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Half of the Toulouse Project Part 5 is completed — Structure theorem for instanton-type solutions of $(P_J)_m$ $(J = \mathrm{I}, \mathrm{II} \text{ or } \mathrm{IV})$ near a simple P-turning point of the first kind

By

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§ 0. Introduction

The purpose of this paper is to announce

Half of Part 5 of the Toulouse Project ([KT2]) is now completed,

that is,

near a simple P-turning point of the first kind, each instanton-type solution of $(P_J)_m$ $(J = \mathrm{I}, \mathrm{II} \text{ or } \mathrm{IV}; m = 1, 2, 3 \cdots)$ can be reduced to an appropriate solution of (P_{I}), the classical Painlevé-I equation with a large parameter η, namely,

$$\frac{d^2\lambda_{\mathrm{I}}}{d\tilde{t}^2} = \eta^2(6\lambda_{\mathrm{I}}^2 + \tilde{t}). \tag{0.1}$$

Here the expression "Half of Part 5" is used to emphasize that only P-turning points of the first kind are studied in this paper: probably we should have divided Part 5 into two parts, like Part 2 and Part 3, which are concerned with 0-parameter solutions.

Let us first recall briefly the current (= as of January, 2007) status of the Toulouse Project. Here and in what follows, we use the same notions and notations as in [KT3], with the exception that the suffix II-2 is now denoted simply by II. In particular, a P-turning point is, by definition, a turning point of a Painlevé equation. This notation was introduced in [KT3] to avoid the possible confusion of a turning point of a Painlevé equation (i.e., in t-space) and that of the underlying linear equation (i.e., in (x, t)-space).

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[1] **Part 1: Stokes geometry of higher order Painlevé equations.**

See [KKoNT1], [KKoNT2] and [N] for $(P_J)_m$ ($J = \mathrm{I}$ or II). See also [Sa1], [Sa2], [AKSaST] and [H] for the Noumi-Yamada system.

[2] **Part 2: Reduction of a 0-parameter solution of $(P_J)_m$ ($J = \mathrm{I}, \mathrm{II}$ or IV) near its turning point of the first kind.**

See [KT3] for $J = \mathrm{I}$ or II and [KT4] for $J = \mathrm{IV}$.

[3] **Part 3: Study of the structure of a 0-parameter solution of $(P_J)_m$ ($J = \mathrm{I}, \mathrm{II}$ or IV) near its turning point of the second kind.**

No Stokes phenomena are observed for 0-parameter solutions there. (Unpublished.)

[4] **Part 4: Construction of $(2m)$-parameter solutions of $(P_J)_m$ ($J = \mathrm{I}, \mathrm{II}$ or IV).**

See [T2] for $J = \mathrm{I}$. As the reasoning there relies only on the existence of the Hamiltonian structure for $(P_{\mathrm{I}})_m$, the recent result of Koike ([Ko]) has enabled us to claim that the construction of such solutions can be done also for $J = \mathrm{II}$ or IV. The $(2m)$-parameter solution constructed in [T2] contains, in parallel with the case of the traditional Painlevé equations ([AKT], [KT1]), terms of the form

$$\alpha_k \exp(\eta \int^t \nu_k dt) \tag{0.2}$$

and hence it is called an instanton-type solution ([T2], [T3]).

Now we announce the result that generalizes the reduction theorem for a 0-parameter solution (Part 2) to that for an instanton-type solution near a P-turning point of the first kind (Main Theorem below). As $(P_{\mathrm{II}})_1$ (resp., $(P_{\mathrm{IV}})_1$) is the traditional (i.e., second order) Painlevé-II (resp., Painlevé-IV) equation, and as every P-turning point of traditional Painlevé equations is of the first kind, our result may also be regarded as a partial generalization of [KT1]. ("A partial generalization" just because it covers only the cases $J = \mathrm{II}$ or IV.)

To clarify and simplify the presentation we consider the case $J = \mathrm{I}$. Let $(P_{\mathrm{I}})_m$ ($m = 1, 2, 3 \cdots$) denote the following system of non-linear differential equations with a large parameter η:

$$\begin{cases} \dfrac{du_j}{dt} = 2\eta v_j \\ \dfrac{dv_j}{dt} = 2\eta(u_{j+1} + u_1 u_j + w_j) \qquad (j = 1, 2, \ldots, m) \\ v_{m+1} = 0, \end{cases} \tag{0.3}$$

where w_j is a polynomial of u_l and v_l $(1 \leq l \leq j)$ that is determined by the following recursive relation:

$$
\begin{aligned}
w_j =& \frac{1}{2}(\sum_{k=1}^{j} u_k u_{j+1-k}) + \sum_{k=1}^{j-1} u_k w_{j-k} \\
&- \frac{1}{2}(\sum_{k=1}^{j-1} v_k v_{j-k}) + c_j + \delta_{jm} t \qquad (j = 1, 2, \ldots, m).
\end{aligned}
\tag{0.4}
$$

Here c_j is a constant and δ_{jm} stands for Kronecker's delta. Then we know ([T2]) the existence of the following instanton-type formal solution of $(P_{\mathrm{I}})_m$:

$$
\begin{cases}
u_j(t, \eta; \alpha) = u_{j,0}(t) + \eta^{-1/2} \sum\limits_{1 \leq k \leq 2m} \alpha_k \exp(\eta \int^t \nu_k dt) u_{jk,1/2}(t) + \cdots , \\
v_j(t, \eta; \alpha) = v_{j,0}(t) + \eta^{-1/2} \sum\limits_{1 \leq k \leq 2m} \alpha_k \exp(\eta \int^t \nu_k dt) v_{jk,1/2}(t) + \cdots .
\end{cases}
\tag{0.5}
$$

Here $\alpha = (\alpha_1, \ldots, \alpha_{2m})$ is a set of free parameters, and ν_k stands for a solution of the characteristic equation of the Fréchet derivative of $(P_{\mathrm{I}})_m$ at a 0-parameter solution. We know ([KKoNT1]) that we can choose ν_j so that

$$\nu_l + \nu_{l+m} = 0 \tag{0.6}$$

holds for $l = 1, \ldots, m$. In parallel with the reasoning of [KT3] we define another set $\{b_j(t, \eta; \alpha)\}$ of instanton-type solutions by considering the solutions $\{b_j\}_{j=1}^m$ of the following equation:

$$x^m - u_1(t, \eta; \alpha) x^{m-1} - \cdots - u_m(t, \eta, \alpha) = 0. \tag{0.7}$$

The function b_j is actually the restriction of a solution of some Garnier system, a multi-dimensional generalization of the Painlevé equation, to an appropriate complex line. This fact is essentially well-known for $J = \mathrm{I}$, and the recent result ([Ko]) of Koike asserts that a similar fact is observed also for $J = \mathrm{II}, \mathrm{IV}$. We will make full use of this fact in our proof to be expounded in our full paper ([KT5]).

Main Theorem. *Let τ be a simple P-turning point of the first kind of $(P_{\mathrm{I}})_m$ that does not coincide with any other P-turning point of $(P_{\mathrm{I}})_m$, and let t_* be a point sufficiently close to τ that lies in a P-Stokes curve emanating from τ. Then there exist an index j_0, formal series*

$$\tilde{x}(x, t, \eta) = \sum_{l \geq 0} \eta^{-l/2} \tilde{x}_{l/2}(x, t, \eta) \tag{0.8}$$

and

$$\tilde{t}(x,t,\eta) = \sum_{l \geq 0} \eta^{-l/2} \tilde{t}_{l/2}(x,t,\eta), \tag{0.9}$$

and a 2-parameter solution $\lambda_{\mathrm{I}}(\tilde{t},\eta;\beta)$ ($\beta = (\beta_1,\beta_2)$) of (0.1) for which the following relations are satisfied on a neighborhood of t_* for an instanton-type solution $b_{j_0}(t,\eta;\alpha)$ with $\alpha_{j_0,0}\alpha_{j_0+m,0}$ different from 0 where α_{j_0} and α_{j_0+m} are coefficients of the instanton terms directly related to the P-turning point τ in the sense specified in the course of our discussion:

$$\tilde{x}(b_{j_0}(t,\eta;\alpha),t,\eta) = \lambda_{\mathrm{I}}(\tilde{t}(t,\eta),\eta;\beta), \tag{0.10}$$

(0.11) *$\alpha_{j_0,0} = 2c\beta_{1,0}$ and $\alpha_{j_0+m,0} = 2c^{-1}\beta_{2,0}$ hold for a constant c that depends only on the product $\alpha_{j_0,0}\alpha_{j_0+m,0}$,*

(0.12) *$\tilde{x}_{1/2}$ and $\tilde{t}_{1/2}$ vanish identically,*

(0.13) *the η-dependence of $\tilde{x}_{l/2}$ and $\tilde{t}_{l/2}$ is only through instanton terms that they contain, and $\tilde{x}_0, \tilde{x}_1, \tilde{t}_0$ and $\tilde{t}_1$ are free from instanton terms.*

In §1 we describe in outline how the proof of Main Theorem goes. In §2 we give a proof of its core part, namely Theorem 1.3 which shows that the principal part (i.e., the top order part) of the Fréchet derivative of $(P_J)_m$ splits into a direct sum of 2×2 systems at the point in question. The final section gives a heuristic description of the relevance of our Main Theorem to the connection formula for solutions of $(P_J)_m$; our argument is only heuristic, as we have not yet found an appropriate method to endow instanton-type formal solutions with their analytic meaning.

The details of this article shall be given in our forthcoming paper ([KT5]).

§ 1. Basic ingredients of the proof of Main Theorem

The flow diagram of our reasoning is basically the same as the reasoning of [KT1] for proving the reduction theorem for 2-parameter solutions of the traditional (i.e., second order) Painlevé equations. As the underlying Lax pair for $(P_J)_m$ ($J = \mathrm{I}, \mathrm{II}$ or IV) is given in a matrix form, we first rewrite it as a system of scalar equations. This part is done by [KT3] for $J = \mathrm{I}, \mathrm{II}$ and by [KT4] for $J = \mathrm{IV}$. The system consists of a Schrödinger equation $(SL_J)_m$ and its deformation equation $(D_J)_m$. For example, $(SL_{\mathrm{I}})_m$ is the following second order equation with a large parameter η:

$$\frac{\partial^2 \psi}{\partial x^2} = \eta^2 Q_{(\mathrm{I},m)} \psi, \tag{1.1}$$

where the potential $Q_{(\mathrm{I},m)}$ is expressed as in (1.2) below in terms of polynomials $U(x), V(x)$ and $W(x)$ given below:

$$Q_{(\mathrm{I},m)} = \frac{1}{4}(2x^{m+1} - xU + 2W)U + \frac{1}{4}V^2 - \frac{\eta^{-1}U_xV}{2U} + \frac{\eta^{-1}V_x}{2} + \frac{3\eta^{-2}U_x^2}{4U^2} - \frac{\eta^{-2}U_{xx}}{2U}, \tag{1.2}$$

with

$$U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j}, \tag{1.3}$$

$$V(x) = \sum_{j=1}^{m} v_j x^{m-j}, \tag{1.4}$$

$$W(x) = \sum_{j=0}^{m} w_j x^{m-j}, \tag{1.5}$$

where (u_j, v_j) $(1 \le j \le m)$ is a solution of $(P_{\mathrm{I}})_m$ and w_j $(1 \le j \le m)$ is a polynomial of (u_l, v_l) $(1 \le l \le j)$ that is given by (0.4). Note that

$$U(b_j) = 0 \qquad (1 \le j \le m) \tag{1.6}$$

holds by the definition of $\{b_j\}$. The deformation equation $(D_{\mathrm{I}})_m$ of $(SL_{\mathrm{I}})_m$ is also described in terms of U as follows:

$$\frac{\partial \psi}{\partial t} = \mathfrak{a}_{(\mathrm{I},m)} \frac{\partial \psi}{\partial x} - \frac{1}{2}\frac{\partial \mathfrak{a}_{\mathrm{I},m}}{\partial x}\psi, \tag{1.7}$$

where

$$\mathfrak{a}_{(\mathrm{I},m)} = \frac{2}{U(x)}. \tag{1.8}$$

Now, a result of [KKoNT1] asserts that a simple turning point and a double turning point coalesce at $t = \tau$ in the Stokes geometry of $(SL_J)_m$. The latter one is given by $x = b_{j_0,0}(t)$ for some j_0. This index j_0 is the one used in the statement of Main Theorem. Then we can prove the following results in the setting of Main Theorem:

Theorem 1.1. *Let V be a sufficiently small neighborhood of t_*. Then there exist a neighborhood U of $x = b_{j_0,0}(t)$, a formal series*

$$z(x,t,\eta) = z_0(x,t,\eta) + \eta^{-1/2} z_{1/2}(x,t,\eta) + \eta^{-1} z_1(x,t,\eta) + \cdots \tag{1.9}$$

whose coefficients $z_{j/2}(x,t,\eta)$ are holomorphic on $U \times V$, and formal series

$$E^{(j_0)}(t,\eta) = E_0^{(j_0)}(t,\eta) + E_{1/2}^{(j_0)}(t,\eta)\eta^{-1/2} + E_1^{(j_0)}(t,\eta)\eta^{-1} + \cdots \tag{1.10}$$

and

$$\rho^{(j_0)}(t,\eta) = \rho_0^{(j_0)}(t,\eta) + \rho_{1/2}^{(j_0)}(t,\eta)\eta^{-1/2} + \rho_1^{(j_0)}(t,\eta)\eta^{-1} + \cdots \tag{1.11}$$

whose coefficients are holomorphic on V, so that the following five conditions are satisfied:

(1.12) z_0 *is free from* η,

(1.13) $\dfrac{\partial z_0}{\partial x}$ *never vanishes on* $U \times V$,

(1.14) $z_0(b_{j_0,0}(t),t) = 0$,

(1.15) $z_{1/2}$ *identically vanishes*,

$$\begin{aligned} Q_{(J,m)}\ (x,t,\eta) = &\left(\frac{\partial z}{\partial x}\right)^2 \Big[4z(x,t,\eta)^2 + \eta^{-1}E^{(j_0)}(t,\eta) \\ &+ \frac{\eta^{-3/2}\rho^{(j_0)}(t,\eta)}{z(x,t,\eta) - z(b_{j_0}(t,\eta),t,\eta)} \\ &+ \frac{3\eta^{-2}}{4(z(x,t,\eta) - z(b_{j_0}(t,\eta),t,\eta))^2}\Big] - \frac{1}{2}\eta^{-2}\{z(x,t,\eta);x\} \end{aligned} \tag{1.16}$$

holds on $U \times V$. Here $\{z;x\}$ denotes the Schwarzian derivative

$$\frac{\partial^3 z/\partial x^3}{\partial z/\partial x} - \frac{3}{2}\left(\frac{\partial^2 z/\partial x^2}{\partial z/\partial x}\right)^2.$$

Furthermore the η-dependence of $z_{j/2}\ (x,t,\eta), E_{j/2}^{(j_0)}\ (t,\eta)$ and $\rho_{j/2}^{(j_0)}(t,\eta)$ is through the instanton terms that $b_{j_0}(t,\eta)$ contains.

The series $E^{(j_0)}(t,\eta)$ and $\rho^{(j_0)}(t,\eta)$ are explicitly given in terms of $\{b_j\}_{j=1}^m$ and $z(x,t,\eta)$ in (1.9):

Theorem 1.2.

(i)

$$\begin{aligned} \rho^{(j_0)}(t,\eta) = &\eta^{-1/2}\left(\frac{\partial z}{\partial x}(b_{j_0}(t,\eta),t,\eta)\right)^{-1} \\ &\times \left[\frac{1}{2}\frac{\partial}{\partial t}(b_{j_0}(t,\eta))\left(\frac{1}{(x-b_{j_0}(t,\eta))\mathfrak{a}_{(J,m)}}\right)\Bigg|_{x=b_{j_0}(t,\eta)} \right.\\ &\left. + \left(\frac{1}{2}\left(\frac{\frac{\partial}{\partial x}(\mathfrak{a}_{(J,m)})}{\mathfrak{a}_{(J,m)}} + \frac{1}{(x-b_{j_0}(t,\eta))}\right) + \frac{3}{4}\left(\frac{\partial^2 z/\partial x^2}{\partial z/\partial x}\right)\right)\Bigg|_{x=b_{j_0}(t,\eta)}\right]. \end{aligned}$$

(ii) $E^{(j_0)}(t,\eta) = (\rho^{(j_0)})^2 - 4(\sigma^{(j_0)})^2$ *holds for*

$$\sigma^{(j_0)} = \eta^{1/2}z(b_{j_0}(t,\eta),t,\eta). \tag{1.17}$$

The proof of Theorems 1.1 and 1.2 can be given in a similar way to the proof of Theorem 3.1 of [AKT]. As is well-known, Theorem 1.1 entails that a WKB solution $\psi(x,t,\eta)$ of $(SL_J)_m$ is expressed as

$$\psi(x,t,\eta) = \left(\frac{\partial z}{\partial x}\right)^{-1/2} \varphi(z(x,t,\eta),t,\eta), \tag{1.18}$$

where φ is a WKB solution of the following Schrödinger equation:

$$\left(-\frac{\partial^2}{\partial z^2} + \eta^2 Q_{\mathrm{can}}(z,t,\eta)\right)\varphi = 0, \tag{1.19}$$

where

$$Q_{\mathrm{can}} = 4z^2 + \eta^{-1}E(\tau,\eta) + \frac{\eta^{-3/2}\rho(t,\eta)}{x - \eta^{-1/2\sigma}(t,\eta)} + \frac{3\eta^{-2}}{4(x-\eta^{-1/2}\sigma(t,\eta))^2}. \tag{1.20}$$

Once we obtain Theorems 1.1 and 1.2, the next thing to do would be to try to extend the domain of definition of the series $z(x,t,\eta)$ so that it may be related to the simple turning point of $(SL_J)_m$ that merges with $b_{j_0,0}(t)$ at $t=\tau$.

However, in order to proceed in that way, we have to confirm that the top order part $\rho_0^{(j_0)}$ and $\sigma_0^{(j_0)}$ of $\rho^{(j_0)}$ and $\sigma^{(j_0)}$ contain instanton terms whose phase functions are related to the P-turning point in question. To be more concrete, we have to confirm Theorem 1.3 below. Before stating it we make a notational preparation: it follows from the definition of a P-turning point of the first kind (cf. [KKoNT1], Section 2) that there exist characteristic roots ν_{j_0} and ν_{j_0+m} of the Fréchet derivative of $(P_J)_m$ such that $\nu_{j_0+m} = -\nu_{j_0}$ and $\nu_{j_0}(\tau) = \nu_{j_0+m}(\tau) = 0$ hold. (Note that in [KKoNT1] ν_{j_0} and ν_{j_0+m} are denoted by $\nu_{j_0,+}$ and $\nu_{j_0,-}$, respectively.) The functions $\int_\tau^t \nu_{j_0} dt$ and $\int_\tau^t \nu_{j_0+m} dt$ are phase functions which appear in the instanton-type solutions. As one might readily surmise, these phase functions are tied up with the P-turning point τ and they are what we really need.

Theorem 1.3. *The top order part* $\rho_0^{(j_0)}$ *and* $\sigma_0^{(j_0)}$ *of* $\rho^{(j_0)}$ *and* $\sigma^{(j_0)}$ *contain only instanton terms* $\exp(\eta\int_\tau^t \nu_{j_0} dt)$ *and* $\exp(\eta\int_\tau^t \nu_{j_0+m} dt)$.

The proof of Theorem 1.3 will be given in §2, where we will use the explicit form of $(P_J)_m$. Another proof which makes use of its Hamiltonian form will be given in our forthcoming paper ([KT5]). We also note that, although $\rho_{j/2}^{(j_0)}$ and $\sigma_{j/2}^{(j_0)}$ $(j \geq 1)$ may contain instanton terms with phase functions other than $\int_\tau^t \nu_{j_0} dt$ and $\int_\tau^t \nu_{j_0+m} dt$, they always contain $\exp(\eta\int_\tau^t \nu_{j_0} dt)$ and $\exp(\eta\int_\tau^t \nu_{j_0+m} dt)$ as their factor. This fact is important in proving our Main Theorem. Theorem 1.1 fortified with Theorem 1.3 enables us to follow the line of the reasoning in the proof of Theorem 4.1 of [KT1]. A crucially important step in our reasoning is to establish Theorem 1.4 below. Here, and in what follows, (Can) designates the following Schrödinger equation

$$\left(-\frac{\partial^2}{\partial z^2}+\eta^2\left(4z^2+\eta^{-1}E_{\text{can}}+\frac{\eta^{-3/2}\rho_{\text{can}}(s,\eta)}{x-\eta^{-1/2}\sigma_{\text{can}}(s,\eta)}+\frac{3\eta^{-2}}{4(x-\eta^{-1/2}\sigma_{\text{can}}(s,\eta))^2}\right)\right)\varphi=0 \tag{1.21}$$

with

$$E_{\text{can}}=\rho_{\text{can}}^2-4\sigma_{\text{can}}^2, \tag{1.22}$$

and (D_{can}) designates the following equation

$$\frac{\partial\psi}{\partial s}=A_{\text{can}}\frac{\partial\psi}{\partial z}-\frac{1}{2}\frac{\partial A_{\text{can}}}{\partial z}\psi \tag{1.23}$$

with

$$A_{\text{can}}=\frac{1}{2(z-\eta^{-1/2}\sigma_{\text{can}})}. \tag{1.24}$$

We note that (Can) and (D_{can}) are in involution if ρ_{can} and σ_{can} satisfy the following (simplest!) Hamiltonian system (H_{can}):

$$\begin{cases}\dfrac{d\rho_{\text{can}}}{ds}=-4\eta\sigma_{\text{can}}\\ \dfrac{d\sigma_{\text{can}}}{ds}=-\eta\rho_{\text{can}}.\end{cases} \tag{1.25}$$

The function ψ given by (1.18) satisfies (SL_J) if $\varphi(z,s,\eta)$ satisfies (Can) (with $(\rho_{\text{can}},\sigma_{\text{can}})=(\rho^{(j_0)},\sigma^{(j_0)})$), but we cannot expect that ψ also solves $(D_J)_m$ even if φ solves both (Can) and (D_{can}); in order to attain such a harmonious situation we need to relate t and s appropriately. The required relation can be obtained by solving

$$\rho_{\text{can}}(s(t;\alpha,A,B;\eta),\eta)=\rho^{(j_0)}(t,\eta) \tag{1.26}$$

and

$$\sigma_{\text{can}}(s(t;\alpha,A,B;\eta),\eta)=\sigma^{(j_0)}(t,\eta) \tag{1.27}$$

under the condition

$$E_{\text{can}}=E^{(j_0)}, \tag{1.28}$$

where

$$\rho_{\mathrm{can}} = -2A(\eta)\exp(2\eta s) + 2B(\eta)\exp(-2\eta s), \tag{1.29}$$

$$\sigma_{\mathrm{can}} = A(\eta)\exp(2\eta s) + B(\eta)\exp(-2\eta s), \tag{1.30}$$

with $A(\eta) = \sum_{j\geq 0} A_{j/2}\eta^{-j/2}$ and $B(\eta) = \sum_{j\geq 0} B_{j/2}\eta^{-j/2}$. The relation (1.28) entails

$$\alpha_{j_0,0}\alpha_{j_0+m,0} = 8A_0B_0, \tag{1.31}$$

but there remains some freedom in the choice of A_0 and B_0; this arbitrariness is got rid of in Main Theorem by considering the problem semi-globally (versus locally near the double turning point $x = b_{j_0,0}(t)$ as in Theorem 1.4 below).

Theorem 1.4. *Let us consider the situation described in Theorem 1.1. In addition to the transformation* (1.9), *we can construct a transformation*

$$s(t,\eta) = s_0(t) + \eta^{-1}s_1(t,\eta) + \eta^{-3/2}s_{3/2}(t,\eta) + \cdots \tag{1.32}$$

so that for a WKB solution $\varphi(z,s,\eta)$ of (Can) that satisfies (D_{can})

$$\psi(x,t,\eta) = \left(\frac{\partial z}{\partial x}\right)^{-1/2} \varphi(z(x,t,\eta), s(t,\eta), \eta) \tag{1.33}$$

satisfies both $(SL_J)_m$ and $(D_J)_m$.

§ 2. Proof of Theorem 1.3

In this section we give the proof of Theorem 1.3 for $(P_{\mathrm{I}})_m$. The cases $J = \mathrm{II}$ and $J = \mathrm{IV}$ can be proved in a similar manner.

We first write down the top order part $\rho_0^{(j_0)}$ and $\sigma_0^{(j_0)}$ of $\rho^{(j_0)}$ and $\sigma_0^{(j_0)}$ in terms of $v_{j,1/2}$, $u_{j,1/2}$, $u_{j,0}$ and $b_{j_0,0}$. Here, and in what follows, $v_{j,k/2}$ $(k = 0, 1, \ldots)$ etc. designate the coefficient of $\eta^{-k/2}$ in the expansion (0.5) of an instanton-type solution $v_j(t,\eta;\alpha)$ etc. (with instanton terms being considered to be order 0 with respect to η). Since $\mathfrak{a}_{(\mathrm{I},m)}$ is given by (1.8) in the case of $(P_{\mathrm{I}})_m$, it follows from Theorem 1.2 (i) that

$$\rho_0^{(j_0)} = \frac{1}{4}\left(\frac{\partial z_0}{\partial x}(b_{j_0,0}(t),t)\right)^{-1} \Delta_{j_0}\left[\eta^{-1}\frac{d}{dt}b_{j_0,1/2}\right]_0, \tag{2.1}$$

where Δ_{j_0} denotes

$$\Delta_{j_0} = \prod_{\substack{1\leq j'\leq m\\ j'\neq j_0}} (b_{j_0,0}(t) - b_{j',0}(t)) \tag{2.2}$$

and $[\eta^{-1}(db_{j_0,1/2}/dt)]_0$ designates the top order part of $\eta^{-1}(db_{j_0,1/2}/dt)$. Note that $[\eta^{-1}(db_{j_0,1/2}/dt)]_0$ does not vanish as $b_{j_0,1/2}(t)$ contains some instanton terms. In view of (1.14) and (1.17) we have also

$$\sigma_0^{(j_0)} = \frac{\partial z_0}{\partial x}(b_{j_0,0}(t),t)b_{j_0,1/2}. \tag{2.3}$$

To seek for more explicit description of $\rho_0^{(j_0)}$ and $\sigma_0^{(j_0)}$ we use the following lemmas.

Lemma 2.1.

$$b_{j,1/2} = (\Delta_j)^{-1}(b_{j,0}^{m-1}u_{1,1/2} + \cdots + u_{m,1/2}). \tag{2.4}$$

Proof. By the definition of b_k

$$x^m - u_1(t,\eta;\alpha)x^{m-1} - \cdots - u_m(t,\eta;\alpha) = \prod_{1\le k\le m}(x - b_k(t,\eta;\alpha)) \tag{2.5}$$

holds. Taking the order $-1/2$ part of both sides of (2.5), we obtain

$$u_{1,1/2}x^{m-1} + \cdots + u_{m,1/2} = \prod_{1\le k\le m} b_{k,1/2}\prod_{\substack{1\le k'\le m\\ k'\neq k}}(x - b_{k',0}). \tag{2.6}$$

Evaluation of (2.6) at $x = b_{j_0,0}$ immediately implies (2.4).

Lemma 2.2.

$$\frac{\partial z_0}{\partial x}(b_{j_0,0},t) = \frac{1}{2}(b_{j_0,0} + 2u_{1,0})^{1/4}(\Delta_{j_0})^{1/2}. \tag{2.7}$$

Proof. It follows from (1.16) that $z_0(x,t)$ satisfies

$$Q_{(\mathrm{I},m),0} = 4\left(\frac{\partial z_0}{\partial x}\right)^2 z_0^2. \tag{2.8}$$

As is observed in [KT3, (1.1.34)], $Q_{(\mathrm{I},m),0}$ is factorized as

$$Q_{(\mathrm{I},m),0} = \frac{1}{4}(x + 2u_{1,0})U_0^2 = \frac{1}{4}(x + 2u_{1,0})\prod_{1\le k\le m}(x - b_{k,0})^2. \tag{2.9}$$

Hence, considering the Taylor expansion of both sides of (2.8) at $x = b_{j_0,0}$ and taking (1.14) into account, we obtain

$$\frac{1}{4}(b_{j_0,0} + 2u_{1,0})(\Delta_{j_0})^2 = 4\left(\frac{\partial z_0}{\partial x}(b_{j_0,0},t)\right)^4. \tag{2.10}$$

Relation (2.7) is an immediate consequence of (2.10).

Lemma 2.3.

$$\left[\eta^{-1}\frac{d}{dt}u_{j,1/2}\right]_0 = 2v_{j,1/2}. \tag{2.11}$$

This lemma readily follows from the first equation of $(P_{\rm I})_m$ (see (0.3)). In particular, combining Lemma 2.1 and Lemma 2.3, we obtain

$$\left[\eta^{-1}\frac{d}{dt}b_{j_0,1/2}\right]_0 = 2(\Delta_{j_0})^{-1}(b_{j_0,0}^{m-1}v_{1,1/2}+\cdots+v_{m,1/2}). \tag{2.12}$$

Using these lemmas together with (2.12), we can deduce the following explicit description of $\rho_0^{(j_0)}$ and $\sigma_0^{(j_0)}$ from (2.1) and (2.3):

$$\rho_0^{(j_0)} = (b_{j_0,0}+2u_{1,0})^{-1/4}(\Delta_{j_0})^{-1/2}(b_{j_0,0}^{m-1}v_{1,1/2}+\cdots+v_{m,1/2}), \tag{2.13}$$

$$\sigma_0^{(j_0)} = \frac{1}{2}(b_{j_0,0}+2u_{1,0})^{1/4}(\Delta_{j_0})^{-1/2}(b_{j_0,0}^{m-1}u_{1,1/2}+\cdots+u_{m,1/2}). \tag{2.14}$$

Making use of the expressions (2.13) and (2.14), we now compute $[\eta^{-1}(d/dt)\rho_0^{(j_0)}]_0$ and $[\eta^{-1}(d/dt)\sigma_0^{(j_0)}]_0$, that is, the differentiation with respect to t of $\rho_0^{(j_0)}$ and $\sigma_0^{(j_0)}$ applied only to their instanton terms.

It follows from the second equation of $(P_{\rm I})_m$ that

$$\left[\eta^{-1}\frac{d}{dt}v_{j,1/2}\right]_0 = 2(u_{j+1,1/2}+u_{1,0}u_{j,1/2}+u_{j,0}u_{1,1/2}+w_{j,1/2}). \tag{2.15}$$

Here, as is verified in [KKoNT1, Lemma 2.1.1], $w_{j,1/2} = u_{1,0}u_{j,1/2}$ holds. Hence we have

$$\left[\eta^{-1}\frac{d}{dt}v_{j,1/2}\right]_0 = 2(u_{j+1,1/2}+2u_{1,0}u_{j,1/2}+u_{j,0}u_{1,1/2}). \tag{2.16}$$

Using (2.13) and (2.16), we can compute $[\eta^{-1}(d/dt)\rho_0^{(j_0)}]_0$ as follows:

$$\begin{aligned}
&\left[\eta^{-1}\frac{d}{dt}\rho_0^{(j_0)}\right]_0 \\
&= (b_{j_0,0}+2u_{1,0})^{-1/4}(\Delta_{j_0})^{-1/2}\sum_{1\le k\le m} b_{j_0,0}^{m-k}\left[\eta^{-1}\frac{d}{dt}v_{k,1/2}\right]_0 \\
&= 2(b_{j_0,0}+2u_{1,0})^{-1/4}(\Delta_{j_0})^{-1/2} \\
&\qquad\times\sum_{1\le k\le m} b_{j_0,0}^{m-k}(u_{k+1,1/2}+2u_{1,0}u_{k,1/2}+u_{k,0}u_{1,1/2}) \\
&= 2(b_{j_0,0}+2u_{1,0})^{-1/4}(\Delta_{j_0})^{-1/2}
\end{aligned} \tag{2.17}$$

$$\times\left\{2u_{1,0}\sum_{1\le k\le m} b_{j_0,0}^{m-k}u_{k,1/2}+b_{j_0,0}\sum_{2\le k\le m} b_{j_0,0}^{m-k}u_{k,1/2}\right.$$
$$\left.+u_{1,1/2}\sum_{1\le k\le m} b_{j_0,0}^{m-k}u_{k,1/2}\right\}$$
$$=2(b_{j_0,0}+2u_{1,0})^{3/4}(\Delta_{j_0})^{-1/2}\sum_{1\le k\le m} b_{j_0,0}^{m-k}u_{k,1/2}.$$

Here we have used the relation

$$b_{j_0,0}^{m}=\sum_{1\le k\le m} b_{j_0,0}^{m-k}u_{k,1/2} \tag{2.18}$$

to obtain the last equality of (2.17). On the other hand, Lemma 2.3 immediately entails

$$\begin{aligned}\left[\eta^{-1}\frac{d}{dt}\sigma_0^{(j_0)}\right]_0 &\\ &=\frac{1}{2}(b_{j_0,0}+2u_{1,0})^{1/4}(\Delta_{j_0})^{-1/2}\sum_{1\le k\le m} b_{j_0,0}^{m-k}\left[\eta^{-1}\frac{d}{dt}u_{k,1/2}\right]_0\\ &=(b_{j_0,0}+2u_{1,0})^{1/4}(\Delta_{j_0})^{-1/2}\sum_{1\le k\le m} b_{j_0,0}^{m-k}v_{k,1/2}.\end{aligned} \tag{2.19}$$

We thus obtain

$$\left[\eta^{-1}\frac{d}{dt}\rho_0^{(j_0)}\right]_0=4(b_{j_0,0}+2u_{1,0})^{1/2}\sigma_0^{(j_0)}, \tag{2.20}$$

$$\left[\eta^{-1}\frac{d}{dt}\sigma_0^{(j_0)}\right]_0=(b_{j_0,0}+2u_{1,0})^{1/2}\rho_0^{(j_0)}. \tag{2.21}$$

Recalling the relations $\nu_{j_0}=2(b_{j_0,0}+2u_{1,0})^{1/2}$ and $\nu_{j_0+m}=-2(b_{j_0,0}+2u_{1,0})^{1/2}$, which were verified in [KKoNT1, Prop. 2.1.3], we conclude that $\rho_0^{(j_0)}$ and $\sigma_0^{(j_0)}$ contain only instanton terms $\exp(\eta\int_\tau^t \nu_{j_0}dt)$ and $\exp(\eta\int_\tau^t \nu_{j_0+m}dt)$ thanks to (2.20) and (2.21). This completes the proof of Theorem 1.3.

§ 3. The relation between structure theorem for instanton-type solutions and the connection problem for higher order Painlevé transcendents

Our Main Theorem asserts that the instanton-type solution $b_{j_0}(t,\eta;\alpha)$ of $(P_J)_m$ is related to $\lambda_{\rm I}(\tilde{t},\eta;\beta)$ by (0.10) near a point t_* on a P-Stokes curve of $(P_J)_m$. In this section we discuss its implication for the analytic structure of solutions of $(P_J)_m$, which we call “higher order Painlevé transcendents”. The vital clue to such a study is the fact that several transformations of underlying Schrödinger equations simultaneously exist in addition to the relation (0.10).

To begin with, let us summarize the geometric situation of our study. In t-plane we find Figure 3.1, where $t(\mathrm{i})$ (resp., $t(\mathrm{ii})$) is a point close to t_* satisfying $\operatorname{Im}\phi_{j_0}(t(\mathrm{i})) > 0$ (resp., $\operatorname{Im}\phi_{j_0}(t(\mathrm{ii})) < 0$) with

$$\phi_{j_0}(t) = \int_\tau^t \nu_{j_0} dt. \tag{3.1}$$

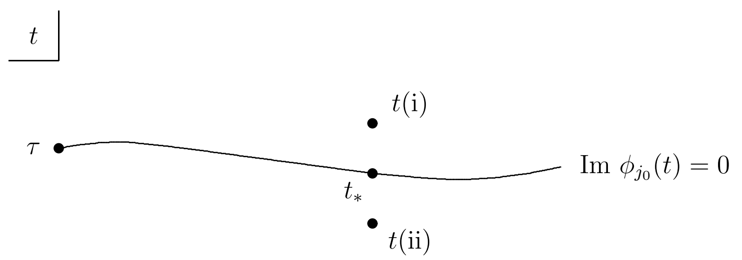

Figure 3.1: P-Stokes curve in question emanating from τ.

As is now well-known ([KKoNT1]), the Stokes geometry of $(SL_J)_m$ is degenerate for $t = t_*$; see Figure 3.2. This degeneration, i.e., the appearance of two turning points

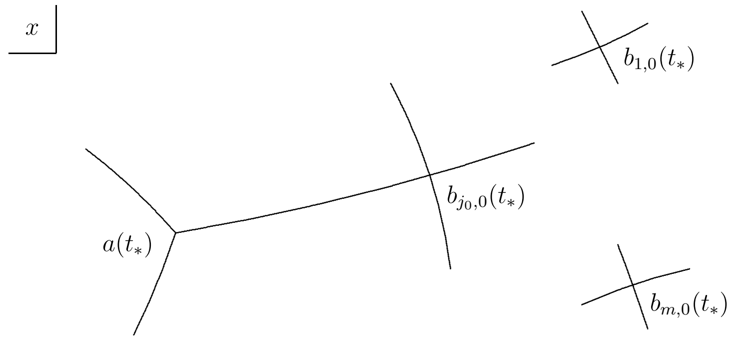

Figure 3.2: Stokes geometry of $(SL_J)_m$ for $t = t_*$, where $b_{j_0,0}(t_*)$ (resp., $a(t_*)$) is a double (resp., simple) turning point.

connected by a Stokes segment, is resolved if the parameter t is away from the P-Stokes curve; the configurations of Stokes curves of $(SL_J)_m$ for $t = t(\mathrm{i})$ and $t = t(\mathrm{ii})$ are respectively shown in Figure 3.3 (i) and (ii). We observe that a topological change of the configuration of Stokes curves is observed only in a neighborhood of the Stokes segment connecting $a(t_*)$ and $b_{j_0,0}(t_*)$: the double turning point $b_{l,0}(t)$ $(l \neq j_0)$ is not accompanied by such a topological change at $t = t(\mathrm{i})$ or $t(\mathrm{ii})$. Note that Theorem 1.1 is applicable to each $b_{l,0}(t)$, regardless of such topological changes. This fact will play an important role in our later discussions.

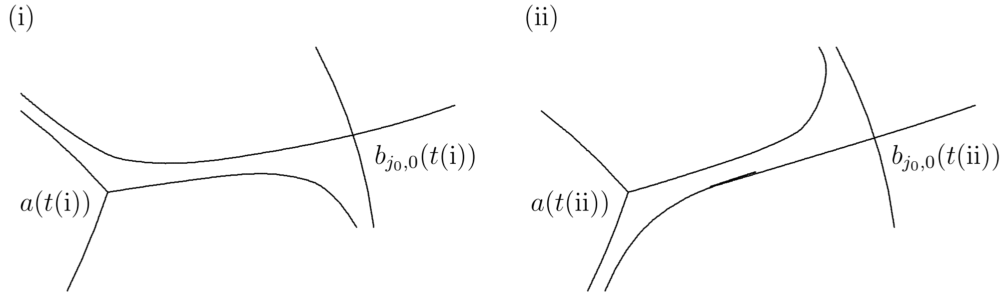

Figure 3.3: Stokes geometry of $(SL_J)_m$ for (i) $t = t(\mathrm{i})$, and (ii) $t = t(\mathrm{ii})$.

Now let us explain the following important implication of our Main Theorem: $(\alpha_{j_0,0}, \alpha_{j_0+m,0})$ inherits the relation that (β_1, β_2) satisfies. In fact, the series $\tilde{x}(x,t,\eta)$ used in (0.10) transforms $(SL_J)_m$ into (SL_{I}) where the Stokes geometry of (SL_{I}) at $\tilde{t} = \tilde{t}(t(\mathrm{i}))$ and $\tilde{t} = \tilde{t}(t(\mathrm{ii}))$ are respectively given in Figure 3.4 (i) and (ii). The Stokes

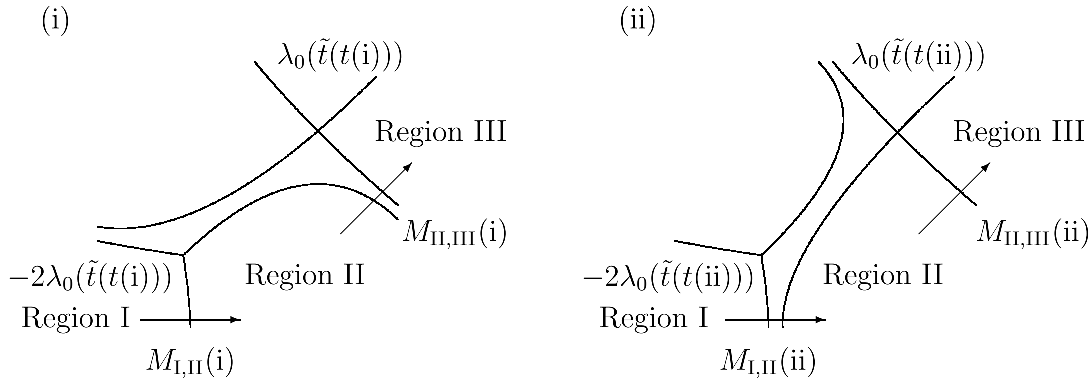

Figure 3.4: Stokes geometry of (SL_{I}) at (i) $\tilde{t} = \tilde{t}(\mathrm{i})$, and (ii) $\tilde{t} = \tilde{t}(\mathrm{ii})$.

multipliers $M_{\mathrm{I,II}}(j)$ and $M_{\mathrm{II,III}}(j)$ ($j =$ i, ii) corresponding respectively to the transfer from Region I to Region II and to that from Region II to Region III for appropriately normalized WKB solutions of (SL_{I}) can be computed in terms of ρ_{I} and σ_{I} (cf. [T3, §4 and §5]). Furthermore they are preserved by the deformation, that is, we have

$$M_{\mathrm{I,II}}(\mathrm{i}) = M_{\mathrm{I,II}}(\mathrm{ii}), \quad M_{\mathrm{II,III}}(\mathrm{i}) = M_{\mathrm{II,III}}(\mathrm{ii}), \tag{3.2}$$

though they have different expressions. Then (3.2) gives relations between $\lambda_{\mathrm{I}}(\tilde{t},\eta;\beta)$ near $\tilde{t} = \tilde{t}(t(\mathrm{i}))$ and its analytic continuation to $\tilde{t} = \tilde{t}(t(\mathrm{ii}))$. The latter one may have a different instanton-type expansion, i.e., $\lambda_{\mathrm{I}}(\tilde{t},\eta;\tilde{\beta})$. The relation (3.2) thus describes the relation between β and $\tilde{\beta}$. Since $\tilde{x}(x,t,\eta)$ defines an invertible transformation between $(SL_J)_m$ and (SL_{I}), the relation between β and $\tilde{\beta}$ is transferred through (0.11) to the top order parts $(\alpha_{j_0,0}, \alpha_{j_0+m,0})$ and $(\tilde{\alpha}_{j_0,0}, \tilde{\alpha}_{j_0+m,0})$, i.e., the top order parts of the coefficients of $\exp(\eta\phi_{j_0})$ and $\exp(\eta\phi_{j_0+m})$ in the instanton-type expansion of the higher order

Painlevé transcendents $(u_j(t,\eta;\alpha), v_j(t,\eta;\alpha))$ near $t = t(\mathrm{i})$ and its analytic continuation $(u_j(t,\eta;\tilde{\alpha}), v_j(t,\eta;\tilde{\alpha}))$ to $t = t(\mathrm{ii})$. Note that we have restricted our consideration to the top order parts in view of Theorem 1.3. It is also true that the explicit calculation of the connection formula for (Can) is available only for the top order parts.

On the other hand, as was already mentioned, $(SL_J)_m$ can be transformed into (Can) near each $b_{l,0}(t)$ $(l \neq j_0)$. To discuss the Stokes phenomena for solutions of (Can) we prepare Figure 3.5. It is readily found from Figure 3.5 that the Stokes geometry of

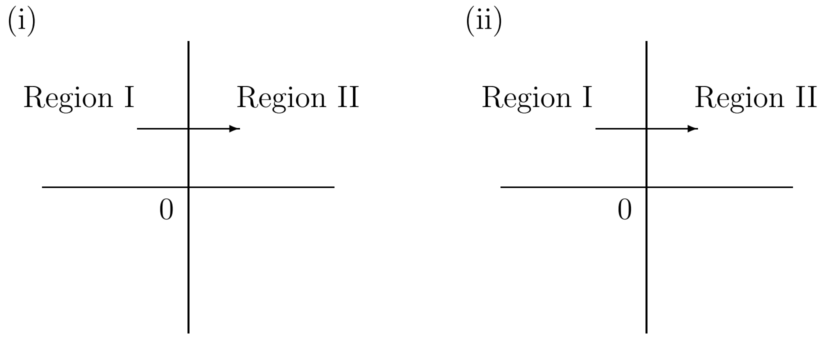

Figure 3.5: Stokes geometry of (Can) at (i) $s = s(t(\mathrm{i}))$, and (ii) $s = s(t(\mathrm{ii}))$.

(Can) is the same for $s = s(t(\mathrm{i}))$ and $s = s(t(\mathrm{ii}))$. Since (Can) can be isomonodromically deformed by (D_{can}), the Stokes multipliers for appropriately normalized WKB solutions of (Can) corresponding to the transfer, say from Region I to Region II remain invariant as we move from $t(\mathrm{i})$ to $t(\mathrm{ii})$. As the Stokes multipliers are computed in terms of ρ_{can} and σ_{can} (cf. [T1]), the invariance of the Stokes multipliers entails the invariance of the coefficients $A(\eta)$ and $B(\eta)$ of ρ_{can} and σ_{can} (cf. (1.29) and (1.30)) and, in particular, the invariance of their top terms A_0 and B_0. Now Theorem 1.3 together with the reasoning in [KT1, §3] again implies, with appropriate labelling of α_j's, that

$$\alpha_{l,0} = 2\sqrt{2}c_l A_0 \ \text{ and } \ \alpha_{l+m,0} = 2\sqrt{2}c_l^{-1} B_0 \tag{3.3}$$

hold with some constant c_l in a neighborhood of $t = t_*$. Hence the top order part $(\alpha_{l,0}, \alpha_{l+m,0})$ of (α_l, α_{l+m}) for $l \neq j_0$ in the instanton-type expansion of solutions of $(P_J)_m$ remains invariant as t moves from $t(\mathrm{i})$ to $t(\mathrm{ii})$.

Summing up, we can conclude that the relation of $(\alpha_{j_0,0}, \alpha_{j_0+m,0})$ inherited from that of (β_1, β_2) together with the invariance of $(\alpha_{l,0}, \alpha_{l+m,0})$ $(l \neq j_0)$ provides the connection formula for instanton-type solutions of $(P_J)_m$ near $t = t_*$. Although the discussion in this section is only heuristic, we hope it will give the reader some insight into the problem how our Main Theorem is related to the connection problem for the higher order Painlevé transcendents.

Remark 3.1. It is better in the context of this article to replace $\tilde{\psi}_\pm$ in [T1, (2.31)] by $\exp(\pm\eta\tilde{t})\tilde{\psi}_\pm$ so that they satisfy (D_{can}) (cf. [T1, p.285, l.2]).

Remark 3.2. We take this opportunity to correct one typographical error in [KT0]: In the second formula of (4.110) (p.102) the exponent of e is $i\pi(E_{\rm I}+1)/2$, not $-i\pi(E_{\rm I}+1)/2$.

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