Limit Elements in the Configuration Algebra for a Discrete Group *

Dedicated to Professor Heisuke Hironaka on the occasion of his seventy-seventh birthday

By

Kyoji Saito **

Contents

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* The present paper is a complete version of the announcement [Sa2]. It is based on the preprint RIMS-726. Compared with them, we rewrote the introduction, left out the filtration by (p, q), improved and divided section §10 into two parts, and updated the references. The new §10 clarifies the previous one. In the newly added §11, we apply the theory to a group Γ with a finite generator system G and introduce the limit set $\Omega(\Gamma, G)$. We prove a trace formula for $\Omega(\Gamma, G)$ and pose problems and conjectures on $\Omega(\Gamma, G)$.

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**RIMS, Kyoto University, Kyoto 606-8502, Japan.

§1. Introduction

The present paper consists of two parts.

In the first part §1-10, we introduce and study certain Hopf algebra, called the *configuration algebra*, generated by all isomorphism classes, called configurations, of finite graphs equipped with colors on edges. The algebra is complete with respect to an adic topology. We construct a space $\mathcal{L}_{\mathbb{R},\infty}$ called the Lie-like space at infinity, complementary to the finite part of the algebra. The goal of the first part is to introduce the set of *limit elements* $\overline{\log(\mathrm{EDP})}_{\infty}$ in $\mathcal{L}_{\mathbb{R},\infty}$.

In the second part §11, we apply the first part to the Cayley graph of any group Γ with a given finite generator system G, and obtain *the set* $\Omega(\Gamma, G)$ *of limit elements for* (Γ, G) inside $\overline{\log(\mathrm{EDP})}_{\infty}$. It turns out that $\Omega(\Gamma, G)$ is non-empty if and only if Γ is infinite, and consists of a single element if Γ is of polynomial growth Conjecturally, $\Omega(\Gamma, G)$ is finite if Γ is a hyperbolic group.

The construction of the limit elements $\Omega(\Gamma, G)$ is inspired by correlation functions (or partition functions) in statistical mechanics ([Gi][I][O][Ba]) where Γ is an abelian lattice and the colors in G are specialized to Boltzmann's weights so that the configurations take on certain values, and then limit elements give rise to partition functions. Our original attempt is to use limit elements in $\Omega(\Gamma, G)$ for the construction of certain "modular functions" on the moduli of anabelian discrete groups Γ ([Sa1,3]). Since the construction of limit elements $\overline{\log(\mathrm{EDP})}_{\infty}$ is independent of each individual group Γ, we separate the general study of the configuration algebra as in the first part of the present paper.

The organization of the first part §1-10 is as follows.

The configuration algebra $\mathbb{A}[\![\mathrm{Conf}]\!]$, as a topological Hopf algebra over an associative algebra $\mathbb{A}$, is introduced in §2,3 and 4. The basis of the space $\mathcal{L}_{\mathbb{A}}$ of its Lie-like elements (and also its group-like elements) are studied in §5,7 and 8. The subspace $\mathcal{L}_{\mathbb{A},\infty}$ of $\mathcal{L}_{\mathbb{A}}$ at infinity is introduced at the end of §8 by the use of kabi-coefficients introduced in §7. Finally, we introduce the set $\overline{\log(\mathrm{EDP})}_{\infty}$ of *limits of logarithmic equal division points* in $\mathcal{L}_{\mathbb{R},\infty}$ in §10, where $\mathbb{A}$ is the real number field $\mathbb{R}$ and we use the *classical topology of* $\mathbb{R}$ in an essential way.

Let us explain the above in more detail. The isomorphism class of a colored oriented finite graph is called a *configuration* (§2). The set of all configurations with a fixed bound of valency and colors, denoted by Conf, has the structure of a monoid (with the disjoint union as the product) and of a partial ordering. In §2, the most basic invariant $\binom{S_1,\ldots,S_m}{S} \in \mathbb{Z}_{\geq 0}$ for $S_1,\ldots,S_m$ and $S \in$ Conf, called a *covering coefficient*, is introduced and studied. The completion $\mathbb{A}\,[\![\mathrm{Conf}]\!]$ of the semigroup ring of Conf with respect to a certain adic topology is called the *configuration algebra* in §3. Using the covering coefficients as structure constants, we introduce a topological Hopf algebra structure on $\mathbb{A}\,[\![\mathrm{Conf}]\!]$ in §4.

For a configuration S, let $1+\mathcal{A}(S)$ be the element of $\mathbb{A}[\mathrm{Conf}]$ given by the sum of all subgraphs of S. Put $\mathcal{M}(S)$:=log$(1+\mathcal{A}(S))$ (§5 and 6). $1+\mathcal{A}(S)$ and $\mathcal{M}(S)$ form a basis of the group-like or Lie-likespace of the non-complete algebra $\mathbb{A}[\mathrm{Conf}]$, respectively. However, they are not a topological basis of the Lie-like space $\mathcal{L}_{\mathbb{A}}$ (resp. group-like space $\mathfrak{G}_{\mathbb{A}}$) of the completed algebra $\mathbb{A}\,[\![\mathrm{Conf}]\!]$. Therefore, we introduce a topological basis, denoted by $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$, for the completed module $\mathcal{L}_{\mathbb{A}}$. The transformation matrix between the basis $\{\mathcal{M}(S)\}_{S\in\mathrm{Conf}_0}$ and $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ is described by certain constants, called *kabi-coefficients*, introduced in §7. The base-change induces a map, called the *kabi-map*, from $\mathcal{L}_{\mathbb{A}}$ to another module, whose kernel $\mathcal{L}_{\mathbb{A},\infty}$ is called the *space at infinity* (§8).

Inside the infinite dimensional group $\mathfrak{G}_{\mathbb{R}}$, the subgroup $\mathfrak{G}_{\mathbb{Z}}$ forms a *lattice*, i.e. it is discrete but dense with respect to the adic topology (§8 and 9). The lattice has a natural *positive cone* generated by $1+\mathcal{A}(S)$ for all configurations S. We are interested in the *equal division points* $(1+\mathcal{A}(S))^{1/\#S}$ of the lattice points in the positive cone (§9), and study the set of *their accumulation points with respect to the classical topology* by specializing the coefficient $\mathbb{A}$ to $\mathbb{R}$. In §10, by taking their logarithms, we describe the accumulation set $\overline{\log(\mathrm{EDP})}$ in $\mathcal{L}_{\mathbb{R}}$. We show that the set decomposes into a joint of the "finite part" and the "infinite part" $\overline{\log(\mathrm{EDP})}_{\infty}$ contained in $\mathcal{L}_{\mathbb{R},\infty}$; this is the object of the first part.

The second part (§11) describes the application of the first part to a group Γ with a fixed finite generator system G. We consider the increasing sequence Γ_n of balls of radius $n\in\mathbb{Z}_{\geq 0}$ in the Cayley graph (Γ, G). The set of accumulation points in $\mathcal{L}_{\mathbb{R},\infty}$ of the sequence of logarithmic equal division points $\frac{\mathcal{M}(\Gamma_n)}{\#\Gamma_n}$ in $\mathcal{L}_{\mathbb{R}}$ is denoted by $\Omega(\Gamma, G)$. If the group Γ is of polynomial growth, then due to a result of Gromov [Gr1] and Pansu [P], $\Omega(\Gamma, G)$ consists of a single element. In order to treat finite accumulating cases, we introduce the concept of a *finite rational accumulation* (11.3). As the goal of the paper, if $\#\Omega(\Gamma, G)<\infty$, we express the traces of $\Omega(\Gamma, G)$ by residues of the growth series $\sum_{n=0}^{\infty}\#\Gamma_n t^n$ and $\sum_{n=0}^{\infty}\mathcal{M}(\Gamma_n)t^n$. We conjecture that hyperbolic groups belong to such a class; however, at present, no examples other than free groups, have been worked out.

§2. Colored graphs and covering coefficients

An isomorphism class of finite graphs with a fixed color-set and a bounded number of edges (valency) at each vertex is called a *configuration*. The set of all configurations carries the structure of an abelian monoid with a partial ordering. The goal of the present section is to introduce a numerical invariant, called the *covering coefficient*, and to show some of its basic properties.

2.1 Colored Graphs.

We first give a definition of colored graph which is used in the present paper.

Definition. 1. A pair (Γ, B) is called a *graph*, if Γ is a set and B is a subset of $\Gamma\times\Gamma\setminus\Delta$ with $\sigma(B)=B$, where σ is the involution $\sigma(\alpha,\beta):=(\beta,\alpha)$ and Δ is the diagonal subset. An element of Γ is called a *vertex* and a σ-orbit in B is called an *edge*. A graph is called *finite* if $\sharp\Gamma<\infty$. We sometimes denote a graph by Γ and the set of its vertices by $|\Gamma|$.

2. Two graphs are *isomorphic* if there is a bijection of vertices inducing a bijection of edges. Any subset $\mathbb{S}$ of $|\Gamma|$ carries a graph structure by taking $B\cap(\mathbb{S}\times\mathbb{S})$ as the set of edges for $\mathbb{S}$. The set $\mathbb{S}$ equipped with this graph structure is called a *subgraph* (or a *full subgraph*) of Γ and is denoted by the same $\mathbb{S}$. In the present paper, the word "subgraph" shall be used only in this sense, and the notation $\mathbb{S}\subset\Gamma$ shall mean also that $\mathbb{S}$ is a subgraph of Γ associated to the subset. Hence, we have the bijection: $\{\text{subgraphs of } \Gamma\} \ \simeq\ \{\text{subsets of } |\Gamma|\}$.

3. A pair (G,σ_G) of a set G and an involution σ_G on G (i.e. a map $\sigma_G: G\to G$ with $\sigma_G^2=id_G$) is called a color set. For a graph (Γ,B), a map $c: B\to G$ is called a (G,σ_G)-*coloring* , or G-*coloring*, if c is equivariant with respect to involutions: $c\circ\sigma=\sigma_G\circ c$. The pair consisting of a graph and a G-coloring is called a G-*colored graph*. Two G-colored graphs are called G-*isomorphic* if there is an isomorphism of the graphs compatible with the colorings. Subgraphs of a G-colored graph are naturally G-colored.

If all points of G are fixed by σ_G, then the graph is called un-oriented. If G consists of one orbit of σ_G, then the graph is called un-colored.

The isomorphism class of a G-colored graph $\mathbb{S}$ is denoted by $[\mathbb{S}]$. Sometimes we will write $\mathbb{S}$ instead of $[\mathbb{S}]$ (for instance, we put $\sharp[\mathbb{S}] := \sharp\mathbb{S}$, and call $[\mathbb{S}]$ *connected* if $\mathbb{S}$ is topologically connected as a simplicial complex).

Example. (Colored Cayley graph). Let Γ be a group and let G be a generator system of Γ with $G=G^{-1}$ and $e\notin G$. Then, Γ carries a graph structure by taking $B:=\{(\alpha,\beta)\in\Gamma\times\Gamma : \alpha^{-1}\beta\in G\}$ as the set of edges. It is

G-colored by taking (G, σ_G) with $\sigma_G(g) := g^{-1}$ as the color set and $c(\alpha, \beta) := \alpha^{-1}\beta$ as the coloring. Let us call the graph, denoted by (Γ, G), the *colored Cayley graph* of the group Γ with respect to G. The left action of $g \in \Gamma$ on Γ is a G-isomorphism of the colored Cayley graph (Γ, G).

2.2 Configuration.

For the remainder of the paper, *we fix a finite color set* (G, σ_G) *(i.e.* $\#G < \infty$*) and a non-negative integer* $q \in \mathbb{Z}_{\geq 0}$*, and consider only the* G*-colored graphs such that the number of edges ending at a vertex (called valency) is at most* q. The isomorphism class $[\mathbb{S}]$ of such a graph $\mathbb{S}$ is called a (G, q)-*configuration* (or, a *configuration*). The set of all (connected) configurations is defined by

$$\text{(2.2.1)} \quad \mathrm{Conf} := \{G\text{-isomorphism classes of } G\text{-colored graphs such that the number of edges ending at any given vertex is at most } q\}$$

$$\text{(2.2.2)} \quad \mathrm{Conf}_0 := \{S \in \mathrm{Conf} \mid S \text{ is connected}\}.$$

The isomorphism class $[\phi]$ of an empty graph is contained in Conf but not in Conf_0. Sometimes it is convenient to exclude $[\phi]$ from Conf. So put:

$$\text{(2.2.3)} \qquad \mathrm{Conf}_+ \ := \ \mathrm{Conf} \setminus \{[\phi]\}.$$

Remark. To be exact, the set of configurations (2.2.1) should have been denoted by $\mathrm{Conf}^{G,q}$. If there is a map $G \to G'$ between two color sets compatible with their involutions and an inequality $q \leq q'$, then there is a natural map $\mathrm{Conf}^{G,q} \to \mathrm{Conf}^{G',q'}$. Thus, for any inductive system $(G_n, q_n)_{n \in \mathbb{Z}_{>0}}$ (i.e. $G_n \to G_{n+1}$ and $q_n \leq q_{n+1}$ for n), we get the inductive limit $\lim\limits_{n \to \infty} \mathrm{Conf}^{G_n, q_n}$. In [S2], we used such limit set. However, in this paper, we fix G and q, since the key limit processes (3.2.2) and (10.1.1) can be carried out for fixed G and q.

2.3 Semigroup structure and partial ordering structure on Conf.

We introduce the following two structures 1. and 2. on Conf.

1. *The set* Conf *naturally has an abelian semigroup structure by putting*

$$[\mathbb{S}] \cdot [\mathbb{T}] \ := \ [\mathbb{S} \amalg \mathbb{T}] \qquad \text{for } [\mathbb{S}], [\mathbb{T}] \in \mathrm{Conf},$$

where $\mathbb{S} \amalg \mathbb{T}$ is the disjoint union of graphs $\mathbb{S}$ and $\mathbb{T}$ representing the isomorphism classes $[\mathbb{S}]$ and $[\mathbb{T}]$. The empty class $[\phi]$ plays the role of the unit and is denoted by 1. It is clear that Conf *is freely generated by* Conf_0. The power S^k $(k \geq 0)$ denotes the class of a disjoint union $\mathbb{S} \amalg \cdots \amalg \mathbb{S}$ of k-copies of $\mathbb{S}$.

2. *The set* Conf *is partially ordered*, where we define for S and $T \in \mathrm{Conf}$ by : $S \leq T \overset{def.}{\Leftrightarrow}$ there exist graphs $\mathbb{S}$ and $\mathbb{T}$ with $S = [\mathbb{S}], T = [\mathbb{T}]$ and $\mathbb{S} \subset \mathbb{T}$.

The unit $1 = [\phi]$ *is the unique minimal element in* Conf *by this ordering.*

2.4 Covering coefficients.

For $S_1, \ldots, S_m$ and $S \in \mathrm{Conf}$, we introduce a non-negative integer:

$$\binom{S_1, \ldots, S_m}{S} \; := \; \sharp \binom{S_1, \ldots, S_m}{\mathbb{S}} \quad \in \mathbb{Z}_{\geq 0} \tag{2.4.1}$$

and call it the *covering coefficient*, where $\binom{S_1,\ldots,S_m}{\mathbb{S}}$ is defined by the following:
i) Fix any G-graph $\mathbb{S}$ with $[\mathbb{S}] = S$.
ii) Define a set:

$$\binom{S_1, \ldots, S_m}{\mathbb{S}} \; := \; \{(\mathbb{S}_1, \ldots, \mathbb{S}_m) \mid \mathbb{S}_i \subset \mathbb{S} \text{ such that } [\mathbb{S}_i] = S_i \; (i = 1, \ldots, m) \text{ and } \cup_{i=1}^m |\mathbb{S}_i| = |\mathbb{S}|.\} \tag{2.4.2}$$

iii) Show: an isomorphism $\mathbb{S} \simeq \mathbb{S}'$ induces a bijection $\binom{S_1,\ldots,S_m}{\mathbb{S}} \simeq \binom{S_1,\ldots,S_m}{\mathbb{S}'}$.

Remark. In the definition (2.4.2), one should notice that
i) Each $\mathbb{S}_i$ in (2.4.2) should be a full subgraph of $\mathbb{S}$ (see (2.1) Def. 2.).
ii) The union of the edges of $\mathbb{S}_i$ $(i = 1, \ldots, k)$ may not cover all edges of $\mathbb{S}$.
iii) The set of vertices $|\mathbb{S}_i|$ $(i = 1, \ldots, k)$ may overlap the set $|\mathbb{S}|$.

Example. Let X_1, X_2 be elements of Conf_0 with $\sharp X_i = i$ for $i = 1, 2$. Then $\binom{X_1 \cdot X_1}{X_2} = 0$ and $\binom{X_1, X_1}{X_2} = 2$.

The covering coefficients are the most basic tool in the present paper. We shall give their elementary properties in 2.5 and their two basic rules: *the composition rule* in 2.6 and *the decomposition rule* in 2.7.

2.5 Elementary properties for covering coefficients.

Some elementary properties of covering coefficients, as immediate consequences of the definition, are listed below. They are used in the study of the Hopf algebra structure on the configuration algebra in §4.

i) $\binom{S_1,\ldots,S_m}{S} = 0$ *unless* $S_i \leq S$ *for* $i = 1, \ldots, m$ *and* $\sum \sharp S_i \geq \sharp S$.
ii) $\binom{S_1,\ldots,S_m}{S}$ *is invariant by the permutation of* S_i *'s.*
iii) For $1 \leq^{\forall} i \leq m$, *one has an elimination rule:*

$$\binom{S_1, \ldots, S_{i-1}, [\phi], S_{i+1}, \ldots, S_m}{S} \; = \; \binom{S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_m}{S}. \tag{2.5.1}$$

iv) For the case $m = 0$, *the covering coefficients are given by*

$$\binom{}{S} \; = \; \begin{cases} 1 & \text{if } S = [\phi], \\ 0 & \text{else,} \end{cases} \tag{2.5.2}$$

v) For the case $m=1$, the covering coefficients are given by

$$\binom{T}{S} = \begin{cases} 1 & \text{if } S=T, \\ 0 & \text{else,} \end{cases} \tag{2.5.3}$$

vi) For the case $S=[\phi]$, the covering coefficients are given by

$$\binom{S_1,\dots,S_m}{[\phi]} = \begin{cases} 1 & \text{if } \cup S_i=\psi, \\ 0 & \text{else.} \end{cases} \tag{2.5.4}$$

2.6 Composition rule.

Assertion. *For $S_1,\dots,S_m,T_1,\dots,T_n,\ S\in\mathrm{Conf}$ $(m,n\in\mathbb{Z}_{\ge 0})$, one has*

$$\sum_{U\in\mathrm{Conf}} \binom{S_1,\dots,S_m}{U}\binom{U,T_1,\dots,T_n}{S} = \binom{S_1,\dots,S_m,T_1,\dots,T_n}{S}. \tag{2.6.1}$$

Proof. If $m=0$, then the formula reduces to 2.5 iii) and iv). Assume $m\ge 1$ and consider the map

$$\begin{array}{ccc} \binom{S_1,\dots,S_m,T_1,\dots,T_n}{\mathbb{S}} & \longrightarrow & \bigsqcup_{U\in\mathrm{Conf}} \binom{U,T_1,\dots,T_n}{\mathbb{S}} \\ (\mathbb{S}_1,\dots,\mathbb{S}_m,\mathbb{T}_1,\dots,\mathbb{T}_n) & \longmapsto & (\cup_{i=1}^m\mathbb{S}_i,\mathbb{T}_1,\dots,\mathbb{T}_n). \end{array}$$

Here, $\cup_{i=1}^m \mathbb{S}_i$ means the subgraph of $\mathbb{S}$ whose vertices are the union of the vertices of the $\mathbb{S}_i$ $(i=1,\dots,m)$ (cf. (2.1) Def. 2.) and the class $[\cup_{i=1}^m \mathbb{S}_i]$ is denoted by U. The fiber over a point $(\mathbb{U},\mathbb{T}_1,\dots,\mathbb{T}_n)$ is bijective to the set $\binom{S_1,\dots,S_m}{\mathbb{U}}$, so that one has the bijection

$$\binom{S_1,\dots,S_m,T_1\dots,T_n}{\mathbb{S}} \simeq \bigsqcup_{U\in\mathrm{Conf}} \binom{S_1,\dots,S_m}{\mathbb{U}}\binom{U,T_1,\dots,T_n}{\mathbb{S}}.$$

$\square$

Note. The LHS of (2.6.1) is a finite sum, since $U\le S$ due to 2.5 i).

2.7 Decomposition rule.

Assertion. *Let $m\in\mathbb{Z}_{\ge 0}$. For $S_1,\dots,S_m,U$ and $V\in\mathrm{Conf}$, one has*

$$\binom{S_1,\dots,S_m}{U\cdot V} = \sum_{\substack{R_1,T_1\in\mathrm{Conf}\\ S_1=R_1\cdot T_1}} \cdots \sum_{\substack{R_m,T_m\in\mathrm{Conf}\\ S_m=R_m\cdot T_m}} \binom{R_1,\dots,R_m}{U}\binom{T_1,\dots,T_m}{V}. \tag{2.7.1}$$

Here R_i and $T_i\in\mathrm{Conf}$ run over all possible decompositions of S_i in Conf.

Proof. If $m=0$, this is (2.5.2). Consider the map

$$\begin{array}{rcl}\binom{S_1,\ldots,S_m}{\mathbb{U}\cdot\mathbb{V}} & \longrightarrow & \bigcup_{S_1=R_1\cdot T_1}\cdots\bigcup_{S_m=R_m\cdot T_m}\binom{R_1,\ldots,R_m}{\mathbb{U}}\times\binom{T_1,\ldots,T_m}{\mathbb{V}}, \\ (\mathbb{S}_1,\cdots,\mathbb{S}_m) & \longmapsto & (\mathbb{S}_1\cap\mathbb{U},\ldots,\mathbb{S}_m\cap\mathbb{U})\times(\mathbb{S}_1\cap\mathbb{V},\ldots,\mathbb{S}_m\cap\mathbb{V}).\end{array}$$

One checks easily that the map is bijective. □

Note. The RHS of (2.7.1) is a finite sum, since $R_i \leq U$ and $T_i \leq V$.

§3. Configuration algebra.

We complete the semigroup ring $\mathbb{A}\cdot\mathrm{Conf}$, where $\mathbb{A}$ is a commutative associative unitary algebra, by use of the adic topology with respect to the grading $\deg(S):=\#S$, and call the completion the configuration algebra. It is a formal power series ring of infinitely many variables $S\in\mathrm{Conf}_0$. We discuss several basic properties of the algebra, including topological tensor products.

3.1 The polynomial type configuration algebra $\mathbb{Z}\cdot\mathrm{Conf}$.

The free abelian group generated by Conf:

(3.1.1) $$\mathbb{Z}\cdot\mathrm{Conf}$$

naturally carries the structure of an algebra by the use of the semigroup structure on Conf (recall 2.3), where $[\phi]=1$ plays the role of the unit element. It is isomorphic to the free polynomial algebra generated by Conf_0, and hence, is called *the polynomial type configuration algebra.* The algebra is graded by taking $\deg(S):=\sharp(S)$ for $S\in\mathrm{Conf}$, since one has additivity:

(3.1.2) $$\sharp(S\cdot T) \quad = \quad \sharp(S)+\sharp(T).$$

3.2 The completed configuration algebra $\mathbb{Z}[\![\mathrm{Conf}]\!]$.

The polynomial type algebra (3.1.1) is not sufficiently large for our purpose, since it does not contain certain limit elements which we want to investigate (cf 4.6 *Remark* 3 and 6.4 *Remark* 2). Therefore, we localize the algebra by the completion with respect to the grading given in 3.1.

For $n\geq 0$, let us define an ideal in $\mathbb{Z}\cdot\mathrm{Conf}$

(3.2.1) $$\mathcal{J}_n \;:=\; \text{the ideal generated by } \{S\in\mathrm{Conf}\mid \sharp(S)\geq n\}.$$

Taking $\mathcal{J}_n$ as a fundamental system of neighborhoods of $0\in\mathbb{Z}\cdot\mathrm{Conf}$, we define the *adic topology* on $\mathbb{Z}\cdot\mathrm{Conf}$ (see Remark below). The completion

(3.2.2) $$\mathbb{Z}[\![\mathrm{Conf}]\!] \;:=\; \varprojlim_n \mathbb{Z}\cdot\mathrm{Conf}/\mathcal{J}_n$$

will be called *the completed configuration algebra*, or, simply, *the configuration algebra*. More generally, for any commutative algebra $\mathbb{A}$ with unit, we put

$$\mathbb{A}\,[\![\mathrm{Conf}]\!] \;:=\; \varprojlim_{n}\; \mathbb{A}\cdot \mathrm{Conf}\,/\mathbb{A}\mathcal{J}_n, \tag{3.2.3}$$

and call it the configuration algebra over $\mathbb{A}$, or, simply, the configuration algebra. The augmentation ideal of the algebra is defined as

$$\begin{aligned} \mathbb{A}\,[\![\mathrm{Conf}]\!]_+ \;:=&\;\; \text{the closed ideal generated by}\,\mathrm{Conf}_+ \\ =&\;\; \text{the closure of } \mathcal{J}_1 \text{ with respect to the adic topology.} \end{aligned}$$

Let us give an explicit expression of an element of the configuration algebra by an infinite series. The quotient $\mathbb{A}[\,\mathrm{Conf}\,]/\mathbb{A}\mathcal{J}_n$ is naturally bijective to the free module $\prod_{\substack{S\in\mathrm{Conf}\\ \sharp S<n}} \mathbb{A}\cdot S$ of finite rank. Taking the inverse limit of the bijection, we obtain

$$\mathbb{A}\,[\![\mathrm{Conf}]\!] \;\simeq\; \prod_{S\in\mathrm{Conf}} \mathbb{A}\cdot S.$$

In the other words, *any element f of the configuration algebra is expressed uniquely by an infinite series*

$$f \;= \textstyle\sum_{S\in\mathrm{Conf}} S\cdot f_S \tag{3.2.4}$$

for some constants $f_S\in\mathbb{A}$ for all $S\in\mathrm{Conf}$. The coefficient $f_{[\phi]}$ of the unit element is called the *constant term of f*. The augmentation ideal is nothing but the collection of f having vanishing constant term.

Remark. The above defined topology on $\mathbb{A}\,[\![\mathrm{Conf}]\!]$ (except for the case $q=0$) is *not equal* to the topology defined by taking the powers of the augmentation ideal as the fundamental system of neighborhoods of 0. More precisely, for $n>1$ and $q\neq 0$, the image of the product map:

$$(\mathbb{A}\,[\![\mathrm{Conf}]\!]_+)^n \;\longrightarrow\; \overline{\mathbb{A}\mathcal{J}}_n \tag{3.2.5}$$

(c.f. (3.5.4) and (3.5.5)) does not generate (topologically) the target ideal on the RHS (= the closure in $\mathbb{A}\,[\![\mathrm{Conf}]\!]$ of the ideal $\mathbb{A}\mathcal{J}_n=\{f\in\mathbb{A}\,[\![\mathrm{Conf}]\!]\mid \deg S\geq n \text{ for } f_S\neq 0\}$), since there exists a connected configuration S with $\deg S=n$, but S, as an element in $\mathcal{J}_n$, cannot be expressed as a function of elements of $\mathcal{J}_m$ for $m<n$. In this sense, the name "adic topology" is *misused* here.

The notation $\mathbb{A}\,[\![\mathrm{Conf}]\!]$ should not be mistaken for the algebra of formal power series in Conf. In fact, it is the set of formal series in Conf_0.

3.3 Finite type element in the configuration algebra.

The support for the series f (3.2.4) is defined as

$$\mathrm{Supp}(f) \ := \ \{S \in \mathrm{Conf} \mid f_S \neq 0\}. \tag{3.3.1}$$

Definition. An element f of a configuration algebra is said to be of *finite type* if $\mathrm{Supp}(f)$ is contained in a finitely generated semigroup in Conf.

Note that f being of finite type does not mean that f is a finite sum, but means that it is expressed only by a finite number of "variables".

3.4 Saturated subalgebras of the configuration algebra.

The configuration algebra is sometimes a bit too large. For later applications, we introduce a class of its subalgebras, called the saturated subalgebras.

A subset $P \subset \mathrm{Conf}$ is called *saturated* if for $S \in P$, any $T \in \mathrm{Conf}_0$ with $T \leq S$ belongs to P. For a saturated set P, let us define a subalgebra

$$\mathbb{A}[\![P]\!] := \ \{f \in \mathbb{A}[\![\mathrm{Conf}]\!] \mid \mathrm{Supp}(f) \subset \text{the semigroup generated by } P\}. \tag{3.4.1}$$

We shall call a subalgebra of the configuration algebra of the form (3.4.1) for some saturated P a *saturated subalgebra.* A saturated algebra R is characterized by the properties: i) R is a closed subalgebra under the adic topology of the configuration algebra, and ii) if $S \in \mathrm{Supp}(f)$ for $f \in R$ then any connected component of S (as a monomial) belongs to R. We call the set

$$\mathrm{Supp}(R) \ := \ \textstyle\bigcup_{f\in R} \mathrm{Supp}(f) \tag{3.4.2}$$

the support of R. Obviously, $\mathrm{Supp}(R)$ is the saturated subsemigroup of Conf generated by P. The algebra R is determined from $\mathrm{Supp}(R)$.

It is clear that if R is a saturated subalgebra of $\mathbb{A}[\![\mathrm{Conf}]\!]$ then $R \cap (\mathbb{A} \cdot \mathrm{Conf})$ is a dense subalgebra of R and that R is naturally isomorphic to the completion of $R \cap (\mathbb{A} \cdot \mathrm{Conf})$ with respect to the induced adic topology.

Example. We give two typical examples of saturated sets.

1. For any any element $S \in \mathrm{Conf}$, we define its saturation by

$$\langle S \rangle \ := \ \{T \in \mathrm{Conf} : T \leq S\}. \tag{3.4.3}$$

2. Let (Γ, G) be a Cayley graph of an infinite group Γ with respect to a finite generator system G. Then, by choosing G as the color set and $q := \#G$, we define a saturated subset of Conf by

$$\langle \Gamma, G \rangle \ := \ \{\text{isomorphism classes of finite subgraphs of } (\Gamma, G)\}. \tag{3.4.4}$$

Apparently, the saturated subalgebra $\mathbb{A}\,[\![\langle S\rangle]\!]$ consists only of finite type elements, whereas the algebra $\mathbb{A}\,[\![\langle \Gamma, G\rangle]\!]$ contains non-finite type elements. This makes the latter algebra interesting when we study limit elements in §11.

3.5 Completed tensor product of the configuration algebra.

The tensor product over $\mathbb{A}$ of m-copies of $\mathbb{A}\cdot\mathrm{Conf}$ for $m \in \mathbb{Z}_{\geq 0}$ is denoted by $\otimes^m(\mathbb{A}\cdot\mathrm{Conf})$. In this section, we describe the completed tensor product $\widehat{\otimes}^m(\mathbb{A}\,[\![\mathrm{Conf}]\!])$ of the completed configuration algebra,

Definition. Let $\mathbb{A}$ be a commutative algebra with unit. For $m \in \mathbb{Z}_{\geq 0}$, the completed m-tensor product $\widehat{\otimes}^m\mathbb{A}\,[\![\mathrm{Conf}]\!]$ of the configuration algebra $\mathbb{A}[\![\mathrm{Conf}]\!]$ is defined by the inverse limit

$$\widehat{\otimes}^m\mathbb{A}\,[\![\mathrm{Conf}]\!] \ := \ \varprojlim_n \ \otimes^m(\mathbb{A}\cdot\mathrm{Conf})/(\otimes^m\mathbb{A}\mathcal{J})_n, \tag{3.5.1}$$

where $(\otimes^m\mathbb{A}\mathcal{J})_n$ is the ideal in $\otimes^m(\mathbb{A}\cdot\mathrm{Conf})$ given by

$$(\otimes^m\mathbb{A}\mathcal{J})_n \ := \sum_{n_1+\cdots+n_m\geq n} \mathbb{A}\mathcal{J}_{n_1}\otimes\cdots\otimes\mathbb{A}\mathcal{J}_{n_m}, \tag{3.5.2}$$

where $\widehat{\otimes}^0\mathbb{A}\,[\![\mathrm{Conf}]\!] = \mathbb{A}$ and $\widehat{\otimes}^1\mathbb{A}\,[\![\mathrm{Conf}]\!] = \mathbb{A}\,[\![\mathrm{Conf}]\!]$.

We list up some basic properties of $\widehat{\otimes}^m\mathbb{A}\,[\![\mathrm{Conf}]\!]$ (proofs are left to the reader). i) Since $\cap_{n=0}^{\infty}(\mathbb{A}\mathcal{J}^{\otimes m})_n = \{0\}$, we have the natural embedding map

$$\otimes^m(\mathbb{A}\cdot\mathrm{Conf}) \ \subset \ \widehat{\otimes}^m(\mathbb{A}\,[\![\mathrm{Conf}]\!]) \tag{3.5.3}$$

whose image (identified with itself) is a dense subalgebra with respect to the (3.5.2)-adic topology. ii) There is a natural algebra homomorphism

$$\otimes^m(\mathbb{A}\,[\![\mathrm{Conf}]\!]) \ \longrightarrow \ \widehat{\otimes}^m(\mathbb{A}\,[\![\mathrm{Conf}]\!]) \tag{3.5.4}$$

with a suitable universal property. The image of an element $f_1\otimes\cdots\otimes f_m$ is denoted by $f_1\widehat{\otimes}\cdots\widehat{\otimes}f_m$. We denote it also by $f_1\otimes\cdots\otimes f_m$ if $f_i\in\mathbb{A}\cdot\mathrm{Conf}$ $(i=1,\cdots,m)$ because of i). iii) If $\Psi_i : \otimes^{m_i}(\mathbb{A}\cdot\mathrm{Conf}) \to \otimes^{n_i}(\mathbb{A}\cdot\mathrm{Conf})$ $(i=1,\cdots,l)$ are continuous homomorphisms. Then, one has the completed homomorphism

$$\widehat{\otimes}_{i=1}^l\Psi_i \ : \ \widehat{\otimes}^{\sum_{i=1}^l m_i}(\mathbb{A}\,[\![\mathrm{Conf}]\!]) \ \longrightarrow \ \widehat{\otimes}^{\sum_{i=1}^l n_i}(\mathbb{A}\,[\![\mathrm{Conf}]\!]) \tag{3.5.5}$$

with some natural characterizing properties. In particular, the completed product map: $\mathbb{A}\,[\![\mathrm{Conf}]\!]\,\widehat{\otimes}\mathbb{A}\,[\![\mathrm{Conf}]\!] \to \mathbb{A}\,[\![\mathrm{Conf}]\!]$ is sometimes denoted by M.

3.6 Exponential and logarithmic maps.

Let $\varphi(t) = \sum_{n=0}^{\infty} \varphi_n t^n \in \mathbb{A}[\![t]\!]$ be a formal power series in the indeterminate t. Then the substitution of t by an element f of $\mathbb{A}[\![\mathrm{Conf}]\!]_+$: $\varphi(f) := \sum_{n=0}^{\infty} \varphi_n f^n \in \mathbb{A}[\![\mathrm{Conf}]\!]$ defines a map $\varphi : \mathbb{A}[\![\mathrm{Conf}]\!]_+ \to \mathbb{A}[\![\mathrm{Conf}]\!]$ (c.f. (3.2.5)). The defined map is equivariant with respect to any continuous endomorphism of the configuration algebra. The map φ can be restricted to any closed subalgebra of the configuration algebra. If f is of finite type, then $\varphi(f)$ is also of finite type.

Assume that $\mathbb{A}$ contains $\mathbb{Q}$. Then we can define the exponential, logarithmic and power (with an exponent $c \in \mathbb{A}$) maps as follows:

$$\exp(\mathcal{M}) \ := \sum_{n=0}^{\infty} \frac{1}{n!} \ \mathcal{M}^n \qquad \text{for } \mathcal{M} \in \mathbb{A}[\![\mathrm{Conf}]\!]_+ , \tag{3.6.1}$$

$$\log(1+\mathcal{A}) \ := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \ \mathcal{A}^n \qquad \text{for } \mathcal{A} \in \mathbb{A}[\![\mathrm{Conf}]\!]_+ , \tag{3.6.2}$$

$$(1+\mathcal{A})^c \ := \sum_{n=0}^{\infty} \frac{c(c-1)\cdots(c-n+1)}{n!} \ \mathcal{A}^n \quad \text{for } \mathcal{A} \in \mathbb{A}[\![\mathrm{Conf}]\!]_+ . \tag{3.6.3}$$

They satisfy the following standard functional relations:

$$\begin{aligned}
\exp(\mathcal{M}+\mathcal{N}) &= \exp(\mathcal{M}) \cdot \exp(\mathcal{N}), \\
\log((1+\mathcal{A})(1+\mathcal{B})) &= \log(1+\mathcal{A}) + \log(1+\mathcal{B}), \\
(1+\mathcal{A})^{c_1} \cdot (1+\mathcal{A})^{c_2} &= (1+\mathcal{A})_1^{c+c_2}, \\
\log((1+\mathcal{A})^c) &= c \cdot \log(1+\mathcal{A}).
\end{aligned}$$

Formulae. Suppose $\mathcal{A}$ and $\mathcal{M} \in \mathbb{A}[\![\mathrm{Conf}]\!]_+$ are related by

$$1+\mathcal{A} = \exp(\mathcal{M}) \quad \Leftrightarrow \quad \mathcal{M} = \log(1+\mathcal{A}). \tag{3.6.4}$$

Then the coefficients of $\mathcal{A} = \sum_{S \in \mathrm{Conf}_+} S \cdot A_S$ and $\mathcal{M} = \sum_{S \in \mathrm{Conf}_+} S \cdot M_S$ are related by

$$A_S \ = \sum_{m=0}^{\infty} \sum_{\substack{S_1,\cdots,S_m \in \mathrm{Conf}_+ \\ S = S_1^{k_1} \cdot \ldots \cdot S_m^{k_m}}} \frac{1}{k_1! \ldots k_m!} M_{S_1}^{k_1} \cdots M_{S_m}^{k_m}, \tag{3.6.5}$$

and

$$M_S \ = \sum_{m=0}^{\infty} \sum_{\substack{S_1,\cdots,S_m \in \mathrm{Conf}_+ \\ S = S_1^{k_1} \cdot \ldots \cdot S_m^{k_m}}} \frac{(k_1 + \cdots + k_m - 1)!(-1)^{k_1+\cdots+k_m^{-1}}}{k_1! \ldots k_m!} A_{S_1}^{k_1} \cdots A_{S_m}^{k_m}. \tag{3.6.6}$$

Here the summation index runs over the set of all decompositions of S:

$$S \quad = \quad S_1^{k_1} \cdot \ \dots \ \cdot S_m^{k_m} \tag{3.6.7}$$

for pairwise distinct $S_i \in \mathrm{Conf}_+$ $(i = 1, \dots, m)$ (which may not necessarily be connected) and for positive integers $k_i \in \mathbb{Z}_{>0}$. Two decompositions $S_1^{k_1} \cdot \dots \cdot S_m^{k_m}$ and $T_1^{l_1} \cdot \dots \cdot T_n^{l_n}$ are regarded as the same if $m = n$ and there is a permutation $\sigma \in G_n$ *such that* $k_i = l_{\sigma(i)}$ and $S_i = T_{\sigma(i)}$ for $i = l, \dots, m$. The RHS's of (3.6.5) and (3.6.6) are finite sums, since the S_i's and k_i's are bounded by S.

Proof. We omit the proof since it is a straightforward calculation of formal power series in the infinite generator system Conf_0. □

Example. Let $T \in \mathrm{Conf}_0$ and let us denote $S_k := T^k \in \mathrm{Conf}$ for $k \in \mathbb{Z}_{\geq 0}$. If $S := S_n$ for some $n > 0$, then the decomposition (3.6.7) is $S = S_1^{k_1} \cdot \dots \cdot S_m^{k_m}$ for $m \geq 1$ and $k_i \geq 0$ such that $n = \sum_{i=1}^m i \cdot k_i$. Thus the summation index for S runs over the set $\{(k_i)_i \in \left(\mathbb{Z}_{\geq 0}\right)^{\mathbb{Z}_{\geq 1}} : n = \sum_{i \in \mathbb{Z}_{\geq 1}} i \cdot k_i\}$.

In particular, for $n = 1$, we have the following important fact, which we shall use repeatedly: **Fact.** let $\mathcal{A}$ and $\mathcal{M} \in \mathbb{A}\,[\![\mathrm{Conf}]\!]_+$ be related by (3.6.4). For any $T \in \mathrm{Conf}_0$, one has the equality: $A_T = M_T$.

§4. The Hopf algebra structure

We construct a topological commutative Hopf algebra structure on the configuration algebra $\mathbb{A}\,[\![\mathrm{Conf}]\!]$. More precisely, we construct in 4.1 a sequence of coproducts Φ_n $(n \in \mathbb{Z}_{\geq 0})$ by the use of the covering coefficients and, in 4.4, the antipode ι, which together satisfy the axioms of a topological Hopf algebra.

4.1 Coproduct Φ_m for $m \in \mathbb{Z}_{\geq 0}$.

For a non-negative integer $m \in \mathbb{Z}_{\geq 0}$ and $U \in \mathrm{Conf}$, define an element

$$\Phi_m(U) \ : = \sum_{S_1 \in \mathrm{Conf}} \cdots \sum_{S_m \in \mathrm{Conf}} \binom{S_1, \dots, S_m}{U} S_1 \otimes \dots \otimes S_m \tag{4.1.1}$$

in the tensor product $\otimes^m(\mathbb{Z} \cdot \mathrm{Conf})$ of m-copies of the polynomial type configuration algebra. Due to 2.5 v), one has,

$$\Phi_m([\phi]) \quad = \quad [\phi] \ \ (= 1). \tag{4.1.2}$$

The map Φ_m is *multiplicative*. That is: for $U, V \in \mathrm{Conf}$, one has

$$\Phi_m(U \cdot V) \quad = \quad \Phi_m(U) \cdot \Phi_m(V). \tag{4.1.3}$$

Proof. The decomposition rule (2.7.1) implies the formula. □

Thus, the linear extension of Φ_m induces an algebra homomorphism from $\mathbb{Z}\cdot\mathrm{Conf}$ to its m-tensor product $\otimes^m(\mathbb{Z}\cdot\mathrm{Conf})$, which we denote by the same Φ_m and call the mth *coproduct*. The coproduct Φ_m can be further extended to a coproduct on the completed configuration algebra.

Assertion. *1. The mth coproduct Φ_m ($m\in\mathbb{Z}_{\geq 0}$) on the polynomial type configuration algebra is continuous with respect to the adic topology. The induced homomorphism is denoted again by Φ_m and called the mth coproduct:*

$$\Phi_m \;:\; \mathbb{A}\,[\![\mathrm{Conf}]\!] \;\longrightarrow\; \widehat{\otimes}^m\mathbb{A}\,[\![\mathrm{Conf}]\!] := \mathbb{A}\,[\![\mathrm{Conf}]\!]\,\widehat{\otimes}\cdots\widehat{\otimes}\mathbb{A}\,[\![\mathrm{Conf}]\!] \tag{4.1.4}$$

2. The completed homomorphism Φ_m has the mulitiplicativity

$$\Phi_m(f\cdot g) \;=\; \Phi_m(f)\cdot\Phi_m(g) \tag{4.1.5}$$

for any $f,g\in\mathbb{A}\,[\![\mathrm{Conf}]\!]$,

3. Any saturated subalgebra R of $\mathbb{A}\,[\![\mathrm{Conf}]\!]$ is preserved by Φ_m:

$$\Phi_m(R) \;\subset\; \widehat{\otimes}^m R. \tag{4.1.6}$$

Proof. 1. Recall the fundamental system $(\otimes^m\mathbb{A}\mathcal{J})_n$ (3.5.2) of neighborhoods of the m-tensor algebra $\otimes^m(\mathbb{Z}\cdot\mathrm{Conf})$. Let us show the inclusion

$$\Phi_m(\mathbb{A}\mathcal{J}_n) \;\subset\; (\otimes^m\mathbb{A}\mathcal{J})_n \tag{4.1.7}$$

for any $m,n\in\mathbb{Z}_{\geq 0}$. The ideal $\mathcal{J}_n$ is generated by $U\in\mathrm{Conf}$ with $\deg(U):=\#U\geq n$, and then $\Phi_m(U)$ is a sum of monomials $S_1\otimes\cdots\otimes S_m$ for $S_i\in\mathrm{Conf}$ such that $\binom{S_1,\dots,S_m}{U}\neq 0$. Then $\sharp S_1+\cdots+\sharp S_m\geq\sharp(U)\geq n$ because of (2.5) i), implying $\Phi_m(U)\in(\otimes^m\mathcal{J})_n$.

2. The multiplicativity of the monomials (4.1.3) implies the multiplicativity of the configuration algebra of polynomial type. Then it extends to multiplicativity on infinite series (3.2.4) because of the continuity of the product with respect to the adic topology.

3. Let f be an element of R and $f=\sum_S Sf_S$ be its expansion. Then $\Phi_m(f)$ is a series of the form $\sum_S S_1\otimes\cdots\otimes S_m\binom{S_1,\dots,S_m}{S}f_S$. Thus, $\binom{S_1,\dots,S_m}{S}f_S\neq 0$ implies each factor S_i satisfies $S_i\leq S$ and $S\in\mathrm{Supp}(f)\subset\mathrm{Supp}(R)$. Then, by the definition of saturatedness, $S_i\in\mathrm{Supp}(R)$ and $\Phi_m(f)\in\widehat{\otimes}^m R$. □

Co-commutativity of the coproduct Φ_m.
The symmetric group $\mathfrak{S}_m$ acts naturally on the m-tensors (3.5.1) by permuting the tensor factors. The image of Φ_m lies in the subalgebra consisting of $\mathfrak{S}_m$-invariant elements, because of 2.5 ii): $\Phi_m(\mathbb{A}\,[\![\mathrm{Conf}]\!])\subset(\widehat{\otimes}^m\mathbb{A}\,[\![\mathrm{Conf}]\!])^{\mathfrak{S}_m}$. We shall call this property *the co-commutativity* of the coproduct Φ_m.

4.2 Co-associativity

Assertion. *For $m, n \in \mathbb{Z}_{\geq 0}$, one has the formula:*

$$(\underbrace{1\widehat{\otimes}\cdots\widehat{\otimes}1}_{n}\ \widehat{\otimes}\Phi_m)\circ\Phi_{n+1} \quad = \quad \Phi_{m+n} \tag{4.2.1}$$

Proof. This follows immediately from the composition rule (2.6.1). $\square$

Using the co-commutativity of Φ_2, Φ_3 can be expressed in two different ways:

$$(\Phi_2\widehat{\otimes}1)\circ\Phi_2 \quad = \quad (1\widehat{\otimes}\Phi_2)\circ\Phi_2.$$

This equality is the *co-associativity* of the coproduct Φ_2. More generally, Φ_m is expressed by a composition of $m-1$ copies of Φ_2's in any order.

4.3 The augmentation map Φ_0.

The augmentation map for the algebra is defined by Φ_0 (recall (2.5.2)):

$$\mathrm{aug} := \Phi_0 :\ \mathbb{A}\,[\![\mathrm{Conf}]\!] \ \longrightarrow\ \mathbb{A}, \quad S \in \mathrm{Conf}_+ \mapsto 0, \quad [\phi] \mapsto 1 \tag{4.3.1}$$

Assertion. *The map* aug *is the co-unit with respect to the coproduct* Φ_2*.*

$$(\mathrm{aug}\,\widehat{\otimes}\,\mathrm{id})\circ\Phi_2 \quad = \quad \mathrm{id}_{\mathbb{Z}[\![\mathrm{Conf}]\!]}. \tag{4.3.2}$$

Proof. This is the case $m=0$ and $n=1$ of the formula (4.2.1). Alternatively, for any $S \in \mathrm{Conf}_+$, using (2.5) iii) and iv), one calculates: $(\mathrm{aug}\,\widehat{\otimes}\,\mathrm{id})\circ\Phi_2(S) = \sum_{T,U\in\mathrm{Conf}}\binom{T,U}{S}T\cdot\mathrm{aug}(U) = \sum_{T\in\mathrm{Conf}}\binom{T,[\phi]}{S} = S.$ $\square$

4.4 The antipodal map ι

The coproduct Φ_2 and the co-unit Φ_0 exist both on the polynomial type and the completed configuration algebras. The co-inverse ι, which we construct in the present section, exists only on the localized configuration algebra.

Assertion. *There exists an algebra automorphism*

$$\iota \ :\ \mathbb{A}\,[\![\mathrm{Conf}]\!] \ \longrightarrow\ \mathbb{A}\,[\![\mathrm{Conf}]\!], \tag{4.4.1}$$

satisfying following properties i)-iv)*. It is characterized uniquely by* ii) *and* iii)*.*

i) *ι is an involutive automorphism. That is:* $\iota^2 = \mathrm{id}_{\mathbb{A}[\![\mathrm{Conf}]\!]}$.

ii) ι *is the co-inverse map with respect to the coproduct* Φ_2*, that is:*

$$M \circ (\iota \widehat{\otimes} \mathrm{id}) \circ \Phi_2 \quad = \quad \mathrm{aug}\,. \tag{4.4.2}$$

where M *is the product defined on the completed tensor product (recall* 4.5*).*
iii) ι *is continuous with respect to the adic topology. More precisely,*

$$\iota(\overline{\mathcal{J}_n}) \quad \subset \quad \overline{\mathcal{J}_n} \tag{4.4.3}$$

for $n \in \mathbb{Z}_{\geq 0}$*, where* $\overline{\mathcal{J}_n}$ *is the closure of the ideal* $\mathcal{J}_n$ (3.2.1)*.*
iv) ι *leaves any saturated subalgebras of* $\mathbb{A}\,[\![\mathrm{Conf}]\!]$ *invariant.*

Proof. First, let us construct the map ι with the properties ii), iii) and iv). The property i) and the uniqueness of ι are shown afterward.

Let us fix a numbering $i \in \mathbb{Z}_{\geq 1} \simeq S_i \in \mathrm{Conf}_0$ such that if $i \leq j$ then $\#S_i \leq \#S_j$ for all $i, j \in \mathbb{Z}_{\geq 1}$. Note that this condition implies that if $S_i \leq S_j$ and $S_i \neq S_j$ then $\#S_i < \#S_j$ and therefore $i < j$. This means that the set $\{[\phi], S_1, \cdots, S_i\}$ is saturated in the sense of 3.4. Consider the increasing sequence $R_0 := \mathbb{A}$, $R_i := \mathbb{A}\,[\![S_1, \cdots, S_i]\!]$ $(i = 1, 2, \cdots)$ of saturated subalgebras of $\mathbb{A}\,[\![\mathrm{Conf}]\!]$. We want to construct a sequence of continuous endomorphisms $\iota_0 = \mathrm{id}_{\mathbb{A}}$, $\iota_i : R_i \to R_i$ $(i = 1, 2, \cdots)$ satisfying the following relations:

a) $\iota_i^2 \ = \ \mathrm{id}_{R_i}$.
b) $\iota_i|_{R_{i-1}} \ = \ \iota_{i-1}$.
c) $(\iota_i \cdot \mathrm{id}) \circ \Phi_2|_{R_i} \ = \ \mathrm{aug}\,|_{R_i}$.
d) $\iota_i(\overline{\mathcal{J}_n} \cap R_i) \ \subset \ \overline{\mathcal{J}_n} \cap R_i$.
e) $\iota_i(S_i) \in \mathbb{Z}\,[\![\langle S_i \rangle]\!]$ (recall (3.4.1) and (3.4.3) for the notation $\mathbb{Z}\,[\![\langle S \rangle]\!]$).

For $i \in \mathbb{Z}_{\geq 1}$, suppose that $\iota_{i-1} : R_{i-1} \to R_{i-1}$ satisfying a)-e) is given. In order to extend ι_{i-1} to ι_i, we first look for $\iota_i(S_i)$ as a solution of the equation:

$$M \circ (\iota_i \widehat{\otimes} 1) \circ \Phi_2(S_i) \quad = \quad 0.$$

For notational simplicity, put $S := S_i$, $\iota := \iota_{i-1}$ and $\iota' := \iota_i$. Let us write down the equation explicitly by using (4.1.1).

$$M \circ (\iota' \widehat{\otimes} \mathrm{id}) \circ \Phi_m(S) \ := \ \textstyle\sum_{U,V \in \mathrm{Conf}} \binom{U,V}{S} \iota'(U) \cdot V \ = \ 0 \ ,$$

where, in fact, the summation index (U, V) runs only over the finite index set $\langle S \rangle^2$ due to 2.5 i). We decompose the index set into three pieces: $\{S\} \times \langle S \rangle$, $(\langle S \rangle \setminus \{S\}) \times (\langle S \rangle \setminus \{S\})$ and $(\langle S \rangle \setminus \{S\}) \times \{S\}$. Accordingly, we decompose the equation into three parts:

$$\iota'(S) \cdot \big(1 + \mathcal{A}(S)\big) + \mathcal{B}(S) + \big(1 + \iota\big(\mathcal{A}(S) - S\big)\big) \cdot S \ = \ 0, \tag{4.4.4}$$

where

(4.4.5)
$$\mathcal{A}(S) \ := \ \textstyle\sum_{V\in\langle S\rangle\setminus\{[\phi]\}} \binom{S,V}{S} V,$$

and

(4.4.5)*
$$\mathcal{B}(S) \ := \ \textstyle\sum_{U,V\in\langle S\rangle\setminus\{S\}} \binom{U,V}{S} \iota(U)\cdot V.$$

We have the following facts concerning the equation (4.4.4).

i) By definition (4.4.5), one has the inclusion $\mathrm{Supp}(\mathcal{A}(S)-S)\subset\langle S\rangle\setminus\{S\}$ and each term of $\mathcal{A}(S)-S$ is a monomial of S_j's such that $j<i$ and $S_j\in\langle S\rangle$. Therefore, by the induction hypothesis e), $\mathrm{Supp}(\iota(\mathcal{A}(S)-S))\subset\langle S\rangle$.

ii) By the induction hypothesis d), $\iota(U)$ belongs to $\overline{\mathcal{J}_{\sharp(U)}}\cap\mathbb{Z}\,[\![\langle U\rangle]\!]$ for $^\forall U\neq S$ with $U\leq S$, and hence, (4.5.5)* implies $\mathcal{B}(S)\in\overline{\mathcal{J}_{\sharp(S)}}\cap\mathbb{Z}\,[\![\langle S\rangle]\!]$.

On the other hand, by definition (4.4.5), $\mathcal{A}(S)$ belongs to the augmentation ideal. Hence, in view of the inclusion (3.2.5), the inverse $(1+\mathcal{A}(S))^{-1} = \sum_{m\geq 0}(-\mathcal{A}(S))^m$ converges in $\mathbb{Z}\,[\![\langle S\rangle]\!]$. Therefore the equation (4.4.4) for $\iota'(S)$ has a unique solution in $\mathbb{Z}\,[\![\langle S\rangle]\!]$:

(4.4.6)
$$\iota'(S) \ := \ \frac{-1}{1+\mathcal{A}(S)}\big(\mathcal{B}(S)+\big(1+\iota\big(\mathcal{A}(S)-S\big)\big)\cdot S\big).$$

As a consequence of the above Facts i) and ii), we have

$*$) *The right hand side of* (4.4.6) *belongs to* $\overline{\mathcal{J}_{\sharp(S)}}\cap\mathbb{Z}\,[\![\langle S\rangle]\!]$.

By using (4.4.6), one can define a homomorphism ι' from $R_{i-1}[S]$ to R_i by extending ι on R_{i-1}. Due to $*$), one has $\iota'(\overline{\mathcal{J}_n}\cap R_{i-1}[S]\subset\overline{\mathcal{J}_n}\ ^\forall n\in\mathbb{Z}_n$. Hence it is continuous in the adic topology and is extended to an endomorphism of $R_i=R_{i-1}\,[\![S]\!]$. We denote the extended homomorphism again by ι'.

Let us show that ι' satisfies a)-e), where b) follows from the construction and e) follows from $*$).

c) and d) : if we restrict the domain to $R_{i-1}[S]$, then the result follows. Then by the continuity of ι', the result extends to the closure $R_i=R_{i-1}\,[\![S]\!]$.

a) : It is enough to show: $(\iota')^2(S)=S$. Let us apply ι' to the equality (4.4.4). Using the induction hypothesis a) and b), one gets

$**$) $$(\iota')^2(S)\big(1+\iota(\mathcal{A}(S)-S)+\iota'(S)\big)+\iota\mathcal{B}(S)+\big(1+\mathcal{A}(S)-S\big)\iota'(S)=0$$

Here, we have $\iota\mathcal{B}(S)=\mathcal{B}(S)$, by applying the symmetry (2.5) ii) and the induction hypothesis a) to the expression (4.4.5)*.

Taking the difference: $**$)- (4.4.4), we obtain an equality:

$$\big((\iota')^2(S)-S\big)\big(1+\iota(\mathcal{A}(S)-S)+\iota'(S)\big)=0.$$

Since $\mathcal{A}(S) \in \mathcal{J}_1$ and $\iota(\mathcal{A}(S) - S) + \iota'(S) \in \mathcal{J}_1$, $1 + \iota(\mathcal{A}(S) - S) + \iota'(S)$ is invertible in the algebra R_i. This implies $(\iota')^2(S) - S = 0$.

Thus the proof of a)-e) for ι_i is completed, and hence, the sequence ι_i of endomorphisms on R_i are constructed. We define the endomorphism ι of the subalgebra $R := \cup_{i=1}^{\infty} R_i$ by $\iota | R_i = \iota_i$. Here, we note that R consists of exactly finite type elements, and is a dense subalgebra of the completed configuration algebra, since it contains the polynomial type configuration algebra. Then d) implies homeomorphicity of ι on R, and therefore ι extends to the completed configuration algebra. The extended homomorphism, denoted by ι again, satisfies i), ii) and iii) due to the continuity of ι.

iv) Let R be any saturated subalgebra of the configuration algebra. We first consider any element $S \in \mathrm{Conf}$. Then by applying e) to each connected component of S, one has $\iota(S) \in \mathbb{Z}[\![\langle S \rangle]\!]$. Next, for any $f = \sum_S S f_S \in R$, by applying the above considerations, one has $\mathrm{Supp}(\iota(f)) \subset \cup_{f_S \neq 0} \mathrm{Supp}(\iota(S)) \subset \cup_{f_S \neq 0}$ semigroup generated by $\langle S \rangle \subset \mathrm{Supp}(R)$. That is, $\iota(f) \in R$.

For the uniqueness of ι: Let ι be the map constructed above, and let ι' be any other endomorphism of the configuration algebra satisfying ii) and iii) of the Lemma. Let us show that $\iota'(S_i) = \iota(S_i)$ by induction on $i \in \mathbb{Z}_{\geq 1}$. Let $i \in \mathbb{Z}_{\geq 1}$, and assume $\iota(S_j) = \iota'(j)$ for $1 \leq j < i$ (there is no assumption if $i = 1$). By ii), $\iota'(S_i)$ should satisfy the same equation as (4.4.4). The uniqueness of the solution (4.4.6) implies $\iota'(S_i) = \iota(S_i)$. This implies the coincidence of ι and ι' on $\mathbb{Z} \cdot \mathrm{Conf}$. Then, by the continuity iii), we have the coincidence of ι and ι' on the completed configuration algebra. $\square$

Equation (4.4.4) for $n = 1$ implies that ι preserves the augmentation ideal of $\mathbb{A}[\![\mathrm{Conf}]\!]$. Hence, we have

$$\mathrm{aug} \circ \iota \;=\; \mathrm{aug}\,. \tag{4.4.7}$$

Let us state an important consequence of our construction.

Assertion. *Any saturated subalgebra of the configuration algebra is a topological Hopf algebra. In particular, for any group Γ with a finite generator system G and commutative ring $\mathbb{A}$ with a unit, $\mathbb{A}[\![\langle \Gamma, G \rangle]\!]$ is a Hopf algebra.*

Proof. We need only to remember that Φ_m ($m \geq 0$) and ι preserves any saturated subalgebra (4.1 Assertion 3. and 4.4 Assertion iv)). $\square$

4.5 Some remarks on ι.

Remark. 1. In section 5., the functions $\mathcal{A}(S)$ $(S \in \mathrm{Conf})$ will be reintroduced and investigated. Particularly we shall show the equality:

$$(4.5.1) \qquad \big(1+\iota(\mathcal{A}(S))\big)\cdot\big(1+\mathcal{A}(S)\big) \;=\; 1$$

for $S \in \mathrm{Conf}$ (5.4.1). This can be also directly shown by use of (2.7.1) and (4.2.1). This relation gives a more natural definition of ι.

2. The polynomial ring $\mathbb{A}\cdot\mathrm{Conf}$ for any $\mathbb{A}$ is not closed under the map ι. For example, let X (resp. Y) be a graph of one (resp. two) vertices. Then,

$$\iota(X) \;=\; -\frac{X}{1+X} \quad \text{and} \quad \iota(Y) \;=\; \frac{-Y+2X^2+XY}{(1+X)(1+2X+Y)}.$$

3. Because of above *Remark* 1., the localization: $\big(\mathbb{Z}\cdot\mathrm{Conf}\,\big)_{\mathfrak{M}} = \{f/g : f \in \mathbb{Z}\cdot\mathrm{Conf}, g \in \mathfrak{M}\}$ for the multiplicative set $\mathfrak{M} := \{1+\mathcal{A}(S) : S \in \mathrm{Conf}\}$ is the smallest necessary extension of the algebra $\mathbb{Z}\cdot\mathrm{Conf}$ to define ι. However, the space $\big(\mathbb{Z}\cdot\mathrm{Conf}\,\big)_{\mathfrak{M}}$ is still too small for our later applications (see 6.3 *Remark*).

4. There is another coalgebra structure studied in combinatorics ([R]).

§5. Growth functions for configurations

For any $S \in \mathrm{Conf}$, the sum of isomorphism classes of all subgraphs of a graph representing S is denoted by $1+\mathcal{A}(S)$. It is a group-like element in the Hopf algebra $\mathbb{A}\,[\![\mathrm{Conf}]\!]$ and shall play a fundamental role in the sequel. We shall call it a growth function (the name is confusing with the terminology [Mi]).

5.1 Growth functions

For S and $T \in \mathrm{Conf}$, we introduce a numerical invariant

$$(5.1.1) \qquad A(S,T) \;:=\; \sharp\mathbb{A}(S,\mathbb{T}),$$

by the following steps i)-iii).

i) Fix a graph $\mathbb{T}$, with $[\mathbb{T}] = T$.

ii) Put

$$(5.1.2) \qquad \mathbb{A}(S,\mathbb{T}) \;:=\; \sharp\{\ \mathbb{S}\ \mid\ \mathbb{S} \text{ is a subgraph of } \mathbb{T} \text{ such that } [\mathbb{S}] = S\}.$$

iii) Show that $\mathbb{A}(S,\mathbb{T}) \simeq \mathbb{A}(S,\mathbb{T}')$ if $[\mathbb{T}] = [\mathbb{T}']$. (The proof is omitted.)

We shall call $A(S,T)$ the growth coefficient of T at $S \in \mathrm{Conf}$.

$$(5.1.3) \qquad A([\phi],T) \;=\; 1 \qquad \text{for} \quad T \in \mathrm{Conf},$$

$$(5.1.4) \qquad A(S,T) \;\neq\; 0 \qquad \text{if and only if} \quad S \in \langle T\rangle.$$

Let us introduce the generating polynomial of the growth coefficients:

$$\mathcal{A}(T) \quad := \quad \textstyle\sum_{S\in\mathrm{Conf}_+} S\cdot A(S,T), \tag{5.1.5}$$

and call it the growth function of T. In fact, this is a finite sum and $\mathcal{A}(T)\in\mathbb{Z}\cdot\mathrm{Conf}$. The definition of $\mathcal{A}(T)$ can be reformulated as:

$$1+\mathcal{A}(T) \quad = \quad \textstyle\sum_{S\in 2^{\mathbb{T}}}[\mathbb{S}], \tag{5.1.6}$$

where $2^{\mathbb{T}}$ denote the set of all subgraphs of $\mathbb{T}$ (cf. 2.1 Definition 2.).

The following multiplicativity follows immediately from the expression (5.1.6). For T_1 and $T_2\in\mathrm{Conf}$

$$(1+\mathcal{A}(T_1\cdot T_2)) \quad = \quad (1+\mathcal{A}(T_1))(1+\mathcal{A}(T_2)). \tag{5.1.7}$$

Remark. 1. By comparing the definition (5.1.1) with (2.4.1), we see immediately $A(S,T)=\binom{T,S}{T}$ for S and $T\in\mathrm{Conf}$. Hence the two definitions (4.4.5) and (5.1.5) for $\mathcal{A}(T)$ coincide.

2. By definition (5.1.1), we have additivity:

$$A(S,T_1\cdot T_2) \quad = \quad A(S,T_1)+A(S,T_2) \tag{5.1.8}$$

for $S\in\mathrm{Conf}_0$ and $T_i\in\mathrm{Conf}$.

5.2 A numerical approximation of the growth coefficients

In our later study on the existence of limit elements in §10, the following estimates of the growth rates of growth coefficients play a crucial role.

Lemma. *For $S,T\in\mathrm{Conf}$, we have*

$$A(S,T) \quad \leq \quad \frac{1}{\sharp\,\mathrm{Aut}(S)}\cdot\sharp T^{n(S)}\cdot(q-1)^{\sharp S-n(S)}. \tag{5.2.1}$$

Here $n(S):=\sharp$ of connected components of S, q is the upper-bound of the number of edges at each vertex of T (recall 2.2), and $\mathrm{Aut}(S)$ means the isomorphism class of $\mathrm{Aut}(\mathbb{S})$ for a representative $\mathbb{S}$ of S and we put $\#\,\mathrm{Aut}(S):=\#\,\mathrm{Aut}(\mathbb{S})$.

Note. In the original version [S2], the factor $q-1$ in (5.2.1) q. The author is grateful to the readers who pointed out this improvement.

Proof. Let $\mathbb{S}$ and $\mathbb{T}$ be representatives by G-colored graphs of S and T respectively. We divide the proof into three steps.

i) Assume S is connected. Let us show:

$$A(S,T) \quad \leq \quad \frac{1}{\sharp\operatorname{Aut}(S)}\sharp T\cdot(q-1)^{\sharp S-1}. \tag{5.2.2}$$

Proof. Let $\mathbb{S}_1,\ldots,\mathbb{S}_a$ be an increasing sequence of connected subgraphs of $\mathbb{S}$ such that $\sharp\mathbb{S}_i = i$ $(i=1,\ldots,a=\sharp\mathbb{S})$. Put $\operatorname{Emb}(\mathbb{S}_i,\mathbb{T}) := \{\varphi : \mathbb{S}_i \to \mathbb{T} \mid$ embeddings as a G-colored graph$\}$. Then, for $i \geq 2$, the natural restriction map $\operatorname{Emb}(\mathbb{S}_i,\mathbb{T}) \to \operatorname{Emb}(\mathbb{S}_{i-1},\mathbb{T})$ has at most $q-1$ points in its fiber. Hence $\sharp\operatorname{Emb}(\mathbb{S}_i,\mathbb{T}) \leq (q-1)\cdot\sharp\operatorname{Emb}(\mathbb{S}_{i-1},\mathbb{T})$ $(i=2,\ldots,a)$. On the other hand, since

$$A(S,T) \;=\; \sharp\operatorname{Emb}(\mathbb{S},\mathbb{T})/\sharp\operatorname{Aut}(\mathbb{S}),$$

one has an approximation:

$$\begin{aligned} A(S,T) &= \sharp\operatorname{Emb}(\mathbb{S}_a,\mathbb{T})/\sharp\operatorname{Aut}(\mathbb{S}_a)\\ &\leq (q-1)^{a-1}\cdot\sharp\operatorname{Emb}(\mathbb{S}_1,\mathbb{T})/\sharp\operatorname{Aut}(\mathbb{S}_a) = (q-1)^{a-1}\cdot\sharp T/\sharp\operatorname{Aut}(S). \end{aligned}$$ □

ii) Assume that S decomposes as: $S = S_1^{k_1}\coprod..\coprod S_m^{k_m}$ for pairwise distinct $S_i \in \operatorname{Conf}_0$ $(i=1,\cdots,m)$ so that $\sum_{i=1}^m k_i = n(S)$. Let us show

$$A(S,T) \quad \leq \quad \frac{1}{k_1!\ldots k_m!}\prod_{i=1}^m A(S_i,T)^{k_i}, \tag{5.2.3}$$

Proof. For $1 \leq i \leq m$, the subgraph of $\mathbb{S} \in \mathbb{A}(S,\mathbb{T})$ corresponding to the factor $S_i^{k_i}$, denoted by $\mathbb{S}|_{S_i^{k_i}}$, defines an off-diagonal element of $(\prod^{k_i}\mathbb{A}(S_i,\mathbb{T}))/\mathfrak{S}_{k_i}$ where $\mathfrak{S}_{k_i}$ is the symmetric group of k_i elements acting freely on the set of off-diagonal elements. Then, the association: $\mathbb{S} \mapsto (\mathbb{S}|_{S_i^{k_i}})_{i=1}^m$ defines an embedding: $\mathbb{A}(S,\mathbb{T}) \to \prod_{i=1}^m\Big(\Big(\prod^{k_i}\mathbb{A}(S_i,\mathbb{T})\Big)/\mathfrak{S}_{k_i}\Big)$ into the off-diagonal part. □

iii) Let S be as in ii). Then, $\operatorname{Aut}(S) = \prod_{i=1}^m \operatorname{Aut}(S_i^{k_i})$ and each factor $\operatorname{Aut}(S_i^{k_i})$ is a semi-direct product of $\operatorname{Aut}(S_i)$ and $\mathfrak{S}_{k_i}$. Then (5.2.1) is a consequence of a combination of (5.2.2) and (5.2.3).

This completes the proof of the Lemma □

5.3 Product-expansion formula for growth coefficients

The coefficients of a growth function of T are not algebraically independent.

Lemma. *Let $S_1,\ldots,S_m$ $(m \geq 0)$ and $T \in \operatorname{Conf}$ be given. Then,*

$$\prod_{i=1}^m A(S_i,T) \quad = \sum_{S\in\operatorname{Conf}}\binom{S_1,\ldots,S_m}{S}A(S,T). \tag{5.3.1}$$

Proof. Let $\mathbb{T}$ be a graph representing T. For $m \in \mathbb{Z}_{\geq 0}$, consider a map

$$(\mathbb{S}_1, \ldots, \mathbb{S}_m) \in \prod_{i=1}^{m} \mathbb{A}(S_i, \mathbb{T}) \longmapsto \mathbb{S} := \bigcup_{i=1}^{m} \mathbb{S}_i \in 2^{\mathbb{T}},$$

whose fiber over $\mathbb{S}$ is $\binom{S_1,\ldots,S_m}{\mathbb{S}}$ so that one has the decomposition

$$\prod_{i=1}^{m} \mathbb{A}(S_i, \mathbb{T}) \simeq \bigcup_{\mathbb{S} \in 2^{\mathbb{T}}} \binom{S_1, \ldots, S_m}{\mathbb{S}}.$$

By counting the cardinality of the both sides, one obtains the formula. □

Remark. The formula (5.3.1) is trivial for $m = 0, 1$, and can be reduced to the case $m = 2$ for $m \geq 2$ by an induction on m as follows.

Multiply $A(S_{m+1}, T)$ to (5.3.1) and apply the formula for $m = 2$.

$$\begin{aligned} \textstyle\prod_{i=1}^{m+1} A(S_i.T) &= \textstyle\sum_{S \in \mathrm{Conf}} \binom{S_1, \ldots, S_m}{S} A(S,T) A(S_{m+1}, T) \\ &= \textstyle\sum_{S \in \mathrm{Conf}} \binom{S_1, \ldots, S_m}{S} \sum_{U \in \mathrm{Conf}} \binom{S, S_{m+1}}{U} A(U,T) \end{aligned}$$

Using the composition rule (2.6.1), this is equal to

$$= \textstyle\sum_{U \in \mathrm{Conf}} \binom{S_1, \ldots, S_{m+1}}{U} A(U,T).$$

5.4 Group-like property of the growth function

An element $g \in \mathbb{A}[\![\mathrm{Conf}]\!]$ is called group-like if it satisfies

$$\Phi_m(g) = \underbrace{g \,\widehat{\otimes} \cdots \widehat{\otimes}\, g}_{m} \tag{5.4.1}$$

for ${}^{\forall} m \in \mathbb{Z}_{\geq 0}$. This in particular implies the conditions $\Phi_0(g) = 1$ and $\iota(g) = g^{-1}$ (c.f. (4.3.1) and (4.4.2)). For any group-like elements g and h, the power product $g^a h^b$ for $a, b \in \mathbb{A}$ (c.f. (3.6.3)) is also group-like. We put

$$\mathfrak{G}_{\mathbb{A}} \ := \ \{\text{the set of all group-like elements in } \mathbb{A}[\![\mathrm{Conf}]\!]\} \tag{5.4.2}$$

$$\mathfrak{G}_{\mathbb{A},\mathrm{finite}} \ := \ \{g \in \mathfrak{G}_{\mathbb{A}} \mid g \text{ is of finite type.}\} \tag{5.4.3}$$

Lemma. *The generating polynomial* $1 + \mathcal{A}(T)$ *for any* $T \in \mathrm{Conf}$ *is group-like. That is: for any* $m \in \mathbb{Z}_{\geq 0}$ *and* $T \in \mathrm{Conf}$*, we have*

$$(1 + \mathcal{A}(T)) \otimes \cdots \otimes (1 + \mathcal{A}(T) \ = \ \Phi_m(1 + \mathcal{A}(T)). \tag{5.4.4}$$

Proof. By the definition of $\mathcal{A}(T)$ (5.1.3), the tensor product of m-copies

$$*) \qquad (1+\mathcal{A}(T))\otimes\cdots\otimes(1+\mathcal{A}(T))$$

can be expanded into a sum of m variables $S_1,\dots,S_m$:

$$**) \qquad \sum_{S_1\in\mathrm{Conf}}\cdots\sum_{S_m\in\mathrm{Conf}} S_1\otimes\cdots\otimes S_m\Big(\prod_{i=1}^m A(S_i,T)\Big).$$

By use of the product-expansion formula (5.3.1), this is equal to

$$\sum_{S_1\in\mathrm{Conf}}\cdots\sum_{S_m\in\mathrm{Conf}} S_1\otimes\cdots\otimes S_m\Big(\sum_{S\in\mathrm{Conf}}\binom{S_1,\cdots,S_m}{S}A(S,T)\Big)$$

Recalling the definition of the map Φ_m (4.1.4), this is equal to

$$***) \quad \sum_{S\in\mathrm{Conf}}\Phi_m(S)\cdot A(S,T) \;=\; \Phi_m\Big(\sum_{S\in\mathrm{Conf}} S\cdot A(S,T)\Big) = \;\Phi_m(1+\mathcal{A}(T)).$$

□

5.5 A characterization of the antipode.

Equation (5.4.4) provides formulae,

$$(5.5.1) \qquad \big(1+\iota(\mathcal{A}(T))\big)\big(1+\mathcal{A}(T)\big) \;=\; 1 \qquad \text{for } T\in\mathrm{Conf},$$

$$(5.5.2) \qquad \Phi_m\circ\iota \;=\; (\iota\widehat{\otimes}\cdots\widehat{\otimes}\iota)\circ\Phi_m \qquad \text{for } m\in\mathbb{Z}_{\ge 0}.$$

Proof of (5.5.1). Apply (5.4.1) to $(\iota\cdot 1)\circ\Phi_2(T)=\mathrm{aug}(T)$ (4.4.2). □

Proof of (5.5.2). It is enough to show the case $m=2$ due to (4.2.1). Apply Φ_2 to (5.5.1). Recalling (5.4.1), one obtains a relation.

$$\big(\Phi_2(1+\iota(\mathcal{A}(T)))\big)\big((1+\mathcal{A}(T))\otimes(1+\mathcal{A}(T))\big) \;=\; 1,$$

Multiply $\big(1+\iota(\mathcal{A}(T))\big)\big(1+\iota(\mathcal{A}(T))\big)$ and apply again (5.5.1) so that one obtains

$$\begin{aligned}\Phi_2(1+\iota(\mathcal{A}(T))) \;&=\; (1+\iota(\mathcal{A}(T))\otimes(1+\iota(\mathcal{A}(T)))\\ &=\; (\iota\otimes\iota)\big((1+\mathcal{A}(T))\otimes(1+\mathcal{A}(T))\big)\\ &=\; (\iota\otimes\iota)\Phi_2(1+\mathcal{A}(T)).\end{aligned}$$

Thus (5.5.2) is true for $\mathcal{A}(T)$ ($T\in\mathrm{Conf}$). Since $\mathcal{A}(T)$ ($T\in\mathrm{Conf}$) span $\mathbb{A}\cdot\mathrm{Conf}$, which is dense in the whole algebra, (5.5.2) holds on $\mathbb{A}[\![\mathrm{Conf}]\!]$. □

§6. The logarithmic growth function

The growth coefficients $\mathcal{A}(S,T)$ in $S\in\langle T\rangle$ was approximated in (5.2.1). However in the sequel, we need to approximate lower terms too. This is achieved by introducing a logarithmic growth coefficients $M(S,T)\in\mathbb{Q}$ in $S\in\langle T\rangle$, and showing linear relations (6.2.2) on it.

6.1 The logarithmic growth coefficient

For $T \in \mathrm{Conf}$, define the logarithm of the growth function:

$$(6.1.1) \qquad \mathcal{M}(T) \quad := \quad \log(1+\mathcal{A}(T)),$$

in $\mathbb{Q}[\![\langle T\rangle]\!]$ (cf (5.1.5) and (3.6.2)). Expand $\mathcal{M}(T)$ in a series

$$(6.1.2) \qquad \mathcal{M}(T) \quad = \quad \sum_{S\in \mathrm{Conf}} S\cdot M(S,T).$$

The coefficient $M(S,T)$ is the *logarithmic growth coefficient* at $S\in\langle T\rangle$.

By definition, $\mathcal{M}(T)$ does not have a constant term, i.e.

$$(6.1.3) \qquad M([\phi],T) \quad := \quad 0 \qquad \text{for} \quad T\in \mathrm{Conf}\,.$$

For later applications, we write the explicit relations among growth-functions and logarithmic growth-functions (cf. (3.6.5) and (3.6.6)).

$$(6.1.4) \qquad A(S,T) \;=\; \sum_{S=S_1^{k_1}\coprod\cdots\coprod S_m^{k_m}} \frac{1}{k_1!\dots k_m!} M(S_1,T)^{k_1}\cdots A(S_m,T)^{k_m}$$

$$(6.1.5) \qquad M(S,T) \;=\; \sum_{S=S_1^{k_1}\coprod\cdots\coprod S_m^{k_m}} \frac{(k_1+\cdots+k_m-1)!(-1)^{k_1+\cdot+k_m-1}}{k_1!\dots k_m!}\times$$

$$\times\, A(S_1,T)^{k_1}\cdots A(S_m,T)^{k_m}.$$

Remark. 1. From the formula, we see that for a connected $S\in\mathrm{Conf}_0$,

$$(6.1.6) \qquad A(S,T) \quad = \quad M(S,T).$$

That is; *the logarithmic growth coefficients coincide with the growth coefficients at connected configurations.* This elementary fact shall be used repeatedly.

2. The multiplicativity of $\mathcal{A}(T)$ (5.1.7) implies the additivity

$$(6.1.7) \qquad \mathcal{M}(T_1\cdot T_2) \quad = \quad \mathcal{M}(T_1)+\mathcal{M}(T_2)$$

for $T_i\in\mathrm{Conf}$ and hence the additivity:

$$(6.1.7)^* \qquad M(S,T_1\cdot T_2) \quad = \quad M(S,T_1)+M(S,T_2) \quad \text{for } S\in\mathrm{Conf}\,.$$

3. The invertibility (5.5.1) implies

$$(6.1.8) \qquad \iota(\mathcal{M}(T)) \quad = \quad -\,\mathcal{M}(T).$$

6.2 The linear dependence relations on the coefficients

Lemma. *The polynomial relation* (5.4.4) *implies the linear relation:*

$$\sum_{i=1}^{m} 1\widehat{\otimes}\cdots\widehat{\otimes}1\widehat{\otimes}\overset{i\text{th}}{\mathcal{M}(T)}\widehat{\otimes}1\widehat{\otimes}\cdots\widehat{\otimes}1 \;=\; \Phi_m(\mathcal{M}(T)), \tag{6.2.1}$$

on the logarithmic growth-function for $T \in \mathrm{Conf}$ *and* $m \in \mathbb{Z}_{\geq 0}$.

Proof. Put $\mathcal{M}_i(T) := 1\widehat{\otimes}\cdots\widehat{\otimes}1\widehat{\otimes}\overset{i\text{th}}{\mathcal{M}(T)}\widehat{\otimes}1\widehat{\otimes}\cdots\widehat{\otimes}1$ so that $\exp(\mathcal{M}_i(T)) = 1\otimes\cdots\otimes 1\otimes(1+\mathcal{A}(T))\otimes 1\otimes\cdots\otimes 1$. Then (5.4.4) can be rewritten as:

$$*) \qquad \exp(\mathcal{M}_1(T))\cdot\ldots\cdot\exp(\mathcal{M}_m(T)) \;=\; \Phi_m(\exp(\mathcal{M}(T)))$$

where the left hand side is equal to $\exp(\mathcal{M}_1(T)+\cdots+\mathcal{M}_m(T))$ due to the commutativity of $\mathcal{M}_i$'s and the addition rule for exp. The right hand side of $*)$ can be rewritten as $\Phi_m(\exp(\mathcal{M}(T))) = \Phi_m(\sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{M}(T)^n) = \sum_{n=0}^{\infty}\frac{1}{n!}\Phi_m(\mathcal{M}(T))^n = \exp(\Phi_m(\mathcal{M}(T)))$. By taking the logarithm of both sides, we obtain (6.2.1). □

Corollary. *Let* $m \geq 2$. *For* $S_1,\ldots,S_m \in \mathrm{Conf}_+$ *and* $T \in \mathrm{Conf}$,

$$\sum_{S\in\mathrm{Conf}} \binom{S_1,\ldots,S_m}{S} M(S,T) = 0. \tag{6.2.2}$$

Proof. Expand both sides of (6.2.1) in a series of the variables $S_i := 1\otimes\ldots\otimes 1\otimes S_i\otimes 1\otimes\ldots\otimes 1$ $(i=1,\ldots,m)$. Since the left hand side of (6.2.1) does not have a mixed term $S_1\otimes\cdots\otimes S_m$ for $S_i \in \mathrm{Conf}_+$ and $m \geq 2$, the corresponding coefficients in the right hand side should vanish. By (4.1.1) and (6.1.2), this implies the formula (6.2.2). □

Remark. 1. The formula (6.2.2) is reduced to the case $m = 2$ with $S_i \neq \phi$ $(i = 1, 2)$ by induction on m. Recalling the composition rule (2.6)

$$\begin{aligned}\sum_{S}\binom{S_1,\ldots,S_m}{S}M(S,T) &= \sum_{S}\Big(\sum_{U\in\mathrm{Conf}}\binom{S_1,\ldots,S_{m-1}}{U}\binom{U,S_m}{U}\Big)M(S,T)\\ &= \sum_{U\in\mathrm{Conf}_+}\binom{S_1,\ldots,S_{m-1}}{U}\Big(\sum_{S}\binom{U,S_m}{U}M(S,T)\Big)\\ &\quad+\binom{S_1,\ldots,S_{m-1}}{\phi}\Big(\sum_{S}\binom{\phi,S_m}{U}M(S,T)\Big) = 0+0=0.\end{aligned}$$

2. The linear dependence relations (6.2.2) among $M(S,T)$'s for $S \in \mathrm{Conf}$ are the key facts of the present paper. The Hopf algebra structure was introduced only to deduce this relation. We shall solve this relation in (8.3.2) by use of kabi coefficients, which we introduce in the next paragraph § 7.

6.3 Lie-like elements

An element $\mathcal{M}$ satisfying (6.2.1) has a name in Hopf algebra theory [9].

Definition. Let $\mathbb{A}$ be a commutative algebra with a unit. An element $\mathcal{M}$ of $\mathbb{A}[\![\mathrm{Conf}]\!]$ is called *Lie-like* if it satisfies the relation:

$$(6.3.1) \qquad \Phi_m(\mathcal{M}) \;=\; \sum_{i=1}^{m} 1\widehat{\otimes}\cdots\widehat{\otimes}1\widehat{\otimes}\overset{i\text{th}}{\mathcal{M}}\widehat{\otimes}1\widehat{\otimes}\cdots\widehat{\otimes}1$$

for ${}^{\forall}m \in \mathbb{Z}_{\geq 0}$. This, in particular, implies the conditions $\Phi_0(\mathcal{M}) = 0$ and $\iota(\mathcal{M}) + \mathcal{M} = 0$ (c.f. (4.3.1) and (4.4.2)). The linear combinations (over $\mathbb{A}$) of Lie-like elements are also Lie-like. We put

$$(6.3.2) \qquad \mathcal{L}_{\mathbb{A}} \;:=\; \{\text{all Lie-like elements in } \mathbb{A}[\![\mathrm{Conf}]\!] .\},$$

and

$$(6.3.3) \qquad \mathcal{L}_{\mathbb{A},\text{ finite}} \;:=\; \{M \in \mathcal{L}_{\mathbb{A}} \mid M \text{ is of finite type}\}.$$

In this terminology, (6.2) Lemma can be rewritten as: *suppose* $\mathbb{Q} \subset \mathbb{A}$, *then one has* $\mathcal{M}(T)\in\mathcal{L}_{\mathbb{A},\mathrm{finite}}$ for $T\in\mathrm{Conf}$.

Remark. We shall see in 8.4 that $\mathcal{L}_{\mathbb{R}}$ is essentially an extension of $\mathcal{L}_{\mathbb{R},\text{ finite}}$ by a space $\mathcal{L}_{\mathbb{R},\infty}$, which is the main objective of the present paper. On the other hand, one has $\mathcal{L}_{\mathbb{A}} \cap \left(\mathbb{A}\cdot\mathrm{Conf}\right)_{\mathfrak{M}} \subset \mathcal{L}_{\mathbb{A},\text{ finite}}$ (actually equality holds, see §8), since $\left(\mathbb{A}\cdot\mathrm{Conf}\right)_{\mathfrak{M}}$ consists only of finite type elements.

§7. Kabi coefficients

We describe the inverse matrix of the infinite matrix: $A \;:=\; (A(S,T))_{S,T\in\mathrm{Conf}_0}$ explicitly in terms of kabi coefficients introduced in (7.2). The construction shows that the inverse matrix has only bounded nonzero entries (7.5). This fact leads to the comparison of the two topologies on $\mathcal{L}_{\mathbb{A},finite}$, which plays a key role in the sequel in construction of the infinite space $\mathcal{L}_{\mathbb{A},\infty}$.

7.1 The unipotency of A

The matrix A is unipotent in the sense that i) $A(S,S) = 1$ and ii) $A(S,T) = 0$ for $S \not\leq T$ (5.1.5). Then a matrix $A^{-1} := E + A^* + A^{*2} + A^{*3} + \cdots$, where $E := (\delta(U,V))_{U,V\in\mathrm{Conf}_0}$ and $A^* := E - A$, is well defined. Precisely,

$$A^{-1}(S,T) \;=\; \begin{cases} 0 & \text{for } S \not\leq T, \\ 1 & \text{for } S = T, \\ \sum\limits_{k>0} (-1)^k \big(\sum\limits_{S=S_0<\cdots<S_k=T} \big(\prod\limits_{i=1}^{k} A(S_{i-1},S_i)\big)\big) & \text{for } S < T. \end{cases}$$

Then the matrix A^{-1} is unipotent in the same sense as A, and, hence, the products $A^{-1} \cdot A$ and $A \cdot A^{-1}$ are well defined and are equal to E.

7.2 Kabi coefficients

Definition. 1. A graph $\mathbb{U}$ is called a *kabi* over its subgraph $\mathbb{S}$ if for all $x \in \mathbb{U} \backslash \mathbb{S}$, there exists $y \in \mathbb{S}$ such that (x, y) is an edge.

2. Let $U \in \mathrm{Conf}_0$ and let $\mathbb{U}$ be a graph with $[\mathbb{U}] = U$. For $S \in \mathrm{Conf}_0$, put

$$
\begin{aligned}
&(7.2.1) \quad \mathbb{K}(S, \mathbb{U}) \ := \ \{ \mathbb{S} \mid \mathbb{S} \subset \mathbb{U} \ \text{ such that } [\mathbb{S}] = S \text{ and } \mathbb{U} \text{ is kabi over } \mathbb{S} \}, \\
&(7.2.2) \quad K(S, U) \ := \ \# \mathbb{K}(S, \mathbb{U}).
\end{aligned}
$$

We call $K(S, U)$ a *kabi-coefficient*. The definition of the coefficient does not depend on the choice of $\mathbb{U}$. If $K(S, U) \neq 0$, we say that U has a kabi structure over S or simply U is kabi over S.

Directly from definition, we have

$$
\begin{aligned}
&(7.2.3) \qquad K(S, U) \ = \ 0 \quad \text{for } S \not\leq U, \\
&(7.2.4) \qquad K(S, S) \ = \ 1 \quad \text{for } S \in \mathrm{Conf}_0.
\end{aligned}
$$

Note. The word "kabi" means "mold" in Japanese.

7.3 Kabi inversion formula

Lemma. *For $S \in \mathrm{Conf}_0$ and $T \in \mathrm{Conf}$, one has the formula:*

$$
(7.3.1) \qquad \sum_{U \in \mathrm{Conf}_0} (-1)^{\# U - \# S} K(S, U) \cdot A(U, T) \ = \ \delta(S, T),
$$

where $\delta(S, T)$ means the $\#$ of connected components of T isomorphic to S.

Proof. The summation index U on the left hand side runs over the range $S \leq U \leq T$ (otherwise $K(S, U) \cdot A(U, T) = 0$). Hence if $S \not\leq T$, then the sum equals 0. If $S = T$, the only term in the sum is $K(S, S) A(S, S)$ which equals 1.

Let $S \in \mathrm{Conf}_0$ and $T \in \mathrm{Conf}$. Assume $S \leq T$ and $S \neq T$. Let $\mathbb{T}$ be a G-colored graph with $T = [\mathbb{T}]$. Applying the definition of $K(S, U)$ and $A(U, T)$

(cf. (5.1.1)), the left hand side of (7.3.1) can be rewritten as

$$\begin{aligned}
&\sum_{U\in \mathrm{Conf}_0} (-1)^{\#U-\#S} K(S,U)\cdot \#\mathbb{A}(U,\mathbb{T})\\
=&\sum_{U\in \mathrm{Conf}_0} (-1)^{\#U-\#S} K(S,U)\cdot \#\{\mathbb{U} \ \mid \ \mathbb{U}\subset\mathbb{T} \ \text{ such that } [\mathbb{U}]=U\}\\
=&\sum_{U\in \mathrm{Conf}_0} (-1)^{\#U-\#S} \# \left\{ (\mathbb{S},\mathbb{U}) \ \mid \ \begin{array}{c} \mathbb{S}\subset\mathbb{U}\subset\mathbb{T} \ \text{ such that}\\ [\mathbb{S}]=S, [\mathbb{U}]=U \text{ and } \mathbb{U} \text{ is kabi over } \mathbb{S}\end{array}\right\}
\end{aligned}$$

Now we make a resummation of this by fixing the subgraph $\mathbb{S}$ in $\mathbb{T}$.

$$= \sum_{\mathbb{S}\in A(S,\mathbb{T})} \left(\sum_{U\in \mathrm{Conf}_0} (-1)^{\#U-\#S} \# \left\{ \mathbb{U} \mid \begin{array}{c} \mathbb{S}\subset\mathbb{U}\subset\mathbb{T} \ \text{ such that}\\ [\mathbb{U}]\simeq U \text{ and } \mathbb{S} \text{ is a kabi over } \mathbb{S}.\end{array}\right\}\right)$$

For a fixed subgraph $\mathbb{S}$ of $\mathbb{T}$, let $\mathbb{U}_{\max}$ be the biggest subgraph of $\mathbb{T}$ such that $\mathbb{U}_{\max}$ is a kabi over $\mathbb{S}$, i.e. $\mathbb{U}_{\max}$ consists of vertexes of $\mathbb{T}$, which is either in $\mathbb{S}$ or connected to $\mathbb{S}$ by an edge. Then a subgraph $\mathbb{U}$ of $\mathbb{T}$ becomes a kabi over $\mathbb{S}$, if and only if it is a subgraph of $\mathbb{U}_{\max}$ containing $\mathbb{S}$. Hence the sum is equal to

$$= \sum_{\mathbb{S}\in A(S,\mathbb{T})} \Big(\sum_{\mathbb{S}\subset\mathbb{U}\subset\mathbb{U}_{\max}} (-1)^{\#\mathbb{U}-\#\mathbb{S}} \Big) \ = \sum_{\mathbb{S}\in A(S,\mathbb{T})} \Big(\sum_{\mathbb{W}\subset\mathbb{U}_{\max}\setminus\mathbb{S}} (-1)^{\#\mathbb{W}} \Big).$$

where the last summation index $\mathbb{W}$ runs over all subsets of $\mathbb{U}_{\max}\setminus\mathbb{S}$. Hence the summation in the parenthesis becomes 1 or 0 according to whether $\mathbb{U}_{\max}\setminus\mathbb{S}$ is ϕ or not. It is clear that $\mathbb{U}_{\max}\setminus\mathbb{S}=\phi$ is equivalent to the fact that $\mathbb{S}$ is a connected component of $\mathbb{T}$. Hence the sum is equal to $\delta(S,T)$. □

7.4 Corollaries to the inversion formula.

The left inverse matrix of $A := (A(S,T))_{S,T\in \mathrm{Conf}_0}$ is given by

$$A^{-1} \ = \ ((-)^{\#T-\#S} K(S,T))_{S,T\in \mathrm{Conf}_0}. \tag{7.4.1}$$

Since the left inverse matrices of A coincides with the right inverse, one has

$$\sum_{U\in \mathrm{Conf}_0} (-1)^{\#T-\#U} A(S,U)\cdot K(U,T) \ = \ \delta(S,T) \tag{7.4.2}$$

for $S\in \mathrm{Conf}_0$. Specializing S in (7.4.2) to $pt :=$ [one point graph], one gets,

$$\sum_{U\in \mathrm{Conf}_0} (-1)^{\#U} \#U\cdot K(U,T) \ = \ (-)^{\#T}\delta(pt,T). \tag{7.4.3}$$

7.5 Boundedness of non-zero entries of K

One of the most important consequences of (7.4.1) is the boundedness of the non-zero entries of the matrix A^{-1}, as follows.

Suppose $K(S,T) \neq 0$. Then, by definition, T must have at least one structure of kabi over S. This implies that for each fixed S and $q \geq 0$, there are only a finite number of $T \in \mathrm{Conf}_0$ with $K(S,T) \neq 0$. Precisely,

Assertion. *For $S \in \mathrm{Conf}_0$, $K(S,T) = 0$ unless $\#T \leq \#S \cdot (q-1) + 2$.*

Proof. Let $\mathbb{T}$ be kabi over $\mathbb{S}$. Every vertex of $\mathbb{S}$ is connected to at most q number of points of $\mathbb{T}$. Since $\mathbb{S}$ is connected, it has at least $\#S - 1$ number of edges. Hence, $\#T - \#S \leq \#\,\{\text{edges connecting } \mathbb{S} \text{ and } \mathbb{T}\backslash\mathbb{S}\} \leq q \cdot \#S - 2\cdot(\#S-1)$. This implies the Assertion. $\square$

Remark. The above boundedness implies that K induces a continuous map between the two differently completed modules of $\mathcal{L}_{\mathbb{A},finite}$ (cf. 8.4).

§8. Lie-like elements $\mathcal{L}_{\mathbb{A}}$

Under the assumption $\mathbb{Q} \subset \mathbb{A}$, we introduce two basis systems $\{\mathcal{M}(T)\}_{T\in\mathrm{Conf}_0}$ and $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ for the module of Lie-like elements $\mathcal{L}_{\mathbb{A},finite}$, where the base change between them is given by the kabi-coefficients. The basis $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ is compatible with the adic topology and gives a topological basis of $\mathcal{L}_{\mathbb{A}}$.

8.1 The splitting map ∂

First, we introduce a useful but somewhat technical map ∂. One reason for its usefulness can be seen from the formula (9.3.6). For $S \in \mathrm{Conf}_0$, let us define an $\mathbb{A}$-linear map $\partial_S : \mathbb{A}[\![\mathrm{Conf}]\!] \to \mathbb{A}$ by associating to a series f its coefficient at S, i.e. $\partial_S f := f_S \in \mathbb{A}$ for f given by (3.2.4). By the use of this, we define

$$
\begin{array}{rcl}
\partial : \mathbb{A}[\![\mathrm{Conf}]\!] & \longrightarrow & \prod_{S\in\mathrm{Conf}_0} \mathbb{A}\cdot e_S. \\
f & \longmapsto & \sum_{S\in\mathrm{Conf}_0} (\partial_S f)\cdot e_S
\end{array}
\tag{8.1.1}
$$

Here, the right hand side is an abstract direct product module of rank one modules $\mathbb{A}\cdot e_S$ with the base e_S for $S \in \mathrm{Conf}_0$. Let us verify that the map is well-defined. First, define the map ∂ from the polynomial ring $\mathbb{A}\cdot\mathrm{Conf}$ to $\bigoplus_{S\in\mathrm{Conf}_0} \mathbb{A}\cdot e_S$. Since $\partial(\mathcal{J}_n) \subset \bigoplus_{\substack{S\in\mathrm{Conf}_0\\ \#S\geq n}} \mathbb{A}\cdot e_S$, the map is continuous with respect to the adic topology (3.2) on the LHS and the direct product topology on the RHS. Then, ∂ (8.1.1) is obtained by completing this polynomial map.

We note that the restriction of the map ∂ (8.1.1) induces a map

$$\partial : \mathbb{A}\,[\![\mathrm{Conf}]\!]_{finite} \longrightarrow \oplus_{S\in\mathrm{Conf}}\mathbb{A}\cdot e_S,$$

even though the domain of this map is not a polynomial ring but the ring of elements of finite type (recall the definition in 3.3).

8.2 Bases $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ of $\mathcal{L}_{\mathbb{A},finite}$ and $\mathcal{L}_{\mathbb{A}}$

Lemma. *Let $\mathbb{A}$ be a commutative algebra containing $\mathbb{Q}$. Then,*

i) *The system $(\mathcal{M}(T))_{T\in\mathrm{Conf}_0}$ give a $\mathbb{A}$-free basis for $\mathcal{L}_{\mathbb{A},finite}$.*

$$\mathcal{L}_{\mathbb{A},finite} \quad\simeq\quad \oplus_{S\in\mathrm{Conf}_0}\mathbb{A}\cdot\mathcal{M}(S). \tag{8.2.1}$$

ii) *The map ∂ (8.1.1) induces a bijection of $\mathbb{A}$-modules:*

$$\partial|_{\mathcal{L}_{\mathbb{A},finite}} : \mathcal{L}_{\mathbb{A},finite} \quad\simeq\quad \oplus_{S\in\mathrm{Conf}_0}\mathbb{A}\cdot e_S \tag{8.2.2}$$

Put $\varphi(S) := \partial|^{-1}_{\mathcal{L}_{\mathbb{A},finite}}(e_S)$ for $S\in\mathrm{Conf}_0$ so that $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ form another $\mathbb{A}$-free basis of $\mathcal{L}_{\mathbb{A},finite}$.

iii) *The two basis systems $\{\mathcal{M}(S)\}_{S\in\mathrm{Conf}_0}$ and $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ for $\mathcal{L}_{\mathbb{A},finite}$ are transformed by the following formula.*

$$\mathcal{M}(T) \;=\; \sum_{S\in\mathrm{Conf}_0}\varphi(S)\cdot A(S,T) \tag{8.2.3}$$

$$\varphi(S) \;=\; \sum_{T\in\mathrm{Conf}_0}\mathcal{M}(T)\cdot(-1)^{\#T-\#S}K(T,S). \tag{8.2.4}$$

iv) *$\mathcal{L}_{\mathbb{A},finite}$ is dense in $\mathcal{L}_{\mathbb{A}}$ with respect to the adic topology on the configuration algebra (cf. (3.2)).*

v) *The map ∂ induces an isomorphism of topological $\mathbb{A}$-modules:*

$$\mathcal{L}_{\mathbb{A}} \quad\simeq\quad \prod_{S\in\mathrm{Conf}_0}\mathbb{A}\cdot e_s. \tag{8.2.5}$$

This means that any $\mathcal{M}\in\mathcal{L}_{\mathbb{A}}$ is expressed uniquely as an infinite sum

$$\mathcal{M} \;=\; \sum_{S\in\mathrm{Conf}_0}\varphi(S)\cdot a_S \tag{8.2.6}$$

for $a_S\in\mathbb{A}$ ($S\in\mathrm{Conf}_0$). That is: $(\varphi(S))_{S\in\mathrm{Conf}_0}$ is a topological basis of $\mathcal{L}_{\mathbb{A}}$. We shall, sometimes, call a_S the coefficient of $\mathcal{M}$ at $S\in\mathrm{Conf}_0$.

Proof. That $\mathcal{M}(T)\in\mathcal{L}_{\mathbb{A},finite}$ for $T\in\mathrm{Conf}$ is shown in (6.2) Lemma.
In the following a), b) and c), we prove i), ii) and iii) simultaneously.

a) *The restriction of the map* ∂ (8.1.1) *on* $\mathcal{L}_{\mathbb{A}}$ *is injective.*

Proof. If $\mathcal{M} = \sum_{S \in \mathrm{Conf}} S \cdot M_S \in \mathbb{A}[\![\mathrm{Conf}]\!]_+$ is Lie-like (6.3), then one has

$$\sum_{S \in \mathrm{Conf}} \binom{S_1, \ldots, S_m}{S} M_S \;=\; 0 \tag{8.2.7}$$

for any $S_1, \ldots, S_m \neq \phi$ and $m \geq 2$ (the proof is the same as that for (6.2.2)). We have to prove that $\partial\mathcal{M} = 0$ implies $M_S = 0$ for all $S \in \mathrm{Conf}$. This will be done by induction on $n(S) = \#\{\text{connected components of } S\}$ as follows. The case $n(S) = 1$ follows from the assumption $\partial\mathcal{M} = 0$. Let $n(S) > 1$ and $S = S_1^{k_1} \amalg \cdots \amalg S_l^{k_l}$ be an irreducible decomposition of S (so $S_i \in \mathrm{Conf}_0$ $(i = 1, \cdots, l)$ are pairwise distinct). Apply (8.2.7) for $m = k_1 + \cdots + k_l (= n(S))$ and take $S_1, \ldots, S_1$ (k_1 times), $\ldots, S_l, \ldots, S_l$ (k_l times) for $S_1, \ldots S_m$.

$$k_1! \ldots k_l! M_S + \sum_{\substack{T \in \mathrm{Conf} \\ n(T) < n(S)}} \binom{S_1, \ldots, S_m}{T} M_T \;=\; 0 \tag{**}$$

By the induction hypothesis, the second term in $**$) is 0, and hence $M_S = 0$. $\square$

b) *For* $T \in \mathrm{Conf}$, *one has the formula:*

$$\partial(\mathcal{M}(T)) \;=\; \sum_{S \in \mathrm{Conf}_0} e_S \cdot A(S, T). \tag{8.2.8}$$

(*Proof.* Recall that $\mathcal{M}(T) = \log(1 + \mathcal{A}(T))$ and the coefficients of $\mathcal{M}(T)$ and $\mathcal{A}(T)$ at a connected $S \in \mathrm{Conf}_0$ coincide (6.1.6). That is: $\partial(\mathcal{M}(T)) = \partial(\mathcal{A}(T))$. By definition, $\partial(\mathcal{A}(T)) =$ the right hand side of (8.2.8) $\square$.

c) *The map* (8.2.2) *is surjective, and hence, is bijective.*

Proof. It was shown in §7 that the infinite matrix $(A(S,T))_{S,T \in \mathrm{Conf}_0}$ is invertible as a unipotent matrix (7.1). Then (8.2.8) implies surjectivity.

Using again (8.2.8), we have that the system $\{\mathcal{M}(T)\}_{T \in \mathrm{Conf}_0}$ is $\mathbb{A}$-linearly independent and span the $\mathcal{L}_{\mathbb{A},finite}$, i.e. i) holds. The formula (8.2.8) can be rewritten as (8.2.3). Then (8.2.4) follows from (8.2.3) and (7.4.2) $\square$.

Proof of iv) and v) is done in the following a), b) and c).

a) $\mathcal{L}_{\mathbb{A}}$ *is closed in* $\mathbb{A}[\![\mathrm{Conf}]\!]$ *with respect to the adic topology*, since the co-product Φ_m is continuous (4.1 Assertion). Thus: $(\mathcal{L}_{\mathbb{A},finite})^{closure} \subset \mathcal{L}_{\mathbb{A}}$.

b) *The map* (8.2.2) *is homeomorphic with respect to the topologies: the induced adic topology on the LHS and the restriction of the direct product topology on the RHS.* (To show this, it is enough to show the bijection:

$$\partial : (\mathcal{L}_{\mathbb{A},finite}) \cap \mathcal{I}_n \;\;\simeq\;\; \bigoplus_{\substack{S \in \mathrm{Conf}_0 \\ \#S \geq n}} \mathbb{A} \cdot e_S, \tag{8.2.9}$$

since the sets on the RHS for $n \in \mathbb{Z}_{\geq 0}$ can be chosen as a system of fundamental neighborhoods for the direct product topology on $\bigoplus_{S \in \mathrm{Conf}_0} \mathbb{A} \cdot e_S$.)

Proof of (8.2.9). Due to the definition of the ideal $\mathcal{I}_n$ (3.2.1), the ∂-image of the left hand side is contained in the right hand side of (8.2.9). Thus, one has only to show surjectivity. For $S \in \mathrm{Conf}_0$, let $\varphi(S)$ be the base of $\mathcal{L}_{\mathbb{A},finite}$ such that $\partial(\varphi(S)) = e_S$ as introduced in ii). It is enough to show that if $\#S \geq n$ and $S \in \mathrm{Conf}_0$, then $\varphi(S)$ belongs to $\mathcal{I}_n$. Expand $\varphi(S) = \sum U \cdot \varphi_U$. We show that $\varphi_U \neq 0$ implies that U is contained in the semi-group generated by $\langle S \rangle$ such that $\#U \geq n$. More precisely, we show $\binom{U_1,\ldots,U_m}{S} \neq 0$, where $U = U_1 \amalg \cdots \amalg U_m$ is an irreducible decomposition of U (cf. (2.5) i)). The proof is achieved by induction on $m = n(U)$. For the case $n(U) = 1$ $\varphi_U \neq 0$ if and only if $U = S$ by the definition of $\varphi(S)$, and hence this is trivial. If $n(U) > 1$, then apply (8.2.7) similarly to $**$) for the irreducible decomposition of U. We get:

$$***) \qquad k_1! \ldots k_l!\ \varphi_U + \sum_{\substack{T \in \mathrm{Conf} \\ n(T) < n(U)}} \binom{U_1, \ldots, U_m}{T} \varphi_T \ = \ 0$$

The fact that $\varphi_U \neq 0$ implies $\varphi_T \cdot \binom{U_1,\ldots,U_m}{T} \neq 0$ for some T. Since $\varphi_T \neq 0$ with $n(T) < n(U)$, we apply the induction hypothesis to T, i.e. $\binom{T_1,\ldots,T_p}{S} \neq 0$ for an irreducible decomposition $T = T_1 \amalg \ldots \amalg T_p$. Since $\binom{U_1,\ldots,U_m}{T} \neq 0$, by composing the maps $U \to T \to S$, we conclude $\binom{U_1,\ldots,U_m}{S} \neq 0$. In particular, $U_i \in \langle S \rangle$ and $\#U = \sum \#U_i \geq \#T \geq \#S$. This completes the proof of b). $\square$)

c) By completing the map (8.2.2), one sees that the composition of the two injective maps $(\mathcal{L}_{\mathbb{A},finite})^{closure} \subset \mathcal{L}_{\mathbb{A}} \to \varinjlim_{p,q} \prod_{S \in \mathrm{Conf}_0} \mathbb{A} \cdot e_s$ is bijective. This shows that all the maps are bijective. Hence, $\mathcal{L}_{\mathbb{A},finite}$ is dense in $\mathcal{L}_{\mathbb{A}}$ and (8.2.5) holds. The formula (8.2.6) is another expression of (8.2.5).

This completes the proof of the Lemma. $\square$

Remark. 1. It was shown in the above proof that for $S \in \mathrm{Conf}_0$

$$(8.2.10) \qquad \varphi(S) \ \in \ \mathbb{Z}[\![\langle S \rangle]\!] \ \cap \ \mathcal{J}_{\#S}.$$

In particular, $\varphi(U,S) = \delta(U,S)$ for $U \in \mathrm{Conf}_0$.

2. It was shown that (8.2.2) is a homeomorphism. But one should note that (8.2.1) is *not* a homeomorphism.

3. In general, an element of $\mathcal{L}_{\mathbb{A}}$ cannot be expressed by an infinite sum of $\mathcal{M}(T)$ $(T \in \mathrm{Conf}_0)$ (cf. (9.4)).

4. The set of Lie-like elements of the localization $\mathbb{A}[\mathrm{Conf}]_{\mathfrak{M}}$ (cf. (4.6) Remark 4.) is equal to $\mathcal{L}_{\mathbb{A},finite}$. This is insufficient for our later application in §10, so we employed the other localization (3.2.2).

8.3 An explicit formula for $\varphi(S)$

Let us expand $\varphi(S)$ for $S \in \mathrm{Conf}_0$ in the series:

(8.3.1) $$\varphi(S) \;=\; \sum_{U \in \mathrm{Conf}} U \cdot \varphi(U,S)$$

for $\varphi(U,S) \in \mathbb{Q}$. The formula (8.2.3) can be rewritten as a matrix relation

(8.3.2) $$M(U,T) \;=\; \sum_{S \in \mathrm{Conf}_0} \varphi(U,S) \cdot A(S,T).$$

We remark that (8.2.3) and (8.3.2) are valid not only for $T \in \mathrm{Conf}_0$ but for all $T \in \mathrm{Conf}$, since both sides are additive with respect to T.

Formula. An explicit formula for the coefficients $\varphi(U,S)$.
(8.3.3)
$$\sum_{\substack{U = U_1^{k_1} \amalg \cdots \amalg U_m^{k_m} \\ V \in \mathrm{Conf},\ W \in \mathrm{Conf}_0}} \frac{(|\underline{k}|-1)!(-1)^{|\underline{k}|-1+|W|+|S|}}{k_1! \cdots k_m!} \underbrace{\left(\overbrace{U_1, \dots U_1}^{k_1}, \dots, \overbrace{U_m, \dots U_m}^{k_m} \right)}_{V} A(V,W) K(W,S).$$

Here the summation index runs over all decompositions $U = U_1^{k_1} \cdot \ldots \cdot U_l^{k_l}$ of U in the same manner explained at (3.6.7), where $|\underline{k}| = k_1 + \cdots + k_k$.

Proof. By use of (6.1.5), rewrite the left hand side of (8.2.3)* into a polynomial of $A(U_i,T)$. Then apply the product expansion formula (5.3.1) to each monomial so that the left hand side is expressed linearly by $A(S,T)$'s. Using the invertibility of $\{A(S,T)\}_{S,T \in \mathrm{Conf}}$ (7.4.2), one deduces (8.3.3). $\square$

Remark. As an application of (8.3.3), we can explicitly determine the coefficients $\{M_U\}_{U \in \mathrm{Conf}}$ of any Lie-like element $\mathcal{M} = \sum_{U \in \mathrm{Conf}} U \cdot M_U$ from the subsystem $\{M_S\}_{S \in \mathrm{Conf}_0}$ by the relation $M_U = \sum_{S \in \mathrm{Conf}_0} \varphi(U,S) \cdot M_S$. Here, the summation index S runs only over the finite set with $\#S \leq \#U$.

8.4 Lie-like elements $\mathcal{L}_{\mathbb{A},\infty}$ at infinity

We introduce the space $\mathcal{L}_{\mathbb{A},\infty}$ of Lie-like elements at infinity for a use after §10.

Recall that the kabi coefficients relate the two basis systems of $\mathcal{L}_{\mathbb{A},finite}$: $\{\varphi(S)\}_{S \in \mathrm{Conf}_0}$ and $\{\mathcal{M}(T)\}_{T \in \mathrm{Conf}_0}$ (cf.(8.2) lemma). The map:

$$\begin{array}{ccccc} & \mathcal{L}_{\mathbb{A},finite} & = & \mathcal{L}_{\mathbb{A},finite} & \\ K: & \displaystyle\sum_{S \in \mathrm{Conf}_0} \varphi(S)\, a_S & \longmapsto & \displaystyle\sum_{T \in \mathrm{Conf}_0} \mathcal{M}(T) \sum_{S \in \mathrm{Conf}_0} (-1)^{\sharp T - \sharp S} K(T,S)\, a_S & \end{array}$$

is the identity homomorphism between the same modules. We define topologies on the modules of both sides: the fundamental system of neighborhoods of 0 are the linear subspaces spanned by the all bases except for finite ones. Actually, the topology on the LHS coincides with the adic topology, which we have been studying (8.2 Lemma). The map K is continuous with respect to the topologies, since for any base $\mathcal{M}(T)$, there are only a finite number of bases $\varphi(S)$ whose image $K(\varphi(S))$ contains the term $\mathcal{M}(T)$, namely $K(T,S) \neq 0$ only for such S satisfying $\sharp T \geq \frac{1}{q-1}(\sharp S - 2)$, 7.5 Assertion. Let us denote by $\overline{K}$ the map between the completed modules and call it the *kabi map*.

$$\overline{K} \ : \ \mathcal{L}_{\mathbb{A}} \ \longrightarrow \prod_{T \in \mathrm{Conf}_0} \mathbb{A} \cdot \mathcal{M}(T). \tag{8.4.1}$$

We consider the set of Lie-like elements which are annihilated by the kabi map:

$$\mathcal{L}_{\mathbb{A},\infty} \ := \ \ker(\overline{K}), \tag{8.4.2}$$

and call it the space of *Lie-like elements at infinity*. In fact, $\mathcal{L}_{\mathbb{A},\infty}$ does not contain a non-trivial finite type element, i.e. $\mathcal{L}_{\mathbb{A},finite} \cap \mathcal{L}_{\mathbb{A},\infty} = \{0\}$. However, the direct sum $\mathcal{L}_{\mathbb{A},finite} \oplus \mathcal{L}_{\mathbb{A},\infty}$ is a small submodule of $\mathcal{L}_{\mathbb{A}}$, and *one looks for a submodule, say $\mathcal{L}'$, of $\mathcal{L}_{\mathbb{A}}$ containing $\mathcal{L}_{\mathbb{A},finite}$, with a splitting $\mathcal{L}_{\mathbb{A}} = \mathcal{L}' \oplus \mathcal{L}_{\mathbb{A},\infty}$.* However, there is some difficulty in finding such $\mathcal{L}'$ for general $\mathbb{A}$: an *infinite sum* $\sum_{T \in \mathrm{Conf}_0} a_T \mathcal{M}(T) \in Im(\overline{K})$ never converges in $\mathcal{L}'$ ($\simeq Im(\overline{K})$) with respect to the adic topology. We shall come back to this problem in (10.2) for the case $\mathbb{A} = \mathbb{R}$, where the classical topology plays the crucial role.

§9. Group-like elements $\mathfrak{G}_{\mathbb{A}}$

We determine the groups $\mathfrak{G}_{\mathbb{A}}$ and $\mathfrak{G}_{\mathbb{A},finite}$ of group-like elements in $\mathcal{L}_{\mathbb{A}}$ and $\mathcal{L}_{\mathbb{A},finite}$, respectively, if $\mathbb{A}$ is $\mathbb{Z}$-torsion free. In particular, if $\mathbb{A} = \mathbb{Z}$, the group $\mathfrak{G}_{\mathbb{Z},finite}$ is, by the correspondence $1 + \mathcal{A}(S) \leftrightarrow S$, isomorphic to $\langle \mathrm{Conf} \rangle$ =the abelian group associated to the semi-group Conf, and it forms a "lattice in the continuous group" $\mathfrak{G}_{\mathbb{R}}$. Then, we introduce the set EDP of equal division points inside the positive cone in $\mathfrak{G}_{\mathbb{R}}$ spanned by the basis $\{\mathcal{M}(S)\}$.

9.1 $\mathfrak{G}_{\mathbb{A},finite}$ and $\mathfrak{G}_{\mathbb{A}}$ for the case $\mathbb{Q} \subset \mathbb{A}$

We start with a general fact: *Assume $\mathbb{Q} \subset \mathbb{A}$. Then one has isomorphisms:*

$$\begin{array}{rcl} \exp \ : \ \mathcal{L}_{\mathbb{A}} & \simeq & \mathfrak{G}_{\mathbb{A}}. \\ \exp \ : \ \mathcal{L}_{\mathbb{A},finite} & \simeq & \mathfrak{G}_{\mathbb{A},finite}. \end{array} \tag{9.1.1}$$

Proof. Since $\mathrm{aug}(g) = 1$, $\log(g)$ (3.6.2) is well defined for $\mathbb{Q} \subset \mathbb{A}$. That g is group-like (5.4.1) implies that $\log(g)$ is Lie-like and belongs to $\mathcal{L}_{\mathbb{A}}$ (cf. proof of (6.2) Lemma). Then g is of finite type, if and only if $\log(g)$ is so (cf. (3.6)). Thus (9.1.1) is shown. The homeomorphism follows from that of exp (3.6). $\square$

9.2 Generators of $\mathfrak{G}_{\mathbb{A},finite}$ and $\mathfrak{G}_{\mathbb{A}}$ for a $\mathbb{Z}$-torsion free $\mathbb{A}$.

Lemma. *Let $\mathbb{A}$ be a commutative $\mathbb{Z}$-torsion-free algebra with unit.*

i) *Any element g of $\mathfrak{G}_{\mathbb{A},finite}$ is uniquely expressed as*

$$g \;=\; \prod_{i\in I}(1+\mathcal{A}(S_i))^{c_i} \tag{9.2.1}$$

for $S_i\in\mathrm{Conf}_0$ and $c_i\in\mathbb{A}$ $(i\in I)$ with $\#I<\infty$. That is, one has an isomorphism:

$$\langle\mathrm{Conf}\rangle\otimes_{\mathbb{Z}}\mathbb{A} \;\simeq\; \mathfrak{G}_{\mathbb{A},finite}\ ,\quad S\leftrightarrow 1+\mathcal{A}(S)\ , \tag{9.2.2}$$

where $\langle\mathrm{Conf}\rangle$ is the group associated to the semi-group Conf.

ii) *$\mathfrak{G}_{\mathbb{A},finite}$ is dense in $\mathfrak{G}_{\mathbb{A}}$ with respect to the adic topology.*

iii) *We have the natural inclusion:*

$$\{\exp(\varphi(S))\mid S\in\mathrm{Conf}_0\}\subset\mathfrak{G}_{\mathbb{Z},finite} \tag{9.2.3}$$

The set $\{\exp(\varphi(S))\}_{S\in\mathrm{Conf}_0}$ is a topological free generator system of $\mathfrak{G}_{\mathbb{A}}$. This means that any element g of $\mathfrak{G}_{\mathbb{A}}$ is uniquely expressed as an infinite product:

$$\prod_{S\in\mathrm{Conf}_0}\exp(\varphi(S)\cdot a_S) \quad :=\quad \lim_{n\to\infty}\Big(\prod\nolimits_{\substack{S\in\mathrm{Conf}_0\\ \#S<n}}\exp(\varphi(S)\cdot a_S)\Big) \tag{9.2.4}$$

for some $a_S\in\mathbb{A}$ $(S\in\mathrm{Conf}_0)$.

Proof. If $\mathbb{Q}\subset\mathbb{A}$, then due to the isomorphisms (9.1.1) and (6.1.1), the Lemma is reduced to the corresponding statements for $\mathcal{L}_{\mathbb{A}}$ and $\mathcal{L}_{\mathbb{A},finite}$ in (8.2) Lemma, where, due to (8.2.4), (8.2.10) and the integrality of kabi K, we have

$$\exp(\varphi(S)) \;=\; \prod_{T\in\mathrm{Conf}_0}(1+\mathcal{A}(T))^{(-1)^{\#T-\#S}K(T,S)} \;\in\mathfrak{G}_{\mathbb{Z},finite}\cap\{1+\mathcal{J}_{\#S}\},$$

where we note $1+\mathcal{A}(T)\in\mathfrak{G}_{\mathbb{Z},finite}$ (c.f. (5.1.6) and (5.4.4)).

Assume $\mathbb{Q}\not\subset\mathbb{A}$ and let $\tilde{\mathbb{A}}$ be the localization of $\mathbb{A}$ with respect to $\mathbb{Z}\setminus\{0\}$. Since $\mathbb{A}$ is torsion free, one has an inclusion $\mathbb{A}\subset\tilde{\mathbb{A}}$, which induces inclusions $\mathfrak{G}_{\mathbb{A}}\subset\mathfrak{G}_{\tilde{\mathbb{A}}}$ and $\mathfrak{G}_{\mathbb{A},finite}\subset\mathfrak{G}_{\tilde{\mathbb{A}},finite}$, and the Lemma is true for $\mathfrak{G}_{\tilde{\mathbb{A}},finite}$ and $\mathfrak{G}_{\tilde{\mathbb{A}}}$.

i) Let us express an element $g\in\mathfrak{G}_{\mathbb{A},finite}$ as $\prod_{i\in I}(1+\mathcal{A}(S_i))^{c_i}$, where $c_i\in\tilde{\mathbb{A}}$ for $i\in I$ and $\#I<\infty$. We need to show that $c_i\in\mathbb{A}$ for $i\in I$. Suppose not. Put $I_1:=\{i\in I: c_i\notin\mathbb{A}\}$ and let S_1 be a maximal element of $\{S_i : i\in I_1\}$ with respect to the partial ordering $\leq$. Put $g_1:=\prod_{i\in I_1\setminus\{1\}}(1+\mathcal{A}(S_i))^{c_i}$ and $g_2:=\prod_{i\in I\setminus I_1}(1+\mathcal{A}(S_i))^{c_i}$. Then $g_1(1+\mathcal{A}(S_1))^{c_1}=g\cdot g_2^{-1}\in\mathfrak{G}_{\mathbb{A},finite}$. In the left hand side, g_1 does not contain the term S_1, whereas $(1+\mathcal{A}(S_1))^{c_1}$ contains the term c_1S_1. Hence, the left hand side contains the term c_1S_1.

ii) Let any $g \in \mathfrak{G}_{\mathbb{A}}$ be given. For a fixed integer $n \in \mathbb{Z}_{\geq 0}$, we calculate

$$\begin{aligned} \log(g) &= \sum_{S \in \mathrm{Conf}_0} \varphi(S) \cdot a_S \qquad \text{for } a_S \in \tilde{\mathbb{A}} \qquad \text{(c.f. (8.2.6))} \\ &= \sum_{T \in \mathrm{Conf}_0} \mathcal{M}(T) \cdot c_{T,n} + R_n, \quad \text{where} \end{aligned}$$

$$*) \qquad c_{T,n} \ := \sum\nolimits_{\substack{S \in \mathrm{Conf}_0 \\ \#S < n}} (-1)^{\#T - \#S} K(T,S) \cdot a_S \in \tilde{\mathbb{A}}, \quad (c.f.\ (8.2.4))$$

$$**) \qquad R_n \ := \sum\nolimits_{\substack{S \in \mathrm{Conf}_0 \\ \#S \geq n}} \varphi(S) \cdot a_S.$$

Here we notice that

$*)$ $c_{T,n} \neq 0$ implies $\#T < n$, since $K(T,S) \neq 0$ implies $T \leq S$ (7.2.3).

$**)$ $R_n \in \mathcal{J}_n$, since $\#S \geq n$ implies $\varphi(S) \in \mathcal{J}_n$ (8.2.10).

Therefore

$$\begin{aligned} g &= \sum\nolimits_{\substack{T \in \mathrm{Conf}_0 \\ \#T < n}} (1 + \mathcal{A}(T))^{c_{T,n}} \cdot \exp(R_n) \\ &= \sum\nolimits_{\substack{T \in \mathrm{Conf}_0 \\ \#T < n}} (1 + \mathcal{A}(T))^{c_{T,n}} \qquad \bmod \mathcal{J}_n. \end{aligned}$$

Let us show that $c_{T,n} \in \mathbb{A}$ for all $T \in \mathrm{Conf}_0$. Suppose not, and let T_1 be a maximal element of $\{T \in \mathrm{Conf}_0 : \#T < n \text{ and } c_{T,n} \notin \mathbb{A}\}$. Then similar to the proof of i), the coefficient of g at $T_1 \equiv c_{T_1,n} \mod \mathbb{A} \not\equiv 0 \mod \mathbb{A}$. This is a contradiction. Therefore, $c_{T,n} \in \mathbb{A}$ for all T and hence, $g \in \mathfrak{G}_{\mathbb{A},finite} \mod \mathcal{J}_n$.

iii) Applying (7.4.2) to the relation $*)$ in the proof of ii), one gets

$$a_S \ = \sum\nolimits_{\substack{T \in \mathrm{Conf}_0 \\ \#T < n}} A(S,T) \cdot c_{T,n}$$

for $\#S < n$. Here the right hand side belongs to $\mathbb{A}$ due to the proof ii). On the other hand, the left hand side ($= a_S$) does not depend on n. Hence, by moving $n \in \mathbb{Z}_{\geq 0}$, one has proven that $a_S \in \mathbb{A}$ for all $S \in \mathrm{Conf}_0$. □

9.3 Additive characters on $\mathfrak{G}_{\mathbb{A}}$

Definition. An *additive character* on $\mathfrak{G}_{\mathbb{A}}$ is an additive homomorphism

$$\mathcal{X} \ : \ \mathfrak{G}_{\mathbb{A}} \ \longrightarrow \ \mathbb{A}, \tag{9.3.1}$$

which is continuous with respect to the adic topology on $\mathfrak{G}_{\mathbb{A}}$ such that

$$\mathcal{X}(g^a) \ = \ \mathcal{X}(g) \cdot a$$

for all $g \in \mathfrak{G}_{\mathbb{A}}$ and $a \in \mathbb{A}$. The continuity of $\mathcal{X}$ (9.3.1) is equivalent to the statement that there exists $n \geq 0$ such that $\mathcal{X}(\exp(\varphi(S))) = \mathcal{X}(1) = 0$ for $S \in \mathcal{J}_n \cap \mathrm{Conf}_0$. Hence it is equivalent to $\#\{S \in \mathrm{Conf}_0 : \mathcal{X}(\exp(\varphi(S))) \neq 0\} < \infty$.

The set of all additive characters will be denoted by

$$\mathrm{Hom}_{\mathbb{A}}(\mathfrak{G}_{\mathbb{A}}, \mathbb{A}). \tag{9.3.2}$$

Assertion. *1. For any fixed $U \in \mathrm{Conf}_0$, the correspondence*

$$\mathcal{X}_U \ : \ 1+\mathcal{A}(S) \in \mathfrak{G}_{\mathbb{Z},finite} \ \longmapsto \ A(U,S) \in \mathbb{Z} \tag{9.3.3}$$

extends uniquely to an additive $\mathbb{A}$-character on $\mathfrak{G}_{\mathbb{A}}$, denoted by $\mathcal{X}_U$. Then

$$\mathcal{X}_U(\exp(\varphi(S))) \ = \ \delta(U,S) \qquad \textit{for } U, S \in \mathrm{Conf}_0. \tag{9.3.4}$$

2. There is a natural isomorphism

$$\begin{array}{lcl} \mathrm{Hom}_{\mathbb{A}}(\mathfrak{G}_{\mathbb{A}}, \mathbb{A}) & \simeq & \underset{U\in \mathrm{Conf}_0}{\otimes} \mathbb{A}\cdot \mathcal{X}_U \\ \mathcal{X} & \longmapsto & \sum_{U\in\mathrm{Conf}_0} \mathcal{X}_U(\exp(\varphi(S)))\cdot \mathcal{X}_U. \end{array} \tag{9.3.5}$$

3. If $\mathbb{Q}\subset \mathbb{A}$, then for any $\mathcal{M} \in \mathcal{L}_{\mathbb{A}}$ and $U \in \mathrm{Conf}_0$ one has

$$\mathcal{X}_U(\exp(\mathcal{M})) \ = \ \partial_U \mathcal{M}. \tag{9.3.6}$$

Proof. 1. First we note that $A(U,S)$ for fixed $U \in \mathrm{Conf}_0$ is additive in S (5.1.8), so that $\mathcal{X}_U$ naturally extends to an additive homomorphism on $\mathfrak{G}_{\mathbb{A},finite}$. For continuity (i.e. the finiteness of S with $\mathcal{X}_U(\exp(\varphi(S))) \neq 0$), it is enough to show (9.3.4). Recalling (8.2.4) and (7.4.2), this proceeds as:

$$\begin{aligned} \mathcal{X}_U(\exp(\varphi(S)) \ &= \ \mathcal{X}_U(\exp(\textstyle\sum_{T\in\mathrm{Conf}_0} \mathcal{M}(T)(-1)^{\#T-\#S}K(T,S))) \\ &= \textstyle\sum_{T\in\mathrm{Conf}_0} \mathcal{X}_U(\exp(\mathcal{M}(T)))\cdot(-1)^{\#T-\#S}K(T,S) \\ &= \textstyle\sum_{T\in\mathrm{Conf}_0} \mathcal{X}_U(1+\mathcal{A}(T))\cdot(-1)^{\#T-\#S}K(T,S) \\ &= \textstyle\sum_{T\in\mathrm{Conf}_0} A(U,T)\cdot(-1)^{\#T-\#S}K(T,S) \ \ = \ \ \delta(U,S). \end{aligned}$$

2. The continuity of $\mathcal{I}$ implies the sum in the target space is finite.
3. Both sides of (9.3.6) take the same values for the basis $(\varphi(S))_{S\in\mathrm{Conf}_0}$. □

9.4 Equal division points of $\mathfrak{G}_{\mathbb{Z},finite}$

Recalling $\langle\mathrm{Conf}\rangle \simeq \mathfrak{G}_{\mathbb{Z},finite}$ (9.2.2), we regard $\langle\mathrm{Conf}\rangle$ as a "lattice" in $\mathfrak{G}_{\mathbb{R},finite}$. In the positive rational cone $\mathfrak{G}_{\mathbb{Q},finite} \cap \big(\prod_{S\in\mathrm{Conf}_0}(1+\mathcal{A}(S))^{\mathbb{R}_{\geq 0}}\big)$, we consider a particular point, which we call the *equal division point* for $S \in \mathrm{Conf}$:

$$\big(1+\mathcal{A}(S)\big)^{1/\sharp(S)}. \tag{9.4.1}$$

Here, the exponent $1/\sharp(S)$ is chosen so that we get the normalization:

$$\mathcal{X}_{pt}\Big(\big(1+\mathcal{A}(S)\big)^{1/\sharp(S)}\Big) \;=\; 1. \tag{9.4.2}$$

The set of all equal division points is denoted by

$$\mathrm{EDP} \;:=\; \{\big(1+\mathcal{A}(S)\big)^{1/\sharp(S)} \mid S\in \mathrm{Conf}\}. \tag{9.4.3}$$

The formulation of (9.4.1) is inspired from the free energy of Helmholtz in statistical mechanics. Instead of treating equal division points in the form (9.4.1) in $\mathfrak{G}_{\mathbb{R}}$, we shall treat their logarithms in $\mathcal{L}_{\mathbb{R}}$ in the next paragraphs.

9.5 A digression to $\mathcal{L}_{\mathbb{A}}$ with $\mathbb{Q}\not\subset\mathbb{A}$

We have determined the generators of $\mathfrak{G}_{\mathbb{A},finite}$ and $\mathfrak{G}_{\mathbb{A}}$ without assuming $\mathbb{Q}\subset\mathbb{A}$ but assuming only $\mathbb{Z}$-torsion freeness of $\mathbb{A}$. The following Assertion seems to suggest that the Lie-like elements $\mathcal{L}_{\mathbb{A}}$ behaves differently than the group-like case. However we do not pursue this subject any further in the present paper.

Assertion. *Let $\mathbb{A}$ be a commutative algebra with unit. If there exists a prime number p such that $\mathbb{A}$ is p-torsion free and $1/p\notin\mathbb{A}$, then $\mathcal{L}_{\mathbb{A}}$ is divisible by p (i.e. $\mathcal{L}_{\mathbb{A}}=p\mathcal{L}_{\mathbb{A}}$). In particular, if $\mathbb{A}$ is noetherian, $\mathcal{L}_{\mathbb{A}}=\{0\}$.*

A sketch of the proof. Consider an element $\mathcal{M}=\sum_{U\in\mathrm{Conf}} U\cdot M_U\in\mathcal{L}_{\mathbb{A}}$. As an element of $\mathcal{L}_{\tilde{\mathbb{A}}}$, it can be expressed as $\sum_{S\in\mathrm{Conf}_0}\varphi(S)\cdot a_S$ where $a_S=\partial_S\mathcal{M}=M_S\in\mathbb{A}$ for $S\in\mathrm{Conf}_0$. Recall the expression (8.3.3) for $\varphi(U,S)$ ($U\in\mathrm{Conf}$) and the remark following it. Then we see that M_U is expressed as:

$$\sum_{\substack{U=U_1^{k_1}\amalg\cdots\amalg U_m^{k_m}\\ V\in\mathrm{Conf},\ W\in\mathrm{Conf}_0\\ S\in\mathrm{Conf}_0}} \frac{(-1)^{|\underline{k}|-1+|W|+|S|}(|\underline{k}|-1)!}{k_1!\cdots k_m!}\left(\begin{matrix}\overbrace{U_1,\ldots,U_1}^{k_1},\ldots,\overbrace{U_m,\ldots,U_m}^{k_m}\\ V\end{matrix}\right)A(V,W)K(W,S)a_S.$$

Apply this formula for $U=T^p$ for a fixed $T\in\mathrm{Conf}_0$. The summation index set is $\{(k_1,k_2,\ldots)\in(\mathbb{Z}_{\ge0})^{\mathbb{Z}_{\ge1}}\mid p=\sum_{i\ge1}i\cdot k_i\}$, as explained in 3.6 *Example*. Except for the case $k_1=p$ and $k_i=0$ ($i>1$), the denominator $k_1!\ldots k_m!$ is a product of prime numbers smaller than p. The coefficient $\left(\begin{matrix}\overbrace{U_1,\ldots U_1}^{k_1},\ldots,\overbrace{U_m,\ldots U_m}^{k_m}\\ V\end{matrix}\right)$ for this case (i.e. $k_1=p, k_i=0$ ($i>1$) and for $V=[\mathbb{V}]$ is equal to the cardinality of the set $\{(\mathbb{U}_1,\ldots,\mathbb{U}_p)\mid \mathbb{U}_i$ is a subgraph of $\mathbb{V}$ such that $[\mathbb{U}_i] = T$ and $\cup_{i=1}^{p} \mathbb{U}_i = \mathbb{V}\}$. Since the cyclic permutation of $\mathbb{U}_1, \dots, \mathbb{U}_p$ acts on the set, and the action has no fixed points except for $V = T$, we see that the covering coefficient is divisible by p except for the case $V = T \in \mathrm{Conf}_0$. In that case $\sum_{W \in \mathrm{Conf}_0} (-1)^{|W|+|S|} A(T, W) K(W, S) = \delta(T, S)$ Therefore $\frac{(-1)^p}{p} a_T \equiv 0 \mod \mathbb{A}_{\mathrm{loc}}$ where $\mathbb{A}_{\mathrm{loc}}$ is the localization of the algebra $\mathbb{A}$ with respect to the prime numbers smaller than p. Hence $a_T \in p\mathbb{A}_{\mathrm{loc}} \cap \mathbb{A} = p\mathbb{A}$. $\square$

§10. Accumulation set of logarithmic equal division points

We consider the space of Lie-like elements $\mathcal{L}_{\mathbb{R}}$ over the real number field $\mathbb{R}$ which is equipped with the classical topology. The set in $\mathcal{L}_{\mathbb{R}}$ of accumulation points of the logarithm of EDP (9.4), denoted by $\overline{\log(\mathrm{EDP})}$, becomes a compact convex set. We decompose $\overline{\log(\mathrm{EDP})}$ into a joint of the finite (absolutely convergent) part $\overline{\log(\mathrm{EDP})}_{abs}$ and the infinite part $\overline{\log(\mathrm{EDP})}_{\infty}$.

10.1 The classical topology on $\mathcal{L}_{\mathbb{R}}$

We equip the $\mathbb{R}$-vector space

$$\mathcal{L}_{\mathbb{R}} = \varprojlim_{n} \mathcal{L}_{\mathbb{R}} / \overline{\mathcal{J}_n} \cap \mathcal{L}_{\mathbb{R}} \tag{10.1.1}$$

with the *classical topology* defined by the projective limit of the classical topology on the finite quotient $\mathbb{R}$-vector spaces. Since the quotient spaces are

$$\mathcal{L}_{\mathbb{R}} / \overline{\mathcal{J}_n} \cap \mathcal{L}_{\mathbb{R}} \simeq \oplus_{S \in \mathrm{Conf}_0, \#S < n} \mathbb{R} \cdot \varphi(S) \simeq \mathcal{L}_{\mathbb{R}, finite} / \mathcal{J}_n \cap \mathcal{L}_{\mathbb{R}, finite},$$

we see that 1) $\mathcal{L}_{\mathbb{R}}$ is homeomorphic to the direct product $\prod_{S \in \mathrm{Conf}_0} \mathbb{R} \cdot \varphi(S)$ (recall (8.2.5)), and 2) $\mathcal{L}_{\mathbb{R}, finite} \simeq \oplus_{S \in \mathrm{Conf}_0} \mathbb{R} \cdot \varphi(S)$ is dense in $\mathcal{L}_{\mathbb{R}}$ with respect to the classical topology. That is, *the classical topology on $\mathcal{L}_{\mathbb{R}}$ is the topology of the coefficient-wise convergence with respect to the basis $\{\varphi(S)\}_{S \in \mathrm{Conf}_0}$*. It is weaker than the adic topology.

Similarly, we equip $\mathbb{R}[\![\mathrm{Conf}]\!]$ with the classical topology defined by

$$\mathbb{R}[\![\mathrm{Conf}]\!] = \varprojlim_{n} \mathbb{R} \cdot \mathrm{Conf} / \mathcal{J}_n = \prod_{S \in \mathrm{Conf}} \mathbb{R} \cdot S. \tag{10.1.2}$$

So, the classical topology on $\mathbb{R}[\![\mathrm{Conf}]\!]$ is the same as the topology of coefficient-wise convergence with respect to the basis $\{S\}_{S \in \mathrm{Conf}}$. The next relation ii) between the two topologies (10.1.1) and (10.1.2) is a consequence of (8.3.3).

Assertion. i) *The product and coproduct on $\mathbb{R}[\![\mathrm{Conf}]\!]$ are continuous with respect to the classical topology.*

ii) *The classical topology on* $\mathcal{L}_{\mathbb{R}}$ *is homeomorphic to the topology induced from that on* $\mathbb{R}[\![\mathrm{Conf}]\!]$.

iii) *Let us equip* $\mathfrak{G}_{\mathbb{R}}$ *with the classical topology induced from that on* $\mathbb{R}[\![\mathrm{Conf}]\!]$. *Then* $\exp : \mathcal{L}_{\mathbb{R}} \to \mathfrak{G}_{\mathbb{R}}$ *is a homeomorphism.*

Proof. i) The product and coproduct are continuous with respect to the adic topology (cf. (3.2) and (4.2)), which implies the statement.

ii) For a sequence in $\mathcal{L}_{\mathbb{R}}$, we need show the equivalence of convergence in $\mathcal{L}_{\mathbb{R}}$ and in $\mathbb{R}[\![\mathrm{Conf}]\!]$. This is true due to (8.3.3).

iii) The maps exp and log are bijective (cf. (9.2) Assertion) and homeomorphic with respect to the adic topology, which implies the statement. $\square$

10.2 Absolutely convergent sum in $\mathcal{L}_{\mathbb{R}}$

Recall the problem posed in 8.4: find a subspace of $\mathcal{L}_{\mathbb{A}}$ containing $\mathcal{L}_{\mathbb{A},finite}$ which is complementary to the subspace at infinity $\mathcal{L}_{\mathbb{A},\infty}$ (8.4.2). In the present paragraph, we answer this problem for the case $\mathbb{A} = \mathbb{R}$ by introducing a sufficiently large submodule $\mathcal{L}_{\mathbb{R},abs}$, which contains $\mathcal{L}_{\mathbb{R},finite}$ but does not intersect with $\mathcal{L}_{\mathbb{R},\infty}$ so that we obtain a splitting submodule $\mathcal{L}_{\mathbb{R},abs} \oplus \mathcal{L}_{\mathbb{R},\infty}$ of $\mathcal{L}_{\mathbb{R}}$.

Definition. We say a formal sum $\sum_{T\in\mathrm{Conf}_0} a_T\mathcal{M}(T) \in \prod_{T\in\mathrm{Conf}_0} \mathbb{R}\cdot\mathcal{M}(T)$ is *absolutely convergent* if, for any $S\in\mathrm{Conf}$, the sum $\sum_{T\in\mathrm{Conf}_0} a_T M(S,T)$ of its coefficients at S is absolutely convergent, i.e. $\sum_{T\in\mathrm{Conf}_0} |a_T| M(S,T) < \infty$ for $\forall S\in\mathrm{Conf}$. Then, any series $\sum_{i=1}^{\infty} a_{Ti}\mathcal{M}T_i)$ defined by any linear ordering $T_1 < T_2 < \cdots$ of the index set Conf_0 converges in $\mathcal{L}_{\mathbb{R}}$ to the same element with respect to the classical topology. We denote the limit by $\sum_{T\in\mathrm{Conf}_0}^{abs} a_T\mathcal{M}(T)$. Define the space of absolutely convergent elements:

$$(10.2.1)\quad \mathcal{L}_{\mathbb{R},abs} \;:=\; \{\text{ all absolutely convergent sums } \textstyle\sum_{T\in\mathrm{Conf}_0}^{abs} a_T\mathcal{M}(T)\ \}.$$

By definition, $\mathcal{L}_{\mathbb{R},abs}$ is an $\mathbb{R}$-linear subspace of $\mathcal{L}_{\mathbb{R}}$ such that $\mathcal{L}_{\mathbb{R},abs} \cap \mathcal{L}_{\mathbb{R},\infty} = \{0\}$ and $\mathcal{L}_{\mathbb{R},abs} \supset \mathcal{L}_{\mathbb{R},finite}$. Hence, the restriction $K|_{\mathcal{L}_{\mathbb{R},abs}}$ of the kabi-map (8.4.1) is injective. We give a criterion for the absolute convergence, which guarantee later that $\mathcal{L}_{\mathbb{R},abs}$ is large enough for our purpose (10.4.3).

Assertion. *A formal sum* $\sum_{T\in\mathrm{Conf}_0} a_T\mathcal{M}(T)$ *is absolutely convergent if and only if the sum* $\sum_{T\in\mathrm{Conf}_0} |a_T|\#(T)$ *is convergent. The* $\mathcal{L}_{\mathbb{R},abs}$ *is a Banach space with respect to the norm* $\left|\sum_{T\in\mathrm{Conf}_0}^{abs} a_T\mathcal{M}(T)\right| := \sum_{T\in\mathrm{Conf}_0} |a_T|\#(T)$.

Proof. The coefficient of $\mathcal{M}(T)$ at [a point] is equal to $\#(T)$. So absolute convergence implies, in particular, the convergence of $\sum_{T\in\mathrm{Conf}_0} |a_T|\#(T)$.

Conversely, under this assumption, let us show the absolute convergence of the sum $\sum_{T\in \mathrm{Conf}_0} a_T M(S,T)$ for any $S \in \mathrm{Conf}$. We prove this by induction on $n(S)$ = the number of connected components of S. If S is connected (i.e. $n(S)=1$), then $A(S,T)=M(S,T)$ and by the use of (5.2.1), we have $\sum_{T\in \mathrm{Conf}_0} |a_T| M(S,T) \le (\sum_{T\in \mathrm{Conf}_0} |a_T| \# T) \frac{(q-1)^{\# S-1}}{\# Aut(S)}$ which converges absolutely. If S is not connected, decompose it into connected components as $S=\prod_{i=1}^m S_i$ and apply (6.2.2). Since $\binom{S_1,\ldots,S_m}{S'} \neq 0$ implies either $n(S') < n(S)$ or $S'=S$, $M(S,T)$ is expressed as a finite linear combination of $M(S',T)$ for $n(S')<n(S)$ (independent of T). Then we are done by the induction hypothesis. □

10.3 Accumulating set $\overline{\log(\mathbf{EDP})}$

Recall that an equal dividing point in $\mathfrak{G}_{\mathbb{Q}}$ (9.4.1) is, by definition, an element of the form $\big(1+\mathcal{A}(S)\big)^{1/\sharp(S)}$ for a $S\in \mathrm{Conf}_+$. Let us consider the set in $\mathcal{L}_{\mathbb{Q}}$ of their logarithms (by use of the homeomorphism in 10.1 Assertion iii):

$$\log(\mathrm{EDP}) \ := \ \{\mathcal{M}(T)/\sharp T \mid T\in \mathrm{Conf}_+\} \tag{10.3.1}$$

and its closure $\overline{\log(\mathrm{EDP})}$ in $\mathcal{L}_{\mathbb{R}}$ with respect to the classical topology. So, any element $\omega \in \overline{\log(\mathrm{EDP})}$ has an expression:

$$\omega \ := \ \lim_{n\to\infty}{}^{cl} \frac{\mathcal{M}(T_n)}{\sharp T_n} \tag{10.3.2}$$

for a sequence $\{T_n\}_{n\in\mathbb{Z}_{>0}}$ in Conf_+, where we denote by $\lim^{cl}$ the limit with respect to the classical topology. Recalling that the topology on $\mathcal{L}_{\mathbb{R}}$ is defined by the coefficient-wise convergence with respect to the basis $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ and using (8.2.3), one has $\omega = \sum_{S\in\mathrm{Conf}_0}\varphi(S)\cdot a_S$ where $a_S=\lim\limits_{n\to\infty}\frac{A(S,T_n}{\# T_n}$.

Assertion. 1. *The set $\overline{\log(\mathrm{EDP})}$ is compact and convex.*

2. *Expand any element $\omega\in\overline{\log(\mathrm{EDP})}$ as $\sum_{S\in\mathrm{Conf}_0}\varphi(S)\cdot a_S$. Then*

i) $0 \le a_S \le (q-1)^{\sharp S-1}/\sharp\,\mathrm{Aut}(S)$ *for* $S\in\mathrm{Conf}_0$,

ii) $(q-1)^{\sharp S-\sharp S'} a_{S'} \ge a_S$ *for* $S'\le S$*. In particular, if* $a_S\neq 0$ *then* $a_{S'}\neq 0$.

Proof. 1. Compactness: it is enough to show that the range of coefficients a_S for $\omega\in\log(\mathrm{EDP})$ is bounded for each $S\in\mathrm{Conf}_0$. Recalling the expansion formula (8.2.3), this is equivalent to the statement that $\{A(S,T)/\sharp T \mid T\in \mathrm{Conf}_0\}$ is bounded for any $S\in\mathrm{Conf}_0$. Applying the inequality (5.2.2), we have

$$0 \ \le \ A(S,T)/\sharp T \ \le \ (q-1)^{\sharp S-1}/\sharp\,\mathrm{Aut}(S),$$

which clearly gives a universal bound for $A(S,T)/\sharp T$ independent of T.

Convexity: since for any T, $T'(\neq [\phi])$ and $r \in \mathbb{Q}$ with $0 < r < 1$, one can find positive integers p and q such that for $T'' := T^p \cdot T'^q$ one has

$$\begin{aligned}\mathcal{M}(T'')/\sharp T'' &= (p \cdot \mathcal{M}(T) + q \cdot \mathcal{M}(T'))/(p \cdot \sharp T + q \cdot \sharp T') \\ &= r \cdot \mathcal{M}(T)/\sharp T + (1-r) \cdot \mathcal{M}(T')/\sharp T'.\end{aligned}$$

2. i) This is shown already in 1.

ii) If $S' \leq S$ and $S \in \mathrm{Conf}_0$, then for any $T \in \mathrm{Conf}$ one has an inequality $(q-1)^{\sharp S - \sharp S'} A(S',T) \geq A(S,T)$. (This can be easily seen by fixing representatives of S and S' as in proof of (5.2.2)). Therefore $(q-1)^{\sharp S - \sharp S'} a_{S'} \geq a_S$. □

Remark. The condition (9.4.2) on EDP implies $a_{pt} = 1$ for any element $\omega \in \overline{\log(\mathrm{EDP})}$. In particular, this implies $0 \notin \overline{\log(\mathrm{EDP})}$.

10.4 Joint decomposition $\overline{\log(\mathbf{EDP})}_{abs} * \overline{\log(\mathbf{EDP})}_{\infty}$

We show that $\overline{\log(\mathrm{EDP})}$ is embedded in $\mathcal{L}_{\mathbb{R},abs} \oplus \mathcal{L}_{\mathbb{R},\infty}$, and, accordingly, decompose $\overline{\log(\mathrm{EDP})}$ into the joint of a finite part and an infinite part, where the finite part is an infinite simplex with the vertex set $\{\frac{\mathcal{M}(T)}{\#T}\}_{T \in \mathrm{Conf}_0}$.

Definition. Define the *finite part* and the *infinite part* of $\overline{\log(\mathrm{EDP})}$ by

$$\overline{\log(\mathrm{EDP})}_{abs} := \overline{\log(\mathrm{EDP})} \cap \mathcal{L}_{\mathbb{R},abs}, \tag{10.4.1}$$

$$\overline{\log(\mathrm{EDP})}_{\infty} := \overline{\log(\mathrm{EDP})} \cap \mathcal{L}_{\mathbb{R},\infty}. \tag{10.4.2}$$

Lemma. 1. $\overline{\log(EDP)}$ *is a joint of the finite part and the infinite part:*

$$\overline{\log(EDP)} = \overline{\log(EDP)}_{abs} * \overline{\log(EDP)}_{\infty}. \tag{10.4.3}$$

Here, the joint of subsets A and B in real vector spaces V and W is defined by
$A * B := \{\lambda p + (1-\lambda) q \in V \oplus W \mid p \in A, q \in B, \lambda \in [0,1]\}$.

2. *The finite part is the infinite simplex of the vertex set* $\{\frac{\mathcal{M}(S)}{\#S}\}_{S \in \mathrm{Conf}_0}$*:*

$$\overline{\log(EDP)}_{abs} = \left\{ \textstyle\sum_{S \in Conf_0}^{abs} \mu_S \frac{\mathcal{M}(S)}{\#S} \mid \mu_S \in \mathbb{R}_{\geq 0} \ \ \textit{and} \ \ \sum_{S \in \mathrm{Conf}_0} \mu_S = 1 \right\}.$$

Proof. We prove 1. and 2. simultaneously in two steps A. and B. We show only the inclusion LHS ⊂ RHS since the opposite inclusion LHS ⊃ RHS is trivial due to the closed compact convexity of $\overline{\log(\mathrm{EDP})}$ (10.3 Assertion 1.).

A. **Finite part.** Let us consider an element $\omega \in \overline{\log(\mathrm{EDP})}$ of the expression (10.3.2). For $S \in \mathrm{Conf}_0$, recall that $\delta(S, T_n)$ is the # of connected components of T_n isomorphic to S. Let us show that *the limit*

$$(10.4.4) \qquad \mu_S \ := \ \#S \lim_{n\to\infty} \frac{\delta(S,T_n)}{\sharp T_n}$$

converges to a finite real number μ_S *such that*

$$(10.4.5) \qquad 0 \ \leq \ \sum\nolimits_{S\in \mathrm{Conf}_0} \mu_S \ \leq \ 1 \ .$$

Note that the kabi-map $\overline{K}$ (8.4.1) is also continuous with respect to the classical topology. So, it commutes with the classical limiting process $\lim\limits_{n\to\infty}^{cl} \frac{\mathcal{M}(T_n)}{\sharp T_n}$. Recalling the kabi-inversion formula (7.3.1), we calculate

$$\overline{K}(\omega) \ = \ \overline{K}(\lim_{n\to\infty}^{cl} \tfrac{\mathcal{M}(T_n)}{\sharp T_n}) \ = \ \lim_{n\to\infty}^{cl} \tfrac{\overline{K}(\mathcal{M}(T_n))}{\sharp T_n} \ = \ \lim_{n\to\infty} \sum\nolimits_{S\in \mathrm{Conf}_0} \tfrac{\delta(S,T_n)}{\sharp T_n} \mathcal{M}(S).$$

Here, the convergence on the RHS is the coefficient-wise convergence with respect to the basis $\mathcal{M}(S)$ for $S\in \mathrm{Conf}_0$. This implies the convergence of (10.4.4).

Let C be any finite subset of Conf_0. For any $n\in \mathbb{Z}_{\geq 0}$, one has

$$\sum\nolimits_{T\in C} \delta(T,T_n)\cdot \sharp T \ \ \leq \ \ \sharp T_n$$

since the LHS is equal to the cardinality of the vertices of the union of connected components of T_n which is isomorphic to an element of C. Dividing both sides by $\sharp T_n$ and taking the limit $n\to\infty$, one has (10.4.5).

Define the *finite part of* ω by the absolutely convergent sum

$$(10.4.6) \qquad \omega_{finite} \ := \ \sum\nolimits_{S\in \mathrm{Conf}_0}^{abs} \mu_S \frac{\mathcal{M}(S)}{\sharp S}$$

(apply 10.2 Assertion to (10.4.5)). We remark that the coefficients μ_S are uniquely determined from ω and is independent of the sequence $\{T_n\}_{n\in\mathbb{Z}_{\geq 0}}$, due to the formula:

$$(10.4.7) \qquad \overline{K}(\omega) \ = \ \sum\nolimits_{S\in \mathrm{Conf}_0} \mu_S \tfrac{\mathcal{M}(S)}{\sharp S}.$$

B. **Infinite part.** Put $\mu_\infty := 1 - \sum_{S\in \mathrm{Conf}_0} \mu_S$. Let us show that

i) *if* $\mu_\infty = 0$*, then we have* $\omega \ = \ \omega_{finite}$, and ii) *if* $\mu_\infty > 0$*, then there exist a unique element* $\omega_\infty \in \mathcal{L}_{\mathbb{R},\infty}$ *so that* $\omega \ = \ \mu_\infty\, \omega_\infty \ + \ \omega_{finite}$

For any $S\in \mathrm{Conf}_0$, let us denote by $T_n(S)$ the isomorphism class of the union of the connected components of T_n isomorphic to $S\in \mathrm{Conf}_0$. Thus, $\sharp T_n(S) = \delta(S,T_n)\sharp S$ and $\sharp T_n(S)/\#T_n \to \mu_S$ as $n\to\infty$. For any finite subset C of Conf_0, put $T_n^*(C^c) := T_n \setminus \bigcup\limits_{S\in C} T_n(S)$ so that one has

$$*)\qquad \frac{\mathcal{M}(T_n)}{\sharp T_n} = \frac{\mathcal{M}(T_n^*(C^c))}{\sharp T_n} + \sum_{S\in C} \frac{\delta(S,T_n)\sharp S}{\sharp T_n} \cdot \frac{\mathcal{M}(S)}{\sharp S}.$$

For the given C and for $\varepsilon > 0$, there exists $n(C,\varepsilon)$ such that

a) $$\sum_{S\in C} |\mu_S - \sharp T_n(S)/\sharp T_n| \ < \ \varepsilon$$

for $n \geq n(C,\varepsilon)$. This implies $|\mu_\infty - \sharp T_n^*(C^c)/\sharp T_n| < \varepsilon + \sum_{S\in \mathrm{Conf}_0 \setminus C} \mu_S$.

Let $\{\varepsilon_m\}_{m\in\mathbb{Z}_{\geq 0}}$ be any sequence of positive real numbers with $\varepsilon_m \downarrow 0$. Choose an increasing sequence $\{C_m\}_{m\in\mathbb{Z}_{\geq 0}}$ of finite subsets of Conf_0 satisfying

b) $$\bigcup_{m\in\mathbb{Z}_{\geq 0}} C_m \ = \ \mathrm{Conf}_0 \quad \text{and} \quad \sum_{S\in \mathrm{Conf}_0 \setminus C_m} \mu_S \ < \ \varepsilon_m.$$

Put $n(m) := n(C_m, \varepsilon_m)$. Then, by definition of μ_∞ and by a) and b), one has

c) $$|\mu_\infty - \sharp T^*_{n(m)}(C^c_m)/\sharp T_{n(m)}| \ < \ 2\varepsilon_m.$$

Substituting n and C in $*)$ by $n(m)$ and C_m, respectively, we obtain a sequence of equalities indexed by $m \in \mathbb{Z}_{\geq 0}$. Let us prove:

i) *the second term of* $*)$ *absolutely converges to* ω_{finite}.

ii) *if* $\mu_\infty = 0$, *then the first term of* $*)$ *converges to 0.*

iii) *if* $\mu_\infty \neq 0$, *then* $T_m^* := T^*_{n(m)}(C^c_m) \neq \phi$ *for large* m *and* $\mathcal{M}(T_m^*)/\sharp T_m^*$ *converges to an element* $\omega_\infty \in \overline{\log(EDP)} \cap \mathcal{L}_{\mathbb{R},\infty}$.

Proof of i). For $m \in \mathbb{Z}_{\geq 0}$, the difference of ω_{finite} and the second term of $*)$ is $\sum_{S\in\mathrm{Conf}_0} c_S \frac{\mathcal{M}(S)}{\sharp TS}$ where $c_S := \mu_S - \frac{\delta(S,T_{n(m)})\sharp S}{\sharp T_{n(m)}}$ for $S \in C_m$ and $c_S := \mu_S$ for $S \in \mathrm{Conf}_0 \setminus C_m$. Therefore, using a) and the latter half of b), one sees that the sum $\sum_{S\in\mathrm{Conf}_0} |c_S|$ is bounded by $2\varepsilon_m$. Then, due to a criterion in 10.2 Assertion, the difference tends to 0 absolutely as $m \uparrow \infty$. □

Proof of ii). Recall *c*) $|\sharp T^*_{n(m)}(C^c_m)/\sharp T_{n(m)}| < 2\varepsilon_m$. The first term of $*)$ is given by $\frac{\mathcal{M}(T^*_{n(m)}(C_m))}{\sharp T_{n(m)}} = \sum_{S\in\mathrm{Conf}_0} \varphi(S) \frac{A(S,T^*_{n(m)}(C_m))}{\sharp T_{n(m)}}$, where the coefficient of $\varphi(S)$ is either 0 if $T^*_{n(m)}(C_m) = \phi$ or equal to $\frac{\sharp T^*_{n(m)}(C_m)}{\sharp T_{n(m)}} \frac{A(S,T^*_{n(m)}(C_m))}{\sharp T_{n(m)}(C_m)}$ otherwise, which is bounded by $2\varepsilon_m q^{\sharp S-1}/\sharp\,\mathrm{Aut}(S)$. So it converges to 0 as $m \uparrow \infty$. □

Proof of iii). The sequence of the first term of the RHS of $*)$ converges to $\omega - \omega_{finite}$, since the LHS of $*)$ and the second term of the RHS of $*)$ converges to ω and ω_{finite}, respectively. On the other hand, due to c), one has $\sharp T^*_{n(m)}(C^c_m)/\sharp T_{n(m)} > \mu_\infty - 2\varepsilon_m$ for sufficiently large m, and hence one has $T^*_{n(m)}(C^c_m) \neq \phi$. Then the first term is decomposed as:

$$\frac{\mathcal{M}(T^*_{n(m)}(C^c_m))}{\sharp T_{n(m)}} \ = \ \frac{T^*_{n(m)}(C^c_m)}{\sharp T_{n(m)}} \frac{\mathcal{M}(T^*_{n(m)}(C^c_m))}{\sharp T^*_{n(m)}(C^c_m)},$$

whose first factor converges to $\mu_\infty \neq 0$ due to *c*). Therefore, the second factor converges to some $\omega_\infty := (\omega - \omega_{finite})/\mu_\infty$, which belongs to $\overline{\log(\mathrm{EDP})}$ by definition. Since $\overline{K}(\omega) = \overline{K}(\omega_\infty)$, ω_∞ belongs to $\ker(\overline{K})$. □

These complete a proof of the Lemma. $\square$

10.5 Extremal points in $\overline{\log(\mathbf{EDP})}_\infty$.

A point ω in a subset A in a real vector space is called an *extremal point* of A if an interval I contained in A contains ω then ω is a terminal point of I.

Assertion. *The extremal point of $\overline{\log(EDP)}$ is one of the following:*
i) $\frac{\mathcal{M}(S)}{\sharp S}$ *for an element $S \in \mathrm{Conf}_0$,*
ii) $\lim\limits_{n\to\infty}^{cl} \frac{\mathcal{M}(T_n)}{\sharp T_n}$ *for a sequence $T_n \in \mathrm{Conf}_0$ with $\# T_n \to \infty$ ($n \to \infty$).*

Proof. For $\omega \in \overline{\log(\mathrm{EDP})}$, if $\mu_\infty \neq 0,\ 1$, then ω cannot be extremal. If $\mu_\infty = 0$, due to Corollary 1, the only possibility for ω to be extremal is when it is of the form $\frac{\mathcal{M}(S)}{\sharp S}$ for an element $S \in \mathrm{Conf}_0$. In fact, using the uniqueness of the expression (Lemma 3.), $\frac{\mathcal{M}(S)}{\sharp S}$ can be shown to be extremal.

Suppose $\mu_\infty = 1$. For any fixed $S \in \mathrm{Conf}_0$ and real $\varepsilon > 0$, let $T_n^+(S,\varepsilon)$ (resp. $T_n^-(S,\varepsilon)$ be the subgraph of T_n consisting of the components T such that $A(S,T)/\sharp T \geq a_S + \varepsilon$ (resp. $\leq a_S - \varepsilon$). Let us show that $\overline{\lim\limits_n} \sharp T_n^\pm(S,\varepsilon)/\sharp T_n = 0$. If not, then there exists a subsequence $\{\widehat{n}\}$ such that $\lim\limits_{\widehat{n}} \sharp T_{\widehat{n}}^\pm(S,\varepsilon)/\sharp T_{\widehat{n}} = \lambda > 0$. Due to the compactness of $\overline{\log(\mathrm{EDP})}$ ((10.3) Assertion 1.), we can choose a subsequence such that $\mathcal{M}(T_{\widehat{n}}^\pm(S,\varepsilon))/\sharp T_{\widehat{n}}^\pm(S,\varepsilon)$ and $\mathcal{M}(T_{\widehat{n}} \backslash T_{\widehat{n}}^\pm(S,\varepsilon))/\sharp(T_{\widehat{n}} \backslash T_{\widehat{n}}^\pm(S,\varepsilon))$ converges to some $\sum_{T\in\mathrm{Conf}_0} \varphi(T)\cdot b_T$ and $\sum_{T\in\mathrm{Conf}_0}\varphi(T)\cdot c_T$, respectively, so that

$$\omega = \lambda \cdot \textstyle\sum_{T\in\mathrm{Conf}_0}\varphi(T)\cdot b_T + (1-\lambda)\cdot \sum_{T\in\mathrm{Conf}_0}\varphi(T)\cdot c_T.$$

In particular, the coefficient of $\varphi(S)$ has the relation $a_S = \lambda \cdot b_S + (1-\lambda)\cdot c_S$. Since $|b_S - a_S| \geq \varepsilon$, λ cannot be 1. This contradicts the extremity of ω.

For any finite subset C of Conf_0, put $T_n^*(C,\varepsilon) := T_n \backslash \bigcup_{S\in C}\big(T_n^+(S,\varepsilon) \cup T_n^-(S,\varepsilon)\big)$. Then $T_n^*(C,\varepsilon) \neq \phi$ for sufficiently large n, since $\lim\limits_{n\to\infty} \sharp T_n^*(C,\varepsilon)/\sharp T_n = 1$ due to the above fact. Let $\{C_m\}_{m\in\mathbb{Z}_{\geq 0}}$ be an increasing sequence of finite subsets of Conf_0 such that $\bigcup_{m\in\mathbb{Z}_{\geq0}} C_m = \mathrm{Conf}_0$ and let$\{\varepsilon_m\}_{m\in\mathbb{Z}_{\geq0}}$ be a sequence of real numbers with $\varepsilon_m \downarrow 0$. For each $m \in \mathbb{Z}_{\geq 0}$, choose any connected component of $T_n^*(C_m,\varepsilon_m)$, say T_m^*, for large n, and put $\omega_m := \mathcal{M}(T_m^*)/\sharp T_m^* = \sum_{S\in\mathrm{Conf}_0}\varphi(S)\cdot a_S^{(m)}$. By definition $|a_S - a_S^{(m)}| < \varepsilon_m$ for $S \in C_m$, which implies $\omega = \lim\limits_{m\to\infty}^{cl} \omega_m$. There are two cases to consider. i) Suppose $\exists$ a subsequence $\{\widehat{m}\}$ such that $\sharp T_{\widehat{m}}^*$ is bounded. Since $\sharp\{T \in \mathrm{Conf}_0\, | \sharp T \leq c\}$ for any constant c is finite, there exists $T \in \mathrm{Conf}_0$ which appears in $\{T_m^*\}_m$ infinitely often. So $\omega = \mathcal{M}(T)/\sharp T$ and $\overline{K}(\omega) = \mathcal{M}(T)/\sharp T \neq 0$. ii) Suppose $\sharp T_m^* \to \infty$. Then the formula (10.4.4) and (10.4.7) imply $\overline{K}(\omega) = 0$. $\square$

10.6 Function value representation of elements of $\overline{\log(\mathbf{EDP})}_\infty$

The coefficients a_S at $S \in \mathrm{Conf}_0$ of the sequential limit $\omega = \lim\limits_{n\to\infty}^{cl} \mathcal{M}(T_n)/\sharp T_n$ (10.3.2) are usually hard to calculate. However, in certain good cases, we represent the coefficient as a special value of a function in one variable t.

Given an expression of the form (10.3.2) of $\omega \in \overline{\log(\mathrm{EDP})}_\infty$ and an increasing sequence of integers $\{n_m\}_{n=0}^\infty$, we consider the following two formal power series in t.

$$P(t) \ := \ \sum_{m=0}^{\infty} \sharp T_m \cdot t^{n_m} \qquad \in \ \mathbb{Z}\,[\![t]\!]\,, \tag{10.6.1}$$

$$P\mathcal{M}(t) \ := \ \sum_{m=0}^{\infty} \mathcal{M}(T_m)\cdot t^{n_m} \ \in \ \mathcal{L}_{\mathbb{Q}}\,[\![t]\!] = \mathcal{L}_{\mathbb{Q}[\![t]\!]}\,, \tag{10.6.2}$$

where, using the basis expansion (8.2.3), the series $P\mathcal{M}(t)$ can be expanded as

$$P\mathcal{M}(t) \ = \ \textstyle\sum_{S\in\mathrm{Conf}_0} \varphi(S) PM(S,t),$$

whose coefficients at $S \in \mathrm{Conf}_0$ are given by

$$PM(S,t) \ := \ \partial_S P\mathcal{M}(t) \ = \ \sum_{m=0}^{\infty} A(S,T_m)\cdot t^{n_m} \in \mathbb{Q}\,[\![t]\!]\,. \tag{10.6.3}$$

Since $T_n \in \mathrm{Conf}_+$, one has $P(t) \neq 0$ and its radius of convergence is at most 1.

Lemma. *Suppose that the series $P(t)$ has a positive radius of convergence r. Then, for any $S \in \mathrm{Conf}_0$ (c.f. Remark), we have*

i) *The series $PM(S,t)$ converges at least in the radius r for $P(t)$. The radius of convergence of $PM(S,t)$ coincides with r, if $a_S := \lim\limits_{m\to\infty} \frac{M(S,T_m)}{\#T_m} \neq 0$.*

ii) *The following two limits in LHS and RHS give the same value:*

$$\lim_{t\uparrow r} \frac{PM(S,t)}{P(t)} \ = \ \lim_{n\to\infty} \frac{M(S,T_n)}{\#T_n}. \tag{10.6.4}$$

Here by the notation $t\uparrow r$ we mean that the real variable t tends to r from below.

iii) *The proportion $P\mathcal{M}(t)/P(t)$ for $t \uparrow r$ converges to ω (10.3.2):*

$$\omega \ = \ \lim_{t\uparrow r}^{cl} \frac{P\mathcal{M}(t)}{P(t)} \ = \ \sum_{S\in\mathrm{Conf}_0} \varphi(S) \ \lim_{t\uparrow r} \frac{PM(S,t)}{P(t)}. \tag{10.6.5}$$

Proof. Before proceeding to the proof, we recall two general properties of power series:

A) The radius of convergence of P(t) is $r := 1/\limsup\limits_{m\to\infty} \sqrt[n_m]{\sharp T_m}$ (Hadamard).

B) Since the coefficients $\sharp T_m$ of $P(t)$ are non-negative real numbers, $P(t)$ *is an increasing positive real function on the interval (0,r) and* $\lim\limits_{t\uparrow r} P(t) = +\infty$.

We now turn to the proof. Due to the linear relations among $M(S,T_m)$ for $S\in\mathrm{Conf}$ (8.3.2), it is sufficient to show the lemma only for the cases $S\in\mathrm{Conf}_0$.

i) Let us show that $PM(S,t)$ for $S\in\mathrm{Conf}_0$ has the radius r of convergence. Since we have $M(S,T_m)=A(S,T_m)$ (6.1 Remark 1), using (5.2.1), we have

$$\limsup_{m\to\infty} \sqrt[n_m]{M(S,T_m)} \ \le\ \limsup_{m\to\infty} \sqrt[n_m]{\#T_m}\sqrt[n_m]{q^{\#S-1}/\#\,\mathrm{Aut}(S)} \ =\ 1/r.$$

This proves the first half of i). The latter half is shown in the next ii).

ii) We show that the convergence of the sequence $A(S,T_m)/\sharp T_m$ to some $a_S\in\mathbb{R}$ implies the convergence of the values of the function $PM(S,t)/P(t)$ to a_S as $t\uparrow r$. The assumption implies that for any $\varepsilon>0$, there exists $N>0$ such that $|A(S,T_m)/\sharp T_m - a_S|\le\varepsilon$ for all $m\ge N$. Therefore,

$$\begin{aligned}\left|\ \tfrac{PM(S,t)}{P(t)}-a_S\ \right| \ &=\ \frac{|\ Q_N(t)+\sum_{m=N}^{\infty}\Big(A(S,T_m)-a_S\cdot\sharp T_m\Big)t^{n_m}\ |}{P(t)}\\ &\le\ \tfrac{|\ Q_N(t)-\varepsilon\sum_{m=0}^{N-1}\#T_m t^{n_m}\ |}{P(t)}+\varepsilon\end{aligned}$$

where $Q_N(t):=\sum_{m<N}\Big(A(S,T_m)-a_S\cdot\sharp T_m\Big)t^{n_m}$ is a polynomial in t. Due to the above B), the first term of the last line tends to 0 as $t\uparrow r$. Hence, $|PM(S,t)/P(t)-a_S|\le 2\varepsilon$ for t sufficiently close to r. This proves (10.6.4).

If $a_S\neq 0$, then $\lim\limits_{t\uparrow r} PM(S,t)=\infty$ since $\lim\limits_{t\uparrow r}P(t)=\infty$. Thus, the radius of convergence of $PM(S,t)$ is less or equal than r. This proves the latter half of i).

iii) We have only to recall that the classical topology on $\mathcal{L}_{\mathbb{R}}$ is the same as coefficient-wise convergence with respect to the basis $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$. $\square$

Corollary. *If $P(t)$ and $PM(S,t)$ $(S\in\mathrm{Conf}_0)$ extend to meromorphic functions at t=r, then $PM(S,t)/P(t)$ is regular at t=r and one has*

$$\omega \ =\ \sum_{S\in\mathrm{Conf}_0}\varphi(S)\ \frac{PM(S,t)}{P(t)}\Big|_{t=r}. \tag{10.6.6}$$

Proof. We have to show that $PM(S,t)/P(t)$ becomes holomorphic at $t=r$ under the assumption. If it were not holomorphic, it would have a pole at $t=r$ and hence $\lim\limits_{t\uparrow r}PM(S,t)/P(t)$ diverges. On the other hand, in view of (5.2.2), one has the inequality $0\le PM(S,t)\le P(t)\cdot q^{\sharp S-1}/\mathrm{Aut}(S)$ for $t\in(0,r)$. Then the positivity of $P(t)$ implies the boundedness: $0\le PM(S,t)/P(t)\le q^{\sharp S-1}/\mathrm{Aut}(S)$ for $t\in(0,r)$. This is a contradiction. $\square$

We sometimes call (10.6.6) a *residual expression of* ω, since the coefficients are given by the proportions of residues of meromorphic functions.

Remark. **1.** The equality (10.6.4) gives the following important replacement. Namely, the RHS, which is a sequential limit of rational numbers and is hard to determine in general, is replaced by the LHS, which is the limit of value of a function in a variable t at the special point $t = r$ where r is often a real algebraic number whose defining equation is easily calculable.

2. The convergence of the sequence $\lim\limits_{n\to\infty}^{cl} \frac{M(S,T_n)}{\#T_n}$ does not imply the convergence of the series $PM(S,t)$ and $P(t)$ in a positive radius. Conversely, the convergence of the series $PM(S,t)$ and $P(t)$ in a positive radius does not imply the convergence of the sequence $\lim\limits_{n\to\infty}^{cl} \frac{M(S,T_n)}{\#T_n}$.

§11. Limit elements for a finitely generated group.

We apply the space $\mathcal{L}_{\mathbb{R},\infty}$ to the study of finitely generated groups.

For any pair (Γ, G) consisting of a group Γ and its finite generator system G, we introduce the limit set $\Omega(\Gamma, G)$ as a subset of $\mathcal{L}_{\mathbb{R},\infty}$. We introduce also another limit set $\Omega(P_{\Gamma,G})$ associated to the Poincare series $P_{\Gamma,G}(t)$ of (Γ, G), and a natural proper surjective map $\pi : \Omega(\Gamma, G) \to \Omega(P_{\Gamma,G})$. Then, the main goal of this section is the trace formula in 11.4, where the trace of a fiber of the map π is represented by residues of the Poincare series $P_{\Gamma,G}(t)$ and $P_{\Gamma,G}\mathcal{M}(t)$.

11.1 The limit set $\Omega(\Gamma, G)$ for a finitely generated group

Let (Γ, G) be a pair consisting of a group Γ and its finite generator system G such that $G = G^{-1}$ and $e \notin G$. We denote also by (Γ, G) the associated colored oriented Cayley graph (2.1 Example 1). In fact, Γ has the bound $\#(G)$ of the valency (c.f. 2.2). In this section, we use this coloring system G and the valency $\#(G)$ as for the definition of Conf in (2.2.1). The set of all isomorphism classes of finite subgraphs of (Γ, G) is denoted by $\langle \Gamma, G\rangle$. Put $\langle \Gamma, G\rangle_0 := \langle \Gamma, G\rangle \cap \mathrm{Conf}_0$.

The length of $\gamma \in \Gamma$ with respect to G is defined (as usual [M]) by

$$(11.1.1)\quad \begin{aligned} \ell_G(\gamma) :=\ & \inf\ \{n \in \mathbb{Z}_{\geq 0} \mid \gamma = g_1 \cdots g_n \text{ for some } g_i \in G\ (i = 1, \ldots, n)\} \\ =\ & \text{the distance between } \gamma \text{ and the unit } e \text{ in Cayley graph } \Gamma. \end{aligned}$$

For $n \in \mathbb{Z}_{\geq 0}$, consider the "balls" of radius n of (Γ, G) defined by

$$(11.1.2)\qquad \Gamma_n\ :=\ \{\ \gamma \in \Gamma \mid \ell_G(\gamma) \leq n\ \}.$$

We shall denote $\dot{\Gamma}_n := \Gamma_n \backslash \Gamma_{n-1}$ for $n \in \mathbb{Z}_{\geq 0}$. So far there is no confusions, we shall denote by Γ_n its isomorphism class $[\Gamma_n] \in \mathrm{Conf}_0$ also.

Definition. The *set of limit elements for* (Γ, G) is defined by

$$(11.1.3)\qquad \Omega(\Gamma, G)\ :=\ \mathcal{L}_{\mathbb{R},\infty} \cap \overline{\left\{ \frac{\mathcal{M}(\Gamma_n)}{\sharp\Gamma_n} \mid n \in \mathbb{Z}_{\geq 0} \right\}}\ ,$$

where $\overline{A}$ is the closure of a subset $A \subset \mathcal{L}_{\mathbb{R}}$ with respect to the classical topology.

Fact. *The limit set $\Omega(\Gamma, G)$ is non-empty if and only if Γ is infinite.*

Proof. Since $\left\{\frac{\mathcal{M}(\Gamma_n)}{\sharp\Gamma_n} \mid n \in \mathbb{Z}_{\geq 0}\right\} \subset \log(EDP)$ and $\overline{\log(EDP)}$ is compact (10.3), the sequence $\left\{\frac{\mathcal{M}(\Gamma_n)}{\sharp\Gamma_n} \mid n \in \mathbb{Z}_{\geq 0}\right\}$ always has accumulation points. Due to (10.4.4) and (10.4.7), an accumulation point ω belongs to $\mathcal{L}_{\mathbb{R},\infty}$, i.e. it satisfies the kabi-condition $\overline{K}(\omega) = 0$, if and only if $\#\Gamma_n \to \infty$. □

Since $\overline{\log(EDP)}$ is metrizable, any element ω in $\Omega(\Gamma, G)$ can be expressed as a sequential limit. That is, there exists a subsequence $n_m \uparrow \infty$ of $n \uparrow \infty$ such that

$$\omega = \lim_{n_m \to \infty}{}^{cl} \frac{\mathcal{M}(\Gamma_{n_m})}{\sharp\Gamma_{n_m}} = \sum_{S \in \langle\Gamma, G\rangle_0} \varphi(S) \lim_{n_m \to \infty} \frac{A(S, \Gamma_{n_m})}{\sharp\Gamma_{n_m}} \tag{11.1.4}$$

where the coefficient of $\varphi(S)$ is convergent for all S.

Definition. We call a finitely generated group (Γ, G) *simple* (resp. *finite*) *accumulating* if $\Omega(\Gamma, G)$ consists of a single (resp. finite number of) element(s).

In the following Examples 1. and 2., we show that any polynomial growth group and any free group is simple accumulating. We first state some general properties of the set Γ_n, which are immediate consequences of the definition.

Fact. 1. For $m, n \in \mathbb{Z}_{\geq 0}$, one has a natural surjection:

$$\Gamma_n \times \Gamma_m \longrightarrow \Gamma_{m+n}, \ \gamma \times \delta \mapsto \gamma\delta \tag{11.1.5}$$

2. For any $S \in \mathrm{Conf}_0$ with $S \leq \Gamma_k$ $(k \in \mathbb{Z}_{\geq 0})$ and for any $n \in \mathbb{Z}_{\geq 0}$, one has:

$$\#\Gamma_{n-k} \leq \#(\mathrm{Aut}(S)) \cdot A(S, \Gamma_n) \leq \#\Gamma_n. \tag{11.1.6}$$

Proof. 1. Obvious by definition.

2. By the assumption on S, there exists a subgraph $\mathbb{S} \subset \Gamma_k$ such that $S = [\mathbb{S}]$. Note that $Aut(S) \simeq Aut(\mathbb{S}) = \{g \in \Gamma \mid g\mathbb{S} = \mathbb{S}\}$ and is finite. Consider a map p from Γ to the set of subgraphs of (Γ, G) defined by $p(g) := g\mathbb{S}$, and define an equivalence relation $\sim$ on Γ by "$g \sim h \Leftrightarrow g\mathbb{S} = h\mathbb{S} \Leftrightarrow g^{-1}h \in Aut(\mathbb{S})$". Then, one has $A(S, \Gamma_n) \geq \#(\mathrm{Image}(p|_{\Gamma_{n-k}})) = \#(\Gamma_{n-k}/\sim) \geq \#(\Gamma_{n-k})/\#(Aut(\mathbb{S}))$. This implies the first inequality.

Suppose further that $e \in \mathbb{S}$, by replacing $\mathbb{S}$ and k if necessary. Consider a set $P := \{g \in \Gamma_n \mid g\mathbb{S} \subset \Gamma_n\}$. Then, by the assumption $e \in \mathbb{S}$, the map $p|_P : P \to \mathbb{A}(S, \Gamma_n)$ is surjective and P is closed under the right multiplication of $Aut(\mathbb{S})$. Then, one has $A(S, \Gamma_n) = \#(P)/\#(Aut(\mathbb{S})) \leq \#(\Gamma_n)/\#(Aut(\mathbb{S}))$. This implies the second inequality. □

Example. **1.** Let (Γ, G) be a group of polynomial growth. This is the case when Γ contains a finitely generated nilpotent group of finite index (Wolf and Gromov [Gr1]). There exist $c, d \in \mathbb{Z}_{>0}$ such that $\#\Gamma_n = cn^d + o(n^d)$ (Pansu [P]). Then, for any $S \in \langle \Gamma, G\rangle_0$, applying (11.1.6), one obtains

$$\lim_{n\to\infty} \frac{A(S,\Gamma_n)}{\#\Gamma_n} = \frac{1}{\#(\mathrm{Aut}(S))}. \tag{11.1.7}$$

Corollary. *If Γ is of polynomial growth, then it is simple accumulating for any generator system G and the limit element is given by*

$$\omega_{\Gamma,G} := \sum_{S\in\langle\Gamma,G\rangle_0} \frac{1}{\#(\mathrm{Aut}(S))}\varphi(S). \tag{11.1.8}$$

2. *Let F_f be a free group with the generator system $G = \{g_1^{\pm1}, \cdots, g_f^{\pm1}\}$ for $f \in \mathbb{Z}_{\geq 2}$. Then (F_f, G) is simple accumulating. The limit element is given by*

$$\omega_{F_f,G} := \sum_{k=0}^{\infty} (2f-1)^{-k} \left(\sum_{\substack{S\in\langle\Gamma,G\rangle_0\\ d(S)=2k}} \varphi(S) + f^{-1} \sum_{\substack{S\in\langle\Gamma,G\rangle_0\\ d(S)=2k+1}} \varphi(S) \right), \tag{11.1.9}$$

where $d(S) := \max\{d(x,y) \mid x, y \in S\}$ is the diameter of S for $S \in \langle\Gamma, G\rangle_0$.

Proof. The induction relation: $\#\Gamma_{n+1} - (2f-1)\#\Gamma_n = 2$ with the initial condition $\#\Gamma_0 = 1$ implies $\#\Gamma_n = \frac{f(2f-1)^n - 1}{f-1}$ for $n \in \mathbb{Z}_{\geq 0}$. Then, for $n \geq [d(S)/2]$,

$$A(S,\Gamma_n) = \begin{cases} \frac{f(2f-1)^{n-[d(S)/2]}-1}{f-1} & \text{if } d(S) \text{ is even,} \\ \frac{(2f-1)^{n-[d(S)/2]}-1}{f-1} & \text{if } d(S) \text{ is odd.} \end{cases}$$

$$\lim_{n\to\infty} \frac{A(S,\Gamma_n)}{\#\Gamma_n} = \begin{cases} (2f-1)^{-[d(S)/2]} & \text{if } d(S) \text{ is even,} \\ f^{-1}(2f-1)^{-[d(S)/2]} & \text{if } d(S) \text{ is odd.} \end{cases}$$

We have only to prove the first formula. Depending on whether $d(S)$ is even or odd, S has either one or two central points. Then it is easy to see the following one to one correspondence: *an embedding of S in Γ_n $\Leftrightarrow$ an embedding of the central point(s) of S in Γ_n such that the distance from the point to the boundary of Γ_n is at least half of the diameter* $[d(S)/2]$. Taking this into account, we can calculate directly the formula. □

11.2 The space $\Omega(P_{\Gamma,G})$ of opposite sequences

We introduce another accumulation set $\Omega(P)$, called *the space of opposite sequences*, associated to certain real power series $P(t)$. Under a suitable assumption on (Γ, G), we have a fibration $\pi : \Omega(\Gamma, G) \to \Omega(P_{\Gamma,G})$ for the Poincare series $P_{\Gamma,G}$ of (Γ, G), which is equivariant with some actions $\tilde{\tau}_\Omega$ and τ_Ω.

We start with a general definition. Consider a power series in t

(11.2.1) $$P(t) \;=\; \sum_{n=0}^{\infty} \gamma_n t^n$$

whose coefficients are real numbers. We assume that there exist positive real numbers u, v (depending on P) such that $u \leq \gamma_{n-1}/\gamma_n \leq v$ for all $n \in \mathbb{Z}_{\geq 1}$. This, in particular, implies that P is convergent of radius r with $u \leq r \leq v$.

Example. If the sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is increasing and semi-multiplicative $\gamma_{m+n} \leq \gamma_m \gamma_n$, we may choose $u = 1/\gamma_1$ and $v = 1$. For example, let $\gamma_n := \#\Gamma_n$ $(n \in \mathbb{Z}_{\geq 0})$ in the setting of 11.1, then (11.1.5) implies semi-multiplicativity.

Associated to P, consider a sequence $\{X_n(P)\}_{n\in\mathbb{Z}_{\geq 0}}$ of polynomials:

(11.2.2) $$X_n(P) \;:=\; \sum_{k=0}^{n} \frac{\gamma_{n-k}}{\gamma_n}\, s^k, \quad n = 0, 1, 2, \cdots,$$

in the space $\mathbb{R}[\![s]\!]$ of formal power series, where $\mathbb{R}[\![s]\!]$ is equipped with the formal classical topology, i.e. the product topology of convergence of every coefficient in classical topology. Since each coefficients of $X_n(P)$ are bounded, i.e. $u^k \leq \frac{\gamma_{n-k}}{\gamma_n} \leq v^k$, the sequence accumulates to a non-empty compact set:

(11.2.3) $\Omega(P) :=$ the set of accumulation points of the sequence (11.2.2).

An element $a(s) = \Sigma_{k=0}^{\infty} a_k s^k$ of $\Omega(P)$ is called an *opposite series.* The coefficients $\{a_k\}_{k=0}^{\infty}$ satisfies $u^k \leq a_k \leq v^k$. We call a_1 the *initial* of the opposite series a, denoted by $\iota(a)$. Let us introduce the space of the initials:

(11.2.4) $\Omega_1(P) :=$ the set of accumulation points of the sequence $\left\{\frac{\gamma_{n-1}}{\gamma_n}\right\}_{n\in\mathbb{Z}_{\geq 0}}$,

which is a compact subset of the positive interval $[u, v]$. The projection map $a \in \Omega(P) \mapsto \iota(a) \in \Omega_1(P)$ is a surjective map.

Assertion. **1.** *If a sequence $\{X_{n_m}(P)\}_{m\in\mathbb{Z}_{\geq 0}}$ converges to an opposite sequence a, then the sequence $\{X_{n_m-1}(P)\}_{m\in\mathbb{Z}_{\geq 0}}$ converges also to an opposite sequence, denoted by $\tau_\Omega(a)$. We have*

(11.2.5) $$\tau_\Omega(a) \;=\; (a-1)/\iota(a)s.$$

2. *Consider a map*

(11.2.5)* $$\tau : \Omega(P) \longrightarrow \overline{\mathbb{R}\Omega}(P), \quad a \;\mapsto\; \iota(a)\tau_\Omega(a)$$

where $\overline{\mathbb{R}\Omega}(P)$ is a closed $\mathbb{R}$-linear subspace of $\mathbb{R}[\![s]\!]$ generated by $\Omega(P)$. Then, the map τ naturally extends to an endomorphism of $\overline{\mathbb{R}\Omega}(P)$.

Proof. 1. By definition, the sequence $\{\gamma_{n_m-1}/\gamma_{n_m}\}_m$ converges to the non-zero initial $\iota(a) \neq 0$. Then, for any fixed $k > 0$, the $k-1$th coefficient of $\tau_\Omega(a)$ is given by the limit of sequence $\{\gamma_{n_m-k}/\gamma_{n_m-1}\}_m$ converging to a_k/a_1.

2. Let $\sum_{i\in I} c_i a_i(s) = 0$ be a linear relation among opposite sequences $a_i(s)$ $(i\in I)$ with $\#I<\infty$, then we also have a linear relation $\sum_{i\in I} c_i a_{i,1}\tau_\Omega(a_i(s)) = 0$, since, using the expression (11.2.5), this follows from the original relation $\sum_{i=1}^{\infty} c_i a_i(s) = 0$ and another one $\sum_{i=1}^{\infty} c_i = 0$, which is obtained by substituting $s=0$ in the first relation. This implies that τ is extended to a linear map: $\mathbb{R}\Omega(P) \to \overline{\mathbb{R}\Omega}(P)$. On the other hand, $a(s) \in \mathbb{R}[\![s]\!] \mapsto (a(s)-a(0))/s \in \mathbb{R}[\![s]\!]$ is a well-defined continuous map, so that it induces a map $\mathrm{End}_{\mathbb{R}}(\overline{\mathbb{R}\Omega}(P))$. □

We return to the setting in 11.1 and consider a Cayley graph (Γ, G). For the sequence $\{\Gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ (11.1.2), we consider two series (10.6.1) and (10.6.2):

(11.2.6) $$P_{\Gamma,G}(t) \ := \ \textstyle\sum_{n=0}^{\infty} \sharp\Gamma_n \cdot t^n,$$

(11.2.7) $$P_{\Gamma,G}\mathcal{M}(t) \ := \ \textstyle\sum_{n=0}^{\infty} \mathcal{M}(\Gamma_n)\cdot t^n.$$

Here (11.2.6) is well known [M] as the growth (or Poincare) series for (Γ, G), and (11.2.7) is the series which we study in the present paper. Due to (11.1.5), it is well known that the growth series converges with positive radius:

(11.2.8) $$r_{\Gamma,G} \ := \ 1 \ / \ \lim_{n\to\infty} \sqrt[n]{\#\Gamma_n} \ \geq \ 1 \ / \ \#\Gamma_1.$$

Due to 10.6 *Lemma* i), the series $P_{\Gamma,G}\mathcal{M}(t)$ converges in the same radius as $P_{\Gamma,G}(t)$. This fact can be directly confirmed by using (11.1.6) for $S \leq [\Gamma_k]$ as $\lim\limits_{n\to\infty} (\sqrt[n-k]{\#\Gamma_{n-k}})^{\frac{n-k}{n}} \leq \lim\limits_{n\to\infty} \sqrt[n]{\#(\mathrm{Aut}(S))A(S,\Gamma_n)} \leq \lim\limits_{n\to\infty} \sqrt[n]{\#\Gamma_n}$.

Let us consider the continuous linear projection map:

(11.2.9) $$\pi \ : \ \mathcal{L}_{\mathbb{R}}\langle\Gamma, G\rangle \longrightarrow \ \mathbb{R}[\![s]\!] , \quad \textstyle\sum_{S\in\mathrm{Conf}_0} \varphi(S)\cdot a_S \mapsto \sum_{k=0}^{\infty} a_{\Gamma_k} s^k.$$

In order that the π induces the map π_Ω (11.2.12), we consider the next two conditions **S** and **I** on the graph (Γ, G). First, recall that an element $g\in\Gamma$ is called *dead* with respect to G if $\ell_G(gx) \leq \ell_G(g)$ $\forall x\in G$ (Bogopolski [Bo], c.f. dead states [E2]).[1] We denote by $D(\Gamma, G)$ the set of dead elements in Γ.

- **S:** The portion $\frac{\#(\Gamma_n\cap D(\Gamma,G))}{\#(\Gamma_n)}$ tends to 0 as $n\to\infty$.
- **I:** For any connected subgraph $\mathbb{S}$ of (Γ, G) and any element $g\in\Gamma$, the equality $\mathbb{S}\Gamma_1 = g\mathbb{S}\Gamma_1$ implies $\mathbb{S} = g\mathbb{S}$, where $\mathbb{S}\Gamma_1 := \cup_{\alpha\in\mathbb{S}}\alpha\Gamma_1$.

Assumption. From now on, we assume the conditions **S** and **I** hold for (Γ, G).

[1] The author is grateful to Takefumi Kondo, who informed him the concept "dead element" and the recent works on it.

Remark. **1.** Bogopolski ([Bo] Question(2)) asked whether **S** holds for arbitrary finite generator system G of a group Γ automatically.

2. Since $\mathrm{Aut}(S)$ is a finite group for any $S\in\langle\Gamma,G\rangle_0$, it is trivial if Γ is torsion free. Then, **I** holds automatically for arbitrary finite generator system.

3. If Γ has a torsion element g of order $d>1$, then for any generator system G, define a new generator system $G' := \cup_{i,j=0,\cdots,d-1}(g^i\Gamma_1 g^j)\setminus\{e\}$. Then, the new unit ball $\Gamma_1' := G'\cup\{e\}$ satisfies $\Gamma_1' = g\Gamma_1'$. That is, the condition **I** fails for $\mathbb{S}:=\{e\}$. It is an open question whether, for any finitely generated infinite group Γ, there always exists a generator system G satisfying **I**, or not.

Notation. For any $S\in\langle\Gamma,G\rangle_0$, we denote by $S\Gamma_1$ the isomorphism class $[\mathbb{S}\Gamma_1]$ for any representative $\mathbb{S}$ of S. We regard $\mathcal{L}_{\mathbb{R}}\langle\Gamma,G\rangle$ as an $\mathbb{R}[\![s]\!]$-module by letting s act on the basis by $\varphi(S)\mapsto\varphi(S\Gamma_1)$ and extending the action formally to $\mathbb{R}[\![s]\!]$. However, *the map* π (11.2.9) *is not an* $\mathbb{R}[\![s]\!]$*-homomorphism.*

Let us state some consequences of the assumptions **S** and **I**. Recall the notation (5.1.1) and (5.1.2)).

Assertion. *For any* $S\in\langle\Gamma,G\rangle_0$*, one has the inequalities:*

$$0\le A(S\Gamma_1,\Gamma_n)-A(S,\Gamma_{n-1})\le \#S\cdot\#(\dot\Gamma_n\cap D(\Gamma,G)). \tag{11.2.10}$$

Proof. Consider a map $\mathbb{S}\in\mathbb{A}(S,\Gamma_{n-1})\mapsto\mathbb{S}\Gamma_1\in\mathbb{A}(S\Gamma_1,\Gamma_n)$. Then the condition **I** implies the injectivity of the map. This implies the first inequality. Next, consider an element $\mathbb{S}\Gamma_1\in\mathbb{A}(S\Gamma_1,\Gamma_n)$ for $\mathbb{S}\subset\Gamma_n$ with $[\mathbb{S}]=S$. It can not be an image of the map (i.e. $\mathbb{S}\not\subset\Gamma_{n-1}$) if and only if $\mathbb{S}\cap\dot\Gamma_n\cap D(\Gamma,G)\neq\emptyset$. Such $\mathbb{S}$ is of the form $ds^{-1}\mathbb{S}_0$ for $d\in\dot\Gamma_n\cap D(\Gamma,G)$ and $s\in\mathbb{S}_0$ for a fixed $\mathbb{S}_0$ with $[\mathbb{S}_0]=S$. Thus the number of such $\mathbb{S}\Gamma_1$ is at most $\#(S)\cdot\#(\dot\Gamma_n\cap D(\Gamma,G))$. This implies the second inequality. □

Corollary. *For* $n,k\in\mathbb{Z}_{\ge0}$ *with* $n-k\ge0$*, one has the inequalities:*

$$0\le A(\Gamma_k,\Gamma_n)-\#(\Gamma_{n-k})\le\#(\Gamma_{k-1})\#(\Gamma_n\cap D(\Gamma,G)). \tag{11.2.11}$$

Proof. We show by induction on k, where $k=0$ is trivial (put $\#\Gamma_{-1}:=0$). Assume for $k-1$. Let n be an integer with $n\ge k$. Applying (11.2.10) for $S=\Gamma_{k-1}$, one has $0\le A(\Gamma_k,\Gamma_n)-\mathbb{A}(\Gamma_{k-1},\Gamma_{n-1})\le\#\Gamma_{k-1}\cdot\#(\dot\Gamma_n\cap D(\Gamma,G))$. This together with the induction hypothesis implies (11.2.11). □

We state about the comparison of $\Omega(\Gamma,G)$ and $\Omega(P_{\Gamma,G})$ by the map π, which is the main result of the present section.

Lemma. **1.** *If* $\lim\limits_{n_m\to\infty}^{cl}\frac{\mathcal{M}(\Gamma_{n_m})}{\sharp\Gamma_{n_m}}$ *converges to an element* $\omega\in\Omega(\Gamma,G)$*, then* $\lim\limits_{n_n\to\infty}^{cl}X_{n_m}(P_{\Gamma,G})$ *converges, too, to the* π*-image of* ω*. We denote by*

$$\pi_\Omega\ :\ \Omega(\Gamma,G)\ \longrightarrow\ \Omega(P_{\Gamma,G}) \tag{11.2.12}$$

the induced map. π_Ω is a continuous map, which is surjective.

2. *If a sequence* $\left\{\frac{\mathcal{M}(\Gamma_{n_m})}{\#\Gamma_{n_m}}\right\}_{m\in\mathbb{Z}_{\geq 0}}$ *converges to an element* $\omega \in \Omega(\Gamma, G)$, *then the sequence* $\left\{\frac{\mathcal{M}(\Gamma_{n_m-1})}{\#\Gamma_{n_m-1}}\right\}_{m\in\mathbb{Z}_{\geq 1}}$ *converges also to an element, denoted by* $\tilde{\tau}_\Omega(\omega)$. *For an element* $\omega = \sum_{S\in\langle\Gamma,G\rangle_0} a_S\varphi(S) \in \Omega(\Gamma, G)$, *one has*

$$\tilde{\tau}_\Omega(\omega) \;=\; \frac{1}{\iota(\pi_\Omega(\omega))}\sum_{S\in\langle\Gamma,G\rangle_0} .a_{S\Gamma_1}\varphi(S). \tag{11.2.13}$$

Using the notation ∂_S and $\partial_{S\Gamma_1}$ for $S \in \langle\Gamma, G\rangle_0$ in §8.1, this is equivalent to

$$\partial_S(\tilde{\tau}_\Omega\omega) \;=\; \frac{1}{\iota(\pi_\Omega(\omega))}\partial_{S\Gamma_1}(\omega). \tag{11.2.13*}$$

Then, π_Ω (11.2.12) is equivariant with respect to the actions $\tilde{\tau}_\Omega$ and τ_Ω.

3. *Let us denote by $\overline{\mathbb{R}\Omega}(\Gamma, G)$ the closed $\mathbb{R}$-linear subspace of $\mathcal{L}_{\mathbb{R},\infty}$ generated by $\Omega(\Gamma, G)$. Define a map $\tilde{\tau}$ from $\Omega(\Gamma, G)$ to $\overline{\mathbb{R}\Omega}(\Gamma, G)$ by*

$$\tilde{\tau}(\omega) \;:=\; \iota(\pi_\Omega(\omega))\;\tilde{\tau}_\Omega(\omega). \tag{11.2.14}$$

Then, $\tilde{\tau}$ naturally extends to an $\mathbb{R}$-linear endomorphism of $\overline{\mathbb{R}\Omega}(\Gamma, G)$.

4. *The restriction of π (11.2.9) (= the $\mathbb{R}$-linear extension of π_Ω):*

$$\pi \;:\; \overline{\mathbb{R}\Omega}(\Gamma, G) \longrightarrow \overline{\mathbb{R}\Omega}(P_{\Gamma,G}). \tag{11.2.15}$$

is equivariant with respect to the endomorphisms $\tilde{\tau}$ and τ, i.e. $\tau\circ\pi = \pi\circ\tilde{\tau}$.

Proof. 1. Using (8.2.3), (11.2.2) and (11.2.6), we see that the difference $\pi\left(\frac{\mathcal{M}(\Gamma_n)}{\#\Gamma_n}\right) - X_n(P_{\Gamma,G})$ is a polynomial in s of degree $\leq n$ whose kth coefficient is $(A(\Gamma_k, \Gamma_n) - \#\Gamma_{n-k})/\#\Gamma_n$. Put $n = n_m$ and take the limit by tending $m \to \infty$. Applying (11.2.11) and the assumption **S**, we see that this tends to 0. That is, kth coefficient of $X_{n_m}(P_{\Gamma,G})$ tends to the coefficient a_{Γ_k} at Γ_k of ω. That is, $X_{n_m}(P_{\Gamma,G})$ converges to the π-image of ω. Thus (11.2.12) is established.

To show the surjectivity, use the compactness of $\overline{\log(EDP)}$ (10.3).

2. For $S\in\langle\Gamma, G\rangle_0$ and $n\in\mathbb{Z}_{\geq 1}$, one has

$$*)\qquad \frac{A(S,\Gamma_{n-1})}{\#\Gamma_{n-1}} \;=\; \left(\frac{A(S\Gamma_1,\Gamma_n)}{\#\Gamma_n} - \frac{A(S\Gamma_1,\Gamma_n) - A(S,\Gamma_{n-1})}{\#\Gamma_n}\right)\Big/\frac{\#\Gamma_{n-1}}{\#\Gamma_n}.$$

Let the sequence $\frac{\mathcal{M}(\Gamma_{n_m})}{\#\Gamma_{n_m}}$ associated to a subsequence $\{n_m\}_{m\in\mathbb{Z}_{\geq 0}}$ of $\mathbb{Z}_{\geq 0}$ converges to an element $\omega = \sum_{S\in\langle\Gamma,G\rangle_0} a_S\varphi(S) \in \Omega(\Gamma, G)$. Put $n = n_m$ in $*$) and let the index m run to ∞. The first (resp. second) term in the bracket in the RHS of $*$) converges to $a_{S\Gamma_1}$ (resp. 0 due to (11.2.10) and the assumption **S**). The denominator of the RHS of $*$) converges to the initial $\iota(\pi(\omega))$. Consequently, the LHS of $*$) converges to $\frac{1}{\iota(\pi(\omega))}a_{S\Gamma_1}$ for all S. This implies the convergence of $\lim\limits_{m\to\infty}{}^{cl}\frac{\mathcal{M}(\Gamma_{n_m-1})}{\#\Gamma_{n_m-1}}$ and the formula (11.2.13) and (11.2.13)*.

Let $a = \pi_\Omega(\omega)$ $(:= \sum_{k=0}^{\infty} a_{\Gamma_k} s^k)$. Then, comparing the formulae (11.2.5) and (11.2.13), one calculates: $\pi_\Omega(\tilde{\tau}_\Omega(\omega)) = \frac{1}{\iota(\pi(\omega))}\sum_{k=0}^{\infty} a_{\Gamma_k\Gamma_1} s^k = \frac{1}{\iota(\pi(\omega))}\sum_{k=0}^{\infty} a_{\Gamma_{k+1}} s^k$ $= \frac{1}{\iota(\pi(\omega))}\sum_{l=1}^{\infty} a_{\Gamma_l} s^{l-1} = \tau_\Omega(a) = \tau_\Omega(\pi_\Omega(\omega))$. This implies the equivariance of π_Ω.

3. Let (r): $\sum_{i\in I} c_i\omega_i = 0$ be a linear relation for $\omega_i \in \Omega(\Gamma, G)$ and $c_i \in \mathbb{R}$ $(i \in I)$ with $\#(I) < \infty$. Let us show the linear relation (s): $\sum_{i\in I} c_i\tilde{\tau}(\omega_i) = 0$. Let us expand $\omega_i = \sum_S a_{S,i}\varphi(S)$. Then, the relation (r) is expressed as the relations $\sum_{i\in I} c_i a_{S,i} = 0$ of coefficients for all $S \in \langle\Gamma, G\rangle_0$. Then the relation (s) is expressed as $\sum_{i\in I} c_i a_{S\Gamma_1,i} = 0$ for all $S \in \langle\Gamma, G\rangle_0$, which are a part of the former relations of the coefficients and are automatically satisfied.

This implies that $\tilde{\tau}$ extends to a linear map $\mathbb{R}\Omega(\Gamma, G) \to \overline{\mathbb{R}\Omega}(\Gamma, G)$. On the other hand, the correspondence $\sum_{S\in\langle\Gamma,G\rangle} a_S\varphi(S) \mapsto \sum_{S\in\langle\Gamma,G\rangle} a_{S\Gamma_1}\varphi(S)$ defines a redefined continuous linear map from a closed subspace of $\mathcal{L}_{\mathbb{R}}$ to itself, which induces the endomorphism $\tilde{\tau} \in \mathrm{End}_{\mathbb{R}}(\overline{\mathbb{R}\Omega}(\Gamma, G))$.

4. Let the notation be as in 1. Comparing (11.2.5)* and (11.2.14), one calculates: $\pi(\tilde{\tau}(\omega)) = \pi(\iota(\pi(\omega))\tilde{\tau}_\Omega(\omega)) = \iota(\pi(\omega))\pi_\Omega(\tilde{\tau}_\Omega(\omega)) = \iota(\pi(\omega))\tau_\Omega(\pi_\Omega(\omega)) =$ $\iota(\pi(\omega))\tau_\Omega(a) = \tau(a) = \tau(\pi(\omega))$. This implies the equivariance of π. $\square$

The map π_Ω (11.2.12) is conjecturally a finite map. We shall see in 11.4 that the "traces" of fibers of π are represented by suitable "residue values" of the functions (11.2.6) and (11.2.7). The key to understand this is the *"duality" between the limit space $\Omega(P_{\Gamma,G})$ and the space of singularities $Sing(P_{\Gamma,G})$ of the series $P_{\Gamma,G}(t)$ on the circle of the convergent radius $r_{\Gamma,G}$.* In next section 11.3, we study the "duality" in case $\Omega(P_{\Gamma,G})$ is finite (see **Theorem** and (11.3.14) and (11.3.15)). However, for general $\Omega(P_{\Gamma,G})$, the "duality" is yet undefined.

In the following, we give an example of (Γ, G), where $\Omega(P_{\Gamma,G})$ consists of two elements $a^{[0]}$ and $a^{[1]}$, and τ_Ω acts on $\Omega(P_{\Gamma,G})$ as their transposition. However, we note that $\tau^2 \neq \mathrm{id}$ and $\det(t\cdot\mathrm{id} - \tau) = t^2 - 1/2$.

Example. (Machì) Let $\Gamma := \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ and $G := \{a, b^{\pm 1}\}$ where a, b are the generators of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, respectively. Then, Machì has shown

$$P_{\Gamma,G}(t) := \sum_{k=0}^{\infty} \#\Gamma_k t^k = \frac{(1+t)(1+2t)}{(1-2t^2)(1-t)},$$

so that $\#\Gamma_{2k} = 7\cdot 2^k - 6$ and $\#\Gamma_{2k+1} = 10\cdot 2^k - 6$ for $k \in \mathbb{Z}_{\geq 0}$. Then, one has

$$\Omega_1(P_{\Gamma,G}) = \Big\{\iota(a^{[0]}) := \lim_{n\to\infty}\frac{\#\Gamma_{2n-1}}{\#\Gamma_{2n}} = \frac{5}{7} \ \&\ \iota(a^{[1]}) := \lim_{n\to\infty}\frac{\#\Gamma_{2n}}{\#\Gamma_{2n+1}} = \frac{7}{10}\Big\},$$

and, hence $h_{\Gamma,G} = 2$. $\Omega(P_{\Gamma,G})$ consists of two opposite sequences:

$$a^{[0]}(s) := \sum_{k=0}^{\infty} 2^{-k}s^{2k} + \frac{5}{7}s\sum_{k=0}^{\infty} 2^{-k}s^{2k} = (1+\tfrac{5}{7}s)/(1-\tfrac{s^2}{2}) = \frac{\frac{7+5\sqrt{2}}{14}}{1-\frac{s}{\sqrt{2}}} + \frac{\frac{7-5\sqrt{2}}{14}}{1+\frac{s}{\sqrt{2}}}.$$

$$a^{[1]}(s) := \sum_{k=0}^{\infty} 2^{-k}s^{2k} + \frac{7}{10}s\sum_{k=0}^{\infty} 2^{-k}s^{2k} = (1+\tfrac{7}{10}s)/(1-\tfrac{s^2}{2}) = \frac{\frac{10+7\sqrt{2}}{20}}{1-\frac{s}{\sqrt{2}}} + \frac{\frac{10-7\sqrt{2}}{20}}{1+\frac{s}{\sqrt{2}}}.$$

11.3 Finite rational accumulation

We introduce the concept of a *finite rational accumulation*, and study the series $P(t)$ (11.2.1) from that view point. First, we start with preliminary definitions.

Definition. **1.** A subset U of $\mathbb{Z}_{\geq 0}$ is called a *rational subset* if the sum $U(t) := \sum_{n\in U} t^n$ is the Taylor expansion at 0 of a rational function in t.

2. A *finite rational partition* of $\mathbb{Z}_{\geq 0}$ is a finite collection $\{U_a\}_{a\in\Omega}$ of rational subsets $U_a \subset \mathbb{Z}_{\geq 0}$ indexed by a finite set Ω such that there is a finite subset D of $\mathbb{Z}_{\geq 0}$ so that one has the disjoint decomposition $\mathbb{Z}_{\geq 0} \setminus D = \amalg_{a\in\Omega}(U_a \setminus D)$.

Assertion. *For any rational subset U of $\mathbb{Z}_{\geq 0}$, there exist a positive integer h, a subset $u \subset \mathbb{Z}/h\mathbb{Z}$ and a finite subset $D \subset \mathbb{Z}_{\geq 0}$ such that $U \setminus D = \cup_{[e]\in u} U^{[e]} \setminus D$, where $[e]$ denotes the element of $\mathbb{Z}/h\mathbb{Z}$ corresponding to $e \in \mathbb{Z}$ and $U^{[e]} := \{n \in \mathbb{Z}_{\geq 0} \mid n \equiv e \bmod h\}$. We call $\cup_{[e]\in u} U^{[e]}$ the standard expression of U.*

Proof. The fact that $U(t)$ is rational implies that the function $\chi : \mathbb{Z}_{\geq 0} \to \{0,1\}$ ($\chi(n) = 1 \leftrightarrow n \in U$) is recursive, i.e. there exist $N \in \mathbb{Z}_{\geq 1}$ and numbers $\alpha_1, \cdots, \alpha_N$ such that one has the recursive relation $\chi(n)+\chi(n-1)\alpha_1+\cdots+\chi(n-N)\alpha_N = 0$ for sufficiently large $n \gg 0$. Since the range of χ is finite, there exists two large numbers $n > m$ such that $\chi(n-i) = \chi(m-i)$ for $i = 0, \cdots, N$. Due to the recursive relation, this means that χ is $h := n-m$-periodic after m. $\square$

Corollary. *Any finite rational partition of $\mathbb{Z}_{\geq 0}$ has a subdivision of the form $\mathcal{U}_h := \{U^{[e]}\}_{[e]\in\mathbb{Z}/h\mathbb{Z}}$ for some $h \in \mathbb{Z}_{>0}$, called a period of the partition. If h is the minimal period, $\mathcal{U}_h$ is called the standard subdivision of the partition.*

Definition. A sequence $\{X_n\}_{n\in\mathbb{Z}_{\geq 0}}$ in a Hausdorff space is *finite rationally accumulating* if the sequence accumulates to a finite set, say Ω, such that for a system of open neighborhoods $\mathcal{V}_a$ for $a \in \Omega$ with $\mathcal{V}_a \cap \mathcal{V}_b = \emptyset$ if $a \neq b$, the system $\{U_a\}_{a\in\Omega}$ for $U_a := \{n \in \mathbb{Z}_{\geq 0} \mid X_n \in \mathcal{V}_a\}$ is a finite rational partition of $\mathbb{Z}_{\geq 0}$. We say also that Ω *is a finite rationally accumulation set of period h.*

The next and 11.4 Lemmas are key facts, which justify the introduction of the concept "rational accumulation". They are also the starting point of the concept of *periodicity* which is the thorough bass of the whole study in sequel.

Lemma. *Let $P(t)$ be a power series in t as given in* (11.2.1). *If $\Omega(P)$ is finite, then it is a finite rationally accumulation set with respect to the standard partition $\mathcal{U}_h$ of $\mathbb{Z}_{\geq 0}$ for some $h>0$, and τ_Ω acts transitively on $\Omega(P)$ of period h.*

Proof. Recall the τ_Ω-action on the set $\Omega(P)$ in 11.2. Since $\Omega(P)$ is finite, there exists a non-empty τ_Ω-invariant subset of $\Omega(P)$. More explicitly, there exists an element $a \in \Omega(P)$ and a positive integer $h \in \mathbb{Z}_{>0}$ such that $(\tau_\Omega)^h a = a \neq (\tau_\Omega)^{h'} a$ for $0 < h' < h$. Put $U_a := \{n \in \mathbb{Z}_{\geq 0} \mid X_n(P) \in \mathcal{V}_a\}$ where $\{\mathcal{V}_a\}_{a \in \Omega(P)}$ is a system of open neighborhoods of points of $\Omega(P)$ such that $\mathcal{V}_a \cap \mathcal{V}_b = \emptyset$ for any $a \neq b \in \Omega(P)$. By the definition of τ_Ω, the relation $(\tau_\Omega)^h a = a$ implies that the sequence $\{X_{n-h}(P)\}_{n \in U_a}$ converges to a. That is; there exists a positive number N such that for any $n \in U_a$ with $n > N$, $n - h$ is contained in U_a. Consider the set $A := \{[e] \in \mathbb{Z}/h\mathbb{Z} \mid$ there are infinitely many elements of U_a which are congruent to $[e]$ modulo h $\}$. Then, U_a is, up to a finite number of elements, equal to the rational set $\cup_{[e] \in A} U^{[e]}$. This implies $A \neq \emptyset$. Further more, $U_{(\tau_\Omega)^i a}$ is also, up to a finite number of elements, equal to the rational set $\cup_{[e] \in A} U^{[e-i]}$. Then, the union $\cup_{i=0}^{h-1} U_{(\tau_\Omega)^i a}$ already covers $\mathbb{Z}_{\geq 0}$ up to finite elements. Since there should not be an overlapping, $\#A = 1$, say $A = \{[0]\}$. If a subsequence $\{X_{n_m}(P)\}$ converges to an element in $\Omega(P)$, then there is at liest one $[e] \in \mathbb{Z}/h\mathbb{Z}$ such that $\#(\{n_m\}_{m=0}^\infty \cap U^{[e]}) = \infty$ so that it converges to $(\tau_\Omega)^{h-e} a$. That is; $\Omega(P)$ is equal to the set $\{a, \tau_\Omega a, \cdots, (\tau_\Omega)^{h-1} a\}$, which is a finite rationally accumulating set with the h-periodic action of τ_Ω. $\square$

In the sequel, we analyze the finite accumulation set $\Omega(P)$ in detail.

Assertion. *Let $P(t)$ be a power series in t as given in* (11.2.1).

1. *$\Omega(P)$ is a finite rationally accumulation set of period $h \in \mathbb{Z}_{\geq 1}$ if and only if $\Omega_1(P)$ is. We say P is finite rationally accumulating of period h.*

2. *Let P be finite rationally accumulating of period $h \in \mathbb{Z}_{\geq 1}$. Then the opposite series $a^{[e]} = \sum_{k=0}^\infty a_k^{[e]} s^k$ in $\Omega(P)$ associated to the rational subset $U^{[e]}$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ of the h-partition of $\mathbb{Z}_{\geq 0}$ converges to a rational function*

$$a^{[e]}(s) \ = \ \frac{A^{[e]}(s)}{1 - r^h s^h}, \tag{11.3.1}$$

where the numerator $A^{[e]}(s)$ is a polynomial in s of degree $h-1$ given by

$$A^{[e]}(s) \ := \ \sum_{j=0}^{h-1} \Big(\prod_{i=1}^{j} a_1^{[e-i+1]}\Big) s^j \quad \& \tag{11.3.2}$$

$$r^h \ := \ \prod_{i=0}^{h-1} a_1^{[i]}. \tag{11.3.3}$$

The hth positive root $r > 0$ of (11.3.3) *is the radius of convergence of $P(t)$.*

3. *If the period h is minimal, then the opposite sequences $a^{[e]}(s)$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ are mutually distinct. That is, $\Omega(P) \simeq \mathbb{Z}/h\mathbb{Z}$, $a^{[e]}(s) \leftrightarrow [e]$ and the standard partition $\mathcal{U}_h$ is the exact partition of $\mathbb{Z}_{\geq 0}$ for the opposite series $\Omega(P)$.*

Proof. 1. The necessity is obvious. To show sufficiency, assume that $\{\gamma_{n-1}/\gamma_n\}_{n \in \mathbb{Z}_{\geq 0}}$ accumulate finite rationally of period h. Let the subsequence $\{\gamma_{n-1}/\gamma_n\}_{n \in U_{[e]}}$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ accumulate to a unique value $a_1^{[e]} =$.

For any $k \in \mathbb{Z}_{\geq 0}$, one has the obvious relation:

$$\frac{\gamma_{n-k}}{\gamma_n} = \frac{\gamma_{n-1}}{\gamma_n}\frac{\gamma_{n-2}}{\gamma_{n-1}}\cdots\frac{\gamma_{n-k}}{\gamma_{n-k+1}}.$$

For $n \in U_{[e]} = \{n \in \mathbb{Z}_{\geq 0} \mid n \equiv e \bmod h\}$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$, we see that the RHS converges to $a_1^{[e]}a_1^{[e-1]}\ldots a_1^{[e-k+1]}$. Then, for $[e]\in\mathbb{Z}/h\mathbb{Z}$ and $k\in\mathbb{Z}_{\geq 0}$, by putting

$$a_k^{[e]} := a_1^{[e]}a_1^{[e-1]}\ldots a_1^{[e-k+1]}, \tag{11.3.4}$$

the sequence $\{X_n(P)\}_{n\in U_{[e]}}$ converges to $a^{[e]} := \sum_{k=0}^{\infty} a_k^{[e]}s^k$ with $a_1^{[e]} = \iota(a^{[e]})$.

2. Define r^h by the relation (11.3.3). Then, the formula (11.3.4) implies the "*periodicity*" $a_{mh+k}^{[e]} = r^{mh}a_k^{[e]}$ for $m\in\mathbb{Z}_{\geq 0}$. This implies (11.3.1).

To show that r is the radius of convergence of $P(t)$, it is sufficient to show:

Fact. *Let $P(t)$ be finite rationally accumulating of period h. Define $r \geq 0$ by the relation* (11.3.3)*. There exist positive real constants c_1 and c_2 such that for any $k \in \mathbb{Z}_{\geq 0}$ there exists $n(k) \in \mathbb{Z}_{\geq 0}$ and for any integer $n \geq n(k)$, one has $c_1r^k \leq \frac{\gamma_{n-k}}{\gamma_n} \leq c_2r^k$.*

Proof. Choose $c_1, c_2 \in \mathbb{R}_{>0}$ satisfying $c_1 < \min\{\frac{a_i^{[e]}}{r^i} \mid [e]\in\mathbb{Z}/h\mathbb{Z}, i\in\mathbb{Z}\cap[0,h-1]\}$ and $c_2 > \max\{\frac{a_i^{[e]}}{r^i} \mid [e] \in \mathbb{Z}/h\mathbb{Z}, i\in\mathbb{Z}\cap[0,h-1]\}$. □

3. Suppose $a^{[e]}(s) = a^{[f]}(s)$ for some $[e], [f] \in \mathbb{Z}/h\mathbb{Z}$. Then, by comparing the coefficients of $A^{[e]}(s)$ and $A^{[f]}(s)$, we get $a_1^{[e-i]} = a_1^{[f-i]}$ for $i=0,\cdots,h-1$. This means $e-f$ is a period. The minimality of h implies $[e-f] = 0$. □

Even if, as in the Assertion, the opposite series $a^{[e]}(s)$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ are mutually distinct for the minimal period h of $P(t)$, they may be linearly dependent. This phenomenon occurs at the zero-loci of the determinant

$$D_h(a_1^{[0]},\cdots,a_1^{[h-1]}) := \det\Big(\big(\textstyle\prod_{i=1}^{f} a_1^{[e-i+1]}\big)_{e,f\in\{0,1,\cdots,h-1\}}\Big). \tag{11.3.5}$$

Regarding $a_1^{[0]},\cdots,a_1^{[h-1]}$ as indeterminates, D_h is an irreducible homogeneous polynomial of degree $h(h-1)/2$, which is neither symmetric nor anti-symmetric, but (anti) invariant under a cyclic permutation (depending on the parity of h). Let us formulate more precise statements for an arbitrary field K.

Assertion. *Let $h\in\mathbb{Z}_{>0}$. For an h-tuple $\bar{a} = (a_1^{[0]},\cdots,a_1^{[h-1]}) \in (K^\times)^h$, define polynomials $A^{[e]}(s)$ ($[e]\in\mathbb{Z}/h\mathbb{Z}$) and $r^h\in K^\times$ by* (11.3.2) *and* (11.3.3)*.*

i) *In the ring $K[s]$, the greatest common divisors* $\gcd(A^{[e]}(s), 1-r^hs^h)$ *and* $\gcd(A^{[e]}(s), A^{[e+1]}(s))$ *for all $[e]\in\mathbb{Z}/h\mathbb{Z}$ are the same up to factors in $K^\times$. Let $\delta_{\bar{a}}(s)$ be the common divisor whose constant term is normalized to 1. Put*

$$\Delta_{\bar{a}}^{op}(s) := (1 - r^hs^h)/\delta_{\bar{a}}(s). \tag{11.3.6}$$

ii) *For* $[e]\in\mathbb{Z}/h\mathbb{Z}$, *let* $a^{[e]}(s)=b^{[e]}(s)/\Delta^{op}_{\bar a}(s)$ *be the reduced expression (i.e.* $b^{[e]}(s)$ *is a polynomial of degree* $<\deg(\Delta^{op}_{\bar a})$ *and* $\gcd(b^{[e]}(s),\Delta^{op}_{\bar a}(s))=1$*). Then, the polynomials* $b^{[e]}(s)$ *for* $[e]\in\mathbb{Z}/h\mathbb{Z}$ *span the space* $K[s]_{<\deg(\Delta^{op}_{\bar a})}$ *of polynomials of degree less than* $\deg(\Delta^{op}_{\bar a})$*. One has the equality:*

$$\text{rank}\left(\left(\textstyle\prod_{i=1}^{f} a_1^{[e-i+1]}\right)_{e,f\in\{0,1,\cdots,h-1\}}\right) = \deg(\Delta^{op}_{\bar a}). \tag{11.3.7}$$

iii) *Let* $K=\mathbb{R}$ *and* $\bar a\in(\mathbb{R}_{>0})^h$*. Then,* $\Delta^{op}_{\bar a}$ *is divisible by* $1-rs$*. Conversely, let* Δ^{op} *be a factor of* $1-r^hs^h$ *which is divisible by* $1-rs$ *for* $r\in\mathbb{R}_{>0}$ *with the constant term 1. Then there exists a smooth non-empty semialgebraic set* $C_{\Delta^{op}}\subset(\mathbb{R}_{>0})^h$ *of dimension* $\deg(\Delta^{op})-1$ *such that* $\Delta^{op}=\Delta^{op}_{\bar a}$ *for* $\forall\bar a\in C_{\Delta^{op}}$*.*

Proof. i) By the definitions (11.3.3) and (11.3.4), we have the relations:

$$a_1^{[e+1]}sA^{[e]}(s)+(1-r^hs^h) = A^{[e+1]}(s) \tag{11.3.8}$$

for $[e]\in\mathbb{Z}/h\mathbb{Z}$. This implies $\gcd(A^{[e]}(s),1-r^hs^h)\mid\gcd(A^{[e+1]}(s),1-r^hs^h)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ so that one concludes that all the elements $\gcd(A^{[e]}(s),1-r^hs^h)$ $=\gcd(A^{[e]}(s),A^{[e+1]}(s))$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ are the same up to a constant factor.

ii) Let us show that the images in $K[s]/(\Delta^{op}_{\bar a})$ of the polynomials $A^{[e]}(s)/\delta_{\bar a}(s)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ span the entire space over K. Let V be the space spanned by the images. The relation (11.3.8) implies that V is closed under the multiplication of s. On the other hand, $A^{[e]}(s)/\delta_{\bar a}(s)$ and $\Delta^{op}_{\bar a}$ are relatively prime so that they generate 1 as a $K[s]$-module. That is, V contains the class $[1]$ of 1, and, hence, V contains the whole $K[s]\cdot[1]$. Since $\deg(A^{[e]}(s)/\delta_{\bar a}(s))<\deg(\Delta^{op}_{\bar a})$, this means that the polynomials $A^{[e]}(s)/\delta_{\bar a}(s)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ span the space of polynomials of degree less than $\deg(\Delta^{op}_{\bar a})$. In particular, one has $\text{rank}_K V=\deg(\Delta^{op}_{\bar a})$.

By definition, $\text{rank}(\left((\prod_{i=1}^{f} a_1^{[e-i+1]})_{e,f\in\{0,1,\cdots,h-1\}}\right))$ is equal to the rank of the space spanned by $A^{[e]}(s)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$, which is equal to the rank of the space spanned by $A^{[e]}(s)/\delta_{\bar a}(s)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ and is equal to $\deg(\Delta^{op}_{\bar a})$.

iii) If $(1-rs)\nmid\Delta^{op}_{\bar a}$, then $1-rs\mid\delta_{\bar a}\mid A^{[e]}(s)$ and $A^{[e]}(1/r)=0$. This is impossible since all coefficients of $A^{[e]}$ and $1/r$ are positive. Conversely, let Δ^{op} be a factor of $1-r^hs^h$ which is divisible by $1-rh$, whose degree is $d>0$. Put $\mathbb{R}[s]_{d-1}:=\{c(s)\in\mathbb{R}[s]\mid\deg(c(s))=d-1,\ c(0)=1\}$. Consider the set

$$\overline{C}_{\Delta^{op}}:=\{c(s)\in\mathbb{R}[s]_{d-1}\ \mid \text{all coefficients of } c'(s):=c(s)\tfrac{1-r^hs^h}{\Delta^{op}} \text{ are positive }\}.$$

Since $\overline{C}_{\Delta^{op}}$ is defined by the strict innequalities, it is an open set of $\mathbb{R}[s]_{d-1}$. Furhter, it is non-empty since it contains $\Delta^{op}/(1-rs)$. For any $c(s)\in\overline{C}_{\Delta^{op}}$, we note that $\deg(c'(s))=h-1$, and hence one can find a unique $\overline{a}\in(\mathbb{R}_{>0})^h$ satisfying $c'(s)=A^{[0]}(s)$ (11.3.2) and (11.3.3). By this correspondence $c(s)\mapsto\overline{a}$, we embedd $\overline{C}_{\Delta^{op}}$ smoothly to a smooth semialgebraic subset of $(\mathbb{R}_{>0})^h$ of dimension d–1. If $\overline{a}$ is the image of $c(s) \in \overline{C}_{\Delta^{op}}$, then $\delta_{\overline{a}} := \gcd\{c'(s), 1-r^h s^h\}$ is divisible by $(1–r^h s^h)/\Delta^{op}$. That is, $\Delta^{op}_{\overline{a}} := (1-r^h s^h)/\delta_{\overline{a}}$ is a factor of Δ^{op}. This implies that the $c(s)$ is a point of the embedded image $C_{\Delta^{op}_{\overline{a}}} \to C_{\Delta^{op}}$ (defined by the multiplication of $\frac{\Delta^{op}}{\Delta^{op}_{\overline{a}}}$). Define the semialgebraic set $C_{\Delta^{op}} := \overline{C}_{\Delta^{op}} \setminus \cup_{\Delta'} \overline{C}_{\Delta'}$, where the index Δ' runs over all factors of Δ^{op} (over $\mathbb{R}$) which are not equal to Δ^{op} and are divisible by $1-rs$. Since $\dim_{\mathbb{R}}(\overline{C}_{\Delta}) = d-1 > \dim_{\mathbb{R}}(\overline{C}_{\Delta'})$ so that the difference C_{Δ} is non-empty. □

Let $\tilde{K}$ be the splitting field of $\Delta^{op}_{\overline{a}}$ with the decomposition $\Delta^{op}_{\overline{a}} = \prod_{i=1}^d (1-x_i s)$ in $\tilde{K}$ for $d := \deg(\Delta^{op}_{\overline{a}})$. Then, one has the partial fraction decomposition:

$$\frac{A^{[e]}(s)}{1-r^h s^h} = \sum_{i=1}^d \frac{\mu^{[e]}_{x_i}}{1-x_i s} \tag{11.3.9}$$

for $[e] \in \mathbb{Z}/h\mathbb{Z}$, where $\mu^{[e]}_{x_i}$ is a constant in $\tilde{K}$ given by the residue:

$$\mu^{[e]}_{x_i} = \frac{A^{[e]}(s)(1-x_i s)}{1-r^h s^h}\Big|_{s=(x_i)^{-1}} = \frac{1}{h} A^{[e]}(x_i^{-1}). \tag{11.3.10}$$

Here, one has the equivariance $\sigma(\mu^{[e]}_{x_i}) = \mu^{[e]}_{\sigma(x_i)}$ with respect to the action of $\sigma \in \mathrm{Gal}(\tilde{K}, K)$. The matrix $(\mu^{[e]}_{x_i})_{[e],x_i}$ is of maximal rank $d = \deg(\Delta^{op}_{\overline{a}})$.

Remark. The index x_i in (11.3.10) may run over all roots x of the equation $x^h - r^h = 0$. However, if x is not a root of $\Delta^{op}_{\overline{a}}$ (i.e. $\Delta^{op}_{\overline{a}}(x^{-1}) \neq 0$), then $\mu^{[e]}_x = 0$.

We return to the series $P(t)$ (11.2.1) with positive radius $r > 0$ of convergence. If $P(t)$ is finite rationally accumulating of period h and $a^{[e]}_1 := \iota(a^{[e]})$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ (recall (11.3.1)), then $\Delta^{op}_{\overline{a}}(s)$ depends only on P but not on the choice of the period h. Therefore, we shall denote it by $\Delta^{op}_P(s)$. The previous Assertion ii) says that we have the $\mathbb{R}$-isomorphism:

$$\overline{\mathbb{R}\Omega}(P) \simeq \mathbb{R}[s]/(\Delta^{op}_P(s)), \quad a^{[e]} \mapsto \Delta^{op}_P \cdot a^{[e]} \bmod \Delta^{op}_P. \tag{11.3.11}$$

Define an endomorphism σ on $\overline{\mathbb{R}\Omega}(P)$ by letting $\sigma(a^{[e]}) := \tau^{-1}(a^{[e]}) = \frac{1}{a^{[e+1]}_1} a^{[e+1]}$. Then, *the action of σ on the LHS and the multiplication of s on the RHS are equivariant with respect to the isomorphism* (11.3.11). Hence, *the linear dependence relations among the generators $a^{[e]}$ ($[e] \in \mathbb{Z}/h\mathbb{Z}$) are obtained by the relations $\Delta^{op}_P(\sigma) a^{[e]} = 0$* for $[e] \in \mathbb{Z}/h\mathbb{Z}$. However, one should note that the σ-action is not the same as the multiplication of s as an element of $\mathbb{R}[\![s]\!]$.

Finally, in this subsection, we introduce some more concepts and notation.

A) For a positive real number r, let us denote by $\mathbb{C}\{t\}_r$ the space consisting of complex powers series $P(t)$ such that i) $P(t)$ converges (at least) on the disc centered at 0 of radius r, and ii) $P(t)$ analytically continues to a meromorphic function on a disc centered at 0 of radius $> r$. Let $\Delta_P(t)$ be the monic polynomial in t of minimal degree such that $\Delta_P(t)P(t)$ is holomorphic in a neighborhood of the circle $|t|=r$. Put $\Delta_P(t) = \prod_{i=1}^N (t-x_i)^{d_i}$ where x_i $(i=1,\cdots,N,\ N\in\mathbb{Z}_{>0})$ are mutually distinct complex numbers with $|x_i|=r$ and $d_i\in\mathbb{Z}_{>0}$ $(i=1,\cdots,N)$. Define the equation for the set of poles of highest order:

$$\Delta_P^{top}(t) := \prod_{i,d_i=d_m}(t-x_i) \quad \text{where} \quad d_m := \max\{d_i\}_{i=1}^N. \tag{11.3.12}$$

B) For a rational set U of $\mathbb{Z}_{\geq 0}$, we define an action T_U on $\mathbb{C}[\![t]\!]$ by letting

$$P = \sum_{n\in\mathbb{Z}_{\geq 0}} \gamma_n t^n \quad \mapsto \quad T_U P := \sum_{n\in U} \gamma_n t^n. \tag{11.3.13}$$

One may regard $T_U P$ as a product of P with the function $U(t)$ in the sense of Hadamard [H]. The radius of convergence of $T_U P$ is not less than that of P.

Fact 1. The action of T_U preserves the space $\mathbb{C}\{t\}_r$ for any $r\in\mathbb{R}_{>0}$.

Proof. Let us expand the meromorphic function $P(t)$ into partial fractions

$$P(t) = \sum_{i=1}^N \sum_{j=0}^{d_i} \frac{c_{i,j}}{(t-x_i)^j} \ + \ Q(t), \tag{$*$}$$

where the coefficients $c_{i,j}$ of the principal part $\sum_{i=1}^N \sum_{j=0}^{d_i} \frac{c_{i,j}}{(t-x_i)^j}$ of $P(t)$ are constants with $c_{i,d_i}\neq 0$ for $\forall i$, and $Q(t)$ is a holomorphic function on a disc of radius $> r$. Then, $T_U P = \sum_{i,j} T_U \frac{c_{i,j}}{(t-x_i)^j} + T_U Q$ where $T_U Q$ is a holomorphic function on a disc of radius $> r$. It is sufficient to show that, for any standard rational set $U^{[e]} := \{n\in\mathbb{Z}_{\geq 0} \mid n\equiv[e] \bmod h\}$ of period $h\in\mathbb{Z}_{>0}$ and $[e]\in\mathbb{Z}/h\mathbb{Z}$, on has $T_{U^{[e]}}\frac{1}{(t-x_i)^j} = \frac{B_{i,j}(t)}{(t^h-x_i^h)^j}$ where $B_{i,j}(t)$ is a polynomial in t. We calculate this explicitly as follows. For the purpose, we claim a "semi-commutativity" $T_{U^{[e]}}\cdot\frac{d}{dt} = \frac{d}{dt}\cdot T_{U^{[e+1]}}$ (proof is trivial and is omitted). Then,

$$\begin{aligned} T_{U^{[e]}}\frac{1}{(t-x_i)^j} &= T_{U^{[e]}}\frac{(-1)^{j-1}}{(j-1)!}\left(\frac{d}{dt}\right)^{j-1}\frac{1}{t-x_i} = \frac{(-1)^{j-1}}{(j-1)!}\left(\frac{d}{dt}\right)^{j-1} T_{U^{[e+j-1]}}\frac{1}{t-x_i} \\ &= \frac{(-1)^{j-1}}{(j-1)!}\left(\frac{d}{dt}\right)^{j-1}\frac{t^f}{t^h-x_i^h} \qquad \text{where } f := e+j-1-h[(e+j-1)/h]. \end{aligned}$$

This gives the required result. $\square$.

The following is the last and the main result of the present subsection.

Theorem. (Duality) *Suppose $P(t)$ (11.2.1) belongs in $\mathbb{C}\{t\}_r$ for $r =$ the radius of convergence of P, and is finite accumulating. Then, we have*

$$t^{\deg(\Delta_P^{op})}\Delta_P^{op}(t^{-1}) \ = \ \Delta_P^{top}(t), \tag{11.3.14}$$

$$\operatorname{rank}(\overline{\mathbb{R}\Omega}(P)) \ = \ \deg(\Delta_P^{op}) \ = \ \deg(\Delta_P^{top}). \tag{11.3.15}$$

Proof. We first show some special case followed by the general case.

Fact 2. *If $P(t)$, above, is simple accumulating (i.e. $\#\Omega(P)=1$), then $\Delta_P^{top}=t–r$.*

Proof. According to the partial fractional expansion $*$) for P, let us split the Taylor coefficients of P into the principal part and that of Q. Since that of Q has the lower order, we may assume that the prinicipal part say P' is simply accumulating. That is, $X_n(P')=\sum_{k=0}^{n}\frac{\sum_{i,j}c_{i,j}x_i^{k-n-1}(n-k;j)/(j-1)!}{\sum_{i,j}c_{i,j}x_i^{-n-1}(n;j)/(j-1)!}s^k$ converges to $\frac{1}{1-rs}=\sum_{k=0}^{\infty}r^ks^k$. Then, we want to show that if $c_{i,d_m}\neq 0$ then $x_i=r$. For a convenience of the proof, we may assume $r=1$ and hence $|x_i|=1$ for all i.

Consider the sequence $v_n:=\sum_{i=1}^{N}c_{i,d_m}x_i^{-n-1}$. Since the range of v_n is bounded, the sequence accumulates to a compact set. Let us, first, show that if it has a unique accumulating value, say v_0 then the result is already true. (*Proof.* Consider the mean sequence: $\{(\sum_{n=0}^{M-1}v_n)/M\}_{M\in\mathbb{Z}_{>0}}$. On one side, it converges to v_0 by the assumption. On the other side, $\sum_{i=1}^{N}c_{i,d_m}\frac{\sum_{n=0}^{M-1}x_i^{-n-1}}{M}$ converges to c_{1,d_m}, where we assume $x_1=1$. That is, the sequence $v_n':=\sum_{i=2}^{N}c_{i,d_m}x_i^{-n-1}$ converges to 0. For a fixed $n_0\in\mathbb{Z}_{>0}$, consider the relations: $v'_{n_0+k}=\sum_{i=2}^{N}(c_{i,d_m}x_i^{-n_0})x_i^{-k+1}$ for $k=1,\cdots,N-1$. Regarding $c_{i,d_m}x_i^{-n_0}$ $(i=2,\cdots,N)$ as the unknown, we can solve the linear equation by a use of the van del Mond determinant for the matrix $(x_i^{-k+1})_{i=2,\cdots,N,k=1,\cdots,N-1}$. So, we obtain a linear approximation: $|c_{i,d_m}x_i^{-n_0}|\leq c\cdot\max\{|v'_{n_0+k}|\}_{k=1}^{N-1}$ $(i=2,\cdots,N)$ for a constant $c>0$ which is independent of n_0. The RHS tend to zero as $n_0\to\infty$, whereas the LHS are unchanged. This implies $|c_{i,d_m}|=0$ $(i=2,\cdots,N)$.

Next, consider the case that the sequence v_n has more than two accumulating values. Suppose the subsequence $\{v_{n_m}\}_{m\in\mathbb{Z}_{>0}}$ converges to a non-zero value, say c. Recall the assumption that the sequence γ_{n-1}/γ_n converges to 1. So, the subsequence $\frac{\gamma_{n_m-1}}{\gamma_{n_m}}=\frac{v_{n_m-1}+\text{lower terms}}{v_{n_m}+\text{lower terms}}$ should converges also to 1 as $m\to\infty$. In the denominator, the first term tends to $c\neq 0$ and the second term tends to zero. Samely, in the numerator, the second term tends to zero. These implies that the first term in the numerator converges also to c. Repeating the same argument, we see that for any $k\in\mathbb{Z}_{\geq 0}$, the subsequence $\{v_{n_m-k}\}_{m\in\mathbb{Z}_{\gg 0}}$ converges to the same c. Then, for each fixed $M\in\mathbb{Z}_{>0}$, the average sequence $\{(\sum_{k=0}^{M-1}v_{n_m-k})/M\}_{m\in\mathbb{Z}_{\gg 0}}$ converges to c, whereas, for sufficiently large M, the values is close to c_{1,d_m}. This implies $c=c_{1,d_m}$. In the other words, the sequences $\{v'_{n_m-k}\}_{m\in\mathbb{Z}_{\gg 0}}$ for any $k\geq 0$ converge to 0. Then, the similar argument as in the previous case implies $|c_{i,d_m}|=0$ $(i=2,\cdots,N)$.

This is the end of the proof of Fact 2. □

We return to the general case, where P is finite rational accumulating of period h. For the standard partition $\{U^{[e]}\mid [e]\in\mathbb{Z}/h\mathbb{Z}\}$, put $T^{[e]}:=T_{U^{[e]}}$.

They decompose the unity: $\sum_{[e]\in\mathbb{Z}/h\mathbb{Z}} T^{[e]}=1$. By the assumption, for each $0\leq f<h$, the series $T^{[f]}P=t^f\sum_{m=0}^{\infty}\lambda_{f+mh}\tau^m$, considered as a series in $\tau=t^h$, is simple accumulating. Then Fact 2. implies that the highest order poles of $T^{[f]}P$ are only at solutions x of the equation $t^h-r^h=0$. In view of the fact that the highest order of poles of $T^{[f]}P$ cannot exceed that of P (recall the explicit expression in *Fact* 1.) and the fact $P=\sum_{[e]\in\mathbb{Z}/h\mathbb{Z}}T^{[e]}P$, the highest order poles of P are also only at solutions x of the equation $t^h-r^h=0$. That is; $\Delta_P^{top}(t)$ is a factor of t^h-r^h. For $0\leq e,f<h$ and a root x of the equation t^h-r^h, we evaluate ((10.6.4) for $\{n_m=e+mh\}_{m=0}^{\infty}$ and $\{n_m=f+mh\}_{m=0}^{\infty}$)

$$\frac{T^{[f]}P}{T^{[e]}P}(t)\Big|_{t=x} \;=\; x^{f-e}\frac{\sum_{m=0}^{\infty}\lambda_{f+mh}\tau^m}{\sum_{m=0}^{\infty}\lambda_{e+mh}\tau^m}\Big|_{\tau=x^h=r^h} \;=\; x^{f-e}\lim_{m\to\infty}\frac{\gamma_{f+mh}}{\gamma_{e+mh}}.$$

Then, a similar argument to that for (11.3.4) shows the formula

$$\frac{T^{[f]}P}{T^{[e]}P}(t)\Big|_{t=x} \;=\; \begin{cases} x^{f-e}/a_1^{[f]}a_1^{[f-1]}\cdots a_1^{[e+1]} & \text{if } e<f\\ \quad 1 & \text{if } e=f\\ x^{f-e}a_1^{[e]}a_1^{[e-1]}\cdots a_1^{[f+1]} & \text{if } e>f. \end{cases} \tag{11.3.16}$$

This implies that the order of poles of $T^{[e]}P(t)$ at a solution x of the equation t^h-r^h is independent of $[e]\in\mathbb{Z}/h\mathbb{Z}$. On the other hand, (11.3.16) implies

$$\frac{T^{[e]}P}{P}(t)\Big|_{t=x} \;=\; \frac{1}{A^{[e]}(x^{-1})}. \tag{11.3.17}$$

(recall the $A^{[e]}(s)$ (11.3.2)). Let x be a solution of $t^h-r^h=0$ but $\Delta_P^{op}(x^{-1})\neq 0$. Then $\delta_a(x^{-1})=0$ (see (11.3.6)) and $A^{[e]}(x^{-1})=0$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ (see Assertion i)). That is; $\frac{T^{[e]}P}{P}(t)$ has a pole at $t=x$. This implies that the pole of $P(t)$ at $t=x$ is order $<d_m$ (otherwise, the pole at $t=x$ of $T^{[e]}P$ is at most of order d_m and can be canceled by dividing by P). That is; $\Delta_P^{top}(t)\mid t^d\Delta_P^{op}(t^{-1})$.

Fact 3. *Let $P(t)$ (11.2.1) belong to $\mathbb{C}\{t\}_r$ and finitely accumulating. Then*

i) There exists a positive constant c such that $\gamma_n\geq cr^{-n}n^{d_m}$ for $n>>0$.

ii) $$t^d\Delta_P^{op}(t^{-1})\mid\Delta_P^{top}(t)\ .$$

Proof. i) Consider the Taylor expansion of the function $*$). Using notation v_n in *Fact* 2., we have $\gamma_n=-v_n\frac{r^{-n-1}(n;d_m)}{(d_m-1)!}$+terms coming from poles of order $<d_m$+ terms coming from $Q(t)$, where $v_n=\sum_i c_{i,d_m}(x_i/r)^{-n-1}$ depends only on n mod h since x_i is the root of the equation $t^h-r^h=0$. Not all of them are zero (otherwise $c_{i,d_m}=0$ for all i). Let us show that non of v_n is zero. Suppose the contrary and $v_e=0\neq v_f$. Then, one observes easily $\lim_{m\to\infty}\frac{\gamma_{e+mh}}{\gamma_{f+mh}}=0$. This contradicts to the assumption $\Omega_1(P)\subset[u,v]$ (positivity of initials).

ii) By definition, the fractional expansion of $\Delta_P^{top}(t)P(t)$ has poles of order at most d_m-1. This means that its $n-k$th Taylor coefficient:

$$**)\quad \gamma_{n-k}\cdot\alpha_l+\gamma_{n-k-1}\cdot\alpha_{l-1}+\cdots+\gamma_{n-k-l}\cdot 1 \quad\sim\quad o((n-k)^{d_m}r^{-(n-k)})$$

as $n-k\to\infty$ $(k,n\in\mathbb{Z}_{\geq 0})$ (here, $\Delta_P^{top}(t)=t^l+\alpha_1t^{l-1}+\cdots+\alpha_l)$. Let $\sum_k a_ks^k\in\Omega(P)$ be the limit of a subsequence $\{X_{n_m}(P)\}_{m\in\mathbb{Z}_{\geq 0}}$ (11.2.2). Divide $**)$ by γ_n. Then, using the part i), one has

$$a_k\alpha_l+a_{k+1}\alpha_{l-1}+\cdots+a_{k+l} \;=\; 0$$

for any $k\geq 0$. Thus $s^l\Delta_P^{top}(1/s)a(s)$ is a polynomial in s of degree $<l$. Thus the denominator $\Delta_P^{op}(s)$ of $a(s)$ divides $s^l\Delta_P^{top}(s^{-1})$. So, ii) is shown. □

We showed (11.3.14). (11.3.15) follows from (11.3.11) and (11.3.14). □

Example. Recall Machì's example 11.2 for the modular group Γ. We have $T_eP(t)=\sum_{k=0}^\infty \#\Gamma_{2k}t^{2k}=\frac{1+5t^2}{(1-2t^2)(1-t^2)}$, $\quad T_oP(t)=\sum_{k=0}^\infty \#\Gamma_{2k+1}t^{2k+1}=\frac{2t(2+t^2)}{(1-2t^2)(1-t^2)}$,

Then the transformation matrix is given by

$$\begin{bmatrix} \frac{T_eP(t)}{P_{\Gamma,G}(t)}=\frac{1+5t^2}{(1+t)^2(1+2t)}\big|_{t=\frac{1}{\sqrt2}} & \frac{T_oP(t)}{P_{\Gamma,G}(t)}=\frac{2t(2+t^2)}{(1+t)^2(1+2t)}\big|_{t=\frac{1}{\sqrt2}} \\ \frac{T_eP(t)}{P_{\Gamma,G}(t)}=\frac{1+5t^2}{(1+t)^2(1+2t)}\big|_{t=\frac{-1}{\sqrt2}} & \frac{T_oP(t)}{P_{\Gamma,G}(t)}=\frac{2t(2+t^2)}{(1+t)^2(1+2t)}\big|_{t=\frac{-1}{\sqrt2}} \end{bmatrix}=\begin{bmatrix} 7(5\sqrt2-7) & 5(10-7\sqrt2) \\ -7(5\sqrt2+7) & 5(10+7\sqrt2)\end{bmatrix}$$

whose determinant is equal to $\frac{5\cdot 7}{\sqrt2}\neq 0$.

11.4 The residual representation of the trace element

As the goal of this section, under the assumption that the limit set $\Omega(\Gamma,G)$ is finite as well as a few other assumptions, we show a trace formula, which states that *the traces of the limit elements are represented by the residues of the series* $P_{\Gamma,G}\mathcal{M}(t)$ (11.2.7) *at the singular points of the Poincare series* $P_{\Gamma,G}(t)$.

Let us first show the following basic fact.

Lemma. *Let (Γ,G) be the pair consisting of a group and its finite generator system. If the limit set $\Omega(\Gamma,G)$ is finite, then it is finite rationally accumulating with respect to the standard partition $\mathcal{U}_{\tilde h}$ of $\mathbb{Z}_{\geq 0}$ for some $\tilde h>0$, and $\tilde\tau_\Omega$ acts transitively on $\Omega(\Gamma,G)$ of period $\tilde h$.*

Proof. Recall the action $\tilde\tau_\Omega$ on $\Omega(\Gamma,G)$ (Lemma in 11.2). Then, finiteness of $\Omega(\Gamma,G)$ implies that there exists an element $\omega\in\Omega(\Gamma,G)$ and an integer $\tilde h\in\mathbb{Z}_{>0}$ such that $(\tilde\tau_\Omega)^{\tilde h}\omega=\omega$ and $(\tilde\tau_\Omega)^{\tilde h'}\omega\neq\omega$ for $0<\tilde h'<\tilde h$. Consider the set $U_\omega:=\{n\in\mathbb{Z}_{\geq 0}\mid \frac{\mathcal{M}(\Gamma_n)}{\#\Gamma_n}\in\mathcal{V}_\omega\}$ (here, $\mathcal{V}_\omega$ is an open neighborhood of ω in

$\mathcal{L}_{\mathbb{R},\infty}$ such that $\overline{\mathcal{V}_\omega} \cap \Omega(\Gamma, G) = \{\omega\}$). Then, the periodicity of the action of $\tilde{\tau}_\Omega$ on ω implies (use the similar argument as in proof of 11.2 Lemma, replacing $a \in \Omega(P)$ by $\omega \in \Omega(\Gamma, G)$ and h by $\tilde{h}$, respectively) that U_ω is, up to a finite number of elements, equal to a rational set $U^{[\tilde{e}]}$ for some $[\tilde{e}] \in \mathbb{Z}/\tilde{h}\mathbb{Z}$, and the following equality holds:

$$\Omega(\Gamma, G) \ = \ \{ \ \omega, \ \tilde{\tau}_\Omega \omega, \ \cdots \ , \ (\tilde{\tau}_\Omega)^{\tilde{h}-1} \omega \ \}.$$

This implies the finite rationality of $\Omega(\Gamma, G)$ and the periodicity of $\tilde{\tau}_\Omega$. $\square$

Let $\Omega(\Gamma, G)$ be finite rationally accumulating of period $\tilde{h}$, which consists of

$$\omega_{\Gamma,G}^{[\tilde{e}]} \ := \ \lim_{m\to\infty}{}^{cl} \ \frac{\mathcal{M}(\Gamma_{\tilde{e}+m\tilde{h}})}{\#\Gamma_{\tilde{e}+m\tilde{h}}} \tag{11.4.1}$$

for $[\tilde{e}] \in \mathbb{Z}/\tilde{h}\mathbb{Z}$. Then, $\Omega(P_{\Gamma,G})$ is also finite rationally accumulating of period h such that $h|\tilde{h}$ (c.f. 11.2 Lemma), since the sequence $\{\pi(\frac{\mathcal{M}(\Gamma_n)}{\sharp\Gamma_n}) = X_n(P_{\Gamma,G})\}_{n\in U^{[\tilde{e}]}}$ for the rational set $U^{[\tilde{e}]} := \{n \in \mathbb{Z}_{\geq 0} \mid n \bmod \tilde{h} \equiv [e]\}$ for any $[\tilde{e}] \in \mathbb{Z}/\tilde{h}\mathbb{Z}$ is convergent to $\pi(\omega_{\Gamma,G}^{[\tilde{e}]})$. Let $\tilde{h}_{\Gamma,G}$ and $h_{\Gamma,G}$ be the minimal period of $\Omega(\Gamma, G)$ and $\Omega(P_{\Gamma,G})$, respectively. Then π is an $m_{\Gamma,G} := \tilde{h}_{\Gamma,G}/h_{\Gamma,G}$-fold covering. We call $m_{\Gamma,G}$ the *inertia* of (Γ, G) and $\mathbb{Z}/m_{\Gamma,G}\mathbb{Z}$ the inertia subgroup.

Let us introduce a $\tilde{\sigma}$-action on the module $\overline{\mathbb{R}\Omega}(\Gamma, G)$, as a consequence of periodicity. For any $[\tilde{e}] \in \mathbb{Z}/\tilde{h}_{\Gamma,G}\mathbb{Z}$, put $[e] \equiv [\tilde{e}] \bmod h_{\Gamma,G}$ and define

$$\tilde{\sigma}\left(\omega_{\Gamma,G}^{[\tilde{e}]}\right) \ := \ \tfrac{1}{a_1^{[e+1]}} \omega_{\Gamma,G}^{[\tilde{e}+1]} = \tilde{\tau}^{-1}\left(\omega_{\Gamma,G}^{[\tilde{e}]}\right). \tag{11.4.2}$$

The endomorphism $\tilde{\sigma}$ is semi-simple since one has $\tilde{\sigma}^{\tilde{h}_{\Gamma,G}} = r_{\Gamma,G}^{\tilde{h}_{\Gamma,G}} \ \mathrm{id}_{\overline{\mathbb{R}\Omega}(\Gamma,G)}$ (c.f. (11.3.3)). The $\mathbb{R}$-linear map π (11.2.15) is equivariant with respect to the endomorphisms $\tilde{\sigma}$ and σ. Our interest is now to investigate the subspaces fixed by the action of $\tilde{\tau}^{h_{\Gamma,G}}$ (the inertia part).

For each opposite sequence $a^{[e]} \in \Omega(P_{\Gamma,G})$ for a $[e] \in \mathbb{Z}/h_{\Gamma,G}\mathbb{Z}$, let us introduce the *trace over the fiber* $\pi^{-1}(a^{[e]})$, as an element of $\overline{\mathbb{R}\Omega}(\Gamma, G)$, by

$$\mathrm{Trace}^{[e]} \ \Omega(\Gamma, G) \ := \textstyle\sum_{[\tilde{e}]\in\mathbb{Z}/\tilde{h}_{\Gamma,G}\mathbb{Z}, \ [\tilde{e}]\subset[e]} \omega_{\Gamma,G}^{[\tilde{e}]} \ = \sum_{i=1}^{m_{\Gamma,G}} \omega_{\Gamma,G}^{[\tilde{e}+ih_{\Gamma,G}]}. \tag{11.4.3}$$

Using (11.4.2), we see that the $\tilde{\sigma}$-action induces a relation among traces:

$$\tilde{\sigma}\left(\mathrm{Trace}^{[e]} \ \Omega(\Gamma, G)\right) \ := \ \tfrac{1}{a_1^{[e+1]}} \mathrm{Trace}^{[e+1]} \ \Omega(\Gamma, G) \tag{11.4.4}$$

for all $[e] \in \mathbb{Z}/h_{\Gamma,G}\mathbb{Z}$. In view of (11.3.3), this, in particular, implies

$$(1 - (r_{\Gamma,G} \ \tilde{\sigma})^h)\left(\mathrm{Trace}^{[e]} \ \Omega(\Gamma, G)\right) \ = \ 0. \tag{11.4.5}$$

After the results of 11.3 and 11.4, the next theorem is now straightforward.

Theorem. *Let (Γ, G) be the pair consisting of a group and its finite generator system with $G = G^{-1}$ and $1 \notin G$. Suppose i) $\Omega(\Gamma, G)$ is finite, and ii) $P_{\Gamma,G} \in \mathbb{C}\{t\}_{r_{\Gamma,G}}$. Then, for any opposite sequence $a^{[e]} \in \Omega(P_{\Gamma,G})$, the following equality holds.*

$$(11.4.6)\qquad \begin{aligned} & h_{\Gamma,G}\mathrm{Trace}^{[e]}\Omega_{\Gamma,G} - \Big(\textstyle\sum_{x^{-1}\in V(\delta_{P_{\Gamma,G}})} \frac{\delta_{P_{\Gamma,G}}(\tilde\sigma)}{1-x\tilde\sigma}\Big)\Delta^{op}_{P_{\Gamma,G}}(\tilde\sigma)\mathrm{Trace}^{[e]}\Omega_{\Gamma,G} \\ = \ & m_{\Gamma,G}\textstyle\sum_{x\in V(\Delta_{P^{top}_{\Gamma,G}})} A^{[e]}(x^{-1})\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\Big|_{t=x}. \end{aligned}$$

where we put $\delta_{P_{\Gamma,G}}(\sigma) := (1 - r^{h_{\Gamma,G}}\sigma^{h_{\Gamma,G}})/\Delta^{op}_{P_{\Gamma,G}}(\sigma)$ (c.f. (11.3.6)).

Proof. Due to Lemma at the beginning of this paragraph, $\Omega(\Gamma, G)$ is finite rationally accumulating of a minimal period $\tilde h_{\Gamma,G}$. Let us, first, express the residue elements by a sum of traces of limit elements. For the purpose, consider the decomposition of unity:

$$*)\qquad \frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)} = \sum_{[\tilde f]\in\mathbb{Z}/\tilde h_{\Gamma,G}\mathbb{Z}} \frac{T^{[\tilde f]}P_{\Gamma,G}(t)}{P_{\Gamma,G}(t)} \cdot \frac{T^{[\tilde f]}P_{\Gamma,G}\mathcal{M}(t)}{T^{[\tilde f]}P_{\Gamma,G}(t)}.$$

where $T^{[\tilde f]} = T_{U^{[\tilde f]}}$ (11.3.13) is the action of the rational set $U^{[\tilde f]}$ of the standard subdivision for $\Omega(\Gamma, G)$ so that $\sum_{\tilde f\in\mathbb{Z}/\tilde h_{\Gamma,G}\mathbb{Z}} T^{[\tilde f]} = 1$. Let x be a root of $\Delta^{top}_{P_{\Gamma,G}}(t) = 0$, and consider the evaluation of both sides of $*)$ at $t = x$. The LHS gives, by definition, the residue element at x. By a slight generalization of the formula (11.3.17), the first factor in the RHS is given by $1/A^{[\tilde f]}(x^{-1}) = 1/(m_{\Gamma,G} A^{[f]}(x^{-1}))$ (note that $A^{[f]}(x^{-1})) \neq 0$ since $\delta_{P_{\Gamma,G}}(x^{-1}) \neq 0$), where $[f] := [\tilde f] \bmod h_{\Gamma,G}$ and $m_{\Gamma,G} = \tilde h_{\Gamma,G}/h_{\Gamma,G}$. The second factor in RHS is

$$\frac{\sum_{m=0}^{\infty}\mathcal{M}(\Gamma_{\tilde f+m\tilde h_{\Gamma,G}})t^{\tilde f+m\tilde h_{\Gamma,G}}}{\sum_{m=0}^{\infty}\#\Gamma_{\tilde f+m\tilde h_{\Gamma,G}}t^{\tilde f+m\tilde h_{\Gamma,G}}}\Bigg|_{t=x} = \frac{\sum_{m=0}^{\infty}\mathcal{M}(\Gamma_{\tilde f+m\tilde h_{\Gamma,G}})\tilde t^{m}}{\sum_{m=0}^{\infty}\#\Gamma_{\tilde f+m\tilde h_{\Gamma,G}}\tilde t^{m}}\Bigg|_{\tilde t=r^{\tilde h_{\Gamma,G}}}$$

where, in the RHS, $\tilde t := t^{\tilde h_{\Gamma,G}}$ is the new variable and $r^{\tilde h_{\Gamma,G}} = x^{\tilde h_{\Gamma,G}}$ is the common singular point of the two power series (the numerator and the denominator) in $\tilde t$ at the crossing of the positive real axis and the circle of the convergent radius (c.f. 10.6 Lemma i)). Then, since the coefficients of the series are non-negative, this proportion of the residue value is equal to the limit of the proportion of the coefficients of the series (c.f. (10.6.4)) $\lim^{cl}_{m\to\infty}\frac{\mathcal{M}(\Gamma_{\tilde f+m\tilde h_{\Gamma,G}})}{\#\Gamma_{\tilde f+m\tilde h_{\Gamma,G}}}$ which is nothing but the limit element $\omega^{[\tilde f]}_{\Gamma,G}$ (11.4.1). Put $\tilde f = f + ih_{\Gamma,G}$ for $0 \le f < h_{\Gamma,G}$ and $0 \le i < m_{\Gamma,G}$. Then the RHS turns into

$$\frac{1}{m_{\Gamma,G}}\sum_{[f]\in\mathbb{Z}/h_{\Gamma,G}\mathbb{Z}}\frac{1}{A^{[f]}(x^{-1})}\sum_{i=0}^{m_{\Gamma,G}-1}\omega^{[f+ih_{\Gamma,G}]}_{\Gamma,G}$$

where the second sum in the RHS gives the trace $\mathrm{Trace}^{[f]}\Omega(\Gamma,G)$. That is:

$$(11.4.7)\qquad \frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\Big|_{t=x} = \frac{1}{m_{\Gamma,G}}\sum_{[f]\in\mathbb{Z}/h_{\Gamma,G}\mathbb{Z}}\frac{1}{A^{[f]}(x^{-1})}\mathrm{Trace}^{[f]}\Omega(\Gamma,G).$$

For a fixed $[e]\in\mathbb{Z}/h_{\Gamma,G}\mathbb{Z}$, we multiply $A^{[e]}(x^{-1})$ to both sides of (11.4.7), and sum over the index x running over the set $V(\Delta^{top}_{P_{\Gamma,G}})$ of all roots of $\Delta^{top}_{P_{\Gamma,G}}(t)=0$, whose LHS is equal to the RHS of (11.4.6). Using (11.3.17), one observes that $\frac{A^{[e]}(x^{-1})}{A^{[f]}(x^{-1})}$ is equal to the LHS of (11.3.16). Replacing the summation index "$[f]\in\mathbb{Z}/h_{\Gamma,G}\mathbb{Z}$" in (11.4.7) by "$[e+i]$ for $i=0,\cdots,h_{\Gamma,G}-1$" for the fixed $[e]$. Using the first line of RHS of (11.3.16) and ith repeated applications of (11.4.4), the sum in RHS turns out to

$$\begin{aligned}
&\frac{1}{m_{\Gamma,G}}\sum_{x\in V(\Delta^{top}_{P_{\Gamma,G}})}\sum_{i=0}^{h_{\Gamma,G}-1}\frac{A^{[e]}(x^{-1})}{A^{[e+i]}(x^{-1})}\mathrm{Trace}^{[e+i]}\ \Omega_{\Gamma,G}\\
&=\frac{1}{m_{\Gamma,G}}\sum_{x\in V(\Delta^{top}_{P_{\Gamma,G}})}\sum_{i=0}^{h_{\Gamma,G}-1}\frac{x^i}{a_1^{[e+i]}a_1^{[e+i-1]}\cdots a_1^{[e+1]}}\prod_{j=1}^{i}(a_1^{[e+j]}\tilde\sigma)\ \mathrm{Trace}^{[e]}\ \Omega_{\Gamma,G}\\
&=\frac{1}{m_{\Gamma,G}}\sum_{x\in V(\Delta^{top}_{P_{\Gamma,G}})}\sum_{i=0}^{h_{\Gamma,G}-1}x^i\tilde\sigma^i\ \mathrm{Trace}^{[e]}\ \Omega_{\Gamma,G}.
\end{aligned}$$

Here, we note that the sum $\sum_{i=0}^{h_{\Gamma,G}-1}x^i\tilde\sigma^i$ is expressed as $\frac{1-(r_{\Gamma,G}\tilde\sigma)^{h_{\Gamma,G}}}{1-x\tilde\sigma}$ and that $x\in V(\Delta^{top}_{P_{\Gamma,G}})$ is equivalent to $x^{-1}\in V(\Delta^{op}_{P_{\Gamma,G}})$ due to the duality (11.3.14). We note further that an identity:

$$\sum_{x^{-1}\in V(1-(r_{\Gamma,G}s)^{h_{\Gamma,G}})}\frac{1-(r_{\Gamma,G}s)^{h_{\Gamma,G}}}{1-xs} = h_{\Gamma,G}$$

holds (in the polynomial ring of s). Therefore, recalling (11.3.6)

$$\delta_{P_{\Gamma,G}}(s)\cdot\Delta^{op}_{P_{\Gamma,G}}(s) = 1-(r_{\Gamma,G}s)^{h_{\Gamma,G}}$$

we calculate further the sum as follows.

$$\begin{aligned}
&=\frac{1}{m_{\Gamma,G}}\left(\sum_{x^{-1}\in V(\Delta^{op}_{P_{\Gamma,G}})}\frac{1-(r_{\Gamma,G}\tilde\sigma)^{h_{\Gamma,G}}}{1-x\tilde\sigma}\right)\ \mathrm{Trace}^{[e]}\ \Omega_{\Gamma,G}\\
&=\frac{1}{m_{\Gamma,G}}\left(h_{\Gamma,G}\cdot id_{\overline{\mathbb{R}\Omega}(\Gamma,G)}-\sum_{x^{-1}\in V(\delta_{P_{\Gamma,G}})}\frac{\delta_{P_{\Gamma,G}}(\tilde\sigma)}{1-x\tilde\sigma}\Delta^{op}_{P_{\Gamma,G}}(\tilde\sigma)\right)\ \mathrm{Trace}^{[e]}\ \Omega_{\Gamma,G}.
\end{aligned}$$

This gives LHS of (11.4.6), and hence Theorem is proven. □

Remark. 1. The second term of the LHS of (11.4.6) belongs to the kernel of π, since one has $\pi(\Delta^{op}_{P_{\Gamma,G}}(\tilde\sigma)\mathrm{Trace}^{[e]}\Omega_{\Gamma,G}) = m_{\Gamma,G}\Delta^{op}_{P_{\Gamma,G}}(\sigma)a^{[e]}=0$. Therefore, we ask whether

$$\Delta^{op}_{P_{\Gamma,G}}(\tilde\sigma)\ \mathrm{Trace}^{[e]}\Omega_{\Gamma,G}=0\ \ ?$$

This is equivalent to the statement that *the module spanned by the traces* $\mathrm{Trace}^{[e]}\Omega_{\Gamma,G}$ *is isomorphic to* $\overline{\mathbb{R}\Omega}(P_{\Gamma,G})$.

2. One can directly calculate the following formula:

$$(11.4.8)\qquad \pi\left(\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\right) = \frac{1}{1-st}.$$

Specializing t to a root x of $\Delta_P(t)=0$ in the formula gives the Cauchy kernel $\frac{1}{1-xs}$. Therefore, the π image of (11.4.6) turns out to be the formula (11.3.9).

3. If (Γ, G) is a group of polynomial growth, then $\Delta_{P_{\Gamma,G}}(t) = (1-t)^{l+1}$ (where l=rank(Γ)>0) is never reduced. However, due to (10.6.4), one sees directly the conclusion of Theorem: $\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\Big|_{t=1} = \sum_{S\in\langle\Gamma,G\rangle_0} \frac{\varphi(S)}{\#Aut(S)}$ (c.f. (11.1.8)).

4. Due to D. Epstein [E3], we know that there is a wide class of groups satisfying Assumption ii). See the remarks and problems in the next paragraph.

Concluding Remarks and Problems.

We are only at the beginning of the study of limit elements for discrete groups. Here are some problems and conjectures for further study.

1. A formula similar to (11.4.6) should be true without assuming the finiteness of $\Omega(\Gamma, G)$, where the formula should be rewritten as an integral formula.

Problem 1.1 Find measures ν_a on $\pi^{-1}(a)$ and μ_a on the set $Sing(P_{\Gamma,G})$ of singularities of the series on the circle of radius r so that the following holds:

$$\frac{\int_{\pi^{-1}(a)} \omega_{\Gamma,G} d\nu_a}{\int_{\pi^{-1}(a)} d\nu_a} = \int_{Sing(P_{\Gamma,G})} \frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\Big|_{t=x} d\mu_{a,x}. \tag{11.4.9}$$

2. Including Machì's example, there are number of examples where $\Omega(P_{\Gamma,G})$ is finite. However, we do not know of an example such that $\Omega(\Gamma, G)$ is finite except for the simple accumulating case (e.g. (11.1.9)). We conjecture the following.

Conjecture 2.1 For any hyperbolic group Γ with any finite generator system G, the limit set $\Omega(\Gamma, G)$ is finite accumulating.

Evidence in favor of the conjecture is provided by a fact due to Coornaert [Co]: *if Γ is hyperbolic, then there exists positive real constants c_1, c_2 such that* $c_1 r_{\Gamma,G}^{-n} \le \#\Gamma_n \le c_2 r_{\Gamma,G}^{-n}$. This implies the Fact. in the proof of the Assertion in (11.2) which is a consequence of the finite rational accumulation of $\Omega(P_{\Gamma,G})$.

3. The following groups are not hyperbolic. However, because of their geometric significance, it is quite interesting to ask the following problems.

Problem 3.1 Are the limit sets $\Omega(\Gamma, G)$ for the following pair of a group and a system of generators finite?

1. Artin groups of finite type with the generator systems given in [BS][Sa4],
2. The fundamental groups of the complement of free divisors with respect to the generator system defining positive monoid structure [IS].

In these examples, the generator system G determines also the positive monoid Γ_+ such that $\Gamma = \cup_{n=0}^{\infty} \Delta^{-n}\Gamma_+$, where Δ is the fundamental element.

Then, using the $\Gamma_{+,n} := \Gamma_+ \cap \Gamma_n$ for $n \in \mathbb{Z}_{\geq 0}$, we define

$$(11.4.10) \qquad \Omega(\Gamma_+, G) \ := \ \mathcal{L}_{\mathbb{R},\infty} \cap \overline{\{\tfrac{\mathcal{M}(\Gamma_{+,n})}{\sharp \Gamma_{+,n}} \mid n \in \mathbb{Z}_{\geq 0}\}},$$

Problem 3.2 Clarify the relationship between $\Omega(\Gamma, G)$ and $\Omega(\Gamma_+, G)$ (c.f. Corner transfer matrices in case $\Gamma = \mathbb{Z}^2$, [Ba] Chap.13.).

4. It is known ([E3]) that, for a wide class of groups, the assumption ii) in the Theorem is satisfied in a stronger (global) form in the following sense. Namely, under suitable conditions on (Γ, G), the Poincare series $P_{\Gamma,G}(t)$ and the growth series $P_{\Gamma,G}\mathcal{M}(t)$ are rational functions, where the denominator polynomial $\Delta_{\Gamma,G}(t)$ for the rational function $P_{\Gamma,G}(t)$ is also the universal denominator for the rational functions $P_{\Gamma,G}\mathcal{M}(t)$.

We remark that denominator polynomial $\Delta_{P_{\Gamma,G}}(t)$ for the Poincare series $P_{\Gamma,G}(t)$ as an element of $\mathbb{C}\{t\}_{r_{\Gamma,G}}$, which we have studied in the present paper, is the factor of $\Delta_{\Gamma,G}(t)$ consisting of the roots whose absolute value is minimal $(= r_{\Gamma,G})$. Therefore, in order to get a global understanding of limit elements for the group (Γ, G), we propose to study the higher residues of $P_{\Gamma,G}\mathcal{M}(t)$ at any root of $\Delta_{\Gamma,G}(t)$, which are defined and shown to belong to $\mathcal{L}_{\mathbb{C},\infty}$ as follows.

Definition. Let $x \in \mathbb{C}$ be a root of $\Delta_{\Gamma,G}(t) = 0$ of the multiplicity $d_x > 0$. Then, for $0 \leq i < d_x$, we define the *higher residue of depth i of the limit function $P_{\Gamma,G}\mathcal{M}(t)$ at x* by the formula

$$(11.4.11) \qquad \left(\frac{d^i}{dx^i}\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\right)\Bigg|_{t=x}.$$

Assertion. *The higher residues belong to the space $\mathcal{L}_{\mathbb{C},\infty}$ at infinity.*

Proof. By the definition (8.4.1), $\overline{K}(\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}) = \sum_{n=0}^{\infty} \mathcal{M}(\Gamma_n)\frac{t^n}{P_{\Gamma,G}(t)}$, whose coefficients $\frac{t^n}{P_{\Gamma,G}(t)}$ are rational functions divisible by $\Delta_{\Gamma,G}$ and have zeros of order d_x at the zero loci x of $\Delta_{\Gamma,G}$. Since $\overline{K}$ is continuous with respect to the classical topology, this implies the vanishing $\overline{K}\left(\left(\frac{d^i}{dt^i}\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\right)\Big|_{t=x}\right) = 0$ for $0 \leq i < d_x$. $\square$

Using them, let us introduce the *global module of limit elements for* (Γ, G):

$$(11.4.12) \quad \mathcal{L}(\Gamma, G) := \oplus_{0<r<\infty} \oplus_{\substack{x:\text{ a root of} \\ \Delta(t)=0 \ s.t. \ |x|=r}} \oplus_{0 \leq i < d_x} \mathbb{C} \cdot \left(\frac{d^i}{dx^i}\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\right)\Bigg|_{t=x},$$

which is doubly filtered: one filtration is given by the absolute values $|x|$ of the roots of $\Delta_{\Gamma,G}(t) = 0$, and the other by the order i of the depth of residues at x.

Theorems in §11 state relationships between the $\tilde{\tau}^{h_{\Gamma,G}}$-invariant part of the module $\overline{\mathbb{R}\Omega}(\Gamma, G)$ with the filter at $|x| = \inf\{r\}$ and the first residues part of the module $\mathcal{L}(\Gamma, G)$. We ask its generalization.

Problem 4.1 What is the relationship between the modules $\overline{\mathbb{R}\Omega}(\Gamma, G)$, $\mathcal{L}(\Gamma, G)$ and $\mathcal{L}_{\mathbb{C},\infty}\langle\Gamma, G\rangle$? Find generalization of Theorems in §11 and, in particular, of (11.3.14), (11.3.15) and (11.4.6) in this context.

5. A particular interest for the space of the residues at $t = 1$ is caused by the following example.

Example.([Sa2]) Consider the infinite cyclic group $(\mathbb{Z}, \pm 1)$. Then, $P_{\mathbb{Z},\pm 1} = \frac{1+t}{(1-t)^2}$ and $P_{\mathbb{Z},\pm 1}\mathcal{M} = \sum_{m=0}^{\infty} \varphi(I_m)\left(\frac{2}{(1-t)^2} - \frac{m}{1-t} + R_m\right)$ where I_m is a linear graph of m-vertices and R_m is a polynomial in t. Then, $\left.\frac{P_{\mathbb{Z},\pm 1}\mathcal{M}}{P_{\mathbb{Z},\pm 1}}\right|_{t=1} = \sum_{m=0}^{\infty} \varphi(I_m)$ and $\left.\left(\frac{d}{dt}\frac{P_{\mathbb{Z},\pm 1}\mathcal{M}}{P_{\mathbb{Z},\pm 1}}\right)\right|_{t=1} = \sum_{m=0}^{\infty} \frac{m-1}{2}\varphi(I_m)$ span the space $\mathcal{L}_{\mathbb{R},\infty}\langle\mathbb{Z}, \pm 1\rangle$.

5.1 What is the meaning of the submodule $\mathcal{L}(\Gamma, G)_1 \coloneqq \oplus_{0 \le i < d_1} \mathbb{R} \left.\left(\frac{d^i}{dt^i} \frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\right)\right|_{t=1}$ at t=1?

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