ON KONTSEVICH'S CHARACTERISTIC CLASSES FOR HIGHER DIMENSIONAL HOMOLOGY SPHERE BUNDLES AND MILNOR'S λ'-INVARIANT

TADAYUKI WATANABE

ABSTRACT. This paper is concerned with M. Kontsevich's universal characteristic classes of smooth bundles with fiber a 'singularly' framed odd-dimensional homology sphere. The main object of the present paper is to show that Kontsevich classes for fiber dimensions greater than 3 are highly non-trivial even after being made framing independent. We have two approaches: (i) explicit framing correction with relative characteristic classes of the vertical tangent bundle, (ii) 'clasper surgery' construction and evaluation on them. By the first approach, an invariant of bundles over spheres that is an 'integral lift' of Milnor's λ'-invariant for exotic spheres is obtained and thus the non-triviality follows. By the second approach, non-triviality of higher classes and new estimates for unstable rational higher homotopy groups of the relative diffeomorphism groups are obtained.

1. INTRODUCTION

In [Kon], M. Kontsevich introduced the notion of graph homology and found important relationships between the graph homology and the cohomologies of various infinite dimensional objects such as the moduli space of Riemann surfaces, certain infinite dimensional Lie algebras and the classifying space for some smooth bundles and so on. In this paper, we focus on his work on the cohomology of the classifying space for smooth bundles. Kontsevich developed the method of configuration space integral to construct cohomology classes of the classifying space of smooth bundles, that will be denoted by $\widetilde{B\mathrm{Diff}}M$, with fiber diffeomorphic to a "singularly framed" odd-dimensional homology sphere M (see [Kon] or §2 for the definition). Here, a singularly framed homology sphere bundle denotes a homology sphere bundle with its fiber framed non singularly outside a fixed point ∞ for which both structures of smooth bundle and fiber tangent bundle are standardly trivialized near ∞.

For bundles with 3-dimensional fibers, all the 3-valent Kontsevich classes are 0-forms, i.e., diffeomorphism invariants of homology 3-spheres. In this case, it has been shown by G. Kuperberg and D. Thurston in [KT] in a purely topological argument that the space of all the 3-valent real valued Kontsevich classes and a certain space of linear functionals on 3-valent graphs are isomorphic in a graded sense. (See also C. Lescop's generalization

Date: June 15, 2007.
2000 *Mathematics Subject Classification.* 57M27, 55R35, 57R20.

[Les2] in a de Rham theoretic approach.) Hence there are very many linearly independent Kontsevich classes in 3-dimension.

For fiber dimensions greater than 3, it has not been understood how finely the Kontsevich classes explain the cohomology of $\widetilde{B\mathrm{Diff}M}$. In this paper, we study the non-triviality of the 3-valent Kontsevich classes after making them framing independent, by two approaches.

The first one is for the simplest class corresponding to the Θ-graph, that is based on the same spirit of Kuperberg–Thurston or of S. Morita's formula for the Casson invariant [Mo]. We will get an invariant $\hat{\zeta}_2$ of unframed higher-dimensional homology sphere bundles over a sphere by adding to the simplest class some multiple of Hirzebruch's signature defect (Theorem 3.2). Moreover, in the case when the fiber is diffeomorphic to a sphere, the total space of a bundle as a smooth manifold is diffeomorphic to a connected sum of the trivial bundle and an exotic sphere. So Milnor's λ'-invariant for exotic spheres [Mil2] gives rise to a bundle invariant. By Novikov in [Nov], and after that by Antonelli-Burghelea-Kahn in [ABK], some higher dimensional sphere-bundles for which the λ'-invariant is non-trivial have been constructed. The λ'-invariant, defined by using the signature of a bounded manifold, is in some sense a higher dimensional analogue of the Rokhlin invariant. We will show that in some cases $\hat{\zeta}_2$ is an "integral lift" of the λ'-invariant and hence conclude that Antonelli–Burghelea–Kahn's constructions give infinite order elements of homotopy groups of the classifying space in that cases, which is unexplainable by traditional approaches, and that those elements are detected by the $\hat{\zeta}_2$-invariant. This is a similar situation as in 3-dimension where the Casson invariant is an integral lift of the Rokhlin invariant. So it is expected that the $\hat{\zeta}_2$-invariant has similar features as for the Casson invariant. We also relate the non-triviality problem of $\hat{\zeta}_2$ to an elementary number theory problem involving numerator of Bernoulli numbers.

Another approach is to construct some framed bundles by using higher-dimensional "graph clasper-bundle surgery" and to show the non-triviality of the Kontsevich classes corresponding to graphs with higher number of vertices (Theorem 4.1). Higher dimensional claspers in a single manifold are introduced in [W, W2] as higher dimensional generalizations of Habiro's claspers in 3-dimension [Hab]. With the idea of Kuperberg–Thurston and Cattaneo–Cotta-Ramusino–Longoni [CCL] in mind, we will prove that the space of all the $\mathbb{R}$-valued Kontsevich classes corresponding to 3-valent graphs with $2n$-vertices is linearly isomorphic to the dual of some space of 3-valent graphs with $2n$-vertices, by computing the configuration space integrals explicitly as "counting the shapes of graphs living in a bundle". Further, when the fiber is D^{8m-1}, each of our construction of bundles is bordant to a bundle over a sphere and consequently, we obtain new non-trivial estimates for the rational homotopy groups of the infinite dimensional Lie group $\mathrm{Diff}(D^{8m-1} \text{ rel } \partial)$.

Let us give some historical remarks for homotopy type of $\mathrm{Diff}(D^n \text{ rel } \partial)$ from the viewpoint of the present paper. A famous result of S. Smale says that $\mathrm{Diff}(D^2 \text{ rel } \partial)$ is contractible [Sm], which implies that any smooth $(D^2 \text{ rel } \partial)$-bundle must be trivial.

Smale further conjectured that the same is true for $\mathrm{Diff}(D^3 \text{ rel } \partial)$ and A. Hatcher has proved Smale's conjecture [Hat]. However, as remarked above, it has been observed after Milnor's discovery of exotic spheres that the same is no longer true for larger n (see [Nov, ABK] etc.). Further, the rational homotopy classification has been completed in a stable range. Namely, by using K-theory, F. Farrell and W. Hsiang [FH] have obtained the stable isomorphism

$$\pi_i B\mathrm{Diff}(D^{2k-1} \text{ rel } \partial) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = 4p \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

for $2k-1 \gg i$. On the other hand, the estimates obtained in the present paper by using Kontsevich classes are rather unstable informations that are disjoint from Farrell–Hsiang's stable range. However, it might be interesting to ask whether there is some relationship between $\hat{\zeta}_2$ and higher FR torsion class. Higher FR torsion class has been deeply studied by K. Igusa [Igu] and the fact that the generator of $\pi_{4p} B\mathrm{Diff}(D^{2k-1} \text{ rel } \partial) \otimes \mathbb{Q}$ for $2k - 1 \gg 4p$ can be detected by higher FR torsion class has been proved by him.

The present paper is organized as follows. In §2, some notations and the definition of the Kontsevich class are given. §3 and §4 correspond to the two approaches above respectively. Some open problems are discussed in §5. In the appendix, proof of a proposition used in §4 is given.

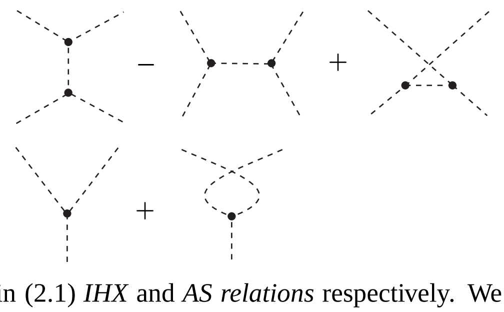
2. Kontsevich's universal characteristic classes

We shall review the definition of Kontsevich's universal characteristic classes.

2.1. **Feynman diagrams.** First we define the space $\mathcal{A}_{2n}$ of trivalent graphs. An *orientation* on a trivalent graph Γ is a choice of an ordering of three edges incident to each trivalent vertex, considered modulo even number of swappings of the orders. We present the orientation in plane diagrams by assuming that the order of three edges incident to each trivalent vertex is always given by anti-clockwise order.

Let $\mathcal{G}_{2n}$ be the vector space over $\mathbb{Q}$ spanned by all connected trivalent graphs with oriented $2n$ vertices. Let $\mathcal{A}_{2n}$ be the quotient space of $\mathcal{G}_{2n} \otimes \mathbb{R}$ by the subspace spanned by the vectors of the following form:

(2.1)



We call the vectors in (2.1) *IHX* and *AS relations* respectively. We will write as $[\Gamma]$ the element of $\mathcal{A}_{2n}$ represented by $\Gamma \otimes 1$ for $\Gamma \in \mathcal{G}_{2n}$. The *degree* of a trivalent graph is defined as the number of vertices. For example, $\mathcal{A}_2 = \mathrm{span}_{\mathbb{R}}\{[\Theta]\}$, where $[\Theta]$ is the Θ-graph.

2.2. **Fulton-MacPherson-Kontsevich compactification of the configuration space.** Let M be a d-dimensional homology sphere with a fixed point $\infty \in M$. Let $C_n(M)$ be the Fulton-MacPherson-Kontsevich compactification ([FM]) of the configuration space

$$M^{\times n} \setminus \text{(diagonals)}.$$

Here we include in the diagonals the set of configurations with some points go infinity. For example, $C_2(M)$ is obtained from $M \times M$ by blowing up first along (∞, ∞) and then along the disjoint diagonals

$$\text{(the diagonal)} \cup ((M \setminus \infty) \times \{\infty\}) \cup (\{\infty\} \times (M \setminus \infty)).$$

Neighborhood of the face of $\partial C_2(M)$ corresponding to $(\infty,\infty) \in M \times M$ in $C_2(M)$ has the same behavior as a neighborhood of the face of $\partial C_2(S^{d-1})$ corresponding to $(\infty,\infty) \in S^{d-1} \times S^{d-1}$, and one has the Gauss map $p_{S^{d-1}}$ from the (∞,∞)-face of $\partial C_2(S^{d-1})$ to S^{d-1}, associating unit relative vectors in $\mathbb{R}^d$. So the Gauss map p_M from the (∞,∞)-face of $\partial C_2(M)$ to S^{d-1} is defined as $p_{S^{d-1}}$. Moreover, the union of the faces corresponding to the above three diagonals is naturally a trivial S^{d-1}-bundle, which is identified with a product S^{d-1}-bundle by the framing. Then one obtains a map $p_M : \partial C_2(M) \setminus ((\infty,\infty)\text{-face}) \to S^{d-1}$ given by the composition of the trivialization and the projection onto the S^{d-1}-factor. Thus we have defined a continuous map $p_M : \partial C_2(M) \to S^{d-1}$. It is known that $p_M^*\omega_{S^{d-1}}$, where $\omega_{S^{d-1}}$ is the SO_d-invariant unit volume form on S^{d-1}, extends to a closed $(d-1)$-form α_M on $C_2(M)$ and it generates $H^{d-1}(C_2(M);\mathbb{R})$ (see e.g., [Coh, Les]).

2.3. **Universal smooth M-bundle.** Let $M^\bullet$ denote M with a puncture at $\infty \in M$. By a *smooth $(M^\bullet$ rel $\partial)$-bundle*, we mean a smooth bundle with fiber $M^\bullet$ such that the bundle is trivialized on $\partial M^\bullet$. We will say that a smooth $(M^\bullet$ rel $\partial)$-bundle has a *vertical framing* if there is a trivialization of its vertical tangent bundle, namely, tangent bundle along the $M^\bullet$-fibers, that is also standard near $\partial M^\bullet$.

Let $\widetilde{\mathrm{Emb}}(M \setminus \{\infty\}, \mathbb{R}^\infty)$ be the space of smooth tangentially framed embeddings $M \setminus \{\infty\} \to \mathbb{R}^\infty$ that are standard near ∞, i.e., coincide with the natural inclusion $\mathbb{R}^d \subset \mathbb{R}^\infty$ near ∞. Here $\mathbb{R}^\infty$ denotes the Hilbert space of square summable sequences. We equip $\widetilde{\mathrm{Emb}}(M \setminus \{\infty\}, \mathbb{R}^\infty)$ with the $\mathcal{FD}$-topology in [Mic]. Then the principal $\mathrm{Diff}(M^\bullet \text{ rel } \partial)$-bundle

$$\pi_{\mathrm{Diff}\,M} : \widetilde{\mathrm{Emb}}(M \setminus \{\infty\}, \mathbb{R}^\infty) \to \widetilde{\mathrm{Emb}}(M \setminus \{\infty\}, \mathbb{R}^\infty)/\mathrm{Diff}(M^\bullet \text{ rel } \partial)$$

is a disjoint union of copies of the universal framed $\mathrm{Diff}(M^\bullet \text{ rel } \partial)$-bundles, each associated to a homotopy class of framings on $M^\bullet$ (in the case $M^\bullet$ is a punctured homology sphere, there are at most $\mathbb{Z}\times$finite-copies)*. We denote the bundle $\pi_{\mathrm{Diff}\,M}$ simply by $\widetilde{E\mathrm{Diff}}\,M \to \widetilde{B\mathrm{Diff}}\,M$. We fix a base point of each component of $\widetilde{B\mathrm{Diff}}\,M$ and fix a standard framing on the fiber of it. $\mathrm{Diff}(M^\bullet \text{ rel } \partial)$ acts on $\widetilde{\mathrm{Emb}}(M \setminus \{\infty\}, \mathbb{R}^\infty)$ from the right by $((\phi, \tilde{\tau}_M) \cdot g)(x) = (\phi(gx), \tilde{\tau}_M(gx))$ for $\phi \in \mathrm{Emb}(M \setminus \{\infty\}, \mathbb{R}^\infty)$ and for $\tilde{\tau}_M : (M^\bullet, \partial M^\bullet) \to (GL_+(\mathbb{R}^d), 1)$ being a difference from the standard framing. $\widetilde{B\mathrm{Diff}}\,M$ is also considered as the base of the universal smooth framed $(M^\bullet \text{ rel } \partial)$-bundle

$$\pi_M : M \ltimes \widetilde{E\mathrm{Diff}}\,M \to \widetilde{B\mathrm{Diff}}\,M,$$

associated to $\pi_{\mathrm{Diff}\,M}$†. Here the expression $F \ltimes \widetilde{E\mathrm{Diff}}\,M$ means the Borel construction $F \times_{\mathrm{Diff}(M^\bullet \text{ rel } \partial)} \widetilde{E\mathrm{Diff}}\,M$. From general theory of bundles, an isomorphism class of a smooth framed $(M^\bullet \text{ rel } \partial)$-bundle $E \to B$ is determined by the homotopy class of a classifying map $f : B \to \widetilde{B\mathrm{Diff}}\,M$ (see e.g. [Mo2]). We will often identify a classifying map f with the induced bundle $f^*\pi_{\mathrm{Diff}\,M}$ and in the light of this identification we will identify each fiber of a bundle with a point of $\widetilde{B\mathrm{Diff}}\,M$.

From the result of Appendix A, there exists a closed $(d-1)$-form $\alpha_{\mathrm{Diff}\,M}$ on the universal $C_2(M)$-bundle

$$\pi_{C_2(M)} : C_2(M) \ltimes \widetilde{E\mathrm{Diff}}\,M \to \widetilde{B\mathrm{Diff}}\,M$$

associated to π_M, whose restriction on each fiber represents $[\alpha_M]$.

2.4. **Kontsevich's characteristic classes.** Let Γ be a connected trivalent graph of degree $2n$ up to automorphism without a part like $-\!\circ$ and let $\omega(\Gamma)$ be the $3n(d-1)$-form on $C_{2n}(M) \ltimes \widetilde{E\mathrm{Diff}}\,M$ defined by

$$\omega(\Gamma) \stackrel{\mathrm{def}}{=} \bigwedge_{e:\text{ edge of }\Gamma} \phi_e^*\alpha_{\mathrm{Diff}\,M}$$

where we fix a bijective correspondence between the set of vertices of Γ and the set of $2n$ points in a configuration, and

$$\phi_e : C_{2n}(M) \ltimes \widetilde{E\mathrm{Diff}}\,M \to C_2(M) \ltimes \widetilde{E\mathrm{Diff}}\,M$$

is the projection corresponding to picking of the two endpoints of e. Note that the choice of the form $\alpha_{\mathrm{Diff}\,M}$ and therefore of $\omega(\Gamma)$ depends on the framing on $M^\bullet$. Then the pushforward $(\pi_{C_{2n}(M)})_*\omega(\Gamma)$ along the fiber of $\pi_{C_{2n}(M)}$ yields an $n(d-3)$-form on $\widetilde{B\mathrm{Diff}}\,M$. See Appendix B for the definition of the pushforward. Note that the choice of the orientation on $C_{2n}(M)$-fiber has ambiguity which is canonically definable by the orientation of Γ.

---

*Here we say universal framed bundle in the sense that it is contractible into the space of framings on M that are standard near ∞ and that there is a bijection between the set of isomorphism classes of vertically framed $(M^\bullet \text{ rel } \partial)$-bundles over B and the homotopy set $[B, \widetilde{B\mathrm{Diff}}\,M]$.

†In fact, $\widetilde{B\mathrm{Diff}}\,M$ is a kind of an infinite dimensional smooth manifold for which the de Rham theorem holds. See [Mic, Mic2] for details about it.

According to [Kon], the form

$$\zeta_{2n} \stackrel{\text{def}}{=} \sum_{\Gamma} \frac{(\pi_{C_{2n}(M)})_* \omega(\Gamma)[\Gamma]}{|\mathrm{Aut}\,\Gamma|} \in \Omega^{n(d-3)}(\widetilde{B\mathrm{Diff}}M;\mathcal{A}_{2n}),$$

where the sum is over all connected trivalent graphs without $\multimap$ and where $|\mathrm{Aut}\,\Gamma|$ is the order of the group of automorphisms of Γ, is closed and thus descends to an $\mathcal{A}_{2n}$-valued universal characteristic class of framed smooth M-bundles. Here the ambiguity of the orientation on $C_{2n}(M)$ is canceled by taking the product with $[\Gamma]$ and hence the resulting class is independent of the choice of the orientations of Γ. Further, $\mathbb{R}$-valued Kontsevich classes are also defined by composing ζ_{2n} with any linear functional on $\mathcal{A}_{2n}$.

That ζ_{2n} is closed may be seen as follows. By a similar argument as in [KT, Les] by means of the generalized Stokes theorem (B.1), we have

$$d\zeta_{2n} = \sum_{\Gamma} \frac{[\Gamma]}{|\mathrm{Aut}\,\Gamma|} \int_{S_{2n}(TM)_b} \omega(\Gamma) \quad (b \in \widetilde{B\mathrm{Diff}}M), \tag{2.2}$$

which vanishes on $\widetilde{B\mathrm{Diff}}M$. Here $S_{2n}(TM)_b \to M_b^\bullet = C_1(M_b) \subset \partial C_2(M)$ denotes the face diffeomorphic to the bundle associated to $TM_b^\bullet$ with fiber the space of configurations of $2n$ points in a d-dimensional plane considered modulo overall translations and dilations. Indeed, $d\zeta_{2n}$ evaluated on any $n(d-3)$-chain σ can be expressed as an integral of a pulled back form from the fiber of a point of σ. Then the integral vanishes by a dimensional reason.

In the case M is a 3-dimensional homology sphere, the set of all ζ_{2n}'s gives rise to a universal $\mathbb{R}$-valued finite type invariants, according to Kuperberg–Thurston [KT].

2.5. **Alternative definition of the Kontsevich classes.** The above definition of ζ_{2n} relies on the de Rham theorem for infinite dimensional manifolds. Of course it is the most universal way of construction, but since we consider only bundles over finite dimensional compact manifolds, we could avoid the de Rham theorem in infinite dimension just by replacing the universal bundle π_M with a given vertically framed bundle $(\pi : E \to B; \tau_E)$ over a finite dimensional compact manifold B. This time one has a cohomology class $\zeta_{2n}(\pi;\tau_E) \in H^{n(d-3)}(B;\mathcal{A}_{2n})$ that can be defined completely in finite dimensional manifolds. The naturality of the pushforward with respect to bundle morphisms implies that $\zeta_{2n}(\pi;\tau_E)$ is a characteristic class of framed ($M^\bullet$ rel ∂)-bundles. Then the evaluation on bundles over $n(d-3)$-dimensional manifolds gives rise to a framed bordism invariant

$$\langle \zeta_{2n}(\pi;\tau_E), \cdot \rangle \in \mathrm{Hom}(\Omega_{n(d-3)}(\widetilde{B\mathrm{Diff}}M), \mathcal{A}_{2n}).$$

Bordism invariance can also be proved in finite dimension. This is indeed enough for the arguments of the present paper.

3. An invariant of unframed S^{2k-1}-bundles and Milnor's λ'-invariant

In this section, we restrict our study mainly to smooth $(D^{2k-1}\ \mathrm{rel}\ \partial)$-bundles over S^{2k-4}. The dimension $(2k-4)$ of the sphere coincide with the degree of ζ_2. We will show that the simplest Kontsevich class ζ_2 after an addition of a certain rational multiple of the signature defect invariant becomes an invariant of *unframed* $(D^{2k-1}\ \mathrm{rel}\ \partial)$-bundles and that it may be considered as an "integral lift" of Milnor's λ'-invariant of homotopy spheres.

For a $(D^{2k-1}\ \mathrm{rel}\ \partial)$-bundle $\pi : E \to S^{2k-4}$, let τ_E denote a vertical framing, if exists. In the following, we assume that all vertically framed bundle π has a base point on its base space and that a diffeomorphism between the fiber of the base point of π and that of $\pi_M = \pi_{S^{2k-1}}$ is fixed so that π represents an element of the homotopy group $\pi_{2k-4}\widetilde{B\mathrm{Diff}}\,S^{2k-1}$. So integer multiplication to a bundle is defined.

Each element of $\pi_{2k-4}\widetilde{B\mathrm{Diff}}\,S^{2k-1}$ may also be represented by a $(D^{2k-1}\ \mathrm{rel}\ \partial)$-bundle $\pi_D : E_D \to D^{2k-4}$ over D^{2k-4} that is standardly trivialized on $\pi_D^{-1}(\partial D^{2k-4})$. Then ∂E_D is canonically diffeomorphic to S^{4k-6} after a smoothing and $-D^{4k-5}$ can be glued along ∂E_D in a natural way to obtain a closed $(4k-5)$-manifold and denote it by $\mathrm{cl}(E_D)$.

Now we shall choose a framing on $\mathrm{cl}(E_D)$. Note that if the vertical framing τ_{E_D} on π_D that is standard on $\pi_D^{-1}(\partial D^{2k-4})$ exists, then together with the standard framing on D^{2k-1} one has a trivialization of TE_D. Then one can show that $T\mathrm{cl}(E_D)\oplus\varepsilon$, where ε denotes the trivial 1-dimensional outward normal bundle, is trivial and that the trivialization of TE_D together with the canonical trivialization of ε extends to whole of $T\mathrm{cl}(E_D)\oplus\varepsilon$ since $\pi_{4k-6}SO_{4k-4} = \pi_{4k-6}SO = 0$. We denote the extended trivialization by $\tau'_{E_D} : T\mathrm{cl}(E_D)\oplus\varepsilon \overset{\sim}{\to} \mathrm{cl}(E_D)\times\mathbb{R}^{4k-4}$. Also, one obtains an extension τ''_{E_D} of the partial $(2k-1)$-frame τ_{E_D} over $\mathrm{cl}(E_D)$ as the partial $(2k-1)$-frame of τ'_{E_D}. Here we assume that *the choice of extension τ''_{E_D} is always equal to some standardly fixed one (independent of the class of $\pi_{2k-4}\widetilde{B\mathrm{Diff}}\,S^{2k-1}$)*, that is possible since the behavior of τ_{E_D} on the boundary is standard. Moreover, we assume that *τ'_{E_D} is fixed so that when π_D represents $0\in\pi_{2k-4}\widetilde{B\mathrm{Diff}}\,S^{2k-1}$ and hence $\mathrm{cl}(E_D)=S^{4k-5}$, τ'_{E_D} is homotopic to the one induced from the standard (Euclidean) trivialization of D^{4k-4} where $\partial D^{4k-4}=\mathrm{cl}(E_D)$.*

Since $\mathrm{cl}(E_D)$ is a homology $(4k-5)$-sphere, it bounds a compact oriented $(4k-4)$-manifold W. Then the relative L-class is defined by Hirzebruch's L-polynomial:

$$L_j(TW;\tau'_{E_D}) \overset{\mathrm{def}}{=} L_j(p_1,\ldots,p_j)$$

where $p_j = p_j(TW;\tau'_{E_D})$ is the j-th relative Pontrjagin class. It is known that the relative p_{k-1}-class can be interpreted as the obstruction class in $H^{4k-4}(W,\partial W;\pi_{4k-5}U_{4k-4}/U_{2k-3})$ to extend the partial $(2k-1)$-framing τ''_{E_D} on ∂W to the partial $(2k-1)$-framing over the complexified tangent bundle $TW\otimes\mathbb{C}$. Then the $(k-1)$-st signature defect $\Delta_{k-1}(E;\tau_{E_D})$

is defined by

$$\Delta_{k-1}(E;\tau_{E_D}) \stackrel{\mathrm{def}}{=} L_{k-1}(TW;\tau'_{E_D})[W] - \mathrm{sign}\, W.$$

Proposition 3.1. *$\Delta_{k-1}(E;\tau_{E_D})$ is well-defined, i.e., independent of the choices of the manifold W and of the extended framing τ'_{E_D}. Further, it is a group homomorphism $\pi_{2k-4}\widetilde{B\mathrm{Diff}}\, S^{2k-1} \to \mathbb{Q}$.*

Proof. Proof that $\Delta_{k-1}(E;\tau_{E_D})$ does not depend on W is the same as [Mo, Proposition 7.3]. By the assumptions for the choice of τ'_{E_D} above, the ambiguity of the choice can be given by an overall twisting of the 'horizontal' framing by an element of $GL_+(\mathbb{R}^{2k-4})$, that does not affect $\Delta_{k-1}(E;\tau_{E_D})$ since the two can be connected by a path.

By the additivity of the relative L_{k-1} numbers, it is enough to show that Δ_{k-1} at the unit of $\pi_{2k-4}\widetilde{B\mathrm{Diff}}\, S^{2k-1}$ vanishes. But by the second assumption for the choice of τ'_{E_D}, all the relative Pontrjagin numbers and the signature vanish when π_D represents 0. □

The main theorem of this section is the following

Theorem 3.2. *Let $k \geq 3$ and let $\pi : E \to S^{2k-4}$ be a $(D^{2k-1}$ rel $\partial)$-bundle over S^{2k-4}. Then there exists a positive integer q_k such that $q_k\pi : q_kE \to S^{2k-4}$ can be vertically framed for all π. Further, if τ_E is a vertical framing on $q_k\pi$, then the number*

$$\hat{\zeta}_2(E) \stackrel{\mathrm{def}}{=} 12\,\zeta_2(q_kE;\tau_E)|_{[\Theta]=1} + \frac{(-1)^{k-1}(2k-2)!}{2^{2k}(2^{2k-3}-1)B_{k-1}}\Delta_{k-1}(q_kE;\tau_{E_D}) \in \mathbb{Q},$$

where B_{k-1} is the $(k-1)$-st Bernoulli number and where $\zeta_2(q_kE;\tau_E)$ is $\zeta_2(q_k\pi;\tau_E)$ evaluated on $[S^{2k-4}]$, does not depend on the choice of τ_E, and is a group homomorphism $\pi_{2k-4}B\mathrm{Diff}(D^{2k-1}$ rel $\partial) \otimes \mathbb{Q} \to \mathbb{Q}$ that is a homotopy invariant of unframed $(D^{2k-1}$ rel $\partial)$-bundles.

Remark 3.3. Since α can be chosen in $\mathbb{Z}$ coefficient, one may define $|\mathrm{Aut}\,\Theta|\,\zeta_2|_{[\Theta]=1} = 12\,\zeta_2|_{[\Theta]=1} = (\pi_{C_2(S^{2k-1})})_*\omega(\Theta)$ completely in the singular cochain complex in $\mathbb{Z}$ coefficients by replacing the pushforward $(\pi_{C_2(S^{2k-1})})_*$ with the Gysin homomorphism $(\pi_{C_2(S^{2k-1})})_!$. So one has $12\,\zeta_2(q_kE;\tau_E)|_{[\Theta]=1} \in \mathbb{Z}$.

The formula for $\hat{\zeta}_2$ is similar to Morita's splitting formula for the Casson invariant (for homology 3-spheres) in terms of the signature defect [Mo]:

$$\lambda(M) = \frac{1}{6}\int_{C_2(M)} \alpha_M^3 - \frac{1}{8}\Delta_1(M;\tau_M).$$

(This is the version described in [KT, Les2]). So the existence of analogous properties and constructions for the Casson invariant may be expected for $\hat{\zeta}_2$. From this formula, it seems likely that the Casson invariant is an integral lift of the Rokhlin invariant and in fact it actually is, as is well known.

Now we shall discuss about a similar correspondence in higher dimensions. Since the closure of the total space E_D of a $(D^{2k-1}$ rel $\partial)$-bundle may be obtained by gluing a

$(4k-5)$-disk to another $(4k-5)$-disk by some diffeomorphism, it is homeomorphic to S^{4k-5}, namely an exotic sphere (it is a well known fact that can be seen by using a Morse theoretic argument). Hence diffeomorphism invariants of exotic spheres can be applied to $(D^{2k-1}$ rel $\partial)$-bundles.

Let Θ^d denote the group of h-cobordism classes (equivalently, diffeomorphism classes, by the h-cobordism theorem) of d-dimensional homotopy spheres. In [Mil], Milnor constructed homotopy spheres by introducing the bilinear pairings:

$$M(\cdot,\cdot):\pi_p SO_{q+1}\otimes\pi_q SO_{p+1}\to\Theta^{p+q+1}$$

defined by surgery along a (p,q)-dimensional Hopf link in S^{p+q+1} with a normal framing given by an element of $\pi_p SO_{q+1}\times\pi_q SO_{p+1}$.

Let $\Theta^d(\partial\pi)$ denote the subgroup of Θ^d consisting of elements which are boundaries of parallelizable manifolds (i.e., manifolds with trivial tangent bundle). It is known that both $\Theta^d/\Theta^d(\partial\pi)$ and $\Theta^d(\partial\pi)$ are finite abelian groups (see [KM]) and computations in [KM] shows that for $d=4k-1\leq 15$, $\Theta^{4k-1}(\partial\pi)$ occupies most of Θ^{4k-1}. As a higher dimensional analogue of the Rokhlin invariant, Milnor defined in [Mil2] a homomorphism $\lambda':\Theta^{4k-1}(\partial\pi)\to\mathbb{Z}_{b_k}$ (λ'-invariant) by

$$\lambda'(\partial W^{4k})\equiv\frac{\mathrm{sign}\,W^{4k}}{8}\pmod{b_k}$$

where W^{4k} parallelizable and $b_k\stackrel{\mathrm{def}}{=}2^{2k-2}(2^{2k-1}-1)\,\mathrm{numerator}\left(\frac{4B_k}{k}\right)$, that has been proved to be an isomorphism by Kervaire and Milnor in [KM]. The following theorem has been proved in [ABK] by means of the λ'-invariant.

Theorem 3.4 (Antonelli–Burghelea–Kahn)**.** *For any* $0\leq a\leq q$, $0\leq b\leq p$, *there is a homomorphism*

$$s_{a,b}:\pi_p SO_{q-a+1}\otimes\pi_q SO_{p-b+1}\to\pi_{a+b+2}B\mathrm{Diff}(D^{p+q-a-b-1}\ \textit{rel}\ \partial)$$

that makes the following diagram commutative:

$$\begin{array}{ccc}\pi_p SO_{q-a+1}\otimes\pi_q SO_{p-b+1} & \xrightarrow{s_{a,b}} & \pi_{a+b+2}B\mathrm{Diff}(D^{p+q-a-b-1}\ \textit{rel}\ \partial)\\ \big\downarrow{\scriptstyle \mathrm{incl}_*\otimes\mathrm{incl}_*} & & \big\downarrow{\scriptstyle \mathrm{cl}}\\ \pi_p SO_{q+1}\otimes\pi_q SO_{p+1} & \xrightarrow{\quad M\quad} & \Theta^{p+q+1}\end{array}$$

Moreover, for all $t\geq 13$, $\mathrm{image}\,(\mathrm{cl}\circ s_{a,b})\cap\Theta^{4t-1}(\partial\pi)\cap\mathrm{image}\,M$ $(a+b+2=2t-2)$ *contains an element of non zero order.*

As a corollary to Theorem 3.2, it turns out that $\hat{\zeta}_2$ gives a refinement of some integer multiple of λ' applied to bundles. For each finite abelian group G, we define

$$o(G)\stackrel{\mathrm{def}}{=}\min\{d\in\mathbb{Z}_{>0}\,|\,dx=0\text{ for all }x\in G\}$$

and let π_ℓ^S denote the stable homotopy group $\pi_{n+\ell}S^n$ $(n>\ell+1)$.

Corollary 3.5. *Let $k \geq 3$ and let $\pi : E \to S^{2k-4}$ be a $(D^{2k-1}$ rel $\partial)$-bundle. If $\mathrm{cl}(q_k E) \in \Theta^{4k-5}(\partial\pi)$, then*

$$c_{k-1}\,\lambda'(\mathrm{cl}(q_k E)) \equiv (-1)^k\, b_{k-1}\,\hat{\zeta}_2(E) \qquad (\mathit{mod}\ b_{k-1}) \tag{3.1}$$

where $c_{k-1} = 4(2k-3)!\,\mathrm{denom}\left(\frac{4B_{k-1}}{k-1}\right)$.

If moreover $14 \leq k \leq 31$, then there exists a $(D^{2k-1}$ rel $\partial)$-bundle $\pi : E \to S^{2k-4}$ for which $\mathrm{cl}(q_k E) \in \Theta^{4k-5}(\partial\pi)$ and $c_{k-1}\lambda'(\mathrm{cl}(q_k E)) \not\equiv 0$. Therefore $(-1)^k\, b_{k-1}\,\hat{\zeta}_2(E)$ is an integral lift of a non-trivial invariant $c_{k-1}\lambda'(\mathrm{cl}(q_k E)) \in \mathbb{Z}_{b_{k-1}}$ if $14 \leq k \leq 31$. More generally, if the number

(3.2)

$$\begin{cases} \dfrac{2(4m)!(2^{2m-1}-1)^2\mathrm{num}\left(\frac{4B_m}{m}\right)^2\prod_{\ell=1}^{4m-2} o(\pi_\ell^S)}{(2^{4m-1}-1)B_{2m}} & k = 2m+1 \\[2ex] \dfrac{2(4m-2)!(2^{2m-1}-1)(2^{2m-3}-1)\mathrm{num}\left(\frac{4B_m}{m}\right)\mathrm{num}\left(\frac{4B_{m-1}}{m-1}\right)\prod_{\ell=1}^{4m-4} o(\pi_\ell^S)}{(2^{4m-3}-1)B_{2m-1}} & k = 2m \end{cases}$$

for $m \geq 7$ is not integral, then the same is true for such k. Hence it turns out that the element constructed in [ABK] *has infinite order and*

$$\dim \pi_{2k-4} B\mathrm{Diff}(D^{2k-1}\ \mathit{rel}\ \partial) \otimes \mathbb{Q} \geq 1$$

in those cases.

Note that $12\,\zeta_2(q_k E; \tau_E)|_{[\Theta]=1} \in \mathbb{Z}$ implies $b_{k-1}\,\hat{\zeta}_2(E) \in \mathbb{Z}$. The restriction $14 \leq k$ is equivalent to $t \geq 13$ of Theorem 3.4. The range $k \leq 31$ is such that our PC replies immediately.

Corollary 3.6. *If $m \geq 7$ is such that*

- *$4m-1$ is a prime and*
- *$2^{4m-1}-1$ has a prime factor p with $p \nmid \mathrm{num}(B_m)$,*

then $(-1)^k b_{k-1}\hat{\zeta}_2$ for $k = 2m+1$ is an integral lift of a non-trivial invariant $c_{k-1}\lambda'(\mathrm{cl}(q_k E)) \in \mathbb{Z}_{b_{k-1}}$.

In particular, m for which $2^{4m-1}-1$ itself is a prime satisfies the first condition. For example, the first 6 examples $m = 8, 27, 32, 152, 320, 551$ ($\Leftrightarrow$ $k = 17, 55, 65, 305, 641, 1103$), for which $2^{4m-1}-1$ is prime, all satisfy both conditions of Corollary 3.6. Proofs of Corollary 3.5 and 3.6 will be given after the proof of Theorem 3.2.

3.1. **Obstructions for vertical framings on $(D^{2k-1}$ rel $\partial)$-bundles.** We shall discuss about the obstructions for the existence of vertical framings on $(D^{2k-1}$ rel $\partial)$-bundles over S^{2k-4} and we prove the first part of Theorem 3.2 here so that the Kontsevich classes can be defined. In the rest of this section, we will denote D^{2k-1} simply by $S^\bullet$.

Proposition 3.7. *There exists a positive integer q_k so that any $(S^\bullet$ rel $\partial)$-bundle over S^{2k-4} can be vertically framed after multiplied by q_k.*

Proof. Let $\pi : E \to S^{2k-4}$ be an ($S^\bullet$ rel ∂)-bundle over S^{2k-4} and choose the obvious cell decomposition of S^{2k-1} with one 0-cell e^0 and one $(2k-4)$-cell e^{2k-4}. We choose the standard vertical framing on $\pi^{-1}(e^0)$.

To see that the vertical framing extends over the whole of S^{2k-4}, we consider a trivial ($S^\bullet$ rel ∂)-bundle $\pi_{e^{2k-4}} : E(\pi_{e^{2k-4}}) = E(\phi^*(\pi^{-1}e^{2k-4})) \to D^{2k-4}$ over D^{2k-4} pulled back by the characteristic map $\phi : D^{2k-4} \to S^{2k-4}$. Since $\pi_{e^{2k-4}}$ is trivial, one may choose a vertical framing

$$\tau_1 : T^{\text{fib}}E(\pi_{e^{2k-4}}) \overset{\sim}{\to} \mathbb{R}^{2k-1} \times S^\bullet \times D^{2k-4}$$

such that

- it is standard on $\pi^{-1}_{e^{2k-4}}(q)$ where q is the base point fixed on ∂D^{2k-4},
- it is also standard on the sub $\partial S^\bullet$-bundle.

Since the bundle π is assumed standard on $\pi^{-1}(e^0)$, the pullback bundle $\pi_{e^{2k-4}}$ restricted to ∂D^{2k-4} is also standard. But τ_1 may not be standard there.

Now let $\tau_0 : T^{\text{fib}}E(\pi_{e^{2k-4}})|\partial D^{2k-4} \overset{\sim}{\to} \mathbb{R}^{2k-1} \times S^\bullet \times \partial D^{2k-4}$ be the vertical framing on $\pi^{-1}_{e^{2k-4}}(\partial D^{2k-4})$ that is everywhere standard, and consider the difference of the two vertical framings τ_0 and τ_1:

$$g = \tau_1 \circ \tau_0^{-1} : S^\bullet \times \partial D^{2k-4} \to GL_+(\mathbb{R}^{2k-1})$$

which is trivial on $(S^\bullet \times \{q\}) \cup (\partial S^\bullet \times \partial D^{2k-4})$. Moreover, this map can be transformed into a map $\bar{g} : S^\bullet \times \partial D^{2k-4} \to SO_{2k-1} \subset GL_+(\mathbb{R}^{2k-1})$ by the deformation retraction given by the Gram-Schmidt orthonormalization. So it suffices to prove the vanishing of the obstruction for homotoping $\bar{g}$ into the constant map.

Choose a cell decomposition of $S^\bullet \times \partial D^{2k-4}$ with respect to $(S^\bullet \times \{q\}) \cup (\partial S^\bullet \times \partial D^{2k-4})$ naturally determined by a cell decomposition of $S^\bullet$ with respect to the boundary. By Lemma 3.8 below, we have

$$H^j(S^\bullet \times \partial D^{2k-4}, (S^\bullet \times \{q\}) \cup (\partial S^\bullet \times \partial D^{2k-4}); \pi_j SO_{2k-1}) = 0$$

for $j \le 4k-7$, which implies that the homotopy extends over the $(4k-7)$-skeleton of $S^\bullet \times \partial D^{2k-4}$. By using Lemma 3.8 again, we see that the first obstruction for the homotopy may lie in the group

$$H^{4k-6}(S^\bullet \times \partial D^{2k-4}, (S^\bullet \times \{q\}) \cup (\partial S^\bullet \times \partial D^{2k-4}); \pi_{4k-6} SO_{2k-1}) \cong \pi_{4k-6} SO_{2k-1}.$$

It is known that the group $\pi_{4k-6}SO_{2k-1}$ is finite and hence one can choose q_k that kills this obstruction so that $q_k\pi$ can be vertically framed. □

Lemma 3.8. *Let $\pi : E \to B$ be a ($S^\bullet$ rel ∂)-bundle over a closed $(2k-6)$-connected oriented manifold B of dimension $\le 2k-4$. Then*

$$H^i(E, \partial E \cup E_q; \mathbb{Z}) \cong \begin{cases} 0 & \text{if } 0 \le i \le 4k-7 \\ H^{4k-5}(B;\mathbb{Z}) & \text{if } i = 4k-6 \end{cases}$$

where $E_q = \pi^{-1}(q)$. □

3.2. **Framing dependence of ζ_2.** To prove Theorem 3.2 one needs to compare the framing dependences of both ζ_2 and Δ_{k-1} and to see that they differ by some non zero constant. In this subsection we shall study the difference of ζ_2 for two different vertical framings.

Lemma 3.9. *Let $(\pi : E \to B; \tau_E)$ be a vertically framed null bordant $(S^\bullet$ rel $\partial)$-bundle over a closed $(2k-4)$-manifold B not necessarily connected where we say that a $(S^\bullet$ rel $\partial)$-bundle is framed null bordant if it represents the null element of the bordism group $\Omega_{2k-4}(\widetilde{B\mathrm{Diff}}\, S^{2k-1})$. Then $\zeta_2(E;\tau_E) = 0$.*

Proof. Since ζ_2 is a cocycle on $\widetilde{B\mathrm{Diff}}\, S^{2k-1}$, it is a framed bordism invariant. Thus the result follows. □

Lemma 3.10. *$\zeta_2(E;\tau_E)$ depends only on the homotopy class of τ_E.*

Proof. Let τ_E and σ_E be two mutually homotopic vertical framings. We prove that $\zeta_2(E;\tau_E) = \zeta_2(E;\sigma_E)$.

The homotopy gives rise to a cylinder $E \times I$ with a vertical framing $\tilde\tau_E(t)$ $(t \in I)$ such that $\tilde\tau_E(0) = \tau_E$ and $\tilde\tau_E(1) = \tau'_E$. This framed cylinder $E \times I$ is a vertically framed bordism between the two vertically framed bundles $(E;\tau_E)$ and $(E;\sigma_E)$. Hence Lemma 3.9 concludes the proof. □

Lemma 3.11. *Let $\pi : E \to S^{2k-4}$ denote a $(S^\bullet$ rel $\partial)$-bundle over S^{2k-4}. Then there is a homotopy deforming any continuous map $g : E \to SO_{2k-1}$ that is trivial on $\partial E \cup E_q$ into a map that is trivial outside a $(4k-5)$-ball embedded in E.*

Proof. Lemma 3.8 implies that the homotopy extends from $\partial E \cup E_q$ over the $(4k-6)$-skeleton of E since $H^{2k-5}(S^{2k-4};\mathbb{Z}) = 0$. □

For any map $G : (E, \partial E \cup E_q) \to (SO_{2k-1}, \mathrm{id})$, let $\psi(G) : \mathbb{R}^{2k-1} \times E \to \mathbb{R}^{2k-1} \times E$ be the continuous map defined by $\psi(G)(v,x) \stackrel{\mathrm{def}}{=} (G(x)v, x)$.

Lemma 3.12. *Let $\pi : E \to S^{2k-4}$ be a $(S^\bullet$ rel $\partial)$-bundle over S^{2k-4} and let τ_E be a vertical framing. Then $\zeta_2(E;\psi(G)\circ\tau_E) - \zeta_2(E;\tau_E)$ does not depend on τ_E. Namely, it depends only on the homotopy class of $\psi(G)$.*

Proof. Let $\tilde\pi : \widetilde{E} = E \times I \to \widetilde{S^{2k-4}} = S^{2k-4} \times I$ be the $(S^\bullet$ rel $\partial)$-bundle over the cylinder pulled back from π by the projection onto the first factor $S^{2k-4} \times I \to S^{2k-4}$ where we identify $E \times \{0\}$ with E. Suppose that $\widetilde{E}$ is partially vertically framed on $E \times \{1\}$ and $E \times \{0\}$ by the framings $\psi(G)\circ\tau_E$ and τ_E respectively.

By Lemma 3.11, we may assume after a homotopy that $\psi(G)\circ\tau_E$ and τ_E coincide outside $\pi^{-1}(B^{2k-4})$ where $B^{2k-4} \subset S^{2k-4}$ is an embedded $(2k-4)$-disk. In other words, the vertical framing over $\partial\widetilde{S^{2k-4}} = S^{2k-4}\times\{1\} \sqcup -S^{2k-4}\times\{0\}$ extends to $\widetilde{S^{2k-4}}$

minus an embedded $(2k-3)$-ball $B^{2k-3} \subset \widetilde{S^{2k-4}}$. Further we may consider $\widetilde{E}^\circ \stackrel{\text{def}}{=} \tilde{\pi}^{-1}(\widetilde{S^{2k-4}} \setminus \mathrm{Int}(B^{2k-3}))$ as a $(S^\bullet \text{ rel } \partial)$-bundle bordism between $E \sqcup (S^\bullet \times S^{2k-4})$ and $-E$ with some vertical framing $\tau_{\widetilde{E}^\circ}$ extending $(\psi(G) \circ \tau_E) \sqcup \tau_E$, where $S^\bullet \times S^{2k-4}$ is a trivial bundle. We denote by τ_G the induced vertical framing on the trivial $(S^\bullet \text{ rel } \partial)$-bundle $S^\bullet \times S^{2k-4}$ over $\partial B^{2k-3} = S^{2k-4}$. Note that τ_G is homotopically canonical.

By Lemma 3.9, we have $\zeta_2(E; \psi(G) \circ \tau_E) + \zeta_2(S^\bullet \times S^{2k-4}; \tau_G) - \zeta_2(E; \tau_E) = 0$. Namely, by Lemma 3.10, $\zeta_2(E; \psi(G) \circ \tau_E) - \zeta_2(E; \tau_E) = -\zeta_2(S^\bullet \times S^{2k-4}; \tau_G)$ does not depend on τ_E. □

The last proposition allows us to define

$$\zeta_2'(E; G) \stackrel{\text{def}}{=} \zeta_2(E; \psi(G) \circ \tau_E) - \zeta_2(E; \tau_E).$$

For a $(S^\bullet \text{ rel } \partial)$-bundle $\pi : E \to B$, we denote by $[E, SO_{2k-1}]^\bullet$ the set of homotopy classes of continuous maps

$$G : (E, \partial E \cup E_q) \to (SO_{2k-1}, \mathrm{id}).$$

It is known that $\pi_{4k-5} SO_{2k-1}/\text{torsion} \cong \mathbb{Z}$. We fix a map $\rho : (S^{4k-5}, *) \to (SO_{2k-1}, \mathrm{id})$ representing an infinite order generator of $\pi_{4k-5} SO_{2k-1}$ such that its image under the natural map $\pi_{4k-5} SO_{2k-1} \to \pi_{4k-5} U_{4k-4}/U_{2k-3} \cong \mathbb{Z}$ represents positive multiple of the usual choice of generator to define the relative Pontrjagin class as in [MS]. Then let $G_E(\rho) : (E, \partial E \cup E_q) \to (SO_{2k-1}, \mathrm{id})$ be a map that coincides with id outside an embedded $(4k-5)$-ball B^{4k-5} in $\mathrm{Int}(E)$ and that the image of B^{4k-5} under $G_E(\rho)$ is homotopic to ρ.

The following proposition is a key to prove Theorem 3.2, describing the structure of the set of homotopy classes of vertical framings.

Proposition 3.13. *Let $\pi : E \to S^{2k-4}$ be a vertically framed $(S^\bullet$ rel $\partial)$-bundle over S^{2k-4}. Then $[E, SO_{2k-1}]^\bullet/\text{torsion} = \langle [G_E(\rho)] \rangle$, the free abelian group generated by $G_E(\rho)$. Thus the degree $[E, SO_{2k-1}]^\bullet \to \mathbb{Z}$ is defined with respect to $[G_E(\rho)]$.*

Proof. By Lemma 3.11, the obstruction to homotoping G into the constant map over whole of E is described by a homotopy class of a map $\partial(B^{4k-5} \times I) \cong S^{4k-5} \to SO_{2k-1}$, which can be considered as an element of $\pi_{4k-5} SO_{2k-1}$. □

Lemma 3.14. *Let $G \in [E, SO_{2k-1}]^\bullet$. Then we have*

$$\zeta_2'(E; G) = \zeta_2'(E; G_E(\rho)) \deg G.$$

Proof. By Lemma 3.12, we have

$$\begin{aligned} \zeta_2'(g) + \zeta_2'(h) &= (\zeta_2(E; \psi(g) \circ \psi(h) \circ \tau_E) - \zeta_2(E; \psi(h) \circ \tau_E)) \\ &\quad + (\zeta_2(E; \psi(h) \circ \tau_E) - \zeta_2(E; \tau_E)) = \zeta_2'(E; gh). \end{aligned}$$

Therefore $\zeta_2' : [E, SO_{2k-1}]^\bullet \to \mathcal{A}_2$ is a group homomorphism into a torsion free abelian group. Then Proposition 3.13 implies that it must be $\deg G$ times the image of the infinite order generator $G_E(\rho)$. □

3.3. **Framing dependence of relative Pontrjagin numbers.** As for ζ_2, we compute the difference between Δ_{k-1}'s for two different vertical framings. We only need to see the framing dependence of the $(k-1)$-st relative Pontrjagin number, the only indecomposable term in L_{k-1}, because

$$H^{4p}(E \times I, \partial(E \times I); \mathbb{Z}) \otimes H^{4(k-1-p)}(E \times I, \partial(E \times I); \mathbb{Z}) = 0$$

unless $p = 0$ or $k-1$.

Lemma 3.15. *Let $\pi : E \to S^{2k-4}$ be a $(S^\bullet$ rel $\partial)$-bundle over S^{2k-4} that can be vertically framed by τ_E. Then $p_{k-1}(E; \psi(G) \circ \tau_E) - p_{k-1}(E; \tau_E)$ does not depend on τ_E. It depends only on the homotopy class of $\psi(G)$.*

Proof. The difference computes the $(k-1)$-st relative Pontrjagin number of $E \times I$ with respect to the vertical framings $\psi(G) \circ \tau_E$ and τ_E on $E \times \{0, 1\}$ together with the standard vertical framing on $\partial E \times I$. Then the proof is similar as Lemma 3.12 by the fact that the $(k-1)$-st relative Pontrjagin number vanishes on vertically framed cobordisms. □

Lemma 3.15 allows us to define

$$p'_{k-1}(E; G) \stackrel{\text{def}}{=} p_{k-1}(E; \psi(G) \circ \tau_E) - p_{k-1}(E; \tau_E).$$

Lemma 3.16. *Let $\pi : E \to S^{2k-4}$ be a vertically framed $(S^\bullet$ rel $\partial)$-bundle over S^{2k-4}. Then*

$$p'_{k-1}(E; G) = -2^{\beta_k} a_{k-1} (2k-3)! \deg G \tag{3.3}$$

where $a_n = 1$ if $n \equiv 0$ (mod 2) and $a_n = 2$ if $n \equiv 1$ (mod 2), and

$$\beta_k = \begin{cases} 3 & \text{if } k = 3 \\ 1 & \text{if } k = 5 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since $p'_{k-1}(E; G) : [E, SO_{2k-1}]^\bullet \to \mathbb{Z}$ is a group homomorphism, it follows from Proposition 3.13 that

$$p'_{k-1}(E; G) = p'_{k-1}(E; G_E(\rho)) \deg G.$$

So it suffices to prove that $p'_{k-1}(E; G_E(\rho)) = -2^{\beta_k} a_{k-1}(2k-3)!$.

Recall that the $(k-1)$-st relative Pontrjagin class p'_{k-1} is considered as the obstruction to extend the vertical framing on $\partial(E \times I)$ to the complexified vertical tangent bundle of $E \times I$. (Note that now the orientation induced on $E \times \{1\}$ is opposite to the one induced from that of W appeared in the definition of Δ_{k-1}. Hence the minus sign appears in the above equation.) This obstruction lies in $H^{4k-5}(E, \partial E; \pi_{4k-5} U_{4k-4}/U_{2k-3}) = H^{4k-5}(E, \partial E; \mathbb{Z})$. In the case $G = G_E(\rho)$, the obstruction corresponds to the image of $[\rho] \in \pi_{4k-5} SO_{2k-1}$ under the map $\pi_{4k-5} SO_{2k-1} \to \pi_{4k-5} U_{4k-4}/U_{2k-3}$. This map factors through $\pi_{4k-5} U_{2k-1} \cong \mathbb{Z}$ and the following two lemmas conclude the proof. □

Lemma 3.17. *The inclusion $i : U_{2k-1} \to U_{4k-4}/U_{2k-3}$ sends the generator of $\pi_{4k-5} U_{2k-1} \cong \mathbb{Z}$ to $\pm(2k-3)!$ times the generator of $\pi_{4k-5} U_{4k-4}/U_{2k-3} \cong \mathbb{Z}$.*

Proof. This is a direct consequence of the following homotopy sequence of the bundle:

$$\begin{array}{ccccccc} \pi_{4k-5}U_{2k-1} & \stackrel{i_*}{\to} & \pi_{4k-5}\dfrac{U_{4k-4}}{U_{2k-3}} & \to & \pi_{4k-5}\dfrac{U_{4k-4}}{U_{2k-1}\times U_{2k-3}} & \to & 0 \\ \wr\| & & \wr\| & & \wr\| & & \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_{(2k-3)!} & & \end{array}$$

The last isomorphism follows from Bott–Milnor [BM]. □

The following lemma is a special case of a result in [Lun].

Lemma 3.18 (Lundell)**.** *The natural inclusion* $c : SO_{2k-1} \to U_{2k-1}$ *sends the generator* $[\rho] \in \pi_{4k-5}SO_{2k-1}$ *to* $\pm 2^{\beta_k} a_{k-1}$ *times the generator of* $\pi_{4k-5}U_{2k-1} \cong \mathbb{Z}$*.*

3.4. **Computation of** $\zeta_2'(E; G_E(\rho))$ **and framing correction.** Let $p : E_\rho \to S^{4k-4}$ be the real $(2k-1)$-dimensional vector bundle over $S^{4k-4} = B^{4k-4} \cup_{\partial = S^{4k-5}} (-B^{4k-4})$ defined by

$$E_\rho \stackrel{\text{def}}{=} (\mathbb{R}^{2k-1} \times B^{4k-4}) \cup_h (\mathbb{R}^{2k-1} \times -B^{4k-4})$$

where the gluing diffeomorphism $h : \mathbb{R}^{2k-1} \times \partial B^{4k-4} = \mathbb{R}^{2k-1} \times S^{4k-5} \to \mathbb{R}^{2k-1} \times S^{4k-5}$ is given by the twist $(v, x) \mapsto (\rho(x)^{-1}v, x)$.

For an $\mathbb{R}^{2k-1}$ vector bundle $p : E \to B$, we denote by $S_2(p) : S_2(E) \to B$ the S^{2k-2}-bundle associated to E. Let α_T be the Thom class of the S^{2k-2}-bundle $S_2(E_\rho)$ whose restriction to a fiber is the generator of $H^{2k-2}(S^{2k-2}; \mathbb{Z})$. Let

$$\delta_2(E_\rho) \stackrel{\text{def}}{=} \alpha_T^3[S_2(E_\rho)] \cdot [\Theta]/12.$$

To compute this number, we use the following lemma proved in [BC].

Lemma 3.19 (Bott–Cattaneo)**.** *Let* $\pi : E \to B$ *be an* $\mathbb{R}^{2k-1}$*-vector bundle over a closed manifold* B *and let* $S(E)$ *be its associated sphere bundle with the canonical Euler class* $e \in H^{2k-2}(S(E); \mathbb{Z})$ *of fiber tangent bundle. Then*

$$\pi_! e^3 = (-1)^{k-1} 2 p_{k-1}(E),$$

twice of the $(k-1)$*-st Pontrjagin class.*

Since the Euler number of S^{2k-2} is 2, e restricts to twice the generator of $H^{2k-2}(S^{2k-2}; \mathbb{Z})$. Hence we have the following

Corollary 3.20. $\delta_2(E_\rho) = (-1)^{k-1} p_{k-1}(E_\rho)[S^{4k-4}] \cdot [\Theta]/48.$

Lemma 3.21. $\zeta_2'(E; G_E(\rho)) = \delta_2(E_\rho).$

Proof. Throughout this proof we work with de Rham complexes although the resulting value is in $\frac{[\Theta]}{12}\mathbb{Z}$. We shall prove that the RHS can be reduced to the LHS. Consider the decomposition

$$E_\rho \to S^{4k-4} = B^{4k-4} \cup_\partial (S^{4k-5} \times I) \cup_\partial (-B^{4k-4})$$

where $S^{4k-5} \times I$ is the mapping cylinder corresponding to h. Since B^{4k-4} and $-B^{4k-4}$ are obviously parallelizable and also the trivialization extends to the partial gluing $B' \stackrel{\text{def}}{=} B^{4k-4} \cup_{\partial|D^{4k-5}} (D^{4k-5} \times I) \cup_{\partial|D^{4k-5}} (-B^{4k-4})$ where $D^{4k-5} \subset S^{4k-5} = \partial B^{4k-4}$ is an embedded disk near the base point of S^{4k-5}, α_T can be chosen so that it is an extension of the fiber volume form on B' determined via the trivialization. Hence over $S_2(p)^{-1}(B')$, the integral of α_T^3 vanishes because the corresponding triple product on S^{2k-2} vanishes, and one has

$$\delta_2(E_\rho) = \frac{[\Theta]}{12} \int_{S_2(E_\rho|D_1^{4k-5} \times I)} \alpha_T^3$$

where $D_1^{4k-5} \stackrel{\text{def}}{=} S^{4k-5} \setminus \mathrm{Int} D^{4k-5}$.

On the other hand, recall that the two different vertical framings $\psi(G_E(\rho)) \circ \tau_E$ and τ_E may be assumed coincide outside some embedded disk $B^{4k-5} \subset \mathrm{Int}(E)$ after a homotopy. By Stokes' theorem and by (2.2), we have

$$\begin{aligned} \zeta_2'(E; G_E(\rho)) &= \zeta_2(E; \psi(G_E(\rho)) \circ \tau_E) - \zeta_2(E; \tau_E) \\ &= \frac{[\Theta]}{12} \int_{S_2(\mathbb{R}^{2k-1} \times I \times B^{4k-5})} \alpha_T(\mathbb{R}^{2k-1} \times I \times B^{4k-1})^3 \end{aligned}$$

where $\mathbb{R}^{2k-1} \times I \times B^{4k-5}$ denotes a trivial $\mathbb{R}^{2k-1}$-vector bundle over $I \times B^{4k-5}$ and where $\alpha_T(\mathbb{R}^{2k-1} \times I \times B^{4k-5})$ denotes a $(2k-2)$-form on $S_2(\mathbb{R}^{2k-1} \times I \times B^{4k-5})$ representing the Thom class of the associated trivial S^{2k-2}-bundle $S^{2k-2} \times I \times B^{4k-5}$ extending $(\psi(G_E(\rho)) \circ \tau_E)^* \omega_{S^{2k-2}}$ and $(\tau_E)^* \omega_{S^{2k-2}}$ on $S^{2k-2} \times \partial I \times B^{4k-5}$. Existence of such a $(2k-2)$-form is because the restriction induces an isomorphism from $H^{2k-2}(S^{2k-2} \times I \times B^{4k-5}; \mathbb{R})$ to $H^{2k-2}(\partial(S^{2k-2} \times I \times B^{4k-5}); \mathbb{R})$.

Since we can choose an S^{2k-2}-bundle isomorphism between $S_2(E_\rho|D_1^{4k-5} \times I)$ and $S_2(\mathbb{R}^{2k-1} \times I \times B^{4k-5})$ sending the trivialization of $S_2(E_\rho|\partial(D_1^{4k-5} \times I))$ to that of $S_2(\mathbb{R}^{2k-1} \times \partial(I \times B^{4k-5}))$, the result follows. □

Lemma 3.22. $\zeta_2'(E; G_E(\rho)) = (-1)^{k-1} 2^{\beta_k} a_{k-1} (2k-3)! \cdot [\Theta]/48$.

Proof. We use the notations appeared in the proof of Lemma 3.12. Let $S_2(\widetilde{E})$ denote the associated S^{2k-2}-bundle to the vertical tangent bundle $T^{\text{fib}}\widetilde{E}$ extending the trivial vertical bundles on $E \times (\{1\} \sqcup \{0\})$ with framings $\psi(G_E(\rho)) \circ \tau_E$ and τ_E respectively.

By a similar argument as in the proof of Lemma 3.21, one can see that

$$p_{k-1}(E_\rho)[S^{4k-4}] = -p_{k-1}'(E; G_E(\rho)).$$

Then by Lemma 3.21, Corollary 3.20 and Lemma 3.16,

$$\begin{aligned} \zeta_2'(E; G_E(\rho)) &= \delta_2(E_\rho) = (-1)^{k-1} p_{k-1}(E_\rho)[S^{4k-4}] \cdot [\Theta]/48 \\ &= (-1)^{k-1} 2^{\beta_k} a_{k-1} (2k-3)! \cdot [\Theta]/48. \end{aligned}$$

□

Proof of Theorem 3.2. By Lemma 3.16 and by

$$L_{k-1}(p_1,\ldots,p_{k-1}) = \frac{2^{2k-2}(2^{2k-3}-1)B_{k-1}}{(2k-2)!}p_{k-1} + (\text{terms of } p_{k-2},\ldots,p_1),$$

one has

$$\begin{aligned}(3.4)\qquad &\frac{(-1)^{k-1}(2k-2)!}{2^{2k}(2^{2k-3}-1)B_{k-1}}\big(\Delta_{k-1}(q_kE;\psi(G)\circ\tau_{E_D}) - \Delta_{k-1}(q_kE;\tau_{E_D})\big)\\ &= (-1)^k 2^{\beta_k} a_{k-1}(2k-3)!\deg G/4\end{aligned}$$

On the other hand, we know from Lemma 3.14 and Lemma 3.22 that

$$(-1)^k 2^{\beta_k} a_{k-1}(2k-3)!\deg G/4 = -12\,\zeta_2'(q_kE;G)|_{[\Theta]=1}$$

This completes the proof. □

3.5. **Integral lift of Milnor's λ'-invariant.** In this subsection, we give a proof of Corollary 3.5 and 3.6. First we need some computations to estimate q_k. The following elementary fact is fundamental in the computation below.

Lemma 3.23. *Suppose we have an exact sequence of finite abelian groups:*

$$0 \to H \to G \to K \to 0.$$

Then $o(G)|o(H)o(K)$.

From the proof of Proposition 3.7, we need to estimate the order of $\pi_{4k-6}SO_{2k-1}$. We shall now estimate more generally o of π_pSO_j for p even and for $\frac{p+3}{2} \leq j \leq p+1$, for the next section. It is known that π_pSO_j for p even is finite for all j. Let $o_p^j \overset{\text{def}}{=} o(\pi_pSO_j)$.

Lemma 3.24. *Let p be an even integer. Then*

$$o_p^{p+1}|4 \quad \textit{if } p\equiv 0 \textit{ (mod 8)}, \qquad o_p^{p+1}|2 \quad \textit{if } p\equiv 2,4,6 \textit{ (mod 8)}$$

Proof. First the following exact sequence of a principal SO_{p+1}-bundle implies that π_pSO_{p+1} is isomorphic to $\pi_{p+1}SO_{n+p+1}/(SO_n\times SO_{p+1})$ for sufficiently large $n \gg p+1$:

$$0 = \pi_{p+1}\frac{SO_{n+p+1}}{SO_n} \to \pi_{p+1}\frac{SO_{n+p+1}}{SO_n\times SO_{p+1}} \overset{\sim}{\to} \pi_pSO_{p+1} \to \pi_p\frac{SO_{n+p+1}}{SO_n} = 0$$

So it suffices to determine the o of the latter group. Consider another exact sequence:

$$\mathbb{Z}_2 = \pi_{p+1}\frac{SO_{n+p+1}}{SO_{p+1}} \to \pi_{p+1}\frac{SO_{n+p+1}}{SO_n\times SO_{p+1}} \to \pi_pSO_n = \begin{cases}\mathbb{Z}_2 & \text{if } p\equiv 0\ (8)\\ 0 & \text{if } p\equiv 2,4,6\ (8)\end{cases}$$

Then by Lemma 3.23 we have the desired result. □

Lemma 3.25. *Let p be an integer of the form $4p'+1$. Then*

$$o_p^p|4 \quad \textit{if } p\equiv 1 \textit{ (mod 8)}, \qquad o_p^p|2 \quad \textit{if } p\equiv 5 \textit{ (mod 8)}$$

Proof. Proof is similar to the proof of Lemma 3.24. This time we use the exact sequences of the following principal bundles for sufficiently large n:

$$SO_p \to SO_{n+p}/SO_n \to SO_{n+p}/(SO_n \times SO_p)$$
$$SO_n \to SO_{n+p}/SO_p \to SO_{n+p}/(SO_n \times SO_p)$$

□

Lemma 3.26. *Let p be an integer not of the form $4p'-1$. Then for $\frac{p+3}{2} \le j \le p+1$,*

$$o_p^j | 4\prod_{\ell=1}^{p-j+1} o(\pi_\ell^S) \quad \text{if } p \equiv 0 \text{ (mod 8)}, \qquad o_p^j | 2\prod_{\ell=1}^{p-j+1} o(\pi_\ell^S) \quad \text{if } p \equiv 2,4,6 \text{ (mod 8)}$$
$$o_p^j | 4\prod_{\ell=2}^{p-j+1} o(\pi_\ell^S) \quad \text{if } p \equiv 1 \text{ (mod 8)}, \qquad o_p^j | 2\prod_{\ell=2}^{p-j+1} o(\pi_\ell^S) \quad \text{if } p \equiv 5 \text{ (mod 8)}$$

Proof. For $\frac{p+3}{2} \le j \le p+1$, the exact sequence of the bundle $SO_j \to SO_{j+1} \to S^j$ partially looks as follows:

$$\pi_{p-j+1}^S = \pi_{p+1}S^j \to \pi_p SO_j \to \pi_p SO_{j+1}$$

Note that the stability condition of the leftmost term is given by $p-j+3 \le j \Leftrightarrow \frac{p+3}{2} \le j$. Applying Lemma 3.23, we have $o_p^j | o_p^{j+1} o(\pi_{p-j+1}^S)$. Then starting from the result of Lemma 3.24 or 3.25 and proceeding inductively in this manner, we obtain the desired result. □

By Lemma 3.26, we obtain the following

Lemma 3.27. $q_k | 2\prod_{\ell=1}^{2k-4} o(\pi_\ell^S)$.

To estimate $o(\pi_\ell^S)$, we will need the following theorem due to H. Toda.

Theorem 3.28 (H. Toda [To]). *Let p be an odd prime. The p-primary component (the subgroup of all elements of order power of p) of π_k^S is isomorphic to*

$$\begin{cases} \mathbb{Z}_p & \text{for } k = 2i(p-1)-1,\ i = 1,2,\ldots,p-1 \\ 0 & \text{otherwise for } k < 2p(p-1)-2 \end{cases}$$

Proof of Corollary 3.5. First we prove (3.1). Namely, we prove that the RHS reduced modulo b_{k-1} is equal to the LHS. Since $X = \mathrm{cl}(q_k E)$ bounds a parallelizable manifold W, the relative L_{k-1}-number in Δ_{k-1} vanishes for the induced framing $\tau_{\partial W}$ on $TX \oplus \varepsilon$ from that of TW. The difference of the Δ_{k-1} terms for $\tau_{\partial W}$ and for τ'_{E_D} may be given by the relative $(k-1)$-st Pontrjagin class on the cylinder $X \times I$, that is integral. Indeed, by stabilizing $T(X \times I)$ if necessary, the difference equals

(3.5)
$$\begin{aligned}
&\frac{-(-1)^{k-1}(2k-2)!}{2^{2k}(2^{2k-3}-1)B_{k-1}} \frac{2^{2k-2}(2^{2k-3}-1)B_{k-1}}{(2k-2)!} p_{k-1}(T(X\times I);\tau_{\partial W} * \tau'_{E_D})[X\times I, \partial] \\
&\qquad = -\frac{(-1)^{k-1}}{4} p_{k-1}(T(X\times I);\tau_{\partial W} * \tau'_{E_D})[X\times I, \partial] \\
&\qquad = \pm\frac{a_{k-1}(2k-3)!}{4} \mathfrak{o}(T(X\times I);\tau_{\partial W} * \tau'_{E_D})[X\times I, \partial] \in \mathbb{Z}
\end{aligned}$$

where $\tau_{\partial W} * \tau'_{E_D}$ denotes the framing on $(X\times\{0,1\})\cup(\{*\}\times I)$ naturally extended from $\tau_{\partial W}\sqcup\tau'_{E_D}$ and where $\mathfrak{o}(T(X\times I);\tau_{\partial W}*\tau'_{E_D})$ is the obstruction class to extending the stable framing $\tau_{\partial W}*\tau'_{E_D}$ over $X\times I$ with values in $\pi_{4k-5}SO=\mathbb{Z}$. The second equality follows from a similar argument as in Lemma 3.16, or from [MK, Lemma 2]. Hence it can be ignored when considered modulo b_{k-1} after multiplied by $(-1)^k b_{k-1}$. Also the term of ζ_2 can be ignored because it is integral. Then we may only need to consider the term

$$\begin{aligned}-(-1)^k 2^{2k-4}(2^{2k-3}-1)\,\mathrm{num}\Big(\frac{4B_{k-1}}{k-1}\Big)\times\frac{(-1)^{k-1}(2k-2)!}{2^{2k}(2^{2k-3}-1)B_{k-1}}\,\mathrm{sign}\,W\\ =4(2k-3)!\,\mathrm{denom}\Big(\frac{4B_{k-1}}{k-1}\Big)\frac{\mathrm{sign}\,W}{8}\end{aligned}$$

that is congruent to the LHS and hence (3.1) is proved.

In [ABK, Theorem 1.5.1], the λ'-invariant of the non-trivial elements in $\mathrm{image}\,(\mathrm{cl}\circ s_{a,b})\cap\Theta^{4k-5}(\partial\pi)\cap\mathrm{image}\,M$ (see Theorem 3.4) are explicitly computed by means of [Mil, Lemma 3]. Namely, one has a $(D^{2k-1}$ rel $\partial)$-bundle $\pi:E_k\to S^{2k-4}$ such that

$$\lambda'(\mathrm{cl}(E_k))\equiv\begin{cases}\frac{I_m^2}{8} & (\mathrm{mod}\,\frac{I_{2m}}{8}) \quad k=2m+1\\ \frac{I_mI_{m-1}}{8} & (\mathrm{mod}\,\frac{I_{2m-1}}{8}) \quad k=2m\end{cases}$$

where $I_t=8b_t$.

Let $q'_k\stackrel{\mathrm{def}}{=}2\prod_{\ell=1}^{2k-4}o(\pi_\ell^S)$. Then by Lemma 3.27, if $c_{k-1}\lambda'(\mathrm{cl}(q_kE_k))$ were trivial, then $c_{k-1}\lambda'(\mathrm{cl}(q'_kE_k))$ must be trivial too. But if we could prove that the latter value is not integral, we would have a contradiction. We have

$$\begin{aligned}&c_{2m}\lambda'(\mathrm{cl}(q'_{2m+1}E_k))\\ &\equiv 4(4m)!\,\mathrm{denom}\Big(\tfrac{4B_{2m}}{2m}\Big)\cdot 2\prod_{\ell=1}^{4m-2}o(\pi_\ell^S)\cdot\tfrac{I_m^2}{8}\ (\mathrm{mod}\ \tfrac{I_{2m}}{8})\\ &c_{2m-1}\lambda'(\mathrm{cl}(q'_{2m}E_k))\\ &\equiv 4(4m-2)!\,\mathrm{denom}\Big(\tfrac{4B_{2m-1}}{2m-1}\Big)\cdot 2\prod_{\ell=1}^{4m-4}o(\pi_\ell^S)\cdot\tfrac{I_mI_{m-1}}{8}\ (\mathrm{mod}\ \tfrac{I_{2m-1}}{8})\end{aligned}$$

This is non-zero if

$$\begin{cases}8(4m)!\,\mathrm{denom}\Big(\tfrac{4B_{2m}}{2m}\Big)\cdot\prod_{\ell=1}^{4m-2}o(\pi_\ell^S)\cdot\tfrac{I_m^2}{I_{2m}}\notin\mathbb{Z} & k=2m+1\\ 8(4m-2)!\,\mathrm{denom}\Big(\tfrac{4B_{2m-1}}{2m-1}\Big)\cdot\prod_{\ell=1}^{4m-4}o(\pi_\ell^S)\cdot\tfrac{I_mI_{m-1}}{I_{2m-1}}\notin\mathbb{Z} & k=2m\end{cases}\tag{3.6}$$

These numbers are precisely the numbers of the statement.

The claim for $14\le k\le 31$ is by direct computation. For example, for $k=14$,

$$\mathrm{denom}\Big\{8(26)!\,\mathrm{denom}\Big(\frac{4B_{13}}{13}\Big)\times\frac{I_7I_6}{I_{13}}\Big\}=31\cdot 601\cdot 1801\cdot 657931$$

Theorem 3.28 for $p=31$ implies that the 31-primary component of π_ℓ^S for $\ell<1858$ is zero unless $\ell=59,119,179,\ldots,1799$. Hence $o(\pi_1^S)\cdots o(\pi_{24}^S)$ does not have a prime divisor 31 and the number (3.6) must not be integral. For $15\le k\le 31$, we can compute similarly. □

Proof of Corollary 3.6. Suppose that m satisfies the required properties and let p be a prime factor of $2^{4m-1}-1$ with $p\nmid \mathrm{num}(B_m)$. We prove that

$$p\nmid 2(4m)!(2^{2m-1}-1)^2\mathrm{num}\left(\frac{4B_m}{m}\right)^2\mathrm{denom}\,(B_{2m})\,\prod_{\ell=1}^{4m-2}o(\pi_\ell^S).$$

First by hypothesis we have $p\nmid \mathrm{num}\big(\frac{4B_m}{m}\big)^2$. Since p is prime and since $p=1+a(4m-1)>4m$ by Fact 3.29(1) below, $p\nmid 2(4m)!$. Further, by the von Staudt–Clausen theorem (e.g., [MS]), which implies that $\mathrm{denom}\,(B_{2m})=\prod_{\substack{q-1|4m\\ q:\mathrm{prime}}}q$, any prime factor of $\mathrm{denom}\,(B_{2m})$ is less than $4m$. Hence $p\nmid \mathrm{denom}\,(B_{2m})$.

By Fact 3.29(2), one has $\gcd(2^{4m-1}-1,2^{2m-1}-1)=1$ from $\gcd(4m-1,2m-1)=1$. Hence $p\nmid(2^{2m-1}-1)^2$.

So it is enough to prove that $\prod_{\ell=1}^{4m-2}o(\pi_\ell^S)$ does not have a prime divisor p. This follows from Theorem 3.28 since the first ℓ for which π_ℓ^S has non zero p-primary component is $\ell=2(p-1)-1=2p-3$ and all ℓ involved in the product is at most $4m-2(<p-2<2p-3)$. Hence the proof is completed. □

The following facts may be well known.

Fact 3.29. (1) *If q is an odd prime and an odd prime p divides 2^q-1, then $p=1+aq$ for some positive integer a.*

(2) $\gcd(p,q)=1\Rightarrow\gcd(2^p-1,2^q-1)=1.$

Proof. (1) is a consequence of a special case of Fermat's theorem: $2^{p-1}\equiv 1$ (mod p) for any odd prime p. Namely, if p,q are odd primes and $p\,|\,2^q-1$, then $2^q\equiv 1\,(p)$. Since q is prime, $q\,|\,p-1$.

(2) If $\gcd(p,q)=1$ and $\gcd(2^p-1,2^q-1)$ has an odd prime factor r, then $2^p-1\equiv 2^q-1\equiv 0\,(r)\Leftrightarrow 2^p\equiv 2^q\equiv 1\,(r)$. If $s>1$ is the minimum positive integer for which $2^s\equiv 1\,(r)$, then s divides both p and q. This is a contradiction. □

Remark 3.30. In [ABK], non-integrality of $\lambda'(\mathrm{cl}(E_k))/(I_{k-1}/8)$ has been proved by numerical estimations of the value. But similar argument does not work for non-integrality of $c_{k-1}q_k'\lambda'(\mathrm{cl}(E_k))/(I_{k-1}/8)$ since the value may be too large.

4. GRAPH CLASPER-BUNDLES

For a $(4m-1)$-dimensional homology sphere M, we shall construct many smooth framed $(M^\bullet$ rel $\partial)$-bundles associated to trivalent graphs, what we will call graph clasper-bundles. We will show that they are in some sense dual to the Kontsevich classes, which implies the non-triviality of the classes.

More precisely, we shall construct a linear map

$$\psi_{2n}:\mathcal{G}_{2n}\to\Omega_{4n(m-1)}(\widetilde{B\mathrm{Diff}}\,M)\otimes\mathbb{Q}$$

by using families of higher-dimensional claspers surgery, and will prove the following

Theorem 4.1. *Let $m \geq 2$ and let M be a $(4m-1)$-dimensional homology sphere, then*

(1) *There exists a non-zero integer r_m that makes the diagram*

$$\begin{array}{ccc} \mathcal{G}_{2n} & \xrightarrow{\psi_{2n}} & \Omega_{4n(m-1)}(\widetilde{B\mathrm{Diff}}\,M)\otimes\mathbb{Q} \\ {\scriptstyle \mathrm{proj.}\otimes 1}\downarrow & & \downarrow{\scriptstyle \langle\zeta_{2n},\cdot\rangle} \\ \mathcal{A}_{2n} & \xrightarrow{\times r_m^{2n}} & \mathcal{A}_{2n} \end{array}$$

commutative.

(2) *If moreover m is even, then $\mathrm{Im}\,\psi_{2n}$ is included in the image of the Hurewicz homomorphism $\pi_{4n(m-1)}\widetilde{B\mathrm{Diff}}\,M \otimes \mathbb{Q} \to \Omega_{4n(m-1)}(\widetilde{B\mathrm{Diff}}\,M) \otimes \mathbb{Q}$.*

Composed with any linear functional on the space of disjoint union of graphs satisfying the AS and the IHX relations, any monomial of the form $\zeta_{2n_1}^{p_1} \cdots \zeta_{2n_r}^{p_r}$ yields $\mathbb{R}$-valued characteristic classes. Recall that the degree of a trivalent graph is the number of vertices.

Corollary 4.2. *Suppose that $m \geq 2$ and that M is a $(4m-1)$-dimensional homology sphere.*

(1) *There exist $\dim \mathbb{R}[\mathcal{A}_2^*, \mathcal{A}_4^*, \ldots, \mathcal{A}_{2n}^*]^{(\deg 2n)}$ linearly independent $\mathbb{R}$-valued characteristic classes of degree $4n(m-1)$ where $\mathbb{R}[\mathcal{A}_2^*, \mathcal{A}_4^*, \ldots, \mathcal{A}_{2n}^*]$ is the polynomial ring generated by elements of the dual spaces $\mathcal{A}_2^*, \mathcal{A}_4^*, \ldots, \mathcal{A}_{2n}^*$.*

(2) $\dim \mathrm{Im}\,\psi_{2n} \geq \dim \mathcal{A}_{2n}$.

Remark 4.3. The dimensions of the spaces $\mathcal{A}_{2n}$ for degrees up to 22 are computed in [BN] as follows:

degree $(2n)$	2	4	6	8	10	12	14	16	18	20	22
$\dim \mathcal{A}_{2n}$	1	1	1	2	2	3	4	5	6	8	9
$\dim \mathbb{R}[\mathcal{A}_2, \mathcal{A}_4, \ldots]^{(\deg\, 2n)}$	1	2	3	6	9	16	25	42	50	90	146

Corollary 4.4. *For $n \geq 2$ and $m \geq 2$ even, we have*

$$\begin{aligned}\dim \pi_{4n(m-1)-1}\mathrm{Diff}(D^{4m-1}\ \textit{rel}\ \partial) \otimes \mathbb{Q} &= \dim \pi_{4n(m-1)} B\mathrm{Diff}(D^{4m-1}\ \textit{rel}\ \partial) \otimes \mathbb{Q} \\ &\geq \dim \mathcal{A}_{2n}.\end{aligned}$$

Proof. Theorem 4.1(2) and Corollary 4.2 imply that $\dim \pi_{4n(m-1)}\widetilde{B\mathrm{Diff}}\,S^{4m-1} \otimes \mathbb{Q} \geq \dim \mathcal{A}_{2n}$ if m is even. Further, one can show that *if $n \geq 2$ and if τ and τ' are two different vertical framings on E that coincide on $\partial E \cup E_q$, then $[(E, \tau)] = [(E, \tau')]$ in $\pi_{4n(m-1)}\widetilde{B\mathrm{Diff}}\,S^{4m-1} \otimes \mathbb{Q}$.* Indeed, similarly as Lemma 3.8, we have

$$H_i(E, \partial E \cup E_q; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 4n(m-1) + 4m - 1 \\ 0 & \text{if } 0 \leq i \leq 4n(m-1) + 4m - 2 \end{cases}$$

Thus we have $[E, SO_{4m-1}]^\bullet \otimes \mathbb{Q} \cong \pi_{4n(m-1)+4m-1} SO_{4m-1} \otimes \mathbb{Q}$ where $[E, SO_{4m-1}]^\bullet$ denotes the set of homotopy classes of continuous maps $(E, \partial E \cup E_q) \to (SO_{4m-1}, 1)$. Further, we have $[E, SO_{4m-1}]^\bullet \otimes \mathbb{Q} = 0$ because it is known that $\pi_{4n(m-1)+4m-1} SO_{4m-1}$

is finite if $n \geq 2$ and $m \geq 2$. So there exists a positive integer p such that $p[(E,\tau)]$ is equivalent to $p[(E,\tau')]$. Therefore $[(E,\tau)] = [(E,\tau')]$ in $\pi_{4n(m-1)}\widetilde{B\mathrm{Diff}}\, S^{4m-1} \otimes \mathbb{Q}$.

Then it follows that

$$\pi_{4n(m-1)}\widetilde{B\mathrm{Diff}}\, S^{4m-1} \otimes \mathbb{Q} = \pi_{4n(m-1)}B\mathrm{Diff}(D^{4m-1} \text{ rel } \partial) \otimes \mathbb{Q}$$

for $n \geq 2$ and $m \geq 2$ even, thus we also have

$$\dim \pi_{4n(m-1)}B\mathrm{Diff}(D^{4m-1} \text{ rel } \partial) \otimes \mathbb{Q} \geq \dim \mathcal{A}_{2n}.$$

□

For $n = 1$, we have a partial result (compare Corollary 3.5 and 3.6).

Corollary 4.5. *Suppose that $m \geq 2$ is even. If the number*

$$\frac{3 \cdot 2^{4m+5}(2^{4m-3}-1)B_{2m-1}o(\pi^S_{8m-5})o(\pi^S_{2m-1})^2 \prod_{\ell=1}^{2m-2} o(\pi^S_\ell)^4}{(4m-2)!} \tag{4.1}$$

is not integral, then $\hat{\zeta}_2$ for $k = 2m$ is non-trivial. Hence the estimate of Corollary 4.4 holds also for $n = 1$ for $k = 2m$ and $\pi^\Theta_{1,1} : E^\Theta_{1,1} \to (S^{4m-2})^{\times 2}$ corresponds to a generator of the 1-dimensional subspace.

Corollary 4.6. *If an even integer $m \geq 2$ is such that*

- *$4m-3$ is prime and*
- *$4m-3 \nmid \mathrm{num}\,(B_{2m-1})$,*

then $\hat{\zeta}_2$ for $k = 2m$ is non-trivial.

The proofs of Corollary 4.5 and 4.6 will be given after the proof of Theorem 4.1. We have checked by a computer test (by using Maxima) that the first 71 examples

$m = 2, 4, 8, 10, 14, 16, 26, 28, 38, 40, 44, 46, 50, 58, 68, 70, 74, 80, 88, 94, 98, 100, 106, 116,$
$128, 136, 140, 154, 164, 166, 170, 176, 178, 184, 190, 194, 206, 208, 214, 220, 236, 250, 254,$
$256, 266, 268, 274, 278, 280, 296, 304, 308, 310, 320, 326, 344, 346, 358, 364, 374, 388, 400,$
$403, 404, 406, 410, 418, 424, 428, 434, 436$

for which $4m-3$ is prime all have the required properties of Corollary 4.6.

4.1. **Claspers and family of claspers.** From now on we construct the homomorphism ψ_{2n}. First we give a definition of higher dimensional claspers, which are generalizations of Habiro's clasper defined in [Hab, Hab2]. They will be used as elementary pieces in the constructions below. For the details about higher dimensional claspers, see [W], though we will describe here self-contained definitions of them.

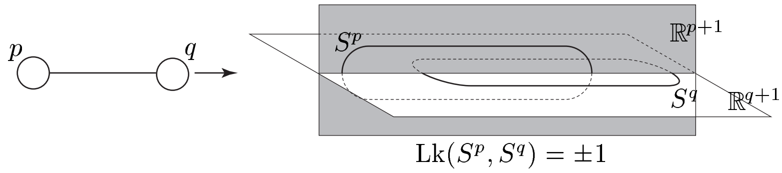


FIGURE 1. $I_{p,q}$-clasper and the associated Hopf link

4.1.1. *I-claspers.* An *$I_{p,q}$-clasper* is a normally framed null-homotopic embeddings of two disjoint spheres $S^p \sqcup S^q \subset M^{p+q+1}$ with $p, q \geq 1$ connected by an arc, equipped with a trivialization of the normal SO_{p+q}-bundle over the arc for which the first p-frame is parallel to the S^p near the intersection of the arc with S^p, and the last q-frame is parallel to the S^q near the intersection of the arc with S^q. We call each of the two spheres a *leaf* and call the arc an *edge*. With the given normal framing, we can canonically associate to an $I_{p,q}$-clasper a normally framed two component link by replacing with an embedded Hopf link as in Figure 1 so that the S^p lies in the $(p+1)$-plane spanned by the first p-frame in the normal frame together with the vector parallel to the direction of the edge, and the S^q lies in the $(q+1)$-plane spanned by the last q-frame in the normal frame together with the vector parallel to the direction of the edge. We orient the two leaves so that the linking number $\mathrm{Lk}(S^p, S^q)$ of the associated Hopf link is 1 if both p and q are odd. By a surgery along an $I_{p,q}$-clasper, we mean a surgery along its associated framed link.

4.1.2. *Family of claspers.* Consider the trivial $(M^\bullet \text{ rel } \partial)$-bundle $E \to B$ in which a trivial sub $I_{p,q}$-bundle with a structure of a family of B-parametrized embeddings of $I_{p,q}$-claspers into $\mathrm{Int}(M^\bullet)$ given.

Further we extend the notion of surgery to family of claspers. Simultaneous surgery along a family of claspers, i.e., attaching of (handles) $\times B$ followed by smoothing of corners so that the two trivial bundle structures are correctly glued together, yields a possibly non-trivial smooth $(M^\bullet \text{ rel } \partial)$-bundle. A *clasper-bundle* is an $(M^\bullet \text{ rel } \partial)$-bundle obtained by a sequence of surgeries along families of claspers.

4.2. **Graph claspers.** Now we briefly review the definition of a higher dimensional graph clasper. The notion of graph clasper in 3-dimension was first introduced by Habiro in [Hab]. Details about higher dimensional graph clasper will be described in [W][‡]. Graph clasper itself is not necessary to define graph clasper-bundles below. But it motivates the definition of the graph clasper-bundle. Also, we aim to explain that naive generalization of 3-valent graph claspers in 3-dimension to higher dimensions does not work.

In [Hab, Hab2], the Borromean rings in 3-dimension plays an important role. In higher dimensions, the higher dimensional Borromean rings play a similar role. When three

[‡]As mentioned in [W], the definition of the higher dimensional (unsuspended) graph clasper was suggested to the author by Kazuo Habiro, after the author's [W2].

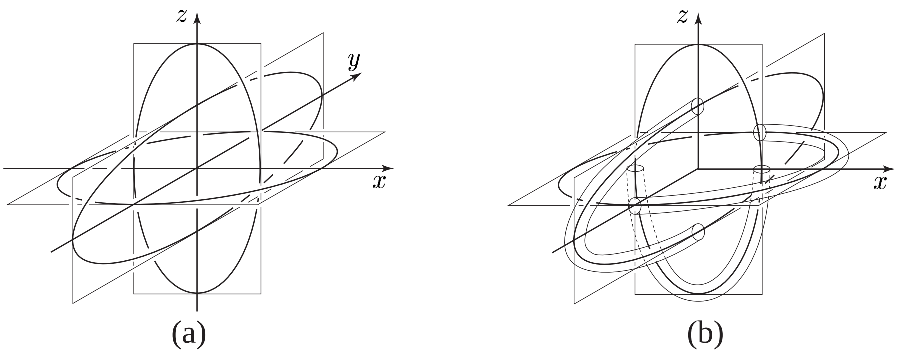


FIGURE 2

integers $0 < p, q, r < d$ satisfy the identity:

$$p + q + r = 2d - 3, \tag{4.2}$$

one can form higher dimensional Borromean rings $S^p \sqcup S^q \sqcup S^r \to \mathbb{R}^d$ as follows. Let p', q', r' be integers such that $p + p' = d - 1, q + q' = d - 1, r + r' = d - 1$. Then $p' + q' + r' = d$. Identify $\mathbb{R}^d$ with $\mathbb{R}^{p'} \times \mathbb{R}^{q'} \times \mathbb{R}^{r'}$. Then the union of the subsets

$$\begin{cases} S_p \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^d \mid |y|^2/4 + |z|^2 = 1,\, x = 0\} \cong S^p \\ S_q \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^d \mid |z|^2/4 + |x|^2 = 1,\, y = 0\} \cong S^q \\ S_r \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^d \mid |x|^2/4 + |y|^2 = 1,\, z = 0\} \cong S^r \end{cases} \tag{4.3}$$

of $\mathbb{R}^d$ forms a non-trivial 3 component link (see Figure 2(a)). Non-triviality of this link can be proved by computing the Massey product of its complement. We shall fix an orientation on the Borromean rings. The three components S_p, S_q, S_r lie on the planes $P_{yz} \stackrel{\text{def}}{=} \mathbb{R}^{q'} \times \mathbb{R}^{r'}$, $P_{zx} \stackrel{\text{def}}{=} \mathbb{R}^{r'} \times \mathbb{R}^{p'}$, $P_{xy} \stackrel{\text{def}}{=} \mathbb{R}^{p'} \times \mathbb{R}^{q'}$ respectively. Then consider $\mathbb{R}^{p'}$, $\mathbb{R}^{q'}$, $\mathbb{R}^{r'}$ are equipped with the induced orientations from that of $\mathbb{R}^d$ and introduce orientations on S_p, S_q, S_r determined from those of P_{yz}, P_{zx}, P_{xy} by the outward normal first convention.

The following property of Borromean rings will be used later in the proof of Theorem 4.1.

Proposition 4.7. *Let $X \stackrel{\text{def}}{=} \mathbb{R}^d \setminus (N(S_p) \cup N(S_q) \cup N(S_r))$ where N denotes a tubular neighborhood, and let $1, \alpha, \beta, \gamma \in \Omega^*(X)$ be differential forms representing the integral generators of $H^*(X)$ such that the supports of α, β, γ are disjoint from $\partial X \setminus \partial\overline{N(S_p)}$, $\partial X \setminus \partial\overline{N(S_q)}$, $\partial X \setminus \partial\overline{N(S_r)}$ respectively, and restrictions of α, β, γ to ∂X are ε-Thom forms about parallels of S_p, S_q, S_r on $\partial\overline{N(S_p)}, \partial\overline{N(S_q)}, \partial\overline{N(S_r)}$. Then the triple product $H^{p'}(X) \times H^{q'}(X) \times H^{r'}(X) \to H^d(X, \partial X)$ is defined in terms of such representatives, and one has $(\alpha \wedge \beta \wedge \gamma)[X, \partial X] = \pm 1$.*

Proof. By the hypotheses of α, β on ∂X, one has a well-defined product

$$H^{p'}(X) \times H^{q'}(X) \to H^{p'+q'}(X, \partial X)$$

given by a wedge. This is shown as follows: We see the change of the product $\alpha \wedge \beta$ when α is replaced with $\alpha' = \alpha + d\eta$ where η is a $(p'-1)$-form on X such that $d\eta|\partial X$ is supported on a subset of $\partial\overline{N(S_p)}$. Take a smooth function χ supported on a thin cylinder $[0, \varepsilon] \times \partial\overline{N(S_p)}$ disjoint from $\mathrm{Supp}(\beta)$ such that $\chi = 0$ on $[0, \varepsilon/2] \times \partial\overline{N(S_p)}$ and $\chi = 1$ on $\{\varepsilon\} \times \partial\overline{N(S_p)}$, and let $\alpha'' = \alpha + d(\chi\eta)$. Then the product $[d(\chi\eta) \wedge \beta]$ is equal to $[d\eta \wedge \beta]$ and is in the image of the usual product $H^{p'}(X, \partial X) \times H^{q'}(X) \to H^{p'+q'}(X, \partial X)$ that is obviously zero. Hence $[\alpha' \wedge \beta] = [\alpha \wedge \beta] + [d(\chi\eta) \wedge \beta] = [\alpha \wedge \beta]$ in $H^{p'+q'}(X, \partial X)$. The case β is replaced is similar. Thus one has a well-defined product

$$H^{p'}(X) \times H^{q'}(X) \times H^{r'}(X) \to H^{p'+q'+r'}(X, \partial X) = H^d(X, \partial X).$$

We shall take Poincaré-duals to α, β, γ and calculate the triple product by means of the intersection theory. Let

$$\begin{cases} D_1 \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^d \mid |y|^2/4 + |z|^2 \leq 1,\ x = 0\} \cong D^{p+1} \\ D_2 \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^d \mid |z|^2/4 + |x|^2 \leq 1,\ y = 0\} \cong D^{q+1} \\ D_3 \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^d \mid |x|^2/4 + |y|^2 \leq 1,\ z = 0\} \cong D^{r+1} \end{cases}$$

so that $\partial D_1 = S_p, \partial D_2 = S_q, \partial D_3 = S_r$. One may see that $D_1 \cap D_2 \cap D_3 = \mathrm{pt}$. But their ε-Thom forms restricted to X do not satisfy the hypotheses for α, β, γ since, for example, D_1 intersects not only S_p but also S_r transversely. To rid the superfluous intersections, attach handles parallel to the components to the disks (see Figure 2(b)). Then we obtain three cycles D_1', D_2', D_3' from D_1, D_2, D_3 respectively and ε-Thom forms of them satisfy the required hypotheses. Thus the triple product evaluated on the fundamental class is given by the triple intersection $D_1' \cap D_2' \cap D_3'$. One may check that there are only double intersections over the side-faces of the attached handles. Hence $D_1' \cap D_2' \cap D_3' = D_1 \cap D_2 \cap D_3 = \mathrm{pt}$. □

We define a *modeled graph clasper* as a connected uni-trivalent graph with

(1) vertex orientation on each trivalent vertex, namely, choice of an order of three incident edges to the trivalent vertex,
(2) decomposition of each edge into a pair of half edges,
(3) a positive integer $p(h)$ on each half edge h so that if $e = (h_0, h_1)$ is a decomposition of an edge e, $p(h_0) + p(h_1) = d - 1$ and if $p = p(h_1), q = p(h_2), r = p(h_3)$ are numbers of three incident half edges of a trivalent vertex, then they satisfy the condition (4.2),
(4) a $p(h_v)$-sphere attached to each univalent vertex v where h_v is the half edge containing v.

A *graph clasper* is a framed embedding of a modeled graph clasper into a d-dimensional manifold together with structures (vertex orientations, $p(\cdot)$). A framed link associated

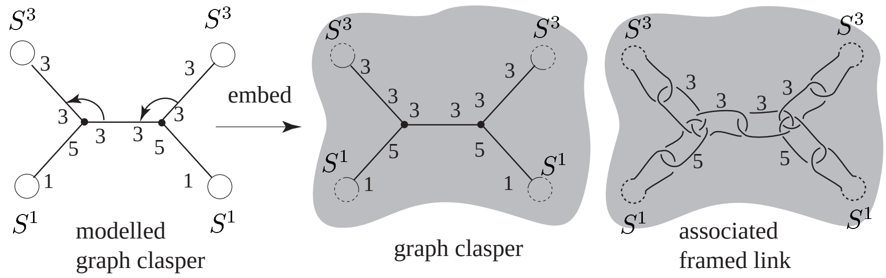


FIGURE 3

with a graph clasper G is a normally framed link in a regular neighborhood of G obtained by replacing each edge labeled (p,p') with a Hopf link associated to an $I_{p,p'}$-clasper so that the three spheres grouped together at a trivalent vertex form a Borromean rings.

Example 4.8. An obvious example is a graph clasper without trivalent vertices. This is just an $I_{p,q}$-clasper. Another example of a graph clasper for $d=7$ is depicted in Figure 3.

One may check that graph clasper with cycles exist only if the label $p(h) = 1$ is allowed. This condition is always satisfied when $d=3$ or 4 (see [W2] for related results in this case). In the case $d \geq 5$, it may happen that $p(h) > 1$ for all h. So in that case, graph claspers with cycles do not exist, that is, only the tree shaped graph claspers exist.

In the case $d = 3$, there are many graph claspers so that any trivalent graph gives rise to a graph clasper. However, in the case $d \geq 4$, no trivalent graph gives rise to a graph clasper! For example, assume that the Θ-shaped graph clasper with labels (p_1,p_2), (q_1,q_2), (r_1,r_2) on the three edges does exist. Then the set of linear equations in those numbers $p_1+p_2 = q_1+q_2 = r_1+r_2 = d-1$, $p_1+q_1+r_1 = 2d-3$, $p_2+q_2+r_2 = 2d-3$, $0 < p_i, q_i, r_i < d$ has the unique solution $(p_1,q_1,r_1,p_2,q_2,r_2,d) = (1,1,1,1,1,1,3)$. In order to construct 'dual' objects to the Kontsevich classes for trivalent graphs in high dimensions though, we need to consider family of claspers as in the next subsection.

4.3. **Graph clasper-bundles.** We shall define graph clasper-bundles here. More precisely, the object of this subsection is to define the announced homomorphism ψ_{2n} at the beginning of this section. Let $d = 4m-1 \geq 3$. In the following, we restrict only to the $I_{2m-1,2m-1}$-claspers in d-dimensional manifolds for simplicity.

4.3.1. *Certain family of three component links.* The following claim is the key observation motivating the definition of the graph clasper-bundle. By an *almost B-parametrized embedding*, we mean a B-parametrized family of smooth maps that are embeddings on $B \setminus \{\text{the base point}\}$.

Observation 4.9. *There exists an almost S^{2m-2}-parametrized embedding of an isotopically trivial 3 component link into an m-ball $B^d(2)$ with radius 2:*

$$\phi_t : S^{2m-1} \sqcup S^{2m-1} \sqcup S^{2m-1} \to B^d(2) \subset \mathbb{R}^d,\ t \in S^{2m-2},$$

such that the locus of their images, projected into a single $B^d(2)$, is isotopic to a Borromean rings of dimensions $(2m-2, 2m-2, 4m-3)$ with the fixed orientation so that the three components lie on P_{yz}, P_{zx}, P_{xy} respectively.

Proof. For a 3-component link embedding ϕ, let $\phi^{(i)}$, $i = 1, 2, 3$, denote ϕ restricted to i-th single component. Since the triple $(2m-1, 2m-1, 4m-3)$ for $d = 4m-1$ satisfies the condition (4.2), we can form a Borromean rings ϕ_L in $B^d(2)$ of dimensions $(2m-1, 2m-1, 4m-3)$ as in the previous subsection. The $(4m-3)$-sphere L_3 in $\mathrm{Im}\ \phi_L$ can be considered as a $(2m-2)$-fold loop suspension of a $(2m-1)$-sphere. Namely, by composing with

$$(S^{2m-2} \times S^{2m-1}, S^{2m-2} \vee S^{2m-1}) \to (S^{4m-3}, *),$$

we may represent $\phi_L^{(3)}$ as an S^{2m-2}-parametrized maps $\tilde{\phi}_t : S^{2m-1} \to B^d(2)$. Therefore, $\phi_t^{(i)} = \phi_L^{(i)}$ (constant over t) for $i = 1, 2$, and $\phi_t^{(3)} = \tilde{\phi}_t$ ($t \in S^{2m-2}$) gives the desired family. □

We will consider a family of embeddings $f_t : S^p \to F$ parametrized by B as a trivial sub bundle embedded in a trivial bundle $F \times B \to B$ so that the restriction to F_t ($t \in B$) is the embedding f_t. For usual graph claspers in [Hab, W] and in the previous subsection, the Borromean rings may be inserted at trivalent vertices. For the definition of the graph clasper-bundles, we will use the sub bundle representation of the Borromean rings $\{\phi_t\}_t$ (with a little modification) at trivalent vertices.

4.3.2. *Surgery along the family of three component links.* Now we want to define correctly a surgery along such a three component parametrized link embeddings. In order for such surgery to be well-defined, we need to overcome the following matter:

- The image of the almost parametrized embedding $\tilde{\phi}_t$ defined above degenerates into a point in the fiber of the base point t^0 of S^{2m-2}.

To overcome this, we define a parametrized embedding $\tilde{\varphi}_t : S^{2m-1} \to B^d(2)$ by modifying $\tilde{\phi}_t$ so that it is non-degenerate everywhere over S^{2m-2}.

Let $Q^d \subset B^d(2)$ be a small neighborhood of the base point of L_3 where L_3 is the third component of the image of ϕ_L appeared in Observation 4.9. First we make an embedding of $S^{2m-2} \times S^{2m-1}$ into $B^d(2)$ by attaching a small $(2m-1)$-handle to the $(4m-2)$-disk bounded by L_3 along the trivially embedded $(2m-2)$-sphere on L_3 (see Figure 4). Then we collapse the $(2m-1)$-handle into its core $(2m-1)$-disk so that

- the (limiting) boundary of the resulting object is a smooth embedding outside the part collapsed, and
- after the collapsing, the image from $\{t\} \times S^{2m-1} \subset S^{2m-2} \times S^{2m-1}$ for each $t \in S^{2m-2}$ is a smooth embedding.

Such a family indeed exists and one may describe a tangential structure of it explicitly. Here we assume that all the changes are included in $\mathrm{Int}(Q^d)$. Then the resulting family

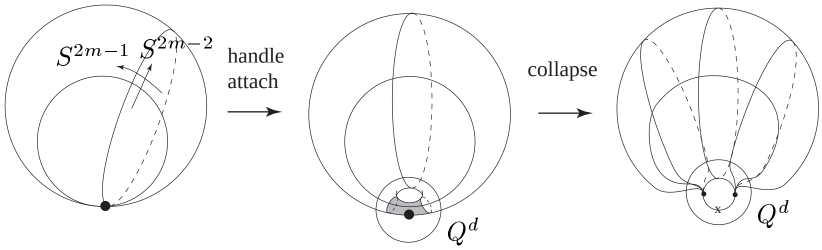


FIGURE 4

of embeddings by the above construction is the desired one and we will denote it by $\tilde{\varphi}_t$. See Figure 4 for an explanation of this construction.

Proposition 4.10. *The parametrized embedding $(\phi_L^{(1)}, \phi_L^{(2)}, \tilde{\varphi}_t)$ can be obtained (up to isotopy) by surgery along a (unsuspended) Y-graph clasper embedded in $B^d(2)$ from the trivial one $(\phi_L^{(1)}, \phi_L^{(2)}, \phi_0^{(3)})$ where*

- *the Y-graph clasper is associated with the Borromean rings of dimensions $(4m-3, 4m-3, 4m-3)$,*
- *$\phi_0^{(3)} : S^{2m-1} \to B^d(2)$ is a parameter-independent embedding disjoint from $\phi_L^{(1)}$ and $\phi_L^{(2)}$.*

Proof. After a suitable isotopy, one can push most of $\mathrm{Im}\,\tilde{\varphi}_t \subset B^d(2) \times S^{2m-2}$ into the fiber $B^d(2)_{t^0}$ of the base point $t^0 \in S^{2m-2}$. Then the image of $(\phi_L^{(1)}, \phi_L^{(2)}, \tilde{\varphi}_t)$ restricts in $B^d(2)_{t^0}$ to a Borromean rings of dimensions $(2m-1, 2m-1, 4m-3)$, with something small change near the base point of the third component, that is disjoint from all other components. Then the first two components trivially arranged over S^{2m-2} together with the modified $(4m-3)$-sphere in $B^d(2)_{t^0}$, may be seen as a part of the Borromean rings of dimensions $(4m-3, 4m-3, 4m-3)$ in $(6m-3)$-dimension. Since the Borromean rings can be obtained by a Y-surgery, the result follows. □

Remark 4.11. Of course the isotopy used in the proof of Proposition 4.10 may break the bundle structure. Proposition 4.10 is just a claim about a manifold structure of the total space.

4.3.3. *Graph clasper-bundle.* We denote by ϕ_t^Y the parametrized embedding

$$(\phi_L^{(1)}, \phi_L^{(2)}, \tilde{\varphi}_t) : S^{2m-1} \sqcup S^{2m-1} \sqcup S^{2m-1} \to \mathrm{Int}\, B^d(2), \quad t \in S^{2m-2}$$

defined above with the fixed orientation that yields the orientation of Observation 4.9. Note that ϕ_t^Y can be chosen so that the base point of each S^{2m-1} component is fixed. By using this parametrized embedding, we shall construct graph clasper-bundles.

Let V be a d-dimensional handlebody obtained from a d-disk by attaching three $(2m-1)$-handles along 3-component isotopically trivial framed link embedded in the boundary of the d-disk. Here we fix an order of the three $(2m-1)$-handles.

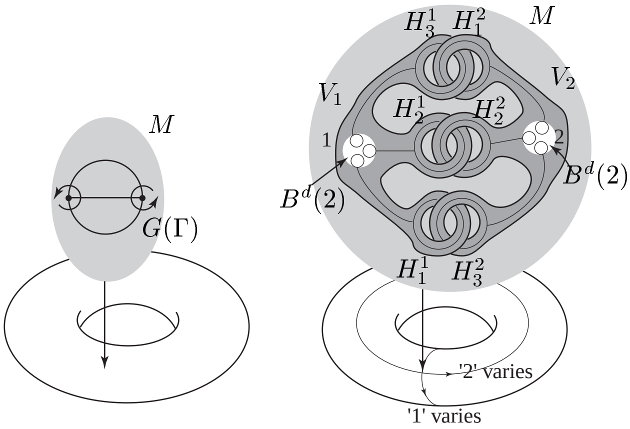


FIGURE 5

First we shall define the $(V \text{ rel } \partial)$-bundle $\pi^Y : V^Y \to S^{2m-2}$. Let us assume that $B^d(2)$ is embedded in the interior of V. Then we make a direct product bundle $(V, B^d(2)) \times S^{2m-2} \to S^{2m-2}$ to obtain a trivial sub $B^d(2)$-bundle $\widehat{B}^d(2) (\cong B^d(2) \times S^{2m-2})$ embedded in the trivial V-bundle $\widehat{V} \cong V \times S^{2m-2} \to S^{2m-2}$. Now let

$$\phi_I : (I_{2m-1,2m-1} \sqcup I_{2m-1,2m-1} \sqcup I_{2m-1,2m-1}) \times S^{2m-2} \to V$$

be the three disjoint union of families of claspers parametrized by S^{2m-2} such that

(1) For each $t \in S^{2m-2}$, one of the two leaves of the i-th component ($i = 1, 2, 3$) of $\text{Im}\,(\phi_I)_t$ is standardly embedded parallel to the core of the i-th $(2m-1)$-handle of $B^d(2)$, and the other leaf is embedded in V isotopically trivially. We consider the latter leaf as an embedding $(S_i)_t : S^{2m-1} \to B^d(2)$.
(2) $\widehat{B}^d(2) \cap \text{Im}\,\phi_I$ is precisely a graph (of a function on t) of an S^{2m-2}-parametrized embedding of the three leaves into $B^d(2)$. Thus $\partial \widehat{B}^d(2) \cap \text{Im}\,\phi_I \cong (\text{pt} \sqcup \text{pt} \sqcup \text{pt}) \times S^{2m-2}$, the intersection points of edges and leaves of claspers.
(3) $(\phi_I)_t$ is standard, i.e., independent of t, on $V \setminus B^d(2)$.
(4) $(\phi_I)_t$ restricted to the leaves $S_1 \sqcup S_2 \sqcup S_3$ coincides with ϕ_t^Y. Also assume that the orientations of the three components coincide with that of ϕ_t^Y. Then the orientations on the three leaves embedded in the cores of the handles are automatically determined.

There is a picture for ϕ_I in Figure 5(ii). Then simultaneous surgeries on $\widehat{V}$ along the family of claspers ϕ_I yield another $(V \text{ rel } \partial)$-bundle. We denote the resulting bundle by $\pi^Y : V^Y \to S^{2m-2}$.

Definition 4.12 (V^Y-surgery)**.** For a given $(M^\bullet \text{ rel } \partial)$-bundle $\pi : E \to B$, we assume that a trivial V-bundle $\widehat{V} \cong V \times B$ is embedded in E as a trivial sub V-bundle of π. Then the V^Y*-surgery on* π *along* $\widehat{V}$, denoted by $\pi^{Y(\widehat{V},\varphi)} : E^{Y(\widehat{V},\varphi)} \to B$, is defined with a choice of a C^∞-map $\varphi : B \to S^{2m-2}$ as follows:

$$\pi^{Y(\widehat{V},\varphi)} \stackrel{\text{def}}{=} \pi|(E \setminus \mathrm{Int}(\widehat{V})) \cup_\partial \varphi^*\pi^Y$$
$$E^{Y(\widehat{V},\varphi)} \stackrel{\text{def}}{=} E \setminus \mathrm{Int}(\widehat{V}) \cup_\partial (\varphi^* V^Y)$$

where $\varphi^*\pi^Y$ denotes the pullback bundle structure (not to be seen as a pullback of a map) and where the trivial bundle structures on the boundaries are glued together correctly. □

Definition 4.13 (Graph clasper-bundle)**.** Let $\Gamma \in \mathcal{G}_{2n}$ be a trivalent graph with $2n$ vertices and $3n$ edges not having the part like $\multimap$ and let $G(\Gamma) \subset M$ be a fixed *irregular* graph clasper for Γ trivially embedded in a d-dimensional manifold M with all labels equal to $2m-1$. Here 'irregular' means that only the condition (4.2) for the three labels at trivalent vertices fails to be a graph clasper. Then replace $G(\Gamma)$ with $2n$ disjointly embedded handlebodies $V_1 \sqcup \cdots \sqcup V_{2n}$ each diffeomorphic to V and satisfying the following conditions.

(1) Decompose each handlebody V_i into a 0-handle B_i and three $(2m-1)$-handles $H_1^i \sqcup H_2^i \sqcup H_3^i$ so that the order corresponds to the order of the three handles of V and so that B_i includes the i-th vertex of $G(\Gamma)$. Then H_j^i and $H_k^{i'}$ are included inside $N_{ii'} \setminus B_i$, where j and k are determined by the vertex orientation of $G(\Gamma)$, and where $N_{ii'}$ denotes a thin tubular neighborhood of the edge of $G(\Gamma)$ connecting the i-th and the i'-th vertices (if exists). The cores of the handles H_j^i and $H_k^{i'}$ link with the linking number 1, and their positions are canonically determined by the framing of the edge similarly as the associated Hopf link to an $I_{p,q}$-clasper. Here the orientation of the cores are assumed to be those induced from V.

(2) Each edge of $G(\Gamma)$ has just one associated pair of handles $(H_j^i, H_k^{i'})$ as above.

Then the $(M^\bullet \text{ rel } \partial)$-bundle $\pi^\Gamma_{d_1,d_2,\ldots,d_{2n}} : E^\Gamma_{d_1,d_2,\ldots,d_{2n}} \to (S^{2m-2})^{\times 2n}$ is defined as follows. First by taking a direct product $(M^\bullet, V_1 \sqcup \cdots \sqcup V_{2n}) \times (S^{2m-2})^{\times 2n}$, we obtain $2n$ disjointly embedded trivial sub V-bundles in the trivial $(M^\bullet \text{ rel } \partial)$-bundle $\pi_{\text{triv}} : E_{\text{triv}} (\cong M^\bullet \times (S^{2m-2})^{\times 2n}) \stackrel{\text{proj.}}{\to} (S^{2m-2})^{\times 2n}$. Then we define

$$\pi^\Gamma_{d_1,d_2,\ldots,d_{2n}} \stackrel{\text{def}}{=} \pi_{\text{triv}}^{Y(V_1 \times S^{2m-2},\varphi_1)\cdots Y(V_{2n} \times S^{2m-2},\varphi_{2n})}$$
$$E^\Gamma_{d_1,d_2,\ldots,d_{2n}} \stackrel{\text{def}}{=} E_{\text{triv}}^{Y(V_1 \times S^{2m-2},\varphi_1)\cdots Y(V_{2n} \times S^{2m-2},\varphi_{2n})}$$

where $\varphi_i : (S^{m-2})^{\times 2n} \to S^{2m-2}$ is the C^∞-map for the V^Y-sugery along V_i, that is the i-th projection followed by a degree d_i map $S^{2m-2} \to S^{2m-2}$. We will call such constructed π^Γ a *graph clasper-bundle* associated to Γ. (See Figure 5.) □

Remark 4.14. 1. The above definition of graph clasper-bundles is also valid for $m=1$, i.e., for graph clasper-bundles consisting of $I_{1,1}$-claspers in a 3-manifold. In this case, the bundle is over $S^0 \times \cdots \times S^0$, namely an alternating sum of Y-clasper surgeries, which appeared in the context of finite type theory of 3-manifolds [Hab2].

2. One may generalize the notion of the graph clasper-bundles to arbitrary base B with general choices for φ_i. In fact there are possibly non isomorphic V^Y-surgeries as many as $[B, S^{2m-2}] \stackrel{1-1}{\leftrightarrow} \Omega^{\mathrm{fr}}_{\dim B-(2m-2)}(B)$, the set of bordism classes of normally framed $(\dim B - (2m-2))$-dimensional submanifolds of B, by the Pontrjagin–Thom construction.

4.3.4. *Existence of vertical framings.* To complete the definition of ψ_{2n}, we give each graph clasper-bundle a certain vertical framing.

Proposition 4.15. *For $m \geq 2$ and for any $\Gamma \in \mathcal{G}_{2n}$, there is a positive integer r_m for which the graph clasper-bundle $\pi^\Gamma_{r_m,\ldots,r_m} : E^\Gamma_{r_m,\ldots,r_m} \to (S^{2m-2})^{\times 2n}$ can be vertically framed and it is standard outside $V_1 \sqcup \cdots \sqcup V_{2n}$.*

The statement given here is stronger than just for saying the existence of the vertical framing because it is needed in the proof of Proposition 4.16.

Proof. Consider r_m as a degree r_m map $S^{2m-2} \to S^{2m-2}$. Then it is enough to prove the claim for the $(V \text{ rel } \partial)$-bundle $r_m^*\pi^Y : r_m^*V^Y \to S^{2m-2}$ since then $\pi^\Gamma_{r_m,\ldots,r_m}$ can be obtained from it by V^Y-surgeries for framed bundles.

Consider the trivial $(V \text{ rel } \partial)$-bundle $\phi^*\pi^Y : \widehat{V} \to D^{2m-2}$ pulled back from π^Y by the characteristic map $\phi : D^{2m-2} \to S^{2m-2}$ of the $(2m-2)$-cell of the cell decomposition $S^{2m-2} = e^0 \cup e^{2m-2}$. Then after a homotopy, we may assume that the pullback bundle $\phi^*\pi^Y$ is standard on $(\phi^*\pi^Y)^{-1}(D^{2m-2} \setminus B)$ for a small $(2m-2)$-disk B embedded in $\mathrm{Int}(D^{2m-2})$, i.e., the holonomy group is reducible to $\{\mathrm{id}\}$ there.

Now we fix a deformation retraction $H : D^{2m-2} \times I \to D^{2m-2}$ so that $H(D^{2m-2} \times \{0\}) = D^{2m-2}$ and $H(D^{2m-2} \times \{1\}) = \{q\} \in \partial D^{2m-2}$. Then for each point $x \in \partial D^{2m-2} \setminus \{q\}$, $\gamma_x \stackrel{\mathrm{def}}{=} H(x, \cdot) : I \to D^{2m-2}$ defines a path from x to q. Along the path γ_x^{-1}, diffeomorphisms relative-∂ between fibers $\varphi_{x,t} : \widehat{V}_{\gamma_x^{-1}(t)} \stackrel{\sim}{\to} \widehat{V}_q$ are induced as the results of the deformation. Then by the pullback $\varphi^*_{x,t}\tau_{E_q}$ a vertical framing on $\widehat{V}$ is defined. In particular, a possibly non standard vertical framing τ_∂ is defined on the standard product bundle $(\phi^*\pi^Y)^{-1}(\partial D^{2m-2})$ as the result. Since the bundle $\phi^*\pi^Y$ is standard outside B in an unframed sense, τ_∂ is non standard only inside $(\phi^*\pi^Y)^{-1}(D^{2m-3})$ where $D^{2m-3} \subset \partial D^{2m-2}$ is an embedded disk.

Thus the problem is reduced to the vanishing of the obstruction to homotoping the framing τ_∂ to the standard one τ_{std} for some choice of positive integer r_m. To see that, we shall fix a relative CW cochain complex $C^*(V \times D^{2m-3}, \partial(V \times D^{2m-3}); \mathbb{Z})$ where the obstruction cocycle may lie, as follows. We can choose a relative cell decomposition of $(V \times D^{2m-3}, \partial(V \times D^{2m-3}))$ with three $(4m-3)$-cells and one $(6m-4)$-cell, the corresponding chain complex is

$$C_*(V\times D^{2m-3},\partial(V\times D^{2m-3});\mathbb{Z})=\begin{cases}\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \text{if } *=4m-3\\ \mathbb{Z} & \text{if } *=6m-4\\ 0 & \text{if otherwise}\end{cases}$$

The obstruction cocycle $c_{4m-3}(\tau_\partial,\tau_{\rm std})$ to homotoping the framings τ_∂ and $\tau_{\rm std}$ up to the $(4m-3)$-skeleton lies in the group:

$$\mathrm{Hom}(\mathbb{Z}^{\oplus 3},\pi_{4m-3}SO_{4m-1})\cong(\pi_{4m-3}SO)^{\oplus 3}\cong\begin{cases}0 & \text{if } m\text{: even}\\ \mathbb{Z}_2^{\oplus 3} & \text{if } m\text{: odd}\end{cases}$$

Hence $c_{4m-3}(\tau_\partial,\tau_{\rm std})$ vanishes if r_m is even.

Similarly, the obstruction cocycle $c_{6m-4}(\tau_\partial,\tau_{\rm std})$ lies in the group

$$\mathrm{Hom}(\mathbb{Z},\pi_{6m-4}SO_{4m-1})\cong\pi_{6m-4}SO_{4m-1}$$

that is finite. Thus we may take $r_m>0$ with $r_m|2o(\pi_{6m-4}SO_{4m-1})$ so that $r_m^*\pi^Y$ and thus $\pi^\Gamma_{r_m,\ldots,r_m}$ have vertical framings with the required property. ☐

4.4. **Duality between graph clasper-bundles and characteristic classes.** Let $m\geq 2$ and let M be a $(4m-1)$-dimensional homology sphere. Let

$$\psi_{2n}:\mathcal{G}_{2n}\to\Omega_{4n(m-1)}(\widetilde{B\mathrm{Diff}\,M})\otimes\mathbb{Q}$$

be the linear map defined for each connected trivalent graph Γ as follows:

- **if Γ does not have $-\!\!\circ$:** ψ_{2n} is defined as the bordism class of the classifying map for $\pi^\Gamma_{r_m,\ldots,r_m}$ with choices of an orientation on Γ and of a vertical framing $\tau(\Gamma)$ which is standard outside $V_1\sqcup\cdots\sqcup V_{2n}\subset M$. (Such a choice of $\tau(\Gamma)$ is possible by Proposition 4.15.)
- **if Γ has $-\!\!\circ$:** ψ_{2n} is defined as 0.

We will write $[E]$ for the bordism class of the classifying map into $\widetilde{B\mathrm{Diff}}M$ for a bundle $E\to B$. Here ψ_{2n} may depend on choices made. But the final claim is the same.

4.4.1. *A choice of the fundamental form in graph clasper-bundles.* The choice of the framing made in Proposition 4.15 allows one to make the fundamental $(4m-2)$-form on $C_2(M)$-bundles more accessible. Let $C(\pi^\Gamma):C(E^\Gamma)\to B$ be the $C_2(M)$-bundle associated to the $(M^\bullet$ rel $\partial)$-bundle $\pi^\Gamma_{r_m,\ldots,r_m}$ and let $\beta_M\stackrel{\rm def}{=}f^*\alpha_{\mathrm{Diff}\,M}$ where

$$f:C(E^\Gamma)\to C_2(M)\ltimes\widetilde{E\mathrm{Diff}}M$$

is a bundle morphism whose pullback is isomorphic to $C(\pi^\Gamma)$. To simplify the proof of Theorem 4.1, we replace β_M with another one within a cohomology class.

One may check that the value $\langle\zeta_{2n},[E^\Gamma_{r_m,\ldots,r_m}]\rangle$ does not change if the form β_M is replaced with another one β'_M satisfying

- $[(\beta_M)_t]=[(\beta'_M)_t]$ in $H^{4m-2}(C_2(M)_t;\mathbb{R})$ for every $t\in B$,
- $\iota^*\beta'_M=-\beta'_M$ and

- $\beta_M|\partial C_2(M)_t = \beta'_M|\partial C_2(M)_t$ for every $t \in B$.

So we shall replace β_M with such a β'_M satisfying some more conditions so that we can compute the integral explicitly.

For any $i \in \{1, \ldots, 2n\}$, fix disjoint simple S^{2m-1}-cycles $(a^i_j)_{j=1,2,3}$ and simple S^{2m-1}-cycles $(b^i_j)_{j=1,2,3}$ on $\partial V_i \cong (S^{2m-1} \times S^{2m-1})^{\#3}$ such that

- a^i_j bounds a $2m$-disk in V_i and b^i_j bounds a $2m$-disk in $M \setminus \mathrm{Int}(V_i)$.
- $\langle a^i_j, b^i_k \rangle_{\partial V_i} = \delta_{jk}$.

Let $\eta(a^i_j, t)$ be a closed $(2m-1)$-form on $\cup_t (V_i)_t$, the sub V_i-bundle of $E^\Gamma_{r_m,\ldots,r_m}$, such that its support intersects the thin collar $I \times (\partial V_i)_t$ inside $I \times (a^i_j \times D^{2m-1})$ where $a^i_j \times D^{2m-1}$ is a fixed tubular neighborhood of a^i_j in $(\partial V_i)_t$, and where the restriction of $\eta(a^i_j, t)$ here is the ε-Thom form on $I \times a^j_i$. We can show that $\eta(a^i_j, t)$ can indeed be chosen to be smooth with respect to t (See Appendix C).

In [Les2], a useful proposition has been proved, that allows to compute 3-dimensional configuration space integrals explicitly in some cases and that is implicit in [KT]. We prove and use the following proposition, a higher dimensional version of Lescop's proposition in some restricted cases, which is enough for graph clasper-bundles.

Proposition 4.16. *Suppose that $m \geq 2$ and that the vertical framing of $E^\Gamma_{r_m,\ldots,r_m}$ is chosen as $\tau(\Gamma)$ (as in Proposition 4.15). The form β_M on $C(E^\Gamma)$ can be replaced without affecting the resulting value $\langle \zeta_{2n}, [E^\Gamma_{r_m,\ldots,r_m}] \rangle$ satisfying the following conditions:*

- *Let $I(t) \subset \{1, \ldots, 2n\}$ be the subset of labels such that $i \in I(t)$ if and only if $t_i \neq t^0_i$. Then for any $t = (t_1, \ldots, t_{2n}), t' = (t'_1, \ldots, t'_{2n}) \in (S^{2m-2})^{\times 2n}$ with $t_i = t'_i$ $(\forall i \in I(t) \cap I(t'))$, we have $\beta_M(t_1, \ldots, t_{2n}) = \beta_M(t'_1, \ldots, t'_{2n})$ where it makes sense, namely on*
$$C_2\Big((M \setminus \textstyle\bigcup_{i \in I(t) \cup I(t')} \mathrm{Int}(V_i)) \cup \bigcup_{j \in I(t) \cap I(t')} (V_j)_t\Big).$$
- *On $(V_i)_t \times (V_k)_t$,*
$$\beta_M(t) = \sum_{j,l \in \{1,2,3\}} \mathrm{Lk}(b^i_j, b^k_l)\, p_1^* \eta(a^i_j, t) \wedge p_2^* \eta(a^k_l, t).$$
where $p_1, p_2 : C_2(M) \to C_1(M)$ denote the first and the second projection, respectively.

Proposition 4.16 follows from the observation above and from an analogous argument as in [Les2, Proposition 3.3]. Proof will be given in Appendix C.

Remark 4.17. In Proposition 4.16, we can choose $\eta(a^i_j, t)$ so that it depends only on t_i (see Appendix C).

Proof of Theorem 4.1. First we assume that the form β_M on $C(E^\Gamma)$ has been chosen as in Proposition 4.16.

(1) The commutativity of the diagram is a consequence of the following identity:

$$\langle \zeta_{2n}, [E^{\Gamma}_{r_m,\ldots,r_m}] \rangle = r_m^{2n}[\Gamma]$$

for any choice of the vertical framing $\tau(\Gamma)$ that is standard outside $V_1 \sqcup \cdots \sqcup V_{2n}$. So we shall prove this identity.

Let $(t_1, \ldots, t_{2n})$ denote the coordinate of $(S^{2m-2})^{\times 2n}$ and let $\omega(\Gamma')(t_1, \ldots, t_{2n})$ be the integrand form for the integral associated to Γ', restricted to the configuration space fiber of $(t_1, \ldots, t_{2n})$.

First we see that the computation can be simplified to the one for a bundle with fiber a direct product of some simple manifolds. Let $U_i \subset C_{2n}(M)$ be the subset consisting of configurations such that no points are included in V_i. We show that the fiber integration restricted to U_i-fiber degenerates. We consider the case $i = 1$ for simplicity. Let

$$\pi_1 : S^{2m-1} \times S^{2m-2} \times \cdots \times S^{2m-2} \to \{t_1^0\} \times S^{2m-2} \times \cdots \times S^{2m-2}, \quad (t_1^0 : \text{base point})$$

be the projection defined by $(t_1, t_2, \ldots, t_{2n}) \mapsto (t_1^0, t_2, \ldots, t_{2n})$. Then π_1 can be extended to a bundle morphism $\hat{\pi}_1$ between the sub U_1-bundles of $\pi^{\Gamma}_{r_m,\ldots,r_m}$ and of its restriction to $\{t_1^0\} \times (S^{2m-2})^{\times 2n-1}$. Since $\omega(\Gamma')(t_1, t_2, \cdots, t_{2n}) = \hat{\pi}_1^* \omega(\Gamma')(t_1^0, t_2, \cdots, t_{2n})$ over U_1 by Proposition 4.16, we have

$$\begin{aligned}
&\int_{(t_1,\ldots,t_{2n}) \in (S^{2m-2})^{\times 2n}} \int^f_{U_1} \omega(\Gamma')(t_1, t_2, \cdots, t_{2n}) \\
&= \int_{(S^{2m-2})^{\times 2n}} \int^f_{U_1} \hat{\pi}_1^* \omega(\Gamma')(t_1^0, t_2, \cdots, t_{2n}) \\
&= \int_{\{t_1^0\} \times (S^{2m-2})^{\times 2n-1}} \int^f_{U_1} \omega(\Gamma')(t_1^0, t_2, \cdots, t_{2n}) = 0
\end{aligned}$$

by a dimensional reason, where $\int^f$ is an integral along the fibers. So it suffices to compute the integral over $\widetilde{C} \stackrel{\text{def}}{=} C_{2n}(M) \setminus \bigcup_i U_i$-fibers. Since at least one point is included in each V_i for any configuration in $\widetilde{C}$, $\widetilde{C}$ is a disjoint union of the spaces of the form $V_1 \times \cdots \times V_{2n}$.

We show that the integration domain can be reduced further into a direct product of some closed manifolds. Let $\widetilde{V}_i \to S^{2m-2}$ be the (V_i rel ∂)-bundle induced from $\pi^{\Gamma}_{r_m,\ldots,r_m}$ by the inclusion $\iota_i : S^{2m-2} \hookrightarrow (S^{2m-2})^{\times 2n}$ given by $t_i \mapsto (t_1^0, \ldots, t_i, \ldots, t_{2n}^0)$, followed by restriction to the V_i-fiber (this is precisely isomorphic to $r_m^* \pi^Y$). Recall from Remark 4.17 that in Proposition 4.16, we can choose $\eta(a_j^i, t)$ so that it depends only on t_i. Hence the integral can be rewritten as

$$\int_{(S^{2m-2})^{\times 2n}} \int^f_{V_1 \times \cdots \times V_{2n}} \omega(\Gamma') = \int_{\widetilde{V}_1 \times \ldots \times \widetilde{V}_{2n}} \omega(\Gamma'). \tag{4.4}$$

It would be best to explain this identity for the simplest case $\Gamma' = \Theta$ as the other cases are completely analogous. By Proposition 4.16, one has on $(V_1 \times V_2)_{t=(t_1,t_2)}$,

$$\beta_M(t) = p_1^* \eta(a_1^1, t_1) \wedge p_2^* \eta(a_1^2, t_2) + p_1^* \eta(a_2^1, t_1) \wedge p_2^* \eta(a_2^2, t_2) + p_1^* \eta(a_3^1, t_1) \wedge p_2^* \eta(a_3^2, t_2).$$

Hence on $(V_1 \times V_2)_{t=(t_1,t_2)}$,

$$\begin{aligned}\omega(\Theta) &= \beta_M(t)^{\wedge 3}\\ &= 3!\, p_1^*(\eta(a_1^1,t_1)\wedge\eta(a_2^1,t_1)\wedge\eta(a_3^1,t_1))\wedge p_2^*(\eta(a_1^2,t_2)\wedge\eta(a_2^2,t_2)\wedge\eta(a_3^2,t_2)).\end{aligned}$$

Write $\omega(\Theta)|(V_1\times V_2)_{t=(t_1,t_2)} = A(t_1)\wedge B(t_2)$. Then the LHS of (4.4) can be written as

$$\begin{aligned}&\int_{(t_1,t_2)\in(S^{2m-2})^{\times 2}}\int^f_{V_1\times V_2} A(t_1)\wedge B(t_2) = \int_{(S^{2m-2})^{\times 2}}\int^f_{V_1} A(t_1)\wedge\int^f_{V_2} B(t_2)\\ &= \int_{t_1\in S^{2m-2}}\int^f_{V_1} A(t_1)\int_{t_2\in S^{2m-2}}\int^f_{V_2} B(t_2) = \int_{\widetilde{V}_1} A(t_1)\int_{\widetilde{V}_2} B(t_2) = (\mathrm{RHS}).\end{aligned}$$

In Proposition 4.16, all the η-forms are standard near $\partial\widetilde{V}_i$ and hence the integral of (4.4) is equal to the integral over $\widetilde{V}_1'\times\cdots\times\widetilde{V}_{2n}'$, where $\widetilde{V}_i'$ denotes the closed manifold obtained from $\widetilde{V}_i$ by collapsing $\partial\widetilde{V}_i\cong\partial V_i\times S^{2m-2}$ into $\partial V_i\times\{t_i^0\}$. Thus the integral can be given by a homological evaluation with the fundamental class.

Now triple cup product evaluated on the fundamental homology class of the closed manifold $\widetilde{V}_i'$

$$H^{2m-1}(\widetilde{V}_i';\mathbb{Z})\wedge H^{2m-1}(\widetilde{V}_i';\mathbb{Z})\wedge H^{2m-1}(\widetilde{V}_i';\mathbb{Z})\to\mathbb{Z}$$

considered up to a sign, coincide with triple intersection among the Poincaré duals in $H_{4m-2}(\widetilde{V}_i';\mathbb{Z})$. In particular, if $\alpha_i,\beta_i,\gamma_i$ are the integral homology classes representing the cores of the three $(2m-1)$-handles of a fiber of $\widetilde{V}_i'$ and if $\alpha_i^*,\beta_i^*,\gamma_i^*$ are the duals of $\alpha_i,\beta_i,\gamma_i$ with respect to the evaluation, then

$$\langle\alpha_i^*\cup\beta_i^*\cup\gamma_i^*,[\widetilde{V}_i']\rangle = \mathrm{PD}(\alpha_i^*)\cdot\mathrm{PD}(\beta_i^*)\cdot\mathrm{PD}(\gamma_i^*) = r_m$$

since the suspended Y-clasper over an S^{2m-2} component can be replaced with an unsuspended Y-clasper by Proposition 4.10, and by Proposition 4.7 with the spheres replaced by the family of $(2m-1)$-handles in $\widetilde{V}_i'$. Note that $H^{6m-3}(\widetilde{V}_i';\mathbb{Z})\cong H^{2m-1}(\widetilde{V}_i';\mathbb{Z})^{\wedge 3}$ is one dimensional and spanned by $\alpha_i^*\cup\beta_i^*\cup\gamma_i^*$.

On the other hand, the $(4m-2)$-form $\theta_{e=(i,j)}\stackrel{\mathrm{def}}{=}\phi_e^*\beta_M\in\Omega^{4m-2}(C(E^\Gamma))$ is considered as an element of $H^{2m-1}(\widetilde{V}_i';\mathbb{R})\otimes H^{2m-1}(\widetilde{V}_j';\mathbb{R})$ corresponding to the linking form and thus is in the image from $H^{2m-1}(\widetilde{V}_i';\mathbb{Z})\otimes H^{2m-1}(\widetilde{V}_j';\mathbb{Z})$. Here ϕ_e is defined as in §2.4.

Therefore, the integral is obtained by contractions of the tensors and we get

$$\int_{\widetilde{V}_1'\times\cdots\times\widetilde{V}_{2n}'}\widetilde{\omega}(\Gamma') = \langle\prod_e\theta_e,[\widetilde{V}_1'\times\cdots\times\widetilde{V}_{2n}']\rangle = \begin{cases}|\mathrm{Aut}_e\Gamma|\cdot r_m^{2n} & \text{if }\Gamma'=\Gamma\\ 0 & \text{otherwise}\end{cases}$$

where $|\mathrm{Aut}_e\Gamma|$ denotes the order of the automorphisms of Γ fixing all vertices. See Figure 6 for an explanation of this for the Θ-graph. Here, $\theta_{12}^* = \alpha_1^*\otimes\alpha_2^*+\beta_1^*\otimes\beta_2^*+\gamma_1^*\otimes\gamma_2^*$ and thus $\langle\theta_{12}^3,[\widetilde{V}_1'\times\widetilde{V}_2']\rangle = 3!\,(\mathrm{PD}(\alpha_1^*)\cdot\mathrm{PD}(\beta_1^*)\cdot\mathrm{PD}(\gamma_1^*))\times(\mathrm{PD}(\alpha_2^*)\cdot\mathrm{PD}(\beta_2^*)\cdot\mathrm{PD}(\gamma_2^*)) = 3!\,r_m^2$.

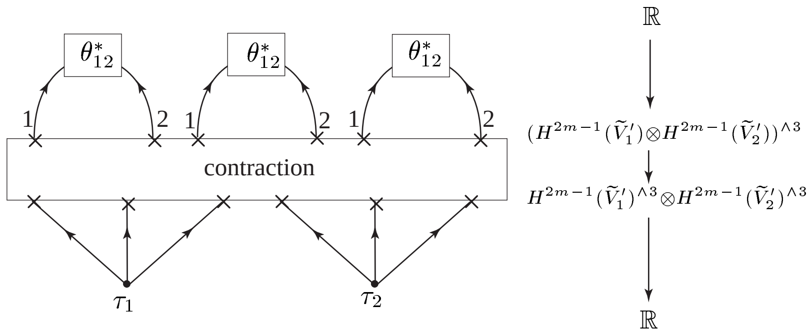


FIGURE 6

More generally, exactly $|\mathrm{Aut}_v\Gamma| \stackrel{\mathrm{def}}{=} |\mathrm{Aut}\,\Gamma|/|\mathrm{Aut}_e\Gamma|$ connected components in $\widetilde{C}$ contribute to the term of Γ as r_m^{2n} and the other parts do not contribute. Therefore,

$$\begin{aligned}&\langle \zeta_{2n}, [E^\Gamma_{r_m,\ldots,r_m}; \tau(\Gamma)]\rangle = |\mathrm{Aut}_v\Gamma|\,\zeta_{2n}(\widetilde{V}_1'\times\cdots\times\widetilde{V}_{2n}')\\ &= |\mathrm{Aut}_v\Gamma| \sum_{\Gamma'}\frac{[\Gamma']}{|\mathrm{Aut}\,\Gamma'|}\int_{\widetilde{V}_1'\times\cdots\times\widetilde{V}_{2n}'}\widetilde{\omega}(\Gamma') = |\mathrm{Aut}_v\Gamma|\,\frac{|\mathrm{Aut}_e\Gamma|\cdot r_m^{2n}[\Gamma]}{|\mathrm{Aut}\,\Gamma|} = r_m^{2n}[\Gamma].\end{aligned}$$

(2) Let $\widetilde{E}_i \stackrel{\mathrm{def}}{=} \iota_i^* E^\Gamma_{r_m,\ldots,r_m}$. One may check by a property of clasper that $\widetilde{E}_i$ is a trivial $(M^\bullet \text{ rel } \partial)$-bundle as an unframed one. Thus the image of $(S^{2m-2})^{\times 2n}$ in $B\mathrm{Diff}(M^\bullet \text{ rel } \partial)$ can be made into the one homotopy equivalent to $S^{4n(m-1)}$ by attaching $2n$ $(2m-1)$-cells along each S^{2m-2}-component of $\iota_1\vee\cdots\vee\iota_{2n} : S^{2m-2}\vee\cdots\vee S^{2m-2} \subset (S^{2m-2})^{\times 2n}$ and that the unframed $(M^\bullet \text{ rel } \partial)$-bundle structure extends over the resulting complex $X \simeq S^{4n(m-1)}$. So we need to consider the obstruction to extend the vertical framing on $E^\Gamma_{r_m,\ldots,r_m}$ over X. To do this, we consider the standardly vertically framed trivial $(M^\bullet \text{ rel } \partial)$-bundle $E_i^{\mathrm{cell}} \stackrel{\mathrm{def}}{=} M^\bullet \times D^{2m-1}$ over a $(2m-1)$-disk. Here we may assume that the vertical framing restricted to the boundary of $\widetilde{E}_i$, i.e., sub $\partial M^\bullet$-bundle of $\widetilde{E}_i$, coincides with that of $\partial E_i^{\mathrm{cell}}$ restricted to ∂D^{2m-1}. We consider the obstruction for the existence of the homotopy between the vertical framings of $\widetilde{E}_i$ and of $E_i^{\mathrm{cell}}|\partial D^{2m-1}$.

By the Poincaré-Lefschetz duality, we can show that

$$H_j(M^\bullet\times D^{2m-2}, \partial(M^\bullet\times D^{2m-2});\mathbb{Z}) \cong \begin{cases}\mathbb{Z} & \text{if } j = 6m-3\\ 0 & \text{otherwise}\end{cases}$$

where $D^{2m-2}\subset \partial D^{2m-1}$ is an embedded disk where the obstruction may be included. So the only obstruction may lie in

$$H^{6m-3}(M^\bullet\times D^{2m-2}, \partial(M^\bullet\times D^{2m-2}); \pi_{6m-3}SO_{4m-1}) \cong \pi_{6m-3}SO_{4m-1}$$

that is finite if m is even. So if we replace $E^\Gamma_{r_m,\ldots,r_m}$ with $E^\Gamma_{r_m,\ldots,p_m r_m,\ldots,r_m}$, where $p_m = o(\pi_{6m-3}SO_{4m-1})$, the vertical framing extends to $E^\Gamma_{r_m,\ldots,p_m r_m,\ldots,r_m}\cup_{E_i^{\mathrm{cell}}|\partial D^{2m-1}}(M^\bullet\times D^{2m-1})$. Therefore, the vertical framing on $E^\Gamma_{p_m r_m,\ldots,p_m r_m}$ can be extended over X.

Finally, by collapsing the attached $(2m-1)$-cells into the base point by a homotopy, we obtain a vertically framed bundle associated to a class in $\pi_{4n(m-1)}\widetilde{B\mathrm{Diff}}M$. Since $[E^{\Gamma}_{p_m r_m,\ldots,p_m r_m}] = p_m^{2n}[E^{\Gamma}_{r_m,\ldots,r_m}]$ in $\Omega_{4n(m-1)}(\widetilde{B\mathrm{Diff}}M)$ and the attaching of a $(2m-1)$-cell followed by the collapsing corresponds to a bordism in $\widetilde{B\mathrm{Diff}}M$, the result follows. $\square$

Proof of Corollary 4.5. In the light of Theorem 4.1(2), we can choose a framed $(D^{4m-1}$ rel $\partial)$-bundle $\mathrm{Hu}^{\Theta}_m \to S^{4m-4}$ for which

$$[\mathrm{Hu}^{\Theta}_m] = [E^{\Theta}_{p_m r_m, p_m r_m}] \text{ in } \Omega_{4m-4}(\widetilde{B\mathrm{Diff}}\, S^{4m-1}).$$

Now we shall compute the value $\hat{\zeta}_2(o(\Theta^{8m-5})\mathrm{Hu}^{\Theta}_m)$. Note that the closure of the total space of $o(\Theta^{8m-5})\mathrm{Hu}^{\Theta}_m$ is diffeomorphic to the standard $(8m-5)$-sphere. So the signature defect term vanishes for a choice of framing by means of the assumption for the choice of τ'_{E_D} that we have made in §3, and thus the framing correction term contributes just by an integer multiple of the jump (3.5).

By making use of Theorem 4.1 and (3.5), we obtain:

$$\hat{\zeta}_2(o(\Theta^{8m-5})\mathrm{Hu}^{\Theta}_m) = 12\, r_m^2 p_m^2 o(\Theta^{8m-5}) + \frac{a_{2m-1}(4m-3)!}{4} N_m$$

for some $N_m \in \mathbb{Z}$. (Here we ignore the number q_k in Theorem 3.2 (now $k = 2m$) because Hu^{Θ}_m is already vertically framed. The result differs only by overall multiple of q_k. So it causes no problem.) The last row is non-zero if $\frac{48 r_m^2 p_m^2 o(\Theta^{8m-5})}{a_{2m-1}(4m-3)!} \notin \mathbb{Z}$. This number is precisely the number of the statement for the following data: By Lemma 3.26, we can choose $r_m = 4\prod_{j=1}^{2m-2} o(\pi^S_j)$ and $p_m = o^{4m-1}_{6m-3} = 4\prod_{j=2}^{2m-1} o(\pi^S_j) = 2\prod_{j=1}^{2m-1} o(\pi^S_j)$ for even m. Further, it is known that

$$|\Theta^{8m-5}| = |\pi^S_{8m-5}|\, 2^{4m-6}\,(2^{4m-3}-1)\frac{B_{2m-1}a_{2m-1}}{2m-1}$$

(see Kervaire–Milnor [KM]). $\square$

Proof of Corollary 4.6. We prove that the number (4.1) is not integral when m satisfies the required properties. Since $4m-3$ is prime, $(4m-2)!$ has a prime factor $4m-3$. Since B_{2m-1} does not have a divisor $4m-3$ by the hypothesis, it is enough to prove that $4m-3 \nmid (2^{4m-3}-1)o(\pi^S_{8m-5})o(\pi^S_{2m-1})^2\prod_{\ell=1}^{2m-2} o(\pi^S_i)^4$. As in the proof of Corollary 3.6, one may see that Fact 3.29(1) implies $4m-3 \nmid 2^{4m-3}-1$. Further, by Theorem 3.28, $o(\pi^S_{8m-5})o(\pi^S_{2m-1})^2\prod_{\ell=1}^{2m-2} o(\pi^S_i)^4$ does not have the prime divisor $p = 4m-3$ because $2i(p-1)-1 = 8m-9, 16m-17, \ldots$ and $2m-1 < 8m-9 < 8m-5 < 16m-7$ for $m > 1$. This completes the proof. $\square$

Remark 4.18. Though we have proved the non-triviality of $\hat{\zeta}_2$ for some m, we can not still say that it gives an integral lift of λ'-invariant as in Corollary 3.5 because λ'-invariant of $\mathrm{cl}(o(\Theta^{8m-5})\mathrm{Hu}^{\Theta}_m)$ is trivial.

5. FURTHER DIRECTIONS

Now we shall briefly remark some directions expected to be studied after the present paper.

Conjecture 5.1. *For each $k \geq 3$, $(-1)^k b_{k-1}\hat{\zeta}_2(E)$ is an integral lift of a non-trivial invariant $c_{k-1}\lambda'(\mathrm{cl}(q_k E)) \in \mathbb{Z}_{b_{k-1}}$ where the notations are those of Corollary 3.5.*

If this conjecture is true, then some of the Antonelli–Burghelea–Kahn constructions turn out to have infinite order. The following conjecture implies Conjecture 5.1.

Conjecture 5.2. *For all $m \geq 2$, the numbers (3.2) are not integral.*

The following conjecture may be a slightly different direction (see Corollary 4.5), but seems easier.

Conjecture 5.3. *For all $m \geq 2$ even, the number (4.1) is not integral.*

Problem 5.4. Determine the subgroup $\mathrm{Im}\,(\mathrm{cl}) \subset \Theta^d$.

Fine partial results are obtained in [ABK, ABK2], but it still seems open. By Corollary 4.6, our bundle $\pi^{\Theta}_{1,1} : E^{\Theta}_{1,1} \to (S^{4m-2})^{\times 2}$ gives a non-trivial element of the group $\pi_{8m-4}B\mathrm{Diff}(D^{8m-1}\ \mathrm{rel}\ \partial)$. On the other hand, in 3-dimension, Habiro's graph clasper construction for the Θ-graph gives 2 times the boundary of the E_8-plumbing manifold. If a similar statement is also true in higher dimensional situations, then it may give a finer estimate of the order of $\mathrm{Im}\,(\mathrm{cl})$ since the boundary of the E_8-plumbing manifold gives a generator of the cyclic group $\Theta^{4t-1}(\partial\pi)$ (see [KM]). Of course, $\hat{\zeta}_2$-invariant of the two differ by some rational multiple. For this one might need to calculate λ'-invariant of $\mathrm{cl}(E^{\Theta}_{1,1})$.

Conjecture 5.5. *The images of the IHX and the AS relations under ψ_{2n} vanish.*

If this conjecture is true, then it suggests that the equivalence relation "bordant" corresponds to "equivalent modulo higher order elements" in 3-dimension.

In 3-dimension, there is an explicitly computable theory that is called the LMO invariant [LMO]. It is known that the LMO invariant is also a universal Ohtsuki finite type invariant [Le] and conjectured that the LMO invariant 'coincides' with the Kontsevich–Kuperberg–Thurston's configuration space invariant. To get an analogous computable theory as the LMO, one might need the following problem.

Problem 5.6. Give a smooth bundle analogue of the Kirby theorem [Kir].

In [MY], T. Moriyama developed some cobordism theory and obtained a $\mathbb{Q}$-valued invariant of rational homology 3-spheres, that is an integral lift of the Rokhlin invariant for integral homology spheres. It would be interesting to ask whether his theory is generalizable to higher dimensional sphere bundles considered in this paper.

APPENDIX A. THE CLOSED FORM α ON $C_2(M)$-BUNDLE

Let $\pi : E \to B$ be a vertically framed ($M^\bullet$ rel ∂)-bundle with the base space B a smooth manifold for which the de Rham theorem holds. The construction of the Kontsevich classes requires a 'fundamental' closed form α on the associated $C_2(M)$-bundle $C(\pi) : C(E) \to B$ to π. We shall give a proof that there exists such a well-defined closed form α, which is omitted in [Kon].

The Serre spectral sequence of the fibration

$$(C_2(M), \partial C_2(M)) \to (C(E), C_\partial(E)) \to B,$$

where $C_\partial(\pi) : C_\partial(E) \to B$ be the sub $\partial C_2(M)$-bundle of $C(\pi)$, gives the following

Lemma A.1. *There exists a spectral sequence with*

$$\begin{aligned} &E_2^{p,q} \cong H^p(B; \{H^q(C_2(M)_b, \partial C_2(M)_b; \mathbb{Z})\}_{b\in B}) \\ &\Rightarrow H^{p+q}(C(E), C_\partial(E); \mathbb{Z}). \end{aligned}$$

The following lemma can be proved by exactly the same way as [Les, Lemma 2.1].

Lemma A.2. $H_*(C_2(M); \mathbb{Z}) \cong H_*(S^{d-1}; \mathbb{Z})$.

Lemma A.3. *For any $b \in B$ and for $0 \le q \le d$, $H^q(C_2(M)_b, \partial C_2(M)_b; \mathbb{Z}) \cong 0$.*

Proof. In this proof, all the (co)homology coefficients are assumed in $\mathbb{Z}$. By the Poincaré-Lefschetz duality and Lemma A.2, we have

$$H^q(C_2(M)_b, \partial C_2(M)_b) \cong H_{2d-q}(C_2(M)) \cong H_{2d-q}(S^{d-1}) \cong 0 \ (0 \le q \le d).$$

□

Lemma A.4. *For $0 \le q \le d$, $H^q(C(E), C_\partial(E); \mathbb{Z}) \cong 0$.*

Proof. This follows immediately from Lemma A.1 and Lemma A.3. □

Lemma A.5. *The inclusion induces an isomorphism*

$$H^{d-1}(C(E); \mathbb{Z}) \cong H^{d-1}(C_\partial(E); \mathbb{Z}).$$

Proof. This follows from the cohomology exact sequence of the pair $(C(E), C_\partial(E))$ and from Lemma A.4. □

Since we have a closed $(d-1)$-form $\tau_E^*\omega_{S^{d-1}}$ on $C_\partial(E)$ uniquely determined by the framing, there exists a well-defined closed $(d-1)$-form α on $C(E)$ extending $\tau_E^*\omega_{S^{d-1}}$ by Lemma A.5 and by the de Rham theorem. Note that the vertical framing on $C(E)$ determines a trivial S^{d-1}-bundle structure on $C_\partial(E)$ and thus the closed $(d-1)$-form on $C(E)$ is non-trivial in cohomology.

APPENDIX B. PUSHFORWARD

Let $\pi : E \to B$ be a bundle with d-dimensional fiber F. Then the *push-forward* (or *integral along the fiber*) $\pi_*\omega$ of an $(d+p)$-form ω on E is a p-form on B defined by

$$\int_c \pi_*\omega = \int_{\pi^{-1}(c)} \omega,$$

where c is a p-dimensional chain in B.

Let $\pi^\partial : \partial_F E \to B$ be the restriction of π to ∂F-bundle with the orientation induced from $\mathrm{Int}(F)$, i.e., $O_{\partial F} = i(n)O_F$ where n is the in-going normal vector field over ∂F. Then the generalized Stokes theorem for the pushforward is

$$d\pi_*\omega = \pi_* d\omega + (-1)^{\deg \pi^\partial_* \omega} \pi^\partial_* \omega. \tag{B.1}$$

APPENDIX C. SIMULTANEOUS NORMALIZATION OF THE β_M-FORMS

Here we normalize the closed $(4m-2)$-form β_M on $C_2(M)$ for $(4m-1)$-dimensional homology spheres, based on the line of a part of [Les2, Proposition 3.3]. The proof below seems quite simpler than Lescop's one in 3-dimension, due to the fact that the involved surgery is restricted and that for higher dimension, some homology classes involved become different dimensional while for 3-dimension they are not (and the proof is surprisingly difficult). In this section, we denote the fiber of the base point $t^0 \in (S^{2m-2})^{\times 2n}$ of $E^\Gamma_{r_m,\ldots,r_m}$ simply by M. We identify a regular neighborhood of $\partial V_i \subset M$ with $[-4,4] \times \partial V_i$ and for $s \in [-4,4]$, set

$$V_i[s] \overset{\mathrm{def}}{=} \begin{cases} V_i \cup ([0,s] \times \partial V_i) & \text{if } s \geq 0 \\ V_i \setminus ((s,0] \times \partial V_i) & \text{if } s \leq 0 \end{cases}$$

Let $S(a^i_j) \subset V_i[4]$ and $S(b^i_k) \subset M \setminus \mathrm{Int}\,(V_i)$ be the $2m$-disks bounded by $4 \times a^i_j$ and b^i_k respectively, such that if $\mathrm{Lk}(a^i_j, a^{i'}_{j'}) = 1$ for $i \neq i'$, then $S(b^i_j) \cap V_{i'} = S(a^{i'}_{j'})$, and if $\mathrm{Lk}(a^i_j, a^{i'}_{j'}) = 0$, then $S(b^i_j) \cap V_{i'} = \emptyset$.

Let $\eta(b^i_j)$ be the closed $(2m-1)$-form supported in an ε-tubular neighborhood $N_\varepsilon S(b^i_j)$ of $S(b^i_j)$ which is restricted to the Thom class in $H^{2m-1}(N_\varepsilon S(b^i_j)_x, \partial(N_\varepsilon S(b^i_j)_x); \mathbb{R})$, $x \in S(b^i_j)$, and $\eta(a^i_j)$ is defined by the pullback by the inclusion $N_\varepsilon S(a^i_j) \to N_\varepsilon S(b^{i'}_{j'})$ for some i', j'.

Fix a base point p^i on ∂V_i and let $\omega(p^i)$ be a closed $(4m-2)$-form supported in a tubular neighborhood of the union of the path $[p^i, \infty]$ and $\partial C_1(M)$ such that it restricts as the usual volume form on $\partial C_1(M) = S^{4m-2}$ and such that the support is disjoint from all $V_i[4]$ and from all the supports of the above forms. First we shall normalize β_M on the subset $V_i \times (C_1(M) \setminus V_i[3]) \subset C_2(M)$.

Proposition C.1. *For any subset $N \subset \{1, \ldots, 2n\}$, we can choose β_M on $C_2(M)$ so that:*

(1) *For every $i \in N$, the restriction of β_M to $V_i \times (C_1(M) \setminus V_i[3]) \subset C_2(M)$ equals*

$$\sum_{(j,k)\in\{1,2,3\}^2} \mathrm{Lk}(b^i_j, a^i_k[4])\, p_1^*\eta(a^i_j) \wedge p_2^*\eta(b^i_k) + p_2^*\omega(p^i)$$

where $p_1, p_2 : C_2(M) \to C_1(M)$ denote the first and the second projection, respectively.

(2) *β_M is antisymmetric with respect to ι and fundamental, that is closed and $\beta_M|\partial C_2(M) = p_M^*\omega_{S^{4m-2}}$ where $p_M : \partial C_2(M) \to S^{4m-2}$ is the projection onto the S^{4m-2} factor determined by the framing.*

Assume Proposition C.1 for the moment. Let $E^\Gamma(i)$ be the pullback bundle from $E^\Gamma_{r_m,\ldots,r_m}$ by the inclusion $S^{2m-2} \hookrightarrow (S^{2m-2})^{\times 2n}$ and let $\widetilde{V}_i[s]$ be the sub $(V_i[s]$ rel $\partial)$-bundle of $E^\Gamma(i)$. We extend the forms $\eta(a^i_j)$ and $\eta(b^i_k)$ to the globally defined ones $\eta(a^i_j, t)$ and $\eta(b^i_k, t)$ on $\widetilde{V}_i[4]$ and $E^\Gamma(i) \setminus \mathrm{Int}(\widetilde{V}_i)$ respectively, as follows.

Observe that there exists a $(2m + 2m - 2 = 4m - 2)$-manifold $\widetilde{S}(a^i_j)$ included in $\widetilde{V}_i[4]$, bounded by $(4 \times a^i_j) \times S^{2m-2} \subset \partial\widetilde{V}_i[4]$, such that it restricts to $S(a^i_j)$ in the fiber of t^0. Indeed, the third component of the locus of the parametrized link of Observation 4.9 bounds a $(4m-2)$-disk if we ignore the other two components. This $(4m-2)$-disk can be considered as a collection of bounded $2m$-disks parametrized by $t \in S^{2m-2}$. So this collection can be suspended over S^{2m-2} with some intersections with the other components. Those intersections can be removed by suitable attachings of handles parallel to the other two components. The resulting $(4m-2)$-manifold is as required. Then $\eta(a^i_j, t)$ is defined as the restriction of the ε-Thom form over $\widetilde{S}(a^i_j)$ to the fiber of t . $\eta(b^i_k, t)$ may be naturally extended from $\eta(a^i_j, t)$'s by using $\eta(b^i_k)$'s.

For $I \subset \{1, \ldots, 2n\}$ and for $t \in (S^{2m-2})^{\times 2n}$ such that $I(t) \subset I$, define $\beta^0_{M_t}$ on

$$D_I(\beta^0_{M_t}) \stackrel{\mathrm{def}}{=} \big(C_2(M_t) \setminus \bigcup_{i\in I}(V_i[-1]_t \times V_i[3]_t) \cup (V_i[3]_t \times V_i[-1]_t)\big) \cup p_{12}^{-1}\Delta_{M_t\setminus\{\infty\}}$$

where $p_{12} : C_2(M_t) \to M_t \times M_t$ be the projection, so that

- $\beta^0_{M_t} = \beta_M$ on $C_2(M_t \setminus \cup_{i\in I} V_i[-1]_t) = C_2(M \setminus \cup_{i\in I} V_i[-1])$,
- $$\beta^0_{M_t} = \sum_{(j,k)\in\{1,2,3\}^2} \mathrm{Lk}(b^i_j, a^i_k[4])\, p_1^*\eta(a^i_j, t) \wedge p_2^*\eta(b^i_k, t) + p_2^*\omega(p^i)$$
  on $p_{12}^{-1}((V_i)_t \times (M_t \setminus V_i[3]_t))$ when $i \in I$.
- $\beta^0_{M_t} = -\iota^*\beta^0_{M_t}$ on $p_{12}^{-1}((M_t \setminus V_i[3]_t) \times (V_i)_t)$ when $i \in I$.
- $\beta^0_{M_t} = p_{M_t}^*\omega_{S^{4m-2}}$ on $\partial C_2(M_t)$.

Note that this condition is consistent. In particular, by Proposition 4.15, the first and the fourth conditions are compatible. Let $C(E^\Gamma(i)) \stackrel{\mathrm{def}}{=} \cup_{t=(t^0_1,\ldots,t_i,\ldots,t^0_{2n})} C_2(M_t)$ denote the $C_2(M)$-bundle over S^{2m-2} associated with $E^\Gamma(i)$. In the following we shall see that the form $\beta^0_{M_t}$ defined over the bundle $D(E^\Gamma(i)) \stackrel{\mathrm{def}}{=} \cup_{t=(t^0_1,\ldots,t_i,\ldots,t^0_{2n})} D_{\{i\}}(\beta^0_{M_t})$ extends to

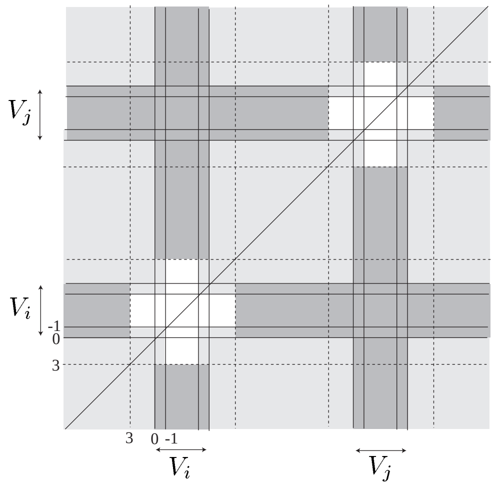


FIGURE 7. Area of $C_2(M)$ where β_M is normalized

a fundamental $(4m-2)$-form on $C(E^\Gamma(i))$. The homology of $D(E^\Gamma(i))$ up to dimension $(4m-2)$ is given by the following lemmas.

Lemma C.2. *There exists a homology spectral sequence with*

$$E^2_{p,q} \cong H_p(S^{2m-2}; H_q(D_{\{i\}}(\beta^0_M); \mathbb{R})) \Rightarrow H_{p+q}(D(E^\Gamma(i)); \mathbb{R})$$

such that $E^2_{p,q} = E^\infty_{p,q}$ *if* $p+q \leq 4m-2$ *and such that* $E^\infty_{p,q} = 0$ *if moreover* $p \notin \{0, 2m-2\}$ *or* $q \notin \{0, 2m, 4m-2\}$. *In particular,* $H_{4m-2}(D(E^\Gamma(i)); \mathbb{R}) = E^\infty_{0,4m-2} \oplus E^\infty_{2m-2,2m} = E^2_{0,4m-2} \oplus E^2_{2m-2,2m}$.

Lemma C.3. (1) $E^\infty_{2m-2,2m}$ *coincides with the kernel of the map induced by the inclusion*

$$H_{4m-2}(D(E^\Gamma(i)); \mathbb{R}) \to H_{4m-2}(C(E^\Gamma(i)); \mathbb{R}).$$

(2) $\beta^0_{M_t}$ *evaluated on* $E^\infty_{2m-2,2m}$ *vanishes.*

Proofs of Lemma C.2 and C.3 will be given later. It follows from these lemmas that the form $\beta^0(i)_t \stackrel{\mathrm{def}}{=} \beta^0_{M_t}$ $(t = (t^0_1, \ldots, t_i, \ldots, t^0_{2n}))$ on $D(E^\Gamma(i))$ is in the image of the map

$$H^{4m-2}(C(E^\Gamma(i)); \mathbb{R}) \to H^{4m-2}(D(E^\Gamma(i)); \mathbb{R}).$$

Namely, $\beta^0(i)$ extends to a closed form $\beta^1(i)$ on $C(E^\Gamma(i))$ by the de Rham theorem, and

$$\beta(i) \stackrel{\mathrm{def}}{=} \frac{\beta^1(i) - \iota^*\beta^1(i)}{2}$$

is a fundamental form.

For any $t \in (S^{2m-2})^{\times 2n}$, we define

$$\beta_{M_t} = \begin{cases} \beta^0_{M_t} & \text{on } C_2(M_t) \setminus \bigcup_{i \in I(t)} (V_i[-1]_t \times V_i[3]_t) \cup (V_i[3]_t \times V_i[-1]_t) \\ \beta(i)_t & \text{on } C_2(V_i[4]_t) \text{ for } i \in I(t) \end{cases}$$

Then β_{M_t} is the required form of Proposition 4.16.

Proof of Proposition C.1. We first prove the proposition for $N = \{1\}$. Let β_0 be a fundamental $(4m-2)$-form on $C_2(M)$ and let β be the closed $(4m-2)$-form on $V_1[1] \times (C_1(M) \setminus \operatorname{Int} V_1[2])$ defined by the statement. Since integrals for both β_0 and β coincide on $H_{4m-2}(V_1[1] \times (C_1(M) \setminus \operatorname{Int} V_1[2]); \mathbb{R})$, there exists a $(4m-3)$-form η on $V_1[1] \times (C_1(M) \setminus \operatorname{Int} V_1[2])$ such that

$$\beta = \beta_0 + d\eta.$$

Here we may assume that $\eta = 0$ on $V_1[1] \times \partial C_1(M)$ because η is closed on $V_1[1] \times \partial C_1(M)$ and hence exact there.

We further modify β so to coincide with β_0 outside $V_1[1] \times (C_1(M) \setminus \operatorname{Int} V_1[2])$. Let χ be a smooth function on $C_2(M)$ supported in $V_1[1] \times (C_1(M) \setminus \operatorname{Int} V_1[2])$, and constant equal to 1 on $V_1 \times (C_1(M) \setminus V_1[3])$. Then set

$$\beta_a \stackrel{\text{def}}{=} \beta_0 + d(\chi\eta).$$

β_a is as required on $V_1 \times (C_1(M) \setminus V_1[3])$ and coincides with β_0 on $\partial C_2(M)$ because $d(\chi\eta) = 0$ there.

Similar modification to β_a for $(C_1(M) \setminus V_1[3]) \times V_1$, that can be done disjointly from the previous ones, yields another $(4m-2)$-form β_b that is as required on

$$\partial C_2(M) \cup (V_1 \times (C_1(M) \setminus V_1[3])) \cup ((C_1(M) \setminus V_1[3]) \times V_1).$$

Thus $\beta_M \stackrel{\text{def}}{=} (\beta_b - \iota^*\beta_b)/2$ is the required form for $N = \{1\}$.

Now we prove the proposition for general N by induction on $|N| = i$. Let β_0 be the $(4m-2)$-form satisfying all the hypotheses for $N = \{1, \ldots, i-1\}$, and let β be the $(4m-2)$-form satisfying the hypotheses on $\{i\}$ obtained by the first step from β_0, replacing V_i with $V_i[1]$. Then there exists a $(4m-3)$-form η such that $\beta = \beta_0 + d\eta$ where η may be assumed to vanish on $\partial C_2(M)$ because $H^{4m-3}(\partial C_2(M); \mathbb{R}) = 0$.

Let χ be a smooth function χ supported in $V_i[1] \times (C_1(M) \setminus \operatorname{Int} V_i[2])$, that is constant equal to 1 on $V_i \times (C_1(M) \setminus V_i[3])$, and let $\beta_a \stackrel{\text{def}}{=} \beta_0 + d(\chi\eta)$. Then β_a is as required on

$$\partial C_2(M) \cup \bigcup_{k \in N} (V_k \times (C_1(M) \setminus V_k[3])) \cup \bigcup_{k \in N \setminus \{i\}} ((C_1(M) \setminus V_k[3]) \times V_k).$$

Still we need to prove that β_a is as required in $V_i[1] \times (\partial C_1(M) \cup \bigcup_{k=1}^{i-1} V_k)$, where the support of χ intersects the previous changes for β_0. By the assumptions, η may be assumed to vanish on $V_i[1] \times \partial C_1(M)$ and is closed on $V_i[1] \times V_k$ for $i \neq k$. Further by $H^{4m-3}(V_i[1] \times V_k; \mathbb{R}) = 0$, we may assume that η vanishes on $V_i[1] \times V_k$.

Finally, by similar modifications as in the first step, we can modify β_a so that it integrates correctly as required, and antisymmetric with respect to ι^*. $\square$

Proof of Lemma C.2. First we compute the homology of $D_{\{i\}}(\beta_M^0)$. For any submanifold X of M, we denote by STX the face of $\partial C_2(X)$ corresponding to the blow up along the main diagonal $\Delta_X \subset X^{\times 2}$. Since the inclusion from $D_{\{i\}}(\beta_M^0)$ to $(C_2(M) \setminus C_2(V_i[-1])) \cup STV_i$ is a homotopy equivalence, it suffices to compute the homology of the latter space.

Let $\overline{M} = C_1(M)$ and $V = V_i$. We compute the homology of $C_2(M) \setminus C_2(V) \simeq \breve{C}_2(M) \setminus \breve{C}_2(V)$ where $\breve{C}_2(X) \stackrel{\text{def}}{=} X^{\times 2} \setminus \{\text{diagonal}\}$.

Now we shall first compute the homology of $M^{\times 2} \setminus V^{\times 2}$. Observe that

$$H_*(\overline{M} \setminus V) = \begin{cases} \mathbb{R}[\partial\overline{M}] & \text{if } * = 4m-2 \\ \mathbb{R}[a_1^i[4]] \oplus \mathbb{R}[a_2^i[4]] \oplus \mathbb{R}[a_3^i[4]] & \text{if } * = 2m-1 \\ \mathbb{R}[\text{pt}] & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then the Mayer-Vietoris sequence involving the homology of $\overline{M}^{\times 2} \setminus V^{\times 2} = (\overline{M} \times (\overline{M} \setminus V)) \cup ((\overline{M} \setminus V) \times \overline{M})$ is as follows.

		$(\overline{M}\setminus V)^{\times 2}$		$\overline{M}\times(\overline{M}\setminus V)+(\overline{M}\setminus V)\times\overline{M}$		$\overline{M}^{\times 2}\setminus V^{\times 2}$
H_{4m-2}	$\to$	$\mathbb{R}[\partial\overline{M}\otimes 1]+\mathbb{R}[1\otimes\partial\overline{M}]$ $+\sum_{j,k}\mathbb{R}[\breve{a}_j^i\otimes\breve{a}_k^i]$	$\twoheadrightarrow$	$\mathbb{R}[\partial\overline{M}\otimes 1]+\mathbb{R}[1\otimes\partial\overline{M}]$	$\stackrel{0}{\to}$	?
$H_{4m-3\sim 2m}$	$\to$	0	$\to$	0	$\to$	?
H_{2m-1}	$\to$	$\sum_j(\mathbb{R}[1\otimes\breve{a}_j^i]+\mathbb{R}[\breve{a}_j^i\otimes 1])$	$\hookrightarrow$	$\sum_j(\mathbb{R}[1\otimes\breve{a}_j^i]+\mathbb{R}[\breve{a}_j^i\otimes 1])$	$\stackrel{0}{\to}$	?
$H_{2m-2\sim 1}$	$\to$	0	$\to$	0	$\to$	?
H_0	$\to$	$\mathbb{R}$	$\to$	$\mathbb{R}+\mathbb{R}$	$\to$	$\mathbb{R}$

Here $\breve{a}_j^i \stackrel{\text{def}}{=} a_j^i[4]$. Therefore the homology of $\overline{M}^{\times 2} \setminus V^{\times 2}$ of dimensions at most $(4m-2)$ is

$$H_*(\overline{M}^{\times 2} \setminus V^{\times 2}) = \begin{cases} 0 & \text{if } 1 \leq * \leq 4m-2 \\ \mathbb{R} & \text{if } * = 0 \end{cases}$$

The homology of $C_2(M) \setminus C_2(V)$ is computed by the exact sequence:

$$\to H_*(\breve{C}_2(\overline{M})\setminus\breve{C}_2(V)) \to H_*(\overline{M}^{\times 2}\setminus V^{\times 2}) \to H_*(\overline{M}^{\times 2}\setminus V^{\times 2}, \breve{C}_2(\overline{M})\setminus\breve{C}_2(V)) \to \cdots$$

By excision, we have

$$\begin{aligned} &H_*(\overline{M}^{\times 2} \setminus V^{\times 2}, \breve{C}_2(\overline{M}) \setminus \breve{C}_2(V)) \\ &\cong H_*((\overline{M} \setminus V) \times \mathbb{R}^{4m-1}, (\overline{M} \setminus V) \times (\mathbb{R}^{4m-1} \setminus \{0\})) \\ &\cong H_{*-(4m-1)}(\overline{M} \setminus V) \otimes H_{4m-2}(S^{4m-2}). \end{aligned}$$

In particular, $H_*(\overline{M}^{\times 2} \setminus V^{\times 2}, \breve{C}_2(\overline{M}) \setminus \breve{C}_2(V)) = 0$ for $0 \leq * \leq 4m-2$. Thus the above exact sequence turns out to be as follows and the homology of $\breve{C}_2(\overline{M}) \setminus \breve{C}_2(V)$ is determined up to dimension $(4m-3)$.

		$\check{C}_2(\overline{M})\backslash\check{C}_2(V)$		$\overline{M}^{\times 2}\backslash V^{\times 2}$		$(\overline{M}^{\times 2}\backslash V^{\times 2}, \check{C}_2(\overline{M})\backslash\check{C}_2(V))$
H_{4m-2}	$\to$	$?$	$\to$	0	$\to$	0
$H_{4m-3\sim 1}$	$\to$	0	$\to$	0	$\to$	0
H_0	$\to$	$\mathbb{R}$	$\to$	$\mathbb{R}$	$\to$	0

Then the homology of $C_2(M)\setminus C_2(V)\cup STV$ is computed as follows. Note that this space can be obtained by gluing $ST\overline{M}\cong\overline{M}\times S^{4m-2}$ and $C_2(M)\setminus C_2(V)$ along $ST(\overline{M}\setminus V)\cong(\overline{M}\setminus V)\times S^{4m-2}$. The Mayer-Vietoris sequence is as follows.

		$(\overline{M}\backslash V)\times S^{4m-2}$		$\overline{M}\times S^{4m-2}$ $+C_2(M)\backslash C_2(V)$		$C_2(M)\backslash C_2(V)\cup STV$
$H_{4m-3\sim 2m+1}$	$\to$	0	$\to$	0	$\to$	0
H_{2m}	$\to$	0	$\to$	0	$\to$	$?$
H_{2m-1}	$\to$	$\sum_j \mathbb{R}[\check{a}_j^i\otimes 1]$	$\to$	0	$\to$	0
$H_{2m-2\sim 1}$	$\to$	0	$\to$	0	$\to$	0
H_0	$\to$	$\mathbb{R}$	$\to$	$\mathbb{R}+\mathbb{R}$	$\to$	$\mathbb{R}$

Hence $H_*(D_{\{i\}}(\beta_M^0))$ vanishes at $*=1\sim 2m-1, 2m-2\sim 4m-3$. This shows that $E^2_{p,q}=0$ if $p+q\leq 4m-2$ and ($p\notin\{0,2m-2\}$ or $q\notin\{0,2m,4m-2\}$). Moreover, all differentials $E^r_{*,*}\to E^r_{*-r,*+r-1}$, $r\geq 2$ involving $E^r_{p,q}$ ($p+q\leq 4m-2$) are zero and hence $E^2_{p,q}=E^\infty_{p,q}$ there. □

Lescop's cycles $F(a)$**.** In order to prove Lemma C.3, we shall give a higher dimensional analogue of Lescop's cycles, which were constructed by Lescop in 3-dimension [Les2], to get the generator of $E^\infty_{2m-2,4m-2}(D(E^\Gamma(i)))$. Namely, for each $a=a^i_j$, we consider a $(4m-2)$-cycle $F(a)$ on the configuration space bundle $D(E^\Gamma(i))$ of the form:

$$
\begin{aligned}
F(a) &\stackrel{\mathrm{def}}{=}(C(a)\times S^{2m-2})\\
&\quad\cup -(\widetilde{S}(a)\widetilde{\times}(4\times p(a)))\cup -((4\times p(a))\widetilde{\times}\widetilde{S}(a))\\
&\quad\cup\,\mathrm{diag}(n)(\widetilde{S}(a))\qquad (p(a)\text{: base point of } a)
\end{aligned}
$$

where $\widetilde{S}(a)\widetilde{\times}(4\times p(a))\stackrel{\mathrm{def}}{=}\cup_t\{x_t\times(4\times p(a)_t)\,|\,x_t\in\widetilde{S}(a)_t\}$ and $(4\times p(a))\widetilde{\times}\widetilde{S}(a)$ is its symmetric. The other chains $C(a),\mathrm{diag}(n)(\widetilde{S}(a))$ involved are defined below.

First we choose a vector field n that is a section of the trivial S^{4m-2}-bundle $\widetilde{STV}[4]$ (the sub $STV[4]$-bundle of $\widetilde{V}[4]$) restricted to $\widetilde{S}(a)$ such that near $\partial\widetilde{S}(a)$ it is normal to $\widetilde{S}(a)$ and tangent to $\partial\widetilde{V}[4]$. Moreover we assume that the map

(C.1) $$(\widetilde{S}(a),\partial\widetilde{S}(a))\to(S^{4m-2},*)$$

which is a composition of the trivialization and the projection given by n to the S^{4m-2}-factor, has mapping degree 0 so that $F(a)$ represents a class in $E^\infty_{2m-2,4m-2}$. Then we introduce a local coordinate $a\times[0,1]\subset\partial V$ where the second coordinate determined by the direction of n.

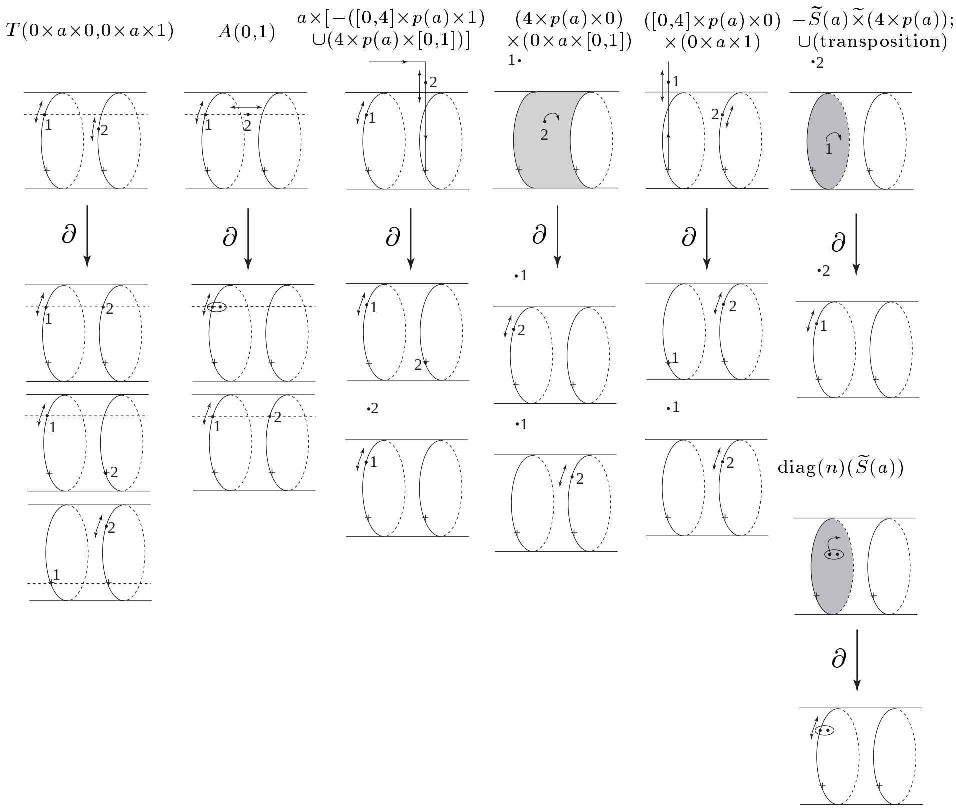


FIGURE 8. Lescop's cycle $F(a)$

$\underline{C(a)}$**:** The $C(a)$ is a $2m$-chain on $C_2([0,4]\times a\times[0,1])\subset C_2(M_t)\setminus C_2(V[-1]_t)\cup STV_t$ defined as a sum of the following chains:

- $T(0\times a\times 0, 0\times a\times 1)$
- $A(0,1)$
- $(0\times a\times 0)\times\big[-([0,4]\times p(a)\times 1)\cup(4\times p(a)\times[0,1])\big]$
- $\big((4\times p(a)\times 0)\times(0\times a\times[0,1])\big)\cup\big(([0,4]\times p(a)\times 0)\times(0\times a\times 1)\big)$

To describe $T(0\times a\times 0, 0\times a\times 1)$ and $A(0,1)$, we identify S^{2m-1} with the reduced suspension $\Sigma S^{2m-2}=(S^{2m-2}\times I)/(S^{2m-2}\times\{0,1\}\cup\{\infty\}\times I)$, $I=[-1,1]$ and introduce a corresponding coordinate $(x,z)\in S^{2m-2}\times I$. Then we consider the $2m$-dimensional submanifold T of $(S^{2m-2}\times I)\times(S^{2m-2}\times I)$ defined by

$$T\stackrel{\text{def}}{=}\{(x,z)\times(x,z')\,|\,x\in S^{2m-2}, z,z'\in I,\ z\geq z'\}\subset(S^{2m-2}\times I)\times(S^{2m-2}\times I)$$

with

$$\partial T = \{(x,z)\times(x,z)\}\cup\{(x,1)\times(x,z)\}\cup\{(x,z)\times(x,-1)\}.$$

Consider a pair of parallel cycles $0\times a\times 0$ and $0\times a\times 1$ and identify $(0\times a\times 0)\times(0\times a\times 1)$ by the base point preserving ($p(a)\leftrightarrow\{\infty\}$) diffeomorphism

$$\varphi:(0\times a\times 0)\times(0\times a\times 1)\overset{\sim}{\to} S^{2m-1}\times S^{2m-1}\overset{\mathrm{pr}}{\leftarrow}(S^{2m-2}\times I)\times(S^{2m-2}\times I).$$

Then we set

$$T(0\times a\times 0, 0\times a\times 1)\overset{\mathrm{def}}{=}\varphi^{-1}\mathrm{pr}(T)$$

$$A(0,1)\overset{\mathrm{def}}{=}\overline{\{(x\times 0)\times(x\times s)\,|\,x\in a, s\in(0,1]\}}$$
$$\subset C_2(M_t)$$

Note that $\mathrm{pr}(T)$ has the boundary of the form

$$\Delta_{S^{2m-1}}\cup(\{\infty\}\times S^{2m-1})\cup(S^{2m-1}\times\{\infty\})\subset S^{2m-1}\times S^{2m-1}.$$

$\underline{\mathrm{diag}(n)(\widetilde{S}(a))}$**:** The chain $\mathrm{diag}(n)(\widetilde{S}(a))$ denotes the image of $\widetilde{S}(a)$ in the trivial S^{4m-2}-bundle $\widetilde{STV}[4]$ under the section n.

See Figure 8 for the form of $F(a)$. Lemma C.3 follows from Lemma C.4 and C.5 described in the following.

Lemma C.4. (1) $[F(a)]$ *spans* $E^{\infty}_{2m-2,2m}(D(E^{\Gamma}(i)))$.
(2) $F(a)$ *is null in* $H_{4m-2}(C(E^{\Gamma}(i));\mathbb{R})$.

Proof. According to the proof of Lemma C.2 and from the definition of $F(a)$, the image of $[F(a)_t]$ under the Mayer-Vietoris boundary homomorphism is $[\check{a}\otimes 1]$ in $ST(\overline{M}\setminus V_i)$ and moreover the collection of this element over the S^{2m-2} is $[\check{a}\otimes S^{2m-2}]$ in $H_{2m-2}(S^{2m-2};H_{2m-1}(ST(\overline{M}\setminus V_i)))$. Hence $[F(a)]$ spans $E^{\infty}_{2m-2,2m}(D(E^{\Gamma}(i)))$.

The second assertion follows from the naturality of the Serre spactral sequences (see e.g., [HatSS]). Namely, together with Lemma C.2, it implies that there are homomorphisms between $E^{\infty}_{*,*}$'s induced by the inclusion

$$E^{\infty}_{0,4m-2}(D(E^{\Gamma}(i)))\to E^{\infty}_{0,4m-2}(C(E^{\Gamma}(i)))$$
$$E^{\infty}_{2m-2,2m}(D(E^{\Gamma}(i)))\to E^{\infty}_{2m-2,2m}(C(E^{\Gamma}(i)))=0$$

which is isomorphism on $E^{\infty}_{0,4m-2}$ and is zero map on $E^{\infty}_{2m-2,2m}$. □

Lemma C.5. *The* $(4m-2)$*-form* $\beta^0_{M_t}$ *on* $D(E^{\Gamma}(i))$ *evaluated on any cycle of* $E^{\infty}_{2m-2,2m}(D(E^{\Gamma}(i)))$ *vanishes.*

Proof. We prove that

$$\int_{F(a)}\beta^0_{M_t}=0.$$

First extend the form β_M on $C_2(M)$ obviously to a fundamental $(2m-2)$-form on the trivial bundle $C_2(M)\times S^{2m-2}$ and denote it also by β_M. We have $\int_{C(a)\times S^{2m-2}}\beta^0_{M_t}=$

$\int_{C(a)\times S^{2m-2}} \beta_M = 0$ since $C(a)$ lives inside $C_2([0,4]\times a\times[0,1]) \subset C_2(M_t)$ where $\beta^0_{M_t}$ and β_M coincide.

The normalization of Proposition C.1 and the partial extension followed imply that the integrals vanish on

$$-(\widetilde{S}(a)\widetilde{\times}(4\times p(a))) \cup -((4\times p(a))\widetilde{\times}\widetilde{S}(a)).$$

Since $F(a)$ is null homologous in $C(E^\Gamma(i))$ by Lemma C.4, it is enough to prove that

$$\int_{\mathrm{diag}(n_0)(\widetilde{S}_0(a))} \beta_M = \int_{\mathrm{diag}(n)(\widetilde{S}(a))} \beta^0_{M_t}$$

where $\widetilde{S}_0(a)$ is any embedding of a $(4m-2)$-manifold diffeomorphic to $\widetilde{S}(a)$ into the trivial sub bundle $V[4]\times S^{2m-2}$ of $M\times S^{2m-2}$ with the same behavior as $\widetilde{S}(a)$ near $\partial V\times S^{2m-2}$, and n_0 is any vector field on $\widetilde{S}_0(a)$ tangent to the fibers of $V[4]\times S^{2m-2}$ which coincides with n near the boundary and which satisfies the same constraint as n on mapping degree of the map (C.1). Then the relative-∂ homology classes of the images of the sections n and n_0 coincide and hence the integrals also coincide. $\square$

Acknowledgments. The author would like to thank Professor Tomotada Ohtsuki for useful comments and encouragements. The author would also like to thank Professors Kazuo Habiro and Tetsuhiro Moriyama for comments and for valuable discussions.

REFERENCES

[ABK] P. L. Antonelli, D. Burghelea, P. J. Kahn, *The non-finite homotopy type of some diffeomorphism groups*, Topology **11** (1972), pp. 1–49.

[ABK2] P. L. Antonelli, D. Burghelea, P. J. Kahn, *Gromoll groups,* $\mathrm{Diff}\, S^n$ *and bilinear constructions of exotic spheres*, Bull. Amer. Math. Soc. **76** (1970), p. 772–777.

[BN] D. Bar-Natan, *Some computations related to Vassiliev invariants*, electronic publication, `http://www.math.toronto.edu/~drorbn`.

[BC] R. Bott, A. Cattaneo, *Integral invariants of 3-manifolds*, J. Diff. Geom. **48** (1998) 91–133.

[BM] R. Bott, J. Milnor, *On the parallelizability of the spheres*, Bull. Amer. Math. Soc. **64** (1958), 87–89.

[Coh] F. R. Cohen, *The homology of* C_{n+1}*-spaces,* $n \geq 0$, from: "The homology of iterated loop spaces", Lecture Notes in Math. **533**, Springer-Verlag (1976) 207–351.

[CCL] A. Cattaneo, P. Cotta-Ramusino, R. Longoni, *Configuration spaces and Vassiliev classes in any dimension*, Algebr. Geom. Topol. **2** (2002), no. 39, pages 949–1000.

[CT] F. R. Cohen, L. R. Taylor, *Computations of Gelfand-Fuks cohomology, the cohomology of function spaces, and the cohomology of configuration spaces*, in Lecture Notes in Math. **657**, Springer-Verlag (1978) 106–143.

[FH] F. T. Farrell, W. C. Hsiang, *On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds*, Proc. Sympos. in Pure Math. **32** (1978) 325–337.

[FM] W. Fulton, R. MacPherson, *A Compactification of Configuration Spaces*, Ann. of Math. **139** (1994) 183–225.

[Hab] K. Habiro, *Claspers and the Vassiliev skein modules*, Ph. D. thesis, University of Tokyo (1997).

[Hab2] K. Habiro, *Claspers and finite type invariants of links*, Geom. Topol., **4** (2000) 1–84.

[Hat] A. Hatcher, *A proof of a Smale conjecture,* $\mathrm{Diff}(S^3) \simeq O(4)$, Ann. of Math. (2) **117** (1983), no.3, 553–607.

[HatSS] A. Hatcher, *Spectral Sequences in Algebraic Topology*, `http://www.math.cornell.edu/~hatcher/`

[KM] M. Kervaire, J. Milnor, *Groups of homotopy spheres: I*, Ann. of Math. **77** (1963), 504–537.

[Kir] R. C. Kirby, *A calculus for framed links in S^3*, Invent. Math. **45** (1978) 35–56.

[Kon] M. Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992), Birkhäuser, 1994, 97–121.

[Kon2] M. Kontsevich, *Vassiliev's knot invariants*, Adv. Soviet Math., **16**(2) (1993), 137–150.

[KT] G. Kuperberg, D. Thurston, *Perturbative 3-manifold invariants by cut-and-paste topology*, preprint, math.GT/9912167.

[Igu] K. Igusa, *Higher Franz-Reidemeister Torsion*, Studies in Adv. Math. **31** (2002) AMS/IP.

[Le] T. T. Q. Le, *An invariant of integral homology 3-sphere which is universal for all finite type invariants*, in "Soliton, Geometry and Topology: On the Crossroad", AMS Transl. ser. 2 **179** (1997), Eds. V. Buchstaber and S. Novikov, 75–100.

[LMO] T. T. Q. Le, J. Murakami, T. Ohtsuki, *On a universal perturbative invariant of 3-manifolds*, Topology **37** (1998) 539–574.

[Les] C. Lescop, *On the Kontsevich-Kuperberg-Thurston construction of a configuration-space invariant for rational homology 3-spheres*, math.GT/0411088, Prépublication de l'Institut Fourier 655 (2004), `http://www-fourier.ujf-grenoble.fr/prepublications.html`

[Les2] C. Lescop, *Splitting formulae for the Kontsevich-Kuperberg-Thurston invariant of rational homology 3-spheres*, math.GT/0411431, Prépublication de l'Institut Fourier 656 (2004), `http://www-fourier.ujf-grenoble.fr/prepublications.html`

[Lun] A. T. Lundell, *The embeddings $O(n) \subset U(n)$ and $U(n) \subset Sp(n)$ and a Samelson product*, Michigan Math. J. **13** (2) (1966) 133–145.

[Mic] P. Michor, *Manifolds of differentiable mappings*, Shiva Mathematics Series 3, Shiva Publ., Orpington, (1980), iv+158 pp, `http://www.mat.univie.ac.at/~michor/listpubl.html`

[Mic2] P. Michor, *Manifolds of smooth mappings IV: The theorem of De Rham*, Cahiers Topol. Geo. Diff., **24** (1983), 57–86, `http://www.mat.univie.ac.at/~michor/listpubl.html`

[Mil] J. Milnor, *Differential structures on spheres*, Amer. J. Math. **81** (1959), 962–971.

[Mil2] J. Milnor, *Differentiable manifolds which are homotopy spheres*, (mimeographed), Princeton University, 1959.

[MK] J. Milnor, M. Kervaire, *Bernoulli numbers, homotopy groups and a theorem of Rohlin*, Proceedings of the International Congress of Mathematicians, Edinburgh, 1958.

[MS] J. Milnor, J. Stasheff, *Characteristic classes*, Ann. of Math. Stud., No. 76, Princeton Univ. Press, 1974.

[Mo] S. Morita, *Casson invariant, signature defect of framed manifolds and the secondary characteristic classes of surface bundles*, J. Diff. Geom. **47** (1997) 560–599.

[Mo2] S. Morita, *Geometry of Characteristic Classes*, Iwanami Ser. in Modern Math, AMS Transl. of Math. Monogr. Vol. 199 2001.

[MY] T. Moriyama, *Casson–Walker invariant and the signature of 4-manifolds: I*, preprint.

[Nov] S. P. Novikov, *Differentiable sphere bundles*, Izv. Akad. Nauk SSSR Mat **29** (1965), 1–96 (Amer. Math. Soc. Transl, Ser. 2, Vol. 63).

[Oh] T. Ohtsuki, *Finite type invariants of integral homology 3-spheres*, J. Knot Theory and its Ramifications **5** (1996) 101–115.

[Sm] S. Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621–626.

[T] D. Thurston, *Integral expressions for the Vassiliev knot invariants*, A.b. thesis, Harvard University, 1995, arXiv:math.QA/9901110.

[To] H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Stud., No. 49, Princeton Univ. Press (1963).

[W] T. Watanabe, *Higher dimensional claspers and nilpotent manifolds*, in preparation.

[W2] T. Watanabe, *Clasper-moves among ribbon 2–knots characterizing their finite type invariants*, J. Knot Theory Ramifications, **15**, No. 9 (2006) 1163–1199.

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO, JAPAN
E-mail address: tadayuki@kurims.kyoto-u.ac.jp