Nichols-Woronowicz model of coinvariant algebra of complex reflection groups

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Dedicated to Professor Shoji on the occasion of his 60th birthday

Abstract

We give a model of the coinvariant algebra of the complex reflection groups as a subalgebra of a braided Hopf algebra called Nichols-Woronowicz algebra.

Introduction

Let V be a finite dimensional complex vector space. A finite subgroup $G \subset GL(V)$ is called *a complex reflection group*, if G can be generated by the set of pseudo-reflections, i.e., transformations that fix a complex hyperplane in V pointwise. Any real reflection group becomes a complex reflection group if one extends the scalars from $\mathbb{R}$ to $\mathbb{C}$. In particular all Coxeter groups give examples of complex reflection groups. We refer the reader to [3] for general background of the theory of complex reflection groups. Below we recall a few facts about real and complex reflection groups which appeared to be a motivation for our paper.

In 1954, G. C. Shephard and J. A. Todd [16] had obtained a complete classification of finite irreducible complex reflection groups. They found that there exist an infinite family of irreducible complex reflection groups $G(e,p,n)$ depending on three positive integer parameters (with p dividing e), and 34 exceptional groups $G_4, \ldots, G_{37}$. The group $G(e,p,n)$ has the order $e^n n!/p$. It also has a normal abelian subgroup of order e^n/p, and the corresponding quotient is the symmetric group on n points. The family of groups $G(e,p,n)$ includes the cyclic group $C_{e/p}$ of order e/p, namely, $C_{e/p} = G(e,p,1)$; the symmetric group on n points $S_n = G(1,1,n)$; the Weyl groups of types B_n, C_n, and D_n, namely, $W_{B_n} = W_{C_n} = G(2,1,n)$ and $W_{D_n} = G(2,2,n)$; and the dihedral groups $I_2(e) = G(e,e,2)$.

The fundamental fact characterizing the finite complex reflection subgroups in $GL(V)$ is the following theorem by G. C. Shephard and J. A. Todd.

Theorem (Shephard-Todd [16]) *A subgroup $G \subset GL(V)$ is a finite complex reflection group if and only if the subring P^G of the G-invariant elements in the symmetric algebra $P = S(V)$ of the space V is generated by n algebraically independent homogeneous elements.*

On the other side, in the case of Coxeter groups that form a part of the complex reflection groups, there is a remarkable result by C. F. Dunkl which states that the algebra generated by the truncated Dunkl operators is isomorphic to the coinvariant algebra of the corresponding Coxeter group. An analogue of Dunkl operators for finite complex reflection groups have been introduced by C. F. Dunkl and E. M. Opdam [5]. So it seems an interesting problem to extend the result by C. F. Dunkl mentioned above, to the case of complex reflection groups.

It is well-known that the cohomology ring of a flag variety has a presentation as the coinvariant algebra of the corresponding Weyl group. Some combinatorial problems on the intersection theory over flag varieties can be formulated for the coinvariant algebra of a finite Coxeter group [7]. In view of Shephard and Todd's theorem, the coinvariant algebra of a finite complex reflection group gives a natural generalization of the framework where one can study problems related to the Schubert calculus, see e.g. [17].

S. Fomin and the first author [6] have given a model of the cohomology ring of the flag variety of type A as a commutative subalgebra in a certain noncommutative quadratic algebra. Their construction has applications to Pieri's formula, quantization and so on [6], [13]. Similar construction for other root systems has been given in [9]. Yu. Bazlov [2] has realized the coinvariant algebra of a finite Coxeter group as a commutative subalgebra in a braided Hopf algebra, called the Nichols-Woronowicz algebra, to give a new mode of thought on the construction in [6]. The quantization operator on the Nichols-Woronowicz algebra and the model of the quantum cohomology ring of the flag varieties are given in [10].

The Nichols-Woronowicz algebra $\mathcal{B}(M)$, which is called the Nichols algebra in [1], associated to a braided vector space M is a braided graded Hopf algebra characterized by the following condition which appeared originally in the work of W. D. Nichols [12]:
(1) $\mathcal{B}^0(M) = \mathbb{C}$,
(2) $\mathcal{B}^1(M) = M = \{\text{primitive elements in } \mathcal{B}(M)\}$,
(3) $\mathcal{B}^1(M)$ generates $\mathcal{B}(M)$ as an algebra.
It is known that the algebra $\mathcal{B}(M)$ has an alternative definition as the braided analogue of the symmetric (or exterior) algebras introduced by S. L. Woronowicz [18] for the study of differential calculus on quantum groups, see [15].

In the present paper, we give a generalization of Bazlov's construction to the case of finite complex reflection groups. Having this aim in mind, we define the Yetter-Drinfeld module M_G corresponding to a finite complex reflection group

G. It is similar to the case of finite Coxeter groups that a linear basis of the Yetter-Drinfeld module M_G is parametrized by the set of pseudo-reflections in the group G. However, the G-grading and the G-module structure on M_G essentially depend on the properties of the hyperplane arrangement $\mathcal{A} = \mathcal{A}_G$ consisting of the reflection hyperplanes corresponding to the group G, see Section 2 for details. With the Yetter-Drinfeld module M_G in hand, the construction of the Woronowicz symmetrizers σ_n and the corresponding Nichols-Woronowicz algebra $\mathcal{B}(M_G)$ is done in the standard manner, see [2], [11]. In Proposition 3.2 we compute the set of quadratic relations in the algebra $\mathcal{B}(M_G)$ in the case $G = G(e, 1, n)$. In Section 4 we construct a realization of the coinvariant algebra P_G of a finite complex reflection group G as a commutative subalgebra in the corresponding Nichols-Woronowicz algebra $\mathcal{B}(M_G)$. The basis for our construction is the definition of the $\mathbb{C}$-linear map μ, see Definition 4.1. The map μ can be treated as "a truncated version" of the Dunkl operators for complex reflection groups introduced in [5]. Section 4 contains the main result of our paper, Theorem 4.1, which states that the subalgebra of $\mathcal{B}(M_G)$ generated by the image of the map μ is isomorphic to the coinvariant algebra of a finite complex reflection group in question. Note that there is a duality between the corresponding NilCoxeter and coinvariant algebras in the case of finite real reflection groups, see e.g. [2]. In Section 5 we study an analogue of such a duality for the group $G(e, 1, n)$. Our results in Section 5 essentially depend on those obtained in [14].

1 Coinvariant algebra of complex reflection group

Let G be a finite complex reflection group and V the reflection representation of G. We fix a G-invariant hermitian inner product $\langle\ ,\ \rangle$ on V. Let $\mathcal{A}$ be the set of reflection hyperplanes $H \subset V$ of G. The stabilizer $G_H \subset G$ of $H \in \mathcal{A}$ is isomorphic to a finite cyclic group $\mathbb{Z}/e_H\mathbb{Z}$, $e_H \in \mathbb{Z}_{>0}$. Each element $g \in G_H$ acts on V as a complex reflection with respect to H. We can assume that for $g \in G_H$ and $\xi \in V$ the action of g is of form

$$g(\xi) = \xi - (1 - \zeta)\frac{\langle \xi, v_H \rangle v_H}{\|v_H\|^2},$$

where v_H is a normal vector to H and ζ is some e_H-th root of the unity. Denote by χ_H the character of the cyclic group G_H defined as a restriction of $\det(g; V)$ to G_H. For $H \in \mathcal{A}$, there exists a unique element $g_H \in G_H$ such that $\chi_H(g) = \exp(2\pi\sqrt{-1}/e_H)$.

Consider the symmetric algebra $S(V) = \mathrm{Sym}_{\mathbb{C}} V$ of V. The G-invariant subalgebra $S(V)^G$ is generated by algebraically independent homogeneous elements $f_1, \ldots, f_r$, $r = \dim V$, by Shephard and Todd's theorem [16]. The coinvariant algebra P_G is the quotient algebra of $S(V)$ by the ideal I_G generated by the fundamental G-invariants $f_1, \ldots, f_r$. It has been shown by Chevalley [4] that the algebra P_G is isomorphic to the regular representation $\mathbb{C}\langle G\rangle$ as a left G-module.

Let us fix a set of vectors $\{v_H\}_{H\in\mathcal{A}}$ such that $v_H \in H^\perp \setminus \{0\}$. Then the action of $g \in G$ can be written as $g(v_H) = \lambda(g, H)v_{gH}$ for a constant $\lambda(g, H) \in \mathbb{C}^\times$ determined by g and H. The constants $\lambda(g, H)$ satisfy $\lambda(\mathrm{id}, H) = 1$ and the cocycle condition $\lambda(gg', H) = \lambda(g, g'H)\lambda(g', H)$. The constants $\{\lambda(g, H)\}_{g\in G, H\in\mathcal{A}}$ determine an element in $H^1(G, (\mathbb{C}^\times)^{\mathcal{A}})$. The family $\{\alpha_H\}_{H\in\mathcal{A}}$ of defining linear forms of the reflection hyperplanes is also determined by $\alpha_H(x) = \langle x, v_H\rangle$. Note that $g^*\alpha_H = \overline{\lambda(g^{-1}, H)}\alpha_{g^{-1}H}$ for $g \in G$.

Definition 1.1 *We define the divided difference operators $\Delta_{H,k} : S(V) \to S(V)$ as a $\mathbb{C}$-linear map defined by the formula*

$$\Delta_{H,k}(f) = \frac{f - g_H^k(f)}{v_H}$$

for $H \in \mathcal{A}$ and $1 \le k \le e_H - 1$.

The divided difference operators $\Delta_{H,k}$ satisfy the twisted Leibniz rule:

$$\Delta_{H,k}(f_1 f_2) = \Delta_{H,k}(f_1)f_2 + g_H^k(f_1)\Delta_{H,k}(f_2).$$

It is easy to see the following.

Lemma 1.1 *A polynomial $f \in S(V)$ is a G-invariant if and only if $\Delta_{H,k}(f) = 0$ for any $H \in \mathcal{A}$ and $1 \le k \le e_H - 1$.*

2 Nichols-Woronowicz algebra over complex reflection group

In this section we introduce the Nichols-Woronowicz algebra associated to a Yetter-Drinfeld module M_G over the complex reflection group G. In general, the Yetter-Drinfeld module over a finite group Γ is defined as follows:

Definition 2.1 *A vector space M is called a Yetter-Drinfeld module over Γ, if the following conditions are satisfied:*
(1) *M is a Γ-module,*
(2) *M is Γ-graded, i.e. $V = \bigoplus_{g\in\Gamma} M_g$, where M_g is a linear subspace of M,*
(3) *for $h \in \Gamma$ and $v \in M_g$, $h(v) \in M_{hgh^{-1}}$.*

Note that the category ${}^{\Gamma}_{\Gamma}YD$ of the Yetter-Drinfeld modules over a fixed finite group Γ is naturally braided by the braiding

$$\begin{array}{rccc} \psi_{M_1,M_2}: & M_1 \otimes M_2 & \to & M_2 \otimes M_1 \\ & x \otimes y & \mapsto & g(y) \otimes x, \end{array}$$

where $M_1, M_2 \in {}^{\Gamma}_{\Gamma}YD$ and $x \in (M_1)_g$.

Fix a set of normal vectors v_H, $H \in \mathcal{A}$. Then we can consider the corresponding constants $\lambda(g,H)$, $g \in G$, $H \in \mathcal{A}$. Let M_G be a $\mathbb{C}$-vector space generated by the symbols $[H;k]$, $H \in \mathcal{A}$ and $1 \le k \le e_H - 1$.

Definition 2.2 *We define a structure of the Yetter-Drinfeld module over G on the space M_G as follows:*
(i) (G-action) $g([H;k]) = \overline{\lambda(g,H)}^{-1}[gH;k]$,
(ii) (G-grading) $\deg_G([H;k]) = g^k_H$.

Lemma 2.1 *The G-action and G-grading defined above satisfy the condition for the Yetter-Drinfeld module, i.e.,* $\deg_G(h([H;k])) = hg^k_H h^{-1}$ *for* $h \in G$.

Proof. Since

$$g(h([H;k])) = g(\overline{\lambda(h,H)}^{-1}[hH;k]) = \overline{\lambda(g,hH)}^{-1}\overline{\lambda(h,H)}^{-1}[ghH;k]$$

$$= \overline{\lambda(gh,H)}^{-1}[ghH;k] = (gh)([H;k]),$$

the formula in (i) defines a G-action on M_G. Let us check the condition (2). From the definition of the G-action, we have

$$\deg_G(h([H;k])) = \deg_G(\overline{\lambda(h,H)}^{-1}[hH;k]) = \deg_G([hH;k]) = g^k_{hH} = hg^k_H h^{-1}.$$

Remark 2.1 Our definition of the Yetter-Drinfeld module M_G is analogous to the construction for the Coxeter group given in [11, Section 5]. In the case of finite Coxeter groups, we can choose the constants $\lambda(g,H)$ to take the values ± 1 by the normalization $\|v_H\| = 1$ as in the construction of the Yetter-Drinfeld module V_W used in [2]. However, it is essential to specify the cocycle $\{\lambda(g,H)\}$ in our case because of the appearance of the multiplication by some root of the unity.

For a braided vector space M with a braiding $\Psi : M \otimes M \to M \otimes M$, consider the linear endomorphism Ψ_i on $M^{\otimes n}$ obtained by applying the braiding $\Psi : M \otimes M \to M \otimes M$ on the i-th and $(i+1)$-st components of $M^{\otimes n}$. The endomorphisms Ψ_i satisfy the braid relation $\Psi_{i+1}\Psi_i\Psi_{i+1} = \Psi_i\Psi_{i+1}\Psi_i$. Denote by s_i the simple transposition $(i, i+1) \in S_n$. For any reduced expression $w = s_{i_1} \cdots s_{i_l} \in S_n$, the

endomorphism $\Psi_w := \Psi_{i_1} \cdots \Psi_{i_l} : M^{\otimes n} \to M^{\otimes n}$ is well-defined. The Woronowicz symmetrizer ([18]) is given by

$$\sigma_n := \sum_{w \in S_n} \Psi_w.$$

Definition 2.3 *The Nichols-Woronowicz algebra associated to a braided vector space M is*

$$\mathcal{B}(M) := \bigoplus_{n \geq 0} M^{\otimes n} / \mathrm{Ker}(\sigma_n),$$

where $\sigma_n : M^{\otimes n} \to M^{\otimes n}$ is the braided symmetrizer.

The braided vector space M naturally acts on $\mathcal{B}(M^*)$ from the right via the right braided derivations $\overleftarrow{D}_x$, $x \in M$. When $\Psi^{-1}_{M,T(M^*)}(\psi \otimes x) = \sum_i x_i \otimes \psi_i$, denote by $\Psi^{-1}_{M,T(M^*)}(\psi \otimes \overleftarrow{D}_x)$ the operator $\phi \mapsto \sum_i (\phi \overleftarrow{D}_{x_i})\psi_i$. The operators $\overleftarrow{D}_x$ are determined by the braided Leibniz rule

$$(\phi\psi)\overleftarrow{D}_x = \phi(\psi \overleftarrow{D}_x) + \phi \Psi^{-1}_{M,T(M^*)}(\psi \otimes \overleftarrow{D}_x),$$

and the condition $\varphi \overleftarrow{D}_x = \varphi(x)$, $\varphi \in M^*$, $x \in M$, see [2, 2.5]. In the subsequent construction, we identify the Yetter-Drinfeld module M_G with its dual M_G^* via the G-invariant symmetric inner product on M_G given by $([H;k],[H';k']) = \delta_{H,H'}\delta_{k,k'}$. In our case, we have $\Psi_{M_G,T(M_G)}([H;k] \otimes \phi) = g_H^k(\phi) \otimes [H;k]$. Hence, the braided Leibniz rule can be written as

$$(\phi\psi)\overleftarrow{D}_{H,k} = \phi(\psi \overleftarrow{D}_{H,k}) + (\phi \overleftarrow{D}_{H,k}) g_H^{-k}(\psi).$$

Lemma 2.2 ([2, Criterion 3.2])

$$\bigcap_{H \in \mathcal{A}, 1 \leq k \leq e_H - 1} \mathrm{Ker}(\overleftarrow{D}_{H,k}) = \mathcal{B}^0(M_G)$$

The linear map

$$\begin{array}{rccc} \nu : & M_G & \to & \mathrm{End}_{\mathbb{C}}(\mathcal{B}(M_G)) \\ & [H;k] & \mapsto & \overleftarrow{D}_{H,e_H-k} \end{array}$$

extends to the algebra homomorphism from the opposite algebra $\mathcal{B}(M_G)^{op}$ of $\mathcal{B}(M_G)$ to $\mathrm{End}_{\mathbb{C}}(\mathcal{B}(M_G))$. The homomorphism $\nu : \mathcal{B}(M_G)^{op} \to \mathrm{End}_{\mathbb{C}}(\mathcal{B}(M_G))$ gives a nondegenerate pairing

$$\begin{array}{rccc} \langle\!\langle\ ,\ \rangle\!\rangle : & \mathcal{B}(M_G) \times \mathcal{B}(M_G)^{op} & \to & \mathcal{B}^0(M_G) = \mathbb{C} \\ & \langle\!\langle \phi, \psi \rangle\!\rangle & \mapsto & \mathrm{CT.}(\nu(\psi)(\phi)), \end{array}$$

where CT. stands for the part of degree zero.

3 Relations in the Nichols-Woronowicz algebra

Denote by $d(N_1, N_2) > 0$ the greatest common divisor of integers N_1 and N_2.

Proposition 3.1 *In the algebra* $\mathcal{B}(M_G)$,

$$[H;k]^{e_H/d(e_H,k)} = 0.$$

Proof. Take a permutation $w \in S_n$ with $l(w) = l$. Then

$$\Psi_w([H;k]^{\otimes n}) = \zeta_H^{kl}[H;k]^{\otimes n},$$

where $\zeta_H = \exp(2\pi\sqrt{-1}/e_H)$. Hence

$$\sigma_n([H;k]^{\otimes n}) = \left(\sum_{w\in S_n} \zeta_H^{kl(w)}\right) \cdot [H;k]^{\otimes n} = \prod_{j=1}^{n-1}(1 + \zeta_H^k + \zeta_H^{2k} + \cdots + \zeta_H^{jk}) \cdot [H;k]^{\otimes n}.$$

If $n = e_H/d(e_H, k)$, then $\sigma_n([H;k]^{\otimes n}) = 0$.

Relations for $G(e, 1, n)$, $(e > 1)$

Take an n-dimensional hermitian vector space $V = \bigoplus_{i=1}^n \mathbb{C}\epsilon_i$ with an orthonormal basis $(\epsilon_i)_{i=1}^n$. Let $(x_i)_{i=1}^n$ be the coordinate system with respect to the basis $(\epsilon_i)_{i=1}^n$. All the reflection hyperplanes for $G(e, 1, n) \subset GL(V)$ are given by

$$H_{ij}(a) : x_i - \zeta^a x_j = 0, \quad H_i : x_i = 0,$$

where $1 \le i < j \le n$, $a \in \mathbb{Z}/e\mathbb{Z}$, $\zeta = \exp(2\pi\sqrt{-1}/e)$. Choose the normal vectors $v_{H_{ij}(a)} := \epsilon_i - \zeta^{-a}\epsilon_j$ and $v_{H_i} := \epsilon_i$. The algebra $\mathcal{B}(M_G)$ is generated by the symbols $[H_{ij}(a)] := [H_{ij}(a); 1]$, $a \in \mathbb{Z}/e\mathbb{Z}$, and $[H_i; s]$, $1 \le s \le e-1$. We put $[H_{ji}(a)] := -\zeta^a[H_{ij}(-a)]$.

Proposition 3.2 *For* $1 \le s, t \le e-1$ *and distinct* $1 \le i, j, k \le n$, *we have the following relations in* $\mathcal{B}(M_G)$.

(1) $[H_{ij}(a)][H_{jk}(b)] - \zeta^a[H_{ik}(a+b)][H_{ij}(a)] - [H_{jk}(b)][H_{ik}(a+b)] = 0$,

(2) $\sum_{p=1}^{e_H/d(e_H,2(a-b))} \zeta^{2p(a-b)}[H_{ij}(a+2p(a-b))][H_{ij}(b+2p(a-b))] =$
$\sum_{q=1}^{e_H/d(e_H,2(a-b))} \zeta^{(2q-1)(a-b)}[H_{ij}(b+2q(a-b))][H_{ij}(a+2(q-1)(a-b))]$,

(3) $[H_{ij}(a)][H_i; s] - \zeta^{-a}[H_j; s][H_{ij}(a)] + \zeta^{-a}[H_{ij}(a-s)][H_j; s] - \zeta^{-s}[H_i; s][H_{ij}(a-s)] = 0$,

(4) $[H_{ij}(a)][H_{kl}(b)] = [H_{kl}(b)][H_{ij}(a)]$, *if* $\{i, j\} \cap \{k, l\} = \emptyset$,

(5) $[H_i; s][H_j; t] = [H_j; t][H_i; s]$,

(6) $[H_{ij}(a)][H_k; s] = [H_k; s][H_{ij}(a)]$, *if* $k \ne i, j$.

These relations follow from straightforward computation of the images of the braided symmetrizers.

Remark 3.1 In the case of $S_n = G(1,1,n)$, the relations (1), (4) and $[H_{ij}(0)]^2 = 0$ cover all the independent quadratic relations in $\mathcal{B}(M_{S_n})$, see [6] and [2, 7.1]. In the case of the Weyl groups $W_{B_n} = W_{C_n} = G(2,1,n)$, the relations in Proposition 3.2 coincide with the quadratic relations given in [9].

4 Model of coinvariant algebra

Definition 4.1 *Fix a set of G-invariant constants $\kappa = (\kappa_{H,i})_{H\in\mathcal{A}, 1\le i\le e_H-1}$. We define the $\mathbb{C}$-linear map $\mu = \mu_\kappa : V \to M_G$ by*

$$\mu(\xi) = -\sum_{H\in\mathcal{A}} \sum_{i,k=1}^{e_H-1} \alpha_H(\xi)\kappa_{H,i}\zeta_H^{-ik}[H;k],$$

where $\zeta_H = \exp(2\pi\sqrt{-1}/e_H)$.

Proposition 4.1 *The map μ is a G-homomorphism, i.e., $\mu(g(\xi)) = g(\mu(\xi))$.*

Proof. Note that $e_{gH} = e_H$ for any $g \in G$ and $H \in \mathcal{A}$. We have

$$\begin{aligned}
\mu(g(\xi)) &= -\sum_{H\in\mathcal{A}} \sum_{i,k=1}^{e_H-1} \alpha_H(g(\xi))\kappa_{H,i}\zeta_H^{-ik}[H;k] \\
&= -\sum_{H\in\mathcal{A}} \sum_{i,k=1}^{e_H-1} g^*\alpha_H(\xi)\kappa_{H,i}\zeta_H^{-ik}[H;k] \\
&= -\sum_{H\in\mathcal{A}} \sum_{i,k=1}^{e_H-1} \overline{\lambda(g^{-1},H)}\alpha_{g^{-1}H}(\xi)\kappa_{H,i}\zeta_H^{-ik}[H;k] \\
&= -\sum_{H\in\mathcal{A}} \sum_{i,k=1}^{e_H-1} \alpha_H(\xi)\kappa_{gH,i}\zeta_{gH}^{-ik}\overline{\lambda(g^{-1},gH)}[gH;k] \\
&= -\sum_{H\in\mathcal{A}} \sum_{i,k=1}^{e_H-1} \alpha_H(\xi)\kappa_{H,i}\zeta_H^{-ik}\overline{\lambda(g,H)}^{-1}[gH;k] \\
&= g\left(-\sum_{H\in\mathcal{A}} \sum_{i,k=1}^{e_H-1} \alpha_H(\xi)\kappa_{H,i}\zeta_H^{-ik}[H;k]\right) = g(\mu(\xi)).
\end{aligned}$$

Here we have used that $\lambda(g^{-1},gH) = \lambda(g,H)^{-1}$.

Proposition 4.2

$$[\mu(\xi), \mu(\eta)] = 0 \quad \text{in } \mathcal{B}(M_G)$$

Proof. Let us show $(\mathrm{id} + \Psi)(\mu(\xi) \otimes \mu(\eta)) = (\mathrm{id} + \Psi)(\mu(\eta) \otimes \mu(\xi))$. The left-hand side equals

$$\sum_{H,H' \in \mathcal{A}} \sum_{i,k=1}^{e_H - 1} \sum_{j,l=1}^{e_{H'} - 1} \alpha_H(\xi)\alpha_{H'}(\eta)\kappa_{H,i}\kappa_{H',j}\zeta_H^{-ik}\zeta_{H'}^{-jl}[H;k] \otimes [H';l]$$

$$+ \sum_{H,H' \in \mathcal{A}} \sum_{i,k=1}^{e_H - 1} \sum_{j,l=1}^{e_{H'} - 1} \alpha_H(\xi)\alpha_{H'}(\eta)\kappa_{H,i}\kappa_{H',j}\zeta_H^{-ik}\zeta_{H'}^{-jl} g_H^k([H';l]) \otimes [H;k].$$

Here the second term is

$$\begin{aligned}
& \sum_{H,H' \in \mathcal{A}} \sum_{i,k=1}^{e_H - 1} \sum_{j,l=1}^{e_{H'} - 1} \alpha_H(\xi)\alpha_{H'}(\eta)\kappa_{H,i}\kappa_{H',j}\zeta_H^{-ik}\zeta_{H'}^{-jl}\overline{\lambda(g_H^k, H')}^{-1}[g_H^k(H');l] \otimes [H;k] \\
= & \sum_{H,H' \in \mathcal{A}} \sum_{i,k=1}^{e_H - 1} \sum_{j,l=1}^{e_{H'} - 1} \alpha_H(\xi)\overline{\lambda(g_H^{-k}, g_H^k(H'))}\alpha_{H'}(\eta)\kappa_{H,i}\kappa_{H',j}\zeta_H^{-ik}\zeta_{H'}^{-jl}[g_H^k(H');l] \otimes [H;k] \\
= & \sum_{H,H' \in \mathcal{A}} \sum_{i,k=1}^{e_H - 1} \sum_{j,l=1}^{e_{H'} - 1} \alpha_H(\xi)(g_H^k)^*(\alpha_{g_H^k(H')})(\eta)\kappa_{H,i}\kappa_{H',j}\zeta_H^{-ik}\zeta_{H'}^{-jl}[g_H^k(H');l] \otimes [H;k] \\
= & \sum_{H,H' \in \mathcal{A}} \sum_{i,k=1}^{e_H - 1} \sum_{j,l=1}^{e_{H'} - 1} \alpha_H(\xi)\alpha_{H'}(g_H^k(\eta))\kappa_{H,i}\kappa_{H',j}\zeta_H^{-ik}\zeta_{H'}^{-jl}[H';l] \otimes [H;k].
\end{aligned}$$

Since

$$g_H^k(\eta) = \eta - (1 - \zeta_H^k)\frac{\alpha_H(\eta)v_H}{\|v_H\|^2},$$

we obtain the following expression of $(\mathrm{id} + \Psi)(\mu(\xi) \otimes \mu(\eta))$ which is symmetric in ξ and η :

$$\sum_{H,H' \in \mathcal{A}} \sum_{i,k=1}^{e_H - 1} \sum_{j,l=1}^{e_{H'} - 1} \alpha_H(\xi)\alpha_{H'}(\eta)\kappa_{H,i}\kappa_{H',j}\zeta_H^{-ik}\zeta_{H'}^{-jl}[H;k] \otimes [H';l]$$

$$+ \sum_{H,H' \in \mathcal{A}} \sum_{i,k=1}^{e_H - 1} \sum_{j,l=1}^{e_{H'} - 1} \alpha_{H'}(\xi)\alpha_H(\eta)\kappa_{H,i}\kappa_{H',j}\zeta_H^{-ik}\zeta_{H'}^{-jl}[H;k] \otimes [H';l]$$

$$+\sum_{H,H'\in\mathcal{A}}\sum_{i,k=1}^{e_H-1}\sum_{j,l=1}^{e_{H'}-1}\alpha_{H'}(\xi)\alpha_{H'}(\eta)\frac{\alpha_H(v_{H'})}{\|v_{H'}\|^2}\kappa_{H,i}\kappa_{H',j}\zeta_H^{-ik}\zeta_{H'}^{-jl}(1-\zeta_{H'}^{l})[H;k]\otimes[H';l].$$

This completes the proof.

The proposition above shows that the map μ extends to an algebra homomorphism

$$\tilde{\mu}: S(V)\to\mathcal{B}(M_G).$$

Remark 4.1 Dunkl and Opdam [5] have introduced the Dunkl operator for the complex reflection groups. Their operator is defined by the following formulas:

$$\begin{aligned}T_\xi(\kappa) &= \partial_\xi+\sum_{H\in\mathcal{A}}\sum_{i=1}^{e_H-1}\sum_{g\in G_H}\alpha_H(\xi)\kappa_{H,i}\alpha_H^{-1}\chi_H^i(g)g\\ &= \partial_\xi-\sum_{H\in\mathcal{A}}\sum_{i=1}^{e_H-1}\sum_{k=1}^{e_H-1}\alpha_H(\xi)\kappa_{H,i}\zeta_H^{-ik}\alpha_H^{-1}(1-g_H^{-k}),\end{aligned}$$

where χ_H^i is the restriction of $\det^i$ to G_H. Hence, our homomorphism μ can be regarded as a truncated version of the operator $T_\xi(\kappa)$ that means the operator without the differential part ∂_ξ after replacing the brackets $[H;k]$ by the divided difference operators $\alpha_H^{-1}(1-g_H^{-k})$ acting on $S(V^*)$.

Below we quote a lemma from [14], which is an analogue of Lemma 1.9 in [8, Chapter IV]. The proof of this lemma given in [14] is applicable to the general finite complex reflection groups. Let us consider the polynomial $Q:=\prod_{H\in\mathcal{A}}v_H^{e_H-1}\in S(V)$.

Lemma 4.1 ([14, Lemma 2.16]) *If a graded ideal $I\subset S(V)$ contains I_G, but does not contain Q, then $I=I_G$.*

We define the divided difference operator $\overleftarrow{\Delta}_{H,k}$ acting from the right by

$$f\overleftarrow{\Delta}_{H,k}:=\frac{f-g_H^{-k}(f)}{v_H}.$$

Proposition 4.3 *For $f\in S(V)$,*

$$C_{H,k}\tilde{\mu}(f\overleftarrow{\Delta}_{H,k})=\tilde{\mu}(f)\overleftarrow{D}_{H,k},$$

where

$$C_{H,k}=\sum_{i=1}^{e_H-1}\kappa_{H,i}\frac{\zeta_H^{-ik}\|v_H\|^2}{\zeta_H^{-k}-1}.$$

Proof. This follows from

$$[H';k']\ \overleftarrow{D}_{H,k} = \delta_{H,H'}\delta_{k,k'},$$

$$\mu(\xi)\overleftarrow{D}_{H,k} = (-\sum_{i=1}^{e_H-1}\kappa_{H,i}\zeta_H^{-ik})\alpha_H(\xi) = \left(\sum_{i=1}^{e_H-1}\kappa_{H,i}\frac{\zeta_H^{-ik}\|v_H\|^2}{\zeta_H^{-k}-1}\right)\xi\overleftarrow{\Delta}_{H,k}, \quad \text{for } \xi \in V,$$

and the braided Leibniz rule

$$(\phi\psi)\overleftarrow{D}_{H,k} = \phi(\psi\overleftarrow{D}_{H,k}) + (\psi\overleftarrow{D}_{H,k})g_H^{-k}(\phi).$$

The constants $(\kappa_{H,i})$ are said to be *generic* if $C_{H,k} \neq 0$ for any $H \in \mathcal{A}$ and $1 \leq k \leq e_H - 1$. We also need the following lemma to prove the main theorem.

Lemma 4.2 *Let $(\kappa_{H,i})$ be generic. There exist sequences $H_1, \ldots, H_p \in \mathcal{A}$ and $k_1, \ldots, k_p$ such that $Q\overleftarrow{\Delta}_{H_1,k_1}\cdots\overleftarrow{\Delta}_{H_p,k_p}$ is a nonzero constant.*

Proof. Take a nonzero homogeneous element $f \in P_G$. If $f\overleftarrow{\Delta}_{H,k} = 0$ for any H and k, then f is a G-invariant element of P_G by Lemma 1.1. Since P_G is isomorphic to the regular representation of G and contains only one copy of the trivial representation at degree zero, f must be in $P_G^0 = \mathbb{C}$. Hence, if the degree of f is positive, then there exist $H \in \mathcal{A}$ and k such that $f\overleftarrow{\Delta}_{H,k} \neq 0$. The polynomial Q affording the character $\det^{-1}$ is a generator of the part of the highest degree in P_G, so we can find the desired sequences $H_1, \ldots, H_p \in \mathcal{A}$ and $k_1, \ldots, k_p$ by induction on the degree.

Theorem 4.1 *For any generic choice of the constants $\kappa = (\kappa_{H,i})$, we have the isomorphism*

$$\mathrm{Im}(\tilde{\mu}) \cong P_G.$$

Proof. If f is a G-invariant polynomial of positive degree, we get $\tilde{\mu}(f)\overleftarrow{D}_{H,k} = 0$ for any H and k from Lemma 1.2 and Proposition 4.3. Hence $\tilde{\mu}(f) \in \mathcal{B}^0(M_G) = \mathbb{C}$ from Lemma 2.2. Since $\tilde{\mu}$ preserves the degree, we have $\tilde{\mu}(f) = 0$ and $\mathrm{Ker}(\tilde{\mu}) \supset I_G$. On the other hand, it follows from Lemma 4.2 that one can find sequences $H_1, \ldots, H_p \in \mathcal{A}$ and $k_1, \ldots, k_p$ such that $Q\overleftarrow{\Delta}_{H_1,k_1}\cdots\overleftarrow{\Delta}_{H_p,k_p}$ is a nonzero constant. This means, see Proposition 4.3, that

$$\tilde{\mu}(Q)\overleftarrow{D}_{H_1,k_1}\cdots\overleftarrow{D}_{H_p,k_p} = C_{H_1,k_1}\cdots C_{H_p,k_p}\tilde{\mu}(Q\overleftarrow{\Delta}_{H_1,k_1}\cdots\overleftarrow{\Delta}_{H_p,k_p}) \in \mathcal{B}^0(M_G)$$

is a nonzero constant. Hence $\mathrm{Ker}(\tilde{\mu})$ does not contain Q. Now we get $\mathrm{Ker}(\tilde{\mu}) = I_G$ from Lemma 4.1.

5 Complex reflection group of type $G(e,1,n)$

We use the notation in Section 3. In the following we identify V^* with V via the G-invariant hermitian form $\langle\ ,\ \rangle$, so that the left G-action on V^* defined by $g(\alpha) = (g^{-1})^*\alpha$, $\alpha \in V^*$, coincides with the left G action on V. Put $\kappa_{ij,a} = \kappa_{H_{ij}(a),1}$ and $\kappa_{i,s} = \kappa_{H_i,s}$. For a generic choice of the constants κ, the subalgebra $\mathrm{Im}(\tilde{\mu}) \subset \mathcal{B}(M_G)$, which is generated by the elements

$$\mu(\epsilon_i) = \sum_{j\neq i} \sum_{a\in\mathbb{Z}/m\mathbb{Z}} \kappa_{ij,a}[H_{ij}(a)] - \sum_{s,t=1}^{e-1} \zeta^{-st}\kappa_{i,s}[H_i;s], \quad 1 \leq i \leq n,$$

is isomorphic to the coinvariant algebra

$$P_G = \mathbb{C}[x_1,\dots,x_n]/(E_1(x_1^e,\dots,x_n^e),\dots,E_n(x_1^e,\dots,x_n^e)),$$

where E_i is the i-th elementary symmetric polynomial. Rampetas and Shoji [14] introduced a family of operators Δ_w, $w \in G$, acting on the polynomial ring $P = \mathbb{C}[x_1,\dots,x_n]$ based on particular choice of the reduced expression of $w \in G$. The pseudo-reflections $s_{ij}(a) := g_{H_{ij}(a)}$ and $t_i := g_{H_i}$ generate the group $G(e,1,n)$. In particular, the pseudo-reflections $s_i = s_{i-1\,i}(0)$ and t_1 play the role of simple reflections. In the following, we use the divided difference operators $\Delta_{s_i} := \Delta_{H_{i-1\,i}(0),1}$ and $\Delta_{t_i} := \Delta_{H_i,1}$. Note that the braid relations among the divided difference operators do not hold in general. Put $\tilde{\Delta}_{s_i} := \Delta_{t_i}^{e-2}\Delta_{s_i}$. Let us consider the operator

$$\Delta_n(k,a) = \begin{cases} \Delta_{s_{k+1}}\cdots\Delta_{s_{n-1}}\Delta_{s_n}, & \text{if } a = 0, \\ \tilde{\Delta}_{s_k}\cdots\tilde{\Delta}_{s_2}\Delta_{t_1}^a\Delta_{s_2}\cdots\Delta_{s_n}, & \text{if } a \neq 0. \end{cases}$$

In [14], it is shown that any element $w \in G$ has a unique decomposition of form

$$w = \omega_n(k_n,a_n)\omega_{n-1}(k_{n-1},a_{n-1})\cdots\omega_1(k_1,a_1),$$

where

$$\omega_n(k,a) = \begin{cases} s_{k+1}\cdots s_{n-1}s_n, & \text{if } a = 0, \\ s_k\cdots s_2t_1^as_2\cdots s_n, & \text{if } a \neq 0. \end{cases}$$

The operators Δ_w are defined by the following formula:

$$\Delta_w := \Delta_n(k_n,a_n)\Delta_{n-1}(k_{n-1},a_{n-1})\cdots\Delta_1(k_1,a_1).$$

Define the evaluation map $\varepsilon : P \to \mathbb{C}$ by $\varepsilon(f) = f(0)$. Denote by $\bar{\mathcal{D}}_G$ the subspace of P^* spanned by the operators $\varepsilon\Delta_w$, $w \in G$.

Proposition 5.1 ([14, Theorem 2.18]) *The coinvariant algebra P_G is naturally isomorphic to the dual space of $\bar{\mathcal{D}}_G$.*

Put $[s_i] := [H_{i-1\,i}(0)]$, $[t_i] := [H_i; e-1]$ and $\widetilde{[s_i]} := [t_i]^{e-2}[s_i]$. We also define the elements $[w] \in \mathcal{B}(M_G)^{op}$, $w \in G$, in a similar way:

$$[w] := [\omega_n(k_n, a_n)][\omega_{n-1}(k_{n-1}, a_{n-1})] \cdots [\omega_1(k_1, a_1)],$$

where

$$[\omega_n(k,a)] := \begin{cases} [s_{k+1}] \cdots [s_{n-1}][s_n], & \text{if } a = 0, \\ \widetilde{[s_k]} \cdots \widetilde{[s_2]}[t_1]^a [s_2] \cdots [s_n], & \text{if } a \neq 0. \end{cases}$$

Consider $\mathbf{D}_G$ the subspace of $\mathcal{B}(M_G)^{op}$ spanned by the elements $[w]$, $w \in G$. Now it is easy to get the following analogue to [2, Theorem 6.1].

Theorem 5.1 *Assume that $\kappa_{ij,a} = 1$ and $\kappa_{i,s} = 1 - \zeta^{-s}$.*
(1) *The linear map*

$$(\mathrm{CT}.)_* \circ \tilde{\mu}^* \circ \nu : \mathcal{B}(M_G)^{op} \to \mathrm{End}_{\mathbb{C}}(\mathcal{B}(M_G)) \to \mathrm{Hom}_{\mathbb{C}}(P, \mathcal{B}(M_G)) \to P^*$$

induces an isomorphism between $\mathbf{D}_G$ and $\bar{\mathcal{D}}_G$.
(2) *The subalgebra $\mathrm{Im}(\tilde{\mu})$ is isomorphic to the dual space of $\mathbf{D}_G$ via the pairing $\langle\!\langle\ ,\ \rangle\!\rangle$. Furthermore, the pairing $\langle\!\langle\ ,\ \rangle\!\rangle$ restricted to $\mathrm{Im}(\tilde{\mu}) \times \mathbf{D}_G$ coincides with the pairing between P_G and $\bar{\mathcal{D}}_G$, i.e.,*

$$\langle\!\langle \tilde{\mu}(f), [w] \rangle\!\rangle = \varepsilon \Delta_w(f).$$

Proof. From Theorem 4.1 and Proposition 5.1, we have the factorization

$$(\mathrm{CT}.)_* \circ \tilde{\mu}^* : \mathrm{End}_{\mathbb{C}}(\mathcal{B}(M_G)) \to P_G^* = \bar{\mathcal{D}}_G \subset P^*.$$

For the choice of the constants κ as assumed, we have $C_{H,k} = 1$ for all $H \in \mathcal{A}$ and $1 \leq k \leq e_H - 1$. Proposition 4.3 shows that

$$\langle\!\langle \tilde{\mu}(f), [w] \rangle\!\rangle = \mathrm{CT}.(\nu([w])(\tilde{\mu}(f))) = \mathrm{CT}.(\tilde{\mu}(\Delta_w(f))) = \varepsilon \Delta_w(f), \quad f \in P,$$

so we obtain $(\mathrm{CT}.)_* \tilde{\mu}^*(\nu(\mathbf{D}_G)) = \bar{\mathcal{D}}_G$. Since $\mathbf{D}_G$ is spanned by $|G|$ elements, $\mathbf{D}_G$ is isomorphic to $\bar{\mathcal{D}}_G$.

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