Even Factors, Jump Systems, and Discrete Convexity

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Abstract

A jump system, which is a set of integer lattice points with an exchange property, is an extended concept of a matroid. Some combinatorial structures such as the degree sequences of the matchings in an undirected graph are known to form a jump system.

On the other hand, the maximum even factor problem is a generalization of the maximum matching problem into digraphs. When the given digraph has a certain property called odd-cycle-symmetry, this problem is polynomially solvable.

The main result of this paper is that the degree sequences of all even factors in a digraph form a jump system if and only if the digraph is odd-cycle-symmetric. Furthermore, as a generalization, we show that the weighted even factors induce M-convex (M-concave) functions on jump systems. These results suggest that even factors are a natural generalization of matchings and the assumption of odd-cycle-symmetry of digraphs is essential.

1 Introduction

In the study of combinatorial optimization, extensions of matroids are introduced as abstract concepts including many combinatorial objects. A number of optimization problems on matroidal structures can be solved in polynomial time. One of the extensions of matroids is a *jump system* of Bouchet and Cunningham [2]. A jump system is a set of integer lattice points with an exchange property (to be described in Section 2.1); see also [18, 22]. It is a generalization of a matroid [4], a delta-matroid [1, 3, 8], and a base polyhedron of an integral polymatroid (or a submodular system) [14]. Typical examples of jump systems are the degree sequences of all subgraphs in an undirected graph and those of all matchings in an undirected graph.

The concept of an *M-convex* (*M-concave*) *function on a jump system* is a quantitative extension of a jump system, which was introduced by Murota [26] as a common generalization of a valuated matroid [9, 11], a valuated delta-matroid [10], and an M-convex function on a base polyhedron [24, 25]. A separable convex function on the degree sequences of an undirected graph is a typical example of an M-convex function on a constant-parity jump system. In what follows, we refer to "M-convex (resp. M-concave) functions on constant-parity jump systems" simply as "M-convex (resp. M-concave) functions." An M-convex function has a virtue that global optimality (minimality) is

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guaranteed by local optimality in the neighborhood of ℓ_1-distance two [26], and several efficient algorithms minimizing M-convex functions [27, 31] follow from this optimality criterion. A recent work of Kobayashi, Murota, and Tanaka [20] showed that M-convex functions are closed under several basic operations. Meanwhile, it is also known that we cannot establish duality results such as the discrete separation theorems.

On the other hand, as a common generalization of the matching and matroid intersection problems, Cunningham and Geelen [6] defined the *basic path-matching problem* in an undirected graph. The basic path-matching problem with free matroids is said to be the *path-matching problem*. Cunningham [5] claimed that the degree sequences of the path-matchings form a jump system.

As a further generalization of the path-matching problem, Cunningham and Geelen [7] introduced the *even factor problem*, which is defined in directed graphs (digraphs). Recent works on even factors [19, 28–30, 32] suggest that the even factor problem generalizes the non-bipartite matching problem in a combinatorially tractable direction. While finding the maximum even factor in a general digraph is NP-hard, some polynomial algorithms [7, 16, 28, 29] are known for a certain class of digraphs, called *odd-cycle-symmetric* (to be defined in Section 2.2). Among them, Pap's alternating path algorithm [28, 29] is a combinatorial one that extends Edmonds' maximum matching algorithm [12].

A common generalization of the even factors and matroid intersection has been considered [7,17]. Cunningham and Geelen [7] proposed a polynomial reduction of the *basic even factor problem* to matroid intersection, in which an even factor algorithm is required for each matroid oracle. Iwata and Takazawa [17] dealt with the *independent even factor problem*, which is essentially equivalent to the basic even factor problem. They devised a combinatorial independent even factor algorithm, which combines Pap's even factor algorithm and the matroid intersection algorithm [13, 21], and exhibited a structure theorem that commonly generalizes the Edmonds-Gallai decomposition for matchings and the principal partition for matroid intersection. We remark that these works on basic/independent even factors also need the assumption of the odd-cycle-symmetry of the digraph.

The *weighted path-matching problem* and the *weighted even factor problem* are considered as natural quantitative extensions. Cunningham and Geelen [6] presented a linear inequality system describing the weighted basic path-matching problem and proved its total dual integrality. Cunningham and Geelen [7] clarified the integrality of a polytope associated with the weighted even factor problem in a certain class called *weakly symmetric weighted digraphs* and proposed a combinatorial primal-dual method of finding a maximum weight even factor by solving the unweighted problems repeatedly. Király and Makai [19] showed that the integrality also holds in the class of *odd-cycle-symmetric weighted digraphs*, a broader class than that of weakly symmetric weighted digraphs. For an odd-cycle-symmetric weighted digraph, they presented a linear program that describes the maximum weight even factor problem and proved its dual integrality. They also provided a characterization of odd-cycle-symmetric digraphs. Based on Király and Makai's description and Pap's alternating path framework, Takazawa [32] gave a combinatorial weighted even factor algorithm for odd-cycle-symmetric weighted digraphs.

In this paper, we reveal that the set of the degree sequences of all even factors in a digraph forms a jump system if and only if the digraph is odd-cycle-symmetric. The sufficiency is immediately derived from an algebraic approach in [5,7]. After accomplishing this approach to a proof, we give an alternative combinatorial proof with the aid of an alternating path algorithm. This approach is a natural extension of a constructive proof showing that the degree sequences all subgraphs/matchings in an undirected graph form a jump system. We also prove that the assumption of odd-cycle-symmetry is necessary for the degree sequences of the even factors to form a jump system.

Moreover, this relation is extended to a weighted version. A function on the degree sequences of the even factors naturally arises from weighted even factors. We exhibit that the function is M-concave if and only if the weighted digraph is odd-cycle-symmetric. The sufficiency can also be proved in two ways: one is an algebraic approach utilizing valuated matroids; and the other is applying the alternating path algorithm to an odd-cycle-symmetric weighted digraph. We analyze the necessity through the characterization of odd-cycle-symmetric weighted digraphs.

Through these results, one would see that even factors, as well as matchings, have a matroidal structure and the notion of odd-cycle-symmetry is essential in dealing with even factors as a solvable generalization of matchings. These results also give a new example of a jump system that does not consist of $\{0,1\}$-vectors.

This paper is organized as follows. Section 2 introduces the definitions and prior works on jump systems and even factors. Section 3 proves that the degree sequences of the even factors in a digraph form a jump system if and only if the digraph is odd-cycle-symmetric. This result is extended to a weighted version in Section 4, where we show that a function defined by the weighted even factors is M-concave if and only if the weighted digraph is odd-cycle-symmetric. Finally, in Section 5, we discuss an undirected version of these results and present a new example of a jump system which is not a delta-matroid.

2 Preliminaries

2.1 Jump systems

Let V be a finite set. For $x = (x(v)), y = (y(v)) \in \mathbf{R}^V$, define

$$x(U) = \sum_{v \in U} x(v) \quad (U \subseteq V),$$
$$[x, y] = \{z \mid z \in \mathbf{R}^V, \min(x(v), y(v)) \le z(v) \le \max(x(v), y(v)), \forall v \in V\}.$$

For $U \subseteq V$, we denote by χ_U the *characteristic vector* of U, with $\chi_U(v) = 1$ for $v \in U$ and $\chi_U(v) = 0$ for $v \in V \setminus U$. For $u \in V$, we denote $\chi_{\{u\}}$ simply by χ_u. For $x, y \in \mathbf{Z}^V$, a vector $s \in \mathbf{Z}^V$ is called an (x, y)*-increment* if $s = \chi_u$ or $s = -\chi_u$ for some $u \in V$ and $x + s \in [x, y]$.

Definition 2.1 (Jump system)**.** A nonempty set $J \subseteq \mathbf{Z}^V$ is said to be a *jump system* if it satisfies an exchange axiom, called the 2*-step axiom*:

> For any $x, y \in J$ and for any (x, y)-increment s with $x + s \notin J$, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$.

A set $J \subseteq \mathbf{Z}^V$ is a *constant-sum system* if $x(V) = y(V)$ for any $x, y \in J$, and a *constant-parity system* if $x(V) - y(V)$ is even for any $x, y \in J$. For constant-parity jump systems, J. F. Geelen pointed out a stronger exchange property:

(EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$ and $y - s - t \in J$.

This property characterizes a constant-parity jump system (see Murota [26] for details).

Theorem 2.2. *A nonempty set J is a constant-parity jump system if and only if it satisfies* (EXC).

In what follows, we refer to "constant-parity jump systems" simply as "jump systems."

Typical examples of a jump system are the degree sequences of all subgraphs and those of all matchings in an undirected graph. That is, for an undirected graph G with vertex set V and edge set E, define $J_{\mathrm{SG}}(G) \subseteq \mathbf{Z}^V$ and $J_{\mathrm{M}}(G) \subseteq \{0, 1\}^V$ by

$$J_{\mathrm{SG}}(G) = \{x \mid \exists F \subseteq E,\ x(v) = |\{e \mid e \in F,\ e \text{ is incident to } v\}|\},$$
$$J_{\mathrm{M}}(G) = \{x \mid \exists \text{matching } M \subseteq E,\ x(v) = |\{e \mid e \in M,\ e \text{ is incident to } v\}|\}.$$

Then, both $J_{\mathrm{SG}}(G)$ and $J_{\mathrm{M}}(G)$ are jump systems, provided that G has no loops.

The index sets of the nonsingular submatrices also form a jump system. For a matrix T with row set V^+ and column set V^-, we denote the submatrix of T with row set $U^+ \subseteq V^+$ and column set $U^- \subseteq V^-$ by $T[U^+, U^-]$.

Lemma 2.3 (Cunningham [5])**.** *Let T be a matrix with rows and columns indexed by V^+ and V^-, respectively. Then, $J \subseteq \mathbf{Z}^{V^+ \cup V^-}$ defined by*

$$J = \{\chi_{U^+ \cup U^-} \mid U^+ \subseteq V^+,\ U^- \subseteq V^-,\ |U^+| = |U^-|,\ \det T[U^+, U^-] \neq 0\}.$$

forms a jump system.

One of the most important operations on a jump system is *elementary aggregation.* For a jump system $J \subseteq \mathbf{Z}^V$, its *elementary aggregation* $\tilde{J} \subseteq \mathbf{Z}^{\tilde{V}}$ at $v_1 \in V$ and $v_2 \in V$ is defined by

$$\tilde{J} = \{(x_0, x(v_1) + x(v_2)) \mid (x_0, x(v_1), x(v_2)) \in J\},$$

where $\tilde{V} = (V \setminus \{v_1, v_2\}) \cup \{v\}$ and $x_0 \in \mathbf{Z}^{V \setminus \{v_1, v_2\}}$. Then, $\tilde{J}$ also forms a jump system [5, 20].

Lemma 2.4. *An elementary aggregation of a jump system is a jump system.*

An *M-concave* (*M-convex*) *function on a jump system* of Murota [26] is a quantitative extension of a jump system.

Definition 2.5 (M-concave function)**.** For $J \subseteq \mathbf{Z}^V$, we call $f : J \to \mathbf{R}$ an *M-concave function* if it satisfies the following exchange axiom:

(M-EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$, $y - s - t \in J$, and $f(x) + f(y) \leq f(x + s + t) + f(y - s - t)$.

It directly follows from (M-EXC) that J satisfies (EXC), and hence J is a jump system. We call a function $f : J \to \mathbf{R}$ an *M-convex function* if $-f$ is an M-concave function. If the domain of an M-concave function f is required to be $\mathbf{Z}^V$, we adopt the convention of $f(x) = -\infty$ for $x \in \mathbf{Z}^V \setminus J$.

The definition of an M-concave (M-convex) function is consistent with the previously considered special cases where (i) J is a constant-sum jump system, and (ii) J is a constant-parity jump system contained in $\{0,1\}^V$. Case (i) is equivalent to J being the set of integer points in the base polyhedron of an integral submodular system [14], and then M-concave (M-convex) function is the same as the M-concave (M-convex) function on base polyhedra investigated in [24,25]. Case (ii) is equivalent to J being an even delta-matroid [33,34], and then an M-concave function is equivalent to a valuated delta-matroid in the sense of [10].

The operation of elementary aggregation is extended to M-concave functions. For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$, the *elementary aggregation* of f at $v_1 \in V$ and $v_2 \in V$ is a function $\tilde{f} : \mathbf{Z}^{\tilde{V}} \to \mathbf{R} \cup \{+\infty, -\infty\}$ defined by

$$\tilde{f}(x_0; \xi) = \sup\{f(x_0; x(v_1), x(v_2)) \mid \xi = x(v_1) + x(v_2)\},$$

where $\tilde{V} = (V \setminus \{v_1, v_2\}) \cup \{v\}$ and $x_0 \in \mathbf{Z}^{V \setminus \{v_1, v_2\}}$. A recent work of Kobayashi, Murota, and Tanaka [20] proved that if f is M-concave, then so is $\tilde{f}$.

Lemma 2.6 (Kobayashi, Murota, and Tanaka [20])**.** *If f is M-concave, then its elementary aggregation $\tilde{f}$ is M-concave, provided that $\tilde{f}(x) < +\infty$ for any $x \in \mathbf{Z}^{\tilde{V}}$.*

2.2 Even factors

Let $G = (V, A)$ be a digraph with vertex set V and arc set A. Throughout this paper, we assume that digraphs have neither loops nor multiple arcs. For an arc a, we refer to its initial vertex u and terminal vertex v as $\partial^+ a$ and $\partial^- a$, respectively, and denote $a = (u, v)$. The reverse arc of a is denoted by $\bar{a}$, that is, $\bar{a} = (v, u)$. We say that $a \in A$ is *symmetric* if $\bar{a} \in A$, and G is *symmetric* if every arc in A is symmetric. For an arc set $F \subseteq A$, we denote $\partial^+ F = \{v \mid v \in V,\ \exists a \in F,\ \partial^+ a = v\}$ and $\partial^- F = \{v \mid v \in V,\ \exists a \in F,\ \partial^- a = v\}$. For a vertex $v \in V$, define $\delta^+ v = \{a \mid a \in A,\ \partial^+ a = v\}$ and $\delta^- v = \{a \mid a \in A,\ \partial^- a = v\}$.

For $v_0, v_1, \ldots, v_k \in V$ and $a_1, \ldots, a_k \in A$, a sequence $(v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_k)$ is said to be a *walk* if $a_i = (v_{i-1}, v_i)$ for $i = 1, \ldots, k$. A walk $W = (v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_k)$ is said to be a *path* if $v_0, \ldots, v_k$ are pairwise distinct, and a *cycle* if $v_0, \ldots, v_{k-1}$ are pairwise distinct and $v_0 = v_k$. The reverse walk of W is denoted by $\bar{W}$, that is, $\bar{W} = (v_k, \bar{a}_k, v_{k-1}, \ldots, v_1, \bar{a}_1, v_0)$. We denote the vertex set $\{v_0, \ldots, v_k\} = V(W)$ and the arc set $\{a_1, \ldots, a_k\} = A(W)$. The length of W, denoted by $|W|$, is defined by k. A walk W is said to be *odd* (resp. *even*) if $|W|$ is odd (resp. even). For paths $P_1 = (v_0, a_1, v_1, \ldots, a_k, v_k)$ and $P_2 = (v_k, a_{k+1}, v_{k+1}, \ldots, a_l, v_l)$, we denote by $P_1 \cup P_2$ the walk $(v_0, a_1, \ldots, a_k, v_k, a_{k+1}, v_{k+1}, \ldots, v_l)$. If P_1 and P_2 share no common vertices except for v_k, $P_1 \cup P_2$ is a path.

If a digraph $G = (V, A)$ is accompanied with a weight vector $w \in \mathbf{R}^A$, we say (G, w) is a weighted digraph. For an arc set $F \subseteq A$, the weight of F is defined by $w(F)$, the total weight of the constituent arcs of F.

Definition 2.7 (Even factor)**.** An arc set $M \subseteq A$ is an *even factor* if it forms a vertex-disjoint collection of paths and even cycles.

Definition 2.8. For two vertex sets $U^+, U^- \subseteq V$, an arc set $M \subseteq A$ is an *even* (U^+, U^-)*-factor* if M is an even factor with $\partial^+ M = U^+$ and $\partial^- M = U^-$.

One easily sees that $|M \cap \delta^+ v| \leq 1$ and $|M \cap \delta^- v| \leq 1$ hold for any even factor M and every $v \in V$, and if an even (U^+, U^-)-factor M exists, it holds that $|M| = |U^+| = |U^-|$.

The even factor problem is to find an even factor with the maximum number of arcs. One can reduce the maximum matching problem to the even factor problem in the following manner. Let $\bar{G} = (V, E)$ be an undirected graph in which we want to find the maximum matching. Then, construct a symmetric digraph $G = (V, A)$, where $A = \{(u, v), (v, u) \mid u \text{ and } v \text{ are adjacent in } \bar{G}\}$. One easily sees that the maximum even factor M in G is composed of even length cycles. Then one can take up arcs in M along these cycles alternately to obtain a vertex-disjoint collection of $|M|/2$ arcs, which corresponds to a matching in $\bar{G}$. Conversely, given a matching $\bar{M}$ in $\bar{G}$, one can replace the edges in $\bar{M}$ by 2-length cycle to get an even factor of size $2|\bar{M}|$ in G.

The maximum even factor problem is known to be NP-hard. Prior works on even factors assume the digraph to have a certain property, called *odd-cycle-symmetric*.

Definition 2.9 (Odd-cycle-symmetric digraph)**.** A digraph $G = (V, A)$ is called *odd-cycle-symmetric* if every arc in any odd cycle is symmetric.

Of course, symmetric digraphs are odd-cycle-symmetric, which means that the maximum even factor problem in the odd-cycle-symmetric digraphs includes the maximum matching problem. Several polynomial algorithms [7,16,28,29] are proposed that are applicable to odd-cycle-symmetric digraphs. Among them, Pap's [28,29] is a combinatorial one that extends the maximum matching algorithm of Edmonds [12]. His algorithm increases the size of the even factor one by one with the aid of *alternating paths* until it attains the maximum.

A weighted version of the maximum even factor problem is also considered. Let $G = (V, A)$ be a digraph and $w : A \to \mathbf{R}$ be a weight vector. The maximum weight even factor problem is to find an even factor M that maximizes $w(M)$ among all even factors in G.

In dealing with the maximum weight even factor problem, Király and Makai [19] defined an extension of odd-cycle-symmetry to weighted digraphs.

Definition 2.10 (Odd-cycle-symmetric weighted digraph)**.** A weighted digraph (G, w) is *odd-cycle-symmetric* if G is odd-cycle-symmetric in the unweighted sense and $w(A(C)) = w(A(\bar{C}))$ holds for any odd cycle C.

Király and Makai [19] showed a linear inequality system that describes the maximum weight even factor problem in odd-cycle-symmetric weighted digraphs and proved its dual integrality. Some polynomial algorithms are proposed to solve the problem in odd-cycle-symmetric weighted digraphs [7,32].

In proving the dual integrality, Király and Makai utilized a characterization of odd-cycle-symmetric digraphs. We say that a digraph is *cycle-connected* if it is strongly connected and its underlying graph is biconnected. Note that a digraph can be decomposed into cycle-connected components. A digraph is *bipartite* if its underlying graph is bipartite. Using these notions, Z. Király characterized odd-cycle-symmetric digraphs (see [19] for details).

Lemma 2.11. *Each cycle-connected component of an odd-cycle-symmetric digraph is either bipartite or symmetric.*

A characterization of odd-cycle-symmetric weighted digraphs follows from Lemma 2.11. A potential function π on V is said to be *valid* if $w(a) - w(\bar{a}) = \pi(v) - \pi(u)$ holds for each symmetric arc $a = (u, v) \in A$. The characterization below immediately follows from that of Király and Makai [19, Lemma 2], which claims that $w(A(C)) = w(A(\bar{C}))$ holds for every cycle C in a non-bipartite cycle-connected component of an odd-cycle-symmetric weighted digraph.

Lemma 2.12. *Let (G, w) be a weighted digraph such that G is cycle-connected but not bipartite. Then, (G, w) is an odd-cycle-symmetric weighted digraph if and only if there exists a valid potential function π.*

3 Even factors and jump systems

In this section, we show that the degree sequences of the even factors in a digraph form a jump system if and only if the digraph is odd-cycle-symmetric. In dealing with this topic, the notion of degree sequences is extended into digraphs.

3.1 Main theorem

First of all, let us define the degree sequences of digraphs. Make two copies V^+ and V^- of V. The copy of $v \in V$ in V^+ (resp. V^-) is denoted by v^+ (resp. v^-).

Definition 3.1 (Degree sequence of digraphs)**.** For a digraph $G = (V, A)$ and its arc set $F \subseteq A$, the *degree sequence* of F is a vector $d_F \in \mathbf{Z}^{V^+ \cup V^-}$ defined by

$$d_F(v^+) = \left|F \cap \delta^+ v\right|, \ d_F(v^-) = \left|F \cap \delta^- v\right| \quad (v \in V).$$

Let $J_{\mathrm{EF}}(G) \subseteq \mathbf{Z}^{V^+ \cup V^-}$ be the set of the degree sequences of all even factors in G. That is,

$$J_{\mathrm{EF}}(G) = \{d_M \mid M \text{ is an even factor in } G\}.$$

By the definition of even factors, one would easily see that $J_{\mathrm{EF}}(G) \subseteq \{0, 1\}^{V^+ \cup V^-}$. The main result of this paper is that $J_{\mathrm{EF}}(G)$ is a jump system if and only if G is odd-cycle-symmetric.

Theorem 3.2. *The set of degree sequences of all even factors in a digraph G form a jump system if and only if G is odd-cycle-symmetric.*

As we stated in the previous section, the set of the degree sequences of all the matchings in an arbitrary undirected graph is a jump system. Theorem 3.2 exhibits a necessary and sufficient condition for the even factors to have the same structure. This means that the assumption of the odd-cycle-symmetry is reasonable and essential when we deal with optimization problems on even factors.

We devote the rest of this section to proving Theorem 3.2. The sufficiency (Proposition 3.4) and the necessity (Proposition 3.10) are proved separately.

Remark 3.3. As mentioned above, $J_{\mathrm{EF}}(G)$ consists of $\{0,1\}$-vectors, so Theorem 3.2 (and the arguments below) can be stated in terms of even delta-matroids. However, in Section 5, we will derive jump systems that are not delta-matroids as a consequence of Theorem 3.2. For this reason, we allow ourselves to say that $J_{\mathrm{EF}}(G)$ is a "jump system," instead of an "even delta-matroid."

3.2 Sufficiency of odd-cycle-symmetry

In this subsection, we prove the sufficiency in Theorem 3.2.

Proposition 3.4. *$J_{\mathrm{EF}}(G)$ is a jump system if G is odd-cycle-symmetric.*

We exhibit two proofs for Proposition 3.4. One proof follows from an algebraic approach in [5,7], and the other employs an alternating path algorithm.

3.2.1 Algebraic proof

The even factors are related to a certain matrix called *Tutte matrix*. Let $\{t_{uv} \mid (u,v) \in A\}$ be a set of indeterminates associated with A such that $t_{uv} = -t_{vu}$ if $(u,v),(v,u) \in A$. The Tutte matrix $T = (T_{u^+v^-})$ of G, whose rows and columns are indexed by V^+ and V^-, is defined by

$$T_{u^+v^-} = \begin{cases} t_{uv} & \text{(if } (u,v) \in A\text{)}, \\ 0 & \text{(otherwise)}. \end{cases}$$

The following theorem represents the relation between the Tutte matrix and the even factors in an odd-cycle-symmetric digraph.

Theorem 3.5 (Cunningham and Geelen [7])**.** *Let $G = (V,A)$ be an odd-cycle-symmetric digraph and $T = (T_{u^+v^-})$ be the Tutte matrix of G. For $U^+, U^- \subseteq V$ with $|U^+| = |U^-|$, it holds that $\det T[U^+,U^-] \neq 0$ if and only if there exists an even (U^+,U^-)-factor in G.*

Cunningham and Geelen [7] showed that this theorem holds in *weakly symmetric* digraphs, which is a subclass of odd-cycle-symmetric digraphs. However, we can prove this theorem for odd-cycle-symmetric digraphs without any modification to the argument in [7].

A proof for Proposition 3.4 follows from Theorem 3.5.

Algebraic proof for Proposition 3.4. By the definition of $J_{\mathrm{EF}}(G)$ and Theorem 3.5, it holds that

$$\begin{aligned} J_{\mathrm{EF}}(G) &= \{d_M \mid M \text{ is an even factor in } G\} \\ &= \{\chi_{U^+\cup U^-} \mid U^+ \subseteq V^+,\ U^- \subseteq V^-,\ |U^+| = |U^-|,\ \det T[U^+,U^-] \neq 0\}. \end{aligned}$$

Hence, by Lemma 2.3, $J_{\mathrm{EF}}(G)$ is a jump system. □

3.2.2 Constructive proof

Here we present an alternative constructive proof for Proposition 3.4 with the aid of alternating paths. That is, we prove that $J_{\mathrm{EF}}(G)$ satisfies (EXC) by presenting an algorithm that finds an $(x+s,y)$-increment t for any $x,y \in J_{\mathrm{EF}}(G)$ and any (x,y)-increment s. We remark here that this approach is easily extensible to a weighted version discussed in Section 4.

For a digraph $G=(V,A)$, construct an auxiliary bipartite graph $G^\circ = (V^\circ, A^\circ)$ as follows. The vertex set $V^\circ = V^+ \cup V^-$, and the arc set A° is defined by

$$A^\circ = \{(u^+, v^-) \mid (u,v) \in A\}.$$

For an arc set $F \subseteq A$, denote its corresponding arc set in A° by F°, that is, $F^\circ = \{(u^+, v^-) \mid (u,v) \in F\}$. Where it causes no confusion, for $a=(u,v) \in A$ and $a^\circ = (u^+, v^-) \in A^\circ$, we identify them and often denote a° simply by a or (u,v).

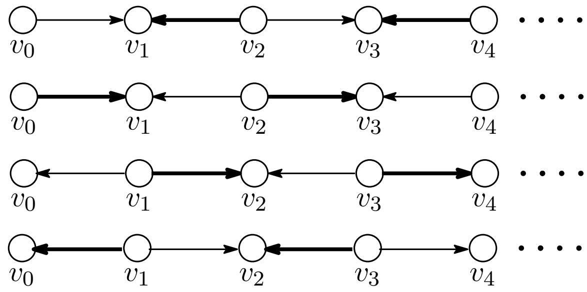
In the auxiliary bipartite graph G°, we introduce the notion of *alternating walk* from the viewpoint of degree sequence. Let $K, L \subseteq A$ be two arc sets in G. We say that a sequence $W = (v_0, a_1, v_1, a_2, v_2, \ldots, v_{k-1}, a_k, v_k)$ in G° is a (K,L)-*alternating walk* if one of the following conditions holds.

- $v_i \in \begin{cases} V^+ & \text{(for even } i\text{)}, \\ V^- & \text{(for odd } i\text{)}, \end{cases} \quad a_i = \begin{cases} (v_{i-1}, v_i) \in K^\circ \setminus L^\circ & \text{(for odd } i\text{)}, \\ (v_i, v_{i-1}) \in L^\circ \setminus K^\circ & \text{(for even } i\text{)}, \end{cases}$
- $v_i \in \begin{cases} V^+ & \text{(for even } i\text{)}, \\ V^- & \text{(for odd } i\text{)}, \end{cases} \quad a_i = \begin{cases} (v_{i-1}, v_i) \in L^\circ \setminus K^\circ & \text{(for odd } i\text{)}, \\ (v_i, v_{i-1}) \in K^\circ \setminus L^\circ & \text{(for even } i\text{)}, \end{cases}$
- $v_i \in \begin{cases} V^- & \text{(for even } i\text{)}, \\ V^+ & \text{(for odd } i\text{)}, \end{cases} \quad a_i = \begin{cases} (v_i, v_{i-1}) \in K^\circ \setminus L^\circ & \text{(for odd } i\text{)}, \\ (v_{i-1}, v_i) \in L^\circ \setminus K^\circ & \text{(for even } i\text{)}, \end{cases}$
- $v_i \in \begin{cases} V^- & \text{(for even } i\text{)}, \\ V^+ & \text{(for odd } i\text{)}, \end{cases} \quad a_i = \begin{cases} (v_i, v_{i-1}) \in L^\circ \setminus K^\circ & \text{(for odd } i\text{)}, \\ (v_{i-1}, v_i) \in K^\circ \setminus L^\circ & \text{(for even } i\text{)}. \end{cases}$

The arc set in W, $\{a_1, \ldots, a_k\}$, is denoted by $A^\circ(W)$, and its corresponding arc set in G by $A(W)$. An illustration of these four patterns of alternating walk is shown in Figure 1. We note that we are inspired to introduce this notion of (K,L)-alternating walk by the alternating paths in Pap's even factor algorithm [28, 29].

Lemma 3.6. *Let K and L be arc sets in a digraph $G=(V,A)$. If s is a (d_K, d_L)-increment, then there exist a (K,L)-alternating walk $(v_0, a_1, v_1, \ldots, v_{k-1}, a_{k-1}, v_k)$ and a $(d_K + s, d_L)$-increment t that satisfy the following* (i) *and* (ii):

(i) $a_1 = \begin{cases} (v_0, v_1) \in K^\circ \setminus L^\circ & \textit{(if } s = -\chi_{v_0} \textit{ and } v_0 \in V^+\textit{)}, \\ (v_0, v_1) \in L^\circ \setminus K^\circ & \textit{(if } s = \chi_{v_0} \textit{ and } v_0 \in V^+\textit{)}, \\ (v_1, v_0) \in K^\circ \setminus L^\circ & \textit{(if } s = -\chi_{v_0} \textit{ and } v_0 \in V^-\textit{)}, \\ (v_1, v_0) \in L^\circ \setminus K^\circ & \textit{(if } s = \chi_{v_0} \textit{ and } v_0 \in V^-\textit{)}, \end{cases}$



($\longrightarrow$: arc in $K^\circ \setminus L^\circ$, $\boldsymbol{\longrightarrow}$: arc in $L^\circ \setminus K^\circ$)

Figure 1: Four patterns of a (K, L)-alternating walk.

$$
\text{(ii)}\ \ a_{k-1} = \begin{cases} (v_k, v_{k-1}) \in K^\circ \setminus L^\circ & (\textit{if } t = -\chi_{v_k} \textit{ and } v_k \in V^+), \\ (v_k, v_{k-1}) \in L^\circ \setminus K^\circ & (\textit{if } t = \chi_{v_k} \textit{ and } v_k \in V^+), \\ (v_{k-1}, v_k) \in K^\circ \setminus L^\circ & (\textit{if } t = -\chi_{v_k} \textit{ and } v_k \in V^-), \\ (v_{k-1}, v_k) \in L^\circ \setminus K^\circ & (\textit{if } t = \chi_{v_k} \textit{ and } v_k \in V^-). \end{cases}
$$

Moreover, if d_K and d_L are $\{0, 1\}$-vectors, then such a (K, L)-alternating walk is unique.

Proof. We prove this lemma by induction on $|A|$. Without loss of generality, we only discuss the case where $s = -\chi_{v_0}$ and $v_0 \in V^+$. Take an arbitrary arc $a_1 = (v_0, v_1) \in K^\circ \setminus L^\circ$.

The case where $-\chi_{v_1}$ is a $(d_K + s, d_L)$-increment is obvious. So, without loss of generality, we assume that $d_K(v_1) \le d_L(v_1)$, which implies that there exists an arc $a_2 \in (L^\circ \setminus K^\circ) \cap \delta^- v_1$. Here, eliminate a_1 and consider arc sets $K^\circ \setminus \{a_1\}$ and L°. It holds that χ_{v_1} is a $(d_{K\setminus\{a_1\}}, d_L)$-increment. Hence, by the induction hypothesis, there exist a $(K \setminus \{a_1\}, L)$-alternating walk $(v_1, a_2, v_2, \dots, v_{k-1}, a_k, v_k)$ and a $(d_{K\setminus\{a_1\}} + s, d_L)$-increment t such that satisfy Condition (ii) in which K° is replaced by $K^\circ \setminus \{a_1\}$. Then, $(v_0, a_1, v_1, a_2, v_2, \dots, v_{k-1}, a_k, v_k)$ is a (K, L)-alternating walk that satisfies Conditions (i) and (ii).

Moreover, if d_K and d_L are $\{0, 1\}$-vectors, we pick up as a_1 the unique arc in $K^\circ \cap \delta^+ v_0$, which implies the uniqueness of the (K, L)-alternating walk. □

For an odd-cycle-symmetric digraph G, let $x, y \in J_{\mathrm{EF}}(G)$ and s be an (x, y)-increment. Here is the description of the algorithm for finding a $(d_K + s, d_L)$-increment t. We remark that a variable τ indicates the time step of the algorithm and τ is used only for the analysis of the algorithm.

Algorithm FIND-INCREMENT

Step 1. Set $i = 1$ and $\tau = 0$, and let u be an element of $V^+ \cup V^-$ such that $s = \chi_u$ or $s = -\chi_u$. Find even factors M and N such that $d_M = x$ and $d_N = y$. Then, go to Step 2.

Step 2. Let $W = (v_0, a_1, v_1, a_2, v_2, \dots, v_{k-1}, a_k, v_k)$ be the unique (M, N)-alternating walk defined in Lemma 3.6. If both of $M' = M \Delta A(W)$ and $N' = N \Delta A(W)$ do not contain odd cycles, then the algorithm terminates, and we have that M' and N' are even factors and χ_{v_k} or $-\chi_{v_k}$ is an $(x + s, y)$-increment t such that $d_{M'} = x + s + t$ and $d_{N'} = y - s - t$. Otherwise, go to Step 3.

Step 3. Update M and N by

$$(M,N) := \begin{cases} (M \setminus \{a_i\},\ N \cup \{a_i\}) & (\text{if } a_i \in M), \\ (M \cup \{a_i\},\ N \setminus \{a_i\}) & (\text{if } a_i \in N). \end{cases}$$

If both of M and N do not contain an odd cycle, then go to Step 4. Otherwise, set $a^* = a_i$ and go to Step 5.

Step 4. If $i = k$, then the algorithm terminates, and we have that M and N are even factors and χ_u or $-\chi_u$ is an $(x+s, y)$-increment t such that $d_M = x + s + t$ and $d_N = y - s - t$. Otherwise, set $\tau := \tau + 1$, $u := v_i$, and $i := i + 1$ and go to Step 3.

Step 5. Let L, either of M and N, be the even factor which came to contain an odd cycle C in Step 3. Update L by replacing $A(C)$ with $A(\bar{C})$. Then, set $\tau := \tau + 1$, $u := v_i$, and find a new (M,N)-alternating walk $(v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_k)$ with $v_0 = u$, $a_1 \in (A(\bar{C}))^\circ \setminus \{\bar{a}^*\}$, and χ_{v_k} or $-\chi_{v_k}$ is an $(x+s, y)$-increment. Then, set $i = 1$ and go to Step 3.

Note that the existence of the (M,N)-alternating in Steps 5 follows from a similar argument to Lemma 3.6 and that of $\bar{C}$ in Step 5 follows from the odd-cycle-symmetry of G.

Here, all that is left is to validate Algorithm FIND-INCREMENT, that is, to show that Algorithm FIND-INCREMENT terminates in finite steps. We denote M, N, and u in $\tau = \tau_0$ by $M(\tau_0)$, $N(\tau_0)$, and $u(\tau_0)$, respectively.

It is not difficult to see that Algorithm FIND-INCREMENT has the following properties.

Lemma 3.7. *After the update of τ and u, it holds that $d_{M(\tau)}(v) = (x+s)(v)$ and $d_{N(\tau)}(v) = (y-s)(v)$ for each $v \in V^+ \cup V^- \setminus \{u(\tau)\}$.*

Lemma 3.8. *In Algorithm FIND-INCREMENT, if M or N comes to contain an odd cycle, the odd cycle contains u.*

In order to prove the finiteness, we now assume to the contrary that $M(\tau_1) = M(\tau_2)$, $N(\tau_1) = N(\tau_2)$, $u(\tau_1) = u(\tau_2)$, and $u(\tau_1 + 1) = u(\tau_2 + 1)$ hold for $\tau_1 < \tau_2$. Consider the pair of (τ_1, τ_2) with τ_1 the smallest in such pairs. Without loss of generality, we assume that we exchanged an arc from N to M in $\tau = \tau_1$, and the arc is in the direction of from $u(\tau_1 + 1)$ to $u(\tau_1)$ in $\tau = \tau_1 + 1$.

Claim 3.9. $\tau_1 \neq 0$.

Proof. Once Algorithm FIND-INCREMENT begins, one of $M(\tau)$ and $N(\tau)$ is not an even factor until the termination. If $\tau_1 = 0$, then both of $M(\tau_1)$ and $N(\tau_1)$ are even factors, and so are $M(\tau_2)$ and $N(\tau_2)$. However, Algorithm FIND-INCREMENT does not terminate in $\tau = \tau_2$, which is a contradiction. □

Now, consider the transitions of $\tau_1 - 1$ to τ_1, and $\tau_2 - 1$ to τ_2 (Claim 3.9 assures the existence of $\tau_1 - 1$). Since we selected the smallest τ_1, different arc-exchanges occurred in $\tau = \tau_1 - 1$ and $\tau = \tau_2 - 1$. Observe that one arc-exchange created an odd cycle, while the other did not. That is, we have two cases:

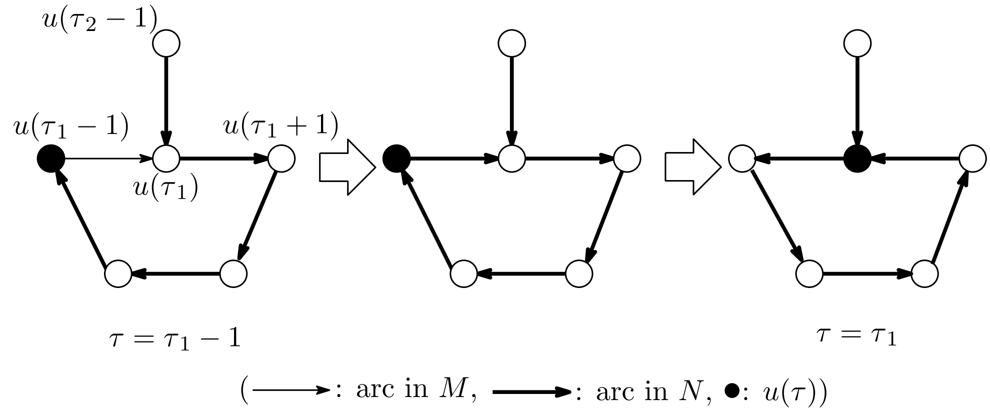


Figure 2: Case 1 in $\tau = \tau_1 - 1$ and $\tau = \tau_1$.

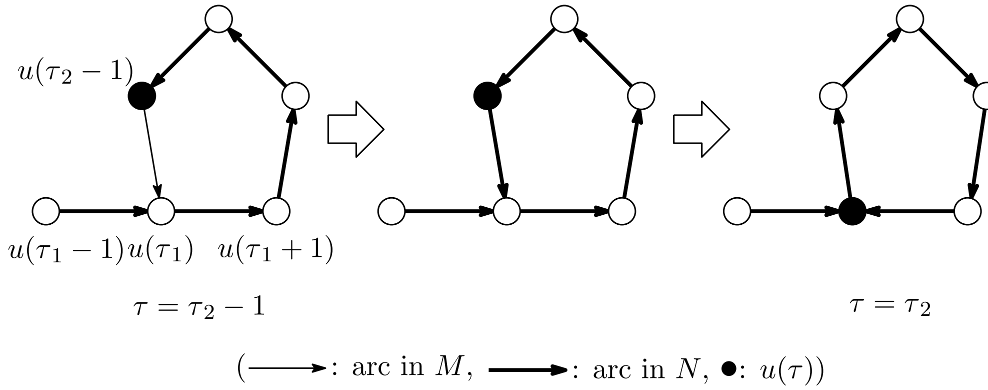


Figure 3: Case 2 in $\tau = \tau_2 - 1$ and $\tau = \tau_2$.

Case 1. The arc-exchange in $\tau = \tau_1 - 1$ resulted in an N-arc odd cycle C.

Case 2. The arc-exchange in $\tau = \tau_2 - 1$ resulted in an N-arc odd cycle C.

We derive a contradiction in both of these two cases.

Case 1 (see Figure 2). In this case, in $\tau = \tau_1 - 1$ an N-arc odd cycle C appeared, which resulted in $\bar{C}$ contained in $N(\tau_1)$, hence also in $N(\tau_2)$. Now consider the arc-exchange in $\tau_2 - 1$. Here, an arc connecting $u(\tau_2 - 1)$ and $u(\tau_2)$ was exchanged from M to N, which did not resulted in an appearance of an odd cycle. Then, we have that $u(\tau_2 - 1) \notin V(C)$ by Lemma 3.7 and $N(\tau_2 - 1)$ also contains $\bar{C}$, which contradicts Lemma 3.8.

Case 2 (see Figure 3). In this case, in $\tau = \tau_2 - 1$ an N-arc odd cycle C appeared, which resulted in $\bar{C}$ contained in $N(\tau_2)$, hence also in $N(\tau_1)$. Now consider the arc-exchange in $\tau_1 - 1$. Here, an arc connecting $u(\tau_1 - 1)$ and $u(\tau_1)$ was exchanged from M to N, which did not resulted in an appearance of an odd cycle. Then, we have that $u(\tau_1 - 1) \notin V(C)$ by Lemma 3.7 and $N(\tau_1 - 1)$ also contains $\bar{C}$, which contradicts Lemma 3.8.

Therefore, Algorithm FIND-INCREMENT terminates in finite steps, which completes the proof for Proposition 3.4.

3.3 Necessity of odd-cycle-symmetry

The objective of this subsection is to prove the necessity in Theorem 3.2.

Proposition 3.10. *For a digraph G, if $J_{\mathrm{EF}}(G)$ is a jump system then G is odd-cycle-symmetric.*

Assume to the contrary that a digraph $G = (V, A)$ is not odd-cycle-symmetric but $J_{\mathrm{EF}}(G)$ is a jump system, and we will derive a contradiction. Let $C = (v_k, a_1, v_1, \dots, v_{k-1}, a_k, v_k)$ be the shortest odd cycle which does not have the reverse cycle $\bar{C}$. Without loss of generality, we assume that $(v_1, v_k) \notin A$. Now we begin with the following lemma.

Lemma 3.11. *For each vertex $v \in V(C)$, the following two properties hold.*

- *There exists an arc $a \in A \setminus A(C)$ such that $\partial^+ a = v$, $\partial^- a \in V(C)$.*
- *There exists an arc $a \in A \setminus A(C)$ such that $\partial^- a = v$, $\partial^+ a \in V(C)$.*

Proof. Suppose that $v \in V(C)$ and $a = (v, u) \in A(C)$. Define $x, y \in J_{\mathrm{EF}}(G)$ by $x = d_{\{a\}}$, $y = d_{A(C)\setminus\{a\}}$. By applying (EXC) to x, y and an (x, y)-increment $s = -\chi_{u^-}$, we have that there exists an $(x+s, y)$-increment t such that $x + s + t \in J_{\mathrm{EF}}(G)$ and $y - s - t \in J_{\mathrm{EF}}(G)$. Then, it holds that $t = \chi_{u_0^-}$ for some $u_0 \in V(C) \setminus \{u\}$ or $t = -\chi_{v^+}$, because $d_F(V^+) = d_F(V^-) = |F|$ holds for any $F \subseteq A$. On the other hand, if an arc set $F \subseteq A$ satisfies $d_F = d_{A(C)}$, then F forms a collection of disjoint cycles which covers $V(C)$, and hence F contains an odd cycle, which implies that $d_{A(C)} \notin J_{\mathrm{EF}}(G)$. For $t = -\chi_{v^+}$, it holds that $y - s - t = d_{A(C)}$, thus $t = -\chi_{v^+}$ is not an $(x + s, y)$-increment that satisfies (EXC). Therefore, there exists a vertex $u_0 \in V(C) \setminus \{u\}$ such that $x + s + t = d_{\{(v,u_0)\}} \in J_{\mathrm{EF}}(G)$ for $t = \chi_{u_0^-}$, which means that $(v, u_0) \in A \setminus A(C)$. We can prove the second property by a similar argument. □

Let $G[C] = (V[C], A[C])$ be the subgraph of G induced by $V(C) = V[C]$. Then, Lemma 3.11 means that for every vertex $v \in V[C]$ at least two arcs in $A[C]$ leave (enter) v. In what follows in this section, we use these properties in stead of using (EXC) directly.

We say that an arc $a = (v_i, v_j) \in A[C]$ is *regular* if $i \not\equiv j \pmod 2$, and it is *irregular* if $i \equiv j \pmod 2$. The following is an easy but useful observation.

Lemma 3.12. *A cycle C' in $G[C]$ is odd if and only if the number of irregular arcs in C' is odd.*

Proof. Suppose that $C' = (v_{i_0}, a'_1, v_{i_1}, a'_2, v_{i_2}, \dots, a'_l, v_{i_l})$ is a cycle in $G[C]$, where $i_0 = i_l$. Let γ be the number of irregular arcs in C'. By the definition of irregular arcs, an arc $(v_{i_{j-1}}, v_{i_j})$ is irregular if and only if $i_j - i_{j-1} + 1 \equiv 1 \pmod 2$. Hence, $l = \sum_{j=1}^{l} (i_j - i_{j-1} + 1) \equiv \gamma \pmod 2$, which completes the proof. □

For $1 \le i < j \le k$, we denote by $P_{i,j}$ the path from v_i to v_j along C, that is, $P_{i,j} = (v_i, a_{i+1}, v_{i+1}, \dots, v_j)$. We say that an arc $a = (v_i, v_j) \in A[C]$ is *forward* if $i < j$, and it is *backward* if $i > j$. Let A_{F} and A_{B} be sets of arcs defined as

$$A_{\mathrm{F}} = \{a \mid a \in A[C] \setminus A(C),\ a \text{ is a forward arc}\},$$
$$A_{\mathrm{B}} = \{a \mid a \in A[C] \setminus A(C),\ a \text{ is a backward arc}\}.$$

Now we prove that arcs in A_{F} satisfy the following conditions by using the minimality of $|C|$.

Claim 3.13. *Every arc in A_{F} is irregular.*

Proof. Suppose that $a = (v_i, v_j) \in A_{\mathrm{F}}$ is a regular arc. Since $i + 2 \leq j$, a cycle C' which consists of $P_{1,i}, a, P_{j,k}$, and a_1 is shorter than C. Furthermore, a_1 is the only irregular arc in C', and hence C' is odd by Lemma 3.12. The existence of such a cycle C' contradicts the minimality of $|C|$. □

Claim 3.14. *For every pair of arcs $(v_{i_1}, v_{j_1}), (v_{i_2}, v_{j_2}) \in A_{\mathrm{F}}$, it holds that $v_{j_1} > v_{i_2}$ and $v_{j_2} > v_{i_1}$.*

Proof. Suppose that $v_{i_2} \geq v_{j_1}$ holds. Then, a cycle C' which consists of P_{1,i_1}, (v_{i_1}, v_{j_1}), P_{j_1,i_2}, (v_{i_2}, v_{j_2}), $P_{j_2,k}$, and a_1 is shorter than C. On the other hand, by Claim 3.13, (v_{i_1}, v_{j_1}) and (v_{i_2}, v_{j_2}) are irregular arcs and so is a_1, but not the other arcs in C'. Hence C' is odd by Lemma 3.12, which contradicts the minimality of $|C|$. We can prove $v_{j_2} > v_{i_1}$ in the same way. □

Let $m = \max\{i \mid 1 \leq i \leq k,\ \exists a \in A_{\mathrm{F}},\ \partial^+ a = v_i\}$. Note that such m always exists, since Lemma 3.11 suggests that there exists an arc in A_{F} leaving v_1. Then we can prove the following claim.

Claim 3.15. *For any integer i with $1 \leq i \leq m$, there exists an arc in A_{B} which enters v_i, and for any integer i with $m < i \leq k$ there exists an arc in A_{B} which leaves v_i.*

Proof. First, by Claim 3.14, no arcs in A_{F} enter v_i for $1 \leq i \leq m$. Hence, by Lemma 3.11, there exists an arc in A_{B} which enters v_i for each $1 \leq i \leq m$. Second, by the definition of m, no arcs in A_{F} leave v_i for $m < i \leq k$. Hence, by Lemma 3.11, there exists an arc in A_{B} which leaves v_i for each $m < i \leq k$. □

Let p and q be integers such that $(v_1, v_p) \in A_{\mathrm{F}}$ and $(v_q, v_k) \in A_{\mathrm{F}}$. Note that Lemma 3.11 assures the existence of p and q. Since (v_1, v_p) and (v_q, v_k) are irregular by Claim 3.13, we have $3 \leq q$ and $p \leq k - 2$. Furthermore, by Claim 3.14, we have $3 \leq q \leq m < p \leq k - 2$.

Now we consider two sequences $p_1, p_2, \ldots, p_\alpha$ and $q_1, q_2, \ldots, q_\beta$ of integers satisfying the following conditions:

- $p = p_1 > p_2 > \cdots > p_{\alpha-1} > m \geq p_\alpha$ such that $(v_{p_i}, v_{p_{i+1}}) \in A_{\mathrm{B}}$ for any $1 \leq i \leq \alpha - 1$, and
- $q = q_1 < q_2 < \cdots < q_{\beta-1} < p_\alpha \leq q_\beta$ such that $(v_{q_{i+1}}, v_{q_i}) \in A_{\mathrm{B}}$ for any $1 \leq i \leq \beta - 1$.

Note that Claim 3.15 assures the existence of these sequences. An example of such sequences is shown in Figure 4. Then the following claim holds for these sequences.

Claim 3.16. *For any integer i with $p_\alpha \leq i \leq p$, there exists a path P from v_p to v_i such that contains only regular arcs and that $V(P) \subseteq \{v_j \mid p_\alpha \leq j \leq p\}$.*

Proof. First, we show that such a path exists for i with $p_2 \leq i \leq p$. If (v_p, v_{p_2}) is regular, a path which consists of (v_p, v_{p_2}) and $P_{p_2,i}$ is a desired path. Otherwise, since (v_p, v_{p_2}) is irregular, a cycle C' which consists of (v_p, v_{p_2}) and $P_{p_2,p}$ is odd by Lemma 3.12. Hence, by the minimality

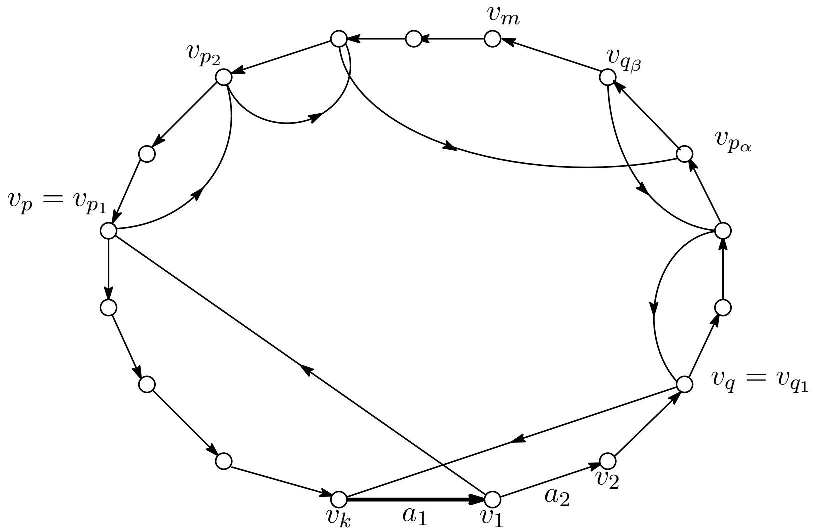


Figure 4: Two sequences.

of $|C|$, every arc in C' has the reverse arc and $\bar{P}_{i,p}$ is a desired path. Note that the path P found above satisfies that $V(P) \subseteq \{v_j \mid p_2 \leq j \leq p\}$.

Similarly, for any α' with $1 \leq \alpha' \leq \alpha - 1$ and for any i with $p_{\alpha'+1} \leq i \leq p_{\alpha'}$, there exists a path from $v_{p_{\alpha'}}$ to v_i which is composed of only regular arcs and satisfies that if v_j belongs to the path then $p_{\alpha'+1} \leq j \leq p_{\alpha'}$. Denote such a path from $v_{p_{\alpha'}}$ to $v_{p_{\alpha'+1}}$ by $P_{\alpha'}$.

Then, for any i and α' satisfying $p_{\alpha'+1} \leq i \leq p_{\alpha'}$, a path which consists of $P_1, P_2, \ldots, P_{\alpha'-1}$ and the path from $v_{p_{\alpha'}}$ to v_i is a desired path. □

In the same way as Claim 3.16, we obtain the following claim.

Claim 3.17. *For any integer i with $q \leq i \leq q_\beta$, there exists a path P from v_i to v_q which contains only regular arcs and satisfies that $V(P) \subseteq \{v_j \mid q \leq j \leq q_\beta\}$.*

Next we claim the following.

Claim 3.18. *$p_\alpha \neq 1$ and $q_\beta \neq k$.*

Proof. Suppose that $p_\alpha = 1$. Then, by Claim 3.16, there exists a path P from v_p to $v_{p_\alpha} = v_1$ containing only regular arcs. Since a cycle which consists of (v_1, v_p) and P is odd and shorter than C, this cycle has the reverse cycle. Hence, there exists an irregular arc (v_p, v_1) in $A[C]$. Then, since a cycle which consists of (v_p, v_1) and $P_{1,p}$ is odd, this cycle has the reverse cycle. In particular, a path $P_{q,p}$ has the reverse path $\bar{P}_{q,p}$. Then, a cycle C' which consists of (v_1, v_p), $\bar{P}_{q,p}$, (v_q, v_k), and a_1 is odd, because C' has three irregular arcs (v_1, v_p), (v_q, v_k), and a_1. As we have $3 \leq q$ and $p \leq k - 2$, C' is shorter than C, which contradicts the minimality of $|C|$. We can prove $q_\beta \neq k$ by a similar argument using Claim 3.17. □

Now we are ready to prove Proposition 3.10.

Proof for Proposition 3.10. By Claim 3.16, there exists a path P from v_p to v_{p_α} containing only regular arcs. On the other hand, by Claim 3.17, there exists a path P' from v_{p_α} to v_q that contains only regular arcs. Note that if $p_\alpha < q$ then $P_{p_\alpha,q}$ is a path from v_{p_α} to v_q that contains only regular arcs.

Thus, $P \cup P'$ forms a walk W from v_p to v_q, hence we obtain a path P'' from v_p to v_q by canceling cycles in W if necessary. Since P'' contains only regular arcs, a cycle C' which consists of $a_1, (v_1, v_p), P''$, and (v_q, v_k) is odd. Note that C' does not visit any vertices more than once, because P'' passes through neither v_1 nor v_k by Claim 3.18. Furthermore, since C' passes through at most one of v_2 and v_{k-1}, C' is shorter than C, which contradicts the minimality of $|C|$. $\square$

4 Weighted even factors and M-concave functions

In the previous section, we have seen that the degree sequences of the even factors in a digraph G induce a jump system $J_{\mathrm{EF}}(G)$ if and only if G is odd-cycle-symmetric. This section considers a weighted generalization of this relation. We introduce a function on $J_{\mathrm{EF}}(G)$ that is naturally defined by weighted even factors and prove that the function is M-concave if and only if (G, w) is odd-cycle-symmetric.

For a weighted digraph (G, w), define $f_{\mathrm{EF}} : J_{\mathrm{EF}}(G) \to \mathbf{R}$ by

$$f_{\mathrm{EF}}(x) = \max\{w(M) \mid M \text{ is an even factor, } d_M = x\}.$$

As a generalization of Theorem 3.2, in this section we prove that f_{EF} is an M-concave function if and only if (G, w) is an odd-cycle-symmetric weighted digraph.

Theorem 4.1. *The function f_{EF} is M-concave if and only if (G, w) is an odd-cycle-symmetric weighted digraph.*

As we proved Theorem 3.2, we prove the sufficiency (Proposition 4.3) and the necessity (Proposition 4.7) of odd-cycle-symmetry separately in the rest of this section.

Remark 4.2. For the same reason as Theorem 3.2, we refer to f_{EF} as an "M-concave function," in stead of a "valuated delta-matroid."

4.1 Sufficiency

This subsection is devoted to proving the sufficiency in Theorem 4.1.

Proposition 4.3. *The function f_{EF} is M-concave if (G, w) is an odd-cycle-symmetric weighted digraph.*

We present two proofs for f_{EF}'s M-concavity which extends those for Proposition 3.4. One is algebraic and the other is constructive.

4.1.1 Algebraic Proof

Let us consider a weighted version of the Tutte matrix. For a weighted digraph (G, w) with $G = (V, A)$ and $w \in \mathbf{R}^A$, let $\{t_{uv} \mid (u, v) \in A\}$ be a set of indeterminates associated with $a \in A$ such that $t_{uv} = -t_{vu}$ if $(u, v), (v, u) \in A$ and let z be another indeterminate. We define the *weighted Tutte matrix* $T = (T_{u^+v^-}(z))$ of (G, w), whose rows and columns are indexed by V^+ and V^-, respectively, by

$$T_{u^+v^-}(z) = \begin{cases} t_{uv} z^{w(a)} & (\text{if } a = (u, v) \in A), \\ 0 & (\text{otherwise}). \end{cases}$$

The degree of determinant of the weighted Tutte matrix relates to the weight of the corresponding even factors.

Theorem 4.4 (Cunningham and Geelen [7]). *Let (G, w) be an odd-cycle-symmetric weighted digraph with $G = (V, A)$ and $w \in \mathbf{R}^A$, and let T be the weighted Tutte matrix of (G, w). If U^+ and U^- are subsets of V such that admit an even (U^+, U^-)-factor, then*

$$\deg_z(\det T[U^+, U^-]) = \max\{w(M) \mid M \text{ is an even } (U^+, U^-)\text{-factor}\}.$$

The lemma below, which implies the relation between the weighted Tutte matrix and M-concave functions, directly follows from Murota [23] (see also Cunningham and Geelen [7]).

Lemma 4.5. *Let $T = (T_{u^+v^-}(z))$ be a matrix whose components are polynomials in z, and whose rows and columns are indexed by V^+ and V^-, respectively. Then, a function $f : \{0, 1\}^{V^+ \cup V^-} \to \mathbf{R} \cup \{-\infty\}$ defined by*

$$f(\chi_{U^+ \cup U^-}) = \begin{cases} -\infty & (\text{if } |V^+ \setminus U^+| \neq |U^-| \text{ or } \det T[V^+ \setminus U^+, U^-] = 0), \\ \deg_z(\det T[V^+ \setminus U^+, U^-]) & (\text{otherwise}) \end{cases}$$

for $U^+ \subseteq V^+, U^- \subseteq V^-$ is a valuated matroid.

Note that since a valuated matroid is a special case of M-concave functions, f is an M-concave function, and hence $f' : \{0, 1\}^{V^+ \cup V^-} \to \mathbf{R} \cup \{-\infty\}$ defined by

$$f'(\chi_{U^+ \cup U^-}) = \begin{cases} -\infty & (\text{if } |U^+| \neq |U^-| \text{ or } \det T[U^+, U^-] = 0), \\ \deg_z(\det T[U^+, U^-]) & (\text{otherwise}) \end{cases}$$

for $U^+ \subseteq V^+, U^- \subseteq V^-$ is also an M-concave function.

We now prove Proposition 4.3 using the weighted Tutte matrix.

Algebraic Proof for Proposition 4.3. Let T be the weighted Tutte matrix of (G, w). By the definition of f_{EF} and Theorem 4.4,

$$\begin{aligned} f_{\text{EF}}(\chi_{U^+ \cup U^-}) &= \max\{w(M) \mid M \text{ is an even } (U^+, U^-)\text{-factor}\}, \\ &= \deg_z(\det T[U^+, U^-]) \end{aligned}$$

for each $U^+ \subseteq V^+$ and $U^- \subseteq V^-$ satisfying $\chi_{U^+ \cup U^-} \in J_{\text{EF}}$. Thus, by Lemma 4.5, f_{EF} is an M-concave function. □

4.1.2 Constructive Proof

Next we give another proof for Proposition 4.3 based on our constructive proof for Proposition 3.4.

Lemma 4.6. *Let (G, w) be an odd-cycle-symmetric digraph, M, N be even factors in G and s be a (d_M, d_N)-increment. If we execute Algorithm FIND-INCREMENT to obtain new even factors M', N' and a $(d_M + s, d_N)$-increment t such that $d_{M'} = d_M + s + t$ and $d_{N'} = d_N - s - t$, then it holds that $w(M) + w(N) = w(M') + w(N')$.*

Proof. An arc-exchange in Step 3 does not change $w(M) + w(N)$. Moreover, $w(M) + w(N)$ does not change in Step 5, since $w(A(C)) = w(A(\bar{C}))$ follows from the odd-cycle-symmetry of (G, w). Therefore, $w(M) + w(N)$ stays constant through the algorithm. □

Constructive Proof for Proposition 4.3. For $x, y \in J_{\mathrm{EF}}(G)$ and an (x, y)-increment s, let M and N be even factors such that $d_M = x$, $d_N = y$, $w(M) = f_{\mathrm{EF}}(x)$, and $w(N) = f_{\mathrm{EF}}(y)$.

Then, execute Algorithm FIND-INCREMENT to find new even factors M' and N' and an $(x + s, y)$-increment t that satisfy $d_{M'} = x + s + t$ and $d_{N'} = y - s - t$. By Lemma 4.6, we have

$$\begin{aligned} f_{\mathrm{EF}}(x) + f_{\mathrm{EF}}(y) &= w(M) + w(N) \\ &= w(M') + w(N') \\ &\leq f_{\mathrm{EF}}(x + s + t) + f_{\mathrm{EF}}(y - s - t). \end{aligned}$$

Hence f_{EF} is M-concave. □

4.2 Necessity

The necessity of odd-cycle-symmetry in Theorem 4.1 is proved in this subsection.

Proposition 4.7. *For a weighted digraph (G, w), if $f_{\mathrm{EF}}(G)$ is M-concave then (G, w) is an odd-cycle-symmetric weighted digraph.*

If f_{EF} is M-concave, it holds that $J_{\mathrm{EF}}(G)$ is a jump system. Hence, Proposition 3.10 suggests that G is odd-cycle-symmetric in the unweighted sense. In order to prove that (G, w) is odd-cycle-symmetric in the weighted sense, it suffices to consider each non-bipartite cycle-connected component of G. Hence, in the rest of this section, we assume that G is cycle-connected but not bipartite, which is symmetric by Lemma 2.11.

We prove Proposition 4.7 by giving a valid potential function π on V. First, we observe that $w(A(C)) = w(A(\bar{C}))$ holds for an odd cycle C without "chords." For a cycle C, an arc a is said to be a *chord* of C if $a \in A[C] \setminus (A(C) \cup A(\bar{C}))$. Recall that $A[C]$ is the set of all arcs whose end vertices are both in $V(C)$.

Lemma 4.8. *Suppose that (G, w) is a weighted digraph such that $f_{\mathrm{EF}}(G)$ is M-concave. If an odd cycle C has no chords, it holds that $w(A(C)) = w(A(\bar{C}))$.*

Proof. Denote $C = (v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_0)$. Then, consider even factors $M = \{a_1\}, N = A(C) \setminus \{a_1\}$ and their degree sequence $x = d_M, y = d_N$. Since C has no chords, N is the unique even factor that achieves y. Hence, we have $f_{\mathrm{EF}}(x) = w(M)$ and $f_{\mathrm{EF}}(y) = w(N)$. Then, pick up an (x, y)-increment $s = -\chi_{v_0^+}$ and consider to apply (M-EXC). Since C has no chords, we have that $t = \chi^+_{v_2}$ is the unique $(x + s, y)$-increment such that $x+s+t, y-s-t \in J_{\mathrm{EF}}(G)$, and $x+s+t$ (resp. $y-s-t$) is achieved only by $M' = \{\bar{a}_2\}$ (resp. $N' = A(\bar{C}) \setminus \{\bar{a}_2\}$). Hence, we have $f_{\mathrm{EF}}(x+s+t) = w(M')$ and $f_{\mathrm{EF}}(y-s-t) = w(N')$. Then, (M-EXC) suggests that $f_{\mathrm{EF}}(x)+f_{\mathrm{EF}}(y) \leq f_{\mathrm{EF}}(x+s+t)+f_{\mathrm{EF}}(y-s-t)$, that is, $w(A(C)) \leq w(A(\bar{C}))$.

The inequation $w(A(C)) \geq w(A(\bar{C}))$ can be proved by a similar argument. □

By Lemma 4.8, we can define a valid potential function on $V(C)$. Beginning with $G_0 = C \cup \bar{C}$, we add "ears" until we obtain the original digraph G. For a subgraph G' of G, a path $P = (v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_k)$ is said to be a *proper ear* of G' if the distinct two vertices v_0 and v_k belong to G' but the other vertices in $V(P)$ do not. The following lemma assures that G can be obtained by addition of proper ears.

Lemma 4.9 (Grötschel [15])**.** *Let G be a cycle-connected digraph, and G' a subgraph of G with at least two vertices. If $G' \neq G$, then G' has a proper ear.*

The following lemma implies a method to compose G by adding proper ears, which defines a valid potential function in each step of addition of a proper ear. We remark that this composition of G refers to that of an odd-cycle-symmetric digraph in [19].

Lemma 4.10. *Let (G, w) be a weighted digraph such that G is cycle-connected but not bipartite, and $f_{\mathrm{EF}}(G)$ is M-concave. There exists a sequence $G_0, G_1, \ldots, G_m = G$ of subgraphs such that satisfies the following* (i)–(iv)*.*

(i) *G_0 consists of an odd cycle C without chords and its reverse cycle $\bar{C}$.*

(ii) *G_{i+1} is obtained from G_i by adding P_i and $\bar{P}_i$, where P_i is a proper ear of G_i, for $i = 0, 1, \ldots, m-1$.*

(iii) *There exist both an even path and an odd path from u to v in $G_i = (V_i, A_i)$ for every vertex pair $u, v \in V_i$.*

(iv) *G_i has a valid potential function for $i = 0, 1, \ldots, m$.*

Proof. By Proposition 3.10 and Lemma 2.11, we have that G is symmetric. In a non-bipartite symmetric digraph G, there exists an odd cycle without chords. For, if an odd cycle C has a chord a, the symmetry of G implies a shorter odd cycle that contains a. Take an arbitrary odd cycle C without chords and define $G_0 = C \cup \bar{C}$. It directly follows from Lemma 4.8 that G_0 satisfies Conditions (i), (iii), and (iv).

Next, we show the existence of $\{G_i \mid i = 1, \ldots, m\}$ that satisfy Conditions (ii), (iii), and (iv) by induction on i. Suppose that G_i satisfies (iii) and (iv). Let k be the minimum length of a proper ear of G_i, the existence of which is assured by Lemma 4.9. Consider an odd cycle $C = (v_0, a_1, v_1, \dots, v_{k-1}, a_k, v_k, \dots, v_{l-1}, a_l, v_0)$ such that $P = (v_0, a_1, v_1, \dots, v_{k-1}, a_k, v_k)$ is a proper ear of G_i and $Q = (v_k, a_{k+1}, \dots, v_{l-1}, a_l, v_0)$ is a path in G_i. Among such cycles, we take an odd cycle C^* with minimum length l^*. We remark here that, by Condition (iii) in G_i, for any proper ear P' of G_i there exists a path Q' in G_i such that $P' \cup Q'$ forms an odd cycle. Denote $C^* = (v_0, a_1, v_1, \dots, v_{k-1}, a_k, v_k, \dots, v_{l^*-1}, a_{l^*}, v_0)$, $P^* = (v_0, a_1, v_1, \dots, v_{k-1}, a_k, v_k)$, and $Q^* = (v_k, a_{k+1}, \dots, v_{l^*-1}, a_{l^*}, v_0)$, where P^* is a proper ear of G_i and Q^* is a path in G_i. If there exist more than one path in G_i from v_k to v_0 with length $l^* - k$, let Q^* be a path with maximum weight among them. We prove that $G_{i+1} = G_i \cup P^* \cup \bar{P}^*$ satisfies (iii) and (iv).

First, we consider Condition (iii).

Case 1. Suppose $u, v \in V_i$. Then it follows from the induction hypothesis that there exist both an even path and an odd path from u to v in G_{i+1}.

Case 2. Suppose $u \in V_i$ and $v \in V_{i+1} \setminus V_i$. Let P' be a path from v_0 to v along P^*. Since there exist both an even path P_{e} and an odd path P_{o} from u to v_0 in G_i, one of $P_{\mathrm{e}} \cup P'$ and $P_{\mathrm{o}} \cup P'$ is an even path, and the other is an odd path. In the case where $v \in V_i$ and $u \in V_{i+1} \setminus V_i$, we can prove that there exist both an even path and an odd path from u to v in a similar way.

Case 3. Suppose $u, v \in V_{i+1} \setminus V_i$. Without loss of generality we assume v_0, v, u, and v_k appear on P^* in this order. Let P' be a path from u to v_k along P^*, and P'' a path from v_0 to v along P^*. Since there exist both an even path P_{e} and an odd path P_{o} from v_k to v_0 in G_i, one of $P' \cup P_{\mathrm{e}} \cup P''$ and $P' \cup P_{\mathrm{o}} \cup P''$ is an even path, and the other is an odd path.

Thus there exist both an even path and an odd path from u to v in G_{i+1} for every vertex pair $u, v \in V_{i+1}$.

Next, we prove that G_{i+1} satisfies (iv).

Case A ($k \geq 2$). P^* has a vertex $v_j \in V_{i+1} \setminus V_i$. Consider even factors $M = \{a_j\}$ and $N = A(C^*) \setminus \{a_j\}$. Obviously, M is the unique even factor that achieves the degree sequence $x = d_M$, and $f_{\mathrm{EF}}(x) = w(M)$. The minimality of $|P^*|$ and $|C^*|$ implies that $y = d_N$ is achieved by even factors that consist of $(A(P^*) \setminus \{a_j\}) \cup A(Q)$, where Q is a path from v_k to v_0 with $V(Q) = V(Q^*)$. Since Q^* maximizes $w(A(Q))$ among such paths, we have that $f_{\mathrm{EF}}(y) = w(N)$. Then, pick up an (x, y)-increment $-\chi_{v_{j-1}^+}$ and consider to apply (M-EXC). By the minimality of $|P^*|$ and $|C^*|$, there does not exist an chord of C^* that is incident to v_j, which implies that $t = \chi_{v_{j+1}^+}$ is the unique $(x + s, y)$-increment such that $x + s + t, y - s - t \in J_{\mathrm{EF}}(G)$. The degree sequence $x + s + t$ is achieved only by $M' = \{\bar{a}_{j+1}\}$, whereas $y - s - t$ is achieved by even factors that consist of $(A(\bar{P}^*) \setminus \{\bar{a}_{j+1}\}) \cup A(\bar{Q})$, where Q is a path from v_k to v_0 with $V(Q) = V(Q^*)$. Since G_i has a valid potential π, it holds that $w(A(\bar{Q})) = w(A(Q)) + \pi(v_k) - \pi(v_0)$ for each path Q from v_k to v_0. Hence, the maximality of $w(A(Q^*))$ implies that $N' = (A(\bar{P}^*) \setminus \{\bar{a}_{j+1}\}) \cup A(\bar{Q}^*) = A(\bar{C}^*) \setminus \{\bar{a}_{j+1}\}$ maximizes the weight among the even factors that achieve $y - s - t$. Therefore, we have that $f_{\mathrm{EF}}(x + s + t) = w(M')$ and $f_{\mathrm{EF}}(y - s - t) = w(N')$. Then, (M-EXC) suggests that $f_{\mathrm{EF}}(x) + f_{\mathrm{EF}}(y) \leq f_{\mathrm{EF}}(x + s + t) + f_{\mathrm{EF}}(y - s - t)$, that is, $w(A(C^*)) \leq w(A(\bar{C}^*))$.

A similar argument shows $w(A(C^*)) \geq w(A(\bar{C}^*))$, and hence $w(A(C^*)) = w(A(\bar{C}^*))$. Therefore, we can assign a value $\pi(v_j)$ to v_j for $j = 1, \dots, k - 1$ so that the potential function π is valid in

G_{i+1}.

Case B ($k = 1$). We have $P^* = (v_0, a_1, v_1)$, where both v_0 and v_1 are in G_i. In this case, either v_0 or v_1 has no incident chord of C^*. For, if both v_0 and v_1 have an incident chord, we have an odd cycle such that contains P^* and has less number of arcs than $|C^*|$. Without loss of generality, we assume that v_1 has no incident chords. Then, consider even factors $M = \{a_1\}$ and $N = A(C^*) \setminus \{a_1\}$. Applying (M-EXC) to degree sequences $x = d_M, y = d_N$ and an (x, y)-increment $-\chi_{v_0^+}$, we obtain an inequation of $w(A(C^*)) \leq w(A(\bar{C}^*))$ by a similar argument to that in Case A. An inequation $w(A(C^*)) \geq w(A(\bar{C}^*))$ also follows from an analogous argument. Hence, we have $w(A(C^*)) = w(A(\bar{C}^*))$, which implies that the potential function π in G_i is also valid in G_{i+1}. □

By Lemma 4.10, G has a valid potential function, which proves Proposition 4.7

5 Degree sequences of underlying graphs

In this section, we discuss the degree sequence of the underlying edge set of even factors.

5.1 Undirected degree sequences

In this paper, we introduced Definition 3.1 for the degree sequence in digraphs. One would naturally come up with another kind of degree sequences in digraphs.

Definition 5.1 (Undirected degree sequence)**.** For a digraph $G = (V, A)$ and its arc set $F \subseteq A$, the *undirected degree sequence* of F is a vector $\bar{d}_F \in \mathbf{Z}^V$ defined by

$$\bar{d}_F(v) = \left|F \cap \delta^- v\right| + \left|F \cap \delta^- v\right| \quad (v \in V).$$

The undirected degree sequences focus on the number of incident arcs, and do not consider their directions. In other words, the undirected degree sequence is exactly the degree sequence (in the usual manner) of the underlying graph.

Let $\bar{J}_{\mathrm{EF}}(G) \subseteq \mathbf{Z}^V$ be the set of the undirected degree sequences of all even factors in G. If $J_{\mathrm{EF}}(G)$ is a jump system, one can see that $\bar{J}_{\mathrm{EF}}(G)$ is also a jump system by Lemma 2.4 as follows. For a jump system $J_{\mathrm{EF}}(G)$, consider an elementary aggregation at v^+ and v^- that correspond to the same vertex $v \in V$. Applying such elementary aggregations for every $v \in V$, one obtains $\bar{J}_{\mathrm{EF}}(G)$, which is a jump system by Lemma 2.4. Therefore, the corollary below follows from Proposition 3.4.

Corollary 5.2. *$\bar{J}_{\mathrm{EF}}(G)$ is a jump system if G is odd-cycle-symmetric.*

Observe that in general $\bar{J}_{\mathrm{EF}}(G) \in \{0, 1, 2\}^V$. Hence, $\bar{J}_{\mathrm{EF}}(G)$ of an odd-cycle-symmetric digraph G is a new example of a jump system that is not a delta-matroid.

Let us consider a weighted generalization. Let (G, w) be a weighted digraph. Define $\bar{f}_{\mathrm{EF}} : \bar{J}_{\mathrm{EF}}(G) \to \mathbf{R}$ by

$$\bar{f}_{\mathrm{EF}}(x) = \max\{w(M) \mid M \text{ is an even factor}, \bar{d}_M = x\} \quad (x \in \bar{J}_{\mathrm{EF}}(G)).$$

Similarly to the argument above, provided that $f_{\rm EF}$ is M-concave, $\bar{f}_{\rm EF}$ is shown to be M-concave by the operation of elementary aggregation on $f_{\rm EF}$. For $f_{\rm EF}$, apply elementary aggregation at $v^+ \in V^+$ and $v^- \in V^-$ for every $v \in V$ to obtain $\bar{f}_{\rm EF}$. If $f_{\rm EF}$ is M-concave, Lemma 2.6 tells that $\bar{f}_{\rm EF}$ is also M-concave. Therefore, by Proposition 4.3, we have the following corollary.

Corollary 5.3. *For a weighted digraph (G, w), $\bar{f}_{\rm EF}$ is M-concave if (G, w) is odd-cycle-symmetric.*

Remark 5.4. Corresponding to Propositions 3.4 and 4.3, Corollaries 5.2 and 5.3 claim that odd-cycle-symmetry of the digraph is sufficient for the undirected degree sequences to have a matroidal structure. One would expect that the necessity of odd-cycle-symmetry is also obtained from Propositions 3.10 and 4.7. However, odd-cycle-symmetry is *not* necessary for $\bar{J}_{\rm EF}(G)$ (resp. $\bar{f}_{\rm EF}$) to be a jump system (resp. an M-concave function). A digraph $G = (V, A)$ defined by $V = \{v_1, v_2, v_3\},\ A = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$ is a counter-example. In fact,

$$\bar{J}_{\rm EF}(G) = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1), (2,1,1), (1,2,1), (1,1,2)\}$$

is a jump system, while G is not odd-cycle-symmetric.

5.2 Degree sequences of path-matchings

This subsection deals with the degree sequences of path-matchings, which can be viewed as a special case of the undirected degree sequences in odd-cycle-symmetric digraphs.

The path-matching problem is defined as follows. Let $\bar{G} = (V, E; S^+, S^-)$ be an undirected graph with a pair of disjoint stable sets $S^+, S^- \subseteq V$ of the same size. We denote $V \setminus (S^+ \cup S^-)$ by R. An arc set $M \subseteq E$ is a *path-matching* in $\bar{G}$ if it consists of a vertex-disjoint collection of paths from $S^+ \cup R$ to $S^- \cup R$ with their internal vertices in R. For a path-matching M, an edge that forms a 1-length path with both ends in R is said to be a *matching-edge*. We denote the set of matching-edges in M by $N(M)$, and that of the other edges in M by $P(M)$. A distinct feature of path-matchings is that the matching-edges can be counted twice, that is, the size of M is defined by $|P(M)| + 2|N(M)|$.

In the case where the undirected graph is accompanied with a weight vector $\bar{w} \in \mathbf{R}^E$, we can consider the weighted path-matching problem, where the weight of a path-matching M is defined by $\bar{w}(P(M)) + 2\bar{w}(N(M))$.

As we mentioned in Section 1, we can reduce the path-matching problem to the even factor problem. In order to deal with the path-matching problem in $\bar{G} = (V, E; S^+, S^-)$, it suffices to consider the even factor problem in a digraph $\vec{G} = (V, A)$, where $A = \{(u, v) \mid u \in S^+ \cup R,\ v \in S^- \cup R,\ (u, v) \in E\}$. In the weighted case $(\bar{G}, \bar{w})$, consider $\vec{G}$ together with the weight vector $w \in \mathbf{R}^A$ with $w(a) = \bar{w}(u, v)$ for $a = (u, v) \in A$. Note that $\vec{G}$ is an odd-cycle-symmetric digraph and $(\vec{G}, w)$ is an odd-cycle-symmetric weighted digraph.

As for the degree sequences, Cunningham [5] claimed that the set of the degree sequences of all path-matchings is a jump system. In order to confirm this claim, we have to modify the definition of the degree sequence in [5].

Definition 5.5 (Degree sequence of a path-matching)**.** For a path-matching M and a set of matching-edges $H \subseteq N(M)$, the degree sequence $d_{M,H}$ is a vector in $\mathbf{Z}^V$ defined by

$$d_{M,H}(v) = |\{a \mid a \in M \setminus H,\, a \text{ is incident to } v\ \}| + 2|\{a \mid a \in H,\, a \text{ is incident to } v\ \}|,$$

for $v \in V$.

This definition means that matching-edges can be counted either once or twice. Remark that it was not needed to count the matching-edges once in optimizing the size/weight of path-matchings.

For an instance of the path-matching problem $\bar{G} = (V, E; S^+, S^-)$, define $J_{\mathrm{PM}}(\bar{G}) \subseteq \{0, 1, 2\}^V$ by

$$J_{\mathrm{PM}}(\bar{G}) = \{d_{M,H} \mid M \text{ is a path-matching in } \bar{G},\, H \subseteq N(M)\}.$$

Then, $J_{\mathrm{PM}}(\bar{G})$ forms a jump system.

Corollary 5.6. *For $\bar{G}$, $J_{\mathrm{PM}}(\bar{G})$ is a jump system.*

Proof. We show $J_{\mathrm{PM}}(\bar{G}) = \bar{J}_{\mathrm{EF}}(\vec{G})$ by proving that $J_{\mathrm{PM}}(\bar{G}) \subseteq \bar{J}_{\mathrm{EF}}(\vec{G})$ and $\bar{J}_{\mathrm{EF}}(\vec{G}) \subseteq J_{\mathrm{PM}}(\bar{G})$. For $x \in J_{\mathrm{PM}}(\bar{G})$, there exists a path-matching M and its matching-edges $H \subseteq N(M)$ such that $d_{M,H} = x$. Associated with M and H, define an even factor M° in $\vec{G}$ as follows: replace $P(M)$ with corresponding paths in $\vec{G}$; replace an edge $(u, v) \in H$ with a 2-length cycle that consists of (u, v) and (v, u); and replace an edge $(u, v) \in N(M) \setminus H$ with (u, v). Then, it holds that $\bar{d}_{M^\circ} = d_{M,H}$, and hence $x \in \bar{J}_{\mathrm{EF}}(\vec{G})$. Conversely, for $y \in \bar{J}_{\mathrm{EF}}(\vec{G})$, take an even factor M° such that $\bar{d}_{M^\circ} = y$. Associated with M°, define a path-matching M in $\bar{G}$ as follows: for a cycle C in M°, pick up arcs in $A(C)$ along C alternately to obtain a matching H in $\bar{G}$; and for a path in M°, take the corresponding path in $\bar{G}$. Then, it holds that $d_{M,H} = \bar{d}_{M^\circ} = y$, which implies $y \in J_{\mathrm{PM}}(\bar{G})$.

Therefore, we have $J_{\mathrm{PM}}(\bar{G}) = \bar{J}_{\mathrm{EF}}(\vec{G})$. Since $\vec{G}$ is odd-cycle-symmetric, $\bar{J}_{\mathrm{EF}}(\vec{G})$ is a jump system by Corollary 5.2, and hence so is $J_{\mathrm{PM}}(\bar{G})$. □

As a quantitative extension, for $(\bar{G}, \bar{w})$, define $f_{\mathrm{PM}} : J_{\mathrm{PM}}(\bar{G}) \to \mathbf{R}$ by

$$f_{\mathrm{PM}}(x) = \max\{\bar{w}(M \setminus H) + 2\bar{w}(H) \mid M \text{ is a path-matching in } \bar{G} \text{ and } H \subseteq N(M) \text{ with } d_{M,H} = x\}$$

for $x \in J_{\mathrm{PM}}(\bar{G})$. Then, we have that f_{PM} is an M-concave function.

Corollary 5.7. *For $(\bar{G}, \bar{w})$, f_{PM} is M-concave.*

Proof. We can show that f_{PM} of $(\bar{G}, \bar{w})$ is exactly the same function as $\bar{f}_{\mathrm{EF}}$ of the weighted even factor problem in $(\vec{G}, w)$ in the same way as the proof for Corollary 5.6. Since $(\vec{G}, w)$ is an odd-cycle-symmetric weighted digraph, we have $\bar{f}_{\mathrm{EF}}$ is M-concave by Corollary 5.3, which completes the proof. □

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References

[1] A. Bouchet, *Greedy algorithm and symmetric matroids*, Math. Program., 38 (1987), pp. 147–159.

[2] A. Bouchet and W. H. Cunningham, *Delta-matroids, jump systems, and bisubmodular polyhedra*, SIAM J. Discrete Math., 8 (1995), pp. 17–32.

[3] R. Chandrasekaran and S. N. Kabadi, *Pseudomatroids*, Discrete Math., 71 (1988), pp. 205–217.

[4] W. J. Cook, W. H. Cunningham, W. R. Pulleyblank, and A. Schrijver, *Combinatorial Optimization*, John Wiley and Sons, New York, 1998.

[5] W. H. Cunningham, *Matching, matroids, and extensions*, Math. Program., 91 (2002), pp. 515–542.

[6] W. H. Cunningham and J. F. Geelen, *The optimal path-matching problem*, Combinatorica, 17 (1997), pp. 315–337.

[7] W. H. Cunningham and J. F. Geelen, *Vertex-disjoint dipaths and even dicircuits*, manuscript, 2001.

[8] A. W. M. Dress and T. Havel, *Some combinatorial properties of discriminants in metric vector spaces*, Adv. Math., 62 (1986), pp. 285–312.

[9] A. W. M. Dress and W. Wenzel, *Valuated matroid: A new look at the greedy algorithm*, Appl. Math. Lett., 3 (1990), pp. 33–35.

[10] A. W. M. Dress and W. Wenzel, *A greedy-algorithm characterization of valuated Δ-matroids*, Appl. Math. Lett., 4 (1991), pp. 55–58.

[11] A. W. M. Dress and W. Wenzel, *Valuated matroids*, Adv. Math., 93 (1992), pp. 214–250.

[12] J. Edmonds, *Paths, trees, and flowers*, Canadian J. Math., 17 (1965), pp. 449–467.

[13] J. Edmonds, *Matroid intersection*, Ann. Discrete Math., 4 (1979), pp. 39–49.

[14] S. Fujishige, *Submodular Functions and Optimization*, 2nd ed., Annals of Discrete Mathematics, 58, Elsevier, 2005.

[15] M. Grötschel, *On minimal strong blocks*, J. Graph Theory, 3 (1979), pp. 213–219.

[16] N. J. A. Harvey, *Algebraic structures and algorithms for matching and matroid problems*, Proc. 47th FOCS, IEEE, 2006, pp. 531–542.

[17] S. Iwata and K. Takazawa, *The independent even factor problem*, Proc. 18th SODA, ACM-SIAM, 2007, pp. 1171–1180.

[18] S. N. Kabadi and R. Sridhar, *Δ-matroid and jump system*, J. Appl. Math. Decision Sci., 9 (2005), pp. 95–106.

[19] T. Király and M. Makai, *On polyhedra related to even factors*, Proc. 10th IPCO, LNCS 3064, Springer-Verlag, pp. 416–430.

[20] Y. Kobayashi, K. Murota, and K. Tanaka, *Operations on M-convex functions on jump systems*, SIAM J. Discrete Math., 21 (2007), pp. 107–129.

[21] E. L. Lawler, *Matroid intersection algorithms*, Math. Program., 9 (1975), pp. 31–56.

[22] L. Lovász, *The membership problem in jump systems*, J. Combin. Theory, Ser. B, 70 (1997), pp. 45–66.

[23] K. Murota, *Finding optimal minors of valuated bimatroids*, Appl. Math. Lett., 8 (1995), pp. 37–41.

[24] K. Murota, *Convexity and Steinitz's exchange property*, Adv. Math., 124 (1996), pp. 272–311.

[25] K. Murota, *Discrete Convex Analysis*, Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[26] K. Murota, *M-convex functions on jump systems: a general framework for minsquare graph factor problem*, SIAM J. Discrete Math., 20 (2006), pp. 213–226.

[27] K. Murota and K. Tanaka, *A steepest descent algorithm for M-convex functions on jump systems*, IEICE Trans. Fundamentals of Electr., Commun., Computer Sci., E89-A (2006), pp. 1160–1165.

[28] G. Pap, *A combinatorial algorithm to find a maximum even factor*, Proc. 11th IPCO, LNCS 3509, Springer-Verlag, 2005, pp. 66–80.

[29] G. Pap, *Combinatorial algorithms for matchings, even factors and square-free 2-factors*, Math. Program., 110 (2007), pp. 57–69.

[30] G. Pap and L. Szegő, *On the maximum even factor in weakly symmetric graphs*, J. Combin. Theory, Ser. B, 91 (2004), pp. 201–213.

[31] A. Shioura and K. Tanaka, *Polynomial-time algorithms for linear and convex optimization on jump systems*, SIAM J. Discrete Math., to appear.

[32] K. Takazawa, *A weighted even factor algorithm*, Math. Program., to appear.

[33] W. Wenzel, *Pfaffian forms and* Δ*-matroids*, Discrete Math., 115 (1993), pp. 253–266.

[34] W. Wenzel, Δ*-matroids with the strong exchange conditions*, Appl. Math. Lett., 6 (1993), pp. 67–70.