THE "HOT SPOTS" CONJECTURE FOR NEARLY CIRCULAR PLANAR CONVEX DOMAINS

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ABSTRACT. We prove the "hot spots" conjecture of J. Rauch in the case that the domain Ω is a planar convex domain satisfying $\mathrm{diam}(\Omega)^2/|\Omega| < 1.378$. Specifically, we show that an eigenfunction corresponding to the lowest non-zero eigenvalue of the Neumann Laplacian on Ω attains its maximum (minimum) at points on $\partial\Omega$. When Ω is a disk, $\mathrm{diam}(\Omega)^2/|\Omega| \approx 1.273$. Hence, the above condition indicates that Ω is a nearly circular planar convex domain. However, symmetries of the domain are not assumed.

1. Introduction and Main results

Let Ω be an open bounded domain with smooth boundary. We consider the eigenvalue problem

$$\Delta\Phi_n + \lambda_n\Phi_n = 0 \ \text{ in } \ \Omega, \qquad \partial_\nu\Phi_n = 0 \ \text{ on } \ \partial\Omega, \tag{1.1}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ are the eigenvalues which satisfy

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to +\infty.$$

Hereafter, u denotes an eigenfunction of (1.1) corresponding to λ_2 which is not identically 0. In this paper, under certain conditions on Ω, we prove the "hot spots" conjecture of the form:

Conjecture 1.1 (J. Rauch[R75]). *For all $p \in \Omega$,*

$$\inf_{q\in\partial\Omega} u(q) < u(p) < \sup_{q\in\partial\Omega} u(q).$$

For a brief history and the physical meaning of Conjecture 1.1, see [BB99, AB04]. This conjecture does not hold in this level of generality. There are counterexamples which are planar non-convex domains with hole(s) [BW99, B05]. On the other hand, the conjecture holds for a planar convex domain with two axes of symmetries [BB99, JN00] ([BB99] also proves the case of another class of domains with one axis of symmetry) and for the so-called *lip domain* [AB04]. However, the case of the convex domains seems to remain open, even if the domain is planar. Our main result is a positive answer for a certain class of planar domains which is not completely included in [BB99, JN00, AB04].

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Theorem A. *Let Ω be a planar convex domain. Let $|\Omega|$ denote the area of Ω, and let $d := \sup_{p,q\in\Omega} |p-q|$ $(:= \mathrm{diam}(\Omega))$. If*

$$\frac{d^2}{|\Omega|} < \frac{j_0^2}{\pi j_1^2} \approx 1.378, \tag{1.2}$$

then Conjecture 1.1 holds. Here j_k $(k=0,1)$ denotes the first positive zero of the derivative of the Bessel function of the first kind of order k, $J_k'(\,\cdot\,)$, namely, $j_0 \approx 3.831$, $j_1 \approx 1.841$.

When Ω is a disk, $d^2/|\Omega| = 4/\pi \approx 1.273$. Therefore, (1.2) indicates that Ω is a nearly circular domain. However, we do not assume symmetries of the domain.

Theorem A is derived from the following:

Lemma B. *Let Ω be a planar convex domain. If*

$$\sqrt{\lambda_2} < \frac{j_0}{d}, \tag{1.3}$$

then Conjecture 1.1 holds.

We briefly show that (1.3) holds if (1.2) holds. Let R_* denote the radius of the disk which has the same area as Ω. Using an upper bound of λ_2 obtained by Szegö [S54] and Weinberger [W56]

$$\sqrt{\lambda_2} \le \frac{j_1}{R_*} = \sqrt{\frac{\pi}{|\Omega|}}\, j_1,$$

we have

$$\sqrt{\lambda_2} \le \sqrt{\frac{\pi}{|\Omega|}}\, j_1 < \frac{j_0}{d}.$$

Hence, Lemma B derives Theorem A.

We prove Lemma B in an indirect way.

Lemma 1.2. *Suppose the same assumptions of Lemma B are satisfied. Then u does not have a critical point in Ω. Here we say that p_0 is a critical point of u, if $u_x(p_0) = u_y(p_0) = 0$.*

We show by contradiction that Lemma 1.2 derives Lemma B. If Conjecture 1.1 fails, then u attains the maximum or minimum at a point in Ω which is a critical point. This contradicts Lemma 1.2.

We will see that Lemma 1.2 is a special case of the following conjecture:

Conjecture 1.3 (E. Yanagida)**.** *Let $\Omega \subset \mathbb{R}^N$ be a convex domain, and let f be a function of class C^2. Let v be a non-constant solution to*

$$\Delta v + f(v) = 0 \ \text{ in } \ \Omega, \qquad \partial_\nu v = 0 \ \text{ on } \ \partial\Omega. \tag{1.4}$$

If v has a critical point p_0 in Ω (i.e., $v_{x_k}(p_0) = 0$ for all $k \in \{1,2,\ldots,N\}$), then the second eigenvalue μ_2 of the eigenvalue problem

$$\Delta\Psi_n + f'(v)\Psi_n = -\mu_n\Psi_n \ \text{ in } \ \Omega, \qquad \partial_\nu\Psi_n = 0 \ \text{ on } \ \partial\Omega \tag{1.5}$$

is strictly negative.

Conjecture 1.3 is posed by E. Yanagida, and he points out that Conjecture 1.3 is a nonlinear version of Conjecture 1.1 [Y06]. Conjecture 1.3 holds for a disk [Mi07b] and for a domain of the form $[0,1] \times D(\subset \mathbb{R}\times\mathbb{R}^N)$ [GM88] (The setting of the problem of [GM88] is, however, different from ours). We consider a more general problem. When the Morse index of a solution to (1.4) on a convex domain is m, what shape is the solution? It is well-known that the solution of (1.4) on a convex domain should be constant when $m = 0$ [CH78, Ma79]. The contrapositive of Conjecture 1.3 is an answer for the case $m = 1$. Specifically, when $m = 1$, a solution on a convex domain attains the maximum (minimum) at points on the boundary.

In the present paper, we consider Lemma 1.2 as the linear case of Conjecture 1.3, i.e., $f(v) = \lambda_2 v$. In this case,

$$\mu_2 = 0. \tag{1.6}$$

However, in Section 3 we show that $\mu_2 < 0$ if u has a critical point in Ω. This contradicion proves Lemma 1.2.

We should mention the strategy. [BW99, BB99, AB04, B05] use probabilistic methods. [JN00] uses an analytic method. Our method is an analytic one which is partially similar to the one of [JN00]. Let $U \in C^\infty(\mathbb{R}^2)$ be the radially symmetric solution to

$$\Delta U + \lambda_2 U = 0 \text{ in } \mathbb{R}^2, \qquad U(0) = \max_{q\in\mathbb{R}^2} U(q) = 1.$$

Then U can be written explicitly as follows:

$$U(x,y) = J_0\left(\sqrt{\lambda_2} r\right) \quad (r = \sqrt{x^2+y^2}),$$

where $J_k(\,\cdot\,)$ is the Bessel function of the first kind of order k. By z we define

$$z := w - u, \tag{1.7}$$

where $w = u(p_0)U_{p_0}$, and U_{p_0} is the translation of U such that the center of U is p_0. Note that $\Delta z + \lambda_2 z = 0$ in Ω. In the proof of Lemma 1.2, we use a contradiction. Suppose that $p_0 \in \Omega$ is a critical point of u. Then p_0 is a degenerate point of z. Using this degeneracy of z, we analyze the zero-level sets of z and construct a good test function which derives a contradiction ($\mu_2 < 0$). Roughly speaking, we compare the eigenfunction u with the radially symmetric solution w in spite that w does not necessarily satisfy the Neumann boundary condition on $\partial\Omega$. The techniques used in the proof are developed in [Mi07a, Mi07b].

The rest of the paper consists as follows: In Section 2, we recall known results about the zero-level sets (the nodal curves) of eigenfunctions in $\mathbb{R}^2$, namely, the theory of Carleman, Hartman, and Wintner. In Section 3, we prove Lemma 1.2.

2. Preliminaries

In this section, we recall the theory of Carleman, Hartman, and Wintner. From now on, let $\{\phi = 0\}$, $\{\phi \neq 0\}$, and $\{\phi > 0\}$ denote the zero-level sets, the non-zero level sets, and the positive regions of ϕ respectively.

Proposition 2.1. *Let $V(x,y) \in C^1(\Omega)$, and let $\phi(x,y)$ be a function satisfying that $\Delta\phi + V\phi = 0$ in Ω. Then $\phi \in C^2(\Omega)$. Furthermore, ϕ has the following properties:*
(i) *If ϕ has a zero of any order at p_0 in Ω, then $\phi \equiv 0$ in Ω.*
(ii) *If ϕ has a zero of order l at p_0 in Ω, then the Taylor expansion of ϕ is*

$$\phi(p) = H_l(p - p_0) + O(|p - p_0|^{l+1}),$$

where H_l is a real valued, non-zero, harmonic, homogeneous polynomial of degree l. Therefore, $\{\phi = 0\}$ has exactly $2l$ branches at p_0.

This proposition is due to Hartman-Wintner [HW53] and generalizes a result by Carleman [C33]. This statement is a slight modification of the statement of [HHHO99]. We see by the proposition that $\{\phi = 0\}$ consists of smooth curves and intersections among them. In particular, the curves do not terminate, unless they hit other curves or the boundary of the domain.

We modify this proposition so that Proposition 2.1 can be applied in our proof.

Corollary 2.2. *Let $V(x, y) \in C^1(\Omega)$, and let $\phi(x, y) \in C^2(\Omega)$ be a function satisfying that $\Delta\phi + V\phi = 0$ in Ω. If ϕ has a degenerate point p_0 in Ω, i.e., $\phi(p_0) = \phi_x(p_0) = \phi_y(p_0) = 0$, then either* (i) *or* (ii) *holds:*
(i) *$\phi \equiv 0$ in Ω.*
(ii) *$\{\phi = 0\}$ has at least four branches at p_0. Moreover, $\{\phi > 0\}$ has at least two connected components near p_0 (However, they may be connected globally in Ω).*

3. Proof

In this section we prove Lemma 1.2 by contradiction. Hereafter, let Ω be a planar convex domain, and let p_0 be a critical point of u in Ω.

First, we study the zero-level sets of $z = w - u$, where z is defined by (1.7).

Lemma 3.1. *p_0 is a degenerate point of z, i.e., $z(p_0) = z_x(p_0) = z_y(p_0) = 0$.*

Proof. Because of the definition of z, $z(p_0) = w(p_0) - u(p_0) = 0$. Since p_0 is a critical point of w and u, we have $z_x(p_0) = w_x(p_0) - u_x(p_0) = 0$ and $z_y(p_0) = w_y(p_0) - u_y(p_0) = 0$. □

Before stating the next lemma, we recall the following relation for Dirichlet and Neumann eigenvalues of the Laplacian:

Proposition 3.2. *The lowest Neumann nonzero eigenvalue of a planar domain is strictly less than the lowest Dirichlet eigenvalue for any domain with the same area.*

The next lemma is well-known for specialists (However, the function z in the statement is not necessarily an eigenfunction of (1.1), because z does not necessarily satisfy the Neumann boundary condition on $\partial\Omega$). See [K85, JN00] for details of Proposition 3.2 and Lemma 3.3. We repeat the proof for the sake of the readers' convenience.

Lemma 3.3. *$\{z = 0\}$ has no loop on $\Omega \cup \partial\Omega$.*

Proof. We prove the lemma by contradiction. Assume the contrary, i.e., $\{z = 0\}$ has a loop. Let D be the open subset of Ω enclosed by the loop. If the sign of z changes in D, then there is another loop on $D \cup \partial D$ such that z does not change the sign in the set enclosed by the loop. Thus we can assume without loss of generality that the sign of z does not change in D. Then z satisfies

$$\Delta z + \lambda_2 z = 0 \text{ in } D, \qquad z = 0 \text{ on } \partial D.$$

Therefore, λ_2 is the first eigenvalue of the Dirichlet Laplacian on D. Let $\kappa_1(D)$ (resp. $\kappa_1(\Omega)$) denote the first eigenvalue of the Dirichlet Laplacian on D (resp. Ω). Using Proposition 3.2 and the fact that $D \subset \Omega$, we have

$$\lambda_2 = \kappa_1(D) \geq \kappa_1(\Omega) > \lambda_2,$$

which is a contradiction. □

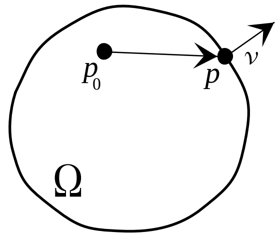

FIGURE 1. The vectors $p - p_0$ and ν.

Lemma 3.4. *Let $\Gamma \subset \partial\Omega$ be an open subset of the boundary with non-zero measure, and let $\delta := \sup_{\xi\in\Gamma,\eta\in\partial\Omega} |\xi - \eta|$. If $z > 0$ on Γ, $u(p_0) > 0$, and*

$$\sqrt{\lambda_2} < \frac{j_0}{\delta}, \tag{3.1}$$

then

$$\int_\Gamma z\partial_\nu z d\sigma < 0.$$

Proof. Let $p \in \Gamma$ be a point on Γ, and let ν denote the outer normal of $\partial\Omega$ at p. Since Ω is convex, we see $\nu \cdot (p - p_0) > 0$, where $\cdot$ denotes the inner product of $\mathbb{R}^2$. See Figure 1. Since $(\partial_\nu u)(p) = 0$ and w is radially symmetric with respect to p_0, we have

$$(\partial_\nu z)(p) = (\partial_\nu w)(p) - (\partial_\nu u)(p) = u(p_0)J_0'\left(\sqrt{\lambda_2}|p - p_0|\right) \nu \cdot \frac{p - p_0}{|p - p_0|}. \tag{3.2}$$

We see that $\sqrt{\lambda_2}|p - p_0| < j_0$, combining (3.1) and the fact that $|p - p_0| \le \delta$. Since $J_0'(\,\cdot\,) < 0$ in the interval $(0, j_0)$, the right-hand side of (3.2) is negative. Hence $(z\partial_\nu z)(p) < 0$, and we have

$$\int_\Gamma z\partial_\nu z d\sigma \le 0.$$

The strictness of the inequality is clear. □

We are in a position to prove the main lemma.

Proof of Lemma 1.2. We prove the lemma by contradiction. Suppose the contrary, i.e., u has a critical point p_0 in Ω. First, we consider the case $u(p_0) = 0$. Then p_0 is a degenerate point of u, i.e., $u(p_0) = u_x(p_0) = u_y(p_0) = 0$. Since $u \not\equiv 0$ in Ω, (ii) of Corollary 2.2 says that $\{u = 0\}$ has at least four branches at p_0. Each branch does not terminate and should connect to another branch or to the boundary $\partial\Omega$. Hence, it follows from an elementary argument that $\{u \neq 0\}$ has at least three connected components. We obtain a contradiction, since u should have exactly two nodal domains.

If $u(p_0) < 0$, then we consider $-u$. Therefore, we can assume without loss of generality that $u(p_0) > 0$. We see by Lemma 3.1 that p_0 is a degenerate point of z. By Corollary 2.2 we can divide the possibilities in two cases.

Case 1: We consider the case that (i) of Corollary 2.2 occurs, i.e., $z \equiv 0$ in Ω. Then u is radially symmetric with respect to p_0, since w is radially symmetric. Ω should be a disk, because u satisfies the Neumann boundary condition on $\partial\Omega$, Ω is convex, and $p_0 \notin \partial\Omega$. In this case, u can be written explicitly,

$$u(r,\theta) = c_1 J_1\left(\frac{j_1 r}{R}\right)\sin\theta + c_2 J_1\left(\frac{j_1 r}{R}\right)\cos\theta \quad (c_1, c_2 \in \mathbb{R},\ c_1^2 + c_2^2 \neq 0), \tag{3.3}$$

where R is the radius of the disk and (r,θ) is a polar coordinate system of the disk. In particular, (3.3) is not radially symmetric, which contradicts that u is radially symmetric. Thus the case does not occur.

Case 2: (ii) of Corollary 2.2 occurs. Then $\{z = 0\}$ has at least four branches at p_0. Each branch does not connect to another branch. If it does, then $\{z = 0\}$ has a loop, which contradicts Lemma 3.3. Hence each branches should connect to the boundary $\partial\Omega$. Moreover, it follows from (ii) of Corollary 2.2 that $\{z > 0\}$ has at least two connected components. They are not connected globally in Ω, since all the branches emanated from p_0 are separated each other. Let D_1, D_2 $(D_1 \cap D_2 = \emptyset)$ denote connected components of $\{z > 0\}$. By z_k $(k = 1, 2)$ we define

$$z_k := \begin{cases} z & \text{in } D_k, \\ 0 & \text{in } \Omega \backslash D_k. \end{cases}$$

Note that $\partial D_k \cap \partial\Omega$ $(k = 1, 2)$ has non-zero measure, because of Lemma 3.3. By ψ we define

$$\psi := z_1 - \alpha z_2 \quad (\alpha \in \mathbb{R}).$$

We take α such that $\langle \psi, \Phi_1 \rangle = 0$, where $\Phi_1 (= 1)$ is the first eigenfunction of (1.1), and $\langle \cdot\,, \cdot \rangle$ denotes the inner product of $L^2(\Omega)$. Specifically, $\alpha = \langle z_1, \Phi_1 \rangle / \langle z_2, \Phi_2 \rangle$. Here $\langle z_2, \Phi_2 \rangle > 0$, because $z_2 > 0$ on a set with non-zero measure. Then we have

$$\begin{aligned} \mathcal{H}[\psi] &:= \iint_\Omega \left(|\nabla\psi|^2 - \lambda_2 \psi^2 \right) dxdy \\ &= -\iint_{\{\psi \neq 0\}} \psi \left(\Delta\psi + \lambda_2 \psi \right) dxdy + \int_{\partial\{\psi \neq 0\}} \psi \partial_\nu \psi d\sigma \\ &= \int_{\partial\{z_1 \neq 0\}} z_1 \partial_\nu z_1 d\sigma + \alpha^2 \int_{\partial\{z_2 \neq 0\}} z_2 \partial_\nu z_2 d\sigma, \end{aligned} \tag{3.4}$$

where we use $\Delta\psi + \lambda_2\psi = 0$ in $\{\psi \neq 0\}$. Because of the assumption (1.3), (3.1) is satisfied. We see by Lemma 3.4 that the right-hand side of (3.4) is negative. From a variational characterization of the second eigenvalue of (1.5), we have

$$\mu_2 := \inf_{\phi \in \mathrm{span}\langle \Phi_1 \rangle^\perp \cap H^1(\Omega)} \frac{\mathcal{H}[\phi]}{\langle \phi, \phi \rangle} \leq \frac{\mathcal{H}[\psi]}{\langle \psi, \psi \rangle} < 0, \tag{3.5}$$

where $\mathrm{span}\langle \Phi_1 \rangle^\perp := \left\{ \phi \in L^2(\Omega);\ \langle \phi, \Phi_1 \rangle = 0 \right\}$. (3.5) contradicts (1.6). □

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