. The singular part of the distribution $\Psi_m(t)$ is given by a linear combination of the distribution t^{-k} $(k = 1, 2, \ldots, m)$ (see Theorem 6.2.1).

A rigorous definition of Ψ_m^+ and Ψ_m is given in Section 6.1 by means of the Mellin–Barnes type integral for distributions (see also Remark 6.2.3). We shall also discuss in Section 6 other aspects of Bessel distributions such as differential equations that $\Psi_m(t)$ and $\Psi_m^+(t)$ satisfy (see Proposition 6.3.3).

1.5 The unitary inversion formula

Let p, q be integers satisfying the following condition:

$$p, q \geq 2, \quad p+q \text{ is even}, \quad \text{and} \quad (p,q) \neq (2,2). \tag{1.5.1}$$

Let $\langle \zeta, \zeta' \rangle$ be the standard (positive definite) inner product of $\mathbb{R}^{p+q-2}$. We define a generalized function $K(\zeta, \zeta')$ (depending on p and q) on the direct product manifold $C \times C$ by

$$K(\zeta, \zeta') \equiv K(p, q; \zeta, \zeta') := c_{p,q} \Phi_{p,q}(\langle \zeta, \zeta' \rangle), \tag{1.5.2}$$

where the constant $c_{p,q}$ and the distribution $\Phi_{p,q}(t)$ are determined as follows:

$$c_{p,q} := \frac{2(-1)^{\frac{(p-1)(p+2)}{2}}}{\pi^{\frac{p+q-4}{2}}}, \tag{1.5.3}$$

$$\Phi_{p,q}(t) := \begin{cases} \Phi^+_{\frac{p+q-6}{2}}(t) & \text{if } \min(p,q) = 2, \\ \Psi^+_{\frac{p+q-6}{2}}(t) & \text{if } p,q > 2 \text{ are both even}, \\ \Psi_{\frac{p+q-6}{2}}(t) & \text{if } p,q > 2 \text{ are both odd}. \end{cases} \tag{1.5.4}$$

Then, here is our main result.

Main Theorem (see Theorem 5.1.1)**.** *Let $p, q \geq 2$ and $p+q \geq 6$ be even. Then the unitary operator $\pi(w_0) : L^2(C) \to L^2(C)$ is given by the following integro-differential operator:*

$$\pi(w_0)u(\zeta) = \int_C K(\zeta,\zeta')u(\zeta')d\mu(\zeta'), \qquad u \in L^2(C). \tag{1.5.5}$$

We list some distinguishing new features of our results on the minimal representation $(\pi, L^2(C))$ of $G = O(p,q)$ from the known results on the Schrödinger model $(\varpi, L^2(\mathbb{R}^n))$ of the Weil representation of $G' = Mp(n,\mathbb{R})$.

P1 (singular integral) We note that π is a non-highest weight representation iff $p, q > 2$. Further, it is a minimal representation in the sense that the annihilator is the Joseph ideal iff $p+q \geq 8$. Now, suppose $p, q > 2$ and $p+q \geq 8$. Then, the kernel function $K(\zeta,\zeta')$ for $\pi(w_0)$ is not locally integrable, whereas, in the case of the Weil representation, the kernel function $\left(\frac{\sqrt{-1}}{2\pi}\right)^{\frac{n}{2}} e^{\sqrt{-1}\langle\zeta,\zeta'\rangle}$ for $\varpi(w_0') = e^{\frac{\sqrt{-1}n\pi}{4}}\mathcal{F}$ (the Fourier transform) is locally integrable. In other words, the unitary inversion operator $\pi(w_0)$ is given as a *singular integral.* For instance, for $p, q > 2$ both even, the kernel function $K(\zeta,\zeta') = c_{p,q}\Psi^+_{\frac{p+q-6}{2}}(\langle\zeta,\zeta'\rangle)$ involves the $\frac{p+q-8}{2}$th derivatives of a measure. Here, the derivatives are taken as normal derivatives with respect to the hyperplane $\{(\zeta,\zeta') \in C \times C : \langle\zeta,\zeta'\rangle = 0\}$.

P2 (support of the kernel) The supports of the kernel function $K(\zeta,\zeta')$ differ according to the parity of p, q, as one observes from (1.5.4)

$$\operatorname{supp}\Phi_{p,q} = \begin{cases} \mathbb{R}_+ & \text{if } p,q \text{ both even}, \\ \mathbb{R} & \text{if } p,q \text{ both odd}. \end{cases}$$

In particular,

$$\operatorname{supp}\Phi_{p,q} \subsetneqq \mathbb{R} \quad \text{and} \quad \operatorname{supp}K(\zeta,\zeta') \subsetneqq C \times C$$

if both p and q are even. This is a distinguishing feature from the known cases for highest weight representations such as the Weil representation, where the kernel function $K'(\zeta,\zeta')$ is given by $\Phi'_n(\langle\zeta,\zeta'\rangle)$ if we set $\Phi'_n(t) := (\frac{\sqrt{-1}}{2\pi})^{\frac{n}{2}} e^{\sqrt{-1}t}$. Thus,

$$\operatorname{supp}\Phi'_n = \mathbb{R} \quad \text{and} \quad \operatorname{supp}K'(\zeta,\zeta') = \mathbb{R}^n \times \mathbb{R}^n$$

for any $n \in \mathbb{N}$.

On the other hand, here are some similarities of $\pi(w_0)$ for our minimal representation π of $G = O(p,q)$ to $\varpi(w_0')$ for the Weil representation of $G' = Mp(n,\mathbb{R})$.

P3 (Plancherel and reciprocal formula) Denote by S the integral transform in (1.5.5). Then the integro-differential operator S has the following Plancherel and reciprocal formulas (see Corollaries 5.1.2 and 5.1.3):

$$\|Su\|_{L^2(C)} = \|u\|_{L^2(C)} \quad \text{for } u \in L^2(C),$$
$$S^2 = \mathrm{id} \qquad \text{in } L^2(C).$$

Analogous results for the Weil representation are well-known properties for the Fourier transform $\mathcal{F}$:

$$\|\mathcal{F}u\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} \quad \text{for } u \in L^2(\mathbb{R}^n),$$
$$\mathcal{F}^4 = \mathrm{id} \qquad \text{in } L^2(\mathbb{R}^n).$$

Here, we adopt the normalization of the Fourier transform $\mathcal{F}$ on $\mathbb{R}^n$ as follows:

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(x) e^{\sqrt{-1}\langle x,\xi\rangle} dx, \tag{1.5.6}$$

where $\langle x,\xi\rangle = \sum_{i=1}^n x_i\xi_i$ and $dx = dx_1 \cdots dx_n$. Then the inverse Fourier transform $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}F(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} F(\xi) e^{-\sqrt{-1}\langle x,\xi\rangle} d\xi. \tag{1.5.7}$$

In our context, in view of $\pi(w_0) = S$ and $\varpi(w_0') = e^{\frac{\sqrt{-1}n\pi}{4}}\mathcal{F}$, the identities $S^2 = \mathrm{id}$ and $\mathcal{F}^4 = \mathrm{id}$ corresponds to the group laws $w_0^2 = 1 \in G$ and $(w_0')^4 = \xi_0 \in G'$, respectively. It is noteworthy that an integro-differential operator S arises as a unitary operator on $L^2(C)$.

P4 (action of the whole group) The action of $\overline{P^{\max}}$ on $L^2(C)$ is of a simple form. In light of the Bruhat decomposition $G = \overline{P^{\max}} \amalg \overline{P^{\max}} w_0 \overline{P^{\max}}$, we can get directly the concrete form of the action of the whole group G once we know $\pi(w_0)$ explicitly.

In the case $q = 2$ (likewise $p = 2$), π splits into the direct sum of a highest weight module and a lowest weight module when restricted to the identity component $SO_0(p, 2)$ of the conformal group $O(p, 2)$ of the Minkowski space $\mathbb{R}^{p-1,1}$, namely, the Euclidean space $\mathbb{R}^p$ equipped with the flat Lorentz metric of signature $(p-1, 1)$. In this case our representation π has been studied also in physics, π may be interpreted as the solution space to the mass-zero spin-zero wave equation. π may be also regarded as the bound states of the Hydrogen atom. For $q = 2$, π extends to a holomorphic semigroup, and we can regard $\pi(w_0)$ as the *boundary value* of a holomorphic semigroup. This was the approach taken in [31]. The approach in this paper (the proof of Main Theorem in the special case $q = 2$) gives a new proof of the formula of $\pi(w_0)$.

1.6 Special functions and minimal representations

Yet another theme of this paper is special functions arising from the minimal representation.

As we explained, the representation space of a minimal representation is "small" relative to the original group itself. For example, in the Schrödinger model $L^2(C)$ of $G = O(p, q)$, we observe

$$\dim C = p + q - 3$$

is strictly smaller than the dimension of any manifold on which G acts non-trivially. This suggests that we could expect a lot of relations among functions in $L^2(C)$ reflected by the group structure of G.

We shall see that various special functions arise from the minimal representation. For example, K-Bessel functions arise in describing K-finite vectors in the Schrödinger model $L^2(C)$. Meijer's G-function appear as the "K-component" of the integral kernel of our unitary inversion operator $\pi(w_0)$ as explained below. Appell's hypergeometric functions bridge two models of the minimal representation, namely, the Schrödinger model and the conformal model. All together, we initiate a new line of investigation on various special functions arising from the minimal representation by group theoretic approaches.

We shall also find in Theorem 4.1.1 the integral kernel for $\pi(w_0)$ when restricted to each component of the following decomposition:

$$L^2(C) \simeq \sum_{l,k=0}^{\infty}{}^{\oplus} L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr) \otimes \mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1}),$$

by using the polar coordinate $C \simeq \mathbb{R}_+ \times S^{p-2} \times S^{q-2}$. Here, $\mathcal{H}^l(\mathbb{R}^{p-1})$ is the space of spherical harmonics (see Appendix). It is easy to see that the unitary inversion operator $\pi(w_0)$ stabilizes each component and acts trivially on $\mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1})$. Hence $\pi(w_0)$ induces the integral transform on the function space $L^2(\mathbb{R}_+, r^{p+q-5}dr)$ of one variable, which we denote by $T_{l,k}$. Then, $T_{l,k}$ is a unitary operator depending on p, q and degrees l, k of spherical harmonics. $T_{l,k}$ is nothing but the Hankel transform given by Bessel functions if $\min(p, q) = 2$. For general $p, q \geq 3$, the integral kernel of $T_{l,k}$ is given by Meijer's G-function. We think it is interesting that Meijer's G-functions arise in the representation theory of reductive Lie groups. We note that G-functions G^{20}_{04} solve ordinary differential equations of order four (see (7.6.6)), that is explained from our viewpoint by the fact that the Casimir element acts on $L^2(C)$ as a fourth order differential operator. As a consequence of the fact that $\pi(w_0)$ is a unitary operator of order two, we get the Plancherel and reciprocal formulas for the G-functions. This gives a group theoretic proof of Fox's theorem [12] on G-functions.

1.7 Organization of this paper

This article is organized as follows. After a quick review of the L^2-realization (a generalization of the classic Schrödinger model) of the minimal representation of $O(p, q)$ in Section 2, we find some K-finite vectors on $L^2(C)$ explicitly by means of K-Bessel function $K_\nu(z)$ in Section 3. Section 4 is devoted entirely to the integral formula of the unitary operator $T_{l,k}$ on $L^2(\mathbb{R}_+, r^{p+q-5}dr)$ (see Theorem 4.1.1). In Section 5, by using the integral formula on the Gegenbauer polynomials, we prove our main theorem (see Theorem 5.1.1). In Appendix, we collect the formulas and the properties of various special functions used in this article.

A part of the results here was announced in [32] with a sketch of proof.

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Notation: $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$, $\mathbb{N} := \{0, 1, 2, \cdots\}$.

2 Review of the minimal representation of $O(p,q)$

In this section, we review from [33, 35] two concrete realizations of the minimal representation of the group $G = O(p,q)$, namely, the conformal model $(\varpi^{p,q}, \overline{V^{p,q}})$ by using the Yamabe operator (2.1.2) in Subsection 2.1 and the L^2-model $(\pi, L^2(C))$ (the Schrödinger model) in Subsection 2.2. In the terminology of representation theory of reductive Lie groups (e.g. [27]), the former corresponds to the K-picture, whereas the latter corresponds to the Fourier transform of the N-picture.

The intertwining operator $\mathcal{T}$ between these two models will be given in (2.2.8), which is summarized as the following diagram:

$$\begin{array}{ccc} L^2(C) & \overset{T}{\hookrightarrow} & \mathcal{S}'(\mathbb{R}^{p+q-2}) \\ \mathcal{T} \downarrow & & \uparrow \mathcal{F} \\ K\text{-picture} & \underset{\widetilde{\Psi}^*}{\rightarrow} & N\text{-picture.} \end{array}$$

Here, $\widetilde{\Psi}^*$ is the twisted pull-back for the conformal map $\Psi : \mathbb{R}^{p+q-2} \to S^{p-1} \times S^{q-1}$.

2.1 K-picture — realization via conformal geometry

In this subsection, we give a brief review of the conformal model of the minimal representation of the indefinite orthogonal group $G = O(p,q)$ ($p + q$: even). See [33] for details. See also [29] for an elementary exposition from viewpoints of conformal transformation groups. In the terminology of representation theory, this model corresponds to a subrepresentation of the most degenerate principal series representations (with a very special parameter). See [5, 23, 36] for this approach. The same subrepresentation can be also captured by the theta correspondence arising from the dual pair $O(p,q) \cdot SL(2,\mathbb{R}) \subset Sp(p+q,\mathbb{R})$ (see [24]).

We denote by $\mathbb{R}^{p,q}$ the Euclidean space $\mathbb{R}^{p+q}$ equipped with the pseudo-Riemannian structure $g_{\mathbb{R}^{p,q}}$ of signature (p,q):

$$ds^2 = dx_1^2 + \cdots + dx_p^2 - dy_1^2 - \cdots - dy_q^2.$$

Then, the restriction of ds^2 to the submanifold

$$\begin{aligned} M &:= \{(x,y) \in \mathbb{R}^{p+q} : |x| = |y| = 1,\ x \in \mathbb{R}^p,\ y \in \mathbb{R}^q\} \\ &\simeq S^{p-1} \times S^{q-1} \end{aligned} \tag{2.1.1}$$

is non-degenerate, and defines a pseudo-Riemannian structure on M of signature $(p-1, q-1)$. Here, $|\cdot|$ stands for the usual Euclidean norm. The resulting pseudo-Riemannian structure g_M on M is nothing but the direct product of the standard unit sphere S^{p-1} (positive definite metric) and the unit sphere S^{q-1} equipped with the negative definite metric ($(-1)\times$ the standard metric).

The Yamabe operator $\widetilde{\Delta}_X$ on an n-dimensional Riemannian (or more generally, pseudo-Riemannian) manifold X is defined to be

$$\widetilde{\Delta}_X := \Delta_X + \frac{n-2}{4(n-1)}\kappa \tag{2.1.2}$$

where Δ_X is the Laplace–Beltrami operator, and κ is the scalar curvature of X. The second factor $\frac{n-2}{4(n-1)}\kappa$ acts as a multiplication. In the case $X = M$, the Yamabe operator $\widetilde{\Delta}_M$ of M takes the following form (see [35, (3,4,1)]):

$$\widetilde{\Delta}_M = \Delta_{S^{p-1}} - \Delta_{S^{q-1}} - \left(\frac{p-2}{2}\right)^2 + \left(\frac{q-2}{2}\right)^2, \tag{2.1.3}$$

where $\Delta_{S^{p-1}}$ and $\Delta_{S^{q-1}}$ are the Laplace–Beltrami operators on S^{p-1} and S^{q-1} respectively.

The indefinite orthogonal group $G = O(p, q)$ acts naturally on $\mathbb{R}^{p,q}$ as isometries. This action preserves the cone

$$\Xi := \{(x, y) \in \mathbb{R}^{p,q} : |x| = |y| \neq 0\}$$

but does not preserve M. In order to let G act on M, we set a function ν on $\mathbb{R}^{p,q}$ by

$$\nu : \mathbb{R}^{p,q} \to \mathbb{R}, \quad (x, y) \mapsto |x|.$$

If $v \in M(\subset \Xi)$ and $h \in G$, then $h \cdot v \in \Xi$, and consequently $\frac{h\cdot v}{\nu(h\cdot v)} \in M$. Thus, we can define the action of G on M by

$$L_h : M \to M, \qquad v \mapsto \frac{h \cdot v}{\nu(h \cdot v)} \quad (h \in G).$$

Then, we have $L_h^* g_M = \frac{1}{\nu(h\cdot v)^2} g_M$ at $T_v M$ and thus the diffeomorphism L_h is conformal with respect to the pseudo-Riemannian metric on M. Conversely, any conformal diffeomorphism of M is of the form L_h for some $h \in G$ (see [28, Chapter IV]).

By the general theory of conformal geometry (see [33, Theorem A]), we can construct a representation, denoted by $\varpi^{p,q}$, of G on the solution space to $\widetilde{\Delta}_M$ in $C^\infty(M)$:

$$V^{p,q} := \operatorname{Ker} \widetilde{\Delta}_M = \{f \in C^\infty(M) : \widetilde{\Delta}_M f = 0\},$$

where we set

$$(\varpi^{p,q}(h^{-1})f)(v) := \nu(h \cdot v)^{-\frac{p+q-4}{2}} f(L_h v), \tag{2.1.4}$$

for $h \in G,\ v \in M$, and $f \in V^{p,q}$. The following theorem was proved in [35] in this geometric framework. There are also algebraic proofs (see Remark 2.1.2).

Fact 2.1.1 (see [35, Theorem 3.6.1]). *Let $p, q \geq 2$ and $p+q \geq 6$ be even.*

1) *(irreducibility) $(\varpi^{p,q}, V^{p,q})$ is an irreducible unitary representation of G.*

2) *(unitarizability) There exists a G-invariant inner product $(\ ,\ )_M$ on $V^{p,q}$.*

We write $\overline{V^{p,q}}$ for its Hilbert completion, and use the same letter $\varpi^{p,q}$ to denote the resulting irreducible unitary representation.

3) *(K-type formula) The K-type formula of $(\varpi^{p,q}, \overline{V^{p,q}})$ is given as follows:*

$$\overline{V^{p,q}} \simeq \sum_{\substack{a+\frac{p-q}{2}=b, \\ a,b\in\mathbb{N}}}^{\infty}{}^{\oplus} \mathcal{H}^a(\mathbb{R}^p) \otimes \mathcal{H}^b(\mathbb{R}^q). \tag{2.1.5}$$

4) *(Parseval–Plancherel formula) On each K-type $\mathcal{H}^a(\mathbb{R}^p) \otimes \mathcal{H}^b(\mathbb{R}^q)$ for $(a,b) \in \mathbb{N}^2$ such that $a + \frac{p-q}{2} = b$, or equivalently, $a + \frac{p-2}{2} = b + \frac{q-2}{2}$, the unitary inner product $(\cdot,\cdot)_M$ is of the form:*

$$(F,F)_M = \Big(a + \frac{p-2}{2}\Big) \|F\|^2_{L^2(M)}. \tag{2.1.6}$$

Remark 2.1.2. *Our manifold M is a double cover of the generalized flag variety $G/\overline{P^{\max}}$ by a maximal parabolic subgroup $\overline{P^{\max}}$ (see (2.3.6)). Then, $(\varpi^{p,q}, V^{p,q})$ is identified with a subrepresentation of the degenerate principal series representation induced from a certain one-dimensional representation of $\overline{P^{\max}}$. In this framework, Fact 2.1.1 was proved by Kostant [36] for $p = q = 4$ and by Binegar–Zierau [5], for general p, q satisfying* (1.5.1)*. Huang and Zhu [24] identified this subrepresentation with the local theta correspondence associated to the dual pair $O(p,q) \times SL(2,\mathbb{R})$ in $Sp(p+q,\mathbb{R})$ (to be more precise, its metaplectic cover) and the trivial one-dimensional representation of $SL(2,\mathbb{R})$.*

Remark 2.1.3. *If $p+q \geq 8$, then $\varpi^{p,q}$ becomes a minimal representation in the sense that the annihilator of $\varpi^{p,q}$ in the enveloping algebra is the Joseph ideal (see [5]).*

2.2 L^2-model (the Schrödinger model)

We recall from Introduction that the quadratic form

$$Q(\zeta) := \zeta_1^2 + \cdots + \zeta_{p-1}^2 - \zeta_p^2 - \cdots - \zeta_{p+q-2}^2 \tag{2.2.1}$$

is the defining polynomial of the cone C in $\mathbb{R}^{p+q-2}$. The substitution $\delta(Q)$ of Q into the Dirac delta function δ of one variable defines a distribution on $\mathbb{R}^{p+q-2} \setminus \{0\}$, which is given as the measure on C defined by the volume form $\omega|_C$. Here, ω is an $(p+q-3)$ form such that $dQ \wedge \omega = d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{p+q-2}$ (see [14, Chapter III, Section 2]).

In the polar coordinate:

$$\mathbb{R}_+ \times S^{p-2} \times S^{q-2} \xrightarrow{\sim} C, \quad (r, \omega, \eta) \mapsto \begin{pmatrix} r\omega \\ r\eta \end{pmatrix}, \tag{2.2.2}$$

the distribution $\delta(Q)$ is given by

$$\langle \delta(Q), \varphi \rangle = \frac{1}{2} \int_0^\infty \int_{S^{p-2}} \int_{S^{q-2}} \varphi(\begin{pmatrix} r\omega \\ r\eta \end{pmatrix}) r^{p+q-5} dr d\omega d\eta \tag{2.2.3}$$

for a test function $\varphi \in C_0(C)$. Here, $d\omega$ and $d\eta$ denote the standard measures on S^{p-2} and S^{q-2}, respectively. By this formula, we see that if $p+q > 4$ then $r^{p+q-5}dr$ is locally integrable. Thus, $\delta(Q)$ gives a Schwartz distribution on $\mathbb{R}^{p+q-2}$ of measure class if $p+q > 4$. Obviously, we have

$$\operatorname{supp} \ \delta(Q) = C \cup \{0\}.$$

We shall write $d\mu$ for the measure $\frac{1}{2} r^{p+q-5} dr d\omega d\eta$ on C, and $L^2(C) \equiv L^2(C, d\mu)$ for the Hilbert space consisting of square integrable functions on C. Thus, for a function φ on C,

$$\|\varphi\|_{L^2(C)} = \frac{1}{2} \int_0^\infty \int_{S^{p-2}} \int_{S^{q-2}} |\varphi(r\omega, r\eta)|^2 r^{p+q-5} dr d\omega d\eta. \tag{2.2.4}$$

Correspondingly to the coordinates, we have an isomorphism of Hilbert spaces:

$$L^2(\mathbb{R}_+, \frac{1}{2} r^{p+q-5} dr) \mathbin{\widehat{\otimes}} L^2(S^{p-2}) \mathbin{\widehat{\otimes}} L^2(S^{q-2}) \simeq L^2(C). \tag{2.2.5}$$

If $p+q > 4$, then $u \mapsto u\delta(Q)$ defines a continuous, injective map from the Hilbert space $L^2(C)$ into the space $\mathcal{S}'(\mathbb{R}^{p+q-2})$ of tempered distributions on $\mathbb{R}^{p+q-2}$:

$$T : L^2(C) \to \mathcal{S}'(\mathbb{R}^{p+q-2}), \quad u \mapsto u\delta(Q). \tag{2.2.6}$$

See [35, §3.4]. Following [35, (2.8.2)], we define an injective map by

$$\Psi : \mathbb{R}^{p+q-2} \to M, \qquad z \mapsto \tau(z)^{-1}\iota(z),$$

where for $z = (z', z'') \in \mathbb{R}^{p-1} \oplus \mathbb{R}^{q-1}$ we set

$$\tau(z) := \Big(1 + \big(\frac{|z'| + |z''|}{z}\big)^2\Big)^{\frac{1}{2}} \Big(1 + \big(\frac{|z'| - |z''|}{2}\big)^2\Big)^{\frac{1}{2}},$$

$$\iota : \mathbb{R}^{p+q-2} \to \mathbb{R}^{p+q},\ (z', z'') \mapsto \Big(1 - \frac{|z'|^2 - |z''|^2}{4}, z', z'', 1 + \frac{|z'|^2 - |z''|^2}{4}\Big).$$

Then, Ψ is a conformal map such that $\Psi^* g_M = \tau(z)^{-2} g_{\mathbb{R}^{p-1,q-1}}$. The image M_+ of Ψ is roughly the half of M:

$$M_+ := \{u = (u_0, u', u'', u_{p+q-1}) \in M : u_0 + u_{p+q-1} > 0\}.$$

We note that Ψ induces a conformal compactification of the flat space $\mathbb{R}^{p-1,q-1}$:

$$\mathbb{R}^{p-1,q-1} \hookrightarrow (S^{p-1} \times S^{q-1})/\sim \mathbb{Z}_2,$$

where $\sim \mathbb{Z}_2$ denotes the equivalence relation in $M = S^{p-1} \times S^{q-1}$ defined by $u \sim -u$. The inverse of $\Psi : \mathbb{R}^{p+q-2} \xrightarrow{\sim} M_+$ is given by

$$\Psi^{-1}(u_0, u', u'', u_{p+q-1}) = \Big(\frac{u_0 + u_{p+q-1}}{2}\Big)^{-1}(u', u'').$$

We note that Ψ^{-1} is the ordinary stereographic projection of the sphere S^{p-1} if $q = 1$.

We write $(\tilde{\Psi}^*)^{-1} = \widetilde{(\Psi^{-1})}^*$ for the twisted pull-back (in the sense of [33, Definition 2.3]) of the conformal map $\Psi^{-1} : M_+ \to \mathbb{R}^{p+q-2}$, that is,

$$(\tilde{\Psi}^*)^{-1} : C^\infty(\mathbb{R}^{p+q-2}) \to C^\infty(M_+)$$

is given by

$$(\tilde{\Psi}^*)^{-1}(F)(v) := (\frac{v_0 + v_{p+q-1}}{2})^{-\frac{p+q-4}{2}} F(\frac{2}{v_0 + v_{p+q-1}} \begin{pmatrix} v' \\ v'' \end{pmatrix}), \tag{2.2.7}$$

where $v = {}^t(v_0, v', v'', v_{p+q-1}) \in M$, $v_0, v_{p+q-1} \in \mathbb{R}$, $v' \in \mathbb{R}^{p-1}$, $v'' \in \mathbb{R}^{q-1}$. In the group language, this is the standard intertwining operator from the N-picture to the K-picture.

Now, we are ready to introduce a key map which will give an intertwining operator between the conformal model and the L^2-model.

$$\mathcal{T} := (\tilde{\Psi}^*)^{-1} \circ \mathcal{F}^{-1} \circ T. \tag{2.2.8}$$

Here, we recall from (1.5.7) that $\mathcal{F}^{-1}$ is the inverse Fourier transform on $\mathbb{R}^{p+q-2}$ given by

$$f(\zeta) \mapsto \Big(\frac{1}{2\pi}\Big)^{\frac{p+q-2}{2}} \int_{\mathbb{R}^{p+q-2}} f(\zeta) e^{-\sqrt{-1}\langle z,\zeta\rangle} d\zeta.$$

For $u \in C_0^\infty(C)$, $\mathcal{T}u \in C^\infty(M_+)$. We extend $\mathcal{T}u$ to a function on $M_+ \cup (-M_+)$ by

$$(\mathcal{T}u)(-v) = (-1)^{\frac{p-q}{2}} (\mathcal{T}u)(v) \quad (v \in M_+).$$

We recall from Fact 2.1.1 that the inner product on $\overline{V^{p,q}}$ is given by the formula (2.1.6). Then, the main ingredient of [35, Theorem 4.9] can be restated as:

Fact 2.2.1. *$\mathcal{T}$ extends to a unitary operator (up to scalar) from $L^2(C)$ onto $\overline{V^{p,q}}$.*

Remark 2.2.2. *The definition* (1.5.6) *of the Fourier transform adopted here involves the scalar multiplication by* $(2\pi)^{-\frac{n}{2}}$*. Accordingly, the normalization of $\mathcal{T}$ is different from that of [35] by a scalar multiplication. In our normalization, we have*

$$\|\mathcal{T}u\|^2 = \frac{1}{2}\|u\|^2_{L^2(C)} \qquad (u \in L^2(C))$$

as we shall observe in (3.1.10).

Through the unitary operator $\mathcal{T}$, we can transfer the unitary representation $(\varpi^{p,q}, \overline{V^{p,q}})$ of $G = O(p,q)$ to a unitary representation on the Hilbert space $L^2(C)$ by

$$\pi^{p,q}(g) := \mathcal{T}^{-1} \circ \varpi^{p,q}(g) \circ \mathcal{T}, \quad g \in G. \tag{2.2.9}$$

Hereafter we shall write π for $\pi^{p,q}$ for simplicity. Then, π is irreducible because so is $\varpi^{p,q}$ (see Subsection 2.1). We note that the unitary inner product of π is nothing but the L^2-inner product of $L^2(C)$. Naming after the classical Schrödinger model $L^2(\mathbb{R}^n)$ for the Weil representation of the metaplectic group (e.g. [11]), we shall say the resulting irreducible unitary representation $(\pi, L^2(C))$ is the *Schrödinger model* for the minimal representation of $G = O(p,q)$.

We have explained two models of $(\varpi^{p,q}, \overline{V^{p,q}})$ and $(\pi, L^2(C))$ for the minimal representation of G. In the realization of $\overline{V^{p,q}}$, the K-structure is very clear to see, while on $L^2(C)$, it is not clear *a priori*. If it is possible to compute the intertwining operator $\mathcal{T}$ (see (2.2.8)) explicitly, then we can transfer the information from one to the other. Along this line, we shall explicitly find in the next Section 3 explicit K-finite vectors of $L^2(C)$ by computing the Hankel transform of the K-Bessel functions.

2.3 Explicit formulas for the group action on $L^2(C)$

In this subsection, we summarize the known results on the Schrödinger model of the minimal representation $(\pi, L^2(C))$ of G.

First, we define subgroups $M_+^{\max}, M^{\max}$, a maximal compact subgroup K, and a compact subgroup K' of $G = O(p,q)$ as follows:

$$\begin{aligned}
m_0 &:= -I_{p+q}, \\
M_+^{\max} &:= \{g \in G : g \cdot e_0 = e_0,\ g \cdot e_{p+q-1} = e_{p+q-1}\} && \simeq O(p-1, q-1), \\
M^{\max} &:= M_+^{\max} \cup m_0 M_+^{\max} && \simeq O(p-1, q-1) \times \mathbb{Z}_2, \\
K &:= G \cap O(p+q) && \simeq O(p) \times O(q), \\
K' &:= K \cap M_+^{\max} && \simeq O(p-1) \times O(q-1).
\end{aligned}$$

We note that the group $M_+^{\max}$ acts on the cone C transitively, and leaves the measure $d\mu$ (see Subsection 2.2) invariant.

Next we set

$$\epsilon_j := \begin{cases} 1 & (1 \le j \le p-1), \\ -1 & (p \le j \le p+q-2). \end{cases}$$

Let $N_j, \overline{N}_j$ $(1 \le j \le p+q-2)$ and E be elements of the Lie algebra $\mathfrak{g} = \mathfrak{o}(p,q)$ given by

$$\begin{aligned}
N_j &:= E_{j,0} - E_{j,p+q-1} - \epsilon_j E_{0,j} - \epsilon_j E_{p+q-1,j}, && (2.3.1)\\
\overline{N}_j &:= E_{j,0} + E_{j,p+q-1} - \epsilon_j E_{0,j} + \epsilon_j E_{p+q-1,j}, && (2.3.2)\\
E &:= E_{0,p+q-1} + E_{p+q-1,0}.
\end{aligned}$$

Then, we define abelian Lie subgroups $N^{\max}, \overline{N^{\max}}$ and A by

$$\begin{aligned}
&\mathbb{R}^{p+q-2} \simeq N^{\max}, \quad a = (a_1, a_2, \cdots, a_{p+q-2}) \mapsto n_a := \exp(\sum_{j=1}^{p+q-2} a_j N_j), \\
&\mathbb{R}^{p+q-2} \simeq \overline{N^{\max}}, \quad a = (a_1, a_2, \cdots, a_{p+q-2}) \mapsto \bar{n}_a := \exp(\sum_{j=1}^{p+q-2} a_j \overline{N}_j), \qquad (2.3.3)\\
&A := \exp(\mathbb{R}E).
\end{aligned}$$

Then, we have

$$M_+^{\max} N^{\max} = \{g \in G : g(e_0 + e_{p+q-1}) = e_0 + e_{p+q-1}\} \qquad (2.3.4)$$

and a diffeomorphism

$$G/M_+^{\max}N^{\max} \simeq \{(\zeta_0, \cdots, \zeta_{p+q-1}) \in \mathbb{R}^{p+q} : \sum_{j=0}^{p-1} \zeta_j^2 - \sum_{j=p}^{p+q-1} \zeta_j^2 = 0\}. \quad (2.3.5)$$

The Lie algebras of the above subgroups will be denoted by their corresponding German lowercase letters.

We review from [35, §3.3] how G acts on $L^2(C)$. The action of the maximal parabolic subgroup

$$\overline{P^{\max}} := M^{\max}A\overline{N^{\max}} \quad (2.3.6)$$

on $L^2(C)$ is described explicitly as follows:

$$(\pi(m)\psi)(\zeta) = \psi({}^t m\zeta) \qquad (m \in M_+^{\max}), \quad (2.3.7)$$

$$(\pi(m_0)\psi)(\zeta) = (-1)^{\frac{p-q}{2}}\psi(\zeta), \quad (2.3.8)$$

$$(\pi(e^{tE})\psi)(\zeta) = e^{-\frac{p+q-4}{2}t}\psi(e^{-t}\zeta) \qquad (t \in \mathbb{R}), \quad (2.3.9)$$

$$(\pi(\overline{n}_a)\psi)(\zeta) = e^{2\sqrt{-1}(a_1\zeta_1+\cdots+a_{p+q-2}\zeta_{p+q-2})}\psi(\zeta) \qquad (a \in \mathbb{R}^{p+q-2}). \quad (2.3.10)$$

Here, we remark that the maximal parabolic subgroup $\overline{P^{\max}}$ of G plays an analogous role to the Siegel parabolic subgroup P_{Siegel} of the metaplectic group $G' = Mp(n,\mathbb{R})$. $L := M^{\max}A$ is a Levi subgroup of $\overline{P^{\max}}$.

It follows from (2.3.10) that the differential action of $\overline{N^{\max}}$ is given as

$$d\pi(\overline{N}_j) = 2\sqrt{-1}\zeta_j \qquad (1 \le j \le p+q-2). \quad (2.3.11)$$

On the other hand, the $N^{\max}$-action on $L^2(C)$ is not simple to describe. Reflecting the fact that the action $\overline{P^{\max}}$ on C does not extend to G, the differential action of $N^{\max}$ on $L^2(C)$ is given not as vector fields but as differential operators of second order. To describe its explicit form, it is convenient to write the differential operator in the ambient space $\mathbb{R}^{p+q-2}$ by using the inclusion map $T : L^2(C) \hookrightarrow \mathcal{S}'(\mathbb{R}^{p+q-2}), u \mapsto u\delta(Q)$ (see (2.2.6)). Then the differential action $d\pi(N_j)$ $(1 \le j \le p+q-2)$ is characterized by the commutative diagram:

$$\begin{array}{ccc} L^2(C)_K & \xrightarrow{T} & \mathcal{S}'(\mathbb{R}^{p+q-2}) \\ {\scriptstyle d\pi(N_j)}\big\downarrow & & \big\downarrow{\scriptstyle D_j} \\ L^2(C)_K & \xrightarrow{T} & \mathcal{S}'(\mathbb{R}^{p+q-2}). \end{array} \quad (2.3.12)$$

Here, D_j is a differential operator on $\mathbb{R}^{p+q-2}$ given in [35, Lemma 3.2] as follows (in the notation loc. cit., $D_j = d\hat{\varpi}_{\lambda,\epsilon}(N_j)$ with $\lambda = \frac{p+q-4}{2}$):

$$D_j = \sqrt{-1}\Big(-\frac{p+q}{2}\epsilon_j\frac{\partial}{\partial\zeta_j} - (\sum_{k=0}^{p+q-2}\zeta_k\frac{\partial}{\partial\zeta_k})\epsilon_j\frac{\partial}{\partial\zeta_j} + \frac{1}{2}\zeta_j(\sum_{k=1}^{p+q-2}\epsilon_k\frac{\partial^2}{\partial\zeta_k^2})\Big). \tag{2.3.13}$$

2.4 The conformal inversion w_0

In this subsection, we list some important features of the element

$$w_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

I0 (*Involution*) The element w_0 is of order two. Therefore, w_0 acts as an involution for any action (in particular, any representation) of G.

I1 (*Bruhat decomposition*) Retain the notation as in Subsection 2.3. Then,

$$\mathrm{Ad}(w_0)E = -E, \tag{2.4.1}$$

and therefore $w_0|_{\mathfrak{a}} = -\mathrm{id}$. We see also readily from (2.3.1) and (2.3.2) that

$$\mathrm{Ad}(w_0)\overline{N}_j = \epsilon_j N_j \qquad (1 \le j \le p+q-2), \tag{2.4.2}$$

and therefore

$$\mathrm{Ad}(w_0)\mathfrak{n}^{\max} = \overline{\mathfrak{n}^{\max}}. \tag{2.4.3}$$

On the group level, we have the following Bruhat decomposition of G:

$$G = \overline{P^{\max}}w_0\overline{P^{\max}} \amalg \overline{P^{\max}}. \tag{2.4.4}$$

I2 (*Jordan algebras*) Let $\mathbb{R}^{p-1,q-1} \simeq \mathbb{R}^{p+q-2}$ be the semisimple Jordan algebra with indefinite quadratic form $Q(\zeta)$ (see (1.3.1)). This Jordan algebra is euclidean if $\min(p,q) = 2$, and non-euclidean if $p, q > 2$. The conformal group (*Kantor–Koecher–Tits group*) of $\mathbb{R}^{p-1,q-1}$ is nothing but the group $G = O(p,q)$, and the action of the element w_0 on $\mathbb{R}^{p-1,q-1}$ corresponds to the *conformal inversion* $\zeta \mapsto -\zeta^{-1}$ (see [46]). Thus, hereafter we shall call w_0 the *conformal inversion.* The structure group

$$L_+ := M_+^{\max}A \tag{2.4.5}$$

acts on $\mathbb{R}^{p-1,q-1}$ as $x \mapsto e^t m^{-1}x$ for $(m, e^{tE}) \in M_+^{\max} \times A$, and on its dual space by

$$\zeta \mapsto e^{-t}\,{}^t m\zeta, \quad (m, e^{tE}) \in M_+^{\max} \times A.$$

I3 (*Restricted root system*) Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition and $\mathfrak{b}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Since $\mathrm{Ad}(w_0)$ acts on $\mathfrak{p}$ as $-\mathrm{id}$, w_0 acts on the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{b})$ as $-\mathrm{id}$. This coincides with the longest element in the Weyl group $W(\Sigma(\mathfrak{g}, \mathfrak{b}))$ except for the case $p = q$ is odd, where

$$w_0|_{\mathfrak{b}} = -\mathrm{id} \notin W(\Sigma(\mathfrak{g}, \mathfrak{b})) \simeq \mathfrak{S}_q \ltimes (\mathbb{Z}/2\mathbb{Z})^{q-1} \quad (\text{type } D_q).$$

I4 (*center of K*) w_0 lies in the center of K.

I5 (*The action on the minimal representation*) In the conformal model $(\varpi^{p,q}, \overline{V^{p,q}})$ (see Section 2.1),

$$(\varpi^{p,q}(w_0)h)(v', v'') = h(v', -v'')$$

by (2.1.4). In the Schrödinger model $(\pi, L^2(C))$,

$$\pi(w_0) = \mathcal{T}^{-1} \circ \varpi^{p,q}(w_0) \circ \mathcal{T}$$

by (2.2.9). However, it is not a simple task (as far as we understand) to find the formulas (1.4.1)–(1.4.3) from the definition of $\mathcal{T}$.

3 K-finite eigenvectors in $L^2(C)$

In the conformal model (see Subsection 2.1), we can find readily explicit K-finite vectors. However, it is far from being obvious to find explicit forms of K-finite vectors in the L^2-model for the minimal representation. In this section, generalizing the idea of [35, Theorem 5.8] that dealt with the explicit form of the minimal K-type, we find explicit vectors in $L^2(C)$ for *every* K-type, by computing the integral operator $\mathcal{T} : L^2(C) \xrightarrow{\sim} \overline{V^{p,q}}$ (see Fact 2.2.1).

3.1 Result of this section

Throughout this section, we assume $p \geq q \geq 2$ and $p+q \geq 6$. For $(l,k) \in \mathbb{N}^2$, we consider the following two (non-exclusive) cases:

$$\boxed{\begin{array}{l} \text{Case 1}: \ \frac{p-q}{2} + l - k \geq 0, \\ \text{Case 2}: \ \frac{p-q}{2} + l - k \leq 0. \end{array}} \tag{3.1.1}$$

The case $\frac{p-q}{2} + l - k = 0$ belongs to both Cases 1 and 2. This convention will be convenient later because all the formulas below are the same for both Cases 1 and 2 if (l,k) satisfies $\frac{p-q}{2} + l - k = 0$.

For $(l,k) \in \mathbb{N}^2$, we define real analytic functions $f_{l,k}$ on $\mathbb{R}_+$ by

$$f_{l,k}(r) := \begin{cases} r^{-\frac{q-3}{2}+l} K_{\frac{q-3}{2}+k}(2r) & \text{Case 1}, \\ r^{-\frac{p-3}{2}+k} K_{\frac{p-3}{2}+l}(2r) & \text{Case 2}, \end{cases} \tag{3.1.2}$$

$$= r^{l+k} \times \begin{cases} \widetilde{K}_{\frac{q-3}{2}+k}(2r) & \text{Case 1}, \\ \widetilde{K}_{\frac{p-3}{2}+l}(2r) & \text{Case 2}. \end{cases} \tag{3.1.3}$$

Here, $K_\nu(z)$ is the K-Bessel function, i.e., the modified Bessel function of the second kind (see Appendix 7.2) and $\widetilde{K}_\nu(z) = (\frac{z}{2})^{-\nu} K_\nu(z)$ is the normalized K-Bessel function (see (7.2.6)).

By using the polar coordinate (2.2.2), we define a linear subspace $H_{l,k}$ of $C^\infty(C)$ consisting of linear combinations of the following functions:

$$f_{l,k}(r)\phi(\omega)\psi(\eta) \quad (\phi \in \mathcal{H}^l(\mathbb{R}^{p-1}), \psi \in \mathcal{H}^k(\mathbb{R}^{q-1})). \tag{3.1.4}$$

We recall from Subsection 2.3 that there are two key compact subgroups for the analysis on the minimal representation $L^2(C)$:

$$\begin{aligned} K & \simeq O(p) \times O(q), \\ K' = K \cap M_+^{\max} & \simeq O(p-1) \times O(q-1). \end{aligned}$$

We note that the K'-action on $L^2(C)$ is just the pull-back of the K'-action on C (see (2.3.7)), but the K-action on $L^2(C)$ is more complicated because K cannot act on C. Then, here is our main result of this section:

Theorem 3.1.1. *For each pair $(l,k) \in \mathbb{N}^2$, we have*

1) *(asymptotic behavior) $H_{l,k} \subset L^2(C)$ for any $l,k \in \mathbb{N}$.*

2) *(K-type and K'-type) $H_{l,k} \simeq \mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1})$ as a K'-module. Furthermore, $H_{l,k}$ belongs to the K-type $\mathcal{H}^{a(l,k)}(\mathbb{R}^p) \otimes \mathcal{H}^{a(l,k)+\frac{p-q}{2}}(\mathbb{R}^q)$ of $L^2(C)$. Here, we define a non-negative integer $a(l,k)$ by*

$$a(l,k) := \max(l, k - \frac{p-q}{2}) = \begin{cases} l & \textit{Case 1,} \\ k - \frac{p-q}{2} & \textit{Case 2.} \end{cases} \tag{3.1.5}$$

3) *(eigenspace of $\pi(w_0)$) $\pi(w_0)$ acts on $H_{l,k}$ by the scalar $(-1)^{a(l,k)+\frac{p-q}{2}}$.*

4) *(intertwining operator) Fix any $\phi \in \mathcal{H}^l(\mathbb{R}^{p-1}), \psi \in \mathcal{H}^k(\mathbb{R}^{q-1})$, and set*

$$u_{l,k}(r\omega, r\eta) := f_{l,k}(r)\phi(\omega)\psi(\eta) \in H_{l,k}. \tag{3.1.6}$$

Then $\mathcal{T} : L^2(C) \to \overline{V^{p,q}}$ has the following form on the subspace $H_{l,k}$:

$$\mathcal{T}u_{l,k} = c_{l,k} I^p_{l \to a(l,k)}(\phi) I^q_{k \to a(l,k)+\frac{p-q}{2}}(\psi), \tag{3.1.7}$$

where $I^m_{i \to j} : \mathcal{H}^i(\mathbb{R}^{m-1}) \to \mathcal{H}^j(\mathbb{R}^m) (0 \le i \le j)$ is an $O(m-1)$-homomorphism defined in Fact 7.5.1, and the constant $c_{l,k}$ is given by

$$c_{l,k} := \frac{\sqrt{-1}^{l+k}}{\sqrt{\pi}} \times \begin{cases} \dfrac{\Gamma(\frac{p-q}{2} + l - k + 1)}{2^{\frac{p-q}{2}+l-k+3}\Gamma(\frac{p-2}{2} + l)} & \textit{Case 1,} \\ \dfrac{\Gamma(-\frac{p-q}{2} - l + k + 1)}{2^{-(\frac{p-q}{2}+l-k)+3}\Gamma(\frac{q-2}{2} + k)} & \textit{Case 2.} \end{cases} \tag{3.1.8}$$

5) *(L^2-norm) For any $(l,k) \in \mathbb{N}^2$,*

$$\|\mathcal{T}u_{l,k}\|^2_{L^2(M)} = \frac{1}{2a(l,k) + p - 2} \|u_{l,k}\|^2_{L^2(C)}. \tag{3.1.9}$$

By using the unitary inner product $(\ ,\ )_M$ (see (2.1.6)) for the conformal model $(\varpi^{p,q}, \overline{V^{p,q}})$, the formula (3.1.9) can be restated as

$$(\mathcal{T}u_{l,k}, \mathcal{T}u_{l,k})_M = \frac{1}{2}\|u_{l,k}\|^2_{L^2(C)}. \tag{3.1.10}$$

Remark 3.1.2. *Theorem 3.1.1 (1) for $l = k = 0$ was proved in [35, Theorem 5.8]. Since $p \geq q$, we are dealing with Case 1 if $l = k = 0$ and $a(0,0) = 0$. In this particular case, Theorem 3.1.1(2) asserts that $f_{0,0}(r) = r^{-\frac{q-3}{2}} K_{\frac{q-3}{2}}(2r)$ belongs to the minimal K-type $\mathcal{H}^0(\mathbb{R}^p) \otimes \mathcal{H}^{\frac{p-q}{2}}(\mathbb{R}^q)$ of $(\pi, L^2(C))$.*

Remark 3.1.3. *For $q = 2$, $\mathcal{H}^k(\mathbb{R}^{q-1})$ is non-zero only if $k = 0$ or 1 (see Appendix 7.5). Thus our assumption $p + q \geq 6$ combined with $q = 2$ and $k = 0, 1$ implies $\frac{p-q}{2} + l - k = \frac{p+q}{2} - 2 + l - k \geq 0$. Hence, (l, k) belongs automatically to Case 1. In this case, $f_{l,0}(r) = f_{l,1}(r) = \frac{\sqrt{\pi}}{2} r^l e^{-2r}$ with the notation here coincides with the function $f_{l,l}(r)$ with the notation in [31, Proposition 3.4] up to a constant multiple.*

Our method is based on the technique used in [35, §5.6, §5.7]. The key lemma is Lemma 7.8.1, which gives a formula of the Hankel transform of the K-Bessel function with trigonometric parameters by means of the Gegenbauer polynomial.

The subspace $\bigoplus_{l,k \in \mathbb{N}} H_{l,k}$ is not dense in $L^2(C)$, but is large enough (see Subsection 3.2) that we can make use of Theorem 3.1.1 for the proof of Theorem 4.1.1 in Section 4 (see also Subsection 4.1 for its idea).

3.2 The subspace $H_{l,k}$

The subspace $\bigoplus_{l,k \in \mathbb{N}} H_{l,k}$ is not dense in $L^2(C)$, but serves as a 'skeleton'. In this subsection, we try to clarify its meaning.

We begin with the branching law $G \downarrow K$ (see (2.1.5)) and $K \downarrow K'$ (K' denotes $K' := K \cap M_+^{\max} \simeq O(p-1) \times O(q-1)$):

$$L^2(C)_K \simeq \bigoplus_{a=0}^{\infty} \mathcal{H}^a(\mathbb{R}^p) \otimes \mathcal{H}^{a+\frac{p-q}{2}}(\mathbb{R}^q) \tag{3.2.1}$$

$$\simeq \bigoplus_{a=0}^{\infty} \bigoplus_{l=0}^{a} \bigoplus_{k=0}^{a+\frac{p-q}{2}} \mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1}). \tag{3.2.2}$$

The irreducible decomposition (3.2.1) shows that $L^2(C)_K$ is multiplicity-free as a K-module. Hereafter, we identify the K-module $\mathcal{H}^a(\mathbb{R}^p) \otimes \mathcal{H}^{a+\frac{p-q}{2}}(\mathbb{R}^q)$ with the corresponding subspace of $L^2(C)_K$. Then we observe

S1 (K'-*type*) Fix a pair $(l, k) \in \mathbb{N}^2$. In light of (3.2.1) and (3.2.2), the K'-type $\mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1})$ occurs in the K-module $\mathcal{H}^a(\mathbb{R}^p) \otimes \mathcal{H}^{a+\frac{p-q}{2}}(\mathbb{R}^q)$

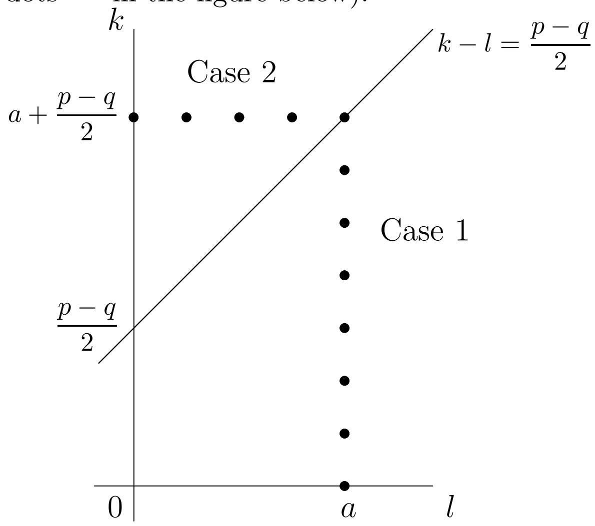
if and only if $a \geq a(l,k)$. Further, $H_{l,k}$ is characterized as a subspace of $L^2(C)$ satisfying the following two conditions:

$$\begin{cases} W \simeq \mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1}) \text{ as } K'\text{-modules}, \\ W \subset \mathcal{H}^{a(l,k)}(\mathbb{R}^p) \otimes \mathcal{H}^{a(l,k)+\frac{p-q}{2}}(\mathbb{R}^q). \end{cases}$$

S2 (K*-type*) Fix $a \in \mathbb{N}$. Then for $(l,k) \in \mathbb{N}^2$,

$$\left(\bigoplus_{l,k \in \mathbb{N}} H_{l,k} \right) \cap \left(\mathcal{H}^a(\mathbb{R}^p) \otimes \mathcal{H}^{a+\frac{p-q}{2}}(\mathbb{R}^q) \right)$$

is non-zero if and only if at least one of l and k attains its maximum in the set $\{(l',k') \in \mathbb{N}^2 : 0 \leq l' \leq a,\ 0 \leq k' \leq a + \frac{p-q}{2}\}$ or equivalently, in the set of the K'-types (l',k') occurring in $\mathcal{H}^a(\mathbb{R}^p) \otimes \mathcal{H}^{a+\frac{p-q}{2}}(\mathbb{R}^q)$ (see the black dots $\bullet$ in the figure below).



3.3 Integral formula for the intertwiner

Before proving Theorem 3.1.1, we prepare, in this subsection, the explicit integral formula for the G-isomorphism $\mathcal{T} : L^2(C) \to \overline{V^{p,q}} (\subset L^2(M))$ (see Fact 2.2.1).

We write $v \in M = S^{p-1} \times S^{q-1} \subset \mathbb{R}^{p+q}$ as

$$v = {}^t(v_0, v', v'', v_{p+q-1}), \qquad v_0, v_{p+q-1} \in \mathbb{R},\ v' \in \mathbb{R}^{p-1},\ v'' \in \mathbb{R}^{q-1} \quad (3.3.1)$$

satisfying

$$v_0^2 + |v'|^2 = |v''|^2 + v_{p+q-1}^2 = 1.$$

Lemma 3.3.1. *Suppose that $u \in L^2(C)$ is of the form*

$$u(r\omega, r\eta) = f(r)\phi(\omega)\psi(\eta),$$

for $\phi \in \mathcal{H}^l(\mathbb{R}^{p-1})$, $\psi \in \mathcal{H}^k(\mathbb{R}^{q-1})$, and $f \in L^2(\mathbb{R}_+, r^{p+q-5}dr)$ with regard to the polar coordinate (2.2.2). Then, $\mathcal{T}u$ is reduced to the following integral transform of one variable: for $v \in M$ such that $v_0 + v_{p+q-1} > 0$,

$$\begin{aligned}(\mathcal{T}u)(v) &= e^{-\frac{\sqrt{-1}(l+k)\pi}{2}} \frac{|v'|^{-\frac{p-3}{2}}|v''|^{-\frac{q-3}{2}}}{v_0 + v_{p+q-1}} \phi\left(\frac{v'}{|v'|}\right)\psi\left(\frac{v''}{|v''|}\right) \\ &\times \int_0^\infty f(r) J_{\frac{p-3}{2}+l}\left(\frac{2|v'|r}{v_0+v_{p+q-1}}\right) J_{\frac{q-3}{2}+k}\left(\frac{2|v''|r}{v_0+v_{p+q-1}}\right) r^{\frac{p+q-4}{2}} dr, \quad (3.3.2)\end{aligned}$$

where $J_\nu(z)$ denotes the Bessel function.

Proof. As $\mathcal{T} = (\widetilde{\Psi}^*)^{-1} \circ \mathcal{F}^{-1} \circ T$, we begin with the computation of $(\mathcal{F}^{-1} \circ Tu)(z)$ for $z \in \mathbb{R}^{p+q-2}$. By the formula (2.2.3) and (2.2.6) for $Tu = u\delta(Q)$, we have

$$\begin{aligned} & 2(\mathcal{F}^{-1} \circ Tu)(z) \\ &= (2\pi)^{-\frac{p+q-2}{2}} \int_0^\infty \int_{S^{p-2}} \int_{S^{q-2}} f(r)\phi(\omega)\psi(\eta) e^{-\sqrt{-1}(\langle z', r\omega\rangle + \langle z'', r\eta\rangle)} r^{p+q-5} dr d\omega d\eta \\ &= \sqrt{-1}^{-(l+k)} |z'|^{-\frac{p-3}{2}} |z''|^{-\frac{q-3}{2}} \phi\left(\frac{z'}{|z'|}\right)\psi\left(\frac{z''}{|z''|}\right) \\ &\qquad \times \int_0^\infty f(r) J_{\frac{p-3}{2}+l}(r|z'|) J_{\frac{q-3}{2}+k}(r|z''|) r^{\frac{p+q-4}{2}} dr. \quad (3.3.3)\end{aligned}$$

Here, in (3.3.3), we used the following formula (see e.g. [20, Introduction, Lemma 3.6]):

$$\int_{S^{m-1}} e^{\sqrt{-1}\lambda\langle\eta,\omega\rangle}\phi(\omega)d\omega = (2\pi)^{\frac{m}{2}} \sqrt{-1}^l \phi(\eta) \frac{J_{l-1+\frac{m}{2}}(\lambda)}{\lambda^{\frac{m}{2}-1}}. \quad (3.3.4)$$

Finally, by the definition (2.2.7) of the pull-back $(\tilde{\Psi}^*)^{-1}$, we get the desired result (3.3.2). □

3.4 K-finite vectors $f_{l,k}$ in $L^2(C)$

In this section, we prove basic properties of the real analytic functions $f_{l,k}$ defined in (3.1.2).

Lemma 3.4.1. *$f_{l,k} \in L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$.*

Proof. The K-Bessel function decays exponentially at infinity. The asymptotic formula (see Fact 7.2.1 (2) in Appendix) implies

$$f_{l,k}(r) \sim \begin{cases} c\, r^{-\frac{q-2}{2}+l} e^{-2r} & \text{Case 1,} \\ c\, r^{-\frac{p-2}{2}+k} e^{-2r} & \text{Case 2,} \end{cases} \tag{3.4.1}$$

as $r \to \infty$. On the other hand, since $\widetilde{K}_\nu(r) = O(r^{-2\nu})$ as r tends to 0 (see (7.2.10)),

$$f_{l,k} = \begin{cases} O(r^{l-k-q+3}) & \text{Case 1,} \\ O(r^{-l+k-p+3}) & \text{Case 2,} \end{cases} \tag{3.4.2}$$

as r tends to 0. In either case, $f_{l,k} = O(r^{\frac{-p-q}{2}+3})$ by the definition (3.1.1) of Cases 1 and 2. Hence, we have $f_{l,k} \in L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$ for $p+q > 4$. $\square$

The explicit formula of the L^2-norm of $f_{l,k}$ is obtained by the integration formula (7.2.13) of K-Bessel functions as follows:

Proposition 3.4.2.

$$\|f_{l,k}\|^2_{L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)} = \begin{cases} \dfrac{\Gamma(\frac{p-1}{2}+l)^2\Gamma(\frac{p+q-4}{2}+l+k)\Gamma(\frac{p-q+2}{2}+l-k)}{16\Gamma(p-1+2l)} & \textit{Case 1,} \\ \dfrac{\Gamma(\frac{q-1}{2}+k)^2\Gamma(\frac{p+q-4}{2}+k+l)\Gamma(\frac{q-p+2}{2}+k-l)}{16\Gamma(q-1+2k)} & \textit{Case 2.} \end{cases} \tag{3.4.3}$$

Lemma 3.4.3.

$$\int_0^\infty r^{\frac{p+q-6}{2}+\sqrt{-1}\zeta} f_{l,k}(r)dr = \begin{cases} \frac{1}{4}\Gamma(\frac{p+q-4}{4}+\frac{l+k+\sqrt{-1}\zeta}{2})\Gamma(\frac{p-q}{4}+\frac{l-k+1+\sqrt{-1}\zeta}{2}) & \textit{Case 1,} \\ \frac{1}{4}\Gamma(\frac{p+q-4}{4}+\frac{k+l+\sqrt{-1}\zeta}{2})\Gamma(\frac{q-p}{4}+\frac{k-l+1+\sqrt{-1}\zeta}{2}) & \textit{Case 2.} \end{cases}$$

Proof. Apply the formula (7.2.11) of the Mellin transform of K-Bessel functions. $\square$

In order to compute $\mathcal{T}u_{l,k}$ explicitly by using the integral formula (3.3.2), we need another lemma:

Lemma 3.4.4. *For a pair* $(l,k) \in \mathbb{N}^2$*, let* $f_{l,k}$ *be the function on* C *defined in* (3.1.2)*. With respect to the coordinate* ${}^t(v_0, v', v'', v_{p+q-1}) \in M \subset \mathbb{R}^{p+q}$ *(see* (3.3.1)*), the integral*

$$\int_0^\infty f_{l,k}(r) J_{\frac{p-3}{2}+l}\left(\frac{2|v'|r}{v_0+v_{p+q-1}}\right) J_{\frac{q-3}{2}+k}\left(\frac{2|v''|r}{v_0+v_{p+q-1}}\right) r^{\frac{p+q-4}{2}} dr \quad (3.4.4)$$

is equal to:

$$\frac{\Gamma(\frac{p-q}{2}+l-k+1)}{2^{\frac{p-q}{2}+l-k+3}\sqrt{\pi}}(v_0+v_{p+q-1})|v'|^{\frac{p-3}{2}+l}|v''|^{\frac{q-3}{2}+k}\widetilde{C}^{\frac{q-2}{2}+k}_{\frac{p-q}{2}+l-k}(v_{p+q-1}) \quad \textit{Case 1},$$

$$\frac{\Gamma(-\frac{p-q}{2}-l+k+1)}{2^{-\frac{p-q}{2}-l+k+3}\sqrt{\pi}}(v_0+v_{p+q-1})|v'|^{\frac{p-3}{2}+l}|v''|^{\frac{q-3}{2}+k}\widetilde{C}^{\frac{p-2}{2}+l}_{-(\frac{p-q}{2}+l-k)}(v_0) \quad \textit{Case 2.}$$

Proof of Lemma 3.4.4. We treat Case 1 first. By the change of variables $t := 2r$, the integral (3.4.4) amounts to

$$\frac{1}{2^{\frac{p+1}{2}+l}}\int_0^\infty t^{\frac{p-1}{2}+l} J_{\frac{p-3}{2}+l}\left(\frac{|v'|t}{v_0+v_{p+q-1}}\right) J_{\frac{q-3}{2}+k}\left(\frac{|v''|t}{v_0+v_{p+q-1}}\right) K_{\frac{q-3}{2}+k}(t)\, dt.$$

Applying Lemma 7.8.1 with

$$\mu := \frac{p-3}{2}+l, \quad \nu := \frac{q-3}{2}+k, \quad \cos\theta := v_0, \quad \cos\phi := v_{p+q-1},$$

we get the formula in Case 1.

The proof for Case 2 goes similarly. In this case, the integral amounts to

$$\frac{1}{2^{\frac{q+1}{2}+k}}\int_0^\infty t^{\frac{q-1}{2}+k} J_{\frac{p-3}{2}+l}\Big(\frac{|v'|t}{v_0+v_{p+q-1}}\Big) J_{\frac{q-3}{2}+k}\Big(\frac{|v''|t}{v_0+v_{p+q-1}}\Big) K_{\frac{p-3}{2}+l}(t)dt$$

by the change of variables $t := 2r$. Now, we substitute $\mu := \frac{q-3}{2}+k$, $\nu := \frac{p-3}{2}+l$, $\cos\theta := v_{p+q-1}$, and $\cos\phi := v_0$ into (7.8.1). $\square$

3.5 Proof of Theorem 3.1.1

In this subsection, we complete the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. 1) By the isomorphism

$$L^2(C) \simeq L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr) \mathbin{\widehat{\otimes}} L^2(S^{p-2}) \mathbin{\widehat{\otimes}} L^2(S^{q-2}) \quad \text{(see (2.2.5))}$$

in the polar coordinate, the first statement follows immediately from Lemma 3.4.1.

4) By (3.3.2) and Lemma 3.4.4, we have

$$\mathcal{T}u_{l,k}(v) = \begin{cases} c_{l,k}\Gamma(\frac{p-2}{2}+l)|v'|^l\phi(\frac{v'}{|v'|})|v''|^k\psi(\frac{v''}{|v''|})\widetilde{C}^{\frac{q-2}{2}+k}_{\frac{p-q}{2}+l-k}(v_{p+q-1}) & \text{Case 1,} \\ c_{l,k}\Gamma(\frac{q-2}{2}+k)|v'|^l\phi(\frac{v'}{|v'|})\widetilde{C}^{\frac{p-2}{2}+l}_{-(\frac{p-q}{2}+l-k)}(v_0)|v''|^k\psi(\frac{v''}{|v''|}) & \text{Case 2,} \end{cases}$$

where $c_{l,k}$ is the constant defined in (3.1.8) and $v = {}^t(v_0, v', v'', v_{p+q-1}) \in S^{p-1} \times S^{q-1}$. Now we use the definition (7.5.1) that

$$I^m_{i\to j}(\phi)(x_0, x') = |x'|^i\phi\Big(\frac{x'}{|x'|}\Big)\widetilde{C}^{\frac{m-2}{2}+i}_{j-i}(x_0)$$

for $(x_0, x') \in S^{m-1}$, and in particular, for $i = j$,

$$I^m_{j\to j}(\phi)(x_0, x') = \Gamma\Big(\frac{m-2}{2}+j\Big)|x'|^j\phi\Big(\frac{x'}{|x'|}\Big).$$

See (7.5.5). Thus, the formula (3.1.7) follows.

2) The first statement is obvious. Since $I^m_{i\to j}$ maps $\mathcal{H}^i(\mathbb{R}^{m-1})$ to $\mathcal{H}^j(\mathbb{R}^m)$ (see Fact 7.5.1), (3.1.7) implies

$$\mathcal{T}H_{l,k} \subset \mathcal{H}^{a(l,k)}(\mathbb{R}^p) \otimes \mathcal{H}^{a(l,k)+\frac{p-q}{2}}(\mathbb{R}^q).$$

Hence, we have proved the second statement.

3) By (2.1.4), the unitary inversion operator $\varpi^{p,q}(w_0)$ on $\overline{V^{p,q}}$ is given by

$$\Big(\varpi^{p,q}(w_0)h\Big)(v', v'') = h(v', -v'').$$

On the other hand, it is easy to see $h(-x) = (-1)^j h(x)$ for $h \in \mathcal{H}^j(\mathbb{R}^q)$ (see Appendix 7.5, H1). Therefore, $\varpi^{p,q}(w_0)$ acts on each K-type component $\mathcal{H}^a(\mathbb{R}^p) \otimes \mathcal{H}^{a+\frac{p-q}{2}}(\mathbb{R}^q)$ by a scalar multiple $(-1)^{a+\frac{p-q}{2}}$. Since $\mathcal{T} : L^2(C) \to \overline{V^{p,q}}$ intertwines π and $\varpi^{p,q}$, $\pi(w_0)$ acts on $H_{l,k}$ as the scalar $(-1)^{a+\frac{p-q}{2}}$ because $H_{l,k}$ belongs to the K-type $\mathcal{H}^{a(l,k)}(\mathbb{R}^p) \otimes \mathcal{H}^{a(l,k)+\frac{p-q}{2}}(\mathbb{R}^q)$. Thus, the third statement is proved.

5) We shall show the following formula:

$$\|\mathcal{T}u_{l,k}\|^2_{L^2(S^{p-1}\times S^{q-1})} = b_{l,k}\Gamma(k+l+\frac{p+q}{2}-2)\|\phi\|^2_{L^2(S^{p-1})}\|\psi\|^2_{L^2(S^{q-1})}, \tag{3.5.1}$$

where

$$b_{l,k} = \begin{cases} \dfrac{\pi\Gamma(p-2+2l)\Gamma(\frac{p-q}{2}+l-k+1)}{2^{2p+4l}\Gamma(l+\frac{p}{2})^2} & \text{Case 1,} \\ \dfrac{\pi\Gamma(q-2+2k)\Gamma(k-l-\frac{p-q}{2}+l)}{2^{2q+4k}\Gamma(k+\frac{q}{2})^2} & \text{Case 2.} \end{cases}$$

To see this, we begin with (3.1.7):

$$\|\mathfrak{T}u_{l,k}\|^2_{L^2(S^{p-1}\times S^{q-1})} = |c_{l,k}|^2 \|I^p_{l\to a(l,k)}(\phi)\|^2_{L^2(S^{p-1})} \|I^q_{k\to a(l,k)+\frac{p-q}{2}}(\psi)\|^2_{L^2(S^{q-1})}.$$

By (3.1.7) and the norm formula (7.5.2), the right-hand side equals

$$\begin{aligned} &|c_{l,k}|^2 \frac{2^{3-p-2l}\pi\Gamma(p-2+l+a(l,k))}{(a(l,k)-l)!(a(l,k)+\frac{p-2}{2})}\|\phi\|^2_{L^2(S^{p-1})} \\ &\times \frac{2^{3-q-2k}\pi\Gamma(q-2+k+a(l,k)+\frac{p-q}{2})}{(a(l,k)+\frac{p-q}{2}-k)!(a(l,k)+\frac{p-q}{2}+\frac{q-2}{2})}\|\psi\|^2_{L^2(S^{q-1})}. \end{aligned}$$

Now, substituting (3.1.5) and (3.1.8) into this formula, we get (3.5.1) by elementary computations.

Finally, comparing (3.5.1) with Proposition 3.4.2, we obtain (3.1.9) by the Gauss duplication formula (7.4.3) of gamma functions. Hence, Theorem 3.1.1 is proved.

$\square$

4 Radial part of the inversion

The goal of this section is to find the 'radial' part $T_{l,k}$ of the unitary inversion operator $\pi(w_0) : L^2(C) \to L^2(C)$. The main result here is Theorem 4.1.1.

4.1 Result of this section

Suppose $p \geq q \geq 2$ and $p+q \geq 6$. We recall from Subsection 2.2 that the polar coordinate of the (generalized) light cone C:

$$\mathbb{R}_+ \times S^{p-2} \times S^{q-2} \simeq C, \qquad (r, \omega, \eta) \mapsto (r\omega, r\eta)$$

induces the isomorphism of Hilbert spaces:

$$L^2(C) \simeq L^2(\mathbb{R}_+, \frac{1}{2} r^{p+q-5} dr) \widehat{\otimes} L^2(S^{p-2}) \widehat{\otimes} L^2(S^{q-2}).$$

Here, we employ the usual notation $\widehat{\otimes}$ for the completion of a tensor product space as a Hilbert space. This isomorphism respects the action of the compact group

$$K' := K \cap M_+^{\max} \simeq O(p-1) \times O(q-1).$$

Since the Hilbert space $L^2(S^{p-2})$ is decomposed into the direct Hilbert sum of spherical harmonics (see H6 in Appendix 7.5):

$$L^2(S^{p-2}) \simeq \sum_{j=0}^{\infty}{}^{\oplus} \mathcal{H}^j(\mathbb{R}^{p-1}),$$

and likewise for $L^2(S^{q-2})$, we have a decomposition of the Hilbert space $L^2(C)$ into the discrete direct sum:

$$L^2(C) \simeq \sum_{l,k=0}^{\infty}{}^{\oplus} L^2(\mathbb{R}_+, \frac{1}{2} r^{p+q-5} dr) \otimes \mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1}). \tag{4.1.1}$$

Each summand of (4.1.1) is a K'-isotypic component.

For each $(l,k) \in \mathbb{N}^2$, we introduce real analytic function $K_{l,k}(t)$ by

$$K_{l,k}(t) := \begin{cases} 4(-1)^{l+\frac{p-q}{2}} G_{04}^{20}(t^2 | \frac{l+k}{2}, \frac{-q+3+l-k}{2}, \frac{-p-q+6-l-k}{2}, \frac{-p+3-l+k}{2}) & \text{Case 1,} \\ 4(-1)^k G_{04}^{20}(t^2 | \frac{l+k}{2}, \frac{-p+3-l+k}{2}, \frac{-p-q+6-l-k}{2}, \frac{-q+3+l-k}{2}) & \text{Case 2.} \end{cases} \tag{4.1.2}$$

Here, $G_{04}^{20}(x|b_1, b_2, b_3, b_4)$ denotes Meijer's G-function (see Appendix 7.6 for definition). For the definition of Cases 1 and 2 with regard to the parameter $(l,k) \in \mathbb{N}^2$, we recall (3.1.1), namely, Case 1: $\frac{p-q}{2} + l - k \geq 0$,

Case 2: $\frac{p-q}{2} + l - k \leq 0$.

The above formulas are the same in Cases 1 and 2 if $\frac{p-q}{2} + l - k = 0$. Later, we shall give an integral expression of $K_{l,k}$ by means of the Mellin–Barnes type integral (see Lemma 4.5.2). This expression looks more natural because the formula is independent of Cases 1 and 2.

Theorem 4.1.1. 1) *The unitary inversion operator $\pi(w_0)$ preserves each summand of* (4.1.1)*, on which $\pi(w_0)$ acts as a form $T_{l,k} \otimes \mathrm{id} \otimes \mathrm{id}$. Here, $T_{l,k}$ is a unitary operator on $L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$.*

2) *For each $l, k \in \mathbb{N}$, the unitary operator $T_{l,k}$ is given by the integral transform against the real analytic function $K_{l,k}$ (see* (4.1.2) *for definition):*

$$(T_{l,k}f)(r) = \frac{1}{2}\int_0^\infty K_{l,k}(rr')f(r')r'^{p+q-5}dr'. \tag{4.1.3}$$

Remark 4.1.2 (Case $q = 2$)**.** *Theorem 4.1.1 covers the case $q = 2$. In this case, $\mathcal{H}^k(\mathbb{R}^1)$ is non-zero only if $k = 0$ or 1 (see Appendix 7.5 for convention). As we saw in Remark 3.1.3, the pair (l, k) belongs to Case 1 for any $l \in \mathbb{N}$ because the inequality $\frac{p-q}{2}+l-k \geq 0$ is implied by $p \geq 6-q = 4$. Hence, by the definition* (4.1.2) *of $K_{l,k}(t)$, we have*

$$K_{l,0}(t) = 4(-1)^{l+\frac{p-2}{2}} G^{20}_{04}(t^2|\frac{l}{2}, \frac{l+1}{2}, \frac{-p+4-l}{2}, \frac{-p+3-l}{2}),$$
$$K_{l,1}(t) = 4(-1)^{l+\frac{p-2}{2}} G^{20}_{04}(t^2|\frac{l+1}{2}, \frac{l}{2}, \frac{-p+3-l}{2}, \frac{-p+4-l}{2}).$$

In view of the symmetric property of the G-function:

$$G^{20}_{04}(x|b_1, b_2, b_3, b_4) = G^{20}_{04}(x|b_2, b_1, b_4, b_3),$$

the above formulas show $K_{l,0}(t) = K_{l,1}(t)$. Applying the reduction formula (7.6.13) *of the G-function of the form $G^{20}_{04}(x|a, a+\frac{1}{2}, b, b+\frac{1}{2})$, we get*

$$K_{l,0}(t) = K_{l,1}(t) = 4(-1)^{l+\frac{p-2}{2}} t^{-\frac{p-3}{2}} J_{p-3+2l}(4\sqrt{t}). \tag{4.1.4}$$

Thus, the integral transform $T_{l,k}$ $(k = 0, 1)$ is the Hankel transform on $\mathbb{R}_+$ (after a suitable change of variables). Therefore, Theorem 4.1.1 in the case $q = 2$ gives the same result with [31, Theorem 6.1.1], but the proof here is different from that of [31].

Remark 4.1.3 (Comparison with the Weil representation)**.** *We record here the corresponding result for the Schrödinger model $L^2(\mathbb{R}^n)$ of the Weil representation ϖ of $Mp(n,\mathbb{R})$ (see also [31, Remark 6.1.3]). We adopt the same normalization of the Weil representation with [25]. Then, the unitary inversion operator $\varpi(w_0') = e^{\frac{\sqrt{-1}n\pi}{4}}\mathcal{F}$.*

1) *According to the $O(n)$-isotypic decomposition of $L^2(\mathbb{R}^n)$*

$$L^2(\mathbb{R}^n) \simeq \sum_{l=0}^{\infty}{}^{\oplus} L^2(\mathbb{R}_+, r^{n-1}dr) \otimes \mathcal{H}^l(\mathbb{R}^n),$$

the unitary inversion operator $\varpi(w_0')$ decomposes as

$$\varpi(w_0') \simeq \sum_{l=0}^{\infty}{}^{\oplus} T_l' \otimes \mathrm{id},$$

by a family of unitary operators $T_l' (l \in \mathbb{N})$ on $L^2(\mathbb{R}_+, r^{n-1}dr)$.

2) *The unitary operator T_l' is given by the Hankel transform*

$$(T_l' f)(r) = \int_0^{\infty} K_l'(rr') f(r') r'^{n-1} dr',$$

where the kernel $K_l'(t)$ is defined by

$$K_l'(t) := e^{-\frac{\sqrt{-1}(n-1+2l)}{4}\pi} t^{-\frac{n-2}{2}} J_{\frac{n-2+2l}{2}}(t).$$

Returning to Theorem 4.1.1, we remark that the group law $w_0^2 = 1$ in G implies $\pi(w_0)^2 = \mathrm{id}$, and consequently $T_{l,k}^2 = \mathrm{id}$ for every $l, k \in \mathbb{N}$. Further, as $\pi(w_0)$ is a unitary operator on $L^2(C)$, so is its restriction $T_{l,k}$ on $L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$ for every l, k. Hence, Theorem 4.1.1 gives a simple group theoretic proof for the Plancherel and reciprocal formulas on the integral transform involving the G-functions due to C. Fox [12]:

Corollary 4.1.4 (Plancherel formula)**.** *Let b_1, b_2, γ be half-integers such that $b_1 \geq 0, \gamma \geq 1, \frac{1-\gamma}{2} \leq b_2 \leq \frac{1}{2} + b_1$. Then the integral transform*

$$S_{b_1,b_2,\gamma} : f(x) \mapsto \frac{1}{\gamma}\int_0^{\infty} G_{04}^{20}((xy)^{\frac{1}{\gamma}} | b_1, b_2, 1-\gamma-b_1, 1-\gamma-b_2) f(y) dy$$

is a unitary operator on $L^2(\mathbb{R}_+)$, namely,

$$\|S_{b_1,b_2,\gamma} f\|_{L^2(\mathbb{R}_+)} = \|f\|_{L^2(\mathbb{R}_+)}. \tag{4.1.5}$$

Proof. Set $b_1 := \frac{l+k}{2}, b_2 := \frac{-q+3+l-k}{2}, \gamma := \frac{p+q-4}{2}, x = r^{2\gamma}, y = r'^{2\gamma}$. Then, (4.1.5) is equivalent to the fact that $T_{l,k}$ is a unitary operator on $L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$. $\square$

Corollary 4.1.5 (Reciprocal formula)**.** *Retain the notation and the assumption as in Corollary 4.1.4. Then, the unitary operator $S_{b_1,b_2,\gamma}$ is of order two in $L^2(\mathbb{R}_+)$, that is, the following reciprocal formula holds for $f \in L^2(\mathbb{R}_+)$:*

$$f(x) = \int_0^\infty G^{20}_{04}((xy)^{\frac{1}{\gamma}}|b_1, b_2, 1-\gamma-b_1, 1-\gamma-b_2) \times \Big(\int_0^\infty G^{20}_{04}((yz)^{\frac{1}{\gamma}}|b_1, b_2, 1-\gamma-b_1, 1-\gamma-b_2)f(z)dz\Big)dy. \quad (4.1.6)$$

Remark 4.1.6 (Fox's reciprocal formula)**.** *The reciprocal formula for the G-transform was found by C. Fox [12] for the following generality:*

$$G^{m,n}_{2n,2m}\Big(x\Big|{a_1, \cdots, a_n, 1-\gamma-a_1, \cdots, 1-\gamma-a_n \atop b_1, \cdots, b_m, 1-\gamma-b_1, \cdots, 1-\gamma-b_m}\Big).$$

Corollary 4.1.5 is a special case of Fox's formula corresponding to the case $(m, n) = (2, 0)$. On the other hand, it follows from Remark 4.1.2 that Corollary 4.1.5 in the special case when $q = 2$ yields the classic reciprocal formula of the Hankel transform ([53, §14.3], and [31]). Our approach gives a new representation theoretic interpretation (and also a proof) of these formulas.

Remark 4.1.7 (Fourth order differential equation)**.** *Let Ω be the Casimir element for the Lie algebra $\mathfrak{k}$. Since an element of the Lie algebra $\mathfrak{n}^{\max}$ acts on smooth vectors of $L^2(C)$ as a differential operator of second order (see* (2.3.13)*), the Casimir operator $d\pi(\Omega)$ acts as a differential operator on C of fourth order. Let us examine what information on $K_{l,k}(t)$ we can obtain from the Casimir operator $d\pi(\Omega)$.*

Since $\mathrm{Ad}(w_0)\Omega = \Omega$, *we have the commutative relation*

$$\pi(w_0) \circ d\pi(\Omega) = d\pi(\Omega) \circ \pi(w_0). \quad (4.1.7)$$

On the other hand, Ω commutes with all the elements in $\mathfrak{k}$, in particular with $\mathfrak{k}'$. This implies that the differential operator $d\pi(\Omega)$ preserves each summand of (4.1.1)*, and the identity* (4.1.7) *can be regarded as the identity on each summand of* (4.1.1)*. Hence,* (4.1.7) *leads us to the fact that the kernel function $K_{l,k}(t)$ for $\pi(w_0)$ solves a certain differential equation of order four for each (l, k). In view of the formula* (4.1.2) *of $K_{l,k}$, this corresponds to the* *fact that Meijer's G-function $G^{20}_{04}(x \mid b_1, b_2, b_3, b_4)$ satisfies the fourth order differential equation (see* (7.6.6)*):*

$$\prod_{j=1}^{4}(x\frac{d}{dx} - b_j)u = 0.$$

For $q = 2$, the situation is simpler because the minimal representation π is essentially a highest weight module. In this case, the Lie algebra $\mathfrak{k} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$ contains the center $\mathfrak{so}(q) = \mathfrak{so}(2)$. Taking a suitable generator Z in $\mathfrak{so}(2)$, we can show

$$d\pi(\Omega) = d\pi(Z)^2 + \text{constant}\,.$$

We note that the differential operator $d\pi(Z)$ is of order two, and this in turn corresponds to the fact that the kernel function $K_{l,k}$ reduces to the Bessel function (see (4.1.4)*) solving the ordinary differential equation of order two.*

The rest of this section is devoted to the proof of Theorem 4.1.1. The key properties of the element w_0 and the unitary inversion operator $\pi(w_0)$ that we use in the proof are listed as follows:

1) w_0 commutes with K'.
2) $\mathrm{Ad}(w_0)E = -E$ (see (2.4.1)).
3) $\pi(w_0)|_{H_{l,k}} = \pm\,\mathrm{id}$ (Theorem 3.1.1 (3)).

The first property (1) gives immediately the proof of Theorem 4.1.1 (1) (see Subsection 4.2).

The second property (2) implies that the kernel function $K_{l,k}$ of $T_{l,k}$ is a function essentially of one variable rr' (see Subsection 4.4).

The non-trivial part of Theorem 4.1.1 is to prove that this one variable function is given by Meijer's G-function, namely, by the formula (4.1.2). The property (3) is crucial to this part. Our trick here is based on Fact 4.3.1, which assures that (4.1.3) holds for any f once we can prove (4.1.3) holds for a specific case $f = f_{l,k}$ (a (K, K')-*skeleton function*). Actual computations for this are carried out in Subsection 4.6. A technical ingredient of the proof is to show that $K_{l,k}$ is a tempered distribution (see Proposition 4.5.6).

Summing up the formulas for $K_{l,k}$ over l, k, we shall give a proof of an integral formula of $\pi(w_0)$ for an arbitrary L^2-function on C in the next Section 5 (see Theorem 5.1.1).

4.2 Proof of Theorem 4.1.1 (1)

As we have explained at the beginning of Subsection 4.1, the decomposition (4.1.1) of $L^2(C)$ corresponds to the branching laws of the restriction of the unitary representation $(\pi, L^2(C))$ from G to $K' \simeq O(p-1)\times O(q-1)$. Thanks to the group law $w_0 m = m w_0$, the relation $\pi(w_0)\circ\pi(m) = \pi(m)\circ\pi(w_0)$ holds for all $m \in K'$. Therefore, the unitary inversion operator $\pi(w_0)$ preserves each summand of the decomposition (4.1.1).

We now observe that the group K' acts irreducibly on $\mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1})$ (see Appendix 7.5 H2). Therefore, it follows from Schur's lemma that $\pi(w_0)$ acts on each summand:

$$L^2(\mathbb{R}_+, \frac{1}{2} r^{p+q-5} dr) \otimes \mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1})$$

as the form $T_{l,k} \otimes \mathrm{id} \otimes \mathrm{id}$ for some operator $T_{l,k}$ on $L^2(\mathbb{R}_+, \frac{1}{2} r^{p+q-5} dr)$. Such an operator $T_{l,k}$ must be unitary because $\pi(w_0)$ is unitary. Now Theorem 4.1.1 (1) has been proved. $\square$

4.3 Preliminary results on multiplier operators

We recall the classic theory of multipliers on $\mathbb{R}$, followed by an observation that the multiplier is determined by the action on an (appropriate) single function.

We write $l(t)$ $(t \in \mathbb{R})$ for the translation operator on $L^2(\mathbb{R})$, namely,

$$(l(t)F)(x) := F(x-t) \qquad \text{for } F \in L^2(\mathbb{R}). \tag{4.3.1}$$

An operator B on $L^2(\mathbb{R})$ is called *translation invariant* if

$$B \circ l(t) = l(t) \circ B \qquad \text{for any } t \in \mathbb{R}.$$

We write $\mathcal{S}(\mathbb{R})$ for the space of rapidly decreasing C^∞-functions on $\mathbb{R}$ (the Schwartz space endowed with the Fréchet topology), and $\mathcal{S}'(\mathbb{R})$ for its dual space consisting of tempered distributions. Then, the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, $g \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x) e^{\sqrt{-1}x\xi} dx$ (see (1.5.6)) extends to $\mathcal{S}'(\mathbb{R})$ by

$$\langle U, \overline{g} \rangle = \langle U, \overline{\mathcal{F}g} \rangle \qquad \text{for } U \in \mathcal{S}'(\mathbb{R}),\ g \in \mathcal{S}(\mathbb{R}). \tag{4.3.2}$$

If U is a tempered distribution and $f \in \mathcal{S}(\mathbb{R})$, then it is easy to see that

$$l(t)(U * f) = U * (l(t)f)$$

for any $t \in \mathbb{R}$. Furthermore,

$$\|U * f\|_{L^2(\mathbb{R})} = \|\mathcal{F}(U * f)\|_{L^2(\mathbb{R})} = \sqrt{2\pi}\|(\mathcal{F}U)(\mathcal{F}f)\|_{L^2(\mathbb{R})}.$$

Therefore, if $\mathcal{F}U$ is a bounded measurable function, then $f \mapsto U * f$ extends to a bounded operator on $L^2(\mathbb{R})$ and its operator norm is given by $\sqrt{2\pi}\|\mathcal{F}U\|_{L^\infty(\mathbb{R})}$. Conversely, it is well-known that the following theorem also holds:

Fact 4.3.1. *Let B be a bounded, translation invariant operator on $L^2(\mathbb{R})$. Then we have*

1) *There exists a unique tempered distribution U whose Fourier transform $\mathcal{F}U$ is a bounded measurable function such that $Bf = U * f$ for any $f \in L^2(\mathbb{R})$.*

2) *If, moreover, $Bf_0 = f_1$ for some function $f_0 \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}f_0(\xi) \neq 0$ for any $\xi \in \mathbb{R}$, then $U = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\big(\frac{\mathcal{F}f_1}{\mathcal{F}f_0}\big)$.*

Proof. 1) See Stein–Weiss [48, Chapter I, Theorem 3.18], for example.

2) It follows from $Bf_0 = f_1$ that $\mathcal{F}(U * f_0) = \mathcal{F}f_1$, and therefore we have

$$\sqrt{2\pi}(\mathcal{F}U)(\mathcal{F}f_0) = \mathcal{F}f_1.$$

Hence, the bounded measurable function $\mathcal{F}U$ is determined by the formula

$$\mathcal{F}U(\xi) = \frac{1}{\sqrt{2\pi}}\frac{\mathcal{F}f_1(\xi)}{\mathcal{F}f_0(\xi)}.$$

□

Next, we introduce two linear maps σ_+ and σ_- by

$$\begin{aligned} \sigma_+ &: L^2(\mathbb{R}_+, \tfrac{1}{2}r^{p+q-5}dr) \to L^2(\mathbb{R}), \quad f(r) \mapsto \tfrac{1}{\sqrt{2}}e^{\frac{p+q-4}{2}x}f(e^x), \\ \sigma_- &: L^2(\mathbb{R}_+, \tfrac{1}{2}r^{p+q-5}dr) \to L^2(\mathbb{R}), \quad f(r) \mapsto \tfrac{1}{\sqrt{2}}e^{-\frac{p+q-4}{2}x}f(e^{-x}). \end{aligned} \tag{4.3.3}$$

Then, both σ_+ and σ_- are unitary operators. Further, clearly we have

$$(\sigma_- f)(x) = (\sigma_+ f)(-x). \tag{4.3.4}$$

The inverse map σ_-^{-1} is given by

$$(\sigma_-^{-1}F)(r) = \sqrt{2}r^{-\frac{p+q-4}{2}}F(-\log r) \quad \text{for } F \in L^2(\mathbb{R}).$$

We define a subspace $\mathcal{S}$ of $L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$ by

$$\mathcal{S} := \sigma_-^{-1}\big(\mathcal{S}(\mathbb{R})\big) = \sigma_+^{-1}(\mathcal{S}(\mathbb{R})), \tag{4.3.5}$$

and endow $\mathcal{S}$ with the topology induced from that of the Schwartz space $\mathcal{S}(\mathbb{R})$. Now let $\mathcal{S}'$ be the dual space of $\mathcal{S}$.

The unitarity of σ_- implies the following identity for $u \in L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$ and $F \in L^2(\mathbb{R})$:

$$\begin{aligned}\langle \sigma_- u, F\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} (\sigma_- u)(x)F(x)dx \\ &= \langle u, \sigma_-^{-1}F\rangle_{L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)}.\end{aligned}$$

Then σ_- extends naturally to an isomorphism from the dual space $\mathcal{S}'$ onto $\mathcal{S}'(\mathbb{R})$ by the formula

$$\langle \sigma_- u, F\rangle := \langle u, \sigma_-^{-1}F\rangle, \qquad \text{for } F \in \mathcal{S}(\mathbb{R}),\ u \in \mathcal{S}'. \tag{4.3.6}$$

Recall from (3.1.2) that we have defined a family of real analytic functions $f_{l,k}$ on $\mathbb{R}_+$ parametrized by $(l,k) \in \mathbb{N}^2$. As we saw in Lemma 3.4.1, $f_{l,k} \in L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$.

For a continuous operator A on $L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$, we define a continuous operator $\tilde{A}$ on $L^2(\mathbb{R})$ by

$$\tilde{A} := \sigma_- \circ A \circ \sigma_+^{-1}. \tag{4.3.7}$$

Thus, the following diagram commutes:

$$\begin{array}{ccc} L^2(\mathbb{R}_+, \dfrac{1}{2}r^{p+q-5}dr) & \xrightarrow{A} & L^2(\mathbb{R}_+, \dfrac{1}{2}r^{p+q-5}dr) \\ \sigma_+ \downarrow & & \sigma_- \downarrow \\ L^2(\mathbb{R}) & \xrightarrow{\tilde{A}} & L^2(\mathbb{R}). \end{array} \tag{4.3.8}$$

Since $\sigma_\pm$ are unitary operators, A is unitary if and only if $\tilde{A}$ is unitary.

For $\kappa \in \mathcal{S}'$, we define an operator A_κ by

$$A_\kappa : \mathcal{S} \to \mathcal{S}', \quad f(r) \mapsto \frac{1}{2}\int_0^\infty \kappa(rr')f(r')r'^{p+q-5}dr'. \tag{4.3.9}$$

It follows from the definition (4.3.3) of σ_+ and σ_- that

$$(\sigma_-\kappa * \sigma_+ f)(x) = \frac{1}{2}e^{-\frac{p+q-4}{2}x}\int_{-\infty}^{\infty} e^{(p+q-4)y}\kappa(e^{-x}e^y)f(e^y)dy.$$

Then, by a change of variables, we have

$$A_\kappa f = \frac{1}{\sqrt{2}}\sigma_-^{-1}(\sigma_-\kappa * \sigma_+ f) \quad \text{for } f \in \mathcal{S}. \tag{4.3.10}$$

The following lemma characterizes operators of the form A_κ:

Lemma 4.3.2. *Let $\{\rho(t) : t \in \mathbb{R}\}$ be a one parameter family of unitary operators on $L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$ defined by*

$$(\rho(t)f)(r) := e^{-\frac{p+q-4}{2}t} f(e^{-t}r). \tag{4.3.11}$$

Suppose that a unitary operator T on $L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$ satisfies the following (anti-)commutative relation:

$$T \circ \rho(t) = \rho(-t) \circ T \quad \textit{for any } t \in \mathbb{R}. \tag{4.3.12}$$

Then, there exists a unique distribution $\kappa \in \mathcal{S}'$ such that $T = A_\kappa$.

Proof. For a general operator A on $L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$, we shall use the symbol $\tilde{A}$ to denote $\sigma_- \circ A \circ \sigma_+^{-1}$ as in (4.3.7). Then we have

$$\begin{aligned}
\widetilde{T \circ \rho(t)} &= \sigma_- \circ (T \circ \rho(t)) \circ \sigma_+^{-1} = \tilde{T} \circ (\sigma_+ \circ \rho(t) \circ \sigma_+^{-1}), \\
\widetilde{\rho(-t) \circ T} &= \sigma_- \circ (\rho(-t) \circ T) \circ \sigma_+^{-1} = (\sigma_- \circ \rho(-t) \circ \sigma_-^{-1}) \circ \tilde{T}.
\end{aligned}$$

On the other hand, by a simple computation, we have the following identities:

$$\sigma_+ \circ \rho(t) \circ \sigma_+^{-1} = \sigma_- \circ \rho(-t) \circ \sigma_-^{-1} = l(t).$$

Here, $l(t)$ denotes the translation operator (4.3.1). Hence, the relation (4.3.12) is equivalent to

$$\tilde{T} \circ l(t) = l(t) \circ \tilde{T} \quad \text{for any } t \in \mathbb{R},$$

that is, $\tilde{T}$ is a translation invariant bounded operator on $L^2(\mathbb{R})$. Therefore, the operator $\tilde{T}$ must be a convolution operator $U*$ for some tempered distribution $U \in \mathcal{S}'(\mathbb{R})$ such that its Fourier transform $\mathcal{F}U$ is a bounded function (see Fact 4.3.1 (1)).

Finally, by setting $\kappa := \sqrt{2}\sigma_-^{-1}U$, we have for any $f \in L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$,

$$\begin{aligned}
Tf &= \sigma_-^{-1} \circ \tilde{T} \circ \sigma_+ f \\
&= \frac{1}{\sqrt{2}}\sigma_-^{-1}(\sigma_-\kappa * \sigma_+ f) \\
&= A_\kappa f
\end{aligned}$$

by (4.3.10). Therefore, $T = A_\kappa$. $\square$

4.4 Reduction to Fourier analysis

The classic theory of multipliers on $\mathbb{R}$ helps us to get some useful information about $T_{l,k}$. The main result of this subsection is Proposition 4.4.4.

We begin with a refinement of Lemma 3.4.1.

Lemma 4.4.1. *$f_{l,k}$ belongs to $\mathcal{S}$. In particular, $f_{l,k} \in L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$.*

Proof. By the definition of $\mathcal{S}$ (see (4.3.5)), it is sufficient to show $\sigma_- f_{l,k} \in \mathcal{S}(\mathbb{R})$. The proof is divided into two steps.

Step 1: *For any $m \in \mathbb{N}$, $x^m(\sigma_- f_{l,k})(x)$ is rapidly decreasing.*

By the definition (3.1.2) of $f_{l,k}$, we have

$$(\sigma_- f_{l,k})(x) = \frac{1}{\sqrt{2}} e^{-(l+k+\frac{p+q}{2}-2)x} \times \begin{cases} \widetilde{K}_{\frac{q-3}{2}+k}(2e^{-x}) & \text{Case 1,} \\ \widetilde{K}_{\frac{p-3}{2}+l}(2e^{-x}) & \text{Case 2.} \end{cases} \tag{4.4.1}$$

Therefore, by the asymptotic behavior of K-Bessel functions (see Fact 7.2.1 (1), (2), respectively), we see

$$(\sigma_- f_{l,k})(x) \sim \frac{e^{-x}}{2\sqrt{2}} \times \begin{cases} \Gamma(\frac{q-3}{2}+k)e^{-(\frac{p-q}{2}+l-k)x} & \text{Case 1,} \\ \Gamma(\frac{p-3}{2}+l)e^{(\frac{p-q}{2}+l-k)x} & \text{Case 2,} \end{cases} \quad \text{as } x \to +\infty, \tag{4.4.2}$$

$$(\sigma_- f_{l,k})(x) \sim \frac{1}{\sqrt{2}} \begin{cases} e^{-(\frac{p-2}{2}+l)x} e^{-e^{-x}} & \text{Case 1,} \\ e^{-(\frac{q-2}{2}+k)x} e^{-e^{-x}} & \text{Case 2,} \end{cases} \quad \text{as } x \to -\infty. \tag{4.4.3}$$

Thus, Step 1 is proved.

Step 2: *For any $n \in \mathbb{N}$, $\frac{d^n}{dx^n}(\sigma_- f_{l,k}(x))$ is rapidly decreasing.*

We use induction on n. We already know from Step 1 that the statement is true for any $l, k \in \mathbb{N}$ in the case $n = 0$. Now assume that the statement is true for n. Then the statement for $n+1$ immediately follows from the next claim. Now Step 2 has been proved, and so has Lemma 4.4.1. $\square$

Claim 4.4.2. *For any $l, k \in \mathbb{N}$, we have the following recurrence formula:*

$$\frac{d}{dx}(\sigma_- f_{l,k}) = -(\frac{p+q-4}{2} + l + k)\sigma_- f_{l,k} + 2\sigma_- f_{l+1,k+1}. \tag{4.4.4}$$

Proof. The formula (4.4.1) says

$$(\sigma_- f_{l,k})(x) = \frac{1}{\sqrt{2}} e^{-ax} \widetilde{K}_\nu(2e^{-x}),$$

where

$$a = \frac{p+q-4}{2} + l + k, \quad \nu = \begin{cases} \frac{q-3}{2} + k & \text{Case 1,} \\ \frac{p-3}{2} + l & \text{Case 2.} \end{cases}$$

By (7.2.1), we have

$$\begin{aligned} \frac{d}{dx}(\sigma_- f_{l,k})(x) &= \frac{1}{\sqrt{2}}(-ae^{-ax}\widetilde{K}_\nu(2e^{-x}) + 2e^{-(a+2)x}\widetilde{K}_{\nu+1}(2e^{-x})) \\ &= -a(\sigma_- f_{l,k})(x) + 2(\sigma_- f_{l+1,k+1})(x). \end{aligned}$$

Here, we have used the observation that (k, l) belongs to Case 1 (i.e. $\frac{p-q}{2} + l - k \geq 0$, see (3.1.1)) if and only if $(k+1, l+1)$ belongs to Case 1, and likewise for Case 2. Therefore, we have proved Claim 4.4.2. $\square$

Observation 4.4.3. *We shall prove in Proposition 4.5.6 that $K_{l,k} \in \mathcal{S}'$, namely, $\sigma_- K_{l,k} \in \mathcal{S}'(\mathbb{R})$. Unlike the proof for $f_{l,k} \in \mathcal{S}$ in Lemma 4.4.1, this is not obvious from the asymptotic behavior of $K_{l,k}$ (see Remark 4.5.1). In fact, it follows from* (4.5.1) *that*

$$\begin{aligned} \limsup_{x \to -\infty} e^{-\epsilon|x|}(\sigma_- K_{l,k})(x) &= \limsup_{x \to -\infty} e^{-\epsilon|x| - \frac{x}{4}} \Big(\frac{4}{\sqrt{2\pi}} \cos(4e^{-\frac{x}{2}} - \frac{2q-3}{4}\pi) \Big) \\ &= +\infty \end{aligned}$$

if $\epsilon < \frac{1}{4}$. Thus, the asymptotic behavior of $K_{l,k}$ is worse than that of polynomials as x tends to $-\infty$. Instead, our proof for $K_{l,k} \in \mathcal{S}'$ uses an explicit computation of the Fourier integral (see Proposition 4.5.6). We note that $(\sigma_- K_{l,k})(x)$ decays exponentially as x tends to $+\infty$.

Proposition 4.4.4. *Let $T_{l,k}$ be the unitary operator defined in Theorem 4.1.1 (1). Then, there exists uniquely a distribution $\kappa_{l,k} \in \mathcal{S}'$ such that*

$$(T_{l,k} f)(r) = \frac{1}{2} \int_0^\infty \kappa_{l,k}(rr') f(r') r'^{p+q-5} dr'. \tag{4.4.5}$$

Namely, $T_{l,k} = A_{\kappa_{l,k}}$ with the notation as in (4.3.9).

Proof of Proposition 4.4.4. We recall from (2.3.9) that the unitary operator $\pi(e^{tE})$ on $L^2(C)$ can be written by means of the unitary operator $\rho(t)$ (see (4.3.11)) on $L^2(\mathbb{R}_+, \frac{1}{2} r^{p+q-5} dr)$ as follows:

$$\pi(e^{tE})\big(f(r)\phi(\omega)\psi(\eta)\big) = \big(\rho(t)f\big)(r)\phi(\omega)\psi(\eta),$$

where $f \in L^2(\mathbb{R}_+, \frac{1}{2} r^{p+q-5} dr), \phi \in \mathcal{H}^l(\mathbb{R}^{p-1})$, and $\psi \in \mathcal{H}^k(\mathbb{R}^{q-1})$.

Applying $\pi(w_0)$ to the both sides, together with the definition of $T_{l,k}$ (see Theorem 4.1.1 (1)), we obtain

$$\pi(w_0) \circ \pi(e^{tE})(f(r)\phi(w)\psi(\eta)) = (T_{l,k}\rho(t)f)(r)\phi(w)\psi(\eta).$$

Similarly, applying $\pi(w_0)$ followed by $\pi(e^{-tE})$, we get

$$\pi(e^{-tE}) \circ \pi(w_0)(f(r)\phi(w)\psi(\eta)) = (\rho(-t)T_{l,k}f)(r)\phi(w)\psi(\eta).$$

On the other hand, it follows from $\mathrm{Ad}(w_0)E = -E$ (see (2.4.1)) that

$$w_0 e^{tE} = e^{-tE} w_0,$$

and then we have

$$\pi(w_0) \circ \pi(e^{tE}) = \pi(e^{-tE}) \circ \pi(w_0).$$

Therefore,

$$T_{l,k} \circ \rho(t) = \rho(-t) \circ T_{l,k}.$$

Now, Proposition 4.4.4 follows from Lemma 4.3.2. $\square$

4.5 Kernel function $K_{l,k}$

As a preparation for the proof of Theorem 4.1.1 (2), we summarize some properties of the real analytic function $K_{l,k}(t)$ defined in (4.1.2). The main result of this subsection is Proposition 4.5.6.

Lemma 4.5.1 (Asymptotic behavior)**.** *$K_{l,k}(t)$ has the following asymptotics as t tends to 0 or ∞:*

1) *As t tends to 0,*

$$K_{l,k}(t) = \begin{cases} O(t^{-q+3+l-k}) & \textit{Case 1} \\ O(t^{-p+3-l+k}) & \textit{Case 2} \end{cases} \qquad (q > 2),$$
$$K_{l,k}(t) = O(t^{l}) \qquad (q = 2).$$

2) *There are some constants $P_1, \cdots, Q_1, \cdots$ such that*

$$K_{l,k}(t) = \frac{4}{\sqrt{2\pi}} t^{-\frac{2p+2q-9}{4}} \cos\big(4t^{\frac{1}{2}} - \frac{2q-3}{4}\pi\big)(1 + P_1 t^{-1} + P_2 t^{-2} + \cdots)$$
$$+ t^{-\frac{2p+2q-9}{4}} \sin\big(4t^{\frac{1}{2}} - \frac{2q-3}{4}\pi\big)(Q_1 t^{-\frac{1}{2}} + Q_2 t^{-\frac{3}{2}} + \cdots), \quad (4.5.1)$$

as t tends to $+\infty$.

Proof. Directly obtained from the asymptotic formula of Meijer's G-function $G^{20}_{04}(x \mid b_1, b_2, 1-\gamma-b_1, 1-\gamma-b_2)$ given in Lemma 7.6.4. $\square$

Next, we give an integral expression of $K_{l,k}(t)$ $(t > 0)$ following the definition (4.1.2).

The integral path L will be taken independently of $l, k \in \mathbb{N}$. We note that the integral formulas (4.5.2) and (4.5.3) are valid both in Cases 1 and 2 (see also Remark 4.5.3).

Lemma 4.5.2. *Fix a real number $\gamma > -\frac{p+q-5}{2}$, and let L be a contour that starts at $\gamma - \sqrt{-1}\infty$ and ends at $\gamma + \sqrt{-1}\infty$ and passes the real axis at a point s_0 satisfying $s_0 < -\frac{p+q-6}{2}$ (see Figure 4.5.1). (Later, we shall assume also that $-\frac{p+q-4}{2} < s_0$ in Section 5.) Then, we have*

$$K_{l,k}(t) = \frac{(-1)^{l+\frac{p-q}{2}}}{\pi\sqrt{-1}} \int_L \frac{\Gamma(\frac{l+k-s}{2})\Gamma(\frac{-q+3+l-k-s}{2})}{\Gamma(\frac{p+q-4+l+k+s}{2})\Gamma(\frac{p-1+l-k+s}{2})} t^s ds \tag{4.5.2}$$

$$= \frac{(-1)^{k}}{\pi\sqrt{-1}} \int_L \frac{\Gamma(\frac{l+k-s}{2})\Gamma(\frac{-p+3-l+k-s}{2})}{\Gamma(\frac{p+q-4+l+k+s}{2})\Gamma(\frac{q-1-l+k+s}{2})} t^s ds. \tag{4.5.3}$$

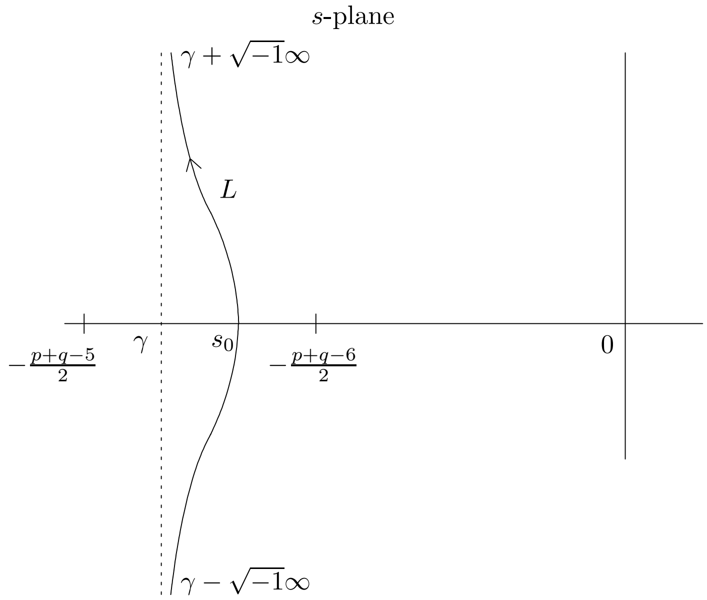


Figure 4.5.1

Proof. The equality (4.5.2) = (4.5.3) is an immediate consequence of the following formula:

$$\frac{\Gamma(\frac{-q+3+l-k-s}{2})}{\Gamma(\frac{p-1+l-k+s}{2})} \cdot \frac{\Gamma(\frac{q-1-l+k+s}{2})}{\Gamma(\frac{-p+3-l+k-s}{2})} = (-1)^{\frac{p-q}{2}+l-k}, \tag{4.5.4}$$

which is derived from

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Let us show (4.5.2) in Case 1, and (4.5.3) in Case 2 (see (3.1.1) for the definition of Cases 1 and 2). First, we note that the poles of the numerators in the integrands (4.5.2) and (4.5.3) with respect to s are given by

$$W_1 := \{l + k + 2a, -q + 3 + l - k + 2a : a \in \mathbb{N}\},$$
$$W_2 := \{l + k + 2a, -p + 3 - l + k + 2a : a \in \mathbb{N}\},$$

respectively. Then,

$$\inf W_1 \geq -\frac{p+q-6}{2} \quad \text{in Case 1} \quad (\text{i.e. } \frac{p-q}{2} + l - k \geq 0),$$
$$\inf W_2 \geq -\frac{p+q-6}{2} \quad \text{in Case 2} \quad (\text{i.e. } \frac{p-q}{2} + l - k \leq 0).$$

Therefore, in either case, the contour L leaves all these sets W_1 and W_2 on the right because our L passes the real axis at some point $s_0 < -\frac{p+q-6}{2}$.

By the definition of Meijer's G-function (see (7.6.2) in Appendix, see also Example 7.6.3), we get (4.5.2) in Case 1 and (4.5.3) in Case 2 by the change of variables $s := 2\lambda$. Hence, Lemma is proved. □

Remark 4.5.3. *We shall use the expression* (4.5.2) *in Case 1 and* (4.5.3) *in Case 2 later. The point here is that there is no cancellation of the poles of the numerator and the denominator of the integrand. For example, the poles of the denominator of the integrand* (4.5.2) *are given by*

$$V_1 := \{-p - q + 4 - l - k - 2b, \ -p + 1 - l + k - 2b : b \in \mathbb{N}\}.$$

Then,

$$\sup V_1 < \inf W_1 \quad \textit{in Case 1},$$

and therefore $V_1 \cap W_1 = \emptyset$. *Likewise, there is no cancellation of the poles of the numerator and the poles of the denominator of the integrand* (4.5.3) *in Case 2.*

As $K_{l,k}$ is a real analytic function on $\mathbb{R}_+$, so is $(\sigma_- K_{l,k})(x)$ on $\mathbb{R}$ (see (4.3.3) for the definition of σ_-), which in turn is a distribution on $\mathbb{R}$. More strongly, we shall see in Proposition 4.5.6 that $(\sigma_- K_{l,k})(x)$ is a tempered distribution.

For this, we define a meromorphic function $\psi(\zeta)$ on $\mathbb{C}$ by

$$\psi(\zeta) := (-1)^{l+\frac{p-q}{2}} \frac{\Gamma\big(\frac{p+q-4}{4} + \frac{l+k-\sqrt{-1}\zeta}{2}\big)\Gamma\big(\frac{p-q}{4} + \frac{l-k+1-\sqrt{-1}\zeta}{2}\big)}{\Gamma\big(\frac{p+q-4}{4} + \frac{l+k+\sqrt{-1}\zeta}{2}\big)\Gamma\big(\frac{p-q}{4} + \frac{l-k+1+\sqrt{-1}\zeta}{2}\big)} \tag{4.5.5}$$

$$= (-1)^{k} \frac{\Gamma\big(\frac{p+q-4}{4} + \frac{l+k-\sqrt{-1}\zeta}{2}\big)\Gamma\big(-\frac{p-q}{4} + \frac{-l+k+1-\sqrt{-1}\zeta}{2}\big)}{\Gamma\big(\frac{p+q-4}{4} + \frac{l+k+\sqrt{-1}\zeta}{2}\big)\Gamma\big(-\frac{p-q}{4} + \frac{-l+k+1+\sqrt{-1}\zeta}{2}\big)} \tag{4.5.6}$$

We shall use (4.5.5) in Case 1, and (4.5.6) in Case 2. The proof of the equality (4.5.5) = (4.5.6) is the same as the proof of the equality (4.5.2) = (4.5.3).

Here, we remark that the letter ζ denoted a point on C in the previous sections, but by a little abuse of notation, ζ in this subsection stands for a complex number.

Lemma 4.5.4. 1) $|\psi(\zeta)| = 1$ *for* $\zeta \in \mathbb{R}$. *In particular, the inverse Fourier transform* $\mathcal{F}^{-1}\psi$ *is defined to be a tempered distribution.*

2) $\psi(\zeta)$ *is a meromorphic function on* $\mathbb{C}$, *and the set of its poles is contained in*

$$\{-\sqrt{-1}m : m = 1, 2, 3, \ldots\}.$$

3) *For* $\eta_1 \le \eta \le \eta_2$,

$$|\psi(\xi - \sqrt{-1}\eta)| \sim \left|\frac{\xi}{2}\right|^{-2\eta} \quad \textit{as } |\xi| \to \infty. \tag{4.5.7}$$

Proof. 1) Since $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ for $z \in \mathbb{C}$, we have $|\psi(\zeta)| = 1$. Therefore $\psi \in \mathcal{S}'(\mathbb{R})$ and thus $\mathcal{F}^{-1}\psi \in \mathcal{S}'(\mathbb{R})$.

2) The proof is straightforward from the definitions (4.5.5) and (4.5.6) in each case.

3) We recall Stirling's asymptotic expansion of the gamma function (see [1, Corollary 1.4.4] for example):

$$\big|\Gamma(a + \sqrt{-1}b)\big| = \sqrt{2\pi}|b|^{a-\frac{1}{2}} e^{-\frac{\pi|b|}{2}}(1 + O(|b|^{-1})) \tag{4.5.8}$$

when $a_1 \le a \le a_2$ and $|b| \to \infty$. Then, for $\alpha \in \mathbb{R}$ and $z = x + \sqrt{-1}y$ $(y_1 \le y \le y_2)$,

$$\left|\frac{\Gamma(\alpha - \sqrt{-1}z)}{\Gamma(\alpha + \sqrt{-1}z)}\right| = |x|^{2y}(1 + O(|x|^{-1})) \quad \text{as } |x| \to \infty, \tag{4.5.9}$$

where the constant implied by O depends only on α, y_1 and y_2. Now, applying (4.5.9) to $z = \frac{1}{2}(\xi - \sqrt{-1}\eta)$ twice, we get (4.5.7). $\square$

By the change of variable $s = \sqrt{-1}\zeta - \frac{p+q-4}{2}$, the integral formula of $K_{l,k}$ (Lemma 4.5.2) can be restated as follows:

Lemma 4.5.5. *Let* $\gamma > -\frac{p+q-5}{2}$ *and* L' *be an integral path starting from* $-(\gamma + \frac{p+q-4}{2})\sqrt{-1} - \infty$ *and ending at* $-(\gamma + \frac{p+q-4}{2})\sqrt{-1} + \infty$ *passing the imaginary axis at some point in the open interval* $(-\frac{\sqrt{-1}}{2}, -\sqrt{-1})$ *(see Figure 4.5.2). Then, we have*

$$K_{l,k}(t) = \frac{1}{\pi} \int_{L'} \psi(\zeta) t^{-\frac{p+q-4}{2}+\sqrt{-1}\zeta} d\zeta, \tag{4.5.10}$$

or equivalently (see (4.3.3) *for the definition of* σ_-*),*

$$(\sigma_- K_{l,k})(x) = \frac{1}{\sqrt{2}\pi} \int_{L'} \psi(\zeta) e^{-\sqrt{-1}x\zeta} d\zeta. \tag{4.5.11}$$

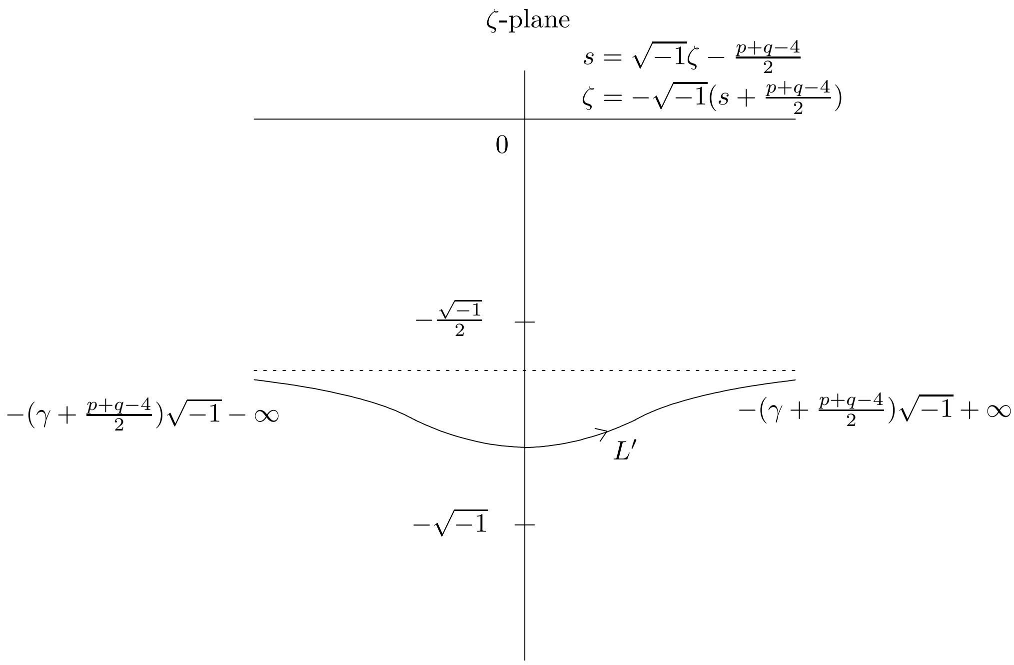


Figure 4.5.2

With these preparations, we recall from Subsection 4.3 that $\mathcal{S}'$ is the dual space of $\mathcal{S} = \sigma_-^{-1}(\mathcal{S}(\mathbb{R}))$ via the following diagram:

$$\begin{array}{ccc}
\mathcal{S}' & \xrightarrow{\sim} & \mathcal{S}'(\mathbb{R}) \\
\cup & & \cup \\
\sigma_- : L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr) & \xrightarrow{\sim} & L^2(\mathbb{R}) \\
\cup & & \cup \\
\mathcal{S} & \xrightarrow{\sim} & \mathcal{S}(\mathbb{R}).
\end{array}$$

The following is the main result of this subsection:

Proposition 4.5.6. 1) *$K_{l,k}$ belongs to $\mathcal{S}'$. That is, $\sigma_- K_{l,k} \in \mathcal{S}'(\mathbb{R})$.*

2) *The Fourier transform $\mathcal{F}(\sigma_- K_{l,k})(\zeta)$ of $\sigma_- K_{l,k}$ is equal to $\frac{1}{\sqrt{\pi}}\psi(\zeta)$ (see (4.5.5) for definition). In particular, $|\mathcal{F}(\sigma_- K_{l,k})(\zeta)| = \frac{1}{\sqrt{\pi}}$ for $\zeta \in \mathbb{R}$.*

Proof. It follows from Lemma 4.5.4 (1) that ψ is a tempered distribution, and therefore, its inverse Fourier transform $\mathcal{F}^{-1}\psi \in \mathcal{S}'(\mathbb{R})$. We also know that $\sigma_- K_{l,k} \in C^\infty(\mathbb{R})$ by definition. Let $\mathcal{D}'(\mathbb{R})$ be the space of distributions, namely, the dual space of $C_0^\infty(\mathbb{R})$. In light of the inclusion

$$\mathcal{S}'(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R}) \supset C^\infty(\mathbb{R}),$$

all the statements of Proposition 4.5.6 will be proved if we show

$$\sqrt{\pi}\sigma_- K_{l,k} = \mathcal{F}^{-1}\psi \quad \text{in } \mathcal{D}'(\mathbb{R}), \tag{4.5.12}$$

that is,

$$\sqrt{\pi}\int_{-\infty}^{\infty} (\sigma_- K_{l,k})(x)\overline{\phi(x)}dx = \int_{-\infty}^{\infty} (\mathcal{F}^{-1}\psi)(x)\overline{\phi(x)}dx \tag{4.5.13}$$

holds for any test function $\phi \in C_0^\infty(\mathbb{R})$. In fact, (4.5.12) will imply that $\sigma_- K_{l,k} \in \mathcal{S}'(\mathbb{R})$ and $\sqrt{\pi}\mathcal{F}(\sigma_- K_{l,k}) = \psi$ as a tempered distribution.

The main machinery of the proof for (4.5.13) is Lemma 4.5.5. By (4.5.11), the left-hand side of (4.5.13) amounts to

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{L'}\psi(\zeta)e^{-\sqrt{-1}x\zeta}d\zeta\right)\overline{\phi(x)}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{L'}\psi(\zeta)\overline{\left(\int_{-\infty}^{\infty}\phi(x)e^{\sqrt{-1}x\bar{\zeta}}dx\right)}d\zeta \\
&= \int_{L'}\psi(\zeta)\overline{(\mathcal{F}\phi)(\bar{\zeta})}d\zeta \\
&= \int_{-\infty}^{\infty}\psi(\zeta)\overline{(\mathcal{F}\phi)(\zeta)}d\zeta \\
&= \text{right-hand side of (4.5.13).}
\end{aligned}$$

In what follows, we give detailed comments on the above equalities:

First equality is by Fubini's theorem. To see this, take $a > 0$ such that $\operatorname{Supp}\phi \subset [-a, a]$. Then,

$$|\overline{\phi(x)}e^{-\sqrt{-1}x\zeta}| \leq \|\phi\|_\infty e^{a\eta}$$

for $\zeta = \xi - \sqrt{-1}\eta$ with $\eta > 0$. Here, $\|\phi\|_\infty$ denotes the L^∞ norm.

Since $\gamma > -\frac{p+q-5}{2}$, we may assume $\zeta = \xi - \sqrt{-1}\eta \in L'$ satisfies

$$\eta_1 \leq \eta \leq \eta_2$$

for some constants η_1 and η_2 such that $\eta_1 > \frac{1}{2}$ if $|\xi|$ is sufficiently large. Then, there exists a constant $C > 0$ such that $|\psi(\zeta)| \leq C|\xi|^{-2\eta}$ as $|\xi| \to \infty$, by Lemma 4.5.4 (3). Thus, if $|\xi|$ is sufficiently large, we have

$$|\psi(\zeta)\overline{\phi(x)}e^{-\sqrt{-1}x\zeta}| \leq C\|\phi\|_\infty |\xi|^{-2\eta_1}.$$

Hence, $\psi(\zeta)\overline{\phi(x)}e^{-\sqrt{-1}x\zeta}$ is absolutely integrable on $L' \times [-a, a]$ (and therefore, on $L' \times (-\infty, \infty)$). Thus, we can apply Fubini's theorem.

Second equality follows immediately from the definition (1.5.6) of the Fourier transform.

Third equality follows from Cauchy's integral formula. To see this, we observe that $\psi(\zeta)\overline{(\mathcal{F}\phi)(\bar{\zeta})}$ is holomorphic in the domain between the two integral paths $(-\infty, \infty)$ and L' since its poles lie only on $\{-\sqrt{-1}m : m = 1, 2, \cdots\}$ (see Lemma 4.5.4(2)). Thus, it is sufficient to show

$$\lim_{|\xi| \to \infty} \int_0^{\eta_0} |\psi(\xi - \sqrt{-1}\eta)\overline{(\mathcal{F}\phi)(\xi + \sqrt{-1}\eta)}| d\eta = 0 \tag{4.5.14}$$

for a fixed η_0 ($\geq \gamma + \frac{p+q-4}{2}$) (see Figure 4.5.2). We suppose as before the support of $\phi \in C_0^\infty(\mathbb{R})$ is given by $\operatorname{Supp}\phi \subset [-a, a]$. Then, by the Paley–Wiener theorem, there exists a constant C such that $|\overline{\mathcal{F}\phi(\xi + \sqrt{-1}\eta)}| \leq Ce^{a\eta}$.

Now combining this with Lemma 4.5.4 (3), we get

$$|\psi(\zeta)\overline{\mathcal{F}\phi(\bar{\zeta})}| \leq C'|\xi|^{-2\eta} \quad \text{as } |\xi| \to \infty$$

for $\zeta = \xi - \sqrt{-1}\eta$ and bounded η. Hence, (4.5.14) is proved. Thus, by Cauchy's integral formula, we get the third equality.

Last equality is by the definition of the Fourier transform for tempered distributions:

$$(f, g) = (\mathcal{F}^{-1}f, \mathcal{F}^{-1}g) \quad f \in \mathcal{S}'(\mathbb{R}),\ g \in \mathcal{S}(\mathbb{R}).$$

Hence, we have proved (4.5.12). Now, the proof of Proposition 4.5.6 is completed. □

4.6 Proof of Theorem 4.1.1 (2)

In this subsection, we complete the proof of Theorem 4.1.1 (2). For this, it is sufficient to show the following proposition:

Proposition 4.6.1. $\kappa_{l,k} = K_{l,k}$.

Here, we recall that the kernel distribution $\kappa_{l,k}$ of $T_{l,k}$ is given in Proposition 4.4.4 and that $K_{l,k}$ is defined in (4.1.2).

Proof. The proof makes use of the following:

Lemma 4.6.2. *Let $\kappa_1, \kappa_2 \in \mathcal{S}'$. If there exists $\phi \in \mathcal{S}$ such that*

$$\mathcal{F}(\sigma_+\phi)(\zeta) \neq 0 \quad \textit{for any } \zeta \in \mathbb{R}, \tag{4.6.1}$$

$$A_{\kappa_1}\phi = A_{\kappa_2}\phi, \tag{4.6.2}$$

then $\kappa_1 = \kappa_2$. Here, we recall from (4.3.9) *for the definition of A_κ.*

Proof of Lemma 4.6.2. The identity (4.6.2) implies

$$\sigma_-\kappa_1 * \sigma_+\phi = \sigma_-\kappa_2 * \sigma_+\phi$$

by the formula (4.3.10) of A_κ. Therefore, we have an identity

$$\mathcal{F}(\sigma_-\kappa_1)(\zeta) \cdot \mathcal{F}(\sigma_+\phi)(\zeta) = \mathcal{F}(\sigma_-\kappa_2)(\zeta) \cdot \mathcal{F}(\sigma_+\phi)(\zeta)$$

in $\mathcal{S}'(\mathbb{R})$ by taking their Fourier transforms.

On the other hand, it follows from the assumption that $\sigma_+\phi \in \mathcal{S}(\mathbb{R})$ and its Fourier transform $\mathcal{F}(\sigma_+\phi)$ does not vanish on $\mathbb{R}$, we can divide the above identity by $\mathcal{F}(\sigma_+\phi)(\zeta)$, and obtain an identity in $\mathcal{S}'(\mathbb{R})$:

$$\mathcal{F}(\sigma_-\kappa_1)(\zeta) = \mathcal{F}(\sigma_-\kappa_2)(\zeta).$$

Hence, $\sigma_-\kappa_1 = \sigma_-\kappa_2$, and in turn, $\kappa_1 = \kappa_2$. ☐

We want to apply Lemma 4.6.2 with $\kappa_1 := \kappa_{l,k}, \kappa_2 := K_{l,k}, \phi := f_{l,k}$ (see (3.1.2) for the definition). The conditions in the lemma will be verified by the following steps.

Step 1. $\kappa_{l,k}, K_{l,k} \in \mathcal{S}'$. These statements have been already proved in Propositions 4.4.4 and 4.5.6.

Step 2. $f_{l,k} \in \mathcal{S}$. This has been proved in Lemma 4.4.1.

Step 3. *$\mathcal{F}(\sigma_+ f_{l,k})(\zeta)$ has no zero points on* $\mathbb{R}$. This statement will follow readily from Claim 4.6.3. We note that we have assumed $p \geq q \geq 2$ and $p + q \geq 6$.

Claim 4.6.3. *We recall from* (3.1.1) *the definitions of Cases 1 and 2. Then,*

$$\begin{aligned}\mathcal{F}(\sigma_+ f_{l,k})(\zeta) &= \frac{1}{8\sqrt{\pi}}\Gamma(\frac{p+q-4}{4}+\frac{l+k+\sqrt{-1}\zeta}{2})\\ &\times\begin{cases}\Gamma(\frac{p-q}{4}+\frac{l-k+1+\sqrt{-1}\zeta}{2}) & \textit{Case 1},\\ \Gamma(-\frac{p-q}{4}+\frac{-l+k+1+\sqrt{-1}\zeta}{2}) & \textit{Case 2}.\end{cases}\end{aligned} \tag{4.6.3}$$

Proof. By the definition (3.1.2) of $f_{l,k}$ and the definition (4.3.3) of σ_+, we have

$$(\sigma_+ f_{l,k})(x) = \frac{1}{\sqrt{2}} e^{(l+k+\frac{p+q}{2}-2)x} \times \begin{cases}\widetilde{K}_{\frac{q-3}{2}+k}(2e^x) & \text{Case 1},\\ \widetilde{K}_{\frac{p-3}{2}+l}(2e^x) & \text{Case 2}.\end{cases}$$

In Case 1, we have

$$\begin{aligned}\mathcal{F}(\sigma_+ f_{l,k})(\zeta) &= \frac{1}{\sqrt{2\pi}\sqrt{2}}\int_{-\infty}^{\infty} e^{(l+k+\frac{p+q}{2}-2)x}\widetilde{K}_{\frac{q-3}{2}+k}(2e^x)e^{\sqrt{-1}x\zeta}dx\\ &= \frac{1}{2\sqrt{\pi}}\int_0^{\infty} r^{(l+k+\frac{p+q}{2}-3+\sqrt{-1}\zeta)}\widetilde{K}_{\frac{q-3}{2}+k}(2r)dr.\end{aligned}$$

Applying the formula (7.2.11) of the Mellin transform for the K-Bessel function, we obtain the right-hand side of (4.6.3). Likewise, in Case 2, $\mathcal{F}(\sigma_+ f_{l,k})(\zeta)$ is equal to

$$\frac{1}{\sqrt{2\pi}\sqrt{2}}\int_{-\infty}^{\infty} e^{(l+k+\frac{p+q}{2}-2)x}\widetilde{K}_{\frac{p-3}{2}+l}(2e^x)e^{\sqrt{-1}x\zeta}dx.$$

Switching the role of (p,l) and (q,k), we see (4.6.3) holds also in Case 2. ☐

Step 4. $A_{\kappa_{l,k}} f_{l,k} = A_{K_{l,k}} f_{l,k}$.

To see this, we shall prove the following explicit formulas: Recall from (3.1.5) the notation $a(l,k)$.

Claim 4.6.4. *We have*

1) $A_{\kappa_{l,k}} f_{l,k} = (-1)^{a(l,k)+\frac{p-q}{2}} f_{l,k}$.
2) $A_{K_{l,k}} f_{l,k} = (-1)^{a(l,k)+\frac{p-q}{2}} f_{l,k}$.

Proof of Claim 4.6.4. As we shall see below, the proof of (1) is algebraic by using the fact that $\pi(w_0)$ acts on each K-type as $\pm\,\mathrm{id}$. On the other hand, the proof of (2) is based on an explicit integral computation.

1) The function $f_{l,k}$ belongs to the K'-invariant subspace $H_{l,k}$ (see (3.1.4)), and therefore, by Theorem 3.1.1 (3), we have

$$\pi(w_0)(f_{l,k}\phi\psi) = (-1)^{a(l,k)+\frac{p-q}{2}} f_{l,k}\phi\psi$$

for $\phi \in \mathcal{H}^l(\mathbb{R}^{p-1})$ and $\psi \in \mathcal{H}^k(\mathbb{R}^{q-1})$. In light of the definition of $T_{l,k}$ (see Theorem 4.1.1 (1)), this implies

$$T_{l,k} f_{l,k} = (-1)^{a(l,k)+\frac{p-q}{2}} f_{l,k}.$$

By the definition of $A_{\kappa_{l,k}}$ (see Proposition 4.4.4), the first statement follows.

2) We consider Case 1, namely, the case $\frac{p-q}{2} + l - k \geq 0$. By the integral expression of $K_{l,k}$ (see Lemma 4.5.5 and the definition (4.3.9) of $A_{K_{l,k}}$), we have

$$\begin{aligned}
(A_{K_{l,k}} f_{l,k})(r) &= \frac{1}{2\pi} \int_0^\infty \Big(\int_{L'} \psi(\zeta)(rr')^{-\frac{p+q-4}{2}+\sqrt{-1}\zeta} d\zeta \Big) f_{l,k}(r') r'^{p+q-5} dr' \\
&= \frac{1}{2\pi} \int_{L'} r^{-\frac{p+q-4}{2}+\sqrt{-1}\zeta} \Big(\int_0^\infty r'^{\frac{p+q-6}{2}+\sqrt{-1}\zeta} f_{l,k}(r') dr' \Big) \psi(\zeta) d\zeta \\
&= \frac{(-1)^{l+\frac{p-q}{2}}}{8\pi} \int_{L'} \Gamma\Big(\frac{p+q-4}{4} + \frac{l+k-\sqrt{-1}\zeta}{2} \Big) \\
&\qquad\qquad \times \Gamma\Big(\frac{p-q}{4} + \frac{l-k+1-\sqrt{-1}\zeta}{2} \Big) r^{-\frac{p+q-4}{2}+\sqrt{-1}\zeta} d\zeta \\
&= \frac{(-1)^{l+\frac{p-q}{2}}}{8\pi\sqrt{-1}} \int_L \Gamma\Big(\frac{l+k-s}{2} \Big) \Gamma\Big(\frac{-q+3+l-k-s}{2} \Big) r^s ds \qquad (4.6.4) \\
&= \frac{1}{2}(-1)^{l+\frac{p-q}{2}} G^{20}_{02}\Big(r^2 \Big| \frac{l+k}{2}, \frac{-q+3+l-k}{2} \Big) \\
&= (-1)^{l+\frac{p-q}{2}} r^{-\frac{q-3}{2}+l} K_{\frac{q-3}{2}+k}(2r) \\
&= (-1)^{a(l,k)+\frac{p-q}{2}} f_{l,k}(r).
\end{aligned}$$

Second equality. We recall the upper estimate of $|\psi(\zeta)|$ given in Lemma 4.5.4 (3) and the asymptotic behavior of $f_{l,k}(r')$ (see (3.4.1) and (3.4.2)). Then, in light of

$$|\psi(\zeta) r'^{\frac{p+q-6}{2}+\sqrt{-1}\zeta} f_{l,k}(r')| \leq |\psi(\zeta)| r'^{\frac{p+q-6}{2}+\eta} |f_{l,k}(r')| \quad \text{for } \zeta = \xi - \sqrt{-1}\eta,$$

the second equality follows from Fubini's theorem.

Third equality is by Lemma 3.4.3.

Fourth equality is from the change of the variable as before: $s = \sqrt{-1}\zeta - \frac{p+q-4}{2}$.

Fifth equality. The poles of the integrand in (4.6.4) are of the form $l+k+2a$ ($a \in \mathbb{N}$) or $-q+3+l-k+2a$ ($a \in \mathbb{N}$). These lie on the right of the contour L because of the assumption $\frac{p-q}{2}+l-k \geq 0$. Hence, the second equality holds by the definition of the G-function.

Sixth equality follows from the reduction formula of the G-function (see (7.6.12)).

Seventh equality is by the definition (3.1.2) of $f_{l,k}$ and the definition (3.1.5) of $a(l,k)$.

Case 2 can be treated in the same manner. In this case, the integral

$$\int_L \Gamma(\frac{l+k-s}{2})\Gamma(\frac{-p+3-l+k-s}{2})r^s ds$$

arises instead of (4.6.4). But again, by the assumption $\frac{p-q}{2}+l-k \leq 0$, this defines the G-function which reduces to $f_{l,k}$ by the same reduction formula. □

Step 5. $\kappa_{l,k} = K_{l,k}$.

Steps 3 and 4 imply $\kappa_{l,k} = K_{l,k}$ by Lemma 4.6.2. Thus, Proposition 4.5.6 is proved. □

Now the proof of Theorem 4.1.1 finishes.

5 Main theorem

This section gives a proof of the main result of this paper, namely, Theorem 5.1.1, where we find an explicit integral kernel for the unitary inversion operator $\pi(w_0)$ on the Schrödinger model $L^2(C)$ of the minimal representation.

5.1 Result of this section

Let C be the conical variety $\{\zeta \in \mathbb{R}^{p+q-2} \setminus \{0\} : Q(\zeta) = 0\}$ where $Q(\zeta) = \zeta_1^2 + \cdots + \zeta_{p-1}^2 - \zeta_p^2 - \cdots - \zeta_{p+q-2}^2$. We recall from Introduction that the generalized function $K(\zeta, \zeta')$ on $C \times C$ is defined by the following formula:

$$K(\zeta, \zeta') \equiv K(p, q; \zeta, \zeta') := c_{p,q} \Phi_{p,q}(\langle \zeta, \zeta' \rangle), \tag{5.1.1}$$

where $\langle \cdot, \cdot \rangle$ stands for the standard (positive definite) inner product on $\mathbb{R}^{p+q-2}$, $c_{p,q} = 2(-1)^{\frac{(p-1)(p+2)}{2}} \pi^{-\frac{p+q-4}{2}}$ (see (1.5.3)), and $\Phi_{p,q}$ is the distribution of one variable introduced in (1.5.4) as *Bessel distributions*.

Now the main result of this paper is stated as follows:

Theorem 5.1.1 (Integral formula for the unitary inversion operator)**.** *Let $(\pi, L^2(C))$ be the Schrödinger model of the minimal representation of $G = O(p, q)$ for $p, q \geq 2$ and $p+q \geq 6$ even, and $w_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Then the unitary operator $\pi(w_0) : L^2(C) \to L^2(C)$ is given by the following integro-differential operator:*

$$\pi(w_0) u(\zeta) = \int_C K(\zeta, \zeta') u(\zeta') d\mu(\zeta'), \qquad u \in L^2(C). \tag{5.1.2}$$

The right-hand side of (5.1.2) involves a singular integral. It factors through the Radon transform (see Subsection 5.2), and we shall see that the right-hand side of (5.1.2) is well-defined for any compactly supported smooth function u on C, and extends as a unitary operator on $L^2(C)$.

As for the Bessel distribution $\Phi_{p,q}$, we shall give a Mellin–Barnes type integral formula for $\Phi_{p,q}$ in Subsection 6.2, and the differential equation satisfied by $\Phi_{p,q}$ in Subsection 6.3.

Since the action of the maximal parabolic subgroup $\overline{P^{\max}}$ on $L^2(C)$ is of a simple form (see (2.3.7)–(2.3.10)), Theorem 5.1.1 gives an explicit action of the whole group G on $L^2(C)$ because $G = \overline{P^{\max}} \coprod \overline{P^{\max}} w_0 \overline{P^{\max}}$.

Theorem 5.1.1 immediately yields two corollaries about the Plancherel formula and the reciprocal formula of our integral transform.

Corollary 5.1.2 (Plancherel formula)**.** *Let $S : L^2(C) \to L^2(C)$ be an integral transform whose kernel function is given by $K(\zeta, \zeta')$ (see (5.1.1)). Then S is unitary:*

$$\|Su\|_{L^2(C)} = \|u\|_{L^2(C)}.$$

Since the group law $w_0^2 = 1$ in $O(p, q)$ implies $\pi(w_0)^2 = \mathrm{id}$ on $L^2(C)$, we immediately obtain the following corollary to Theorem 5.1.1, which can also be viewed as giving the inversion formula $S^{-1} = S$.

Corollary 5.1.3 (Reciprocal formula)**.** *Retain the notation as in Corollary 5.1.2. The unitary operator S is of order two in $L^2(C)$. Namely, we have the following reciprocal formula for $u \in L^2(C)$:*

$$u(\zeta) = \int_C K(\zeta, \zeta'') \Big(\int_C K(\zeta'', \zeta') u(\zeta') d\mu(\zeta') \Big) d\mu(\zeta'').$$

Remark 5.1.4 (Comparison with the Schrödinger model of the Weil representation)**.** *In the case of the Schrödinger model of the Segal–Shale–Weil representation ϖ of the metaplectic group $Mp(n, \mathbb{R})$, the corresponding 'inversion' element w_0' acts on $L^2(\mathbb{R}^n)$ as $e^{\frac{\sqrt{-1}n\pi}{4}} \mathcal{F}$, where $\mathcal{F}$ denotes the Fourier transform. We note that $(w_0')^4$ gives the unique non-trivial element ξ_0 in the kernel of the metaplectic covering $Mp(n, \mathbb{R}) \to Sp(n, \mathbb{R})$, and $\varpi(\xi_0) = -\mathrm{id}$. This fact reflects the identity $\mathcal{F}^4 = \mathrm{id}$ on $L^2(\mathbb{R}^n)$. Thus, the above two corollaries can be interpreted as the counterparts to the Plancherel formula and the equality $\mathcal{F}^4 = \mathrm{id}$ of the Fourier transform $\mathcal{F}$ on $\mathbb{R}^n$.*

Remark 5.1.5. *In [31, Corollary 6.3.1], we gave a different proof of the same Plancherel and reciprocal formulas in the case $q = 2$ based on analytic continuation of holomorphic semigroup of operators.*

This section is organized as follows. In Subsection 5.2, we analyze the integral transform (5.1.2) by means of the (singular) Radon transform. In particular, we prove that the integral transform (5.1.2) is well-defined in the sense of distributions for $u \in C_0^\infty(C)$. The key ingredient of the proof of Theorem 5.1.1 is the restriction from G to $K' = K \cap M_+^{\max}$ (see Subsection 2.3) and is to show the (l, k)-th spectrum of the unitary inversion operator $\pi(w_0)$ coincides with the radial part $T_{l,k}$ of $\pi(w_0)$ when restricted to each K'-isotypic component $\mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1})$ (see Lemma 5.4.1). The latter operator $T_{l,k}$ was studied in details in the previous section (see Theorem 4.1.1). Subsection 5.3 is devoted to the formula giving spectra of a K'-invariant integral operator.

5.2 Radon transform-analog of the plane wave expansion

This subsection studies the kernel $K(\zeta, \zeta')$. The subtle point in defining $K(\zeta, \zeta')$ is that the distribution $\Phi_{p,q}(t)$ is not locally integrable near $t = 0$, whereas the level set $\langle \zeta, \zeta' \rangle = t$ is not a regular submanifold in $C \times C$ if $t = 0$. In fact, $\Phi_{p,q}(t)$ involves a linear combination of distributions $\delta^{(k-1)}(t)$ and t^{-k} $(k = 1, 2, \ldots, \frac{p+q-6}{2})$ as we shall see in Subsection 6.2 on the one hand, and the differential form

$$dQ(\zeta) \wedge dQ(\zeta') \wedge d(\langle \zeta, \zeta' \rangle - t)$$

of ζ, ζ' vanishes if (ζ, ζ') belongs to the submanifold

$$Y := \{(\zeta, \zeta') \in C \times C : \mathbb{R}\zeta = \mathbb{R}(I_{p,q}\zeta')\}.$$

Here, we note that $Y \subset \{(\zeta, \zeta') : \langle \zeta, \zeta' \rangle = 0\}$.

Our idea to give a rigorous definition of $K(\zeta, \zeta')$ is factorize the transform (5.1.2) by using the (singular) Radon transform.

Let δ denote the Dirac delta function of one variable. The *Radon transform* of a function φ on $\mathbb{R}^{p+q-2}$ is defined by the formula (see for example [15, Chapter I]):

$$(R\varphi)(\zeta, t) := \int_{\mathbb{R}^{p+q-2}} \varphi(\zeta')\delta(t - \langle \zeta, \zeta' \rangle)d\zeta', \tag{5.2.1}$$

for $\zeta \in \mathbb{R}^{p+q-2} \setminus \{0\}$, $t \in \mathbb{R}$.

The Radon transform $R\varphi$ is well-defined, for example, for a compactly supported continuous function φ. More generally, $R\varphi$ makes sense if φ is a compactly supported distribution such that the multiplication of two distributions $\varphi(\zeta')$ and $\delta(t - \langle \zeta, \zeta' \rangle)$ makes sense.

Now we recall the injective map (see (2.2.6))

$$T : L^2(C) \to \mathcal{S}'(\mathbb{R}^{p+q-2}), \quad u \mapsto u\delta(Q)$$

yields a compactly supported distribution Tu if $u \in C_0^\infty(C)$. In this context, what we need here is the following result:

Let $C_0^k(\mathbb{R})$ denote the space of compactly supported functions on $\mathbb{R}$ with continuous derivatives up to k.

Lemma 5.2.1. *Suppose $u \in C_0^\infty(C)$.*

0) *The Radon transform $R(Tu)(\zeta, t)$ is well-defined and continuous as a function of $(\zeta, t) \in C \times (\mathbb{R} \setminus \{0\})$. Moreover, there exists $A > 0$ such that*

$$\operatorname{Supp} R(Tu) \subset \{(\zeta, t) \in C \times (\mathbb{R} \setminus \{0\}) : t \leq A|\zeta|\},$$

where $|\zeta| := (\zeta_1^2 + \cdots + \zeta_{p+q-2}^2)^{\frac{1}{2}}$*. In particular,* $R(Tu)(\zeta, t)$ *vanishes if* $|t|$ *is sufficiently large for a fixed* $\zeta \in C$*.*

1) *If* $p, q > 2$ *and* $p + q \geq 8$*, then* $R(Tu)(\zeta, t)$ *extends continuously to* $t = 0$ *and* $R(Tu)(\zeta, \cdot) \in C_0^k(\mathbb{R})$ *where* $k := \frac{p+q-8}{2}$*.*

2) *If* $\min(p, q) = 2$*, then* $R(Tu)(\zeta, t)$ *is bounded on* $C \times (\mathbb{R} \setminus \{0\})$*.*

3) *If* $p, q > 2$ *and* $p + q = 6$ *(namely,* $(p, q) = (3, 3)$*), then there exists* $C \equiv C(\zeta) > 0$ *such that*

$$|R(Tu)(\zeta, t)| \leq C\, |\log |t||$$

if t *is sufficiently small.*

Proof. See [38]. □

We note that $R(Tu)(\zeta, t)$ is well-defined for $(\zeta, t) \in (\mathbb{R}^{p+q} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$, but we need here only the case where $\zeta \in C$.

In order to justify the right-hand side of (5.1.2) for $u \in C_0^\infty(C)$, we consider

$$\begin{aligned}
(Su)(\zeta) &:= \int_C K(\zeta, \zeta')u(\zeta')d\mu(\zeta') \\
&= c_{p,q} \int_{\mathbb{R}^{p+q-2}} \Phi_{p,q}(\langle \zeta, \zeta' \rangle) Tu(\zeta')d\zeta' \qquad \text{by (5.1.1)} \\
&= c_{p,q} \int_{\mathbb{R}} \int_{\langle \zeta, \zeta' \rangle = t} \Phi_{p,q}(\langle \zeta, \zeta' \rangle) Tu(\zeta')d\omega(\zeta')dt \\
&= c_{p,q} \int_{\mathbb{R}} \Phi_{p,q}(t) R(Tu)(\zeta, t)dt. \qquad (5.2.2)
\end{aligned}$$

Lemma 5.2.2. *The right-hand side of* (5.2.2) *is well-defined for* $u \in C_0^\infty(C)$*.*

The above lemma defines a linear map

$$S : C_0^\infty(C) \to C^\infty(C),$$

and defines $K(\zeta, \zeta')$ as a distribution on the direct product manifold $C \times C$. In Subsection 5.4, we shall see that the image is contained in $L^2(C)$, and S extends to a unitary operator on $L^2(C)$, which in turn equals the unitary inversion operator $\pi(w_0)$.

Proof. It follows from Theorem 6.2.1 which we shall prove later and from the definition (1.5.4) of the distribution $\Phi_{p,q}(t)$ that $\Phi_{p,q}(t)$ has the following decomposition:

$$\Phi_{p,q}(t) = \Phi_{p,q}^{\mathrm{reg}}(t) + \Phi_{p,q}^{\mathrm{sing}}(t),$$

where $\Phi_{p,q}^{\mathrm{reg}}(t)$ and $\Phi_{p,q}^{\mathrm{sing}}(t)$ are distributions on $\mathbb{R}$ such that

1) $\Phi_{p,q}^{\mathrm{reg}}(t)|t|^{-\epsilon}$ is a locally integrable function on $\mathbb{R}$ for any sufficiently small $\epsilon \geq 0$,

2)
$$\Phi_{p,q}^{\mathrm{sing}}(t) = \begin{cases} 0 & \text{if } \min(p,q) = 2, \\ -\displaystyle\sum_{k=1}^{\frac{p+q-6}{2}} \frac{(-1)^{k-1}}{2^k(m-k)!}\delta^{(k-1)}(t) & \text{if } p,q > 2 \text{ both even}, \\ -\dfrac{1}{\pi}\displaystyle\sum_{k=1}^{m} \frac{(k-1)!}{2^k(m-k)!}t^{-k} & \text{if } p,q > 2 \text{ both odd}. \end{cases}$$

We note that $\Phi_{p,q}^{\mathrm{sing}}(t) \neq 0$ only if $p,q > 2$ and $p+q \geq 8$. Combining with Lemma 5.2.1, we see that the right-hand side of (5.2.2) is well-defined in all the cases. □

Remark 5.2.3. *The plane wave expansion gives a decomposition of the Fourier transform $\mathcal{F}$ on $L^2(\mathbb{R}^n)$ into the one-dimensional integral transform (Mellin transform) and the Radon transform, namely:*

$$(\mathcal{F}u)(\zeta) = c_n\langle\Psi, (Ru)(\zeta,\cdot)\rangle,$$

where $c_n := \left(\frac{1}{2\pi}\right)^{\frac{n}{2}}$ and $\Psi(t) := e^{\sqrt{-1}t}$ (e.g. [15, Chapter I, §1.2]). In this sense, the formula (5.2.2) can be regarded as an analog of the plane wave expansion for the unitary operator $\pi(w_0)$ on $L^2(C)$.

5.3 Spectra of a K'-invariant operator on $S^{p-2} \times S^{q-2}$

The expansion into spherical harmonics

$$L^2(S^{n-1}) \simeq \sum_{l=0}^{\infty}{}^{\oplus} \mathcal{H}^l(\mathbb{R}^n)$$

gives a multiplicity-free decomposition of $O(n)$ into its irreducible representations (see Appendix 7.5), and consequently, any $O(n)$-intertwining operator on $L^2(S^{n-1})$ acts on $\mathcal{H}^l(\mathbb{R}^n)$ as a scalar multiplication owing to Schur's lemma. The scalar is given by the Funk–Hecke formula (see [1, §9.7], see also [31, Lemma 5.5.1]): for an integrable function h on the interval $[-1,1]$ and for $\phi \in \mathcal{H}^l(\mathbb{R}^n)$,

$$\int_{S^{n-1}} h(\langle\omega,\omega'\rangle)\phi(\omega')d\omega' = c_{l,n}(h)\phi(\omega)$$

where the eigenvalue $c_{l,n}(h)$ is given by

$$c_{l,n}(h) = \frac{2^{n-2}\pi^{\frac{n-2}{2}}l!}{\Gamma(n-2+l)}\int_{-1}^{1} h(x)\widetilde{C}_l^{\frac{n-2}{2}}(x)(1-x^2)^{\frac{n-3}{2}}dx. \tag{5.3.1}$$

Here, $\widetilde{C}_l^{\mu}(x)$ stands for the normalized Gegenbauer polynomial (see Subsection 7.4).

Likewise, any K'-intertwining operator on $L^2(S^{p-2}\times S^{q-2})$ acts on $\mathcal{H}^l(\mathbb{R}^{p-1})\otimes\mathcal{H}^k(\mathbb{R}^{q-1})$ as a scalar multiplication for each $k, l \in \mathbb{N}$ (we recall $K' \simeq O(p-1)\times O(q-1)$). In this subsection, we determine this scalar for specific intertwining (integral) operators. In particular, the scalar in Example 5.3.2 will be used in the proof of our main theorem (Theorem 5.1.1).

We begin with a general setup for a K'-intertwining operator on $L^2(S^{p-2}\times S^{q-2})$. Let h be an integrable function of two variables on $[-1,1]\times[-1,1]$. We consider the following integral transform:

$$\begin{aligned} B_h : C(S^{p-2}\times S^{q-2}) &\to C(S^{p-2}\times S^{q-2}), \\ \varphi(\omega,\eta) &\mapsto \int_{S^{p-2}\times S^{q-2}} h(\langle\omega,\omega'\rangle,\langle\eta,\eta'\rangle)\varphi(\omega',\eta')d\omega'd\eta'. \end{aligned} \tag{5.3.2}$$

Then, B_h becomes a K'-invariant integral operator on $L^2(S^{p-2}\times S^{q-2})$.

Lemma 5.3.1. *B_h acts on each K'-type $\mathcal{H}^l(\mathbb{R}^{p-1})\otimes\mathcal{H}^k(\mathbb{R}^{q-1})$ by a scalar multiplication of $\alpha_{l,k}(h)\in\mathbb{C}$. The spectrum $\alpha_{l,k}(h)$ is given by the following formulas.*

1) *If $\min(p,q)=2$, say $q=2$, then for $k=0,1$,*

$$\alpha_{l,k}(h) = \frac{2^{p-3}\pi^{\frac{p-3}{2}}l!}{\Gamma(p-3+l)}\int_{-1}^{1}(U_k h)(x)\widetilde{C}_l^{\frac{p-3}{2}}(x)(1-x^2)^{\frac{p-4}{2}}dx, \tag{5.3.3}$$

where we set

$$(U_k h)(x) := h(x,1) + (-1)^k h(x,-1). \tag{5.3.4}$$

For $k\geq 2$, $\alpha_{l,k}(h)=0$.

2) *If $p, q > 2$, then*

$$\begin{aligned} \alpha_{l,k}(h) = {} & \frac{2^{p+q-6}\pi^{\frac{p+q-6}{2}}l!\,k!}{\Gamma(p-3+l)\Gamma(q-3+k)} \\ & \times \int_{-1}^{1}\int_{-1}^{1} h(x,y)\widetilde{C}_l^{\frac{p-3}{2}}(x)\widetilde{C}_k^{\frac{q-3}{2}}(y)(1-x^2)^{\frac{p-4}{2}}(1-y^2)^{\frac{q-4}{2}}dxdy. \end{aligned} \tag{5.3.5}$$

Proof. 1) If $q = 2$, then $S^{p-1} \times S^{q-1} = S^{p-1} \coprod S^{p-1}$ (disjoint union), and $\mathcal{H}^k(\mathbb{R}^{q-1}) = 0$ if $k \geq 2$ (see Subsection 7.5). Then, the formula (5.3.3) is essentially the Funk–Hecke formula (5.3.1) for S^{p-1}.

2) Applying (5.3.1) to each factor, we get (5.3.5). $\square$

Here, we give an example of the explicit computation of the spectrum $\alpha_{l,k}(h)$.

Example 5.3.2 (Riesz potential)**.** *Consider the following Riesz potential for* $\operatorname{Re}\lambda > -1$*:*

$$h_\lambda^{\pm}(x,y) := \frac{(x+y)_\pm^\lambda}{\Gamma(\lambda+1)},$$
$$= \begin{cases} \dfrac{(x+y)^\lambda}{\Gamma(\lambda+1)} & \textit{if } \ \epsilon(x+y) > 0, \\ 0 & \textit{if } \ \epsilon(x+y) \leq 0, \end{cases} \tag{5.3.6}$$

where $\epsilon = \pm 1$*. Then, the spectrum* $\alpha_{l,k}(h_\lambda^\pm)$ *for the* K'*-intertwining operator* $B_{h_\lambda^\pm}$ *amounts to*

$$\alpha_{l,k}(h_\lambda^\pm) = \frac{2^{1-\lambda}\pi^{\frac{p+q-2}{2}}(\pm 1)^{l+k}\Gamma(\lambda+\frac{p+q-4}{2})}{\Gamma(\frac{\lambda+p+q-4+l+k}{2})\Gamma(\frac{\lambda+p-1+l-k}{2})\Gamma(\frac{\lambda+q-1-l+k}{2})\Gamma(\frac{\lambda-l-k+2}{2})}$$
$$= \frac{(\pm 1)^{l+k}}{\pi}\Gamma\Big(\lambda+\tfrac{p+q-4}{2}\Big)\sin\Big(\tfrac{\lambda-l-k+2}{2}\pi\Big)\sin\Big(\tfrac{\lambda+q-1-l+k}{2}\pi\Big)\gamma_{l,k}(\lambda),$$

where we set

$$\gamma_{l,k}(\lambda) := 2^{1-\lambda}\pi^{\frac{p+q-4}{2}}\frac{\Gamma(\frac{l+k-\lambda}{2})\Gamma(\frac{-q+3+l-k-\lambda}{2})}{\Gamma(\frac{\lambda+p+q+l+k-4}{2})\Gamma(\frac{\lambda+p-1+l-k}{2})}. \tag{5.3.7}$$

Proof of Example 5.3.2. Use (5.3.5). We postpone the actual computation of the integral (the first equation of $\alpha_{l,k}(h_\lambda^\pm)$) until Appendix (see Lemma 7.9.1 with $\mu = \frac{p-3}{2}, \nu = \frac{q-3}{2}$). In the second equation of $\alpha_{l,k}(h_\lambda^\pm)$, we have used the functional equation $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)}$. $\square$

We define a kernel function $h_\lambda(x,y) \equiv h_\lambda^{p,q}(x,y)$ with parameter λ as follows:

$$h_\lambda(x,y) := \frac{\Gamma(-\lambda)}{\Gamma(\lambda+\frac{p+q-4}{2})} \times \begin{cases} (x+y)_+^\lambda & \text{if } p,q > 2 \text{ both even,} \\ \left(\dfrac{(x+y)_+^\lambda}{\tan\lambda\pi} + \dfrac{(x+y)_-^\lambda}{\sin\lambda\pi}\right) & \text{if } p,q > 2 \text{ both odd.} \end{cases} \tag{5.3.8}$$

Proposition 5.3.3. *Let* $\operatorname{Re}\lambda > -1$*. For a kernel function* h_λ *(see* (5.3.8)*), the spectrum* $\alpha_{l,k}(h_\lambda)$ *given in Lemma 5.3.1 amounts to*

$$\alpha_{l,k}(h_\lambda) = \frac{(-1)^{l+[\frac{q-3}{2}]}\pi^{\frac{p+q-4}{2}}}{2^\lambda} \frac{\Gamma(\frac{l+k-\lambda}{2})\Gamma(\frac{-q+3+l-k-\lambda}{2})}{\Gamma(\frac{\lambda+p+q+l+k-4}{2})\Gamma(\frac{\lambda+p-1+l-k}{2})} \tag{5.3.9}$$

$$= \frac{(-1)^{k+[\frac{p-3}{2}]}\pi^{\frac{p+q-4}{2}}}{2^\lambda} \frac{\Gamma(\frac{l+k-\lambda}{2})\Gamma(\frac{-p+3-l+k-\lambda}{2})}{\Gamma(\frac{p+q-4+l+k+\lambda}{2})\Gamma(\frac{q-1-l+k+\lambda}{2})}. \tag{5.3.10}$$

Proof. The second equation (5.3.10) follows from the identity (4.5.4) of gamma functions. Let us show the first equation (5.3.9). In terms of $h_\lambda^\pm$ defined in (5.3.6), we rewrite h_λ (see (5.3.8)) as

$$h_\lambda = \frac{\pi}{\Gamma(\lambda + \frac{p+q-4}{2})\sin(-\lambda\pi)} \times \begin{cases} h_\lambda^+ & \text{if } p, q \text{ both even,} \\ \frac{h_\lambda^+}{\tan(\lambda\pi)} + \frac{h_\lambda^-}{\sin(\lambda\pi)} & \text{if } p, q \text{ both odd.} \end{cases} \tag{5.3.11}$$

Since $\alpha_{l,k}$ is linear, i.e., $\alpha_{l,k}(ah + bg) = a\alpha_{l,k}(h) + b\alpha_{l,k}(g), a, b \in \mathbb{C}$, by (5.3.11), we have

$$\alpha_{l,k}(h_\lambda) = C_{l,k}(\lambda)\gamma_{l,k}(\lambda),$$

(see (5.3.7)). Here,

$$C_{l,k}(\lambda) := \frac{\sin\frac{\lambda-l-k+2}{2}\pi \sin\frac{\lambda+q-1-l+k}{2}\pi}{\sin(-\lambda\pi)} \times \begin{cases} 1 & \text{if } p, q \text{ both even,} \\ \frac{1}{\tan(\lambda\pi)} + \frac{(-1)^{l+k}}{\sin(\lambda\pi)} & \text{if } p, q \text{ both odd.} \end{cases}$$

Hence, Proposition is proved by the following claim: $\square$

Claim 5.3.4.

$$C_{l,k}(\lambda) = \frac{(-1)^{l+[\frac{q-1}{2}]}}{2}.$$

Proof. Let us first consider the case where both p and q are even. Then, the two integers $-l-k+2$ and $q-1-l+k$ have different parities. Hence,

$$C_{l,k}(\lambda) = (-1)^{l+\frac{q-2}{2}} \frac{\sin\frac{\lambda}{2}\pi\cos\frac{\lambda}{2}\pi}{\sin\lambda\pi} = \frac{(-1)^{l+\frac{q-2}{2}}}{2}.$$

Next, suppose both p and q are odd. Then,

$$\frac{1}{\tan(\lambda\pi)} + \frac{(-1)^{l+k}}{\sin(\lambda\pi)} = \begin{cases} \frac{1}{\tan\frac{\lambda}{2}\pi}, \\ -\tan\frac{\lambda}{2}\pi, \end{cases}$$

$$\frac{\sin\frac{\lambda-l-k+2}{2}\pi\sin\frac{\lambda+q-1-l+k}{2}\pi}{\sin(-\lambda)\pi} = \begin{cases} (-1)^{-l+\frac{q-1}{2}}\frac{\sin^2\frac{\lambda}{2}\pi}{\sin\lambda\pi} = \frac{(-1)^{-l+\frac{q-1}{2}}}{2}\tan\frac{\lambda}{2}\pi, \\ (-1)^{-l+\frac{q+1}{2}}\frac{\cos^2\frac{\lambda}{2}\pi}{\sin\lambda\pi} = \frac{(-1)^{-l+\frac{q+1}{2}}}{2}\frac{1}{\tan\frac{\lambda}{2}\pi}, \end{cases}$$

according as $l+k$ is even (upper row) and odd (lower row). Thus we have

$$C_{l,k}(\lambda) = \frac{(-1)^{\frac{q-1}{2}-l}}{2}$$

in either case. Hence, Claim 5.3.4 is verified. $\square$

Let T be the triangular domain in $\mathbb{R}^2$ given by

$$T := \{(x,y) \in \mathbb{R}^2 : x < 1, y < 1, 0 < x+y\},$$

and define a function $g_\lambda(x,y)$ with parameter $\lambda \in \mathbb{C}$ by

$$g_\lambda(x,y) := \begin{cases} (x+y)^\lambda (1-x^2)^{\frac{p-4}{2}} (1-y^2)^{\frac{q-4}{2}} & (x,y) \in T, \\ 0 & (x,y) \notin T. \end{cases} \tag{5.3.12}$$

Lemma 5.3.5. 1) *For* $\operatorname{Re}\lambda > -1$, g_λ *is a distribution of compact support, and with holomorphic parameter* λ. *That is,* $\langle g_\lambda, \varphi\rangle$ *is holomorphic in* $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -1\}$ *for any* $\varphi \in C^\infty(\mathbb{R}^2)$.

2) g_λ *extends meromorphically to all* $\lambda \in \mathbb{C}$. *That is,* $\langle g_\lambda, \varphi\rangle$ *is a meromorphic function with respect to* $\lambda \in \mathbb{C}$ *for any* $\varphi \in C^\infty(\mathbb{R}^2)$.

Proof. The first statement is clear because $g_\lambda \in L'(\mathbb{R}^2)$ if $\operatorname{Re}\lambda > -1$. For the second statement, we rewrite g_λ as

$$g_\lambda(x,y) = (x+y)_+^\lambda (1-x)_+^{\frac{p-4}{2}} (1+x)_+^{\frac{p-4}{2}} (1-y)_+^{\frac{q-4}{2}} (1+y)_+^{\frac{q-4}{2}}.$$

Then, Lemma follows from Bernstein's theorem [4]. $\square$

5.4 Proof of Theorem 5.1.1

As in Lemma 5.2.2, we shall denote by S the map $u(\zeta) \mapsto \int_C K(\zeta,\zeta')u(\zeta')d\mu(\zeta')$. Since S commutes with the K'-action ($K' \simeq O(p-1) \times O(q-1)$), S preserves each K'-isotypic component of $L^2(C)$ given in the decomposition (see (4.1.1)):

$$L^2(C) \simeq \sum_{l,k=0}^{\infty}{}^{\oplus} L^2(\mathbb{R}_+, \frac{1}{2} r^{p+q-5} dr) \otimes \mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1}).$$

On the other hand, we have seen in Theorem 4.1.1 that $\pi(w_0)$ also preserves each K'-isotypic component, and accordingly has a decomposition:

$$\pi(w_0) = \sum_{l,k=0}^{\infty}{}^{\oplus} T_{l,k} \otimes \mathrm{id} \otimes \mathrm{id},$$

where $T_{l,k}$ is a unitary operator on $L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}dr)$ whose kernel $K_{l,k}(t)$ is explicitly given in (4.1.2).

We shall show the equality $S = \pi(w_0)$ by restricting to each (l,k) component, namely,

Lemma 5.4.1. *For each $l, k \in \mathbb{N}$, we have*

$$S|_{L^2(\mathbb{R}_+, r^{p+q-5}dr)\otimes\mathcal{H}^l(\mathbb{R}^{p-1})\otimes\mathcal{H}^k(\mathbb{R}^{q-1})} = T_{l,k} \otimes \mathrm{id} \otimes \mathrm{id}\,. \tag{5.4.1}$$

Instead of proving Lemma 5.4.1, we shall provide Lemma 5.4.2 on the spectra $\alpha_{l,k}$ and the kernel functions $K_{l,k}$, which turns out to be equivalent to Lemma 5.4.1. For that purpose, we set

$$h_{r,r'}(x,y) := c_{p,q}\Phi_{p,q}(rr'(x+y)), \tag{5.4.2}$$

where $c_{p,q}$ and $\Phi_{p,q}$ are defined in (1.5.3) and (1.5.4). Then, by the definition (5.1.1) of $K(\zeta, \zeta')$, we have

$$\begin{aligned} K(\begin{pmatrix} r\omega \\ r\eta \end{pmatrix}, \begin{pmatrix} r'\omega' \\ r'\eta' \end{pmatrix}) &= c_{p,q}\Phi_{p,q}\big(rr'(\langle w, w'\rangle + \langle \eta, \eta'\rangle)\big) \\ &= h_{r,r'}(\langle \omega, \omega'\rangle, \langle \eta, \eta'\rangle). \end{aligned}$$

Suppose $f(r)u(\omega,\eta) \in L^2(\mathbb{R}_+, r^{p+q-5}dr) \otimes \mathcal{H}^l(\mathbb{R}^{p-1}) \otimes \mathcal{H}^k(\mathbb{R}^{q-1})$. Then, it follows from Theorem 4.1.1(2) that

$$\begin{aligned} &\big((T_{l,k} \otimes \mathrm{id} \otimes \mathrm{id})(fu)\big)(r\omega, r\eta) \\ &= (T_{l,k}f)(r)u(\omega,\eta) \\ &= \frac{1}{2}\int_0^\infty K_{l,k}(rr')f(r')r'^{p+q-5}dr'u(\omega,\eta). \end{aligned}$$

On the other hand,

$$\begin{aligned} &S(fu)(r\omega, r\eta) \\ &= \int_C K\bigg(\begin{pmatrix} r\omega \\ r\eta \end{pmatrix}, \zeta'\bigg)(fu)(\zeta')d\mu(\zeta') \\ &= \frac{1}{2}\int_0^\infty \int_{S^{p-2}} \int_{S^{q-2}} h_{r,r'}(\langle \omega, \omega'\rangle, \langle \eta, \eta'\rangle) f(r')u(\omega', \eta')r'^{p+q-5}dr'd\omega'd\eta' \\ &= \frac{1}{2}\int_0^\infty (B_{h_{r,r'}}u)f(r')r'^{p+q-5}dr'. \end{aligned}$$

Since $B_{h_{r,r'}}u = \alpha_{l,k}(h_{r,r'})u$ by Lemma 5.3.1, we have

$$S(fu)(r\omega, r\eta) = \frac{1}{2}\int_0^\infty \alpha_{l,k}(h_{r,r'})f(r')r'^{p+q-5}dr'u(\omega,\eta).$$

Thus, in light of Theorem 4.1.1 (2), (5.4.1) is equivalent to showing the following formula between kernel functions:

Lemma 5.4.2. *We have*

$$\alpha_{l,k}(h_{r,r'}) = K_{l,k}(rr').$$

The proof of Lemma 5.4.2 will be given in the following two Subsections.

5.5 Proof of Lemma 5.4.2 ($\min(p, q) = 2$)

Suppose $q = 2$. By the definition (1.5.4) of $\Phi_{p,2}(t)$, the definition (5.4.2) of $h_{r,r'}$ amounts to:

$$h_{r,r'}(x, y) = c_{p,2}\Phi^{+}_{\frac{p-4}{2}}(rr'(x + y)).$$

Since $\Phi^{+}_{\frac{p-4}{2}}(t)$ is a locally integrable function supported on $t \geq 0$ (see (1.4.1) and Theorem 6.2.1), we have from the definition (5.3.4) of the operator U_k: for $-1 \leq x \leq 1$,

$$\begin{aligned}
(U_k h_{r,r'})(x) &= c_{p,2}\Phi^{+}_{\frac{p-4}{2}}(rr'(x + 1)) + (-1)^k c_{p,2}\Phi^{+}_{\frac{p-4}{2}}(rr'(x - 1)) \\
&= c_{p,2}\Phi^{+}_{\frac{p-4}{2}}(rr'(x + 1)) \\
&= \frac{2^{-\frac{p-8}{4}}(-1)^{\frac{(p-1)(p+2)}{2}}}{\pi^{\frac{p-2}{2}}}(rr')^{-\frac{p-4}{4}}(x + 1)^{-\frac{p-4}{4}} J_{\frac{p-4}{2}}(2\sqrt{2rr'(x + 1)}).
\end{aligned}$$

Applying the formula (5.3.3), $\alpha_{l,k}(h_{r,r'})$ amounts to

$$\begin{aligned}
\alpha_{l,k}(h_{r,r'}) &= \frac{2^{\frac{3p-4}{4}}(-1)^{\frac{(p-1)(p+2)}{2}} l!}{\sqrt{\pi}\Gamma(p - 3 + l)}(rr')^{-\frac{p-4}{4}} \times \\
&\quad \int_{-1}^{1} J_{\frac{p-4}{2}}(2\sqrt{2rr'(x + 1)})\widetilde{C}_l^{\frac{p-3}{2}}(x)(1 + x)^{\frac{p-4}{4}}(1 - x)^{\frac{p-4}{2}} dx \\
&= 4(-1)^{\frac{(p-1)(p+2)}{2}+l}(rr')^{-\frac{p-3}{2}} J_{p-3+2l}(4\sqrt{rr'}) \\
&= (-1)^{\frac{p^2}{2}} K_{l,k}(rr').
\end{aligned}$$

Here, the second equality follows from (7.4.10) with $\alpha = 2\sqrt{2rr'}$ and $\nu = \frac{p-4}{2}$, and the last equality follows from (4.1.4). Since p is even if $q = 2$, the right-hand side is equal to $K_{l,k}(rr')$. Hence, Lemma 5.4.2 is proved for $q = 2$. $\square$

5.6 Proof of Lemma 5.4.2 ($p, q > 2$)

First, we give an integral formula of Mellin–Barnes type for $h_{r,r'}(x, y)$ (see (5.4.2)) by means of $h_\lambda(x, y)$ (see (5.3.8)): Suppose $p, q > 2$.

Claim 5.6.1. *Let $\gamma > -1$ and L be a contour that starts at $\gamma - \sqrt{-1}\infty$ and ends at $\gamma+\sqrt{-1}\infty$ and that passes the real axis in the interval $(-\frac{p+q-4}{2}, -\frac{p+q-6}{2})$. Then, we have*

$$h_{r,r'}(x, y) = \frac{c_{p,q}}{2\pi\sqrt{-1}} \int_L (2rr')^\lambda h_\lambda(x, y) d\lambda. \tag{5.6.1}$$

Proof. By the definition (5.3.8) of $h_\lambda(x, y) \equiv h_\lambda^{p,q}(x, y)$ and the integral formulas (6.2.2) and (6.2.4) of $\Psi^+_{\frac{p+q-6}{2}}(t)$ and $\Psi_{\frac{p+q-6}{2}}(t)$ respectively, we have

$$\frac{1}{2\pi\sqrt{-1}} \int_L s^\lambda h_\lambda(x, y) d\lambda = \begin{cases} \Psi^+_{\frac{p+q-6}{2}}\left(\frac{s(x+y)}{2}\right) & p, q \text{ even}, \\ \Psi_{\frac{p+q-6}{2}}\left(\frac{s(x+y)}{2}\right) & p, q \text{ odd}, \end{cases}$$

for $s > 0$. In either case, it follows from the definition(1.5.4) of $\Phi_{p,q}(t)$ that

$$\frac{1}{2\pi\sqrt{-1}} \int_L s^\lambda h_\lambda(x, y) d\lambda = \Phi_{p,q}\Big(\frac{s(x + y)}{2}\Big). \tag{5.6.2}$$

Hence, we get Claim5.6.1 by the definition (5.4.2) of $h_{r,r'}(x, y)$. $\square$

Thus,

$$\begin{aligned} \alpha_{l,k}(h_{r,r'}) &= \frac{c_{p,q}}{2\pi\sqrt{-1}} \int_L \alpha_{l,k}(h_\lambda)(2rr')^\lambda d\lambda \\ &= \frac{(-1)^{l+\frac{p-q}{2}}}{\pi\sqrt{-1}} \int_L \frac{\Gamma(\frac{l+k-\lambda}{2})\Gamma(\frac{-q+3+l-k-\lambda}{2})}{\Gamma(\frac{\lambda+p+q+l+k-4}{2})\Gamma(\frac{\lambda+p-1+l-k}{2})} (rr')^\lambda d\lambda \\ &= K_{l,k}(rr'). \end{aligned}$$

Here, in the second equality, we applied Proposition 5.3.3 and then used the equality $(-1)^{l+[\frac{q-1}{2}]}(-1)^{\frac{(p-1)(p+2)}{2}} = (-1)^{l+\frac{p-q}{2}}$, which follows from the congruence equality:

$$\frac{(p-1)(p+2)}{2} + \left[\frac{q-1}{2}\right] \equiv \frac{p-q}{2} \bmod 2$$

under the assumption that $p + q$ is even. The last equality follows from Lemma 4.5.2.

6 Bessel distributions

We have seen in the previous section (see Theorem 5.1.1) that the unitary inversion operator $\pi(w_0) : L^2(C) \to L^2(C)$ is given by the distribution kernel $K(\zeta, \zeta')$ which is the composition of the restriction of the bilinear map

$$C \times C \to \mathbb{R}, \qquad (\zeta, \zeta') \mapsto \langle \zeta, \zeta' \rangle$$

and Bessel distributions (see (1.4.1) – (1.4.3)) of one variable. In this section, we analyze the distribution kernel from three viewpoints: integral formulas, power series expansion (including distributions such as $\delta^{(k)}(x)$ and x^{-k}), and differential equations.

The results of Subsection 6.3 are not used for the proof of our main results, but gives a heuristic account on why $K(\zeta, \zeta')$ is essentially of one variable, and why the Bessel distribution arises in $K(\zeta, \zeta')$. Subsection 6.3 can be read independently of other subsections.

6.1 Meijer's G-distributions

In this subsection, we give a definition of *Meijer's G-distributions* which have the following two properties:

1) They are distributions on $\mathbb{R}$.

2) The restrictions to the positive half line $\{x > 0\}$ are (usual) Meijer's G-functions (see Appendix 7.6).

Let m, n, p and q be integers with $0 \leq m \leq q$ and $0 \leq n \leq p$. Suppose moreover that the complex numbers $a_1, \dots, a_p$ and $b_1, \dots, b_q$ fulfill the condition

$$a_j - b_k \neq 1, 2, 3, \dots \quad (j = 1, \dots, n;\ k = 1, \dots, m).$$

This means that no pole of the gamma function $\Gamma(b_j - \lambda)$ $(j = 1, \dots, m)$ coincides with any pole of $\Gamma(1 - a_k + \lambda)$ $(k = 1, \dots, n)$. We set

$$c^* := m + n - \frac{p+q}{2}, \tag{6.1.1}$$

$$\mu := \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2} + 1. \tag{6.1.2}$$

Throughout this section, we assume $c^* \geq 0$. If $c^* = 0$ then we also assume

$$p - q < 0 \quad \text{or} \quad p - q > \operatorname{Re} \mu. \tag{6.1.3}$$

It is easy to see that the condition (6.1.3) allows us to find $\gamma \in \mathbb{R}$ such that

$$\gamma > -1 \quad \text{and} \quad (q-p)\gamma > \operatorname{Re}\mu. \tag{6.1.4}$$

Remark 6.1.1. *The conditions* (6.1.3) *and* $\gamma > -1$ *will not be used when we define (usual) Meijer's G-function as an analytic function in* $x > 0$ *(see* (7.6.2)*). They will be used in showing that the Mellin–Barnes type integral* (6.1.8) *gives a locally integrable function if we take an appropriate contour* L *(see Proposition 6.1.2 (3)).*

We now take a contour L which starts at the point $\gamma - \sqrt{-1}\infty$ and finishes at $\gamma + \sqrt{-1}\infty$. Later, we shall impose the following conditions on L:

L does not go through any negative integer. (6.1.5)

L leaves b_j $(1 \le j \le m)$ to the right, and $a_j - 1$ $(1 \le j \le n)$ to the left. (6.1.6)

We note that the condition (6.1.6) implies:

L does not go through any point in $\{b_j + k : 1 \le j \le m, k \in \mathbb{N}\} \cup \{a_j - 1 - k : 1 \le j \le n, k \in \mathbb{N}\}$. (6.1.6)′

With these parameters, we define a meromorphic function of λ by

$$\Gamma^{m,n}_{p,q}\left(\lambda \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) := \frac{\prod_{j=1}^{m} \Gamma(b_j - \lambda) \prod_{j=1}^{n} \Gamma(1 - a_j + \lambda)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \lambda) \prod_{j=n+1}^{p} \Gamma(a_j - \lambda)}. \tag{6.1.7}$$

For $\operatorname{Re}\lambda > -1$, we set

$$x_+^\lambda := \begin{cases} x^\lambda & (x > 0) \\ 0 & (x \le 0), \end{cases} \qquad x_-^\lambda := \begin{cases} 0 & (x \ge 0) \\ |x|^\lambda & (x < 0). \end{cases}$$

Then, x_+^λ and x_-^λ are locally integrable functions of the variable x in $\mathbb{R}$, and extend to distributions with meromorphic parameter λ in the entire complex plane (see Appendix 7.1).

Proposition 6.1.2. *Let* L *be a contour satisfying* (6.1.5) *and* (6.1.6)′.

1) *The Mellin–Barnes type integral (cf. [9, §1.19]):*

$$G(x_+)_L \equiv G^{m,n}_{p,q}\left(x_+ \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right)_L$$
$$:= \frac{1}{2\pi\sqrt{-1}} \int_L \Gamma^{m,n}_{p,q}\left(\lambda \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) x_+^{\lambda}\, d\lambda \tag{6.1.8}$$

is well-defined as a distribution on $\mathbb{R}$.

Its support is given by

$$\operatorname{supp} G(x_+)_L = \{x \in \mathbb{R} : x \geq 0\}.$$

2) *If the contour* L *satisfies* (6.1.6), *then the restriction of* $G(x_+)_L$ *to the positive half line* $\{x \in \mathbb{R} : x > 0\}$ *is a real analytic function, which coincides with the (usual) G-function* $G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right)$ *(see* (7.6.2) *for definition).*

3) *If the contour* L *is contained in the half plane* $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -1\}$, *then* $G(x_+)_L$ *is a locally integrable function. More precisely, there exists* $\epsilon_0 > 0$ *such that* $G(x_+)_L\, x_+^{-\epsilon}$ *is locally integrable for any* ϵ *with* $0 \leq \epsilon < \epsilon_0$.

Likewise, we can define the distribution

$$G(x_-)_L \equiv G^{m,n}_{p,q}\left(x_- \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right)_L$$
$$:= \frac{1}{2\pi\sqrt{-1}} \int_L \Gamma^{m,n}_{p,q}\left(\lambda \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) x_-^{\lambda}\, d\lambda$$

by using the same contour L, and the support of $G(x_-)_L$ is equal to the negative half line $\{x \in \mathbb{R} : x \leq 0\}$.

Remark 6.1.3. *The distribution* $G(x_\pm)_L$ *depends on the choice of the contour* L *even when we assume* L *satisfies the conditions* (6.1.5) *and* (6.1.6). *In fact, if* L *and* L' *are contours satisfying* (6.1.5) *and* (6.1.6), *then* $G(x_\pm)_L$ *may differ from* $G(x_\pm)_{L'}$ *by a distribution supported at* 0, *namely, a finite sum of Dirac's delta function and its derivatives. This is because the distribution* $x_\pm^\lambda$ *has simple poles at* $\lambda = -1, -2, \dots,$ *and consequently, its residues (see* (7.1.1) *and* (7.1.2)*) may appear when we move the contour* L *across negative integers. For the uniqueness of the G-distribution, we need to impose an additional constraint on the contour* L. *We shall work with concrete examples for this in Subsection 6.2 where we use Cauchy's integral formula for distributions with meromorphic parameter.*

In order to prove Proposition 6.1.2, we need an asymptotic estimate of the Γ-factors in the integrand of (6.1.8) as follows:

Lemma 6.1.4. *For any* $\epsilon > 0$*, there exists a constant* $C > 0$ *such that*

$$\left| \Gamma^{m,n}_{p,q}\left(\lambda \;\middle|\; \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) \right| \le C e^{-\pi c^* |\operatorname{Im}\lambda|} \, |\operatorname{Im}\lambda|^{\operatorname{Re}\mu + (p-q)\gamma - 1 + \epsilon}$$

for any $\lambda \in L$ *such that* $|\operatorname{Im}\lambda|$ *is sufficiently large. Here,* c^* *and* μ *are defined as in* (6.1.1) *and* (6.1.2)*, and* $\gamma = \lim\limits_{\substack{\lambda \in L \\ |\operatorname{Im}\lambda| \to \infty}} \operatorname{Re}\lambda$.

Proof. Fix $a \in \mathbb{C}$. By Stirling's asymptotic formula (4.5.8) of the gamma function, we have

$$\begin{aligned} |\Gamma(a-\lambda)| &= C_a \, |\operatorname{Im}\lambda|^{\operatorname{Re}a - \operatorname{Re}\lambda - \frac{1}{2}} \, e^{-\frac{\pi}{2}|\operatorname{Im}\lambda|} (1 + O(|\operatorname{Im}\lambda|^{-1})), \\ |\Gamma(1-a+\lambda)| &= C_a \, |\operatorname{Im}\lambda|^{-\operatorname{Re}a + \operatorname{Re}\lambda + \frac{1}{2}} \, e^{-\frac{\pi}{2}|\operatorname{Im}\lambda|} (1 + O(|\operatorname{Im}\lambda|^{-1})), \end{aligned}$$

as $|\operatorname{Im}\lambda|$ tends to infinity with $\operatorname{Re}\lambda$ bounded. Here, the constant C_a is given by

$$C_a = \sqrt{2\pi} \, e^{-\frac{\pi}{2}\operatorname{sgn}(\operatorname{Im}\lambda)|\operatorname{Im}a|}.$$

By the definition (6.1.7) of $\Gamma^{m,n}_{p,q}\left(\lambda \;\middle|\; \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$, we now get the following asymptotic behavior:

$$\left| \Gamma^{m,n}_{p,q}\left(\lambda \;\middle|\; \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) \right| = C' \, |\operatorname{Im}\lambda|^s \, e^{-\frac{\pi}{2}t|\operatorname{Im}\lambda|} (1 + O(|\operatorname{Im}\lambda|^{-1})),$$

as $|\operatorname{Im}\lambda|$ tends to infinity, where C' is a constant depending on $\operatorname{Im}a_j$ and $\operatorname{Im}b_j$, and

$$\begin{aligned} s &= \sum_{j=1}^{m} \operatorname{Re}(b_j - \lambda - \frac{1}{2}) + \sum_{j=1}^{n} \operatorname{Re}(-a_j + \lambda + \frac{1}{2}) \\ &\quad - \sum_{j=m+1}^{q} \operatorname{Re}(\frac{1}{2} - b_j + \lambda) - \sum_{j=n+1}^{p} \operatorname{Re}(a_j - \lambda - \frac{1}{2}) \\ &= \operatorname{Re}\mu + (p-q)\operatorname{Re}\lambda - 1, \\ t &= m + n - (q-m) - (p-n) \\ &= 2c^*. \end{aligned}$$

As $\operatorname{Re}\lambda$ converges to γ when $\lambda \in L$ goes to infinity, we get Lemma 6.1.4. $\square$

We are ready to give a proof of Proposition 6.1.2.

Proof of Proposition 6.1.2. 3) We begin with the proof of the third statement. Suppose L is contained in the half plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -1\}$. We need to show the integral (6.1.8) gives rise to a locally integrable function. The non-trivial part is an estimate in the neighborhood of $x = 0$. Let us consider the interval $0 < x \leq 1$.

Since the contour L has the property:

$$\gamma = \lim_{\substack{\lambda \in L \\ |\operatorname{Im}\lambda| \to \infty}} \operatorname{Re}\lambda > -1,$$

the assumption $L \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -1\}$ implies $\delta > -1$, where we set

$$\delta := \inf_{\lambda \in L} \operatorname{Re}\lambda.$$

Hence, we get

$$|x_+^\lambda| \leq x^\delta \quad \text{for } 0 < x \leq 1.$$

On the other hand, it follows from Lemma 6.1.4 that

$$\left| \Gamma^{m,n}_{p,q}\left(\lambda \;\middle|\; \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) \right| \leq \begin{cases} Ce^{-\pi c^* |\operatorname{Im}\lambda|} & \text{if } c^* > 0, \\ C\,|\operatorname{Im}\lambda|^{-1+\epsilon} & \text{if } c^* = 0, \end{cases}$$

when $|\operatorname{Im}\lambda|$ is sufficiently large. Here, we used the inequality $\operatorname{Re}\mu + (p - q)\gamma < 0$ (see (6.1.4)) in the second case. Hence, $\Gamma^{m,n}_{p,q}\left(\lambda \;\middle|\; \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$ is absolutely integrable on L in either case. Therefore, the integration (6.1.8) converges, giving rise to a function of x which is bounded by a scalar multiple of x^δ on the interval $0 < x \leq 1$, whence a locally integrable function of x. Thus, $G(x_+)_L$ is locally integrable. Similarly, if we set

$$\epsilon_0 := 1 + \delta \quad (> 0),$$

then for any $0 \leq \epsilon < \epsilon_0$, $x^{-\epsilon+\delta}$ is locally integrable, and consequently $G(x_+)_L\, x_+^{-\epsilon}$ is locally integrable. Hence, the third statement of Proposition is proved.

1) We divide the integral (6.1.8) into the sum of the following two integrals

$$\int_L = \int_{L'} + \int_C,$$

where L' is a contour contained in the right half plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -1\}$, and C is the closed oriented curve given by $L - L'$ (see Figure 6.1.1).

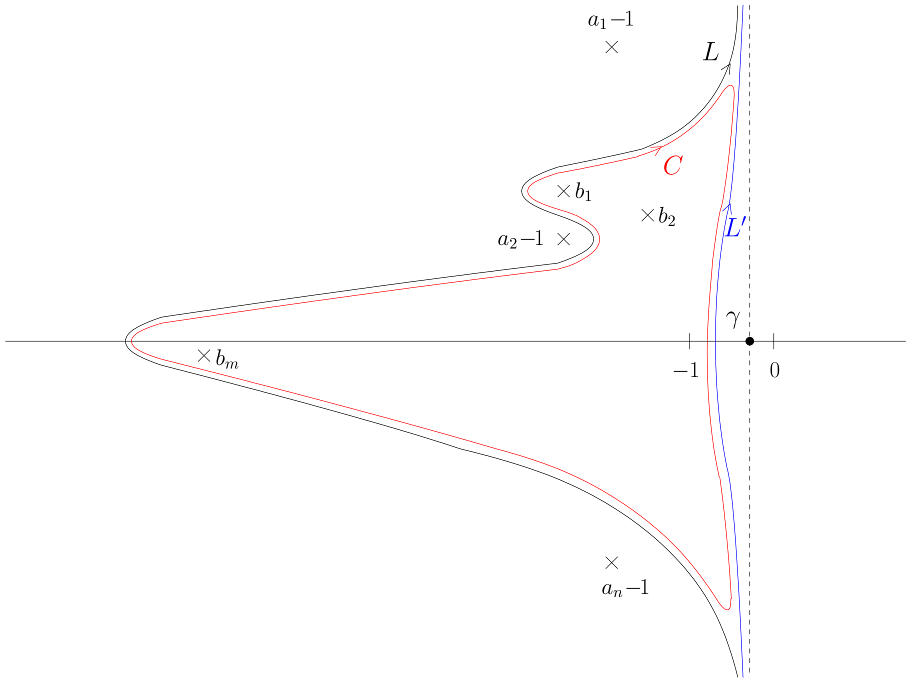


Figure 6.1.1

Then, the second term gives a locally integrable function of x by 3), and the third term is well-defined as a distribution because C is compact and the integrand is a distribution of x that depends continuously on λ as far as λ lies in C. Hence, the first statement is also proved.

2) This statement is well-known. See Appendix 7.6 and references therein. $\square$

6.2 Integral expression of Bessel distributions

In this subsection, we apply general results on Meijer's G-distributions developed in Subsection 6.1 to special cases, and the Mellin–Barnes type integral expression for the distribution kernel of the unitary inversion operator $\pi(w_0)$.

Let m be a non-negative integer. We take a contour L such that

1) L starts at $\gamma - \sqrt{-1}\infty$, passes the real axis at some point s, and ends at $\gamma + \sqrt{-1}\infty$.

2) $-1 < \gamma$ and $-m-1 < s < -m$.

Likewise, we take a contour L_0 (with analogous notation) such that

3) $-1 < \gamma_0$ and $-1 < s_0 < 0$.

For later purpose, we may and do take $\gamma = \gamma_0$. See Figure 6.2.1.

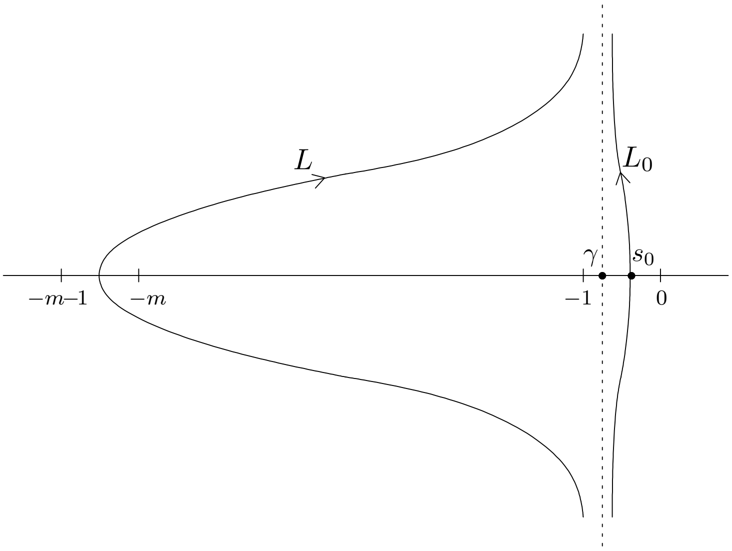


Figure 6.2.1

Then, we consider the following Mellin–Barnes type integrals:

$$\Phi_m^+(t) := \frac{1}{2\pi\sqrt{-1}} \int_{L_0} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)} (2t)_+^\lambda d\lambda, \tag{6.2.1}$$

$$\Psi_m^+(t) := \frac{1}{2\pi\sqrt{-1}} \int_{L} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)} (2t)_+^\lambda d\lambda, \tag{6.2.2}$$

$$\Phi_m(t) := \frac{1}{2\pi\sqrt{-1}} \int_{L_0} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)} \left(\frac{(2t)_+^\lambda}{\tan(\pi\lambda)} + \frac{(2t)_-^\lambda}{\sin(\pi\lambda)} \right) d\lambda, \tag{6.2.3}$$

$$\Psi_m(t) := \frac{1}{2\pi\sqrt{-1}} \int_{L} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)} \left(\frac{(2\lambda)_+^\lambda}{\tan(\pi\lambda)} + \frac{(2t)_-^\lambda}{\sin(\pi\lambda)} \right) d\lambda. \tag{6.2.4}$$

We shall see that these integrals are special cases of (6.1.8) and define distributions on $\mathbb{R}$. The next theorem is the main result of this subsection, which will be derived from Proposition 6.1.2 by applying the reduction formula of Meijer's G-functions.

Theorem 6.2.1. 1) $\Phi_m^+(t)$ *and* $\Phi_m(t)$ *are locally integrable functions on* $\mathbb{R}$. *Furthermore, for a sufficiently small* $\epsilon > 0$, $\Phi_m^+(t)|t|^{-\epsilon}$ *and* $\Phi_m(t)|t|^{-\epsilon}$ *are also locally integrable.*

2)

$$\Psi_m^+(t) = \Phi_m^+(t) - \sum_{k=1}^{m} \frac{(-1)^{k-1}}{2^k(m-k)!}\delta^{(k-1)}(t). \tag{6.2.5}$$

$$\Psi_m(t) = \Phi_m(t) - \frac{1}{\pi}\sum_{k=1}^{m} \frac{(k-1)!}{2^k(m-k)!}t^{-k}. \tag{6.2.6}$$

See (7.1.5) *in Appendix for the definition of the distribution* t^{-k}. *In particular,* Ψ_m^+ *and* Ψ_m *are defined as functionals on the space* $C_0^{m-1}(\mathbb{R})$ *of compactly supported functions on* $\mathbb{R}$ *with continuous derivatives up to* $m-1$ *if* $m \geq 1$.

We shall prove Theorem 6.2.1 after Lemma 6.2.2 below.

Before regarding the integrals (6.2.1)–(6.2.4) as those for distributions, we consider the classical cases, namely, their restrictions to $\mathbb{R} \setminus \{0\}$, which are analytic functions.

Let L_i $(i = 1, 2, 3)$ be contours that start at $\gamma_i - \sqrt{-1}\infty$ and end at $\gamma_i + \sqrt{-1}\infty$, and pass the real axis at some point s_i. We assume

$$\begin{aligned} -\frac{m}{2} &< \gamma_1, & s_1 &< 0, \\ -\frac{m}{2} &< \gamma_2, & s_2 &< -m, \\ & & s_3 &< -m. \end{aligned}$$

Then, we have the following integral expressions of Bessel functions. Although the results are classical, we shall give a proof as a preparation for passing from Bessel functions to *Bessel distributions*. We shall derive these integral expressions by applying those of Meijer's G-functions (see Appendix 7.6, see also Proposition 6.1.2 2)):

Lemma 6.2.2. 1) *For* $t > 0$,

$$\begin{aligned} \widetilde{J}_m(2\sqrt{2t}) &= (2t)^{-\frac{m}{2}} J_m(2\sqrt{2t}) \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{L_1} \frac{\Gamma(-\lambda)}{\Gamma(\lambda + m + 1)} (2t)_+^{\lambda} d\lambda. \end{aligned} \tag{6.2.7}$$

2) *For $t > 0$,*

$$\begin{aligned}\widetilde{Y}_m(2\sqrt{2t}) &= (2t)^{-\frac{m}{2}} Y_m(2\sqrt{2t}) \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{L_2} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+m+1)} \frac{(2t)_+^{\lambda}}{\tan(\pi\lambda)} d\lambda. \end{aligned} \tag{6.2.8}$$

3) *For $t < 0$,*

$$\begin{aligned}\widetilde{K}_m(2\sqrt{2|t|}) &= (2|t|)^{-\frac{m}{2}} K_m(2\sqrt{2|t|}) \\ &= \frac{(-1)^{m+1}}{4\sqrt{-1}} \int_{L_3} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+m+1)} \frac{(2t)_-^{\lambda}}{\sin(\pi\lambda)} d\lambda. \end{aligned} \tag{6.2.9}$$

Proof of Lemma. Each of the first equalities is by the definition of the normalized Bessel functions $\widetilde{J}_m$, $\widetilde{Y}_m$, and $\widetilde{K}_m$ given in (7.2.3), (7.2.5), and (7.2.6), respectively. Let us verify the second equalities (the integral formulas for the Bessel functions).

1) By the reduction formula (7.6.11) of the G-function G_{02}^{10}, we have

$$(2t)^{-\frac{m}{2}} J_m(2\sqrt{2t}) = G_{02}^{10}(2t \mid 0, -m)$$

for $t > 0$. Then, by the integral expression (7.6.2) of the G-function G_{02}^{10}, we have

$$G_{02}^{10}(2t \mid 0, -m) = \frac{1}{2\pi\sqrt{-1}} \int_{L_1} \frac{\Gamma(-\lambda)}{\Gamma(1+m+\lambda)} (2t)^{\lambda} d\lambda$$

for $t > 0$. Hence, (6.2.7) is proved.

2) By the reduction formula (7.6.14) of the G-function G_{13}^{20}, we have

$$(2t)^{-\frac{m}{2}} Y_m(2\sqrt{2t}) = G_{13}^{20}\left(2t \,\middle|\, \begin{matrix} -m-\frac{1}{2} \\ -m, 0, -m-\frac{1}{2} \end{matrix}\right),$$

for $t > 0$. Then, by Example 7.6.2, we have

$$\begin{aligned} & G_{13}^{20}\left(2t \,\middle|\, \begin{matrix} -m-\frac{1}{2} \\ -m, 0, -m-\frac{1}{2} \end{matrix}\right) \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{L_2} \frac{\Gamma(-m-\lambda)\Gamma(-\lambda)}{\Gamma(m+\frac{3}{2}+\lambda)\Gamma(-m-\frac{1}{2}-\lambda)} (2t)^{\lambda} d\lambda. \end{aligned}$$

Now, (6.2.8) is deduced from this formula and the following identity:

$$\frac{\Gamma(-m-\lambda)\Gamma(\lambda+1+m)}{\Gamma(m+\frac{3}{2}+\lambda)\Gamma(-m-\frac{1}{2}-\lambda)} = \frac{1}{\tan \pi\lambda} \quad \text{for any } m \in \mathbb{Z}.$$

Here, the last identity is an elementary consequence of the formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$.

3) By the reduction formula (7.6.12) of the G-function G^{20}_{02}, we have

$$(2|t|)^{-\frac{m}{2}} K_m(2\sqrt{2|t|}) = \frac{1}{2} G^{20}_{02}\Big(2|t| \;\Big|\; 0, -m\Big).$$

Suppose $t < 0$. Then, again by the integral expression (7.6.2) of G^{20}_{02}, the right-hand side amounts to

$$\frac{1}{4\pi\sqrt{-1}} \int_{L_3} \Gamma(-\lambda)\Gamma(-m-\lambda)(2t)^{\lambda}_{-}\, d\lambda.$$

Then, (6.2.9) follows from the identity:

$$\Gamma(\lambda+1+m)\Gamma(-m-\lambda) = \frac{\pi}{\sin(-\pi(\lambda+m))} = \frac{(-1)^{m+1}\pi}{\sin \pi\lambda}.$$

Thus, all the statements of Lemma 6.2.2 are proved. $\square$

The integrals in Lemma 6.2.2 do not depend on the choice of L_i ($i = 1, 2, 3$) as ordinary functions on $\mathbb{R} \setminus \{0\}$. However, as we mentioned in Remark 6.1.3, they may depend on the choice of L_i as distributions on $\mathbb{R}$ because the poles of the distributions $t^{\lambda}_{\pm}$ are located at $\lambda = -1, -2, -3, \ldots$.

To avoid this effect, we need to impose more constraints on the contours L_i. Thus, let us assume further $-1 < s_1$ and $-m-1 < s_i$ ($i = 2, 3$). Moreover, we assume $-1 < \gamma_j$ ($j = 1, 2$). That is, we shall assume from now that the integral paths L_i ($i = 1, 2, 3$) are under the constraints:

$$-1 < \gamma_1, \qquad -1 < s_1 < 0, \tag{6.2.10}$$

$$-1 < \gamma_2, \qquad -m-1 < s_2 < -m, \tag{6.2.11}$$

$$-m-1 < s_3 < -m. \tag{6.2.12}$$

Then, the right-hand sides of (6.2.7)–(6.2.9) define distributions on $\mathbb{R}$, which are independent of the choice of the integral paths L_i ($i = 1, 2, 3$).

Proof of Theorem 6.2.1. The first statement is a special case of Proposition 6.1.2 3).

Let us show the second statement. The contour L used in (6.2.2) and (6.2.4) meets the constraints (6.2.11) and (6.2.12), and can be used as L_2 and L_3. Likewise, the contour L_0 used in (6.2.1) and (6.2.3) can be used as L_1. Further, we shall assume that the contour L_0 coincides with L when $|\mathrm{Im}\,\lambda|$ is sufficiently large.

The integrand of (6.2.1) has poles at $\lambda = -1, -2, \dots, -m$ inside the closed contour $L_0 - L$, and its residue is given by

$$\operatorname*{res}_{\lambda=-k} \frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)}(2t)_+^\lambda = \frac{(-1)^{k-1}}{2^k(m-k)!}\,\delta^{(k-1)}(t)$$

for $k = 1, 2, \dots, m$ by (7.1.1). Therefore, by Cauchy's integral formula, we have

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\left(\int_{L_0} - \int_L\right)\frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)}(2t)_+^\lambda d\lambda \\
&= \sum_{k=1}^m \frac{(-1)^{k-1}}{2^k(m-k)!}\,\delta^{(k-1)}(t)
\end{aligned}$$

as distributions. Hence, (6.2.5) is proved.

Next, let us prove (6.2.6). We recall from (7.1.3) that the Laurent expansion of the distribution $t_\pm^\lambda$ at $\lambda = -k$ $(k = 1, 2, \dots)$ is given by

$$\begin{aligned}
t_+^\lambda &= \frac{1}{\lambda+k}\,\frac{(-1)^{k-1}}{(k-1)!}\,\delta^{(k-1)}(t) + t_+^{-k} + \cdots, \\
t_-^\lambda &= \frac{1}{(\lambda+k)(k-1)!}\,\delta^{(k-1)}(t) + t_-^{-k} + \cdots.
\end{aligned}$$

Combining with the Taylor expansions at $\lambda = -k$ $(k = 1, 2, \dots, m)$:

$$\begin{aligned}
\sin \pi\lambda &= (-1)^k \pi(\lambda+k) + \cdots, \\
\tan \pi\lambda &= \pi(\lambda+k) + \cdots, \\
\frac{\Gamma(-\lambda)2^\lambda}{\Gamma(\lambda+1+m)} &= b_0 + b_1(\lambda+k) + \cdots,
\end{aligned}$$

where $b_0 = \frac{(k-1)!}{2^k(m-k)!}$, we have

$$\begin{aligned}
&\frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)}\left(\frac{(2t)_+^\lambda}{\tan(\pi\lambda)} + \frac{(2t)_-^\lambda}{\sin(\pi\lambda)}\right) \\
&= \frac{b_0\big((-1)^{k-1}\delta^{(k-1)}(t) + (-1)^k\delta^{(k-1)}(t)\big)}{\pi(k-1)!}\,\frac{1}{(\lambda+k)^2} \\
&\quad + \left(\frac{b_0(t_+^{-k} + (-1)^k t_-^{-k})}{\pi} + \frac{b_1\big((-1)^{k-1}\delta^{(k-1)}(t) + (-1)^k\delta^{(k-1)}(t)\big)}{\pi(k-1)^2}\right)\frac{1}{\lambda+k} + \cdots \\
&= \frac{(k-1)!\,t^{-k}}{2^k(m-k)!\,\pi}\,\frac{1}{\lambda+k} + O(1).
\end{aligned}$$

Therefore, by (6.2.3) and (6.2.4), we have

$$\begin{aligned}\Psi_m(t) - \Phi_m(t) &= \frac{1}{2\pi\sqrt{-1}}\left(\int_L - \int_{L_0}\right)\frac{\Gamma(-\lambda)}{\Gamma(\lambda+1+m)}\left(\frac{(2t)_+^\lambda}{\tan(\pi\lambda)} + \frac{(2t)_-^\lambda}{\sin(\pi\lambda)}\right)d\lambda \\ &= -\sum_{k=1}^{m}\frac{(k-1)!}{2^k(m-k)!\,\pi}t^{-k}.\end{aligned}$$

Hence, (6.2.6) is proved. Now, we have completed the proof of Theorem 6.2.1. □

Remark 6.2.3. *We shall use the symbols*

$$\widetilde{J}_m(2\sqrt{2t_+}) = (2t)_+^{-\frac{m}{2}} J_m(2\sqrt{2t_+}), \tag{6.2.13}$$

$$\widetilde{K}_m(2\sqrt{2t_+}) = (2t)_+^{-\frac{m}{2}} K_m(2\sqrt{2t_+}), \tag{6.2.14}$$

$$\widetilde{Y}_m(2\sqrt{2t_-}) = (2t)_-^{-\frac{m}{2}} Y_m(2\sqrt{2t_-}), \tag{6.2.15}$$

to denote the distributions defined by the right-hand sides of (6.2.7)–(6.2.9) *and by the contours* L_i $(i = 1, 2, 3)$ *satisfying* (6.2.10)–(6.2.12)*, respectively.*

It is noteworthy that $\widetilde{J}_m(2\sqrt{2t_+})$ *is locally integrable, but* $\widetilde{K}_m(2\sqrt{2t_+})$ *and* $\widetilde{Y}_m(2\sqrt{2t_-})$ *are not. Then, by the above proof of Theorem 6.2.1, we have*

$$\begin{aligned}\Phi_m^+(t) &= \widetilde{J}_m(2\sqrt{2t_+}) \\ &= (2t)_+^{-\frac{m}{2}} J_m(2\sqrt{2t_+}),\end{aligned} \tag{6.2.16}$$

$$\begin{aligned}\Psi_m^+(t) &= \widetilde{J}_m(2\sqrt{2t_+}) - \sum_{k=1}^{m}\frac{(-1)^{k-1}}{2^k(m-k)!}\delta^{(k-1)}(t) \\ &= (2t)_+^{-\frac{m}{2}} J_m(2\sqrt{2t_+}) - \sum_{k=1}^{m}\frac{(-1)^{k-1}}{2^k(m-k)!}\delta^{(k-1)}(t),\end{aligned} \tag{6.2.17}$$

$$\begin{aligned}\Psi_m(t) &= \widetilde{Y}_m(2\sqrt{2t_+}) + \frac{2(-1)^{m+1}}{\pi}\widetilde{K}_m(2\sqrt{2t_-}) \\ &= (2t)_+^{-\frac{m}{2}} Y_m(2\sqrt{2t_+}) + \frac{2(-1)^{m+1}}{\pi}(2t)_-^{-\frac{m}{2}} K_m(2\sqrt{2t_-}).\end{aligned} \tag{6.2.18}$$

6.3 Differential equation

As defined in Introduction, the kernel function $K(\zeta, \zeta')$ is given by means of the (modified) Bessel distribution $\Phi_{p,q}(t)$. In this subsection, we shall give a heuristic account on why the Bessel function arises in the kernel function. Loosely, it turns out from the L_+-intertwining operator (see (2.4.5) for the definition of $L_+ = M_+^{\max} A$) that the kernel function $K(\zeta, \zeta')$ should be a function of one variable $\langle \zeta, \zeta' \rangle$ because generic L_+-orbits on $C \times C$ are of codimension one (see Lemma 6.3.2). Instead of using the Casimir operator, our idea is to make use of the differential equation arising from $\mathrm{Ad}(w_0)\mathfrak{n}^{\max} = \overline{\mathfrak{n}^{\max}}$ (see (2.4.3)). This argument gives a differential equation of second order since $\mathfrak{n}$ acts on $L^2(C)$ as differential operator of second order. The result here gives a useful information about $K(\zeta, \zeta')$. In fact, we used Proposition 6.3.3 as a clue to find the explicit form of $K(\zeta, \zeta')$. However, we did not use the results of this subsection for the actual proof of our main theorem. We hope that the heuristic argument here will be helpful to find the integral kernel of the inversion $\pi(w_0)$ of the minimal representation of other groups.

Let $\theta : g \mapsto {}^t g^{-1}$ be the Cartan involution of G. Since $g \in G = O(p, q)$ satisfies ${}^t g I_{p,q} g = I_{p,q}$ where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, we have ${}^t g^{-1} = I_{p,q} g I_{p,q}^{-1}$. Since $w_0 = I_{p,q}$, we get

$$\theta(g) = w_0 g w_0^{-1}. \tag{6.3.1}$$

We let $L_+ = M_+^{\max} A$ act on C by

$$m e^{tE} \cdot \zeta = e^t m \zeta$$

for $m \in M_+^{\max} \simeq O(p-1, q-1)$ and $a = e^{tE} \in A$ (see Subsection 2.3 for notation).

Lemma 6.3.1. *The kernel function $K(\zeta, \zeta')$ of the unitary operator $\pi(w_0)$ satisfies the following functional equation:*

$$K(\zeta, \zeta') = K(\theta(l)\zeta, l\zeta') \quad \textit{for all } l \in L_+. \tag{6.3.2}$$

Proof. Building on the unitary representation $(\pi, L^2(C))$, we define another unitary representation π^θ on $L^2(C)$ by the following twist:

$$\pi^\theta(g) := \pi(\theta(g)).$$

Then, (6.3.1) implies that $\pi(w_0) : L^2(C) \to L^2(C)$ is an intertwining operator from $(\pi, L^2(C))$ to $(\pi^\theta, L^2(C))$. In particular, we have,

$$\pi^\theta(l) \circ \pi(w_0) = \pi(w_0) \circ \pi(l) \quad \text{for any } l \in L_+. \tag{6.3.3}$$

For $l = m \in M_+^{\max}$, we recall from (2.3.3) that

$$(\pi(m)u)(\zeta) = u({}^t m \zeta) \quad \text{for } \zeta \in C.$$

Hence, for any $u \in L^2(C)$, we have

$$\int_C K({}^t\theta(m)\zeta, \zeta')u(\zeta')d\mu(\zeta') = \int_C K(\zeta, \zeta'')u({}^t m\zeta'')d\mu(\zeta'').$$

Since $d\mu$ is $M_+^{\max}$-invariant, the right-hand side is equal to

$$\int_C K(\zeta, {}^t m^{-1}\zeta')u(\zeta')d\mu(\zeta').$$

Since u is arbitrary, the kernel function must coincide:

$$K({}^t\theta(m)\zeta, \zeta') = K(\zeta, {}^t m^{-1}\zeta').$$

Replacing ${}^t\theta(m)\zeta$ with ζ, we have

$$K(\zeta, \zeta') = K(m\zeta, {}^t m^{-1}\zeta') \quad \text{for any } m \in M_+^{\max}.$$

Thus, (6.3.2) holds for $l \in M_+^{\max}$. For $l = a := e^{tE} \in A$, we recall from (2.3.9) that

$$(\pi(a)u)(\zeta) = e^{-\frac{p+q-4}{2}t}u(e^{-t}\zeta) \ (\zeta \in C).$$

Since $\pi^\theta(a) = \pi(a^{-1})$, the equation (6.3.3) amounts to

$$\pi(w_0) = \pi(a) \circ \pi(w_0) \circ \pi(a).$$

Hence, for any $u \in L^2(C)$, we have

$$\int_C K(\zeta, \zeta')u(\zeta')d\mu(\zeta') = e^{-(p+q-4)t}\int_C K(e^{-t}\zeta, \zeta'')u(e^{-t}\zeta'')d\mu(\zeta'').$$

By the formula (2.2.3) of the measure $d\mu$ in the polar coordinate, we have

$$d\mu(\zeta'') = e^{(p+q-4)t}d\mu(\zeta') \quad \text{for } \zeta' = e^{-t}\zeta''.$$

Thus, the right-hand side equals $\int_C K(e^{-t}\zeta, e^t\zeta')u(\zeta')d\mu(\zeta')$. Hence, we have

$$K(\zeta, \zeta') = K(e^{-t}\zeta, e^t\zeta') \quad \text{for any } t \in \mathbb{R}$$

and therefore

$$K(\zeta, \zeta') = K(\theta(a)\zeta, a\zeta') \quad \text{for any } a \in A.$$

Now, Lemma 6.3.1 is proved. $\square$

Now let $M_+^{\max}$ act on the direct product manifold $C \times C$ by the formula:

$$M_+^{\max} \times C \times C \to C \times C, \quad (l, \zeta, \zeta') \mapsto (\theta(l)\zeta, l\zeta').$$

Furthermore, we define the level set of $C \times C$ by

$$H_t := \{(\zeta, \zeta') \in C \times C : \langle \zeta, \zeta' \rangle = t\}, \quad t \in \mathbb{R}$$

with respect to the standard positive definite inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{p+q-2}$. Then we have:

Lemma 6.3.2. 1) *The level set H_t is stable under the $M_+^{\max}$-action.*
2) *Moreover H_t is a single $M_+^{\max}$-orbit for any non-zero t.*

Proof. 1) For $\zeta, \zeta' \in \mathbb{R}^{p+q-2}$ $(\subset \mathbb{R}^{p+q})$ and $l \in M_+^{\max} \simeq O(p-1, q-1)$, we have

$$\begin{aligned}
\langle \theta(l)\zeta, l\zeta' \rangle &= \langle w_0 l w_0^{-1} \zeta, l\zeta' \rangle \\
&= {}^t\zeta'{}^t l w_0 l w_0^{-1} \zeta \\
&= {}^t\zeta' w_0 w_0^{-1} \zeta \\
&= \langle \zeta, \zeta' \rangle.
\end{aligned}$$

Hence, H_t is $M_+^{\max}$-stable.
2) We replace $(p-1, q-1)$ by (p, q), and consider the G-action on

$$\widetilde{C} := \{(\zeta_0, \cdots, \zeta_{p+q-1}) : \zeta_0^2 + \cdots + \zeta_{p-1}^2 - \zeta_p^2 - \cdots - \zeta_{p+q-1}^2 = 0\}$$

in place of the $M_+^{\max}$ action on C (this change allows us to use the notation $N_+^{\max}$ and $\overline{N^{\max}}_+$ in Subsection 2.3). Then, we recall from (2.3.5) that G acts transitively on $\widetilde{C}$ and the isotropy subgroup at $e_0 + e_{p+q-1} = {}^t(1, 0, \cdots, 0, 1)$ is given by $M_+^{\max} N^{\max}$ (see (2.3.5)). Let us consider the orbit of $\theta(M_+^{\max} N^{\max}) = M_+^{\max} \overline{N}^{\max}$ on $\widetilde{C}$. In view of (2.3.3), we have

$$\overline{n}_b \begin{pmatrix} x_0 \\ x \\ x_{p+q-1} \end{pmatrix} = \begin{pmatrix} x_0 - {}^t x w_0 b \\ 0 \\ x_{p+q-2} + {}^t x w_0 b \end{pmatrix} + \frac{x_0 + x_{p+q-2}}{2} \begin{pmatrix} -Q(b) \\ 2b \\ Q(b) \end{pmatrix},$$

for $b = {}^t(b_1, \cdots, b_{p+q-2})$ and $x = {}^t(x_1, \cdots, x_{p+q-2}) \in \mathbb{R}^{p+q-2}$. If $x_0 + x_{p+q-1} \neq 0$ and ${}^t(x_0, \cdots, x_{p+q-1}) \in \widetilde{C}$, we set

$$b := \frac{-x}{x_0 + x_{p+q-1}} = \frac{(x_0 - x_{p+q-1})x}{Q(x)}.$$

Then, we have

$$\overline{n}_b \begin{pmatrix} x_0 \\ x \\ x_{p+q-1} \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \\ x_{p+q-1} \end{pmatrix} + \frac{Q(x)}{2(x_0 + x_{p+q-1})} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{x_0 + x_{p+q-1}}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, the second statement is proved. □

Thus it follows from Lemma 6.3.1 and Lemma 6.3.2 that $K(\zeta, \zeta')\big|_{C\times C\setminus H_0}$ is of the form

$$K(\zeta, \zeta') = \Psi(\langle \zeta, \zeta' \rangle) \tag{6.3.4}$$

for some function $\Psi(t)$ defined on $\mathbb{R} \setminus \{0\}$.

By lifting the inversion relation $\mathrm{Ad}(w_0)\mathfrak{n}^{\max} = \overline{\mathfrak{n}^{\max}}$ (see (2.4.3)) in the Lie algebra $\mathfrak{g}$ to the actions on $L^2(C)_K$, we get the differential equation satisfied by Ψ. More precisely,

Proposition 6.3.3. $\Psi(t)$ *satisfies the following ordinary differential equation on* $\mathbb{R} \setminus \{0\}$*:*

$$t\frac{d^2\Psi}{dt^2} + \frac{p+q-4}{2}\frac{d\Psi}{dt} + 2\Psi = 0. \tag{6.3.5}$$

Proof. It follows from $\mathrm{Ad}(w_0)\overline{N}_j = \epsilon_j N_j$ (see (2.4.2)) that

$$\pi(w_0) \circ d\pi(\overline{N}_j) = \epsilon_j d\pi(N_j) \circ \pi(w_0). \tag{6.3.6}$$

We recall from (2.2.6) that $Tu = u\delta(Q)$ for $u \in L^2(C)$ and from (2.3.13) that D_j is a differential operator on $\mathbb{R}^{p+q-2}$. Then, by diagram (2.3.12), (6.3.6) implies:

$$T \circ \pi(w_0) \circ d\pi(\overline{N}_j)u = \epsilon_j D_j \circ T \circ \pi(w_0)u, \quad u \in L^2(C)_K. \tag{6.3.7}$$

Now we recall from (6.3.4) that the kernel function of $\pi(w_0)$ is given by $K(\zeta, \zeta') = \Psi(\langle \zeta, \zeta' \rangle)$. Therefore, (6.3.7) equals

$$\begin{aligned} &\delta(Q(\zeta)) \int_C \Psi(\langle \zeta, \zeta' \rangle) 2\sqrt{-1}\zeta'_j u(\zeta') d\mu(\zeta') \\ &= \epsilon_j D_j \Big(\delta(Q(\zeta)) \int_C \Psi(\langle \zeta, \zeta' \rangle) u(\zeta') d\mu(\zeta') \Big). \end{aligned} \tag{6.3.8}$$

where $\delta(Q(\zeta))$ stands for the delta function $\delta(Q)$ with respect to the variable ζ. Here, the left-hand side follows from (2.3.11). Let us show that the right-hand side of (6.3.8) is equal to

$$-\sqrt{-1}\delta(Q(\zeta)) \int_C \Big(\langle \zeta, \zeta' \rangle \Psi''(\langle \zeta, \zeta' \rangle) + \frac{p+q-4}{2}\Psi'(\langle \zeta, \zeta' \rangle) \Big) \zeta'_j u(\zeta') d\mu(\zeta').$$

Claim 6.3.4. *For $1 \leq j \leq p+q-2$ and for any function f on $\mathbb{R}$, we have*

$$\begin{aligned} & D_j\Big(\delta(Q(\zeta))f(\langle\zeta,\zeta'\rangle)\Big) \\ & = -\sqrt{-1}\epsilon_j\zeta_j'\delta(Q(\zeta))\Big(\langle\zeta,\zeta'\rangle f''(\langle\zeta,\zeta'\rangle) + \frac{p+q-4}{2}f'(\langle\zeta,\zeta'\rangle)\Big). \end{aligned}$$

Sketch of Proof. The actual computation consists of a differentiation such as $\frac{\partial}{\partial\zeta_j}\big(\delta(Q(\zeta))f(\langle\zeta,\zeta'\rangle)\big)$, which can be computed by using, for example, the following formula (see [14, Chapter III, §2.2]),

$$\delta(Q(\zeta)) = \frac{Q(\zeta)_+^\lambda}{\Gamma(\lambda+1)}\Big|_{\lambda=-1}.$$

□

Now since the equation (6.3.8) holds for all $u \in L^2(C)_K$, we have

$$2\Psi(\langle\zeta,\zeta'\rangle) = -\Big(\langle\zeta,\zeta'\rangle\Psi''(\langle\zeta,\zeta'\rangle) + \frac{p+q-4}{2}\Psi'(\langle\zeta,\zeta'\rangle)\Big).$$

Hence, Ψ satisfies the differential equation (6.3.5). □

7 Appendix: special functions

We have seen that various special functions arise naturally in the analysis on the minimal representations. Some of their fundamental properties (e.g. integral formulas, differential equations, etc.) have been used in the proof of the unitary inversion formulas. Conversely, representation theoretic properties are reflected as algebraic relations (e.g. functional equations) of such special functions. Further, different models of the same representation yield formulas connecting special functions arising from each model.

For the convenience of the reader, we collect the formulas and the properties of special functions that were used in the previous sections.

7.1 Riesz distribution x_+^λ

A distribution f_λ on $\mathbb{R}$ with parameter $\lambda \in \mathbb{C}$ is said to be *a distribution with meromorphic parameter* λ if the pairing

$$\langle f_\lambda, \varphi \rangle$$

is a meromorphic function of λ for any test function $\varphi \in C_0^\infty(\mathbb{R})$. We say f_λ has a pole at $\lambda = \lambda_0$ if $\langle f_\lambda, \varphi \rangle$ has a pole at $\lambda = \lambda_0$ for some φ. Then, taking a residue at $\lambda = \lambda_0$, we get a distribution:

$$C_0^\infty(\mathbb{R}) \to \mathbb{C}, \quad \varphi \mapsto \operatorname*{res}_{\lambda=\lambda_0} \langle f_\lambda, \varphi \rangle,$$

which we denote by $\operatorname*{res}_{\lambda=\lambda_0} f_\lambda$.

By Cauchy's integral formula, if C is a contour surrounding $\lambda = \lambda_0$, then we have

$$\operatorname*{res}_{\lambda=\lambda_0} \langle f_\lambda, \varphi \rangle = \frac{1}{2\pi\sqrt{-1}} \int_C \langle f_\lambda, \varphi \rangle d\lambda,$$

and in turn we get an identity of distributions:

$$\operatorname*{res}_{\lambda=\lambda_0} f_\lambda = \frac{1}{2\pi\sqrt{-1}} \int_C f_\lambda \, d\lambda.$$

A classic example of distributions with meromorphic parameter is the Riesz distribution x_+^λ defined as a locally integrable function (and hence a distribution):

$$x_+^\lambda = \begin{cases} x^\lambda & (x > 0) \\ 0 & (x \leq 0) \end{cases}$$

for $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -1$. Then, x_+^λ extends meromorphically to the entire complex plane, and all the poles are located at $\lambda = -1, -2, \ldots$. The residue is given by

$$\operatorname*{res}_{\lambda=-k} x_+^\lambda = \frac{(-1)^{k-l}}{(k-1)!}\,\delta^{(k-1)}(x) \tag{7.1.1}$$

for $k = 1, 2, 3, \ldots$. To see this, we set

$$\varphi_N(x) := \varphi(x) - \sum_{k=1}^{N} \frac{\varphi^{(k-1)}(0)}{(k-1)!}\,x^{k-1}.$$

Then,

$$\begin{aligned}
\langle x_+^\lambda, \varphi\rangle &= \sum_{k=1}^{N} \frac{\varphi^{(k-1)}(0)}{(k-1)!} \int_0^1 x^{\lambda+k-1}dx + \int_0^1 x^\lambda \varphi_N(x)dx \\
&\quad + \int_1^\infty x^\lambda \varphi(x)dx \\
&= \sum_{k=1}^{N} \frac{1}{\lambda+k}\frac{\varphi^{(k-1)}(0)}{(k-1)!} + \int_0^1 x^\lambda \varphi_N(x)dx \\
&\quad + \int_1^\infty x^\lambda \varphi(x)dx.
\end{aligned}$$

The first two term has a simple pole at $\lambda = -k$ with residue

$$\frac{\varphi^{(k-1)}(0)}{(k-1)!} = \left\langle \frac{(-1)^{k-1}}{(k-1)!}\,\delta^{(k-1)}(x), \varphi(x) \right\rangle,$$

the second term is holomorphic if $\operatorname{Re}\lambda > -N-1$ because $\varphi_N(x) = O(x^N)$, and the last term is an entire function of λ because φ is compactly supported. Hence, (7.1.1) is proved. Likewise,

$$x_-^\lambda := \begin{cases} |x|^\lambda & (x < 0) \\ 0 & (x \geq 0) \end{cases}$$

extends a distribution with meromorphic parameter λ and all the poles are located at $\lambda = 0, -1, -2, \ldots$. They are simple poles with

$$\operatorname*{res}_{\lambda=-k} x_-^\lambda = \frac{\delta^{(k-1)}(x)}{(k-1)!}. \tag{7.1.2}$$

We write the Laurent expansions of x_+^λ and x_-^λ at $\lambda = -k$ ($k = 1, 2, 3, \dots$) as follows:

$$x_+^\lambda = \frac{(-1)^{k-1}}{\lambda + k}\delta^{(k-1)}(x) + x_+^{-k} + (\lambda + k)x_+^{-k}\log x_+ + \cdots, \tag{7.1.3}$$

$$x_-^\lambda = \frac{1}{\lambda + k}\delta^{(k-1)}(x) + x_-^{-k} + (\lambda + k)x_-^{-k}\log x_- + \cdots. \tag{7.1.4}$$

Then, x_+^{-k} and x_-^{-k} are tempered distributions supported on the half lines $x \geq 0$ and $x \leq 0$, respectively. We note that they are not homogeneous as distributions. Then, the sum $x_+^\lambda + (-1)^\lambda x_-^\lambda$ becomes a distribution with holomorphic parameter λ in the entire complex plane because

$$\operatorname*{res}_{\lambda=-k}(x_+^\lambda + (-1)^\lambda x_-^\lambda) = 0$$

for $k = 1, 2, 3, \dots$. We now define a distribution

$$x^{-k} := \left(x_+^\lambda + (-1)^\lambda x_-^\lambda\right)\big|_{\lambda=-k}. \tag{7.1.5}$$

This distribution is homogeneous, and coincides with $x_+^{-k} + (-1)^k x_-^k$.

For $k = 1$, x^{-1} is the distribution that gives Cauchy's principal value:

$$\langle x^{-1}, \varphi \rangle = \lim_{\epsilon \downarrow 0}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)\frac{\varphi(x)}{x}dx.$$

This formula is valid for any $\varphi \in C_0(\mathbb{R})$. Likewise, x^{-k} extends to a functional on the space $C_0^{k-1}(\mathbb{R})$ of compactly supported functions on $\mathbb{R}$ with continuous derivatives up to $k-1$. See the textbook [14] of Gelfand and Shilov for a nice introduction to these distributions.

7.2 Bessel functions

For $\nu \geq 0$, the series

$$J_\nu(z) := \left(\frac{z}{2}\right)^\nu \sum_{j=0}^\infty \frac{(-1)^j (\frac{z}{2})^{2j}}{j!\,\Gamma(j + \nu + 1)}$$

converges in the entire complex plane. Its sum $J_\nu(z)$ is called the *Bessel function of the first kind* of order ν (see [53, §3.54]).

We set

$$Y_\nu(z) := \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}.$$

If ν is an integer, say $\nu = m$, then this definition reads as

$$Y_m(z) := \lim_{\nu \to m} \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}.$$

Y_ν is known as the *Bessel function of the second kind* or *Neumann's function.*

$$\begin{aligned} I_\nu(z) &= e^{-\frac{\sqrt{-1}\nu\pi}{2}} J_\nu(e^{\frac{\sqrt{-1}\pi}{2}} z) \\ &= \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(\frac{z}{2})^{2j}}{j!\,\Gamma(j+\nu+1)}, \\ K_\nu(z) &= \frac{\pi}{2\sin\nu\pi}(I_{-\nu}(z) - I_\nu(z)) \end{aligned}$$

solve the differential equation

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2+\nu^2)u = 0.$$

$I_\nu(z)$ is known as the *modified Bessel function of the first kind*, and is real when $\nu \in \mathbb{R}$ and $z > 0$.

$K_\nu(z)$ is known as the *modified Bessel function of the third kind* or *Basset's function.* Clearly we have

$$K_{-\nu}(z) = K_\nu(z).$$

We call Y_ν, I_ν, and K_ν simply as *Y-Bessel*, *I-Bessel*, and *K-Bessel functions.*

The K-Bessel function satisfies the following formula (see [9, **II**, §7.11 (22)]):

$$\left(\frac{d}{z\,dz}\right)^m (z^{-\nu}K_\nu(z)) = (-1)^m z^{-\nu-m} K_{\nu+m}(z).$$

This formula may be stated as

$$\left(-\frac{2d}{z\,dz}\right)^m \widetilde{K}_\nu(z) = \widetilde{K}_{\nu+m}(z)$$

in terms of the normalized K-Bessel function (7.2.6). By the change of variables $z = 2e^{-x}$, the $m = 1$ case amounts to:

$$\begin{aligned} \frac{d}{dx}(e^{-ax}\widetilde{K}_\nu(2e^{-x})) = &-ae^{-ax}\widetilde{K}_\nu(2e^{-x}) \\ &+ 2e^{-(a+2)x}\widetilde{K}_{\nu+1}(2e^{-x}). \end{aligned} \tag{7.2.1}$$

The K-Bessel functions $K_\nu(z)$ reduce to combinations of elementary functions if ν is half of an odd integer. For $n \in \mathbb{N}$ we have

$$\begin{aligned} K_{n+\frac{1}{2}}(z) &= \Big(\frac{\pi}{2z}\Big)^{\frac{1}{2}} e^{-z} \sum_{j=0}^{n} \frac{(n+j)!}{j!(n-j)!} \frac{1}{(2z)^j} \\ &= (-1)^n \Big(\frac{\pi}{2z}\Big)^{\frac{1}{2}} z^{n+1} \Big(\frac{d}{z\,dz}\Big)^n \frac{e^{-z}}{z}. \end{aligned}$$

For instance, if $n=0$, we have

$$K_{\frac{1}{2}}(z) = \Big(\frac{\pi}{2z}\Big)^{\frac{1}{2}} e^{-z}. \tag{7.2.2}$$

The following renormalization is sometimes convenient:

$$\widetilde{J}_\nu(z) := (\frac{z}{2})^{-\nu} J_\nu(z) = \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{z}{2})^{2j}}{j!\,\Gamma(\nu+j+1)}, \tag{7.2.3}$$

$$\widetilde{I}_\nu(z) := (\frac{z}{2})^{-\nu} I_\nu(z) = \sum_{j=0}^{\infty} \frac{(\frac{z}{2})^{2j}}{j!\,\Gamma(j+\nu+1)}, \tag{7.2.4}$$

$$\widetilde{Y}_\nu(z) := (\frac{z}{2})^{-\nu} Y_\nu(z), \tag{7.2.5}$$

$$\widetilde{K}_\nu(z) := (\frac{z}{2})^{-\nu} K_\nu(z). \tag{7.2.6}$$

By the Taylor expansion as above, we see that both $\widetilde{J}_\nu(z)$ and $\widetilde{I}_\nu(z)$ are holomorphic function of z in the entire complex plane.

$\widetilde{J}_\nu(x)$ and $\widetilde{Y}_\nu(x)$ are linearly independent of each other (whether ν is an integer or not) and form a basis of the space of solutions to the following differential equation:

$$x\frac{d^2u}{dx^2} + (\nu+1)\frac{du}{dx} + u = 0. \tag{7.2.7}$$

For $m = 1, 2, 3, \ldots$, the infinite sum expressions of $Y_m(z)$ and $K_m(z)$ (or $\widetilde{Y}_m(z)$ and $\widetilde{K}_m(z)$) at $z=0$ are given in [9, **II**, §7.2, (31) and (37)], which may be stated as follows:

$$\begin{aligned}\widetilde{Y}_m(z) = & -\frac{1}{\pi}\sum_{k=1}^{m}(\frac{z}{2})^{-2k}\frac{(k-1)!}{(m-k)!} \\ & +\frac{2}{\pi}\widetilde{J}_m(z)\log(\frac{z}{2}) \\ & -\frac{1}{\pi}\sum_{l=0}^{\infty}(-1)^l(\frac{z}{2})^{2l}\,\frac{\psi(m+l+1)+\psi(l+1)}{l!(m+l)!}. \end{aligned} \tag{7.2.8}$$

$$\begin{aligned}\widetilde{K}_m(z) = & \frac{1}{2}\sum_{k=1}^{m}(-1)^{m-k}(\frac{z}{2})^{-2k}\frac{(k-1)!}{(m-k)!} \\ & +(-1)^{m+1}\widetilde{I}_m(z)\log(\frac{z}{2}) \\ & +\frac{1}{2}(-1)^m\sum_{l=0}^{\infty}(\frac{z}{2})^{2l}\,\frac{\psi(m+l+1)+\psi(l+1)}{l!(m+l)!}. \end{aligned} \tag{7.2.9}$$

Here, the function $\psi(z)$ is the logarithmic derivative of the gamma function:

$$\psi(z) := \frac{d\log\Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Th ψ function is meromorphic with simple poles at $z = 0, -1, -2, \ldots$.

Next, we summarize the asymptotic behaviors of the Bessel functions:

Fact 7.2.1 (see [1, Chapter 4], [53, Chapter VII])**.** *The asymptotic behaviors of the Bessel functions at $x = 0, \infty$ are given by*

1) *As x tends to 0, $J_\nu(x), I_\nu(x) = O(x^\nu)$.*

For $\nu > 0$,

$$\widetilde{K}_\nu(x) = \frac{\Gamma(\nu)}{2}\left(\frac{x}{2}\right)^{-2\nu} + o(x^{-2\nu}) \quad \textit{as } x \textit{ tends to } 0. \tag{7.2.10}$$

2) *As x tends to infinity*

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\Big(\cos\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big)\sum_{j=0}^{\infty}\frac{(-1)^j(\nu,2j)}{(2x)^{2j}}$$

$$- \sin\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big)\sum_{j=0}^{\infty}\frac{(-1)^j(\nu,2j+1)}{(2x)^{2j+1}}\Big) \qquad (|\arg x| < \pi),$$

$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\Big(\sin\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big)\sum_{j=0}^{\infty}(-1)^j\frac{(\nu,2j)}{(2x)^{2j}}$$

$$+ \cos\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big)\sum_{j=0}^{\infty}(-1)^j\frac{(\nu,2j+1)}{(2x)^{2j+1}}\Big) \qquad (|\arg x| < \pi),$$

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}\sum_{j=0}^{\infty}\frac{(-1)^j(\nu,j)}{(2x)^j} + \frac{e^{-x+(\nu+\frac{1}{2})\sqrt{-1}\pi}}{\sqrt{2\pi x}}\sum_{j=0}^{\infty}\frac{(\alpha,j)}{(2x)^j} \quad (-\frac{\pi}{2} < \arg x < \frac{3}{2}\pi),$$

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}\,e^{-x}\Big(1 + \sum_{j=1}^{\infty}\frac{(\nu,j)}{(2x)^j}\Big) \qquad (|\arg x| < \frac{3\pi}{2}).$$

In particular, we have

$$\widetilde{K}_\nu(2x) = \frac{\sqrt{\pi}}{2}\,e^{-2x}\,x^{-\nu-\frac{1}{2}}(1 + O(\frac{1}{x})) \quad \text{as } x \to \infty.$$

Here, we have used Hankel's notation:

$$\begin{aligned}(\alpha, j) &:= (-1)^j\frac{(\frac{1}{2}-\alpha)_j(\frac{1}{2}+\alpha)_j}{j!} \\ &= \frac{(4\alpha^2-1^2)(4\alpha^2-3^2)\cdots(4\alpha^2-(2j-1)^2)}{2^{2j}j!}.\end{aligned}$$

Finally, we list some integral formulas for the Bessel functions:

B1 (the Mellin transform of K-Bessel functions, see [16, p684]). For $\mathrm{Re}(\mu + 1 \pm \nu) > 0$ and $\mathrm{Re}\, a > 0$,

$$\int_0^\infty x^\mu K_\nu(ax)dx = 2^{\mu-1}a^{-\mu-1}\Gamma(\frac{1+\mu+\nu}{2})\Gamma(\frac{1+\mu-\nu}{2}).$$

Equivalently, we have

$$\int_0^\infty x^s \widetilde{K}_\nu(ax)dx = 2^{s-1}a^{-s-1}\Gamma(\frac{1+s}{2})\Gamma(\frac{1+s}{2} - \nu). \qquad (7.2.11)$$

B2 Formula of the Hankel transform due to W. Bailey [3] (see also [10, § 19.6 (8)]).

$$\int_0^\infty t^{\lambda-1} J_\mu(at) J_\nu(bt) K_\rho(ct) dt$$
$$= \frac{2^{\lambda-2} a^\mu b^\nu \Gamma(\frac{1}{2}(\lambda+\mu+\nu-\rho))\Gamma(\frac{1}{2}(\lambda+\mu+\nu+\rho))}{c^{\lambda+\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)}$$
$$\times F_4(\frac{1}{2}(\lambda+\mu+\nu-\rho), \frac{1}{2}(\lambda+\mu+\nu+\rho); \mu+1, \nu+1; -\frac{a^2}{c^2}, -\frac{b^2}{c^2}). \tag{7.2.12}$$

Here, F_4 is Appell's hypergeometric function of two variables (see (7.7.4)).

B3 (see [10, §7, 14.2 (36)]) For $\mathrm{Re}(\alpha+\beta) > 0$ and $\mathrm{Re}(\rho \pm \mu \pm \nu + 1) > 0$,

$$2^{\rho+2}\Gamma(1-\rho)\int_0^\infty K_\mu(\alpha t) K_\nu(\beta t) t^{-\rho} dt$$
$$= \alpha^{\rho-\nu-1}\beta^\nu {}_2F_1(\frac{1+\nu+\mu-\rho}{2}, \frac{1+\nu-\mu-\rho}{2}; 1-\rho; 1-\frac{\beta^2}{\alpha^2})$$
$$\times \Gamma(\frac{1+\nu+\mu-\rho}{2})\Gamma(\frac{1+\nu-\mu-\rho}{2})\Gamma(\frac{1-\nu+\mu-\rho}{2})\Gamma(\frac{1-\nu-\mu-\rho}{2}).$$

In particular, we have

$$\int_0^\infty K_\mu(2t)^2 t^{2s-1} dt = \frac{\Gamma(s)^2\Gamma(s+\mu)\Gamma(s-\mu)}{8\Gamma(2s)}. \tag{7.2.13}$$

7.3 Associated Legendre functions

The associated Legendre functions on the interval $(-1, 1)$ is defined as the special value of the hypergeometric function:

$$P_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)}\Big(\frac{1+x}{1-x}\Big)^{\frac{\mu}{2}} {}_2F_1\big(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\big). \tag{7.3.1}$$

The associated Legendre functions satisfy the following functional relation:

$$\frac{d}{dx}\Big((1-x^2)^{-\frac{\mu}{2}} P_\nu^\mu(-x)\Big) = (1-x^2)^{-\frac{\mu+1}{2}} P_\nu^{\mu+1}(-x), \tag{7.3.2}$$

which is derived from the following recurrence relation (see [16, §8.733 (1)]):

$$(1-x^2)\frac{d}{dx}P_\nu^\mu(x) = -\sqrt{1-x^2}P_\nu^{\mu+1}(x) - \mu x P_\nu^\mu(x).$$

Integral formulas for the associated Legendre functions:

L1 (see [16, p803]) Formula of the Riemann–Liouville integral: $\operatorname{Re}\lambda < 1$, $\operatorname{Re}\mu > 0$, $0 < y < 1$,

$$\frac{1}{\Gamma(\mu)}\int_0^y (y-x)^{\mu-1}\big(x(1-x)\big)^{-\frac{\lambda}{2}} P_\nu^\lambda(1-2x)dx = \big(y(1-y)\big)^{\frac{\mu}{2}-\frac{\lambda}{2}} P_\nu^{\lambda-\mu}(1-2y). \tag{7.3.3}$$

L2 (see [16, p798]) For $2\operatorname{Re}\lambda > |\operatorname{Re}\mu|$,

$$\int_{-1}^{1}(1-x^2)^{\lambda-1}P_\nu^\mu(x)dx = \frac{\pi 2^\mu \Gamma(\lambda+\frac{\mu}{2})\Gamma(\lambda-\frac{\mu}{2})}{\Gamma(\lambda+\frac{\nu}{2}+\frac{1}{2})\Gamma(\lambda-\frac{\nu}{2})\Gamma(\frac{-\mu+\nu+2}{2})\Gamma(\frac{-\mu-\nu+1}{2})}. \tag{7.3.4}$$

7.4 Gegenbauer polynomials

Definition of the Gegenbauer polynomials: For $l \in \mathbb{N}$, we define

$$C_l^\mu(x) := \frac{(-1)^l}{2^l}\frac{\Gamma(2\mu+l)\Gamma(\mu+\frac{1}{2})}{\Gamma(2\mu)\Gamma(\mu+l+\frac{1}{2})}\frac{(1-x^2)^{\frac{1}{2}-\mu}}{l!}\frac{d^l}{dx^l}\big((1-x^2)^{\mu+l-\frac{1}{2}}\big). \tag{7.4.1}$$

Slightly different from the usual notation in the literature, we adopt the following normalization of the Gegenbauer polynomial:

$$\widetilde{C}_l^\mu(x) := \Gamma(\mu)C_l^\mu(x). \tag{7.4.2}$$

By using Gauss's duplication formula

$$\Gamma(2\mu) = 2^{2\mu-1}\pi^{-\frac{1}{2}}\Gamma(\mu)\Gamma(\mu+\frac{1}{2}), \tag{7.4.3}$$

the definition (7.4.1) may be stated as

$$\widetilde{C}_l^\mu(x) = \frac{(-1)^l\Gamma(2\mu+l)\sqrt{\pi}}{2^{2\mu+l-1}l!\,\Gamma(\mu+l+\frac{1}{2})}(1-x^2)^{-\mu+\frac{1}{2}}\frac{d^l}{dx^l}\big((1-x^2)^{\mu+l-\frac{1}{2}}\big). \tag{7.4.4}$$

The special value at $\mu = 0$ is given by the limit formula (see [9, §3.15.1 (14)]):

$$\widetilde{C}_l^0(\cos\theta) = \lim_{\mu\to 0}\Gamma(\mu)C_l^\mu(\cos\theta) = \frac{2\cos(l\theta)}{l}. \tag{7.4.5}$$

On the other hand, the special value at $l = 0$ is given by

$$\widetilde{C}_0^\mu(x) = \Gamma(\mu).$$

Connection with Gauss' hypergeometric function (see [9, §3.15 (3)]):

$$\begin{aligned}\widetilde{C}_l^{\mu}(x) &= \frac{\Gamma(l+2\mu)\Gamma(\mu)}{\Gamma(l+1)\Gamma(2\mu)} {}_2F_1(l+2\mu, -l; \mu+\frac{1}{2}; \frac{1-x}{2}) \\ &= \frac{\Gamma(l+2\mu)\Gamma(\mu)}{\Gamma(l+1)\Gamma(2\mu)} {}_2F_1(\frac{l+2\mu}{2}, -\frac{l}{2}; \mu+\frac{1}{2}; 1-x^2). \end{aligned} \tag{7.4.6}$$

Here, the second equation is derived from the formula of quadratic transformation for hypergeometric function (see [9, §2.11 (2)]):

$${}_2F_1(a, b; a+b+\frac{1}{2}; 4x(1-x)) = {}_2F_1(2a, 2b; a+b+\frac{1}{2}; x).$$

By using Kummer's transformation formula for the hypergeometric functions:

$${}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma; z),$$

one can obtain the following relationship between the Gegenbauer polynomials and the associated Legendre functions.

$$\widetilde{C}_l^{\mu}(x) = \frac{\sqrt{\pi}\Gamma(2\mu+l)}{2^{\mu-\frac{1}{2}}\Gamma(l+1)} (1-x^2)^{\frac{1}{4}-\frac{\mu}{2}} P_{\mu+l-\frac{1}{2}}^{\frac{1}{2}-\mu}(x), \quad -1 < x < 1. \tag{7.4.7}$$

Integral formulas for the Gegenbauer polynomials:

Ge1 (Orthogonality relations; see [9, §3.15.1 (17)]) For $\operatorname{Re}\mu > -\frac{1}{2}$,

$$\int_{-1}^{1} \widetilde{C}_l^{\mu}(x)\widetilde{C}_m^{\mu}(x)(1-x^2)^{\mu-\frac{1}{2}} dx = \begin{cases} 0 & \text{if } l \neq m, \\ \frac{2^{1-2\mu}\pi\Gamma(l+2\mu)}{(l+\mu)\Gamma(l+1)} & \text{if } l = m. \end{cases} \tag{7.4.8}$$

Ge2 (see [16, §7.321]) For $\operatorname{Re}\mu > -\frac{1}{2}$,

$$\int_{-1}^{1} (1-x^2)^{\mu-\frac{1}{2}} e^{\sqrt{-1}ax} \widetilde{C}_l^{\mu}(x) dx = \frac{\pi 2^{1-\mu}\Gamma(2\mu+l)}{\Gamma(l+1)} a^{-\mu} J_{\mu+l}(a). \tag{7.4.9}$$

Ge3 (see [31, Lemma 8.5.2]) For $\alpha \in \mathbb{C}$, $\operatorname{Re}\nu > -1$, and $l \in \mathbb{N}$,

$$\int_{-1}^{1} J_\nu(\alpha\sqrt{x+1}) \widetilde{C}_l^{\nu+\frac{1}{2}}(x)(1+x)^{\frac{\nu}{2}}(1-x)^{\nu} dx = \frac{2^{\frac{3}{2}}(-1)^l\sqrt{\pi}\Gamma(2\nu+l+1)}{\alpha^{\nu+1} l!} J_{2\nu+2l+1}(\sqrt{2}\alpha). \tag{7.4.10}$$

Ge4 For $\operatorname{Re}\nu > -\frac{1}{2}$ and $\operatorname{Re}\lambda > -1$,

$$\begin{aligned}
&\int_{-x}^{1} (x+y)^{\lambda} \widetilde{C}_k^{\nu}(y)(1-y^2)^{\nu-\frac{1}{2}} dy \\
&= \frac{\sqrt{\pi}\Gamma(2\nu+k)\Gamma(\lambda+1)}{2^{\nu-\frac{1}{2}} k!} (1-x^2)^{\frac{\lambda}{2}+\frac{\nu}{2}+\frac{1}{4}} P_{\nu+k-\frac{1}{2}}^{-\lambda-\nu-\frac{1}{2}}(-x). \qquad (7.4.11)
\end{aligned}$$

This formula (7.4.11) is essentially the integration formula (7.3.3) for the associated Legendre functions. For the sake of completeness, we give a proof:

The left-hand side of (7.4.11)

$$\begin{aligned}
&= \frac{\sqrt{\pi}\Gamma(2\nu+k)}{2^{\nu-\frac{1}{2}} k!} \int_{-x}^{1} (1-y^2)^{\frac{\nu}{2}-\frac{1}{4}} (x+y)^{\lambda} P_{\nu+k-\frac{1}{2}}^{\frac{1}{2}-\nu}(y) dy \qquad \text{by (7.4.7)} \\
&= \frac{2^{\lambda+1}\sqrt{\pi}\Gamma(2\nu+k)}{k!} \int_{0}^{\frac{1+x}{2}} \big((1-t)t\big)^{\frac{\nu}{2}-\frac{1}{4}} \Big(\frac{x+1}{2} - t\Big)^{\lambda} P_{\nu+k-\frac{1}{2}}^{\frac{1}{2}-\nu}(1-2t) dt \\
&= \frac{2^{\lambda+1}\sqrt{\pi}\Gamma(2\nu+k)\Gamma(\lambda+1)}{k!} \Big(\frac{1-x^2}{4}\Big)^{\frac{\lambda}{2}+\frac{\nu}{2}+\frac{1}{4}} P_{\nu+k-\frac{1}{2}}^{-\lambda-\nu-\frac{1}{2}}(-x) \qquad \text{by (7.3.3)}
\end{aligned}$$

= the right-hand side of (7.4.11).

7.5 Spherical harmonics

A *spherical harmonics* f of degree $j = 0, 1, 2, \dots$ is the restriction to the unit sphere $S^{m-1} \subset \mathbb{R}^m$ of a homogeneous harmonic polynomials of degree j in $\mathbb{R}^m$. Equivalently, f is a smooth function satisfying the differential equation:

$$\Delta_{S^{m-1}} f = -j(j+m-2)f.$$

The space of spherical harmonics of degree j is denoted by

$$\mathcal{H}^j(\mathbb{R}^m) := \{f \in C^{\infty}(S^{m-1}) : \Delta_{S^{m-1}} f = -j(j+m-2)f\}.$$

When $m = 1$, it is convenient to set:

$$\mathcal{H}^0(\mathbb{R}^1) := \mathbb{C}\mathbf{1}, \quad \mathcal{H}^1(\mathbb{R}^1) := \mathbb{C}\mathrm{sgn}, \quad \mathcal{H}^j(\mathbb{R}^1) := 0 \quad (j \geq 2).$$

The following facts are well-known (see [20, Introduction], [49]):

H1 For $f \in \mathcal{H}^j(\mathbb{R}^m)$, $f(-x) = (-1)^j f(x)$.

H2 $O(m)$ acts irreducibly on $\mathcal{H}^j(\mathbb{R}^m)$.

H3 $\mathcal{H}^j(\mathbb{R}^m)$ is still irreducible as an $SO(m)$-module if $m \geq 3$.

H4 $\mathcal{H}^j(\mathbb{R}^2) = \mathbb{C}e^{\sqrt{-1}j\theta} \oplus \mathbb{C}e^{-\sqrt{-1}j\theta}$, $j \geq 1$ as $SO(2)$-modules, where $\theta = \tan^{-1}\frac{y}{x}$, $(x, y) \in \mathbb{R}^2$.

H5 $\mathcal{H}^j(\mathbb{R}^m)\big|_{O(m-1)} \simeq \bigoplus_{i=0}^{j} \mathcal{H}^i(\mathbb{R}^{m-1})$ as $O(m-1)$-modules.

H6 The Hilbert space $L^2(S^{m-1})$ decomposes into a direct sum of the space of spherical harmonics:

$$L^2(S^{m-1}) \simeq \sum_{j=0}^{\infty}{}^{\oplus} \mathcal{H}^j(\mathbb{R}^m).$$

Here, $\sum^{\oplus}$ stands for the completion of the algebraic direct sum $\bigoplus_{j=0}^{\infty} \mathcal{H}^j(\mathbb{R}^m)$.

Let $(x_0, x) \in \mathbb{R}^m, x \in \mathbb{R}^{m-1}$ be a coordinate of $\mathbb{R}^m$. Then this branching law H5 is explicitly constructed by the $O(m-1)$-intertwining operator

$$I^m_{i\to j} : \mathcal{H}^i(\mathbb{R}^{m-1}) \to \mathcal{H}^j(\mathbb{R}^m)$$

as follows:

Fact 7.5.1. *(e.g. [49, Chapter III]) For* $0 \leq i \leq j$ *and* $\phi \in \mathcal{H}^i(\mathbb{R}^{m-1})$*, we define a function* $I^m_{i\to j}\phi$ *on* S^{m-1} *by*

$$(I^m_{i\to j}(\phi))(x_0, x) := |x|^i \phi\left(\frac{x}{|x|}\right) \widetilde{C}^{\frac{m-2}{2}+i}_{j-i}(x_0). \tag{7.5.1}$$

Here, $\widetilde{C}^{\nu}_{l}(z)$ *is the normalized Gegenbauer polynomial (see* (7.4.2)*). Then,*

1) $I^m_{i\to j}(\phi) \in \mathcal{H}^j(\mathbb{R}^m)$.

2) $I^m_{i\to j}$ *gives an injective* $O(m-1)$*-homomorphism from* $\mathcal{H}^i(\mathbb{R}^{m-1})$ *to* $\mathcal{H}^j(\mathbb{R}^m)$.

3) *(norm)*

$$\|I^m_{i\to j}(\phi)\|^2_{L^2(S^{m-1})} = \frac{2^{3-m-2i}\pi\Gamma(m-2+i+j)}{(j-i)!\,(j+\frac{m-2}{2})}\|\phi\|^2_{L^2(S^{m-2})}. \tag{7.5.2}$$

Proof. We use the following coordinate:

$$[-1,1]\times S^{m-2}\to S^{m-1},\quad (r,\eta)\mapsto \omega=(r,\sqrt{1-r^2}\eta). \tag{7.5.3}$$

Then, the standard volume form $d\omega$ on the unit sphere S^{m-1} is given by $(1-r^2)^{\frac{m-3}{2}}d\eta dr$. Therefore,

$$\|I^m_{i\to j}(\phi)\|^2_{L^2(S^{m-1})}=\int_{-1}^{1}\int_{S^{m-2}}(1-r^2)^i|\phi(\eta)|^2|\widetilde{C}^{\frac{m-2}{2}+i}_{j-i}(r)|^2(1-r^2)^{\frac{m-3}{2}}d\eta\, dr.$$

Now, apply (7.4.8). $\square$

We illustrate the intertwining operator I_{ij} by the two important cases, $i=0$ and $i=j$:

Example 7.5.2. 1) *The case $i=0$. Then,*

$$(I^m_{0\to j}\mathbf{1})(x_0,x)=\widetilde{C}^{\frac{m-2}{2}}_j(x_0) \tag{7.5.4}$$

is the generator of $O(m-1)$-invariant vectors in $\mathcal{H}^j(\mathbb{R}^m)$, where $\mathbf{1}$ is the constant function on S^{m-1}.

2) *The case $i=j$. Then, we have simply*

$$I^m_{i\to j}(\phi)(x_0,x)=\Gamma(m)|x|^j\phi(\frac{x}{|x|}). \tag{7.5.5}$$

7.6 Meijer's G-functions

Let m,n,p and q be integers with $0\le m\le q$, $0\le n\le p$ and

$$c^*:=m+n-\frac{p+q}{2}\ge 0.$$

Suppose further that the complex numbers $a_1,\dots,a_p$ and $b_1,\dots,b_q$ fulfill the condition:

$$a_j-b_k\ne 1,2,3,\dots\quad (j=1,\dots n;\ k=1,\dots,m). \tag{7.6.1}$$

Then, *Meijer's G-function* of order (m,n,p,q) is defined by the following integral (see [9, §5.3], [39, **I**, §1], [43, §8.2]): for $x>0$,

$$G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1, & \cdots, & a_p\\ b_1, & \cdots, & b_q\end{matrix}\right)$$

$$:=\frac{1}{2\pi\sqrt{-1}}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\lambda)\prod_{j=1}^{n}\Gamma(1-a_j+\lambda)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\lambda)\prod_{j=n+1}^{p}\Gamma(a_j-\lambda)}x^\lambda d\lambda, \tag{7.6.2}$$

where an empty product is interpreted as 1.

The contour L starts at the point $\gamma-\sqrt{-1}\infty$ (γ is a real number satisfying (7.6.4) below if $c^*=0$), leaving all the poles of the integrand of the forms

$$\lambda = b_j, b_j+1, b_j+2, \dots \quad (1 \le j \le m) \tag{7.6.3}$$

to the right, and all the poles of the forms

$$\lambda = a_j-1, a_j-2, a_j-3, \dots \quad (1 \le j \le n)$$

to the left of the contour and finishing at the point $\gamma+\sqrt{-1}\infty$.

Here, the condition on the real number γ is given by

$$(q-p)\gamma > \operatorname{Re}\mu, \tag{7.6.4}$$

where we set

$$\mu := \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2} + 1.$$

It follows from the asymptotic behavior of the gamma factors (see Lemma 6.1.4) that the integral (7.6.2) converges and is independent of γ if one of the following conditions hold:

1) $c^* > 0$, $|\arg x| < c^*\pi$;

2) $c^* \ge 0$, $|\arg x| = c^*\pi$, $(q-p)\gamma > \operatorname{Re}\mu$.

In particular, the G-function extends holomorphically to the complex domain $|\arg x| < c^*\pi$ if $c^* > 0$.

The G-function is symmetric in the parameters $a_1, \dots, a_n$, likewise in $a_{n+1}, \dots, a_p$, in $b_1, \dots, b_m$, and in $b_{m+1}, \dots, b_q$.

Obvious changes of variables in the integral give

$$x^s G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{matrix}\right) = G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1+s, & \cdots, & a_p+s \\ b_1+s, & \cdots, & b_q+s \end{matrix}\right),$$

$$G^{m,n}_{p,q}\left(x^{-1} \,\middle|\, \begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{matrix}\right) = G^{n,m}_{q,p}\left(x \,\middle|\, \begin{matrix} 1-b_1, & \cdots, & 1-b_q \\ 1-a_1, & \cdots, & 1-a_p \end{matrix}\right).$$

The G-function $G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$ satisfies the differential equation (see [9, §5.4 (1)]):

$$\Big((-1)^{p-m-n} x \prod_{j=1}^{p} (x\frac{d}{dx} - a_j + 1) - \prod_{j=1}^{q} (x\frac{d}{dx} - b_i)\Big) u = 0. \tag{7.6.5}$$

If $p < q$, the only singularities of (7.6.5) are $x = 0, \infty$; $x = 0$ is a regular singularity, $x = \infty$ an irregular one. For example, $G^{20}_{04}(x \mid b_1, b_2, b_3, b_4)$ satisfies the fourth order differential equation:

$$\prod_{j=1}^{4}(x\frac{d}{dx} - b_j)u = 0. \tag{7.6.6}$$

The condition (7.6.1) implies that none of the poles of $\Gamma(b_j - \lambda)$ $(j = 1, 2, \dots, m)$ coincides with any of the poles of $\Gamma(1 - a_k + \lambda)$ $(k = 1, \dots, n)$. Suppose further that

$$b_j - b_k \neq 0, \pm 1, \pm 2, \dots \quad (1 \leq j < k \leq m).$$

Then the integrand (as an ordinary function for $x > 0$) has simple poles at the points (7.6.3). (We note that as a distribution of x, x^λ has simple poles at $\lambda = -1, -2, -3, \dots$, and the analysis involved is more delicate; see Subsections 6.1 and 6.2.) For $p \leq q$, by the residue calculus, we obtain (see [39, **I**, (7)]):

$$G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{matrix}\right)$$
$$= \sum_{k=1}^{m} \frac{\prod\limits_{\substack{j=1\\ j\neq k}}^{m} \Gamma(b_j - b_k) \prod\limits_{j=1}^{n} \Gamma(1 + b_k - a_j)}{\prod\limits_{j=m+1}^{q} \Gamma(1 + b_k - b_j) \prod\limits_{j=n+1}^{p} \Gamma(a_j - b_k)} x^{b_k}$$
$$\times {}_pF_{q-1}(1 + b_k - a_1, \dots, 1 + b_k - a_p; 1 + b_k - b_1, \overset{k}{\widehat{\cdots}}, 1 + b_k - b_q; (-1)^{p-m-n}x). \tag{7.6.7}$$

Here, ${}_pF_q$ denotes the (usual) generalized hypergeometric function:

$${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; x) = \sum_{k=0}^{\infty} \frac{x^k \prod\limits_{j=1}^{p} \alpha_j(\alpha_j + 1) \cdots (\alpha_j + k - 1)}{k! \prod\limits_{j=1}^{q} \beta_j(\beta_j + 1) \cdots (\beta_j + k - 1)}.$$

Similarly, for $q \le p$, if $a_j - a_k \ne 0, \pm 1, \pm 2, \dots$ $(1 \le j < k \le n)$, we have

$$G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{matrix}\right)$$
$$= \sum_{k=1}^{n} \frac{\prod\limits_{\substack{j=1 \\ j\ne k}}^{n} \Gamma(a_k - a_j) \prod\limits_{j=1}^{n} \Gamma(b_j - a_k + 1)}{\prod\limits_{j=n+1}^{p} \Gamma(a_j - a_k + 1) \prod\limits_{j=m+1}^{q} \Gamma(a_k - b_j)} x^{a_k - 1}$$
$$\times {}_qF_{p-1}(1 + b_1 - a_k, \dots, 1 + b_q - a_k; 1 + a_1 - a_k, \overset{k}{\widehat{\cdots}}, 1 + a_p - a_k; (-1)^{q-m-n} x^{-1}). \tag{7.6.8}$$

For $p \le q$, it follows from (7.6.7) that

$$G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{matrix}\right) = O(|x|^{\min(\operatorname{Re} b_1, \dots, \operatorname{Re} b_m)}) \tag{7.6.9}$$

as $x \to 0$ (see also [9, **I**, §5.4.1 (8)], but there is a typographical error: $\max \operatorname{Re} b_h$ loc. cit. should be $\min \operatorname{Re} b_h$). On the other hand, the asymptotic expansion of $G^{m,n}_{p,q}(x)$ $(p \le q)$ for large $x > 0$ that we need in this paper is the following case:

Fact 7.6.1 ([39, **VII**, Theorem 17]). *Let m, p and q be integers satisfying*

$$0 \le p \le q - 2 \quad \text{and} \quad p + 1 \le m \le q - 1.$$

Then the G-function $G^{m,0}_{p,q}(x)$ possesses the following asymptotic expansion for large $x > 0$:

$$G^{m,0}_{p,q}(x) \sim A^{m,0}{}_{q} H_{p,q}(x e^{(q-m)\pi\sqrt{-1}}) + \bar{A}^{m,0}{}_{q} H_{p,q}(x e^{(m-q)\pi\sqrt{-1}}).$$

Here, $H_{p,q}(z)$ is a function that possesses the following expansion (see [39, **I**, (25)]):

$$H_{p,q}(z) = \exp\left((p-q) z^{\frac{1}{q-p}}\right) z^{\theta} \left(\frac{(2\pi)^{\frac{q-p-1}{2}}}{\sqrt{q-p}} + \frac{M_1}{z^{\frac{1}{q-p}}} + \frac{M_2}{z^{\frac{2}{q-p}}} + \cdots \right),$$

where $M_1, M_2, \dots$ are constants, and θ is given by

$$\theta := \frac{1}{q-p}\left(\frac{p-q+1}{2} + \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right), \quad [39, \mathbf{I}, (23)].$$

The coefficients $A^{m,0}_{\ q}$ and $\bar{A}^{m,0}_{\ q}$ are given by

$$A^{m,0}_{\ q} := (-2\pi\sqrt{-1})^{m-q}\, e^{-(b_{m+1}+\cdots+b_q)\pi\sqrt{-1}}, \quad [39, \textbf{II}, (45)],$$
$$\bar{A}^{m,0}_{\ q} := (2\pi\sqrt{-1})^{m-q}\, e^{(b_{m+1}+\cdots+b_q)\pi\sqrt{-1}}, \quad [39, \textbf{II}, (46)].$$

Example 7.6.2. *For* $(m, n, p, q) = (2, 0, 1, 3)$, $c^* = 0$. *We take* γ *such that*

$$\gamma > \frac{1}{2}\operatorname{Re}(b_1 + b_2 + b_3 - a_1).$$

Then, we have an integral expression:

$$G^{20}_{13}\left(x \,\middle|\, \begin{matrix} a_1 \\ b_1, b_2, b_3 \end{matrix}\right) = \frac{1}{2\pi\sqrt{-1}} \int_L \frac{\Gamma(b_1-\lambda)\Gamma(b_2-\lambda)}{\Gamma(1-b_3+\lambda)\Gamma(a_1-\lambda)} x^\lambda d\lambda,$$

where the integral path L *starts from* $\gamma - \sqrt{-1}\infty$, *leaves* b_1, b_2 *to the right and ends at* $\gamma + \sqrt{-1}\infty$ *(see Figure 7.6.1).*

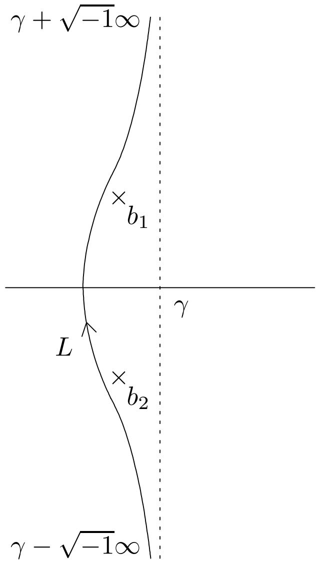


Figure 7.6.1

Example 7.6.3. *If* $p = 0$, *the G-function is denoted by* $G^{m,0}_{0,q}(x \mid b_1, \ldots, b_q)$. *The G-function that we use most frequently in this paper is of type* G^{20}_{04}. *Again, we have* $c^* = 0$. *Then, we have an integral expression:*

$$G^{20}_{04}(x \mid b_1, b_2, b_3, b_4) = \frac{1}{2\pi\sqrt{-1}} \int_L \frac{\Gamma(b_1-\lambda)\Gamma(b_2-\lambda)}{\Gamma(1-b_3+\lambda)\Gamma(1-b_4+\lambda)} x^\lambda d\lambda,$$

where L starts from $\gamma - \sqrt{-1}\infty$, leaves b_1, b_2 to the right, and ends at $\gamma + \sqrt{-1}\infty$ (see Figure 7.6.1) for $\gamma \in \mathbb{R}$ such that

$$\gamma > \frac{1}{4}(\operatorname{Re}(b_1 + b_2 + b_3 + b_4) - 1).$$

In Subsection 4.5, we need the following lemma on the asymptotic behavior:

Lemma 7.6.4. *The asymptotic behavior of the G-functions $G_{04}^{20}(x \mid b_1, b_2, 1-\gamma - b_1, 1-\gamma-b_2)$ are given as follows:*

1) *As x tends to 0, $G_{04}^{20}(x \mid b_1, b_2, 1-\gamma-b_1, 1-\gamma-b_2) = O(x^{\min(b_1,b_2)})$.*

2) *As x tends to ∞,*

$$\begin{aligned} &G_{04}^{20}(x \mid b_1, b_2, 1-\gamma-b_1, 1-\gamma-b_2) \\ = &-\frac{1}{\sqrt{2\pi}} x^{\frac{1-4\gamma}{8}} \cos\bigl(4x^{\frac{1}{4}} - (\gamma + b_1 + b_2 + \frac{1}{4})\pi\bigr)(1 + P_1 x^{-\frac{1}{2}} + P_2 x^{-1} + \cdots) \\ &+ x^{\frac{1-4\gamma}{8}} \sin\bigl(4x^{\frac{1}{4}} - (\gamma + b_1 + b_2 + \frac{1}{4})\pi\bigr)(Q_1 x^{-\frac{1}{4}} + Q_2 x^{-\frac{3}{4}} + \cdots). \end{aligned} \tag{7.6.10}$$

Here, $P_1, \cdots, Q_1, \cdots$ are the constants independent of x.

Proof. 1) This estimate is a special case of (7.6.9).

2) We apply Fact 7.6.1 to the case

$$(m, p, q) = (2, 0, 4), \quad (b_1, b_2, b_3, b_4) = (b_1, b_2, 1-\gamma-b_1, 1-\gamma-b_2).$$

Then, the coefficients A_4^{20}, $\bar{A}_4^{20}$ and the constant θ amount to

$$\begin{aligned} &A_4^{20} = -\frac{1}{4\pi^2} e^{(2\gamma-2+b_1+b_2)\pi\sqrt{-1}}, \quad \bar{A}_4^{20} = -\frac{1}{4\pi^2} e^{-(2\gamma-2+b_1+b_2)\pi\sqrt{-1}}, \\ &\theta = \frac{1-4\gamma}{8}. \end{aligned}$$

The expansion of $H_{0,4}(xe^{\pm 2\pi\sqrt{-1}})$ is given by

$$H_{0,4}(xe^{\pm 2\pi\sqrt{-1}}) = e^{\mp(4x^{\frac{1}{4}} - \frac{1-4\gamma}{4}\pi)\sqrt{-1}} x^{\frac{1-4\gamma}{8}} \left(\frac{(2\pi)^{\frac{3}{2}}}{2} \pm \frac{M_1}{\sqrt{-1}x^{\frac{1}{4}}} + \cdots \right).$$

Hence, $G_{04}^{20}(x)$ has the following asymptotic expansion:

$$\begin{aligned} &-\frac{1}{4\pi^2} e^{-(4x^{\frac{1}{4}} - \pi(\gamma + b_1 + b_2 - \frac{7}{4}))\sqrt{-1}} x^{\frac{1-4\gamma}{8}} \left(\frac{(2\pi)^{\frac{3}{2}}}{2} + \frac{M_1}{\sqrt{-1}x^{\frac{1}{4}}} + \cdots \right) \\ &\qquad -\frac{1}{4\pi^2} e^{(4x^{\frac{1}{4}} - \pi(\gamma + b_1 + b_2 - \frac{7}{4}))\sqrt{-1}} x^{\frac{1-4\gamma}{8}} \left(\frac{(2\pi)^{\frac{3}{2}}}{2} - \frac{M_1}{\sqrt{-1}x^{\frac{1}{4}}} + \cdots \right), \end{aligned}$$

which is expressed as the right-hand side of (7.6.10) by virtue of the formulas $e^{c\pi\sqrt{-1}} + e^{-c\pi\sqrt{-1}} = 2\cos(c\pi)$ and $e^{c\pi\sqrt{-1}} - e^{-c\pi\sqrt{-1}} = 2\sqrt{-1}\sin(c\pi)$. $\square$

Finally, we list the reduction formulas of G-functions that are used in this paper:

$$G^{10}_{02}(x \mid a, b) = x^{\frac{1}{2}(a+b)} J_{a-b}(2x^{\frac{1}{2}}) \qquad [9, §5.6(3)], \tag{7.6.11}$$

$$G^{20}_{02}(x \mid a, b) = 2x^{\frac{1}{2}(a+b)} K_{a-b}(2x^{\frac{1}{2}}) \qquad [9, §5.6(4)], \tag{7.6.12}$$

$$G^{20}_{04}(x \mid a, a+\frac{1}{2}, b, b+\frac{1}{2}) = x^{\frac{1}{2}(a+b)} J_{2(a-b)}(4x^{\frac{1}{4}}) \qquad [9, §5.6(11)], \tag{7.6.13}$$

$$G^{20}_{13}\left(x \,\middle|\, \begin{matrix} a-\frac{1}{2} \\ a, b, a-\frac{1}{2} \end{matrix}\right) = x^{\frac{1}{2}(a+b)} Y_{b-a}(2x^{\frac{1}{2}}) \qquad [9, §5.6(23)]. \tag{7.6.14}$$

7.7 Appell's hypergeometric functions

Appell's hypergeometric functions (in two variables) F_1, F_2, F_3, F_4 are defined by the following double power series:

$$F_1(\alpha, \beta, \beta', \gamma; x, y) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n} m!\, n!} x^m y^n, \tag{7.7.1}$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!\, n!} x^m y^n, \tag{7.7.2}$$

$$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m!\, n!} x^m y^n, \tag{7.7.3}$$

$$F_4(\alpha, \beta, \gamma, \gamma'; x, y) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n m!\, n!} x^m y^n. \tag{7.7.4}$$

Ap1 Reduction from F_3 to F_1 ([9, §5.11, (11)]):

$$F_3(\alpha, \alpha', \beta, \beta', \alpha+\alpha'; x, y) = (1-y)^{-\beta'} F_1(\alpha, \beta, \beta', \alpha+\alpha'; x, \frac{y}{y-1}). \tag{7.7.5}$$

Ap2 Reduction from F_3 to ${}_2F_1$ ([9, §5.10, (4)]):

$$F_3(\alpha, \gamma-\alpha, \beta, \gamma-\beta; \gamma; x, y) = (1-y)^{\alpha+\beta-\gamma} {}_2F_1(\alpha, \beta, \gamma; x+y-xy). \tag{7.7.6}$$

Ap3 Reduction formula of F_4 ([9, §5.10, (8)]):

$$F_4(\alpha, \beta; 1+\alpha-\beta, \beta; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)})$$
$$= (1-y)^{\alpha} {}_2F_1(\alpha, \beta; 1+\alpha-\beta; \frac{-x(1-y)}{1-x}). \tag{7.7.7}$$

Ap4 Single integral of Euler's type for F_1 ([9, §5.8.2, (5)]): For $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}(\gamma-\alpha) > 0$,

$$F_1(\alpha, \beta, \beta', \gamma; x, y) =$$
$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'} du. \tag{7.7.8}$$

Ap5 Double integral of Euler's type for F_3 ([9, §5.8.1, (3)]):

$$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}$$
$$\times \iint_D u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1}(1-ux)^{-\alpha}(1-vy)^{-\alpha'} dudv$$
$$\operatorname{Re}\beta > 0, \operatorname{Re}\beta' > 0, \operatorname{Re}(\gamma-\beta-\beta') > 0, \tag{7.7.9}$$

where $D := \{(u, v) \in \mathbb{R}^2 : u \geq 0, v \geq 0, u+v \leq 1\}$.

7.8 Hankel transform with trigonometric parameters

This subsection presents an integral formula (7.8.1) on the Hankel transform with two trigonometric parameters. In the conformal model [33] (i.e. the solution space to the Yamabe equation) of the minimal representation, K-finite vectors can be explicitly expressed in terms of spherical harmonics (e.g. Gegenbauer's polynomials). On the other hand, in the L^2-model (the Schrödinger model) which is obtained by the Fourier transform of the conformal model (or the N-picture in a terminology of representation theory), it is not easy to find explicit K-finite vectors. The formula (7.8.1) bridges these two models and gives an explicit formula of K-finite vectors in the Schrödinger model (see the proof of Lemma 3.4.4).

Since we have not found this formula in the literature, we give a proof here for the sake of completeness, generalizing the argument in [35, §5.6, 5.7].

Lemma 7.8.1. *The following integral formula on the Hankel transform holds:*

$$\int_0^\infty t^{\mu+1} J_\mu\left(\frac{t\sin\theta}{\cos\theta+\cos\phi}\right) J_\nu\left(\frac{t\sin\phi}{\cos\theta+\cos\phi}\right) K_\nu(t)dt$$
$$= \frac{2^{\nu-1}}{\sqrt{\pi}}\Gamma(\mu-\nu+1)(\cos\theta+\cos\phi)\sin^\mu\theta\sin^\nu\phi\ \widetilde{C}_{\mu-\nu}^{\nu+\frac{1}{2}}(\cos\phi) \tag{7.8.1}$$

Proof. By Baily's formula (7.2.12) of the Hankel transform, the left-hand side of (7.8.1) equals

$$\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\frac{2^\mu\sin^\mu\theta\sin^\nu\phi}{(\cos\theta+\cos\phi)^{\mu+\nu}}$$
$$\times F_4(\mu+1,\mu+\nu+1;\mu+1,\nu+1;-(\frac{\sin\theta}{\cos\theta+\cos\phi})^2,-(\frac{\sin\phi}{\cos\theta+\cos\phi})^2). \tag{7.8.2}$$

Here F_4 denotes Appell's hypergeometric function (see (7.7.4)). Thus, the proof of Lemma 7.8.1 will be completed if we show the following:

Claim 7.8.2. *We have*

$$F_4(\mu+1,\mu+\nu+1;\mu+1,\nu+1;-(\frac{\sin\theta}{\cos\theta+\cos\phi})^2,-(\frac{\sin\phi}{\cos\theta+\cos\phi})^2)$$
$$=\frac{(\cos\theta+\cos\phi)^{\mu+\nu+1}}{2^{\mu-\nu+1}\sqrt{\pi}}\frac{\Gamma(\mu-\nu+1)\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}\widetilde{C}_{\mu-\nu}^{\nu+\frac{1}{2}}(\cos\phi). \tag{7.8.3}$$

□

Claim 7.8.2 is essentially a restatement of [35, Lemma 5.7]. For the convenience of the reader, we include its proof here.

Proof of Claim 7.8.2. We recall a quadratic transformation for hypergeometric functions (see [9, § 2.11 (32)]):

$${}_2F_1(\alpha,\beta;1+\alpha-\beta;z)=(1-z)^{-\alpha}{}_2F_1(\frac{\alpha}{2},\frac{\alpha+1-2\beta}{2};1+\alpha-\beta;\frac{-4z}{(1-z)^2}). \tag{7.8.4}$$

Combining the reduction formula (7.7.7) with (7.8.4), and using the symmetry of a and b; (c,x) and (d,y) in $F_4(a,b;c,d;x,y)$, we have

$$\begin{aligned} & F_4(\alpha,\beta;\alpha,1-\alpha+\beta;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}) \\ &= \left(\frac{(1-x)(1-y)}{1-xy}\right)^{\beta} {}_2F_1(\frac{\beta}{2},\frac{1-2\alpha+\beta}{2};1-\alpha+\beta;\frac{4y(1-x)(1-y)}{(1-xy)^2}). \end{aligned} \tag{7.8.5}$$

Consider the change of variables from (x,y) to (θ,ϕ) by the following identities:

$$\frac{x}{(1-x)(1-y)}=\left(\frac{\sin\theta}{\cos\theta+\cos\phi}\right)^2,\quad \frac{y}{(1-x)(1-y)}=\left(\frac{\sin\phi}{\cos\theta+\cos\phi}\right)^2$$

such that $(x,y)=(0,0)$ corresponds to $(\theta,\phi)=(0,0)$. Then, a simple computation shows

$$\frac{1-xy}{(1-x)(1-y)}=\frac{2}{\cos\theta+\cos\phi},\quad \frac{4y(1-x)(1-y)}{(1-xy)^2}=\sin^2\phi.$$

Now, we set

$$\alpha=\mu+1,\quad \beta=\mu+\nu+1,$$

in (7.8.5). Then, the left-hand side of (7.8.3) amounts to

$$\left(\frac{2}{\cos\theta+\cos\phi}\right)^{-\mu-\nu-1} {}_2F_1(\frac{\mu+\nu+1}{2},-\frac{\mu-\nu}{2};\nu+1;\sin^2\phi). \tag{7.8.6}$$

By using (7.4.6), (7.8.6) is expressed as

$$\left(\frac{\cos\theta+\cos\phi}{2}\right)^{\mu+\nu+1}\frac{\Gamma(\mu-\nu+1)\Gamma(2\nu+1)}{\Gamma(\mu+\nu+1)\Gamma(\nu+\frac{1}{2})}\widetilde{C}_{\mu-\nu}^{\nu+\frac{1}{2}}(\cos\phi).$$

By using Gauss's duplication formula (7.4.3), we get Claim. □

7.9 Formula for the fractional integral

The following formula is used in Subsection 5.3 where we find explicit eigenvalues of intertwining operators on $L^2(S^{p-2}\times S^{q-2})$ by using the Funk–Hecke formula (see Example 5.3.2).

Lemma 7.9.1. *For* $\operatorname{Re}\mu, \operatorname{Re}\nu > -\frac{1}{2}, \operatorname{Re}\lambda > -1$ *and* $l, k \in \mathbb{N}$, *we have the following formula for the fractional integral:*

$$\int_{-1}^{1}\int_{-1}^{1} \frac{(x+y)_{\pm}^{\lambda}}{\Gamma(\lambda+1)} \widetilde{C}_l^{\mu}(x)\widetilde{C}_k^{\nu}(y)(1-x^2)^{\mu-\frac{1}{2}}(1-y^2)^{\nu-\frac{1}{2}}dxdy$$
$$= \frac{b2^{1-\lambda}\Gamma(\lambda+\mu+\nu+1)}{\Gamma(\frac{\lambda+2\mu+2\nu+l+k+2}{2})\Gamma(\frac{\lambda+2\mu+l-k+2}{2})\Gamma(\frac{\lambda+2\nu-l+k+2}{2})\Gamma(\frac{\lambda-l-k+2}{2})}, \tag{7.9.1}$$

where

$$b := \frac{(\pm 1)^{l+k}\pi^2}{2^{2\mu+2\nu}} \frac{\Gamma(2\mu+l)\Gamma(2\nu+k)}{l!\,k!}$$

is a constant independent of λ.

Proof. The left-hand side of (7.9.1) amounts to

$$\int_{-1}^{1}\Big(\int_{-x}^{1} \frac{(x+y)^{\lambda}}{\Gamma(\lambda+1)} \widetilde{C}_k^{\nu}(y)(1-y^2)^{\nu-\frac{1}{2}}dy\Big)\widetilde{C}_l^{\mu}(x)(1-x^2)^{\mu-\frac{1}{2}}dx$$
$$= \frac{\sqrt{\pi}\Gamma(2\nu+k)}{2^{\nu-\frac{1}{2}}k!}\int_{-1}^{1}\Big((1-x^2)^{\frac{\lambda}{2}+\frac{\nu}{2}+\frac{1}{4}}P_{\nu+k-\frac{1}{2}}^{-\lambda-\nu-\frac{1}{2}}(-x)\Big)\widetilde{C}_l^{\mu}(x)(1-x^2)^{\mu-\frac{1}{2}}dx$$
$$= \frac{2^{-\nu-2\mu-l+\frac{3}{2}}\pi}{\Gamma(\mu+l+\frac{1}{2})}\frac{\Gamma(2\nu+k)\Gamma(2\mu+l)}{k!\,l!}\int_{-1}^{1}(1-x^2)^{\frac{\lambda}{2}+\mu+\frac{\nu}{2}-\frac{l}{2}-\frac{1}{4}}P_{\nu+k-\frac{1}{2}}^{-\lambda-\nu+l-\frac{1}{2}}(-x)dx$$
$$= \frac{2^{1-\lambda-2\mu-2\nu}\pi\Gamma(2\mu+l)\Gamma(2\nu+k)\Gamma(\lambda+\mu+\nu+1)}{l!\,k!\,\Gamma(\frac{\lambda+2\mu+2\nu+l+k+2}{2})\Gamma(\frac{\lambda+2\mu+l-k+2}{2})\Gamma(\frac{\lambda+2\nu-l+k+2}{2})\Gamma(\frac{\lambda-l-k+2}{2})}.$$

Hence, the right-hand side of (7.9.1) follows. Some remarks on each equality are given in turn:

First equality follows from Ge4 in Appendix 7.4.

Second equality. First, we made use of the integral by parts because we have (see (7.4.1))

$$\widetilde{C}_l^{\mu}(x)(1-x^2)^{\mu-\frac{1}{2}} = \frac{(-1)^l}{2^{2\mu+l-1}\,l!}\frac{\Gamma(2\mu+l)\sqrt{\pi}}{\Gamma(\mu+l+\frac{1}{2})}\frac{d^l}{dx^l}\Big((1-x^2)^{\mu+l-\frac{1}{2}}\Big).$$

Then, we applied the functional relation

$$\frac{d^l}{dx^l}\Big((1-x^2)^{\frac{\lambda}{2}+\frac{\nu}{2}+\frac{1}{4}}P_{\nu+k-\frac{1}{2}}^{-\lambda-\nu-\frac{1}{2}}(-x)\Big) = (1-x^2)^{\frac{\lambda}{2}+\frac{\nu}{2}-\frac{l}{2}+\frac{1}{4}}P_{\nu+k-\frac{1}{2}}^{-\lambda-\nu+l-\frac{1}{2}}(-x),$$

which follows from the iteration of (7.3.2).

Third equality. We applied the integral formula (7.3.4) after changing the variable $x \mapsto -x$. $\square$

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