# Exact WKB analysis for the degenerate third Painlevé equation of type $(D_8)$

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**Abstract**

Exact WKB analysis for instanton-type solutions of the degenerate third Painlevé equation of type $(D_8)$ is discussed. Explicit connection formulas are obtained through computations of the monodromy data of the underlying linear equations.

## 1 Introduction

In this paper we discuss the exact WKB analysis for instanton-type solutions (i.e., 2-parameter formal solutions) of the following degenerate third Painlevé equation of type $(D_8)$ with a large parameter $\eta$:

$$(P) \qquad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\left(\frac{dq}{dt}\right) + \eta^2\left(\frac{q^2}{t^2} - \frac{1}{t}\right).$$

Exact WKB analysis for instanton-type solutions of Painlevé equations $(P_J)$ $(J = \mathrm{I}, \ldots, \mathrm{VI})$ with a large parameter has been developed in [3], [1], [4], [9] etc. On the other hand, since the work of Sakai [8] on geometrical classification of the space of initial conditions of $(P_J)$, it is considered to be natural to distinguish the degenerate third Painlevé equations of type $(D_7)$ and $(D_8)$ from the generic third Painlevé equation $(P_{\mathrm{III}})$: Separately from $(P_{\mathrm{III}})$, several important properties (such as $\tau$-functions, irreducibility etc.) and asymptotics of solutions of the degenerate third Painlevé equations are studied in [7] and [5], respectively. The above equation $(P)$ is obtained from an equation equivalent to the most degenerate third Painlevé equation of type $(D_8)$ by introducing a large parameter $\eta$ through an appropriate scaling of variables (or through the degeneration from $(P_{\mathrm{III}})$; cf. [10]). From the viewpoint of exact WKB analysis $(P)$ is also very peculiar: There is no turning point of $(P)$ in the sense of [3] while it has two singular points $t = 0$ and $\infty$. In particular, $t = 0$ can be regarded as a non-linear analogue of a "singular point of simple pole type" (i.e., a singular point which also plays the role of turning points) of a second order linear differential equation discussed in [6]. In fact, as we will show in §3 below, a Stokes curve of $(P)$ emanates from $t=0$. The purpose of this paper is to discuss the Stokes phenomenon and connection formula for instanton-type solutions of $(P)$ on a Stokes curve emanating from $t=0$.

To analyze the Stokes phenomena, we make full use of the well-known fact that Painlevé equations govern the isomonodromic deformations of underlying systems of linear differential equations in the sense of [2]. In the case of $(P)$ it is formulated as follows (cf. [7, §3]): Let $(SL)$ and $(D)$ denote the following linear differential equations, respectively.

$$(SL) \qquad \left(-\frac{\partial^2}{\partial x^2}+\eta^2 Q\right)\psi=0,$$

$$(D) \qquad \frac{\partial\psi}{\partial t}=A\frac{\partial\psi}{\partial x}-\frac{1}{2}\frac{\partial A}{\partial x}\psi,$$

where

$$Q=\frac{tK}{x^2}+\frac{1}{2x}+\frac{t}{2x^3}-\eta^{-1}\frac{qp}{x(x-q)}-\eta^{-2}\frac{1}{x(x-q)}+\eta^{-2}\frac{3}{4(x-q)^2},$$
$$tK=q^2p^2-\left(\frac{q}{2}+\frac{t}{2q}\right)+\eta^{-1}qp, \quad A=\frac{xq}{t(x-q)}.$$

Then the compatibility condition of $(SL)$ and $(D)$ is described by the Hamiltonian system

$$(H) \qquad \frac{dq}{dt}=\eta\frac{\partial K}{\partial p}, \quad \frac{dp}{dt}=-\eta\frac{\partial K}{\partial q}$$

which is equivalent to $(P)$. Consequently the monodromy data of $(SL)$ are preserved (i.e., not depending on $t$) if a solution of $(H)$ is substituted into the coefficients of $(SL)$. In this paper, following the argument of [9] where the connection formula for $(P_{\mathrm{I}})$ is discussed, we explicitly compute the monodromy data of $(SL)$ to write down the connection formula for $(P)$.

# 2 Instanton-type solutions of $(P)$

First of all, we introduce instanton-type solutions of $(P)$.

We can readily see that $q=\pm\sqrt{t}$ and $(q,p)=\pm(\sqrt{t},-\eta^{-1}/(4\sqrt{t}))$ respectively satisfy $(P)$ and $(H)$. As these solutions contain no free parameters, they are called 0-parameter solutions. In what follows we adopt

$$(1) \qquad q^{(0)}=\sqrt{t}, \quad (q^{(0)},p^{(0)})=(\sqrt{t},-\eta^{-1}\frac{1}{4\sqrt{t}})$$

as 0-parameter solutions of $(P)$ and $(H)$.

Instanton-type formal solutions $q(t,\eta;\alpha,\beta)$ containing 2 free parameters $(\alpha,\beta)$ (or "2-parameter solutions" for short) are then constructed through the multiple-scale analysis. See [1, §1] for details. In particular, similarly to the case of $(P_{\mathrm{I}})$ (cf. [9, §1]), we can construct the following 2-parameter solutions of $(P)$ with homogeneity:

$$q(t,\eta;\alpha,\beta)=\sqrt{t}+\eta^{-1/2}\sum_{n=0}^{\infty}\eta^{-n/2}L_{n/2}(t,\eta), \tag{2}$$

where $L_0=L_0(t,\eta)$ is given by

$$L_0=t^{3/8}\left(\alpha(t^{1/4}\eta)^{\sqrt{2}\gamma}e^{\phi\eta}+\beta(t^{1/4}\eta)^{-\sqrt{2}\gamma}e^{-\phi\eta}\right)$$

with $\gamma=\alpha\beta$ and $\phi=4\sqrt{2}t^{1/4}$, and $L_{n/2}=L_{n/2}(t,\eta)$ $(n\geq 1)$ is of the form

$$\sum_{k=0}^{n+1}c^{(n/2)}_{n+1-2k}t^{(3-n)/8}\left((t^{1/4}\eta)^{\sqrt{2}\gamma}e^{\phi\eta}\right)^{n+1-2k}$$

with $c^{(n/2)}_l$ being constants depending only on $(\alpha,\beta)$. Note that $q=q(t,\eta;\alpha,\beta)$ has homogeneity to the effect that $t^{-1/2}q$ is a formal series of one variable $t^{1/4}\eta$. Using the first equation of $(H)$, i.e., $p=\eta^{-1}\{t(dq/dt)-q\}/(2q^2)$, we also obtain 2-parameter solutions $(q(t,\eta;\alpha,\beta),p(t,\eta;\alpha,\beta))$ of $(H)$.

# 3 Stokes geometry of $(P)$ and $(SL)$

To define the Stokes geometry (i.e., turning points and Stokes curves) of $(P)$, we consider its Frechét derivative at $q^{(0)}=\sqrt{t}$:

$$-\frac{d^2\varphi}{dt^2}+\eta^2\left(\frac{2}{t^{3/2}}-\eta^{-2}\frac{1}{4t^2}\right)\varphi=0. \tag{3}$$

It is transformed by a change of variables $(t,\varphi)=(\tilde{t}^2,\tilde{t}^{1/2}\tilde{\varphi})$ into

$$-\frac{d^2\tilde{\varphi}}{d\tilde{t}^2}+\eta^2\left(\frac{8}{\tilde{t}}-\eta^{-2}\frac{1}{4\tilde{t}^2}\right)\tilde{\varphi}=0. \tag{4}$$

Note that (4) has the same form as the equation discussed in [6]. As is proved in [6], $\tilde{t}=0$ plays the role of a turning point of (4). Having this result in mind, we consider $\tilde{t}=0$, i.e., $t=0$ as a turning point of $(P)$, though $(P)$ has no ordinary turning point.

**Definition 3.1.** (i) We call $t=0$ a turning point of $(P)$.
(ii) A Stokes curve of $(P)$ is by definition

$$\{t\in\mathbf{C}\mid \mathrm{Im}\int_0^t\sqrt{\frac{2}{t^{3/2}}}dt=\mathrm{Im}(4\sqrt{2}t^{1/4})=0\}. \tag{5}$$

By the definition (5) the Stokes curves of $(P)$ are explicitly given by $\{t \in \mathbf{C} \mid \arg\sqrt{t} = 2n\pi \ (n \in \mathbf{Z})\}$.

We now study the relationship between the Stokes geometry of $(P)$ and that of $(SL)$. Here and in what follows we assume 2-parameter solutions $(q(t,\eta;\alpha,\beta), p(t,\eta;\alpha,\beta))$ are substituted into the coefficients of $(SL)$ and $(D)$. Then $Q$ becomes an infinite series (in $\eta^{-1/2}$) of the form $Q = \sum_{n\geq 0} \eta^{-n/2} Q_{n/2}$ with

$$Q_0 = \frac{(x-\sqrt{t})^2}{2x^3} \quad \text{and} \quad Q_{1/2} \equiv 0.$$

Hence $(SL)$ has only one turning point at $x = \sqrt{t}$, which is double. Furthermore, letting $\gamma$ be a positively oriented circle $\{|x| = \sqrt{t}\}$ starting and ending at $x = \sqrt{t}$, we find

$$\int_\gamma \sqrt{Q_0}\,dx = \frac{1}{\sqrt{2}} \int_\gamma \left( \frac{1}{\sqrt{x}} - \frac{\sqrt{t}}{\sqrt{x^3}} \right) dx = -4\sqrt{2}t^{1/4}.$$

This implies

$$\mathrm{Im} \int_\gamma \sqrt{Q_0}\,dx = 0 \iff \arg\sqrt{t} = 2n\pi \ (n \in \mathbf{Z}).$$

Thus we have

**Proposition 3.2.** (i) $(SL)$ *has a unique turning point at* $x = \sqrt{t}$*, which is double.* (ii) *When and only when* $\arg\sqrt{t} = 2n\pi$ $(n \in \mathbf{Z})$*, there exists a Stokes curve of* $(SL)$ *that starts from* $\sqrt{t}$*, encircles* $t = 0$ *and returns to* $\sqrt{t}$*. It is the circle centered at the origin with radius* $\sqrt{t}$ *(cf. Fig.1).*

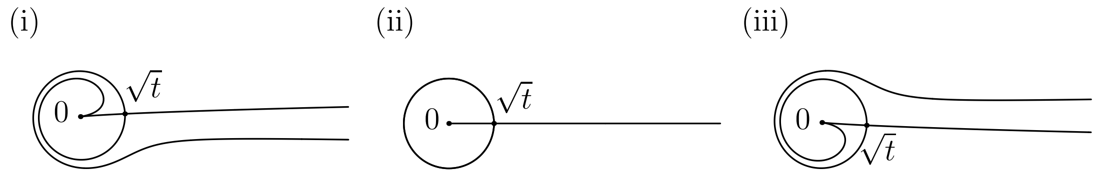


Fig. 1: Stokes curves of $(SL)$ in the case of (i) $\arg\sqrt{t} > 0$, (ii) $\arg\sqrt{t} = 0$ and (iii) $\arg\sqrt{t} < 0$.

# 4 Canonical form of $(SL)$ and $(D)$ near the double turning point

In this section, as a preparation for computations of the monodromy data of $(SL)$, we discuss the transformation of $(SL)$ and $(D)$ near the double turning point $x = \sqrt{t}$ into their canonical form.

We first introduce the following WKB solutions as fundamental systems of solutions of $(SL)$.

$$(6) \quad \psi_{\pm}^{(k)} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp(\pm\eta 2\sqrt{2}t^{1/4}) \exp \pm \left( \eta \int_{\sqrt{t}}^{x} S_{-1} dx + \int_{k}^{x} (S_{\text{odd}} - \eta S_{-1}) dx \right),$$

where $k = 0$ or $\infty$, $S = \sum_{n \geq -2} \eta^{-n/2} S_{n/2}$ is a formal power series solution of the Riccati equation $S^2 + (\partial S/\partial x) = \eta^2 Q$ associated with $(SL)$ and $S_{\text{odd}}$ denotes its odd part in the sense of [1, Def. 3.1]. Note that both WKB solutions (6) are well-defined since $S_{\text{odd}} = \eta S_{-1} + \sum_{n \geq 0} \eta^{-n/2} S_{\text{odd},n/2}$ satisfies that $x^{1/2} S_{\text{odd},n/2}$ (resp., $x^{3/2} S_{\text{odd},n/2}$) are holomorphic at $x = 0$ (resp., $x = \infty$) for $n \geq 0$. Here and in what follows we assume the branch of (6) is chosen so that $(x - \sqrt{t})^{1/2} > 0$ for $x > \sqrt{t}$ and $x^{-3/4} > 0$ for $x > 0$ may hold (in defining $\sqrt{S_{-1}}$) when $\arg\sqrt{t} = 0$. (As we are interested in Stokes phenomena for $(P)$, we may assume that $t$ lies near a Stokes curve $\arg\sqrt{t} = 0$ of $(P)$.) The WKB solutions (6) then become single-valued in a cut plane indicated

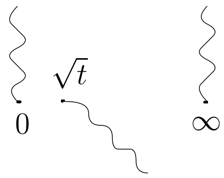


Fig. 2: $x$-plane with cuts. (Wiggly lines designate cuts.)

in Fig.2. We also have the following relation between $\psi_{\pm}^{(0)}$ and $\psi_{\pm}^{(\infty)}$:

$$(7) \qquad \psi_{\pm}^{(\infty)} = \left(\exp \pm (\pi i \operatorname{Res}_{x=\sqrt{t}} S_{\text{odd}})\right) \psi_{\pm}^{(0)}.$$

Now, using $(\partial/\partial t) S_{\text{odd}} = (\partial/\partial x)(A S_{\text{odd}})$ (cf. [1, (2.14)]), we can confirm the following

**Proposition 4.1.** *Both WKB solutions $\psi_{\pm}^{(0)}$ and $\psi_{\pm}^{(\infty)}$ satisfy $(D)$.*

Furthermore, the WKB solutions (6) enjoy the following homogeneity property: Letting $\mathcal{H}$ be a scaling operator defined by $\mathcal{H} : (x, t, \eta) \mapsto (r^{-2}x, r^{-4}t, r\eta)$, we find (6) are homogeneous of degree $-1$ for $\mathcal{H}$ (i.e., $\psi_{\pm}^{(k)}(\mathcal{H}(x, t, \eta)) = r^{-1}\psi_{\pm}^{(k)}(x, t, \eta)$ hold).

To determine the connection formula for the WKB solutions (6) on Stokes curves of $(SL)$ emanating from $x = \sqrt{t}$, we make use of the transformation theorem proved in [4] for Painlevé equations: Let $(SL_{\text{can}})$ and $(D_{\text{can}})$ denote the following equations, respectively.

$$(SL_{\text{can}}) \qquad \left( -\frac{\partial^2}{\partial z^2} + \eta^2 Q_{\text{can}} \right) \phi = 0,$$

$$(D_{\text{can}}) \qquad \frac{\partial \phi}{\partial s} = A_{\text{can}} \frac{\partial \phi}{\partial z} - \frac{1}{2} \frac{\partial A_{\text{can}}}{\partial z} \phi,$$

where

$$Q_{\mathrm{can}} = 4z^2 + \eta^{-1}E + \frac{\eta^{-3/2}\rho}{z-\eta^{-1/2}\sigma} + \frac{3\eta^{-2}}{4(z-\eta^{-1/2}\sigma)^2},$$
$$E = \rho^2 - 4\sigma^2, \quad A_{\mathrm{can}} = \frac{1}{2(z-\eta^{-1/2}\sigma)}.$$

The compatibility condition of $(SL_{\mathrm{can}})$ and $(D_{\mathrm{can}})$ is given by the following Hamiltonian system:

$$(H_{\mathrm{can}}) \qquad \frac{\partial \rho}{\partial s} = -4\eta\sigma, \quad \frac{\partial \sigma}{\partial s} = -\eta\rho$$

(cf. [4, Prop. 2.1]). In what follows $(\rho_{\mathrm{can}}, \sigma_{\mathrm{can}})$ denotes a solution of $(H_{\mathrm{can}})$ and $E_{\mathrm{can}} = \rho_{\mathrm{can}}^2 - 4\sigma_{\mathrm{can}}^2$, that is,

$$(8) \qquad \begin{cases} \sigma_{\mathrm{can}}(s,\eta) = A(\eta)e^{2\eta s} + B(\eta)e^{-2\eta s}, \\ \rho_{\mathrm{can}}(s,\eta) = -2A(\eta)e^{2\eta s} + 2B(\eta)e^{-2\eta s}, \\ E_{\mathrm{can}}(\eta) = -16A(\eta)B(\eta), \end{cases}$$

with $A(\eta) = \sum \eta^{-n/2}A_{n/2}$ and $B(\eta) = \sum \eta^{-n/2}B_{n/2}$ being formal power series with constant coefficients. Then the following theorem holds:

**Theorem 4.2.** *For any given* 2*-parameter* $(\alpha,\beta)$ *and a point* $t_0$ *in question, there exist a neighborhood* $V$ *of* $t_0$*, a neighborhood* $U$ *of* $x=\sqrt{t_0}$*, formal series* $(A(\eta), B(\eta)) = (\sum \eta^{-n/2}A_{n/2}, \sum \eta^{-n/2}B_{n/2})$*, and formal series* $z(x,t,\eta) = \sum \eta^{-n/2}z_{n/2}(x,t)$ *and* $s(t,\eta) = \sum \eta^{-n/2}s_{n/2}(t)$ *whose coefficients* $z_{n/2}$ *and* $s_{n/2}$ *are holomorphic on* $U \times V$ *and* $V$ *respectively, so that the following holds: If* $\phi(z,s,\eta)$ *is a WKB solution of* $(SL_{\mathrm{can}})$ *which also satisfies* $(D_{\mathrm{can}})$*, then*

$$(9) \qquad \psi(x,t,\eta) = \left(\frac{\partial z}{\partial x}\right)^{-1/2} \phi(z(x,t,\eta), s(t,\eta), \eta)$$

*satisfies both* $(SL)$ *and* $(D)$*.*

For the proof see [4, Prop. 3.1]. In our case, by the same reasoning as that used in [9, §2.3] for the underlying linear equations of $(P_{\mathrm{I}})$, we can verify that $z(x,t,\eta)$ and $s(t,\eta)$ are homogeneous for $\mathcal{H}$ of degree $-1/2$ and $-1$ respectively. Furthermore, $A(\eta)$ and $B(\eta)$ can be taken so that

$$(10) \qquad A(\eta) = 2^{-3/4}\alpha, \quad B(\eta) = 2^{-3/4}\beta$$

may hold and $E = E_{\mathrm{can}}(\eta)$ also satisfies

$$(11) \qquad E = -4\sqrt{2}\alpha\beta = 4\,\mathrm{Res}_{x=\sqrt{t}}\, S_{\mathrm{odd}}.$$

See [9, §2.3] for the proof of (10) and (11).

As in [9, §2.3], we take the following WKB solutions of $(SL_{\text{can}})$:

$$\phi_\pm = \frac{1}{\sqrt{T_{\text{odd}}}}(\eta^{1/2}z)^{\pm E/4}\exp\pm\Big(\eta\int_0^z T_{-1}dz + \int_\infty^z (T_{\text{odd}} - \eta T_{-1} - \frac{E}{4z})dz\Big),$$

where $T = \sum_{n\geq -2}\eta^{-n/2}T_{n/2}$ is a solution of the Riccati equation associated with $(SL_{\text{can}})$ and $T_{\text{odd}}$ denotes its odd part. For the fundamental properties (such as the well-definedness) of $\phi_\pm$ we refer the reader to [9, §2.3] and only recall the following important properties here: $e^{\pm\eta s}\phi_\pm$ also satisfy $(D_{\text{can}})$ ([9, Lemma 2]) and further $\phi_\pm$ are homogeneous of degree $-1/4$ for a scaling operator $\tilde{\mathcal{H}} : (z, s, \eta) \mapsto (r^{-1/2}z, r^{-1}s, r\eta)$.

Between $\phi_\pm$ and (6) we have the following

**Proposition 4.3.**

$$\psi_\pm^{(k)} = C_\pm^{(k)}\Big(\frac{\partial z}{\partial x}\Big)^{-1/2}\phi_\pm(z(x,t,\eta), s(t,\eta), \eta) \tag{12}$$

*$(k = 0, \infty)$ hold with*

$$C_\pm^{(\infty)} = 2^{\mp 5E/16}e^{\pm\eta s(t,\eta)}, \quad C_\pm^{(0)} = e^{\mp\pi i E/4}C_\pm^{(\infty)}. \tag{13}$$

Using Theorem 4.2 and the homogeneity of $\psi_\pm^{(\infty)}$, $\phi_\pm$ and $(z(x,t,\eta), s(t,\eta))$, we can prove Proposition 4.3 by the same argument as in the proof of [9, Prop. 3]. Note that the second relation of (13) is an immediate consequence of (7) and (11).

Combining Proposition 4.3 and the connection formula for $\phi_\pm$ ([9, Prop. 4]), we obtain the following connection formulas for the WKB solutions (6): Here we label the Stokes curves and the Stokes regions near $x = \sqrt{t}$ as is indicated in Fig.3. We also use the notation $\psi_\pm^{(k),R}$ $(k = 0, \infty)$ to denote the Borel sum of $\psi_\pm^{(k)}$ in a Stokes region $R$ here and in what follows.

$$\begin{cases} \psi_+^{(k),R_{j-1}} = \psi_+^{(k),R_j} + \frac{C_+^{(k)}}{C_-^{(k)}}a_{j-1j}\psi_-^{(k),R_j} \\ \psi_-^{(k),R_{j-1}} = \psi_-^{(k),R_j} \end{cases} \tag{14}$$

on $C_j$ for $j = 1, 3$ and

$$\begin{cases} \psi_+^{(k),R_{j-1}} = \psi_+^{(k),R_j} \\ \psi_-^{(k),R_{j-1}} = \psi_-^{(k),R_j} + \frac{C_-^{(k)}}{C_+^{(k)}}a_{j-1j}\psi_+^{(k),R_j} \end{cases} \tag{15}$$

on $C_j$ for $j = 2, 4$, where $C_\pm^{(k)}$ $(k = 0, \infty)$ are defined by (13) and $a_{j-1j}$ are given as follows:

$$(-1)^{(j+1)/2}\frac{\rho + 2\sigma}{2}\frac{i\sqrt{2\pi}}{\Gamma(1 - \frac{E}{4})}2^{-E/2}e^{(j-1)\pi i E/4} \tag{16}$$

for $j = 1, 3$ and

$$(17) \qquad (-1)^{(j-2)/2}\frac{\rho - 2\sigma}{2}\frac{\sqrt{2\pi}}{\Gamma(1+\frac{E}{4})}2^{E/2}e^{(1-j)\pi i E/4}$$

for $j = 2, 4$ with $(\rho, \sigma) = (\rho_{\mathrm{can}}, \sigma_{\mathrm{can}})$, $E = -4\sqrt{2}\alpha\beta$.

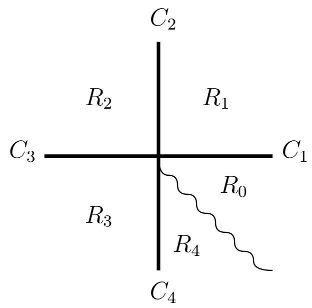


Fig. 3: Stokes curves and Stokes regions near $x = \sqrt{t}$.

# 5 Computation of the monodromy data of $(SL)$

In this section, using the connection formulas (14) and (15), we explicitly compute the monodromy data of $(SL)$.

First, we review the monodromy data of $(SL)$ (cf. [2]). As in Fig.4, let us take

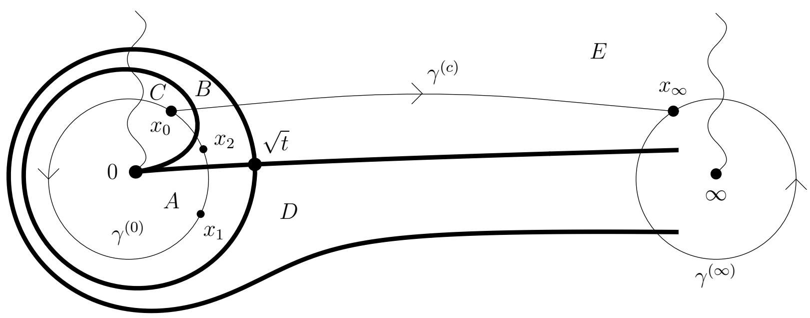


Fig. 4: Paths of analytic continuation $\gamma^{(0)}$, $\gamma^{(c)}$ and $\gamma^{(\infty)}$ and Stokes curves for $\arg\sqrt{t} > 0$. ($A$, $B$, $C$, ... designate the label of Stokes regions.)

base points $x_0, x_\infty$ and paths of analytic continuation $\gamma^{(k)}$ $(k = 0, c, \infty)$. Further, we take fundamental systems $\varphi^k_\pm$ of holomorphic solutions near $x_k$ $(k = 0, \infty)$. Then, according to [2, §2], the monodromy data of $(SL)$ is given by the following set of matrices

$$\{M_0,\ M_c,\ M_\infty\},$$

where the matrices $M_k$ ($k = 0, c, \infty$) are defined by

$$(\gamma_*^{(k)})(\varphi_+^k, \varphi_-^k) = (\varphi_+^k, \varphi_-^k)M_k \quad (\text{for } k = 0, \infty),$$
$$(\gamma_*^{(c)})(\varphi_+^0, \varphi_-^0) = (\varphi_+^\infty, \varphi_-^\infty)M_c.$$

Here and in what follows $\gamma_*(f)$ designates the analytic continuation of $f$ along a path $\gamma$.

**Remark 5.1.** Since $x = 0$ and $x = \infty$ are irregular singular points (with Poincaré rank 1/2) of $(SL)$, the monodromy matrices $M_0$ and $M_\infty$ are expressed in terms of the (triangular) Stokes matrices $S_0$ and $S_\infty$ as

$$M_0 = S_0 J, \quad M_\infty = J S_\infty \ \text{ with } \ J = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Note that the matrix $J$ appears as an effect of crossing a cut emanating from $x = 0$ or $x = \infty$. In the case of $(SL)$ we can also confirm the following

$$-(M_0)^{-1} = (M_c)^{-1} M_\infty M_c. \tag{18}$$

Thanks to (18), if we write

$$S_0 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \ S_\infty = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \ M_c = \begin{pmatrix} c & d \\ e & f \end{pmatrix},$$

we find that all monodromy data are determined once $a$ and $d$ are computed.

In what follows, we adopt the Borel sum of $\psi_\pm^{(k)}$ near $x_k$ ($k = 0, \infty$) as fundamental systems of solutions, i.e., $\varphi_\pm^0 = \psi_\pm^{(0),C}$ and $\varphi_\pm^\infty = \psi_\pm^{(\infty),E}$, and compute $M_0$, $M_c$ when $\arg\sqrt{t} > 0$ and $\arg\sqrt{t} < 0$ respectively. To illustrate how the computations are done, we explain the computation of $M_0$ for $\arg\sqrt{t} > 0$ in details here.

First, we consider the analytic continuation from a point $x_1$ in Region $A$ to a point $x_2$ in Region $B$ (cf. Fig.4). It is described by the connection formula (14) for $j = 3$, that is,

$$(\psi_+^{(0),A}, \psi_-^{(0),A}) = (\psi_+^{(0),B}, \psi_-^{(0),B}) \begin{pmatrix} 1 & 0 \\ -\frac{C_+^{(0)}}{C_-^{(0)}} a_{23} & 0 \end{pmatrix}. \tag{19}$$

Second, we discuss the analytic continuation from $x_2$ to $x_0$. We divide this step of analytic continuation into the following three substeps; (i) from $x_2$ to $x_D$ along $\gamma_{BD}$, (ii) from $x_D$ to $x_A$ across a Stokes curve emanating from $\sqrt{t}$, and (iii) from $x_A$ to $x_0$ along $\gamma_{AC}$ (cf. Fig.5). The substep (i) is described by

$$(\gamma_{BD})_*(\psi_+^{(0),B}, \psi_-^{(0),B}) = (\psi_+^{(0),D}, \psi_-^{(0),D}) \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

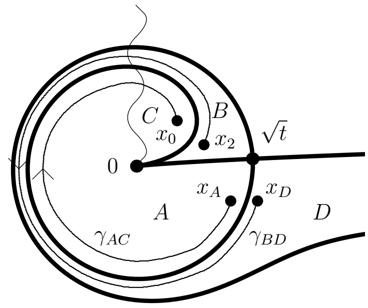


Fig. 5: Paths $\gamma_{BD}$ and $\gamma_{AC}$.

the substep (ii) is described by the connection formula (15) for $j = 4$:

$$(\psi_+^{(0),D}, \psi_-^{(0),D}) = (\psi_+^{(0),A}, \psi_-^{(0),A}) \begin{pmatrix} 1 & -\frac{C_-^{(0)}}{C_+^{(0)}} a_{34} \\ 0 & 1 \end{pmatrix},$$

and the substep (iii) is described by

$$(\gamma_{AC})_*(\psi_+^{(0),A}, \psi_-^{(0),A}) = (\psi_+^{(0),C}, \psi_-^{(0),C}) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{20}$$

Combining these three substeps, we obtain

$$(\psi_+^{(0),B}, \psi_-^{(0),B}) = (\psi_+^{(0),C}, \psi_-^{(0),C}) \begin{pmatrix} 1 & 0 \\ -\frac{C_-^{(0)}}{C_+^{(0)}} a_{34} & 1 \end{pmatrix} \tag{21}$$

for the analytic continuation from $x_2$ to $x_0$. Finally, (20) also entails

$$(\gamma_{AC}^{-1})_*(\psi_+^{(0),C}, \psi_-^{(0),C}) = (\psi_+^{(0),A}, \psi_-^{(0),A}) \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \tag{22}$$

which describes the analytic continuation from $x_0$ to $x_1$. We thus conclude from (19), (21) and (22) that

$$\gamma_*^{(0)}(\psi_+^{(0),C}, \psi_-^{(0),C}) = (\psi_+^{(0),C}, \psi_-^{(0),C}) M_0 \tag{23}$$

with

$$M_0 = \begin{pmatrix} 1 & 0 \\ -\frac{C_+^{(0)}}{C_-^{(0)}} a_{23} - \frac{C_-^{(0)}}{C_+^{(0)}} a_{34} & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{24}$$

for $\arg \sqrt{t} > 0$.

The computation of $M_c$ for $\arg\sqrt{t} > 0$ and that of $M_0$ and $M_c$ for $\arg\sqrt{t} < 0$ can be done in a similar manner. Here, omitting the details of computations, we give only the consequence of them:

$$M_c = \begin{pmatrix} \frac{C_+^{(0)}}{C_+^{(\infty)}} - \frac{(C_-^{(0)})^2}{C_+^{(0)}C_+^{(\infty)}} a_{12}a_{34} & -\frac{C_-^{(0)}}{C_+^{(\infty)}} a_{12} \\ \frac{(C_-^{(0)})^2}{C_+^{(0)}C_-^{(\infty)}} a_{34} & \frac{C_-^{(0)}}{C_-^{(\infty)}} \end{pmatrix} \tag{25}$$

for $\arg\sqrt{t} > 0$,

$$M_0 = \begin{pmatrix} 1 & 0 \\ -\frac{C_+^{(0)}}{C_-^{(0)}} a_{23} - \frac{C_-^{(0)}}{C_+^{(0)}} a_{12} & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \tag{26}$$

$$M_c = \begin{pmatrix} \frac{C_+^{(0)}}{C_+^{(\infty)}} & -\frac{C_-^{(0)}}{C_+^{(\infty)}} a_{12} \\ \frac{(C_-^{(0)})^2}{C_+^{(0)}C_-^{(\infty)}} a_{34} & \frac{C_-^{(0)}}{C_-^{(\infty)}} - \frac{(C_-^{(0)})^3}{(C_+^{(0)})^2 C_-^{(\infty)}} a_{12}a_{34} \end{pmatrix} \tag{27}$$

for $\arg\sqrt{t} < 0$.

Using these results together with (8), (10), (11), (13), (16) and (17), we thus obtain the following formulas (with $E = -4\sqrt{2}\alpha\beta$) for the relevant monodromy data $a$ and $d$ explained in Remark 5.1.

(When $\arg\sqrt{t} > 0$)

$$\begin{aligned} a &= -2^{-9E/8+1/4} \frac{i\sqrt{2\pi}\beta}{\Gamma(-\frac{E}{4}+1)} - 2^{9E/8+1/4} e^{-i\pi E/4} \frac{\sqrt{2\pi}\alpha}{\Gamma(\frac{E}{4}+1)}, \\ d &= 2^{9E/8+1/4} \frac{\sqrt{2\pi}\alpha}{\Gamma(\frac{E}{4}+1)}. \end{aligned} \tag{28}$$

(When $\arg\sqrt{t} < 0$)

$$\begin{aligned} a &= -2^{-9E/8+1/4} \frac{i\sqrt{2\pi}\beta}{\Gamma(-\frac{E}{4}+1)} + 2^{9E/8+1/4} e^{i\pi E/4} \frac{\sqrt{2\pi}\alpha}{\Gamma(\frac{E}{4}+1)}, \\ d &= 2^{9E/8+1/4} \frac{\sqrt{2\pi}\alpha}{\Gamma(\frac{E}{4}+1)}. \end{aligned} \tag{29}$$

# 6 Connection formula for 2-parameter instanton-type solutions of $(P)$

Finally we discuss the connection formula for 2-parameter instanton-type solutions of $(P)$.

Let us now suppose that a 2-parameter solution $q(t,\eta;\alpha,\beta)$ in $\{t; \arg\sqrt{t} > 0\}$ and a 2-parameter solution $q(t,\eta;\tilde{\alpha},\tilde{\beta})$ in $\{t; \arg\sqrt{t} < 0\}$ may represent the same holomorphic solution of $(P)$. Then, thanks to the result of [2], the corresponding monodromy data of $(SL)$ for $\arg\sqrt{t} > 0$ should coincide with that for $\arg\sqrt{t} < 0$. Since the monodromy data is explicitly given by (28) and (29), we thus conclude $(\alpha,\beta)$ and $(\tilde{\alpha},\tilde{\beta})$ should satisfy

$$
\begin{aligned}
&2^{-9E/8}\frac{i\beta}{\Gamma(-\frac{E}{4}+1)}+2^{9E/8}e^{-i\pi E/4}\frac{\alpha}{\Gamma(\frac{E}{4}+1)}\\
&\qquad\qquad = 2^{-9\tilde{E}/8}\frac{i\tilde{\beta}}{\Gamma(-\frac{\tilde{E}}{4}+1)} - 2^{9\tilde{E}/8}e^{i\pi\tilde{E}/4}\frac{\tilde{\alpha}}{\Gamma(\frac{\tilde{E}}{4}+1)},\\
&2^{9E/8}\frac{\alpha}{\Gamma(\frac{E}{4}+1)} = 2^{9\tilde{E}/8}\frac{\tilde{\alpha}}{\Gamma(\frac{\tilde{E}}{4}+1)}
\end{aligned}
\tag{30}
$$

with $E = -4\sqrt{2}\alpha\beta$ and $\tilde{E} = -4\sqrt{2}\tilde{\alpha}\tilde{\beta}$. By (30) we find that $(\alpha,\beta)$ and $(\tilde{\alpha},\tilde{\beta})$ are different in general. This is the Stokes phenomenon for $q(t,\eta;\alpha,\beta)$ and (30) gives their connection formula.

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