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TRIANGULATED CATEGORIES OF MATRIX FACTORIZATIONS FOR REGULAR SYSTEMS OF WEIGHTS WITH $\varepsilon = -1$

HIROSHIGE KAJIURA, KYOJI SAITO, AND ATSUSHI TAKAHASHI

Abstract. We construct a full strongly exceptional collection in the triangulated category of graded matrix factorizations of a polynomial associated to a non-degenerate regular system of weights whose smallest exponents are equal to -1. In the associated Grothendieck group, the strongly exceptional collection defines a root basis of a generalized root system of sign $(l, 0, 2)$ and a Coxeter element of finite order, whose primitive eigenvector is a regular element in the expanded symmetric domain of type IV with respect to the Weyl group.

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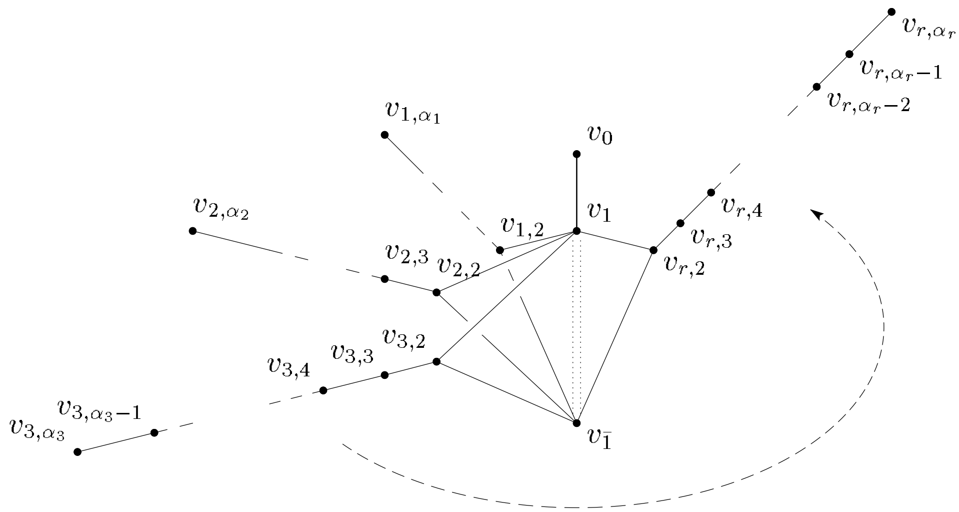
1. Introduction

A quadruple of positive integers $W := (a, b, c; h)$ is called a *regular system of weights* if the rational function $\chi_W = T^{-h}\frac{(T^h-T^a)(T^h-T^a)(T^h-T^c)}{(T^a-1)(T^b-1)(T^c-1)}$ develops in a Laurent polynomial and satisfies a suitable reducedness condition ([Sa1], see subsection 5.1). Then, χ_W is a sum of Laurent monomials and the exponents of the monomials are called the *exponents* of W. The *smallest exponent*, given by $a+b+c-h$, is denoted by ε_W. The regularity condition on W is equivalent to that a degree h weighted homogeneous polynomial $f_W \in A := \mathbb{C}[x, y, z]$ in three variables x, y and z of weights a, b and c, respectively with a generic choice of coefficients defines a hypersurface in $\mathbb{A}^3$ having an isolated singular point at the origin.

Motivated by the theory of primitive forms associated to the polynomial f_W, we asked to construct a generalization of a root system and a Lie algebra for any regular system of weights W ([Sa4, Sa5]). In fact, by taking the set of vanishing cycles in the Milnor fiber of f_W, a finite root system of type ADE or an elliptic root system of type $E_6^{(1,1)}$, $E_7^{(1,1)}$ or $E_8^{(1,1)}$ [Sa2] is associated to a regular system of weights W with $\varepsilon_W = 1$ or 0, respectively, where the set of exponents of the weight system coincides with the set of Coxeter exponents of the root system. However, the vanishing cycles are transcendental object and are hard to study further for the cases of $\varepsilon_W < 0$. Then, based on the duality theory of the weight systems [Sa3, T1] and the (homological) mirror symmetry [Ko], the third author [T2] proposed to use the triangulated category of graded matrix factorizations of f_W introduced by [T2] and Orlov [O] independently, where the root system appears as the set of the isomorphism classes of the exceptional objects via the Grothendieck group of the category.

In our previous paper [KST1], we showed that, for any regular system of weights W of type ADE (i.e., $\varepsilon_W = 1$), the triangulated category $HMF^{gr}_A(f_W)$ of graded matrix factorizations of f_W is equivalent to the bounded derived category of finitely generated modules over the path algebra of a Dynkin quiver of the corresponding ADE-type. Due to a theorem of Gabriel [Ga1], this implies that one gets the root system of type ADE in the Grothendieck group $K_0(HMF^{gr}_A(f_W))$ as expected (see [T2] for A_l case).

The present paper studies the category $HMF^{gr}_A(f_W)$ associated to regular systems of weights W with $\varepsilon_W = -1$ and $a_0 = 0$ (the second condition means, by definition, there are no exponents equal to 0, and we call such W *nondegenerate*). The set of indecomposable objects of $HMF^{gr}_A(f_W)$ is no more simple to describe as opposed to the case of type ADE. However, we can still find a strongly exceptional collection which generates the category and gives a good basis of the generalized root system in $K_0(HMF^{gr}_A(f_W))$. More precisely, the main theorem (Theorem 5.8) states that, for any regular system of weights W with $\varepsilon_W = -1$ and $a_0 = 0$, there exists a strongly exceptional collection in the category $HMF^{gr}_A(f_W)$, the associated quiver (in a generalized sense, see subsection 5.2) to which is given by the following diagram



with appropriate orientations of arrows for the edges (see Figures 1, 2 and 3 in subsection 5.3), where the multi-set $A_W = \{\alpha_1, \ldots, \alpha_r\}$ of positive integers is the *signature of* W (see Definition 5.1).

Let us discuss some background and consequences of Theorem 5.8.

1. There are 14+8 regular systems of weights with $\varepsilon_W = -1$ and $a_0 = 0$. The first 14 cases define exceptional unimodular singularities of Arnold [Ar], who found an involutive bijection, called the strange duality, among their numerical invariants. The strange duality was reconstructed by the $*$-duality among regular systems of weights [Sa3] in terms of the characteristic polynomial, which is understood as a mirror symmetry in [KaYa, T1]. Then, Theorem 5.8 implies that *the lattice of the vanishing cycles of* f_W *is obtained by the Grothendieck group of* $HMF^{gr}_A(f_{W^*})$ *for the dual weight system* W^* as explained in subsection 5.5. In particular, *the set of exponents of* W *coincides with the set of Coxeter exponents of the root system in* $K_0(HMF^{gr}_A(f_{W^*}))$ (see Remark 5.11).

2. For those fourteen regular systems of weights W corresponding to exceptional unimodular singularities, the root lattice $(K_0(HMF^{gr}_A(f_{W^*})), \chi + {}^t\chi)$ is an indefinite lattice of sign $(2, \mu_{W^*} - 2)$, where μ_{W^*} is the Milnor number of the Milnor fiber of f_{W^*} and χ is the Euler pairing. The Coxeter transformation defined as a product of reflections associated to the underlying graph of the above quiver is identified with the Auslander-Reiten translation τ_{AR} and, hence, is of *finite* order h (see Remark 5.11). The Weyl group $\mathcal{W}$ generated by those reflections acts on the expanded symmetric domain $\mathcal{B}_{\mathbb{C}} := \{\varphi \in \mathrm{Hom}_{\mathbb{R}}(K_0(HMF^{gr}_A(f_{W^*})) \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{C}) \mid$

$\mathrm{Ker}(\varphi)>0\}$ of type IV [Sa5, Sa6]. It is shown that the eigenvectors of the Coxeter transformation whose eigenvalues belong to the primitive h-th roots of unity are regular with respect to $\mathcal{W}$ (see [Sa4]). This fact will conjecturally play an important role to construct a flat structure on the quotient variety $\mathcal{B}_{\mathbb{C}}/\mathcal{W}$ which should be the space of stability conditions on $HMF^{gr}_A(f_{W^*})$ in the sense of Bridgeland [Bd].

3. Due to a theorem of Bondal and Kapranov [Bo, BK], we see that *the triangulated category $HMF^{gr}_A(f_W)$ is equivalent to the bounded derived category of finitely generated modules over the path algebra with relations corresponding to the quiver above* (Corollary 5.9). Recently, a parallel statement is proven independently by Lenzing and de la Pena [LP] in the framework of the weighted projective lines by Geigle and Lenzing [GL1] by combining it with Orlov's arguments in [O].

The construction of the present paper is as follows.

Section 2 is devoted to the preparation of the categories of our study in three equivalent formulations. In subsections 2.1 and 2.2, following Orlov [O] (see also Buchweitz [Bu]), we recall the triangulated category $D^{\mathrm{gr}}_{\mathrm{Sg}}(R_W)$ of singularity and the triangulated category $\underline{\mathrm{CM}^{gr}}(R_W)$ of graded maximal Cohen-Macaulay modules over $R_W = A/(f_W)$, respectively. Then, in subsection 2.3, we recall the triangulated category $HMF^{gr}_A(f_W)$ of graded matrix factorizations from [KST1].

In section 3, we show the existence of the Serre duality and the Auslander-Reiten triangles in the triangulated category $\underline{\mathrm{CM}^{gr}}(R)$ (Proposition 3.7). This fact may be well-known among experts. We are grateful to Prof. Iyama who explained the result to us.

In section 4, we show the other basic result which we use in the proof of the main theorem: *a right admissible full triangulated subcategory $\mathcal{T}'$ of $D^{gr}_{Sg}(R)$ satisfying the conditions*:

(i) *the degree shift functor τ is an autoequivalence of $\mathcal{T}'$,*

(ii) *$\mathcal{T}'$ has an object E which is isomorphic to $R/\mathfrak{m}$ in $D^{gr}_{Sg}(R)$,*

is equivalent to the category $D^{gr}_{Sg}(R)$ itself as a triangulated category (Theorem 4.5).

Section 5 is devoted to stating our main results. In subsection 5.1, we recall regular systems of weights and related notion. In subsection 5.2, we prepare a generalized notion of quivers. In this formulation, we define the quivers associated to regular systems of weights W of $\varepsilon_W = -1$ with genus $a_0 = 0$ in subsection 5.3. In subsection 5.4, we state the main Theorem (Theorem 5.8). It is obtained as a consequence of the structure theorem (Theorem 5.10) on the category $HMF^{gr}_A(f_W)$, where we describe the Auslander-Reiten triangles for all objects forming the exceptional collection. The proof of Theorem 5.10, stated in subsection 6.1, is based on explicit data of graded matrix factorizations of f_W for a regular system of weights W with $\varepsilon_W = -1$ and $a_0 = 0$, which are given in section 7. We hope these data can give a well-defined stability condition [Bd], which is one of future directions.

Throughout the paper, we denote by k an algebraic closed field of characteristic zero.

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2. Some equivalent categories

Let k be an algebraic closed field of characteristic zero. For a positive integer h, let $R := \oplus_{s\in\frac{2}{h}\mathbb{Z}_{\geq 0}} R_s$ be a commutative Noetherian $(2\mathbb{Z}/h)$-graded ring of dimension $d(\geq 0)$ with $R_0 = k$. This ring R defines a graded isolated singularity, i.e., the graded localization $R_{(\mathfrak{p})}$ is regular for any graded prime $\mathfrak{p} \neq \mathfrak{m}$, where $\mathfrak{m} := \oplus_{s\in\frac{2}{h}\mathbb{Z}_{>0}} R_s$.

By a graded R-module, we always mean a graded R-module with degrees only in $2\mathbb{Z}/h$. Namely, a graded R-module M decomposes into the direct sum $M = \oplus_{s\in\frac{2}{h}\mathbb{Z}} M_s$. For two graded R-modules M and N, a graded R-homomorphism g of degree $t \in 2\mathbb{Z}/h$ is an R-homomorphism $g : M \to N$ such that $g(M_s) \subset N_{s+t}$ for any $s \in 2\mathbb{Z}/h$.

2.1. Category of graded singularities.

Definition 2.1. Denote by gr$-R$ the abelian category of finitely generated graded R-modules, in which morphisms are R-homomorphism of degree zero. The degree shift of $M \in$ gr$-R$, denoted by τM, is defined by $(\tau M)_s := M_{s+\frac{2}{h}}$. This τ naturally induces an auto-equivalence functor on gr$-R$, which we denote by the same symbol τ.

We have $\mathrm{Ext}^i_R(M,N) \simeq \oplus_{n\in\mathbb{Z}}\mathrm{Ext}^i_{\mathrm{gr}-R}(\tau^{-n}M,N) \simeq \oplus_{n\in\mathbb{Z}}\mathrm{Ext}^i_{\mathrm{gr}-R}(M,\tau^n N)$ since R is Noetherian. In particular, for $i = 0$,

$$\mathrm{Hom}_R(M,N) \simeq \oplus_{n\in\mathbb{Z}}\mathrm{Hom}_{\mathrm{gr}-R}(\tau^{-n}M,N) \simeq \oplus_{n\in\mathbb{Z}}\mathrm{Hom}_{\mathrm{gr}-R}(M,\tau^n N)$$

forms a graded R-module, where the grading of each homogeneous piece is defined as $2n/h$. Note also that any graded projective module is free since R is finitely generated over $R_0 = k$. Denote by grproj$-R$ the exact category (= the extension-closed full additive subcategory in gr$-R$) of graded projective modules.

Definition 2.2 (Orlov [O])**.** The triangulated category $D^{gr}_{Sg}(R)$, called the category of the singularity R, is defined as the quotient $D^b(\text{gr--}R)/D^b(\text{grproj}-R)$. We denote by T the translation functor [1] on the triangulated category $D^{gr}_{Sg}(R)$.

2.2. **Category of graded maximal Cohen-Macaulay modules.**

The above definition of $D^{gr}_{Sg}(R)$ is simple, however, it is not easy to understand morphisms between objects since they are defined in the localized category. Therefore, we recall some categories equivalent to $D^{gr}_{Sg}(R)$. In this subsection, we recall the triangulated category $\underline{\mathrm{CM}^{gr}}(R)$ of graded maximal Cohen-Macaulay modules; we refer to [Y] for terminologies and the statements presented here.

Definition 2.3. An element $M \in \text{gr--}R$ is called a *graded maximal Cohen-Macaulay module* if $\mathrm{Ext}^i_R(R/\mathfrak{m}, M) = 0$ for $i < d$, where d is the dimension of the ring R. We denote the full subcategory of gr–R consisting of all graded maximal Cohen-Macaulay modules over R by $\mathrm{CM}^{gr}(R)$, which forms an exact category.

Recall that an element $K_R \in \mathrm{CM}^{gr}(R)$ is called a *canonical module* of R if $\mathrm{Ext}^i_R(R/\mathfrak{m}, K_R) \simeq 0$ for $i \neq d$ and $\mathrm{Ext}^d_R(R/\mathfrak{m}, K_R) \simeq k$.

Lemma 2.4. *The following conditions are equivalent*:

(i) M *is a graded maximal Cohen-Macaulay module,*

(ii) $H^i_{\mathfrak{m}}(M) = 0$ *for* $i \neq d$*, where* $H^\bullet_{\mathfrak{m}}$ *is the local cohomology functor with support on* $\{\mathfrak{m}\}$ *defined by* $H^i_{\mathfrak{m}}(M) := \varinjlim \mathrm{Ext}^i_R(R/R_{\geq n}, M)$*,* $R_{\geq n} := \oplus_{i \in \frac{2}{h}\mathbb{Z}_{\geq n}} R_i$*,*

(iii) $\mathrm{Ext}^i_R(M, K_R) = 0$ *for* $i > 0$.

□

Definition 2.5. The ring R is called *Gorenstein* if the injective dimension of R is finite and the canonical module K_R is isomorphic to $\tau^{-\varepsilon(R)}R$ for some $\varepsilon(R) \in \mathbb{Z}$. The integer $\varepsilon(R)$ is called the *Gorenstein parameter* of R.

Lemma 2.6. *For a Gorenstein ring* R*,* $\mathrm{CM}^{gr}(R)$ *is a Frobenius category, i.e., it has enough projectives and enough injectives and the projectives coincide with the injectives.* □

Definition 2.7. For a Gorenstein ring R, we define an additive category $\underline{\mathrm{CM}^{gr}}(R)$ as follows: objects of it are graded maximal Cohen-Macaulay modules over R and, for any $M, N \in \underline{\mathrm{CM}^{gr}}(R)$, the space of morphisms $\underline{\mathrm{Hom}}_{\text{gr--}R}(M, N)$ is given by $\mathrm{Hom}_{\text{gr--}R}(M, N)/\mathcal{P}(M, N)$, where $\mathcal{P}(M, N)$ is the subspace consisting of elements factoring through projectives, i.e.,

[1] It is often called the shift functor and denoted by [1]. In this paper, in order to avoid confusions, we always mean by τ the degree shift functor and by T the translation functor. Also, the Auslander-Reiten translation will be denoted by τ_{AR}.

$g \in \mathcal{P}(M, N)$ if and only if $g = g'' \circ g'$ for $g' : M \to P$ and $g'' : P \to N$ with a projective object P.

The stable category of a Frobenius category forms a triangulated category (Happel [Ha])). Since $\underline{\mathrm{CM}}^{gr}(R)$ is the stable category of the Frobenius category $\mathrm{CM}^{gr}(R)$, one obtains that:

Proposition 2.8. *The stable category $\underline{\mathrm{CM}}^{gr}(R)$ forms a triangulated category.* □

The following important fact is implicit in Orlov [O]:

Theorem 2.9 (Section 1.3. in [O] (see also Buchweitz [Bu]))**.** *For a Gorenstein ring R, there is an equivalence $\underline{\mathrm{CM}}^{gr}(R) \simeq D^{gr}_{Sg}(R)$ as triangulated categories.* □

2.3. **Category of graded matrix factorizations.**

Consider the case when R is a quotient algebra $A/(f)$ of a graded Noetherian regular algebra $A = \oplus_{i \in \frac{2}{h}\mathbb{Z}_{\geq 0}} A_i$ with $A_0 = k$ and an element $f \in A_2$ which is a non-zero divisor. Since $A/(f)$ defines a hypersurface, R is Gorenstein. Recall that τ is the degree shifting operator defined in Definition 2.1.

Definition 2.10. For a non-zero element $f \in A_2$, we define an additive category $MF^{gr}_A(f)$ as follows. Objects of it are graded matrix factorizations M of f defined by

$$\overline{F} := \left(F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1 \right),$$

where F_0 and F_1 are graded free A-modules of finite rank, $f_0 : F_0 \to F_1$ is a graded A-homomorphism of degree zero, $f_1 : F_1 \to F_0$ is a graded A-homomorphism of degree two such that $f_1 f_0 = f \cdot \mathrm{Id}_{F_0}$ and $f_0 f_1 = f \cdot \mathrm{Id}_{F_1}$. A morphism $g : \overline{F} \to \overline{F'}$ in the category $MF^{gr}_A(f)$ is a pair $g = (g_0, g_1)$ of graded A-homomorphisms $g_0 : F_0 \to F'_0$ and $g_1 : F_1 \to F'_1$ of degree zero satisfying $g_1 f_0 = f'_0 g_0$ and $g_0 f_1 = f'_1 g_1$.

For a graded matrix factorization $\overline{F}$, by definition, the rank of F_0 coincides with that of F_1, which we call the *rank* of the matrix factorization $\overline{F}$.

Componentwise monomorphisms and epimorphisms equip the additive category $MF^{gr}_A(f)$ with an exact structure. Moreover, we have the following:

Lemma 2.11 ([O])**.** *$MF^{gr}_A(f)$ is a Frobenius category.* □

A morphism $g = (g_0, g_1) : \overline{F} \to \overline{F'}$ is called null-homotopic if there are graded A-homomorphisms $\psi_0 : F_0 \to F'_1$ of degree minus two and $\psi_1 : F_1 \to F'_0$ of degree zero such that $g_0 = f'_1 \psi_0 + \psi_1 f_0$ and $g_1 = \psi_0 f_1 + f'_0 \psi_1$. Morphisms factoring through projectives in $MF^{gr}_A(f)$ are null-homotopic morphisms.

Definition 2.12. We denote by $HMF_A^{gr}(f)$ the stable (homotopy) category of the Frobenius category $MF_A^{gr}(f)$.

Proposition 2.13 (See Eisenbud [E], Orlov [O] and Yoshino [Y], for example.)**.** *$HMF_A^{gr}(f)$ is a triangulated category which is equivalent to $\underline{\mathrm{CM}^{gr}}(A/(f))$. The equivalence is given by the correspondence*

$$\overline{F} = \Big(F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1 \Big) \mapsto M := \mathrm{Coker}(f_1).$$

□

Remark 2.14. An object $M \in HMF_A^{gr}(f)$ is zero if and only if it is a direct sum of the graded matrix factorizations of the forms $(\ \tau^n(A) \underset{f}{\overset{1}{\rightleftarrows}} \tau^n(A)\ ) \in MF_A^{gr}(f)$ and $(\ \tau^{n'}(A) \underset{1}{\overset{f}{\rightleftarrows}} \tau^{n'+h}(A)\ ) \in MF_A^{gr}(f)$ for some $n, n' \in \mathbb{Z}$.

2.4. **Some basic properties of the category of graded matrix factorizations.**

For later necessity, we discuss the structure of $HMF_A^{gr}(f)$ in more detail.

The auto-equivalence functor τ on gr–A induces an auto-equivalence on $HMF_A^{gr}(f)$, which we denote by the same notation τ. Explicitly, the action of τ takes an object $\overline{F}$ to the object

$$\tau\overline{F} := \Big(\tau F_0 \underset{\tau(f_1)}{\overset{\tau(f_0)}{\rightleftarrows}} \tau F_1 \Big),$$

and takes a morphism $g = (g_0, g_1)$ to the morphism $\tau(g) := (\tau(g_0), \tau(g_1))$. The translation functor T on $HMF_A^{gr}(f)$ takes an object $\overline{F}$ to the object

$$T\overline{F} := \Big(F_1 \underset{-\tau^h(f_0)}{\overset{-f_1}{\rightleftarrows}} \tau^h F_0 \Big),$$

and takes a morphism $g = (g_0, g_1)$ to the morphism $T(g) := (g_1, \tau^h(g_0))$.

The following fact is straightforward by definition, but plays an important role in the study of $HMF_A^{gr}(f)$:

Proposition 2.15. *$T^2 = \tau^h$ on $HMF_A^{gr}(f)$.* □

Next, we explain the triangulated structure in $HMF_A^{gr}(f)$. First, we recall the mapping cone.

Definition 2.16. For a morphism $g = (g_0, g_1) \in \mathrm{Hom}_{MF_A^{gr}(f)}(\overline{F}, \overline{F'})$, we define a mapping cone $C(g) \in MF_A^{gr}(f)$ as

$$C(g) := \Big(F_1 \oplus F_0' \underset{c_1}{\overset{c_0}{\rightleftarrows}} \tau^h F_0 \oplus F_1' \Big), \qquad c_0 := \begin{pmatrix} -f_1 & 0 \\ g_1 & f_0' \end{pmatrix}, \quad c_1 := \begin{pmatrix} -\tau^h(f_0) & 0 \\ \tau^h(g_0) & f_1' \end{pmatrix}.$$

We sometimes denote this cone by $C(\overline{F} \to \overline{F'})$ when omit writing the morphism explicitly.

Note that there exist morphisms $\overline{F'} \stackrel{{}^t(\mathrm{id},0)}{\to} C(g)$ and $C(g) \stackrel{(0,-\mathrm{id})}{\to} T\overline{F}$. By definition of the triangulated structure on $HMF^{gr}_A(f)$, one can easily see that

Proposition 2.17. *Each exact triangle in $HMF^{gr}_A(f)$ is isomorphic to a triangle of the form*

$$\overline{F} \stackrel{g}{\longrightarrow} \overline{F'} \stackrel{{}^t(\mathrm{id},0)}{\longrightarrow} C(g) \stackrel{(0,-\mathrm{id})}{\longrightarrow} T\overline{F}$$

for some $\overline{F}, \overline{F'} \in MF^{gr}_A(f)$ and $g \in \mathrm{Hom}_{MF^{gr}_A(f)}(\overline{F}, \overline{F'})$. □

Let $\overline{F} = (F_0 \overset{f_0}{\underset{f_1}{\rightleftarrows}} F_1) \in HMF^{gr}_A(f)$ be a graded matrix factorization of rank r. Choose homogeneous free basis $(b_1, \ldots, b_r; \bar{b}_1, \ldots, \bar{b}_r)$ such that $F_0 = b_1 A \oplus \cdots \oplus b_r A$ and $F_1 = \bar{b}_1 A \oplus \cdots \oplus \bar{b}_r A$. Then, the graded matrix factorization $\overline{F}$ is expressed as a pair (Q, S) of $2r$ by $2r$ matrices, where S is the diagonal matrix of the form $S := \mathrm{diag}(s_1, \ldots, s_r; \bar{s}_1, \ldots, \bar{s}_r)$ such that $s_i = \deg(b_i)$ and $\bar{s}_i = \deg(\bar{b}_i) - 1$, $i = 1, 2, \ldots, r$, and Q is given by

$$Q = \begin{pmatrix} \mathbf{0} & q_0 \\ q_1 & \mathbf{0} \end{pmatrix}, \qquad q_0, q_1 \in \mathrm{Mat}_r(A), \tag{2.1}$$

with q_0 and q_1 the matrix expressions of the graded A-homomorphisms $f_0 : F_0 \to F_1$ and $f_1 : f_1 \to f_0$, respectively. Namely, they are defined as $f_0(b_1, \ldots, b_r) = (\bar{b}_1, \ldots, \bar{b}_r) q_0$ and $f_1(\bar{b}_1, \ldots, \bar{b}_r) = (b_1, \ldots, b_r) q_1$. By definition, (Q, S) satisfies

$$Q^2 = f \cdot \mathbf{1}_{2r}, \qquad -SQ + QS + 2EQ = Q, \tag{2.2}$$

where $E \in \mathrm{Der}_k(A)$ is the derivation corresponding to the infinitesimal generator of $k^\times$-action (see eq.(5.1)). We call this S a *grading matrix* of Q.

This procedure $\overline{F} \mapsto (Q, S)$ gives a triangulated equivalence between the triangulated category $HMF^{gr}_A(f)$ and the triangulated category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ introduced in [T2]. In particular, the latter category $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ is defined as the cohomology of a DG-category of twisted complexes. This implies that $D^b_{\mathbb{Z}}(\mathcal{A}_f)$ and then $HMF^{gr}_A(f)$ are enhanced triangulated categories in the sense of Bondal-Kapranov [BK].

In this paper, we often represent a graded matrix factorization $\overline{F} = (F_0 \overset{f_0}{\underset{f_1}{\rightleftarrows}} F_1)$ by its matrix representation (Q, S).

Definition 2.18. Let $t : \mathrm{Ob}(HMF^{gr}_A(f)) \to \mathrm{Ob}(HMF^{gr}_A(f))$ be the bijection induced by the correspondence $t : (Q, S) \mapsto ({}^tQ, -S)$, where ${}^tQ$ is the transpose of the matrix Q. This t is lifted to be a contravariant equivalence functor on $HMF^{gr}_A(f)$, which we denote by the same notation t.

Proposition 2.19. *On $HMF^{gr}_A(f)$, one has the following identities*:

$$\tau t\tau = t, \qquad TtT = t.$$

□

2.5. **Further remark.**

Let $\mathcal{T}$ be one of the equivalent triangulated categories $D^{gr}_{Sg}(A/(f))$, $\underline{\mathrm{CM}}^{gr}(A/(f))$ and $HMF^{gr}_A(f)$.

Since we assume that the ring $R = A/(f)$ defines an isolated singularity, $\mathcal{T}$ is Krull-Schmidt (see [KST1]), that is,

(a) for any two objects $M, M' \in \mathcal{T}$, $\mathrm{Hom}_{\mathcal{T}}(M, M')$ is of finite rank over k;

(b) for any object $M \in \mathcal{T}$ and any idempotent $e \in \mathrm{Hom}_{\mathcal{T}}(M, M)$, there exists an object $M' \in \mathcal{T}$ and a pair of morphisms $g \in \mathrm{Hom}_{\mathcal{T}}(M, M')$, $g' \in \mathrm{Hom}_{\mathcal{T}}(M', M)$ such that $g'g = e$ and $gg' = \mathrm{Id}_{M'}$.

3. Serre duality

In this section, we assume R is a Gorenstein ring and show the existence of the Serre duality and the Auslander-Reiten triangles in the triangulated category $\underline{\mathrm{CM}}^{gr}(R)$. For terminologies, we again refer to [Y].

Definition 3.1. Consider a finite presentation of $M \in \mathrm{gr}–R$ by graded free modules, $F_1 \xrightarrow{f} F_0 \to M \to 0$. Define $\mathrm{tr}(M)$ by the following exact sequence

$$0 \to \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(F_0, R) \xrightarrow{\mathrm{Hom}_R(f,R)} \mathrm{Hom}_R(F_1, R) \to \mathrm{tr}(M) \to 0,$$

i.e., $\mathrm{tr}(M) = \mathrm{Coker}(\mathrm{Hom}_R(f, R))$. The graded module $\mathrm{tr}(M)$ is called the *Auslander transpose* of M.

The Auslander transpose $\mathrm{tr}(M)$ is unique up to free summands. Since we shall only deal with properties that are independent of free summands of $\mathrm{tr}(M)$, the above definition will be sufficient.

Definition 3.2. For a graded R-module $M \in \mathrm{gr}–R$, consider a long exact sequence

$$0 \to N \to F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to F_0 \to M \to 0$$

in $\mathrm{gr}–R$, where each F_i is graded free. The *reduced n-th syzygy* $\mathrm{syz}^n(M)$ of M is the graded R-module obtained from N by deleting all graded free summands.

The reduced n-th syzygy $\mathrm{syz}^n(M)$ is uniquely determined by M and n up to isomorphism.

Definition 3.3. For a graded R-module $M \in \text{gr}-R$, the *Auslander-Reiten (AR-) translation* $\tau_{AR}(M) \in \text{gr}-R$ is defined by

$$\tau_{AR}(M) := \text{Hom}_R(\text{syz}^d(\text{tr}(M)), K_R).$$

Remark 3.4. For $M \in \text{CM}^{gr}(R)$ which is reduced, i.e., has no free direct summands, we have $\text{syz}^2(\text{tr}(M)) \simeq \text{Hom}_R(M, R)$.

Lemma 3.5. *For $M \in \underline{\text{CM}}^{gr}(R)$, we have*

$$\tau_{AR}(M) \simeq T^{d-2}\tau^{-\varepsilon(R)}M,$$

where $\varepsilon(R)$ is the Gorenstein parameter of R defined in Definition 2.5.

Proof. This easily follows from that R is Gorenstein and the definition of the translation functor T on $\underline{\text{CM}}^{gr}(R)$. □

Corollary 3.6. *Suppose that R defines a weighted homogeneous hypersurface as in subsection 2.3. Let $[\tau_{AR}]$ denotes the induced map of $\tau_{AR} : \underline{\text{CM}}^{gr}(R) \to \underline{\text{CM}}^{gr}(R)$ on the Grothendieck group. Then, $[(\tau_{AR})^h] = (-\text{Id})^{hd}$ holds and hence $[\tau_{AR}]$ is of finite order.* □

Proposition 3.7 (Auslander-Reiten duality [AR2])**.** *Let R be a graded Cohen-Macaulay ring of dimension d which defines an isolated singularity and has the canonical module K_R. Then, there exists the following bi-functorial isomorphism of degree zero*

$$\text{Ext}^d_R(\underline{\text{Hom}}_R(M, N), K_R) \simeq \text{Ext}^1_R(N, \tau_{AR}(M)). \tag{3.1}$$

□

By this Proposition, we see that the triangulated category $\underline{\text{CM}}^{gr}(R) \simeq D^{gr}_{Sg}(R)$ has a Serre functor:

Theorem 3.8. [2] *The functor $\mathcal{S} := T\tau_{AR} = T^{d-1}\tau^{-\varepsilon(R)}$ is the Serre functor on $\underline{\text{CM}}^{gr}(R)$. More precisely, $\mathcal{S}$ is an auto-equivalence functor which induces bi-functorial isomorphisms*

$$\text{Hom}_k(\underline{\text{Hom}}_{\text{gr}-R}(M, N), k) \simeq \underline{\text{Hom}}_{\text{gr}-R}(N, \mathcal{S}M), \quad M, N \in \underline{\text{CM}}^{gr}(R).$$

Proof. Note that $\underline{\text{Hom}}_R(M, N)$ is a graded R-module of finite length since R is an isolated singularity. Hence, we have the following isomorphism of degree zero by the local duality theorem

$$\begin{aligned}\text{Hom}_k(\underline{\text{Hom}}_{\text{gr}-R}(M, N), k) &\simeq \text{Hom}_{\text{gr}-R}(\underline{\text{Hom}}_R(M, N), E_R(R/\mathfrak{m}))\\ &\simeq \text{Hom}_{\text{gr}-R}(H^0_{\mathfrak{m}}(\underline{\text{Hom}}_R(M, N)), E_R(R/\mathfrak{m}))\\ &\simeq \text{Ext}^d_{\text{gr}-R}(\underline{\text{Hom}}_R(M, N), K_R),\end{aligned}$$

[2]We thank O. Iyama for explaining to us that Auslander-Reiten duality implies the Serre duality.

where $E_R(R/\mathfrak{m})$ is the injective envelope of the graded R-module $R/\mathfrak{m}$.

Since R is Gorenstein, by Lemma 2.4 (iii), one has $\mathrm{Ext}^1_R(N, F) = 0$ for $N \in \mathrm{CM}^{gr}(R)$ and a free module F, and hence one sees that $\mathrm{Ext}^1_{\mathrm{gr}-R}(N, \tau_{AR}(M)) \simeq \underline{\mathrm{Hom}}_{\mathrm{gr}-R}(N, T\tau_{AR}(M))$. (Recall that there exists an exact sequence $0 \to \tau_{AR}(M) \to F \to T\tau_{AR}(M) \to 0$ in $\mathrm{CM}^{gr}(R)$.) Therefore, we have the canonical isomorphism $\mathrm{Hom}_k(\underline{\mathrm{Hom}}_{\mathrm{gr}-R}(M, N), k) \simeq \underline{\mathrm{Hom}}_{\mathrm{gr}-R}(N, T\tau_{AR}(M))$ of degree zero. □

Remark 3.9. This theorem holds true even if we replace the $\mathbb{Z}$-grading by $L(p)$-grading in the sense of Geigle-Lenzing [GL1, GL2], since the generalization of the Auslander-Reiten duality (Proposition 3.7) is straightforward.

Recall the notion of Auslander-Reiten (AR-)triangles (see [Ha],[Y]; for an Auslander-Reiten sequence or equivalently an almost split sequence, see[AR1].). A morphism g is called *irreducible* if g is neither a split monomorphism nor a split epimorphism but for any factorization $h = g_1 g_2$ either g_1 is a split epimorphism or g_2 is a split monomorphism.

Definition 3.10. An exact triangle in a Krull-Schmidt triangulated category $\mathcal{T}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \tag{3.2}$$

is called an *Auslander-Reiten (AR-)triangle* if the following conditions are satisfied:

(AR1) X, Z are indecomposable objects in $\mathcal{T}$.

(AR2) $w \neq 0$

(AR3) If $g : W \to Z$ is not a split epimorphism, then there exists $g' : W \to Y$ such that $vg' = g$.

We call such a triangle (3.2) an AR-triangle of Z.

Proposition 3.11 (Happel [Ha, Proposition 4.3])**.** *Suppose given an AR-triangle (3.2) in a Krull-Schmidt triangulated category $\mathcal{T}$.*

(i) *Any AR-triangle of Z is isomorphic to the AR-triangle (3.2) as exact triangles.*

(ii) *The morphisms u and v in the AR-triangle (3.2) are irreducible morphisms.*

□

We say that a Krull-Schmidt triangulated category $\mathcal{T}$ *has AR-triangles* if there exists an Auslander-Reiten (AR-)triangle (3.2) of Z for any indecomposable object $Z \in \mathcal{T}$.

Now, for $\mathcal{T} = \underline{\mathrm{CM}}^{gr}(R)$, Theorem 3.8 implies the followings.

Corollary 3.12. *The triangulated category $\underline{\mathrm{CM}}^{gr}(R)$ has AR-triangles.*

Proof. For any indecomposable object $Z \in \mathcal{T}$, the AR-triangle is given by

$$\tau_{AR}(Z) \to AR(Z) \to Z \to T\tau_{AR}(Z)$$

for some $AR(Z) \in \underline{\mathrm{CM}}^{gr}(R)$, where the morphism $Z \to T\tau_{AR}(Z)$ is given by the Serre dual, in the sense in Theorem 3.8, of the identity morphism on Z. This fact can be shown just by following the same argument as that in [Ha, Chapter I. 4.6]. □

Thus, by definition of AR-triangles, one obtains the following which will be employed in the proof of Theorem 5.10. For $X, Y \in \mathcal{T}$, we denote $\hom_{\mathcal{T}}(X, Y) := \dim_k(\mathrm{Hom}_{\mathcal{T}}(X, Y))$.

Corollary 3.13. *For any indecomposable object $Z \in \mathcal{T} := \underline{\mathrm{CM}}^{gr}(R)$, consider the AR-triangle*

$$\tau_{AR}(Z) \to AR(Z) \to Z \to T\tau_{AR}(Z).$$

Then, for any indecomposable object $W \in \mathcal{T}$, one has

$$\hom_{\mathcal{T}}(W, AR(Z)) = (\hom_{\mathcal{T}}(W, Z) - \sigma) + (\hom_{\mathcal{T}}(W, \tau_{AR}(Z)) - \sigma'),$$

$$\hom_{\mathcal{T}}(AR(Z), W) = (\hom_{\mathcal{T}}(Z, W) - \sigma) + (\hom_{\mathcal{T}}(\tau_{AR}(Z), W) - \sigma'),$$

where $\sigma := 1$ if $W \simeq Z$ and zero otherwise, and $\sigma' := 1$ if $W \simeq T^{-1}(Z)$ and zero otherwise. □

4. Category generating theorem

In this section, we discuss about the generation of the category $D^{\mathrm{gr}}_{\mathrm{Sg}}(R)$. We show Theorem 4.5 and then Corollary 4.7 which is necessary to prove the structure theorem (Theorem 5.10) of $HMF^{gr}_A(f)$.

We first recall some definitions and facts concerning admissible categories and exceptional collections from [Bo, O].

Definition 4.1. Let $\mathcal{T}$ be a triangulated category and $\mathcal{T}' \subset \mathcal{T}$ a full triangulated subcategory. The *right orthogonal* to $\mathcal{T}'$ is a full subcategory $(\mathcal{T}')^{\perp} \subset \mathcal{T}$ consisting of all objects M such that $\mathrm{Hom}_{\mathcal{T}}(N, M) = 0$ for any $N \in \mathcal{T}'$.

Definition 4.2. Let $\mathcal{T}$ be a triangulated category and $\mathcal{T}' \subset \mathcal{T}$ a full triangulated subcategory. We say that $\mathcal{T}'$ is *right admissible* if, for any $X \in \mathcal{T}$, there is an exact triangle $N \to X \to M \to TN$ with $N \in \mathcal{T}'$ and $M \in (\mathcal{T}')^{\perp}$.

Definition 4.3. An object E of a k-linear triangulated category $\mathcal{T}$ is called *exceptional* if $\mathrm{Hom}_{\mathcal{T}}(E, T^n E) = 0$ when $n \neq 0$ and $\mathrm{Hom}_{\mathcal{T}}(E, E) \simeq k$. An *exceptional collection* is a sequence of exceptional objects $\mathcal{E} := (E_1, \ldots, E_l)$ satisfying the condition $\mathrm{Hom}_{\mathcal{T}}(E_i, T^n E_j) = 0$ for all n and $i > j$. Furthermore, an exceptional collection $\mathcal{E} = (E_1, \ldots, E_l)$ is called a *strongly exceptional collection* if $\mathrm{Hom}_{\mathcal{T}}(E_i, T^n E_j) = 0$ for all i, j and all n except for $n = 0$.

We say that a triangulated category $\mathcal{T}$ is of *finite type* if, for any $E, E' \in \mathcal{T}$, $\mathrm{Hom}_{\mathcal{T}}(E, T^n E')$ is of finite rank over k and in particular zero for almost all $n \in \mathbb{Z}$.

Proposition 4.4 (Bondal [Bo, Theorem 3.2] (see also [BP]))**.** *Let $\mathcal{T}$ be a triangulated category of finite type and $\mathcal{T}' := \langle E_1, \ldots, E_l \rangle \subset \mathcal{T}$ a full triangulated subcategory generated by an exceptional collection $(E_1, \ldots, E_l)$. Then, $\mathcal{T}'$ is right admissible.* □

The following result is the key lemma of this paper.

Theorem 4.5. *Let $\mathcal{T}'$ be a right admissible full triangulated subcategory of $D^{\mathrm{gr}}_{\mathrm{Sg}}(R)$, with R a Gorenstein ring, satisfying the following conditions*:

(i) *The shift functor τ on $D^{\mathrm{gr}}_{\mathrm{Sg}}(R)$ induces an autoequivalence of $\mathcal{T}'$,*
(ii) *$\mathcal{T}'$ has an object E which is isomorphic to $R/\mathfrak{m}$ in $D^{\mathrm{gr}}_{\mathrm{Sg}}(R)$.*

Then $\mathcal{T}'$ is equivalent to $D^{\mathrm{gr}}_{\mathrm{Sg}}(R)$ as a triangulated category.

Proof. By the equivalence $\underline{\mathrm{CM}}^{gr}(R) \simeq D^{\mathrm{gr}}_{\mathrm{Sg}}(R)$, we shall often represent objects in $D^{\mathrm{gr}}_{\mathrm{Sg}}(R)$ by the corresponding graded maximal Cohen-Macaulay modules.

First, recall the following characterization of free modules:

Lemma 4.6 ((see [Y]))**.** *An object $M \in \mathrm{CM}^{gr}(R)$ is graded free if and only if $\mathrm{Ext}^i_R(R/\mathfrak{m}, M) = 0$ for $i \neq d$.* □

Take a minimal graded free resolution of $R/\mathfrak{m} \in D^{\mathrm{gr}}_{\mathrm{Sg}}(R)$

$$\cdots \to F_2 \to F_1 \to F_0 \to R/\mathfrak{m} \to 0.$$

By definition of syzygy, one has

$$0 \to \mathrm{syz}^{i+1}(R/\mathfrak{m}) \to F_i \to \mathrm{syz}^i(R/\mathfrak{m}) \to 0 \tag{4.1}$$

for any $i \geq 0$, which implies $\mathrm{syz}^i(R/\mathfrak{m}) \simeq T^{-i}(R/\mathfrak{m})$ in $D^{\mathrm{gr}}_{\mathrm{Sg}}(R)$. For $N \in \mathrm{CM}^{gr}(R)$ and $i \geq 0$, the long exact sequence obtained from eq.(4.1) yields

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}_R(\mathrm{syz}^i(R/\mathfrak{m}), N) & \to & \mathrm{Hom}_R(F_i, N) & \to & \mathrm{Hom}_R(\mathrm{syz}^{i+1}(R/\mathfrak{m}), N) \quad \to \\ & \to & \mathrm{Ext}^1_R(\mathrm{syz}^i(R/\mathfrak{m}), N) & \to & 0 & & \end{array} \tag{4.2}$$

and

$$\mathrm{Ext}^n_R(\mathrm{syz}^{i+1}(R/\mathfrak{m}), N) \simeq \mathrm{Ext}^{n+1}_R(\mathrm{syz}^i(R/\mathfrak{m}), N), \quad n \geq 1 \tag{4.3}$$

since $\mathrm{Ext}^k(F_i, N) = 0$ for $k \geq 1$. Note that $\mathrm{syz}^i(R/\mathfrak{m}) \in \mathrm{CM}^{gr}(R)$ for $i \geq d = \dim R$ and then the exact sequence (4.1) becomes the one in $\mathrm{CM}^{gr}(R)$.

Now, in the exact sequence (4.2) with $i \geq d$ and $N \in (\mathcal{T}')^{\perp}$, any morphism in $\mathrm{Hom}_R(\mathrm{syz}^{i+1}(R/\mathfrak{m}), N)$ factors through a projective-injective object I in the Frobenius category $\mathrm{CM}^{gr}(R)$.[3] Moreover, any morphism in $\mathrm{Hom}_R(\mathrm{syz}^{i+1}(R/\mathfrak{m}), I)$ factors through F_i

[3] In fact, $I \in \mathrm{CM}^{gr}(R)$ is a projective-injective object if and only if I is graded free.

with the injection $\mathrm{syz}^{i+1}(R/\mathfrak{m}) \to F_i$ since I is injective. Therefore, any morphism in $\mathrm{Hom}_R(\mathrm{syz}^{i+1}(R/\mathfrak{m}), N)$ factors through F_i, which implies that the map $\mathrm{Hom}_R(F_i, N) \to \mathrm{Hom}_R(\mathrm{syz}^{i+1}(R/\mathfrak{m}), N)$ in eq.(4.2) is surjective and hence $\mathrm{Ext}^1_R(\mathrm{syz}^i(R/\mathfrak{m}), N) = 0$.

Here, by the isomorphisms (4.3), $\mathrm{Ext}^1_R(\mathrm{syz}^i(R/\mathfrak{m}), N) \simeq \mathrm{Ext}^{i+1}_R(R/\mathfrak{m}, N)$ holds and hence $\mathrm{Ext}^{i+1}_R(R/\mathfrak{m}, N) = 0$ for $N \in (\mathcal{T}')^\perp$ and $i \geq d$. By Lemma 4.6, this means that N is graded free, which is isomorphic to zero in $\underline{\mathrm{CM}}^{gr}(R)$. The theorem follows. □

If R defines a hypersurface singularity $A/(f)$, then, by the isomorphism of functors $T^2 \simeq \tau^h$, we see that $D^{gr}_{Sg}(R)$ is of finite type.

Corollary 4.7. *Let $\langle E_1, \ldots, E_l \rangle$ be a full triangulated subcategory of $\mathcal{T} = D^{gr}_{Sg}(R)$ generated by an exceptional collection $(E_1, \ldots, E_l)$ which is closed under the action of τ and contains an object isomorphic to $R/\mathfrak{m}$. Then $\langle E_1, \ldots, E_l \rangle \simeq \mathcal{T}$ as a triangulated category.* □

Remark 4.8. In this paper, we shall apply this category generating lemma (Theorem 4.5 or Corollary 4.7) together with the Serre functor in Theorem 3.8 to the corresponding triangulated categories associated to the regular systems of weight with $\varepsilon_W = -1$ and $a_0 = 0$ (see subsection 5.1). However, these two theorems theirselves can be applied to the cases of any ε_W with $a_0 = 0$. By these theorems, the proof of the main theorem in [KST1] (ADE case: $\varepsilon_W = 1$ and $a_0 = 0$) can be simplified. Moreover, the category generating lemma (Theorem 4.5) holds true even if we place the $\mathbb{Z}$-grading by $L(p)$-grading as Theorem 3.8 does. Thus, we can apply these two theorems to $a_0 > 0$ cases including the elliptic cases ($\varepsilon_W = 0$ and $a_0 = 1$), which simplifies the proof of the main theorem of [U].

5. Strongly exceptional collections in $\mathcal{T}_W$ and the associated quivers

In this section, we formulate our main result on the structure of the triangulated category ($D^{\mathrm{gr}}_{\mathrm{Sg}}(R_W) \simeq \underline{\mathrm{CM}}^{gr}(R_W) \simeq HMF^{gr}_A(f_W)$) associated to a regular system of weights W of $\varepsilon_W = -1$ and $a_0 = 0$ with a fixed weighted homogeneous polynomial f_W. In subsection 5.1, we recall the definition of the regular system of weights. In subsection 5.2, we prepare a generalized notion of quivers; the quivers associated to regular systems of weights W of $\varepsilon_W = -1$ with genus $a_0 = 0$ are defined in this notion in subsection 5.3. Then, in subsection 5.4, we state the main theorem of the present paper.

5.1. **Regular system of weights.**

In this subsection, we recall the definition and some of basic facts on the regular systems of weights. A quadruple $W := (a, b, c; h)$ of positive integers with $a, b, c < h$ and g.c.d.$(a, b, c) = 1$ is called a *weight system.* For a weight system W, we define the *Euler vector* *field* $E = E_W$ by

$$E := \frac{a}{h}x\frac{\partial}{\partial x} + \frac{b}{h}y\frac{\partial}{\partial y} + \frac{c}{h}z\frac{\partial}{\partial z} \ . \tag{5.1}$$

For a given weight system W, the regular $\mathbb{C}$-algebra $A = \mathbb{C}[x, y, z]$ becomes a graded ring by putting $\deg(x) = 2a/h$, $\deg(y) = 2b/h$ and $\deg(z) = 2c/h$. Let $A = \oplus_{s\in\frac{2}{h}\mathbb{Z}_{\geq 0}} A_s$ be the graded piece decomposition, where $A_s := \{f \in A \mid 2Ef = s \cdot f\}$. A weight system W is called *regular* ([Sa1]) if the following equivalent conditions are satisfied:

(a) $\chi_W(T) := T^{-h}\frac{(T^h-T^a)(T^h-T^b)(T^h-T^c)}{(T^a-1)(T^b-1)(T^c-1)}$ has no poles except at $T = 0$.

(b) A generic element f_W of the space $A_2 = \{f \in A \mid Ef = f\}$ has an isolated critical point at the origin, *i.e.*, the Jacobi ring $A/\left(\frac{\partial f_W}{\partial x}, \frac{\partial f_W}{\partial y}, \frac{\partial f_W}{\partial z}\right)$ is finite rank over $\mathbb{C}$.

(c) There exists a finite sequence of integers $m_1 \leq m_2 \leq \cdots \leq m_{\mu_W}$ for some $\mu_W \in \mathbb{Z}_{>0}$ such that the function $\chi_W(T)$ has a Laurent polynomial expansion:

$$\chi_W(T) = T^{m_1} + T^{m_2} + \cdots + T^{m_{\mu_W}}.$$

Here, the number μ_W, called the rank of W, is given by $(h - a)(h - b)(h - c)/abc$, and $\{m_1, \ldots, m_{\mu_W}\}$ is called the set of *exponents* of W. The smallest exponent m_1 is given by $\varepsilon_W := a + b + c - h$. An element f_W in A_2 as in (b) is called a *polynomial of type* W. The quotient ring $R_W := A/(f_W)$ is Gorenstein, whose Gorenstein parameter $\varepsilon(R_W)$ is given by the smallest exponent ε_W because the canonical module K_{R_W} is given by the residue $Res[A\frac{dxdydz}{f_W}] = \tau^{-\varepsilon_W} R_W$.

The regular systems of weights are classified as follows. Regular systems of weights W with $\varepsilon_W > 0$ automatically have $\varepsilon_W = 1$. They are called of type ADE via the identification of their exponents with those of the root systems of type ADE. In our previous paper [KST1], we studied the triangulated category $HMF^{gr}_A(f_W)$ of this type and obtained the root systems of type ADE, as expected. Next, regular systems of weights with $\varepsilon_W = 0$ are called of *elliptic* since they are associated with simply elliptic singularities. There are three such regular systems of weights, which are often denoted by $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ according to the classification of elliptic root systems [Sa2].

In this paper, we discuss the triangulated category $HMF^{gr}_A(f_W)$ attached to regular systems of weights with $\varepsilon_W = -1$. We concentrate on such regular systems of weights with genus $a_0 = 0$, where the genus a_0 of W is defined as the number of zero exponents of W. There are twenty two regular systems of weights with $\varepsilon_W = -1$ and genus zero, which include fourteen ones, so-called, of *exceptional unimodular* type (see subsection 5.5).

In order to describe our strongly exceptional collections in $HMF^{gr}_A(f_W)$ for a regular system of weights W of $\varepsilon_W = -1$, we recall the notion of the signature.

Definition 5.1 (Signature of a regular system of weights ([Sa1] eq.(5.3.2)))**.** For a given regular system of weights W, consider the following multi-sets of positive integers

$$A'_W := \{a_i \,|\, h/a_i \notin \mathbb{Z}\,,\, i = 1, 2, 3\} \coprod \{\gcd(a_i, a_j)^{(m(a_i,a_j:h)-1)} \,|\, 1 \le i < j \le 3\}$$

where $a_1 = a$, $a_2 = b$, $a_3 = c$, and $\gcd(a_i, a_j)^{(m(a_i,a_j:h)-1)}$ indicates that we include $(m(a_i, a_j : h) - 1)$ copies of $\gcd(a_i, a_j)$ in A'_W with $m(a_i, a_j : h) := \sharp\{(u, v) \in (\mathbb{Z}_{\ge 0})^2 \,|\, a_i u + a_j v = h\}$. We exclude elements equal to one in A'_W, and denote the result by $A_W = (\alpha_1, \dots, \alpha_r)$ for some $r \in \mathbb{Z}_{\ge 0}$, where $\alpha_1, \dots, \alpha_r$ are the remaining elements in A_W so that $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_r$. The pair $(A_W; a_0)$ is called the *signature* of W.

Definition 5.2 (Dual rank ν_W)**.** For a given regular system of weights W, we call the integer

$$\nu_W := \sum_{i=1}^{r} (\alpha_i - 1) + 2(1 - a_0) - \varepsilon_W$$

the *dual rank* of W.

Remark 5.3. Historically, the pair (A_W, a_0) was called the signature (Fricke-Klein, Magnus). We omit a_0 and call A_W the signature of W in this paper, since we discuss regular systems of weights W with $\varepsilon_W = -1$ and $a_0 = 0$ only.

The dual rank ν_W was originally introduced in [Sa3] as a "virtual" rank of the $*$ dual weight system W^* of a regular system of weights W. Since the original formula needs a slight preparation, we employ the above formula. The equivalence between those two formulas shall be discussed elsewhere.

For a given W, consider a polynomial f_W of type W. The quotient of the hypersurface $\{(x, y, z) \in \mathbb{C}^3 \,|\, f_W = 0\}$ by the $\mathbb{C}^\times$-action defined by the weight (a, b, c) turns out to be a curve of genus a_0 having r distinct orbifold points $(\lambda_1, \dots, \lambda_r)$ with order $A_W = (\alpha_1, \dots, \alpha_r)$. See Remark 10.3. of [Sa3]. The relation of these curves with the weighted projective line [GL1, GL2] will be discussed in [KST2].

5.2. **Quivers and path algebras of relations.**

For a triangulated category $\mathcal{T}$, assume there exists a strongly exceptional collection $\mathcal{E} := (E_1, \dots, E_l)$. We denote

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, \mathcal{E}) := \oplus_{i,j=1}^{l} \mathrm{Hom}_{\mathcal{T}}(E_i, E_j)$$

and call it the *homomorphism algebra* of $\mathcal{E}$. Then, it is known that there is a unique quiver such that its path algebra with some relations is isomorphic to the homomorphism algebra $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, \mathcal{E})$ (see Gabriel [Ga2]). Although those terminologies are enough for our purpose, in this subsection, we introduce a modified notion of quivers and give a more explicit rule to attach such a quiver to an exceptional collection $\mathcal{E}$. In particular, when $\mathcal{E}$ is a strongly exceptional collection, we define path algebras of relations of the corresponding quivers.

Let $\mathcal{T}$ be a triangulated category of finite type. For any two objects $X, Y \in \mathcal{T}$, the Euler pairing is defined by

$$\chi(X,Y) := \sum_{n\in\mathbb{Z}} (-1)^n \hom_{\mathcal{T}}(X, T^n(Y)).$$

Suppose there exists an exceptional collection $\mathcal{E} = (E_1, \dots, E_l)$ in $\mathcal{T}$. The l by l matrix

$$\chi := \{\chi_{ij}\}, \qquad \chi_{ij} := \chi(E_i, E_j),$$

is an upper half triangular matrix with $\chi_{ii} = 1$ for any $i = 1, \dots, l$. Then, the inverse matrix $C := \chi^{-1}$ is also an upper half triangular matrix with $C_{ii} = 1$, $i = 1, \dots, l$. Here we define a quiver associated to such a matrix C.

Definition 5.4. Let $C = \{C_{ij}\}_{i,j=1,\dots,l}$ be an upper triangular l by l matrix of integer valued such that $C_{ii} = 1$ for any $i = 1, \dots, l$.

The *quiver* $\vec{\Delta}_C = (\Delta_0, \Delta_1; s, e, d)$ *associated to* C is the set $\Delta_0 = \{1, \dots, l\}$ of vertices and the set Δ_1 of arrows with maps $s : \Delta_1 \to \Delta_0$, $e : \Delta_1 \to \Delta_0$, $d : \Delta_1 \to \{\pm 1\}$ such that $s(\rho) \neq e(\rho)$ for any $\rho \in \Delta_1$ and

$$(\sharp\{\rho \in \Delta_1 \,|\, (s,e,d)(\rho) = (i,j,+1)\}, \sharp\{\rho \in \Delta_1 \,|\, (s,e,d)(\rho) = (i,j,-1)\}) = \begin{cases} (-C_{ij}, 0) & C_{ij} < 0 \\ (0, C_{ij}) & C_{ij} > 0 \\ (0,0) & C_{ij} = 0 \end{cases}$$

for any $i < j$.

Now, suppose that we start from a quiver associated to C and denote the inverse matrix of C by $\chi := \{\chi_{ij}\}$. By definition one has $\chi_{ii} = 1$ for any i and $\chi_{ij} = 0$ for $i > j$.

When $\chi_{ij} \geq 0$ for any i and j, a *path algebra with relation* of the quiver $\vec{\Delta}_C$ is defined as follows. Let $\mathbb{C}\vec{\Delta}_C$ be the path algebra defined by arrows ρ of $d(\rho) = +1$. For each arrow ρ such that $(s,e,d)(\rho) = (i,j,-1)$, a relation I_ρ is given as a $\mathbb{C}$-linear combination of all paths from i to j. Then, the path algebra with relations is the quotient algebra $\mathbb{C}\vec{\Delta}_C/I$, where I is the ideal generated by $\{I_\rho\}_{d(\rho)=-1}$.

Remark 5.5. For a given exceptional collection $\mathcal{E} := (E_1, \dots, E_l)$ in a triangulated category $\mathcal{T}$ of finite type, the above procedure actually gives an explicit way to define a quiver $\vec{\Delta}_C$, and, in particular, if $\mathcal{E}$ is a strongly exceptional collection, one can define a path algebra with relations of the quiver $\vec{\Delta}_C$. However, in general, the homomorphism algebra $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, \mathcal{E})$ of $\mathcal{E}$ is not isomorphic to a path algebra with relations of $\vec{\Delta}_C$. This is related to that Definition 5.4 forbids the existence of both arrows $\rho, \rho' \in \Delta_1$ such that $(s,e,d)(\rho) = (i,j,+1)$ and

$(s, e, d)(\rho) = (i, j, -1)$ for the same $i < j$. As we shall see in subsection 5.4, in our situation, this procedure gives an explicit way to give the correct quiver associated to an exceptional collection $\mathcal{E}$. In particular, the matrix element χ_{ij} of the inverse matrix χ of C is calculated by counting paths from i to j, consisting of arrows $\rho \in \Delta_1$ of both $d(\rho) = \pm 1$, with sign associated to $d = \pm 1$.

Hereafter we call a quiver associated to C just a quiver. We sometimes drop C when we do not give the explicit form of the corresponding upper triangular matrix C.

5.3. Path algebras with relations of quivers $\vec{\Delta}_W$, $\vec{\Delta}_W^T$ and $\vec{\Delta}'_W$.

Now, for a regular system of weights W with $\varepsilon_W = -1$ and $a_0 = 0$, we define quivers $\vec{\Delta}_W$, $\vec{\Delta}_W^T$, $\vec{\Delta}'_W$ and associated path algebras with relations which are necessary to state the main theorem.

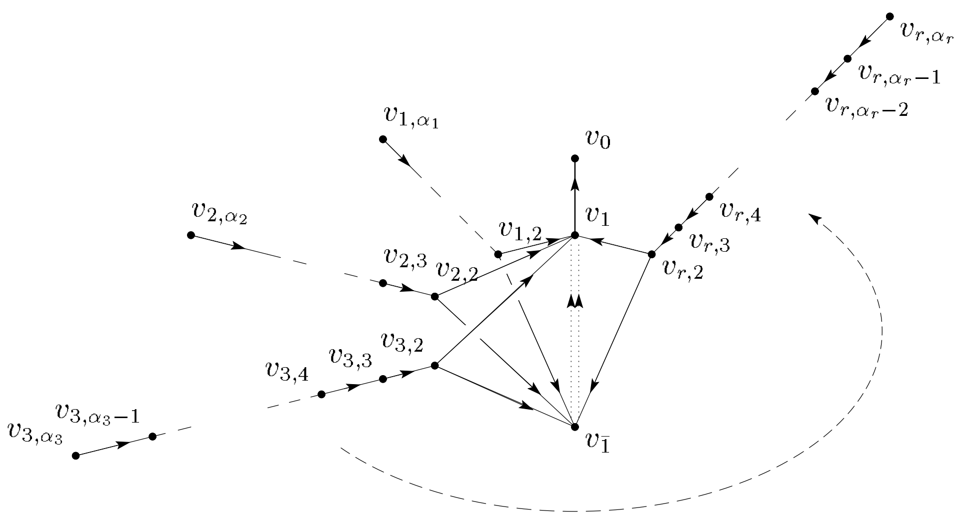

FIGURE 1. Figure of $\vec{\Delta}_W$

Definition 5.6 (Quivers $\vec{\Delta}_W$, $\vec{\Delta}_W^T$, $\vec{\Delta}'_W$)**.** For a regular system of weights W of $\varepsilon_W = -1$ and genus $a_0 = 0$ with signature $A_W = (\alpha_1, \dots, \alpha_r)$, we define quivers $\vec{\Delta}_W$, $\vec{\Delta}_W^T$ and $\vec{\Delta}'_W$ by those in Figure 1, 2 and 3, respectively, where $\Delta_0 = \Pi_W := \coprod_{i=1}^r \{v_{i,2}, \dots, v_{i,\alpha_i}\} \coprod \{v_0, v_1, v_{\bar{1}}\}$ is the vertex set which is isomorphic to $\{1, \dots, \nu_W\}$ as sets, the arrows denote elements $\rho \in \Delta_1$ of $d(\rho) = +1$ and the dotted arrows denote elements $\rho \in \Delta_1$ of $d(\rho) = -1$. Under the identification $\Pi_W \simeq \{1, \dots, \nu_W\}$, for any $v, v' \in \Pi_W$, we sometimes denote by $C(v, v')$ or $\chi(v, v')$ the corresponding matrix elements of C or χ.

If we forget the orientation of the quiver $\vec{\Delta}_W$ or $\vec{\Delta}'_W$, we can recover the diagram Δ_W in Figure 4 which was used to appear as the intersection matrix of the vanishing cycles of an exceptional unimodular singularity (see subsection 5.5). We sometimes denote this diagram

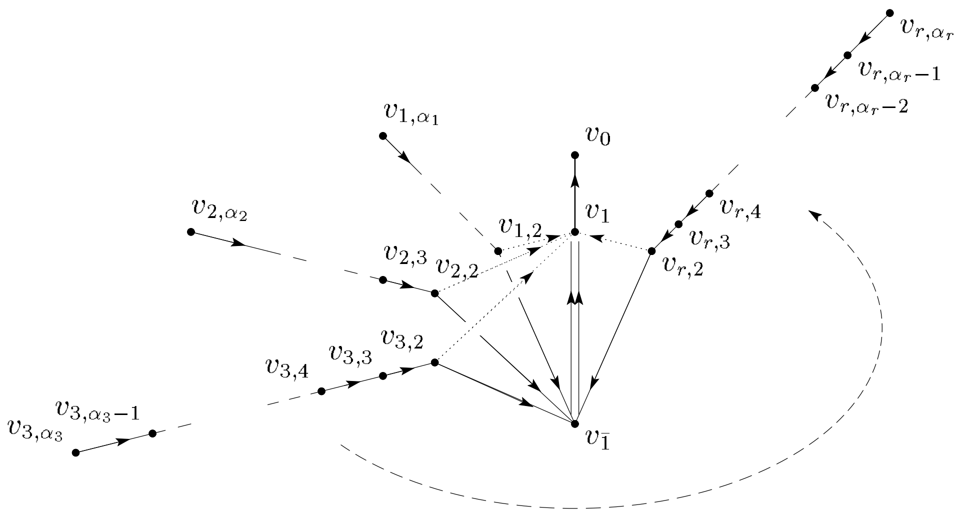

FIGURE 2. Figure of $\vec{\Delta}_W^T$

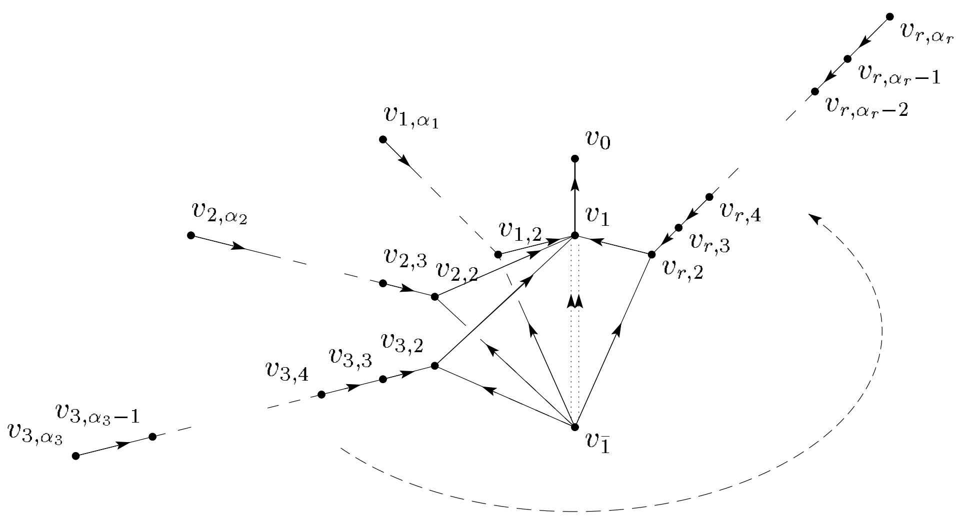

FIGURE 3. Figure of $\vec{\Delta}'_W$

more explicitly by $\Delta_{\alpha_1,\ldots,\alpha_r} := \Delta_W$ for $A_W = (\alpha_1, \ldots, \alpha_r)$.

Note that, for the quiver $\vec{\Delta}_W$, $C(v_{i,2}, v_{\bar{1}}) = C(v_{i,2}, v_1) = -1$ for any $i = 1, \ldots, r$ and $C(v_{\bar{1}}, v_1) = 2$. Thus, $\chi(v_{\bar{1}}, v_1) = -2$ and also $\chi(v_{i,2}, v_1) = -1$. On the other hand, for the quiver $\vec{\Delta}_W^T$, one has $C(v_{i,2}, v_1) = 1$ and $C(v_{\bar{1}}, v_1) = -2$. Then, the inverse matrix χ of C has non-negative elements only. In particular, $\chi(v_{\bar{1}}, v_1) = 2$ and $\chi(v_{i,2}, v_1) = 1$. For the quiver $\vec{\Delta}'_W$, $C(v_{i,2}, v_{\bar{1}}) = 0$ but $C(v_{\bar{1}}, v_{i,2}) = -1$. Then, again the inverse matrix χ has non-negative elements only. In particular, $\chi(v_{\bar{1}}, v_{i,2}) = \chi(v_{i,2}, v_1) = 1$ and $\chi(v_{\bar{1}}, v_1) = r - 2$.

For each of the quivers $\vec{\Delta}_W^T$ and $\vec{\Delta}'_W$, we define a path algebra with relations as follows.

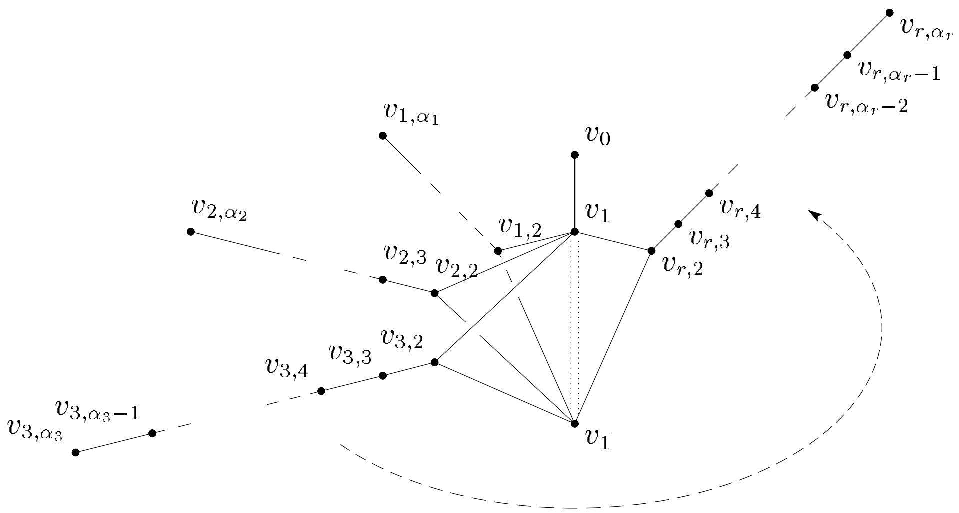

FIGURE 4. The diagram $\Delta_W = \Delta_{\alpha_1,\dots,\alpha_r}$, $A_W = (\alpha_1, \dots, \alpha_r)$, obtained from $\vec{\Delta}_W$ or $\vec{\Delta}'_W$ by removing the orientation of the arrows.

Definition 5.7 ($\mathbb{C}\vec{\Delta}^T_W / I_{W,\Lambda}$, $\mathbb{C}\vec{\Delta}'_W / I'_{W,\Lambda}$)**.** For the quiver $\vec{\Delta}^T_W$, we define r relations corresponding to the dotted arrows $\rho(v_{i,2}, v_1)$ from $v_{i,2}$ to v_1, $i = 1, \dots, r$, by

$$u_{1,i} \cdot (\rho_1(v_{\bar{1}}, v_1) \circ \rho(v_{i,2}, v_{\bar{1}})) + u_{2,i} \cdot (\rho_2(v_{\bar{1}}, v_1) \circ \rho(v_{i,2}, v_{\bar{1}})), \tag{5.2}$$

where $\rho_1(v_{\bar{1}}, v_1)$ and $\rho_2(v_{\bar{1}}, v_1)$ are two arrows from $v_{\bar{1}}$ to v_1, and $[u_{1,i} : u_{2,i}] = \lambda_i$ is a point in $\mathbb{P}^1$. We denote by $I_{W,\Lambda}$, $\Lambda := (\lambda_1, \dots, \lambda_r)$, the ideal generated by these relations, and by $\mathbb{C}\vec{\Delta}^T_W / I_{W,\Lambda}$ the corresponding path algebra with relations.

For the quiver $\vec{\Delta}'_W$, we define two relations corresponding to the dotted arrows $\rho_1(v_{\bar{1}}, v_1)$, $\rho_2(v_{\bar{1}}, v_1)$ from $v_{\bar{1}}$ to v_1 by

$$\sum_{i=1}^{r} u_{1,i} \cdot (\rho(v_{i,2}, v_1) \circ \rho(v_{\bar{1}}, v_{i,2})), \qquad \sum_{i=1}^{r} u_{2,i} \cdot (\rho(v_{i,2}, v_1) \circ \rho(v_{\bar{1}}, v_{i,2})), \tag{5.3}$$

where $\rho(v_{i,2}, v_1)$ is the arrow from $v_{i,2}$ to v_1, and $\rho(v_{\bar{1}}, v_{i,2})$ is the arrow from $v_{\bar{1}}$ to $v_{i,2}$. We denote by $I'_{W,\Lambda}$ the ideal generated by these relations and by $\mathbb{C}\vec{\Delta}'_W / I'_{W,\Lambda}$ the corresponding path algebra with relations.

5.4. **The main theorem (Theorem 5.8).**

Recall that $\mathcal{T}_W$ is one of the equivalent triangulated categories $HMF^{gr}_A(f_W) \simeq D^{\mathrm{gr}}_{\mathrm{Sg}}(R_W) \simeq \underline{\mathrm{CM}^{gr}}(R_W)$ with $R_W = A/(f_W)$ for a fixed polynomial f_W of W.

The following is the main theorem of the present paper.

Theorem 5.8. *Let f_W be a polynomial of type W with $\varepsilon_W = -1$ and $a_0 = 0$. Then, there exist distinct r-points $\Lambda = (\lambda_1, \dots, \lambda_r)$ in $\mathbb{P}^1$ and a full strongly exceptional collection $\mathcal{E}^T$ in*

$\mathcal{T}_W$ such that the homomorphism algebra $\mathrm{Hom}_{\mathcal{T}_W}(\mathcal{E}^T,\mathcal{E}^T)$ is isomorphic to the path algebra with relations $\mathbb{C}\vec{\Delta}^T_W/I_{W,\Lambda}$.

The same statement holds true even if $\mathbb{C}\vec{\Delta}^T_W/I_{W,\Lambda}$ is replaced by $\mathbb{C}\vec{\Delta}'_W/I'_{W,\Lambda}$.

This, together with Bondal-Kapranov's theorem [Bo, BK], implies the following:

Corollary 5.9. *The triangulated category $\mathcal{T}_W$ is triangulated equivalent to the derived category of finitely generated modules over the algebra $\mathbb{C}\vec{\Delta}^T_W/I_{W,\Lambda}$ or $\mathbb{C}\vec{\Delta}'_W/I'_{W,\Lambda}$.*

Recently, this corollary is proved independently by Lenzing and de la Pena [LP] in the framework of the weighted projective lines by Geigle-Lenzing together with Orlov's arguments in [O].

The main theorem is obtained as a consequence of the following structure theorem of the triangulated category $\mathcal{T}_W$.

Theorem 5.10. *Let f_W be a polynomial of type W with $\varepsilon_W=-1$, $a_0=0$ and the virtual rank ν_W. Then, there exist distinct r-points $\Lambda=(\lambda_1,\dots,\lambda_r)$ in $\mathbb{P}^1$ and a full exceptional collection $\mathcal{E}:=(E_1,\dots,E_{\nu_W})$ in $\mathcal{T}_W$ which satisfies the following properties:*

(i) *For the quiver $\vec{\Delta}_W=(\Delta_0=\Pi_W,\Delta_1;s,e,d)$, there exists a map $V:\Pi_W\to\mathrm{Ob}(\mathcal{T}_W)$ and one has*

$$\{\mathcal{E}\}:=\{E_1,\dots,E_{\nu_W}\}=\coprod_{i=1}^{r}\{V_{i,2},\dots,V_{i,\alpha_i}\}\coprod\{V_0,V_1,V_{\bar{1}}\}$$

as sets, where $V_{i,j}:=V(v_{i,j})$, $V_0:=V(v_0)$, $V_1:=V(v_1)$, $V_{\bar{1}}:=V(v_{\bar{1}})$, and ν_W is the virtual rank of W defined in Definition 5.2.

(ii) *For any $v,v'\in\Pi_W$, one has $\mathrm{Hom}_{\mathcal{T}}(V(v),T^n(V(v')))\neq 0$ only if $n=0$ or $n=1$. Thus, $\chi(V(v),V(v'))=\hom_{\mathcal{T}}(V(v),V(v'))-\hom_{\mathcal{T}}(V(v),T(V(v')))$ and then*

$$\chi(V(v),V(v'))=\chi(v,v')$$

holds, where $\chi(v,v')$ is the matrix element in Definition 5.6. We denote by f_i, $i=1,\dots,r$, a basis of $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,2},V_{\bar{1}})$ and by g_i, $i=1,\dots,r$, a basis of $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,2},TV_1)$, where note that $\chi(v_{i,2},v_1)=\chi(V_{i,2},V_1)=-1$.

(iii) *Under the transpose in Definition 2.18, the objects in $\{\mathcal{E}\}$ satisfy*

$$t(V_{i,j})=\tau^{-(j-1)}(V_{i,j}),$$
$$t(V_0)=T\tau^{-1}(V_0),$$
$$t(V_1)=T(V_1),$$
$$t(V_{\bar{1}})=\tau^{-1}(V'_{\bar{1}}),$$

where $T(V'_{\bar{1}})\in\mathrm{Ob}(\mathcal{T})$ is the cone of the morphisms $\oplus_{i=1}^r f_i:\oplus_{i=1}^r V_{i,2}\to V_{\bar{1}}$.

(iv) *For each element $V \in \{\mathcal{E}\}$, the AR-triangle is given as follows. Recall that $\tau_{AR} = \tau$.*

(iv-a) For $V_{i,j} \in \{\mathcal{E}\}$, $i = 1, 2, \dots, r$, $j = 2, 3, \dots, \alpha_i$,

$$\tau_{AR}(V_{i,j}) \to \tau V_{i,j-1} \oplus V_{i,j+1} \to V_{i,j} \to T\tau_{AR}(V_{i,j}), \tag{5.4}$$

where we put $V_{i,\alpha_i+1} = 0$ for $j = \alpha_i$ and $V_{i,1}$ is defined by the above triangle with $j = 2$.

(iv-a') For $V_0 \in \{\mathcal{E}\}$,

$$\tau_{AR}(V_0) \to V_1 \to V_0 \to T\tau_{AR}(V_0), \tag{5.5}$$

(iv-b) For $V_1, V_{\bar{1}} \in \{\mathcal{E}\}$,

$$\tau_{AR}(V_1) \to V_2 \oplus \tau V_0 \to V_1 \to T\tau_{AR}(V_1), \tag{5.6}$$

$$\tau_{AR} V_{\bar{1}} \to AR(V_{\bar{1}}) \to V_{\bar{1}} \to T\tau V_{\bar{1}}. \tag{5.7}$$

Here, in eq.(5.6), $T(V_2)$ is defined by the cone of $(V_{1,2} \oplus V_{2,2} \oplus \cdots \oplus V_{r,2}) \to (V_{\bar{1}})^{\oplus 2}$ with the morphisms given by

$$u_{1,i} \cdot f_i \oplus u_{2,i} \cdot f_i : V_{i,2} \to V_{\bar{1}} \oplus V_{\bar{1}}$$

for $\lambda_i = [u_{1,i} : u_{2,i}] \in \mathbb{P}^1$ and f_i, $i = 1, \dots, r$, a basis of $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, V_{\bar{1}})$. In eq.(5.7), $AR(V_{\bar{1}})$ is defined by the cone of $(\tau V_1)^{\oplus 2} \to (V_{1,2} \oplus V_{2,2} \oplus \cdots \oplus V_{r,2})$ with the morphism given by

$$u_{1,i} \cdot \tau t(g_i) \oplus u_{2,i} \cdot \tau t(g_i) : (\tau V_1)^{\oplus 2} \to V_{i,2}$$

for g_i, $i = 1, \dots, r$, a basis of $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, TV_1)$, where recall that t is the transpose in Definition 2.18.

(iv') *There exists a triangle*

$$V_1 \to V_{i,1} \to V_{\bar{1}} \to TV_1 \tag{5.8}$$

for each $i = 1, 2, \dots r$, where the morphism $V_{\bar{1}} \to TV_1$ is described as $u_{1,i}e_1 + u_{2,i}e_2$ in terms of a basis $\{e_1, e_2\}$ of $\mathrm{Hom}_{\mathcal{T}_W}(V_{\bar{1}}, T(V_1))$.

The AR-triangles (5.4) and (5.5) imply that the morphism $V_{i,j} \to V_{i,j-1}$ for each i and $j = 2, \dots, \alpha_i$ and the morphism $V_1 \to V_0$, respectively, are irreducible. Furthermore, by $\chi(v_{\bar{1}}, v_1) = -2$, one has $\mathrm{hom}_{\mathcal{T}_W}(V_{\bar{1}}, TV_1) = \mathrm{hom}_{\mathcal{T}_W}(T^{-1}(V_{\bar{1}}), V_1) = 2$ and can consider the cone of the morphisms $T^{-1}(V_{\bar{1}}) \to V_1$ parameterized by $\mathbb{P}^1 = \mathbb{P}(\mathrm{Hom}_{\mathcal{T}_W}(V_{\bar{1}}, TV_1))$. Then, the triangle (5.8) implies that this $\mathbb{P}^1$ has r special points $\lambda_i = [u_{1,i} : u_{2,i}]$, $i = 1, \dots, r$, which correspond to $V_{i,1}$.

As we shall discuss in the proof of the main theorem in subsection 6.2, Theorem 5.10 implies the main theorem (Theorem 5.8) since we can obtain strongly exceptional collections corresponding to quivers $\vec{\Delta}_W^T$ and $\vec{\Delta}'_W$ and the AR-triangles for the collections. Also, we can replace quiver $\vec{\Delta}_W^T$ or $\vec{\Delta}'_W$ by the one whose orientation of the arrows between $V_{i,j}$ and $V_{i,j+1}$ are taken arbitrary, and then the parallel statement to the main theorem holds true. The parallel statement further holds true even if the orientation of the arrows of these quivers is reversed, since the contravariant functor $t : \mathcal{T}_W \to \mathcal{T}_W$ is an automorphism on $\mathcal{T}_W$.

Remark 5.11. For a full exceptional collection $\mathcal{E} = (E_1, \dots, E_l)$ in a triangulated category $\mathcal{T}$, let C be the inverse matrix of $\{\chi_{ij}\}_{i=1,\dots,l}$, where $\chi_{ij} = \chi(E_i, E_j)$ is the Euler number. Then, it is known [Bo, BP] that the AR-translation τ_{AR} induces an isomorphism $[\tau_{AR}]$ on the Grothendieck group of $\mathcal{T}$, which is expressed in terms of the basis $[E_1], \dots, [E_l]$ as

$$[\tau_{AR}]([E_1], \dots, [E_l]) = -([E_1], \dots, [E_l])\, C \cdot {}^tC^{-1}, \tag{5.9}$$

where $-C \cdot {}^tC^{-1}$ is the Coxeter transformation originally defined as the product of the reflections associated to each root base of the root lattice $(K_0(\mathcal{T}), \chi + {}^t\chi)$. In our case $\mathcal{T} = \mathcal{T}_W$, by Corollary 3.6, $[(\tau_{AR})^h] = \mathrm{Id}$ holds, which implies that the Coxeter transformation on the Grothendieck group of $\mathcal{T}_W$ is of finite order and in particular of order h.

On the other hand, we can see that the AR-triangles in Theorem 5.10 reduce to the identity (5.9) at the level of the Grothendieck group. First, the reduction of the AR-triangles in Theorem 5.10 gives the following identity

$$[\tau_{AR}(E_i)] + [E_i] = \sum_{j=1}^{\nu_W} (-C_{ij} \cdot [\tau_{AR}(E_j)] - C_{ji} \cdot [E_j]) \tag{5.10}$$

for any $i = 1, \dots, \nu_W$. It is easy to see that this is equivalent to the identity (5.9).

5.5. A categorification of the strange duality.

There exist fourteen regular systems of weights with $\varepsilon_W = -1$ and $a_0 = 0$ such that $r = 3$ for the signature $A_W = (\alpha_1, \dots, \alpha_r)$. The singularity defined by a generic polynomial $f_W \in A_2$ has been called an exceptional unimodular singularity, where the signature A_W coincides with the *Dolgachev numbers* [Ar, D].

For any exceptional unimodular singularity, the intersection diagram of a distinguished basis for the vanishing cycles of the Milnor fiber is given [Gv1] (see also [EbW, Eb, Gv2]) as $\Delta_{\beta_1,\beta_2,\beta_3}$ (see Figure 4) with some triple of positive integers $B_W := (\beta_1, \beta_2, \beta_3)$. This triple B_W is called the *Gabrielov numbers* of the singularity of the polynomial f_W.

The *strange duality*, found by Arnord, is a duality between these fourteen exceptional unimodular singularities stating the existence of an exceptional unimodular singularity $(f_W)^*$, that is, a regular system of weights W^* of exceptional type, satisfying

$$A_W = B_{W^*}, \qquad B_W = A_{W^*}$$

for any W of exceptional unimodular type such that $(W^*)^* = W$.

This duality is now interpreted in various ways: Kawai-Yang [KaYa] explained this strange duality in terms of the duality of orbifoldized Poincaré polynomials, i.e., the topological mirror symmetry. On the other hand, the $*$ duality, introduced as a duality of weight systems in [Sa3], includes the strange duality. Though the $*$ duality was originally defined in terms of the characteristic polynomials of the Milnor monodromy, the relation of it with the topological mirror symmetry is also discussed in [T1].

Now, Theorem 5.8 gives another interpretation, say, a categorification of the strange duality at the level of the Grothendieck group:

Corollary 5.12. *The following isomorphism of abelian group holds*:

$$K_0(\mathcal{T}_W) \simeq (H_2(f_{W^*}^{-1}(1), \mathbb{Z}), -I_{W^*}).$$

This can be thought of the *homological* mirror symmetry at the level of the Grothendieck group. In order to discuss this kind of duality at the level of triangulated categories, we need to define a suitable Fukaya category of the vanishing cycles of the Milnor fiber, which we leave for one of future directions.

6. Proof of Theorems 5.8 and 5.10

6.1. **Proof of Theorem 5.10.**

In this subsection, we give a proof of Theorem 5.10 in the following order.

- We first give a way to construct a collection $\{\mathcal{E}\} := \coprod_{i=1}^{r}\{V_{i,2},\dots,V_{i,\alpha_i}\}\coprod\{V_1, V_{\bar{1}}, V_0\}$ of indecomposable objects which have the grading matrices listed as in section 7 so that $\{\mathcal{E}\}$ has AR-triangles (5.4), (5.5), (5.6), (5.7) and the triangle (5.8). Thus, Statements (i) and (iv) are completed there.
- Using Lemma 6.3 on the existence of morphisms, we show Statement (ii), which implies that $\{\mathcal{E}\}$ forms an exceptional collection $\mathcal{E}$, and also complete Statement (iv').
- Statement (iii) is then clear by construction except for $V_{\bar{1}}$. For $V_{\bar{1}}$, we show Statement (iii) from the AR-triangle (5.7).
- Finally, we show that the exceptional collection $\mathcal{E}$ is full by using Statements (i), (ii), (iv) and (iv') together with Corollary 4.7.

The construction of $\{\mathcal{E}\}$: We first find a candidate for V_0 and V_{i,α_i}, $i = 1,\dots,r$, by hands, which are listed in section 7. We choose V_0 so that it is isomorphic to $R/\mathfrak{m}$ in $D^{\mathrm{gr}}_{\mathrm{Sg}}(R_W)$ up to grading shifts. Then, V_1 is obtained as $AR(V_0)$. Also, given V_{i,α_i}, we obtain $V_{i,\alpha_i-1} = AR(\tau^{-1}V_{i,\alpha_i})$, $V_{i,\alpha_i-2}\oplus\tau(V_{\alpha_i}) = AR(\tau(V_{i,\alpha_i}))$, and repeating this procedure yields $V_{i,2},\dots,V_{i,\alpha_i}$

and also $V_{i,1}$. The remaining object we should find is then $V_{\bar{1}}$. It is obtained as the cone of $V_1 \to V_{i,1}$ for some $i = 1, \dots, r$. In fact, we have the isomorphic object for any i.

By construction, the collection $\{\mathcal{E}\}$ satisfies the AR-triangles (5.4) and (5.5). The existence of the triangle (5.8) also follows from the construction above. We shall discuss on the morphism $V_{\bar{1}} \to TV_1$ in the triangle (5.8) later. The AR-triangles (5.6) and (5.7) can be checked by direct calculations.

At the level of grading matrices (cf. section 7), all these AR-triangles can be checked easily.

Now, Statements (i) and (iv) are completed.

On the structure of $\{\mathcal{E}\}$: We shall need some explicit data of morphisms between two graded matrix factorizations. For this purpose, we introduce the notion of phase.

Definition 6.1 (Phase of an indecomposable graded matrix factorization)**.** A graded matrix factorization $\overline{F} \in HMF_A^{gr}(f_W)$ is called *reduced* if it has minimal rank in its isomorphism class in $MF_A^{gr}(f_W)$. For an indecomposable graded matrix factorization $\overline{F} \in HMF_A^{gr}(f_W)$, the *phase* $\phi(\overline{F})$ of $\overline{F}$ is defined by

$$\phi(\overline{F}) := \frac{1}{\mathrm{rank}(Q,S)}\mathrm{Tr}(S),$$

where (Q,S) is a reduced graded matrix factorization which is isomorphic to $\overline{F}$ in $HMF_A^{gr}(f_W)$ and is reduced.

Note that, for an indecomposable object $\overline{F} \in HMF_A^{gr}(f_W)$, the phase $\phi(\overline{F})$ does not depend on the choice of (Q,S).

Definition 6.2. For two indecomposable objects $\overline{F}, \overline{F'} \in HMF_A^{gr}(f_W)$, we denote $\phi(\overline{F},\overline{F'}) := \phi(\overline{F'}) - \phi(\overline{F})$. Define the *spectrum* $\mathfrak{sp}(\overline{F},\overline{F'})$ by the following multi-set of rational numbers:

$$\mathfrak{sp}(\overline{F},\overline{F'}) := \{\phi(\overline{F},\tau^n(\overline{F'}))^{\hom_{HMF_A^{gr}(f_W)}(\overline{F},\tau^n(\overline{F'}))} \mid n \in \mathbb{Z}\},$$

where $\phi(\overline{F},\tau^n(\overline{F'}))^{\hom_{HMF_A^{gr}(f_W)}(\overline{F},\tau^n(\overline{F'}))}$ indicates that we include $\hom_{HMF_A^{gr}(f_W)}(\overline{F},\tau^n(\overline{F'}))$ copies of $\phi(\overline{F},\tau^n(\overline{F'}))$ in $\mathfrak{sp}(\overline{F},\overline{F'})$. In particular, we denote $\mathfrak{sp}(\overline{F},\overline{F}) = \mathfrak{sp}(\overline{F})$.

By definition, for any two indecomposable objects $\overline{F}, \overline{F'} \in HMF_A^{gr}(f_W)$, one has

$$\mathfrak{sp}(\overline{F},\tau(\overline{F'})) = \mathfrak{sp}(\tau(\overline{F}),\overline{F'}) = \mathfrak{sp}(\overline{F},\overline{F'}),$$

$$\mathfrak{sp}(t(\overline{F'}),t(\overline{F})) = \mathfrak{sp}(\overline{F},\overline{F'}),$$

$$\mathfrak{sp}(T(\overline{F}),T(\overline{F'})) = \mathfrak{sp}(\overline{F},\overline{F'}).$$

However in general $\mathfrak{sp}(\overline{F}, T(\overline{F'})) \neq \mathfrak{sp}(\overline{F}, \overline{F'})$; instead of it, by the Serre duality, the following holds:

$$\mathfrak{sp}(\overline{F'}, T(\overline{F})) = \mathfrak{sp}(\overline{F'}, \mathcal{S}(\overline{F})) = \left\{ \left(1 - \frac{2\varepsilon_W}{h}\right) - p \;\middle|\; p \in \mathfrak{sp}(\overline{F}, \overline{F'}) \right\}$$

(with $\varepsilon_W = -1$).

Lemma 6.3. *Given a triangulated category $\mathcal{T}_W$ of a regular system of weights $W = (a, b, c; h)$ with $\varepsilon_W = -1$, $a_0 = 0$, let the spectrum $\mathfrak{sp}(V, V')$, $V, V' \in \mathcal{T}_W$, be $\{p_0 \leq p_1 \leq \cdots \leq p_k\}$, $i = 0, 1, \ldots, k$, for some $k \in \mathbb{Z}_{\geq 0}$. Then, one has*

$$0 \leq p_0 \leq \cdots \leq p_k \leq 1 - \frac{2\varepsilon_W}{h},$$

for any $V, V' \in \coprod_{i=1}^r \{V_{i,\alpha_i}\} \coprod \{V_0\}$. In particular

(a) $\mathfrak{sp}(V_0) = \{0 \leq 2(a - \varepsilon_W)/h \leq (b - \varepsilon_W)/h \leq 2(c - \varepsilon_W)/h\}$,

(b) *for* $\mathfrak{sp}(V_{i,\alpha_i})$, $p_0 = 0$ *and* $p_1 = 2\alpha_i/h$,

(c) $\mathfrak{sp}(V_0, V_{i,\alpha_i}) = \{p_0 \leq \cdots < 1/2 - \alpha_i/h < 1/2 + (\alpha_i + 2)/h < \cdots \leq p_k\}$, *where* $(2\alpha_i + 2)/h - 1/2 < p_0$,

(d) *for* $\mathfrak{sp}(V_{i,\alpha_i}, V_{j,\alpha_j})$, $i \neq j$, $p_0 = (\alpha_i + \alpha_j - 2\varepsilon_W)/h$.

□

A few remarks about (c) are in order. The rational number $1/2 - \alpha_i/h$ is always greater than zero for any regular system of weights W with $\varepsilon_W = -1$ and $a_0 = 0$. By the transpose of V_0 and V_{i,α_i}, $\mathfrak{sp}(V_0, V_{i,\alpha_i}) = \mathfrak{sp}(V_{i,\alpha_i}, TV_0)$ holds. Thus, by the Serre duality, the condition $(2\alpha_i + 2)/h - 1/2 < p_0$ is equivalent to that $p_k < 3/2 - 2\alpha_i/h$.

Now, we calculate the dimension of the space of morphisms to show Statement (ii) and to complete to show Statement (iv'). Notice that, by construction, the phase of $V_{i,j}$ is $\phi(V_{i,j}) = (j-1)/h$ for any $i = 1, \ldots, r$ and $j = 1, \ldots, \alpha_i$. On the other hand, the phases of V_1 and V_0 are $\phi(V_1) = -1/2$ and $\phi(V_0) = -(1/2) - (1/h)$. (See subsection 7.1). Then, for all $V, V' \in \coprod_{i=1}^r \{V_{i,1}, \ldots, V_{i,\alpha_i}\} \coprod \{V_0, V_1\}$ $(= (\{\mathcal{E}\} \backslash \{V_{\bar{1}}\}) \coprod (\coprod_{i=1}^r \{V_{i,1}\}))$, we can calculate $\hom_{\mathcal{T}_W}(V, T^n V')$ with any $n \in \mathbb{Z}$ due to Corollary 3.13. Consequently, we obtain the followings: for $i, i' \in \{1, \ldots, r\}$, $j \in \{1, \ldots, \alpha_i\}$, $j' \in \{1, \ldots, \alpha_{i'}\}$, and $k, k' \in \{0, 1\}$,

$$\hom_{\mathcal{T}_W}(V_k, T^n(V_{k'})) = \begin{cases} 1 & n = 0 \text{ and } k \geq k' \\ 0 & \text{otherwise} \end{cases}, \tag{6.1}$$

$$\hom_{\mathcal{T}_W}(V_{i,j}, T^n(V_{i',j'})) = \begin{cases} 1 & (n = 0,\ i = i',\ j \geq j') \text{ or } (n = 1,\ i = i',\ j' = 1,\ \forall j) \\ 0 & \text{otherwise} \end{cases}, \tag{6.2}$$

$$\hom_{\mathcal{T}_W}(V_{i,j}, T^n(V_k)) = \begin{cases} 1 & n = 1 \text{ and any } i, j, k \\ 0 & \text{otherwise} \end{cases}, \tag{6.3}$$

$$\hom_{\mathcal{T}_W}(V_k, T^n(V_{i,j})) = \begin{cases} 1 & n = 0 \text{ and any } i, \quad j = 1, \quad k = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{6.4}$$

The equation (6.1) follows from Lemma 6.3 (a). The equations (6.2) and (6.3) follow from Lemma 6.3 (c). The equations (6.4) with $i = i'$ and $i \neq i'$ follow from Lemma 6.3 (b) and (d), respectively. For $k = 1$ and $j = 1$, eq.(6.3) and eq.(6.4) are equivalent under the transpose t. The equation (6.2) implies that $V_{i,1}$ and $T^n(V_{i',1})$ are not isomorphic to each other for any n if $i \neq i'$.

By eq.(6.1) with $k = k'$ and eq.(6.2) with $i = i'$ and $j = j'$, we can see that all elements in $\{\mathcal{E}\}\backslash\{V_{\bar{1}}\}$ are exceptional objects. Furthermore, for any $V \in \coprod_{i=1}^r \{V_{i,1}, \dots, V_{i,\alpha_i}\} \coprod \{V_0, V_1\}$, since $\hom_{\mathcal{T}_W}(V, V) = 1 = \hom_{\mathcal{T}_W}(V, \mathcal{S}(V))$, the spectrums satisfy the following rules

$$\mathfrak{sp}(V, AR(V')) = \{p - 1/h \mid p \in \mathfrak{sp}(V, V'),\ p \neq 0\} \coprod \{p + 1/h \mid p \in \mathfrak{sp}(V, V'),\ p \neq 1 + 2/h\},$$

$$\mathfrak{sp}(AR(V), V') = \{p - 1/h \mid p \in \mathfrak{sp}(V', V),\ p \neq 0\} \coprod \{p + 1/h \mid p \in \mathfrak{sp}(V', V),\ p \neq 1 + 2/h\}$$

for any $V, V' \in \coprod_{i=1}^r \{V_{i,1}, \dots, V_{i,\alpha_i}\} \coprod \{V_0, V_1\}$, which are obtained by rewriting Corollary 3.13 directly.

The remaining thing to show Statement (ii) is to calculate $\hom_{\mathcal{T}_W}(V, V')$ for the case $V = V_{\bar{1}}$ and/or $V' = V_{\bar{1}}$. First, by applying the functor $\mathrm{Hom}_{\mathcal{T}_W}(\,\cdot\,, V_1)$ to the triangle (5.8), we get

$$\hom_{\mathcal{T}_W}(V_{\bar{1}}, T^n V_1) = \begin{cases} 2 & n = 1 \\ 0 & \text{otherwise} \end{cases},$$

where we use eq.(6.3) with $j = 1$, $k = 1$, and eq.(6.1) with $k = k' = 1$. This implies that the morphism $V_{\bar{1}} \to TV_1$ in the triangle (5.8) is described as that stated in Theorem 5.10 since $V_{i,1}$ and $V_{i',1}$ are not isomorphic to each other for $i \neq i'$.

By applying the functor $\mathrm{Hom}_{\mathcal{T}_W}(V_1, \,\cdot\,)$ to the triangle (5.8), we get

$$\hom_{\mathcal{T}_W}(T^n V_1, V_{\bar{1}}) = 0 \qquad \text{for any } \ n \in \mathbb{Z}, \tag{6.5}$$

where we use eq.(6.4) with $j = 1$ and eq.(6.1) with $k = k' = 1$.

In a similar way, by applying $\mathrm{Hom}_{\mathcal{T}_W}(\,\cdot\,, V_0)$ and $\mathrm{Hom}_{\mathcal{T}_W}(V_0, \,\cdot\,)$ to the triangle (5.8) we have

$$\hom_{\mathcal{T}_W}(T^{-n}(V_{\bar{1}}), V_0) = \begin{cases} 2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hom_{\mathcal{T}_W}(V_0, T^n(V_{\bar{1}})) = 0 \qquad \text{for any } \ n \in \mathbb{Z}.$$

Here, the former equation follows from that $\hom_{\mathcal{T}_W}(T^{-n}(V_1), V_0)$ is equal to one for $n = 0$ and zero otherwise by eq.(6.1), and that $\hom_{\mathcal{T}_W}(T^{-n}(V_{i',1}), V_0)$ is equal to one for $n = 1$

and zero otherwise by eq.(6.3). The latter equation follows from that $\hom_{\mathcal{T}_W}(V_0, T^n(V_1)) = \hom_{\mathcal{T}_W}(V_0, T^n(V_{i,1})) = 0$ for any $n \in \mathbb{Z}$ by eq.(6.1) and eq.(6.4), respectively.

In order to compute $\hom_{\mathcal{T}_W}(V_{i,2}, T^n(V_{\bar{1}}))$, we apply the functor $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, \,\cdot\,)$ to the triangle (5.8):

$$V_1 \to V_{i',1} \to V_{\bar{1}} \to TV_1 \tag{6.6}$$

for $i' \neq i$. The resulting exact sequence is

$$\begin{aligned}\cdots \to \mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, V_1) \to \mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, V_{i',1}) \to \mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, V_{\bar{1}}) \to \mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, TV_1)\\ \to \mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, T(V_{i',1})) \to \mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, T(V_{\bar{1}})) \to \mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, T^2(V_1)) \to \cdots,\end{aligned}$$

where, $\hom_{\mathcal{T}_W}(V_{i,2}, T^n(V_{i',1})) = 0$ by eq.(6.2) with $j = 2$, $j' = 1$, and $\hom_{\mathcal{T}_W}(V_{i,2}, T^n(V_1))$ is equal to one for $n = 1$ and zero otherwise by eq.(6.3) with $j = 2$ and $k = 1$. This implies

$$\hom_{\mathcal{T}_W}(V_{i,2}, T^n(V_{\bar{1}})) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, for $j > 2$, by considering the functor $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,j}, \,\cdot\,)$ we obtain that $\hom_{\mathcal{T}_W}(V_{i,j}, T^n(V_{\bar{1}}))$ is equal to one for $n = 0$ and zero otherwise. Conversely, applying the functor $\mathrm{Hom}_{\mathcal{T}_W}(\,\cdot\,, V_{i,j})$ with $j \geq 2$ to the triangle (6.6) with $i' = i$ leads to the result $\hom_{\mathcal{T}_W}(T^n(V_{\bar{1}}), V_{i,j}) = 0$ for any $n \in \mathbb{Z}$, since $\hom_{\mathcal{T}_W}(T^n(V_1), V_{i,j}) = \hom_{\mathcal{T}_W}(T^n(V_{i,1}), V_{i,j}) = 0$ for any $n \in \mathbb{Z}$ by eq.(6.4) and eq.(6.2), respectively.

Finally, $\hom_{\mathcal{T}_W}(V_{\bar{1}}, T^n(V_{\bar{1}}))$ is computed as follows. Applying $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,1}, \,\cdot\,)$ to the triangle (6.6) with $i \neq i'$, we obtain

$$\hom_{\mathcal{T}_W}(V_{i,1}, T^n(V_{\bar{1}})) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.7}$$

since $\hom_{\mathcal{T}_W}(V_{i,1}, T^n(V_1))$ is equal to one for $n = 1$ and zero otherwise (eq.(6.3) with $j = 1$ and $k = 1$), and $\hom_{\mathcal{T}_W}(V_{i,1}, T^n(V_{i',1})) = 0$ (eq.(6.2) with $j = 1$ and $j' = 1$). Then, applying $\mathrm{Hom}_{\mathcal{T}_W}(\,\cdot\,, V_{\bar{1}})$ to the triangle (5.8) yields

$$\hom_{\mathcal{T}_W}(V_{\bar{1}}, T^n(V_{\bar{1}})) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

due to eq.(6.5) and eq.(6.7).

Statement (ii) has been completed.

Statement (iii), that is, the properties under the transpose can be checked for V_0 and V_{i,α_i}, $i = 1, \ldots, r$, by the explicit form of the graded matrix factorizations and for $V_{i,j}$, $j = 2, \ldots, \alpha_i - 1$, by construction above.

The statement $t(V_{\bar{1}}) = \tau^{-1}(V'_{\bar{1}})$ is related to the AR-triangle (5.7) via the octahedral axiom of triangulated categories. Recall that, for a triangulated category $\mathcal{T}$, the octahedral axiom states the existence of the triangle $Z' \to Y' \to X' \to T(Z')$ (with appropriate compatibility of morphisms) for any given objects $X, Y, Z \in \mathcal{T}$ and a composition $X \to Y \to Z$ of morphisms, where X', Y' and Z' are defined by the triangles $X \to Y \to Z' \to T(X)$, $Y \to Z \to X' \to T(Y)$ and $X \to Z \to Y' \to T(X)$, respectively. Now, apply the octahedral axiom for the case $\mathcal{T} = \mathcal{T}_W$ with $X = \oplus_{i=1}^r T^{-1}(V_{i,2})$, $Y = T^{-1}AR(V_{\bar{1}})$ and $Z = T^{-1}(V_{\bar{1}})$. Then, one obtains the triangle $Z' \to Y' \to X' \to T(Z')$ with $Z' \simeq (\tau(V_1))^{\oplus 2}$, $Y' \simeq V'_{\bar{1}}$ and $X' \simeq \tau(V_{\bar{1}})$. Namely, $V'_{\bar{1}}$ is the mapping cone $V'_{\bar{1}} \simeq \tau C(T^{-1}V_{\bar{1}} \to V_1^{\oplus 2})$. Next, apply the octahedral axiom again to the case $X = T^{-1}V_{\bar{1}}$, $Y = V_1^{\oplus 2}$ and $Z = V_1$, where $T^{-1}V_{\bar{1}} \to V_1$ is the map defining the triangle $T^{-1}V_{\bar{1}} \to V_1 \to V_{i,1} \to V_{\bar{1}}$ for some $i \in \{1, \ldots, r\}$. Then, one obtains the triangle

$$\tau^{-1}(V'_{\bar{1}}) \to V_{i,1} \to TV_1 \to T\tau^{-1}(V'_{\bar{1}}) \tag{6.8}$$

as $Z' \to Y' \to X' \to T(Z')$. On the other hand, as the transpose of the triangle (5.8), one obtains

$$t(V_{\bar{1}}) \to V_{i,1} \to TV_1 \to Tt(V_{\bar{1}}). \tag{6.9}$$

By comparing these two triangles (6.8) (6.9), one can see that $t(V_{\bar{1}})$ is isomorphic to $\tau^{-1}(V'_{\bar{1}})$.

The exceptional collection $\mathcal{E}$ is full : Statements (i), (ii), (iv) and (iv') for the collection $\{\mathcal{E}\}$ together with Corollary 4.7 leads that $\mathcal{E}$ forms a *full* exceptional collection in $\mathcal{T}_W$ as follows.

For the exceptional collection $\mathcal{E} = (E_1, \ldots, E_{\nu_W})$, let $\langle \mathcal{E} \rangle := \langle E_1, \ldots, E_{\nu_W} \rangle$ denote the smallest full triangulated category including objects $\{\mathcal{E}\}$. First, in the AR triangle (5.4), one has $V_0, V_1 \in \langle \mathcal{E} \rangle$, which leads that $T^n(\tau(V_0)) \in \langle \mathcal{E} \rangle$ for any $n \in \mathbb{Z}$ since $T\tau V_0$ is isomorphic to the cone of $V_1 \to V_0$. Next, in the AR triangle (5.6), one has $V_0, V_1 \in \langle \mathcal{E} \rangle$ and $V_2 \in \langle \mathcal{E} \rangle$ since $TV_2 \in \langle \mathcal{E} \rangle$. This implies that $T^n(\tau V_1) \in \langle \mathcal{E} \rangle$ for any $n \in \mathbb{Z}$ since $T(\tau V_1)$ is isomorphic to the cone of $V_2 \oplus V_0 \to V_1$. In a similar way, by the AR triangle (5.7), one has $T^n(\tau V_{\bar{1}}) \in \langle \mathcal{E} \rangle$. Then, by the triangle (5.8), one sees that $T^n(\tau V_{i,1}) \in \langle \mathcal{E} \rangle$ for each $i = 1, \ldots, r$. Next, in the AR-triangle (5.4)

$$\tau V_{i,j} \to \tau V_{i,j-1} \oplus V_{i,j+1} \to V_{i,j} \to T\tau V_{i,j}$$

with $j = 2$, we already know that $\tau V_{i,1}, V_{i,3}, V_{i,2} \in \langle \mathcal{E} \rangle$. Thus, one obtains $\tau V_{i,2} \in \langle \mathcal{E} \rangle$. The AR-triangle (5.4) with $j = 3$ then implies that $\tau V_{i,3} \in \langle \mathcal{E} \rangle$, and repeating this argument leads that $\tau V_{i,j} \in \langle \mathcal{E} \rangle$ for any $j = 1, \ldots, \alpha_i$ with any $i = 1, \ldots, r$.

Thus, what we obtained is $\tau V \in \langle \mathcal{E} \rangle$ for any $V \in \{\mathcal{E}\}$. Therefore, one can repeat the same procedure for τV, $V \in \{\mathcal{E}\}$, and then obtain that $\tau^n V \in \langle \mathcal{E} \rangle$ for any $n \in \mathbb{Z}_{\geq 0}$.

Recall that, in this triangulated category $\mathcal{T}_W$, the identity $T^2 = \tau^h$ holds, which implies that $\tau^n V \in \langle \mathcal{E} \rangle$, $V \in \{\mathcal{E}\}$, for any $n \in \mathbb{Z}$.

Now, since V_0 corresponds to $R/\mathfrak{m}$ in $\mathcal{T}_W \simeq D^{\mathrm{gr}}_{\mathrm{Sg}}(R)$ up to grading shift, Theorem 4.5 and in particular Corollary 4.7 can be applied to the exceptional collection $\langle \mathcal{E} \rangle$, which implies that $\langle \mathcal{E} \rangle$ is full. □

6.2. **Proof of Theorem 5.8.**

In this subsection, we give a proof of the main theorem (Theorem 5.8) using the structure theorem (Theorem 5.10) shown in the previous subsection.

Before discussing each case $\vec{\Delta}^T_W$ or $\vec{\Delta}'_W$ separately, let us first prepare the following two lemmas (Lemma 6.4 and Lemma 6.5) which follow from the AR-triangles in the structure theorem (Theorem 5.10).

Lemma 6.4. *For a fixed $i \in \{1, \dots, r\}$ and $j \in \{3, 4, \dots, \alpha_i\}$, the composition of a nonzero element in $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,j}, V_{i,j-1})$ and a nonzero element in $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,j-1}, V)$ gives a nonzero element in $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,j}, V)$ for any $V \in \{V_{i,j-2}, \dots, V_{i,1}\} \coprod \{V_{\bar{1}}, TV_1, TV_0\}$.*

Proof. Applying $\mathrm{Hom}_{\mathcal{T}_W}(\,\cdot\,, \tau_{AR} V)$ to the AR-triangle (5.4) gives arise to the following short exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{T}_W}(V_{i,j}, \tau_{AR}V) \to &\mathrm{Hom}_{\mathcal{T}_W}(V_{i,j+1}, \tau_{AR}V) \\ &\oplus \mathrm{Hom}_{\mathcal{T}_W}(\tau_{AR}V_{i,j-1}, \tau_{AR}V) \to \mathrm{Hom}_{\mathcal{T}_W}(\tau_{AR}V_{i,j}, \tau_{AR}V) \to 0. \end{aligned} \tag{6.10}$$

Consider the case $j = \alpha_j$. Then, since the term $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,j+1}, \tau_{AR}V)$ is absent, the map $\mathrm{Hom}_{\mathcal{T}_W}(\tau_{AR}V_{i,\alpha_i-1}, \tau_{AR}V) \to \mathrm{Hom}_{\mathcal{T}_W}(\tau_{AR}V_{i,\alpha_i}, \tau_{AR}V)$ is surjective and in particular bijective since $\hom_{\mathcal{T}_W}(\tau_{AR}V_{i,\alpha_i-1}, \tau_{AR}V) = \hom_{\mathcal{T}_W}(\tau_{AR}V_{i,\alpha_i}, \tau_{AR}V) = 1$. This gives the statement of this lemma for the case $j = \alpha_i$ together with that $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,\alpha_i}, \tau_{AR}V) = 0$. Next, consider the short exact sequence (6.10) for the case $j = \alpha_i - 1$. As we saw just now, since $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,\alpha_i}, \tau_{AR}V) = 0$, the map $\mathrm{Hom}_{\mathcal{T}_W}(\tau_{AR}V_{i,\alpha_i-2}, \tau_{AR}V) \to \mathrm{Hom}_{\mathcal{T}_W}(\tau_{AR}V_{i,\alpha_i-1}, \tau_{AR}V)$ is bijective, which gives the statement of this lemma for the case $j = \alpha_i - 1$, and also that $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,\alpha_i-1}, \tau_{AR}V) = 0$. Repeating this procedure gives the statement of this lemma for all $j \in \{3, \dots, \alpha_i\}$. □

Lemma 6.5. *For any $V \in \coprod_{i=1}^r \{V_{i,2}, \dots, V_{i,\alpha_i}\} \coprod \{V_{\bar{1}}, V'_{\bar{1}}\}$, the composition of a nonzero element in $\mathrm{Hom}_{\mathcal{T}_W}(V, TV_1)$ and a nonzero element in $\mathrm{Hom}_{\mathcal{T}_W}(TV_1, TV_0)$ gives a nonzero element in $\mathrm{Hom}_{\mathcal{T}_W}(V, TV_0)$. In particular, this induces an isomorphism $\mathrm{Hom}_{\mathcal{T}_W}(V, TV_1) \simeq \mathrm{Hom}_{\mathcal{T}_W}(V, TV_0)$.*

Proof. We may apply $\mathrm{Hom}_{\mathcal{T}_W}(V,\,\cdot\,)$ to the AR-triangle (5.5) and then obtain the short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{T}_W}(V, \tau_{AR}TV_0) \to \mathrm{Hom}_{\mathcal{T}_W}(V, TV_1) \to \mathrm{Hom}_{\mathcal{T}_W}(V, TV_0) \to 0.$$

Since $\hom_{\mathcal{T}_W}(V, TV_1) = \hom_{\mathcal{T}_W}(V, TV_0)$, the map $\mathrm{Hom}_{\mathcal{T}_W}(V, TV_1) \to \mathrm{Hom}_{\mathcal{T}_W}(V, TV_0)$ is bijective, which implies Lemma 6.5. □

Now, we show Theorem 5.8 for the quiver $\vec{\Delta}^T_W$. Consider the collection

$$\{\mathcal{E}^T\} := \coprod_{i=1}^{r} \{V_{i,1}, \ldots, V_{i,\alpha_i}\} \coprod \{TV_0, TV_1, V_{\bar{1}}\}.$$

By Theorem 5.10, this forms a strongly exceptional collection by giving an appropriate ordering.

By Lemma 6.4 and Lemma 6.5, in order to obtain the composition law of morphisms between $\{\mathcal{E}^T\}$, the remaining thing is only to check the relations corresponding to $\rho(v_{i,2}, v_1)$. These relations are obtained by applying $\mathrm{Hom}(V_{i,2},\,\cdot\,)$ to the triangle (5.8); the resulting exact sequence is

$$0 \to \mathbb{C} \to \mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, V_{\bar{1}}) \overset{u_{1,i}e_1 + u_{2,i}e_2}{\longrightarrow} \mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, T(V_1)) \to \mathbb{C} \to 0.$$

Here, $(u_{1,i}e_1 + u_{2,i}e_2) : \mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, V_{\bar{1}}) \to \mathrm{Hom}_{\mathcal{T}_W}(V_{i,2}, T(V_1))$ is a zero map:

$$(u_{1,i}e_1 + u_{2,i}e_2) \circ f_i = 0, \qquad i = 1, \ldots, r, \tag{6.11}$$

since $\hom_{\mathcal{T}_W}(V_{i,2}, V_{\bar{1}}) = 1$ and $\hom_{\mathcal{T}_W}(V_{i,2}, T(V_1)) = 1$. This implies the relation (5.2):

$$(u_{1,i}\rho_1(v_{\bar{1}}, v_1) + u_{2,i}\rho_2(v_{\bar{1}}, v_1)) \circ \rho(v_{i,2}, v_{\bar{1}})),$$

where we identify $\rho_1(v_{\bar{1}}, v_1)$ and $\rho_2(v_{\bar{1}}, v_1)$ with e_1 and e_2, respectively. Thus, Theorem 5.8 has been completed for the quiver $\vec{\Delta}^T_W$.

Next, we show Theorem 5.8 for the quiver $\vec{\Delta}'_W$. Consider the collection

$$\{\mathcal{E}'\} := \coprod_{i=1}^{r} \{V_{i,1}, \ldots, V_{i,\alpha_i}\} \coprod \{TV_0, TV_1, V'_{\bar{1}}\}.$$

Recall that $V'_{\bar{1}} \in \mathcal{T}_W$ is defined by the triangle

$$V'_{\bar{1}} \to \oplus_{i=1}^{r} V_{i,2} \overset{\oplus_i f_i}{\longrightarrow} V_{\bar{1}} \to T(V'_{\bar{1}}) \tag{6.12}$$

and we can set $V'_{\bar{1}} = \tau(t(V_{\bar{1}}))$ due to Theorem 5.10 (iii).

Lemma 6.6. *The transpose of the triangle (6.12) is isomorphic to the triangle (6.12) itself. Equivalently, the morphism $V'_{\bar{1}} \to \oplus_{i=1}^{r} V_{i,2}$ in the triangle (6.12) is given by $\oplus_i \tau^{-1} t(f_i)$.*

Proof. By applying $\tau^{-1}t$ to the triangle (6.12), one obtains the triangle

$$V'_{\bar{1}} \xrightarrow{\oplus_i \tau^{-1}t(f_i)} \oplus_{i=1}^r V_{i,2} \to V_{\bar{1}} \to T(V'_{\bar{1}}),$$

which shows the statement of this lemma. □

Now, by applying $\mathrm{Hom}_{\mathcal{T}_W}(\,\cdot\,, V_{i,j})$ to the triangle (6.12) for any $i = 1, \dots, r$ and $j = 1, \dots, \alpha_i$, one obtains that $\mathrm{hom}_{\mathcal{T}_W}(T^{-n}(V'_{\bar{1}}), V_{i,j})$ is equal to one only for $n = 0$ and $j = 2$, and is zero otherwise. Also, applying $\mathrm{Hom}_{\mathcal{T}_W}(V_{i,j}, \,\cdot\,)$ and $\mathrm{Hom}_{\mathcal{T}_W}(V_k, \,\cdot\,)$, $k = 0, 1$, to the triangle (6.12) leads that $\mathrm{hom}_{\mathcal{T}_W}(V_{i,j}, V'_{\bar{1}}) = \mathrm{hom}_{\mathcal{T}_W}(V_k, V'_{\bar{1}}) = 0$ for any i, j and k.

Let us apply $\mathrm{Hom}_{\mathcal{T}_W}(\,\cdot\,, TV_1)$ to the triangle (6.12). Then, we obtain the following exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{T}_W}(V_{\bar{1}}, TV_1) \xrightarrow{\oplus_i f_i} \mathrm{Hom}_{\mathcal{T}_W}(\oplus_{i=1}^r V_{i,2}, TV_1) \xrightarrow{\oplus_i \tau t(f_i)} \mathrm{Hom}_{\mathcal{T}_W}(V'_{\bar{1}}, TV_1) \to 0 \qquad (6.13)$$

which gives relations of the path algebra $\mathbb{C}\vec{\Delta}'_W/I'_{W,\Lambda}$. Here, in order to obtain the exact sequence above, we used the fact $\mathrm{Hom}_{\mathcal{T}_W}(T^{-n}V'_{\bar{1}}, TV_1) = 0$ for any $n \neq 0$. The $n = -1$ case, $\mathrm{Hom}_{\mathcal{T}_W}(TV'_{\bar{1}}, TV_1) = 0$, is nontrivial, which is equivalent to

$$\mathrm{Hom}_{\mathcal{T}_W}(T\tau(V_1), V_{\bar{1}}) = 0 \qquad (6.14)$$

by the transpose t. This equality (6.14) can be shown in the following two steps. First, by applying the functor $\mathrm{Hom}_{\mathcal{T}_W}(T\tau(V_1), \,\cdot\,)$ to the triangle

$$\oplus_{i=1}^r V_{i,2} \to AR(V_{\bar{1}}) \to T\tau V_1^{\oplus 2} \to T(\oplus_{i=1}^r V_{i,2}),$$

one obtains $\mathrm{Hom}_{\mathcal{T}_W}(T\tau(V_1), AR(V_{\bar{1}})) = 0$ because the map $\mathrm{Hom}_{\mathcal{T}_W}(T\tau(V_1), (T\tau V_1)^{\oplus 2}) \to \mathrm{Hom}_{\mathcal{T}_W}(T\tau(V_1), T(\oplus_{i=1}^r V_{i,2}))$ is injective and

$$\mathrm{Hom}_{\mathcal{T}_W}(T\tau(V_1), (\oplus_{i=1}^r V_{i,2})) \simeq \mathrm{Hom}_{\mathcal{T}_W}(T(V_1), \tau^{-1}(\oplus_{i=1}^r V_{i,2})) \simeq \mathrm{Hom}_{\mathcal{T}_W}((\oplus_{i=1}^r V_{i,2}), V_1) = 0,$$

where the second isomorphism follows from the transpose. Next, by applying the functor $\mathrm{Hom}_{\mathcal{T}_W}(T\tau(V_1), \,\cdot\,)$ to the AR-triangle of $V_{\bar{1}}$ (eq.(5.7)), one obtains eq.(6.14).

Thus, one obtains the exact sequence (6.13). By Lemma 6.6, one has the morphism $\oplus_i \tau^{-1}t(f_i) : \mathrm{Hom}_{\mathcal{T}_W}(\oplus_{i=1}^r V_{i,2}, TV_1) \to \mathrm{Hom}_{\mathcal{T}_W}(V'_{\bar{1}}, TV_1)$. We identify $\tau^{-1}t(f_i)$ with $\rho(v_{\bar{1}}, v_{i,2})$ for $i = 1, \dots, r$. On the other hand, the exact sequence (6.13) implies the following two relations

$$\sum_{i=1}^r (e_1 \circ f_i) \circ \tau^{-1}t(f_i) = 0, \qquad \sum_{i=1}^r (e_2 \circ f_i) \circ \tau^{-1}t(f_i) = 0.$$

We can assume that $|u_{1,i}|^2 + |u_{2,i}|^2 = 1$ for each $i = 1, \dots, i$ since all the triangles in $\mathcal{T}_W$ depend on the ratio $u_{1,i} : u_{2,i}$ only. Then, using the relation (6.11), one obtains

$$e_1 \circ f_i = ((1 - u_{1,i}^* u_{1,i})e_1 - u_{1,i}^* u_{2,i} e_2) \circ f_i = u_{2,i}(u_{2,i}^* e_1 - u_{1,i}^* e_2) \circ f_i,$$

where $u^*_{1,i}$ and $u^*_{2,i}$ are the complex conjugates of $u_{1,i}$ and $u_{2,i}$, respectively. We can set $g_i = (u^*_{2,i}e_1 - u^*_{1,i}e_2) \circ f_i$ and identify g_i with $\rho(v_{i,2}, v_1)$. Thus, one of the relations (5.3), $\sum_i u_{2,i} \cdot (\rho(v_{i,2}, v_1) \circ \rho(v_{\bar{1}}, v_{i,2}))$, is obtained. In a similar way, the other relation of eq.(5.3), $-\sum_i u_{1,i} \cdot (\rho(v_{i,2}, v_1) \circ \rho(v_{\bar{1}}, v_{i,2}))$, is obtained from $\sum_{i=1}^r (e_2 \circ f_i) \circ \tau^{-1} t(f_i) = 0$.

Now, together with Lemma 6.5, we can conclude that $\{\mathcal{E}'\}$ forms a strongly exceptional collection. Furthermore, together with Lemma 6.4, we checked all the composition law. Theorem 5.8 has been completed for the quiver $\vec{\Delta}'_W$. □

Remark 6.7. For the strongly exceptional collection $\mathcal{E}'$ associated to the quiver $\vec{\Delta}'_W$, the AR-triangles can be described in terms of the indecomposable objects $\{\mathcal{E}'\}$ and their grading shifts. In particular, the transpose of the AR-triangles (5.6) and (5.7) yield

$$AR(V_1) \simeq \tau V_1 \oplus C((V'_{\bar{1}})^{\oplus 2} \to \oplus_{i=1}^r V_{i,2}),$$
$$AR(V'_{\bar{1}}) \simeq T^{-1}\tau C(\oplus_{i=1}^r V_{i,2} \to (TV_1)^{\oplus 2}),$$

where the morphisms in the mapping cones are also determined by the transpose.

7. Graded matrix factorizations associated to the vertices of the quiver $\vec{\Delta}_W$

In this section, we list up the graded matrix factorizations which form the exceptional collection $\mathcal{E}$ and are in the reduced form, i.e., representatives having minimum rank in the isomorphism classes.

7.1. The grading matrices.

We list up the grading matrices for the reduced indecomposable graded matrix factorizations of the corresponding quivers $\vec{\Delta}_W$.

For the grading matrix $S := \mathrm{diag}\{s_1, \dots, s_r; \bar{s}_1, \dots, \bar{s}_r\}$, recall that $s_i \in 2\mathbb{Z}/h$ and $\bar{s}_i \in 2\mathbb{Z}/h - 1$ for $i = 1, \dots, r$. For some $\phi \in 2\mathbb{Z}/h - 1/2$, rewrite

$$\mathrm{diag}\{s_1, \dots, s_r; \bar{s}_1, \dots, \bar{s}_r\} = \mathrm{diag}\{s'_1, \dots, s'_r; \bar{s}'_1, \dots, \bar{s}'_r\} + \phi \cdot \mathbf{1}$$

where $s'_i = s_i - \phi$ and $\bar{s}'_i = \bar{s}_i - \phi$ for $i = 1, \dots, r$. Except for $V_{\bar{1}}$, we take this ϕ so that $S - \phi \cdot \mathbf{1}$ is traceless. We describe the data of the grading matrix as

$$[hs'_1, \dots, hs'_r; h\bar{s}'_1, \dots, h\bar{s}'_r]_{h\phi}.$$

In particular, we use the following simplified notation

$$(q_1, \dots, q_\nu; \bar{q}_1, \dots, \bar{q}_\nu)_{h\phi} := [q_1, -q_1, \dots, q_\nu, -q_\nu; \bar{q}_1, -\bar{q}_1, \dots, \bar{q}_r, -\bar{q}_r]_{h\phi}\ ,$$

for a grading matrix of this kind. Furthermore, we denote

$$(q_1, \dots, q_\nu)_{h\phi} := (q_1, \dots, q_\nu; \bar{q}_1, \dots, \bar{q}_\nu)_{h\phi}$$

if $q_i = \bar{q}_i$ for $i = 1, \dots, \nu$. Similarly, if $s'_i = \bar{s}'_i$ for any $i = 1, \dots, r$, we denote

$$[hs'_1, \dots, hs'_r]_{h\phi} := [hs'_1, \dots, hs'_r; h\bar{s}'_1, \dots, h\bar{s}'_r]_{h\phi}.$$

If we attach the grading matrices below to the vertices of the corresponding quiver $\vec{\Delta}_W$, we see some interesting phenomenological rules due to the Auslander-Reiten triangles in Theorem 5.10 at the level of the Grothendieck group (see eq.(5.11)) as in the case that W is of type ADE (see [KST1, section 5, Table 2]).

- $W = (6, 14, 21; 42)$, $f_W = x^7 + y^3 + z^2$, $A_W = (2, 3, 7)$,

  $V_{1,2} : (3, 7, 11)_1$,

  $V_{2,2} : (3, 5, 9, 11)_1$, $V_{2,3} : (4, 10)_2$,

  $V_{3,2} : (3, 5, 7, 9, 11, 13, 27)_1$, $V_{3,3} : (4, 6, 8, 10, 12, 26)_2$, $V_{3,4} : (5, 7, 9, 11, 25)_3$,

  $V_{3,5} : (6, 8, 10, 24)_4$, $V_{3,6} : (7, 9, 23)_5$, $V_{3,7} : (8, 22)_6$,

  $V_0 : (8, 22)_{-22}$, $V_1 : (7, 9, 23)_{-21}$, $V_{\bar{1}} : [28, 14, 12^2, 10, 8, 6, 4, -2^2, -4, -6, -8, -10]$.

- $W = (4, 10, 15; 30)$, $f_W = y^3 + yx^5 + z^2$, $A_W = (2, 4, 5)$,

  $V_{1,2} : (3, 7)_1$, $V_{2,2} : (3, 5, 7)_1$, $V_{2,3} : (4, 6)_2$, $V_{2,4} : (5)_3$,

  $V_{3,2} : (3, 5, 7, 9, 19)_1$, $V_{3,3} : (4, 6, 8, 18)_2$, $V_{3,4} : (5, 7, 17)_3$, $V_{3,5} : (6, 16)_4$,

  $V_0 : (6, 16)_{-16}$, $V_1 : (5, 7, 17)_{-15}$, $V_{\bar{1}} : [20, 10, 8^2, 6, 4, -2^2, -4, -6]$.

- $W = (3, 8, 12; 24)$, $f_W = x^4 z + y^3 + z^2$, $A_W = (3, 3, 4)$,

  $V_{1,2} : (3, 5)_1$, $V_{1,3} : (4)_2$, $V_{2,2} : (3, 5)_1$, $V_{2,3} : (4)_2$,

  $V_{3,2} : (3, 5, 7, 15)_1$, $V_{3,3} : (4, 6, 14)_2$, $V_{3,4} : (5, 13)_3$,

  $V_0 : (5, 13)_{-13}$, $V_1 : (4, 6, 14)_{-12}$, $V_{\bar{1}} : [16, 8, 6^2, 4, -2^2, -4]$.

- $W = (6, 8, 15; 30)$, $f_W = x^5 + xy^3 + z^2$, $A_W = (2, 3, 8)$,

  $V_{1,2} : (3, 7, 11)_1$, $V_{2,2} : (3, 5, 9, 11)_1$, $V_{2,3} : (4, 10)_2$,

  $V_{3,2} : (3, 5, 7, 9, 11, 13, 15)_1$, $V_{3,3} : (4, 6, 8, 10, 12, 14)_2$,

  $V_{3,4} : (5, 7, 9, 11, 13)_3$, $V_{3,5} : (6, 8, 10, 12)_4$, $V_{3,6} : (7, 9, 11)_5$, $V_{3,7} : (8, 10)_6$, $V_{3,8} : (9)_7$,

  $V_0 : (2, 16)_{-16}$, $V_1 : (1, 3, 17)_{-15}$, $V_{\bar{1}} : [16, 14, 12^2, 10, 8, 6, 4, -2^2, -4, -6, -8, -10]$.

- $W = (4, 6, 11; 22)$, $f_W = yx^4 + xy^3 + z^2$, $A_W = (2, 4, 6)$,

  $V_{1,2} : (3, 7)_1$, $V_{2,2} : (3, 5, 7)_1$, $V_{2,3} : (4, 6)_2$, $V_{2,4} : (5)_3$,

  $V_{3,2} : (3, 5, 7, 9, 11)_1$, $V_{3,3} : (4, 6, 8, 10)_2$, $V_{3,4} : (5, 7, 9)_3$, $V_{3,5} : (6, 8)_4$, $V_{3,6} : (7)_5$,

  $V_0 : (2, 12)_{-12}$, $V_1 : (1, 3, 13)_{-11}$, $[12, 10, 8^2, 6, 4, -2^2, -4, -6]$.

- $W=(3,5,9;18)$, $f_W=x^3z+xy^3+z^2$, $A_W=(3,3,5)$,

$$V_{1,2}:(3,5)_1,\quad V_{1,3}:(4)_2,\qquad V_{2,2}:(3,5)_1,\quad V_{2,3}:(4)_2,$$
$$V_{3,2}:(3,5,7,9)_1,\quad V_{3,3}:(4,6,8)_2,\quad V_{3,4}:(5,7)_3,\quad V_{3,5}:(6)_4,$$
$$V_0:(2,10)_{-10},\quad V_1:(1,3,11)_{-9},\quad V_{\bar{1}}:[10,8,6^2,4,-2^2,-4].$$

- $W=(4,5,10;20)$, $f_W=x^5+y^2z+z^2$, $A_W=(2,5,5)$,

$$V_{1,2}:(3,7)_1,$$
$$V_{2,2}:(3,5,7,9)_1,\quad V_{2,3}:(4,6,8)_2,\quad V_{2,4}:(5,7)_3,\quad V_{2,5}:(6)_4,$$
$$V_{3,2}:(3,5,7,9)_1,\quad V_{3,3}:(4,6,8)_2,\quad V_{3,4}:(5,7)_3,\quad V_{3,5}:(6)_4,$$
$$V_0:(1,11)_{-11},\quad V_1:(0,2,12)_{-10},\quad V_{\bar{1}}:[10^2,8^2,6,4,-2^2,-4,-6].$$

- $W=(3,4,8;16)$, $yx^4+y^2z+z^2$, $A_W=(3,4,4)$,

$V_{1,2}:(3,5)_1,\quad V_{1,3}:(4)_2,$

$V_{2,2}:(3,5,7)_1,\quad V_{2,3}:(4,6)_2,\quad V_{2,4}:(5)_3,\qquad V_{3,2}:(3,5,7)_1,\quad V_{3,3}:(4,6)_2,\quad V_{3,4}:(5)_3,$

$V_0:(1,9)_{-9},\quad V_1:(0,2,10)_{-8},\quad V_{\bar{1}}:[8^2,6^2,4,-2^2,-4].$

- $W=(6,8,9;24)$, $f_W=x^4+y^3+xz^2$, $A_W=(2,3,9)$,

$V_{1,2}:(3,7,11;1,3,5)_1,\qquad V_{2,2}:(3,5,9,11;1,3^2,5)_1,\quad V_{2,3}:(4,10;2,4)_2,$

$V_{3,2}:(3,5,7,9,11,13,15,17;1^2,3^2,5^2,7,9)_1,\quad V_{3,3}:(4,6,8,10,12,14,16;0,2^2,4^2,6,8)_2,$

$V_{3,4}:(5,7,9,11,13,15;1^2,3^2,5,7)_3,\quad V_{3,5}:(6,8,10,12,14;0,2^2,4,6)_4,$

$V_{3,6}:(7,9,11,13;1^2,3,5)_5,\quad V_{3,7}:(8,10,12;0,2,4)_6,\quad V_{3,8}:(9,11;1,3)_7,\quad V_{3,9}:(10;2)_8,$

$V_0:[13,-1,-5,-7;7,5,1,-13]_{-13},\quad V_1:[14,0,-2,-4,-6^2,-8;8,6^2,4,2,0,-14]_{-12},$

$V_{\bar{1}}:[18,16,14,12^2,10,8,6,4,-2^2,-4,-6,-8,-10;10,8,6^3,4^3,2^2,0^2,-2^2,-4].$

- $W=(4,6,7;18)$, $f_W=x^3y+y^3+xz^2$, $A_W=(2,4,7)$,

$V_{1,2}:(3,7;1,3)_1,\qquad V_{2,2}:(3,5,7;1^2,3)_1,\quad V_{2,3}:(4,6;0,2)_2,\quad V_{2,4}:(5;1)_3,$

$V_{3,2}:(3,5,7,9,11,13;1^2,3^2,5,7)_1,\quad V_{3,3}:(4,6,8,10,12;0,2^2,4,6)_2,$

$V_{3,4}:(5,7,9,11;1^2,3,5)_3,\quad V_{3,5}:(6,8,10;0,2,4)_4,\quad V_{3,6}:(7,9;1,3)_5,\quad V_{3,7}:(8;2)_6,$

$V_0:[10,0,-4,-6;6,4,0,-10]_{-10},\quad V_1:[11,1,-1,-3,-5^2,-7;7,5^2,3,1,-1,-11]_{-9},$

$V_{\bar{1}}:[14,12,10,8^2,6,4,-2^2,-4,-6;8,6,4^3,2^3,0^2,-2].$

- $W = (3,5,6;15)$, $f_W = x^3z + y^3 + xz^2$, $A_W = (3,3,6)$,

  $V_{1,2} : (3,5;0,2)_1$, $V_{1,3} : (4;1)_2$, $V_{2,2} : (3,5;0,2)_1$, $V_{2,3} : (4;1)_2$,

  $V_{3,2} : (3,5,7,9,11;0,2^2,4,6)_1$, $V_{3,3} : (4,6,8,10;1^2,3,5)_2$,

  $V_{3,4} : (5,7,9;0,2,4)_3$, $V_{3,5} : (6,8;1,3)_4$, $V_{3,6} : (7;2)_5$,

  $V_0 : [17/2, 1/2, -7/2, -11/2; 11/2, 7/2, -1/2, -17/2]_{-17/2}$,

  $V_1 : [19/2, 3/2, -1/2, -5/2, (-9/2)^2, -13/2; 13/2, (9/2)^2, 5/2, 1/2, -3/2, -19/2]_{-15/2}$,

  $V_{\bar{1}} : [12, 10, 8, 6^2, 4, -2^2, -4; 7, 5, 3^3, 1^3, -1]$.

- $W = (4,5,6;16)$, $f_W = x^4 + y^2z + z^2x$, $A_W = (2,5,6)$,

  $V_{1,2} : (3,7;1,3)_1$,

  $V_{2,2} : (3,5,7,9;1^2,3,5)_1$, $V_{2,3} : (4,6,8;0,2,4)_2$, $V_{2,4} : (5,7;1,3)_3$, $V_{2,5} : (6;2)_4$,

  $V_{3,2} : (3,5,7,9,11;1^2,3^2,5)_1$, $V_{3,3} : (4,6,8,10;0,2^2,4)_2$, $V_{3,4} : (5,7,9;1^2,3)_3$,

  $V_{3,5} : (6,8;0,2)_4$, $V_{3,6} : (7;1)_5$,

  $V_0 : [9,-1,-3,-5;5,3,1,-9]_{-9}$, $V_1 : [10,0,-2^2,-4^2,-6;6,4^2,2^2,0,-10]_{-8}$,

  $V_{\bar{1}} : [12, 10^2, 8^2, 6, 4, -2^2, -4, -6; 6^2, 4^3, 2^3, 0^2, -2]$.

- $W = (3,4,5;13)$, $f_W = x^3y + y^2z + z^2x$, $A_W = (3,4,5)$,

  $V_{1,2} : (3,5;0,2)_1$, $V_{1,3} : (4;1)_2$, $V_{2,2} : (3,5,7;0,2^2,4)_1$, $V_{2,3} : (4,6;1,3)_2$, $V_{2,4} : (5;2)_3$,

  $V_{3,2} : (3,5,7,9;0,2^2,4)_1$, $V_{3,3} : (4,6,8;1^2,3)_2$, $V_{3,4} : (5,7;0,2)_3$, $V_{3,5} : (6;1)_4$,

  $V_0 : [15/2, -1/2, -5/2, -9/2; 9/2, 5/2, 1/2, -15/2]_{-15/2}$,

  $V_1 : [17/2, 1/2, (-3/2)^2, (-7/2)^2, -11/2; 11/2, (7/2)^2, (3/2)^2, -1/2, -17/2]_{-13/2}$,

  $V_{\bar{1}} : [10, 8^2, 6^2, 4, -2^2, -4; 5^2, 3^3, 1^3, -1]$.

- $W = (3,4,4;12)$, $f_W = x^4 + yz(y-z)$, $A_W = (4,4,4)$,

  $V_{1,2} : (3,5,7;1^2,3)_1$, $V_{1,3} : (4,6;0,2)_2$, $V_{1,4} : (5;1)_3$,

  $V_{2,2} : (3,5,7;1^2,3)_1$, $V_{2,3} : (4,6;0,2)_2$, $V_{2,4} : (5;1)_3$,

  $V_{3,2} : (3,5,7;1^2,3)_1$, $V_{3,3} : (4,6;0,2)_2$, $V_{3,4} : (5;1)_3$,

  $V_0 : [7,-1,-3,-3;3,3,1,-7]_{-7}$, $V_1 : [8,0,-2^3,-4^2;4^2,2^3,0,-8]_{-6}$,

  $V_{\bar{1}} : [8^3, 6^2, 4, -2^2, -4; 4^3, 2^4, 0^2]$.

- $W = (2,6,9;18)$, $f_W = y(y-x^3)(y-\lambda x^3) + z^2$, $\lambda \neq 0, 1$, $A_W = (2,2,2,3)$,

  $V_{1,2} : (3)_1$, $V_{2,2} : (3)_1$, $V_{3,2} : (3)_1$, $V_{4,2} : (3,5,11)_1$, $V_{4,3} : (4,10)_2$,

  $V_0 : (4,10)_{-10}$, $V_1 : (3,5,11)_{-9}$, $V_{\bar{1}} : [12, 6, 4^2, -2^2]$.

- $W = (2,4,7;14)$, $f_W = xy(y-x^2)(y-\lambda x^2)+z^2$, $\lambda \neq 0,1$, $A_W = (2,2,2,4)$,

  $V_{1,2} : (3)_1,\quad V_{2,2} : (3)_1,\quad V_{3,2} : (3)_1,\quad V_{4,2} : (3,5,7)_1,\quad V_{4,3} : (4,6)_2,\quad V_{4,4} : (5)_3,$

  $V_0 : (2,8)_{-8},\quad V_1 : (1,3,9)_{-7},\quad V_{\bar{1}} : [8,6,4^2,-2^2].$

- $W = (2,4,5;12)$, $f_W = y(y-x^2)(y-\lambda x^2)+xz^2$, $\lambda \neq 0,1$, $A_W = (2,2,2,5)$,

  $V_{1,2} : (3;1)_1,\quad V_{2,2} : (3;1)_1,\quad V_{3,2} : (3;1)_1,$

  $V_{4,2} : (3,5,7,9;1^2,3,5)_1,\quad V_{4,3} : (4,6,8;0,2,4)_2,\quad V_{4,4} : (5,7;1,3)_3,\quad V_{4,5} : (6;2)_4,$

  $V_0 : [7,1,-3,-5;5,3,-1,-7]_{-7},\quad V_1 : [8,2,0,-2,-4^2,-6;6,4^2,2,0,-2,-8]_{-6},$

  $V_{\bar{1}} : [10,8,6,4^2,-2^2;6,4,2^3,0^2].$

- $W = (2,3,6;12)$, $f_W = (y^2-x^3)(y^2-\lambda x^3)+z^2$, $\lambda \neq 0,1$, $A_W = (2,2,3,3)$,

  $V_{1,2} : (3)_1,\quad V_{2,2} : (3)_1,\quad V_{3,2} : (3,5)_1,\quad V_{3,3} : (4)_2,\quad V_{4,2} : (3,5)_1,\quad V_{4,3} : (4)_2,$

  $V_0 : (1,7)_{-7},\quad V_1 : (0,2,8)_{-6},\quad V_{\bar{1}} : [6^2,4^2,-2^2].$

- $W = (2,3,4;10)$, $f_W = x(z-x^2)(z-\lambda x^2)+y^2z$, $\lambda \neq 0,1$, $A_W = (2,2,3,4)$,

  $V_{1,2} : (3;1)_1,\quad V_{2,2} : (3;1)_1,\quad V_{3,2} : (3,5;1,3)_1,\quad V_{3,3} : (4;2)_2,$

  $V_{4,2} : (3,5,7;1^2,3)_1,\quad V_{4,3} : (4,6;0,2)_2,\quad V_{4,4} : (5;1)_3,$

  $V_0 : [6,0,-2,-4;4,2,0,-6]_{-6},\quad V_1 : [7,1,-1^2,-3^2,-5;5,3^2,1^2,-1,-7]_{-5},$

  $V_{\bar{1}} : [8,6^2,4^2,-2^2;4^2,2^3,0^2].$

- $W = (2,3,3;9)$, $f_W = x^3y+z(z-y)(z-\lambda y)$, $\lambda \neq 0,1$, $A_W = (2,3,3,3)$,

  $V_{1,2} : (3;0)_1,\quad V_{2,2} : (3,5;0,2)_1,\quad V_{2,3} : (4;1)_2,$

  $V_{3,2} : (3,5;0,2)_1,\quad V_{3,3} : (4;1)_2,\quad V_{4,2} : (3,5;0,2)_1,\quad V_{4,3} : (4;1)_2,$

  $V_0 : [11/2,-1/2,-5/2,-5/2;5/2,5/2,1/2,-11/2]_{-11/2},$

  $V_1 : [13/2,1/2,(-3/2)^3,(-7/2)^2;(7/2)^2,(3/2)^3,-1/2,-13/2]_{-9/2},$

  $V_{\bar{1}} : [6^3,4^2,-2^2;3^3,1^4].$

- $W = (2,2,5;10)$, $f_W = xy(x-y)(y-\lambda_1 x)(y-\lambda_2 x)+z^2$, $\lambda_1 \neq 0,1$, $\lambda_1 \neq \lambda_2$, $A_W = (2,2,2,2,2)$,

  $V_{1,2} : (3)_1,\quad V_{2,2} : (3)_1,\quad V_{3,2} : (3)_1,\quad V_{4,2} : (3)_1,\quad V_{5,2} : (3)_1,$

  $V_0 : (0,6)_{-6},\quad V_1 : (1^2,7)_{-5},\quad V_{\bar{1}} : [6,4^3,-2^2].$

- $W = (2, 2, 3; 8)$, $f_W = y(y-x)(y-\lambda_1 x)(y-\lambda_2 x)+xz^2$, $\lambda_1 \neq 0, 1$, $\lambda_1 \neq \lambda_2$, $A_W = (2, 2, 2, 2, 3)$,

  $V_{1,2} : (3; 1)_1$, $V_{2,2} : (3; 1)_1$, $V_{3,2} : (3; 1)_1$, $V_{4,2} : (3; 1)_1$, $V_{5,2} : (3, 5; 1^2)_1$, $V_{5,3} : (4; 0)_2$,

  $V_0 : [5, -1^2, -3; 3, 1^2, -5]_{-5}$, $V_1 : [6, 0^2, -2^3, -4; 4, 2^3, 0^2, -6]_{-4}$,

  $V_{\bar{1}} : [6^2, 4^3, -2^2; 4, 2^4, 0^2]$.

7.2. **Matrix factorizations** $V_0, V_{\alpha_i}, V_{\bar{1}}$**.**

We present the explicit form of a reduced graded matrix factorization for V_0, V_{α_i}, and $V_{\bar{1}}$. The remaining reduced graded matrix factorizations are obtained by considering the AR-triangles (5.4) and (5.5).

Since the grading matrices are already listed up in the previous subsection, we present only the part $Q = \left(\begin{smallmatrix} \mathbf{0} & q_0 \\ q_1 & \mathbf{0} \end{smallmatrix}\right)$ of each reduced graded matrix factorization (Q, S).

- $W = (6, 14, 21; 42)$, $A_W = (2, 3, 7)$. For the polynomial $f_W = x^7 + y^3 + z^2$,

$$V_0 : q_0 = q_1 = \begin{pmatrix} z & y^2 & x^6 & 0 \\ y & -z & 0 & x^6 \\ x & 0 & -z & -y^2 \\ 0 & x & -y & z \end{pmatrix}, \quad V_{1,2} : q_0 = q_1 = \begin{pmatrix} z & 0 & 0 & y^2 & x^5 & x^3y \\ 0 & z & 0 & -x^2y & y^2 & -x^5 \\ 0 & 0 & z & x^4 & -x^2y & -y^2 \\ y & 0 & x^3 & -z & 0 & 0 \\ x^2 & y & 0 & 0 & -z & 0 \\ 0 & -x^2 & -y & 0 & 0 & -z \end{pmatrix},$$

$$V_{2,3} : q_0 = q_1 = \begin{pmatrix} z & x^4 & -y^2 & 0 \\ x^3 & -z & 0 & y^2 \\ -y & 0 & -z & x^4 \\ 0 & y & x^3 & z \end{pmatrix}, \qquad V_{3,7} : q_0 = q_1 = q_0(V_0) = q_1(V_0)\ ,$$

$$V_{\bar{1}} : q_0 = q_1 = \begin{pmatrix}
z & -y^2 & 0 & 0 & 0 & 0 & -x^3y & 0 & -x^6 & 0 & 0 & 0 & 0 & 0 \\
-y & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x^5 & 0 & x^3y & 0 \\
0 & xy & z & 0 & 0 & 0 & x^4 & 0 & -y^2 & 0 & 0 & 0 & 0 & 0 \\
0 & -xy & 0 & z & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & x^5 & 0 & x^2y \\
x^2 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 & -y^2 & 0 & -x^5 & 0 \\
0 & x^2 & 0 & 0 & 0 & z & 0 & 0 & 0 & -x^2y & 0 & y^2 & 0 & -x^5 \\
0 & 0 & x^2 & 0 & 0 & 0 & -z & 0 & 0 & 0 & x^2y & 0 & -y^2 & 0 \\
0 & 0 & 0 & 0 & x^3 & 0 & 0 & z & 0 & x^4 & 0 & -x^2y & 0 & -y^2 \\
-x & 0 & -y & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 \\
-x & 0 & 0 & y & 0 & 0 & 0 & x^3 & 0 & -z & 0 & 0 & x^4 & 0 \\
0 & -x^2 & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\
0 & 0 & 0 & x^2 & 0 & y & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 \\
0 & 0 & 0 & 0 & -x^2 & 0 & -y & 0 & -x^3 & 0 & 0 & 0 & z & 0 \\
0 & 0 & 0 & 0 & 0 & -x^2 & 0 & -y & 0 & 0 & -x^3 & 0 & 0 & -z
\end{pmatrix}.$$

- $W=(4,10,15;30)$, $A_W=(2,4,5)$. For the polynomial $f_W=y^3+yx^5+z^2$,

$$V_0: q_0=q_1=\begin{pmatrix} z & y^2 & x^4y & 0 \\ y & -z & 0 & x^4y \\ x & 0 & -z & -y^2 \\ 0 & x & -y & z \end{pmatrix}\ ,\quad V_{1,2}: q_0=q_1=\begin{pmatrix} z & 0 & y^2 & x^3y \\ 0 & -z & x^2y & -y^2 \\ y & x^3 & z & 0 \\ x^2 & -y & 0 & z \end{pmatrix}\ ,$$

$$V_{2,4}: q_0=q_1=\begin{pmatrix} z & x^5+y^2 \\ y & -z \end{pmatrix}\ ,\qquad V_{3,5}: q_0=q_1=q_0(V_0)=q_1(V_0)\ ,$$

$$V_{\bar{1}}: q_0=q_1=\begin{pmatrix} -z & y^2 & 0 & 0 & 0 & 0 & x^4y & 0 & 0 & -x^2y^2 \\ y & z & 0 & 0 & 0 & x^2y & 0 & 0 & 0 & 0 \\ 0 & -xy & -z & 0 & 0 & 0 & y^2 & 0 & 0 & x^3y \\ 0 & -xy & 0 & -z & x^4 & 0 & 0 & -y^2 & 0 & 0 \\ 0 & 0 & 0 & xy & z & 0 & 0 & 0 & -y^2 & 0 \\ 0 & x^3 & 0 & 0 & 0 & -z & -x^2y & 0 & 0 & 0 \\ x & 0 & y & 0 & 0 & 0 & z & 0 & 0 & 0 \\ -x & 0 & 0 & -y & 0 & -x^3 & 0 & z & -x^4 & 0 \\ 0 & -x^2 & 0 & 0 & -y & 0 & 0 & -xy & -z & 0 \\ 0 & 0 & x^2 & 0 & 0 & y & 0 & 0 & 0 & z \end{pmatrix}.$$

- $W=(3,8,12;24)$, $A_W=(3,3,4)$. For the polynomial $f_W=x^4z+y^3+z^2$,

$$V_0: q_0=q_1=\begin{pmatrix} z & y^2 & x^3z & 0 \\ y & -z & 0 & x^3z \\ x & 0 & -z & -y^2 \\ 0 & x & -y & z \end{pmatrix}\ ,$$

$$V_{1,3}: q_0=\begin{pmatrix} z+x^4 & y^2 \\ y & -z \end{pmatrix}\ ,\qquad q_1=\begin{pmatrix} z & y^2 \\ y & -(z+x^4) \end{pmatrix}\ ,$$

$$V_{2,3}: q_0=\begin{pmatrix} z & y^2 \\ y & -(z+x^4) \end{pmatrix}\ ,\qquad q_1=\begin{pmatrix} z+x^4 & y^2 \\ y & -z \end{pmatrix}\ ,$$

$$V_{3,4}: q_0=q_1=q_0(V_0)=q_1(V_0)\ ,$$

$$V_{\bar{1}}: q_0=\begin{pmatrix} -z & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & z+x^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & xy & -z & 0 & 0 & -y^2 & 0 & 0 \\ 0 & -xy & 0 & z+x^4 & 0 & 0 & y^2 & x^3y \\ -x^2 & 0 & 0 & 0 & z+x^4 & 0 & 0 & -y^2 \\ x & 0 & -y & 0 & 0 & z+x^4 & 0 & 0 \\ -x & 0 & 0 & y & x^3 & 0 & -z & 0 \\ 0 & -x^2 & 0 & 0 & -y & 0 & 0 & -(z+x^4) \end{pmatrix},$$

$$q_1 = \begin{pmatrix} -(z+x^4) & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & xy & -(z+x^4) & 0 & 0 & -y^2 & 0 & 0 \\ 0 & xy & 0 & z & 0 & 0 & y^2 & x^3y \\ -x^2 & 0 & 0 & 0 & z & 0 & 0 & -y^2 \\ x & 0 & -y & 0 & 0 & z & 0 & 0 \\ x & 0 & 0 & y & x^3 & 0 & -(z+x^4) & 0 \\ 0 & -x^2 & 0 & 0 & -y & 0 & 0 & -z \end{pmatrix}.$$

- $W = (6,8,15;30)$, $A_W = (2,3,8)$. For the polynomial $f_W = x^5 + xy^3 + z^2$,

$$V_0 : q_0 = q_1 = \begin{pmatrix} z & xy^2 & x^4 & 0 \\ y & -z & 0 & x^4 \\ x & 0 & -z & -xy^2 \\ 0 & x & -y & z \end{pmatrix}, \quad V_{1,2} : q_0 = q_1 = \begin{pmatrix} z & 0 & 0 & xy^2 & x^4 & -x^2y \\ 0 & z & 0 & -x^2y & xy^2 & x^4 \\ 0 & 0 & z & x^3 & -x^2y & xy^2 \\ y & 0 & x^2 & -z & 0 & 0 \\ x & y & 0 & 0 & -z & 0 \\ 0 & x & y & 0 & 0 & -z \end{pmatrix},$$

$$V_{2,3} : q_0 = q_1 = \begin{pmatrix} z & x^3 & xy^2 & 0 \\ x^2 & -z & 0 & -xy^2 \\ y & 0 & -z & x^3 \\ 0 & -y & x & z \end{pmatrix}, \quad V_{3,8} : q_0 = q_1 = \begin{pmatrix} z & x^4+y^3 \\ x & -z \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = q_1 = \begin{pmatrix} z & -y^2 & 0 & 0 & 0 & 0 & -x^2y & 0 & -x^4 & 0 & 0 & 0 & 0 & 0 \\ -xy & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x^4 & 0 & x^2y & 0 \\ 0 & -xy & z & 0 & 0 & 0 & -x^3 & 0 & xy^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & y^2 & 0 & 0 & 0 & 0 & xy^2 & 0 & x^4 & 0 & -x^3y \\ 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 & -xy^2 & 0 & -x^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & z & y^2 & 0 & 0 & -x^2y & 0 & xy^2 & 0 & x^4 \\ 0 & 0 & -x^2 & 0 & 0 & 0 & -z & 0 & 0 & 0 & x^2y & 0 & -xy^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -xy & z & 0 & x^3 & 0 & -x^2y & 0 & xy^2 \\ -x & 0 & y & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -y & y & 0 & 0 & 0 & x^2 & 0 & -z & y^2 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & -y & 0 & 0 & 0 & xy & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & y & 0 & 0 & 0 & 0 & 0 & -z & y^2 & 0 \\ 0 & 0 & 0 & 0 & -x & 0 & -y & 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & y & 0 & 0 & 0 & 0 & 0 & -z \end{pmatrix}.$$

- $W = (4,6,11;22)$, $A_W = (2,4,6)$. For the polynomial $f_W = yx^4 + xy^3 + z^2$,

$$V_0 : q_0 = q_1 = \begin{pmatrix} z & -x^4 & y^3 & 0 \\ y & -z & 0 & y^3 \\ x & 0 & -z & -x^4 \\ 0 & x & -y & z \end{pmatrix}, \quad V_{1,2} : q_0 = q_1 = \begin{pmatrix} z & 0 & xy^2 & x^3y \\ 0 & z & x^2y & -xy^2 \\ y & x^2 & -z & 0 \\ x & -y & 0 & -z \end{pmatrix},$$

$$V_{2,4} : q_0 = q_1 = \begin{pmatrix} z & x(x^3+y^2) \\ y & -z \end{pmatrix}, \qquad V_{3,6} : q_0 = q_1 = \begin{pmatrix} z & y(x^3+y^2) \\ x & -z \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = q_1 = \begin{pmatrix} z & -y^2 & 0 & 0 & -xy & 0 & -x^3y & 0 & 0 & 0 \\ -xy & -z & 0 & 0 & 0 & 0 & 0 & 0 & -x^3y & 0 \\ 0 & xy & z & 0 & -x^3 & 0 & -xy^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & y^2 & 0 & 0 & xy^2 & 0 & x^3y \\ 0 & 0 & -xy & 0 & -z & 0 & 0 & 0 & -xy^2 & 0 \\ 0 & 0 & 0 & 0 & xy & z & 0 & x^2y & 0 & -xy^2 \\ -x & 0 & -y & 0 & 0 & 0 & -z & 0 & 0 & 0 \\ 0 & 0 & y & y & 0 & x^2 & 0 & -z & y^2 & 0 \\ 0 & -x & 0 & 0 & -y & 0 & xy & 0 & z & 0 \\ 0 & 0 & 0 & x & 0 & -y & 0 & 0 & 0 & -z \end{pmatrix}.$$

- $W = (3,5,9;18)$, $A_W = (3,3,5)$. For the polynomial $f_W = x^3z + xy^3 + z^2$,

$$V_0 : q_0 = q_1 = \begin{pmatrix} z & xy^2 & x^2z & 0 \\ y & -z & 0 & x^2z \\ x & 0 & -z & -xy^2 \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,3} : q_0 = \begin{pmatrix} x^3+z & xy^2 \\ y & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & xy^2 \\ y & -(x^3+z) \end{pmatrix},$$

$$V_{2,3} : q_0 = \begin{pmatrix} z & xy^2 \\ y & -(x^3+z) \end{pmatrix}, \qquad q_1 = \begin{pmatrix} x^3+z & xy^2 \\ y & -z \end{pmatrix},$$

$$V_{3,5} : q_0 = q_1 = \begin{pmatrix} z & y^3+x^2z \\ x & -z \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = q_1 = \begin{pmatrix} z & y^2 & 0 & 0 & -xz & 0 & x^2z & 0 \\ 0 & -z & -y^2 & 0 & -x^2y & yz & 0 & -x^2z \\ 0 & 0 & -z & -x^3 & 0 & -xy^2 & 0 & x^3y \\ 0 & -xy & -z & z & 0 & 0 & xy^2 & 0 \\ 0 & 0 & 0 & xy & -z & -xz & 0 & -xy^2 \\ 0 & 0 & -y & 0 & -x^2 & z & 0 & 0 \\ x & 0 & -y & y & -x^2 & 0 & -z & 0 \\ 0 & -x & 0 & 0 & -y & 0 & xy & -z \end{pmatrix}.$$

- $W = (4,5,10;20)$, $A_W = (2,5,5)$. For the polynomial $f_W = x^5 + y^2z + z^2$,

$$V_0 : q_0 = q_1 = \begin{pmatrix} z & yz & x^4 & 0 \\ y & -z & 0 & x^4 \\ x & 0 & -z & -yz \\ 0 & x & -y & z \end{pmatrix}, \quad V_{1,2} : q_0 = q_1 = \begin{pmatrix} z & x^3 & yz & 0 \\ x & -z & 0 & -yz \\ y & 0 & -z & x^3 \\ 0 & -y & x & z \end{pmatrix},$$

$$V_{2,5} : q_0 = \begin{pmatrix} y^2+z & x^4 \\ x & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & x^4 \\ x & -(y^2+z) \end{pmatrix},$$

$$V_{3,5} : q_0 = \begin{pmatrix} z & x^4 \\ x & -(y^2+z) \end{pmatrix}, \qquad q_1 = \begin{pmatrix} y^2+z & x^4 \\ x & -z \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = q_1 = \begin{pmatrix} -z & -y^2 & 0 & 0 & 0 & 0 & x^4 & 0 & x^3y & 0 \\ -y^2 & z & 0 & 0 & -x^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & xy & -z & 0 & 0 & 0 & y^3 & 0 & -x^4 & 0 \\ xy & 0 & 0 & z & 0 & x^3 & 0 & y^3 & 0 & 0 \\ 0 & -x^2 & 0 & 0 & -z & 0 & -xy^2 & 0 & -y^3 & 0 \\ 0 & 0 & 0 & x^2 & 0 & -z & -x^2y & 0 & -xy^2 & -y^3 \\ x & 0 & y & 0 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & x & 0 & y & 0 & 0 & 0 & -z & 0 & x^3 \\ 0 & 0 & -x & 0 & -y & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & x & -y & 0 & x^2 & 0 & z \end{pmatrix}.$$

- $W = (3,4,8;16)$, $A_W = (3,4,4)$. For the polynomial $f_W = yx^4 + y^2z + z^2$,

$$V_0 : q_0 = q_1 = \begin{pmatrix} z & yz & x^3y & 0 \\ y & -z & 0 & x^3y \\ x & 0 & -z & -yz \\ 0 & x & -y & z \end{pmatrix}, \quad V_{1,3} : q_0 = q_1 = \begin{pmatrix} z & x^4+yz \\ y & -z \end{pmatrix},$$

$$V_{2,4} : q_0 = \begin{pmatrix} y^2+z & x^3y \\ x & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & x^3y \\ x & -(y^2+z) \end{pmatrix},$$

$$V_{3,4} : q_0 = \begin{pmatrix} z & x^3y \\ x & -(y^2+z) \end{pmatrix}, \qquad q_1 = \begin{pmatrix} y^2+z & x^3y \\ x & -z \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = q_1 = \begin{pmatrix} z & z & 0 & 0 & x^2y & 0 & -x^3y & 0 \\ y^2 & -z & 0 & 0 & 0 & x^3y & 0 & x^2y^2 \\ xy & 0 & -z & 0 & 0 & -yz & 0 & x^3y \\ 0 & 0 & z & z & x^3 & 0 & yz & 0 \\ 0 & 0 & 0 & xy & -z & 0 & 0 & -yz \\ 0 & x & -y & 0 & 0 & z & 0 & 0 \\ -x & 0 & 0 & y & 0 & z & -z & 0 \\ 0 & 0 & x & 0 & -y & 0 & xy & z \end{pmatrix}.$$

- $W = (6,8,9;24)$, $A_W = (2,3,9)$. For the polynomial $f_W = x^4 + y^3 + xz^2$,

$$V_0 : q_0 = \begin{pmatrix} xz & y^2 & x^3 & 0 \\ y & -z & 0 & x^3 \\ x & 0 & -z & -y^2 \\ 0 & x & -y & xz \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & y^2 & x^3 & 0 \\ y & -xz & 0 & x^3 \\ x & 0 & -xz & -y^2 \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,2} : q_0 = \begin{pmatrix} xz & y^2 & 0 & x^3 & 0 & -x^2y \\ 0 & -xy & xz & y^2 & 0 & x^3 \\ 0 & x^2 & 0 & -xy & xz & y^2 \\ y & -z & 0 & 0 & x^2 & 0 \\ x & 0 & y & -z & 0 & 0 \\ 0 & 0 & x & 0 & y & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & 0 & 0 & y^2 & x^3 & -x^2y \\ y & 0 & x^2 & -xz & 0 & 0 \\ 0 & z & 0 & -xy & y^2 & x^3 \\ x & y & 0 & 0 & -xz & 0 \\ 0 & 0 & z & x^2 & -xy & y^2 \\ 0 & x & y & 0 & 0 & -xz \end{pmatrix},$$

$$V_{2,3} : q_0 = \begin{pmatrix} xz & y^2 & -x^3 & 0 \\ x^2 & 0 & xz & -y^2 \\ y & -z & 0 & x^2 \\ 0 & x & y & z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & x^2 & y^2 & 0 \\ y & 0 & -xz & x^3 \\ -x & z & 0 & y^2 \\ 0 & -y & x^2 & xz \end{pmatrix},$$

$$V_{3,9} : q_0 = \begin{pmatrix} y^2 & x^3 + z^2 \\ x & -y \end{pmatrix}, \qquad q_1 = \begin{pmatrix} y & x^3 + z^2 \\ x & -y^2 \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = \left(\begin{array}{ccccccccccccccc} -y^2 & -yz & z^2 & -x^3 & 0 & 0 & 0 & 0 & 0 & 0 & xz^2 & 0 & 0 & 0 & 0 \\ xz & -y^2 & yz & 0 & 0 & -x^3 & 0 & 0 & 0 & 0 & x^2y & 0 & 0 & 0 & 0 \\ xy & xz & y^2 & 0 & 0 & 0 & 0 & 0 & -x^3 & 0 & 0 & 0 & x^2y & 0 & 0 \\ 0 & -xy & xz & xz & 0 & y^2 & 0 & 0 & 0 & 0 & x^3 & 0 & 0 & 0 & 0 \\ 0 & xy & 0 & 0 & xz & 0 & 0 & y^2 & 0 & 0 & x^3 & x^3 & 0 & 0 & -x^2y \\ x^2 & 0 & xy & 0 & 0 & 0 & xz & 0 & y^2 & 0 & 0 & 0 & x^3 & 0 & 0 \\ 0 & -x^2 & 0 & 0 & 0 & 0 & 0 & -xy & 0 & xz & 0 & y^2 & 0 & 0 & x^3 \\ 0 & 0 & -x^2 & -x^2 & 0 & 0 & 0 & 0 & -xy & 0 & xz & 0 & y^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^2 & x^2 & 0 & 0 & 0 & -xy & 0 & xz & y^2 \\ -x & 0 & 0 & y & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 0 & y & 0 & y & 0 & 0 & -z & 0 & 0 & 0 & 0 & x^2 & x^2 & 0 \\ 0 & -x & 0 & 0 & 0 & 0 & y & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & y & 0 & -z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & x & 0 & 0 & 0 & y & 0 & -z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 & 0 & y & -z \end{array} \right),$$

$$q_1 = \begin{pmatrix} -y & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x^3 & 0 & x^2y & 0 \\ x & 0 & y & 0 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -y & 0 & z & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & x^3 & 0 & -x^2y \\ 0 & -x & 0 & y & 0 & 0 & 0 & 0 & 0 & -xz & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & x^3 & 0 \\ 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & x^2 & 0 & -xz & 0 & 0 & -x^3 & 0 \\ 0 & 0 & -x & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & -xz & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & z & 0 & 0 & 0 & -xy & 0 & y^2 & 0 & x^3 \\ 0 & 0 & 0 & x & 0 & 0 & 0 & z & 0 & 0 & 0 & -xy & 0 & y^2 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & y & 0 & 0 & 0 & 0 & 0 & -xz & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & y & 0 & x^2 & 0 & 0 & 0 & -xz & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & 0 & 0 & z & 0 & x^2 & 0 & -xy & 0 & y^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & x & 0 & 0 & -x^2 & 0 & 0 & -xz \end{pmatrix}.$$

- $W = (4,6,7;18)$, $A_W = (2,4,7)$. For the polynomial $f_W = x^3y + y^3 + xz^2$,

$$V_0 : q_0 = \begin{pmatrix} xz & y^2 & x^2y & 0 \\ y & -z & 0 & x^2y \\ x & 0 & -z & -y^2 \\ 0 & x & -y & xz \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & y^2 & x^2y & 0 \\ y & -xz & 0 & x^2y \\ x & 0 & -xz & -y^2 \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,2} : q_0 = \begin{pmatrix} xz & y^2 & 0 & x^2y \\ 0 & -xy & xz & y^2 \\ y & -z & -x^2 & 0 \\ x & 0 & y & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & 0 & y^2 & -x^2y \\ y & -x^2 & -xz & 0 \\ 0 & z & -xy & y^2 \\ x & y & 0 & -xz \end{pmatrix},$$

$$V_{2,4} : q_0 = \begin{pmatrix} xz & x^3+y^2 \\ y & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & x^3+y^2 \\ y & -xz \end{pmatrix},$$

$$V_{3,7} : q_0 = \begin{pmatrix} x^2y+z^2 & y^2 \\ y & -x \end{pmatrix}, \qquad q_1 = \begin{pmatrix} x & y^2 \\ y & -(x^2y+z^2) \end{pmatrix},$$

$$V_{\bar{1}} \ : q_0 = \begin{pmatrix} -y^2 & -yz & z^2 & -x^2y & 0 & 0 & x^2z & 0 & 0 & 0 & 0 \\ xz & -y^2 & yz & 0 & 0 & -x^2y & x^2y & 0 & 0 & 0 & 0 \\ xy & xz & y^2 & 0 & 0 & 0 & 0 & 0 & x^2y & 0 & 0 \\ 0 & -xy & xz & xz & 0 & y^2 & x^3 & 0 & 0 & 0 & 0 \\ 0 & -xy & 0 & 0 & xz & xy & 0 & y^2 & 0 & 0 & x^2y \\ 0 & 0 & -xy & -xy & 0 & 0 & xz & 0 & y^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -xy & -xy & 0 & xz & y^2 \\ -x & 0 & 0 & y & 0 & -z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -y & -y & y & 0 & 0 & -z & 0 & -x^2 & 0 \\ 0 & x & 0 & 0 & 0 & -y & y & 0 & -z & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & -y & y & -z \end{pmatrix},$$

$$q_1 = \begin{pmatrix} -y & z & 0 & 0 & 0 & 0 & 0 & -x^2y & 0 & 0 & 0 \\ 0 & -y & z & 0 & 0 & 0 & 0 & 0 & 0 & x^2y & 0 \\ x & 0 & y & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & z & 0 & 0 & 0 & y^2 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & z & 0 & 0 & 0 & y^2 & 0 & x^2y \\ 0 & -x & 0 & y & 0 & 0 & 0 & -xz & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & z & 0 & 0 & 0 & y^2 & 0 \\ 0 & -x & 0 & 0 & y & 0 & -x^2 & 0 & -xz & 0 & 0 \\ 0 & 0 & x & 0 & 0 & y & 0 & xy & 0 & -xz & 0 \\ 0 & 0 & 0 & 0 & 0 & y & z & xy & -xy & 0 & y^2 \\ 0 & 0 & 0 & 0 & x & 0 & y & 0 & 0 & xy & -xz \end{pmatrix}.$$

- $W = (3, 5, 6; 15)$, $A_W = (3, 3, 6)$. For the polynomial $f_W = x^3z + y^3 + xz^2$,

$$V_0 : q_0 = \begin{pmatrix} xz & y^2 & x^2z & 0 \\ y & -z & 0 & x^2z \\ x & 0 & -z & -y^2 \\ 0 & x & -y & xz \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & y^2 & x^2z & 0 \\ y & -xz & 0 & x^2z \\ x & 0 & -xz & -y^2 \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,3} : q_0 = \begin{pmatrix} xz & y^2 \\ y & -(x^2+z) \end{pmatrix}, \qquad q_1 = \begin{pmatrix} x^2+z & y^2 \\ y & -xz \end{pmatrix},$$

$$V_{2,3} : q_0 = \begin{pmatrix} x(x^2+z) & y^2 \\ y & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & y^2 \\ y & -x(x^2+z) \end{pmatrix},$$

$$V_{3,6} : q_0 = \begin{pmatrix} y^2 & z(x^2+z) \\ x & -y \end{pmatrix}, \qquad q_1 = \begin{pmatrix} y & z(x^2+z) \\ x & -y^2 \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = \begin{pmatrix} -y^2 & -yz & x^z + z^2 & 0 & 0 & 0 & 0 & 0 & xyz \\ xz & -y^2 & x^y + yz & x^2y & 0 & -x^2 & 0 & 0 & xy^2 \\ xy & xz & y^2 & 0 & 0 & 0 & 0 & 0 & -x^2z \\ 0 & -xy & x^3 + xz & x^3 + xz & 0 & y^2 & 0 & 0 & x^2y \\ 0 & 0 & -x^3 & -x^3 & xz & 0 & 0 & y^2 & -x^2y \\ x^2 & 0 & xy & 0 & 0 & 0 & xz & 0 & y^2 \\ -x & 0 & 0 & y & 0 & -z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & -x^2 & x^2 & -x^2 - z & 0 \\ 0 & -x & 0 & 0 & 0 & 0 & y & 0 & -z \end{pmatrix},$$

$$q_1 = \begin{pmatrix} -y & z & 0 & 0 & 0 & 0 & -x^2z & 0 & 0 \\ 0 & -y & z & 0 & 0 & xy & x^2y & 0 & -x^2z \\ x & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & z & 0 & 0 & y^2 & 0 & 0 \\ 0 & 0 & 0 & x^2 & x^2 + z & 0 & 0 & y^2 & -x^2y \\ 0 & -x & 0 & y & 0 & 0 & -xz & 0 & 0 \\ 0 & -x & 0 & 0 & 0 & x^2 + z & x^3 & 0 & y^2 \\ 0 & 0 & 0 & 0 & y & x^2 & x^3 & -xz & 0 \\ 0 & 0 & -x & 0 & 0 & y & 0 & 0 & -xz \end{pmatrix}.$$

- $W = (4, 5, 6; 16)$, $A_W = (2, 5, 6)$. For the polynomial $f_W = x^4 + y^2z + xz^2$,

$$V_0 : q_0 = \begin{pmatrix} xz & yz & x^3 & 0 \\ y & -z & 0 & x^3 \\ x & 0 & -z & -yz \\ 0 & x & -y & xz \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & yz & x^3 & 0 \\ y & -xz & 0 & x^3 \\ x & 0 & -xz & -yz \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,2} : q_0 = \begin{pmatrix} xz & yz & x^3 & 0 \\ -x^2 & 0 & xz & yz \\ -y & z & 0 & -x^2 \\ 0 & x & -y & z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & -x^2 & -yz & 0 \\ y & 0 & xz & x^3 \\ x & z & 0 & -yz \\ 0 & y & -x^2 & xz \end{pmatrix},$$

$$V_{2,5} : q_0 = \begin{pmatrix} y^2 + xz & x^3 \\ x & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & x^3 \\ x & -(y^2 + xz) \end{pmatrix},$$

$$V_{3,6} : q_0 = \begin{pmatrix} yz & x^3 + z^2 \\ x & -y \end{pmatrix}, \qquad q_1 = \begin{pmatrix} y & x^3 + z^2 \\ x & -yz \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = \begin{pmatrix} -yz & yz & x^3+z^2 & -x^3 & 0 & 0 & 0 & 0 & -x^2z & 0 & 0 \\ -xz & xz & -yz & 0 & 0 & 0 & -x^3 & 0 & 0 & 0 & 0 \\ xz & y^2 & yz & 0 & 0 & -x^3 & 0 & 0 & -x^2y & 0 & 0 \\ 0 & -xy & 0 & -xz & 0 & -yz & 0 & 0 & x^3 & 0 & 0 \\ 0 & xy & 0 & 0 & xz & 0 & 0 & yz & -x^3 & x^3 & 0 \\ 0 & x^2 & 0 & 0 & 0 & xz & xz & 0 & yz & 0 & 0 \\ 0 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 & -xz & xz & yz \\ -x & 0 & -y & y & 0 & -z & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & -y & 0 & 0 & z & 0 & 0 & -x^2 \\ 0 & 0 & 0 & x & 0 & 0 & -y & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & -x & x & 0 & -y & z \end{pmatrix},$$

$$q_1 = \begin{pmatrix} -y & 0 & z & 0 & 0 & 0 & 0 & -x^3 & 0 & 0 & 0 \\ 0 & z & z & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 \\ x & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^3 & 0 \\ 0 & -y & 0 & -z & 0 & 0 & 0 & yz & 0 & x^3 & 0 \\ 0 & 0 & 0 & -z & z & 0 & -x^2 & 0 & -yz & 0 & 0 \\ 0 & 0 & -x & -y & 0 & 0 & 0 & -xz & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & 0 & z & 0 & xz & 0 & -yz & 0 \\ 0 & -x & 0 & 0 & y & 0 & 0 & xz & xz & 0 & x^3 \\ 0 & 0 & 0 & x & 0 & y & 0 & 0 & 0 & xz & 0 \\ 0 & 0 & 0 & 0 & x & 0 & z & 0 & 0 & xz & -yz \\ 0 & 0 & 0 & 0 & 0 & x & y & 0 & -x^2 & 0 & xz \end{pmatrix}.$$

- $W = (3,4,5;13)$, $A_W = (3,4,5)$. For the polynomial $f_W = x^3y + y^2z + xz^2$,

$$V_0 : q_0 = \begin{pmatrix} xz & yz & x^2y & 0 \\ y & -z & 0 & x^2y \\ x & 0 & -z & -yz \\ 0 & x & -y & xz \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & yz & x^2y & 0 \\ y & -xz & 0 & x^2y \\ x & 0 & -xz & -yz \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,3} : q_0 = \begin{pmatrix} xz+y^2 & x^3 \\ y & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & x^3 \\ y & -(xz+y^2) \end{pmatrix},$$

$$V_{2,4} : q_0 = \begin{pmatrix} y^2 & x^2y+z^2 \\ x & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & x^2y+z^2 \\ x & -y^2 \end{pmatrix},$$

$$V_{3,5} : q_0 = \begin{pmatrix} yz & x^2y+z^2 \\ x & -y \end{pmatrix}, \qquad q_1 = \begin{pmatrix} y & x^2y+z^2 \\ x & -yz \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = \begin{pmatrix} yz & yz & 0 & -x^2y - z^2 & 0 & 0 & 0 & xy^2 & 0 \\ xz & -y^2 & 0 & yz & 0 & -x^2y & 0 & 0 & xy^2 \\ xz & xz & 0 & yz & 0 & 0 & 0 & x^2y & 0 \\ 0 & -xy & xz & xz & 0 & yz & 0 & 0 & x^2y \\ 0 & 0 & 0 & 0 & xz + y^2 & 0 & x^3 & 0 & -x^2y \\ 0 & -x^2 & -xy & -xy & 0 & xz & 0 & y^2 + xz & 0 \\ -x & 0 & y & 0 & 0 & -z & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & y & z & -z & z & 0 \\ 0 & 0 & -x & -x & 0 & 0 & 0 & y & z \end{pmatrix},$$

$$q_1 = \begin{pmatrix} y & z & 0 & 0 & 0 & 0 & -x^2y & 0 & -xy^2 \\ 0 & -z & z & 0 & 0 & -xy & 0 & 0 & xy^2 \\ x & 0 & 0 & z & 0 & 0 & yz & 0 & -x^2y \\ -x & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & -x^2 & 0 & x^3 & x^2y \\ 0 & -x & 0 & y & 0 & 0 & -xz & 0 & 0 \\ 0 & 0 & 0 & 0 & y & z & 0 & -y^2 - xz & 0 \\ 0 & 0 & x & 0 & 0 & z & xz & 0 & 0 \\ 0 & 0 & 0 & x & 0 & -y & 0 & 0 & y^2 + xz \end{pmatrix}.$$

- $W = (3, 4, 4; 12)$, $A_W = (4, 4, 4)$. For the polynomial $f_W = x^4 + yz(y - z)$,

$$V_0 : q_0 = \begin{pmatrix} -yz & yz & x^3 & 0 \\ y & -z & 0 & x^3 \\ x & 0 & -z & -yz \\ 0 & x & -y & -yz \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & yz & x^3 & 0 \\ y & yz & 0 & x^3 \\ x & 0 & yz & -yz \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,4} : q_0 = \begin{pmatrix} yz & x^3 \\ x & -(y-z) \end{pmatrix}, \qquad q_1 = \begin{pmatrix} (y-z) & x^3 \\ x & -yz \end{pmatrix},$$

$$V_{2,4} : q_0 = \begin{pmatrix} (y-z)z & x^3 \\ x & -y \end{pmatrix}, \qquad q_1 = \begin{pmatrix} y & x^3 \\ x & -(y-z)z \end{pmatrix},$$

$$V_{3,4} : q_0 = \begin{pmatrix} y(y-z) & x^3 \\ x & -y \end{pmatrix}, \qquad q_1 = \begin{pmatrix} y & x^3 \\ x & -y(y-z) \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = \begin{pmatrix} -(y-z)z & 0 & 0 & -x^3 & 0 & 0 & 0 & -x^2z & 0 \\ 0 & y(y-z) & 0 & 0 & -x^3 & 0 & 0 & -x^2y & 0 \\ 0 & -y^2 & yz & 0 & 0 & 0 & x^3 & 0 & x^2y \\ xz & xy & 0 & -yz & yz & 0 & 0 & -x^3 & 0 \\ 0 & xy & 0 & 0 & 0 & yz & 0 & 0 & -x^3 \\ -x^2 & 0 & 0 & xy & 0 & 0 & 0 & yz & 0 \\ -x & -x & 0 & y & -z & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & y & -(y-z) & 0 & 0 \\ 0 & 0 & 0 & 0 & x & -x & 0 & y & -(y-z) \end{pmatrix},$$

$$q_1 = \begin{pmatrix} -y & 0 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & x^2 & -x^3 & 0 & 0 \\ 0 & y & y-z & 0 & 0 & 0 & 0 & x^3 & x^2y \\ -x & 0 & 0 & z & 0 & 0 & yz & 0 & 0 \\ 0 & -x & 0 & y & 0 & 0 & yz & 0 & 0 \\ 0 & -x & 0 & 0 & y-z & 0 & 0 & 0 & -x^3 \\ 0 & 0 & x & 0 & y & 0 & 0 & -yz & 0 \\ 0 & 0 & 0 & -x & 0 & y-z & -xy & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & y & -xy & 0 & -yz \end{pmatrix} .$$

• $W = (2,6,9;18)$, $A_W = (2,2,2,3)$. For the polynomial $f_W = yy'y'' + z^2$, $y' := y - x^3$, $y'' := y - \lambda x^3$, $\lambda \neq 0, 1$,

$$V_0 : q_0 = q_1 = \begin{pmatrix} z & yy'' & -x^2yy'' & 0 \\ y & -z & 0 & -x^2yy'' \\ x & 0 & -z & -yy'' \\ 0 & x & -y & z \end{pmatrix} ,$$

$$V_{1,2} : q_0 = q_1 = \begin{pmatrix} z & y'y'' \\ y & -z \end{pmatrix} , \quad V_{2,2} : q_0 = q_1 = \begin{pmatrix} z & yy'' \\ y' & -z \end{pmatrix} , \quad V_{3,2} : q_0 = q_1 = \begin{pmatrix} z & yy' \\ y'' & -z \end{pmatrix} ,$$

$$V_{4,3} : q_0 = q_1 = q_0(V_0) = q_1(V_0) ,$$

$$V_{\bar{1}} : q_0 = q_1 = \begin{pmatrix} z & -y^2 & 0 & 0 & x^2yy'' & 0 \\ -y'' & -z & x^2y'' & 0 & 0 & 0 \\ 0 & -xy & z & 0 & yy'' & 0 \\ 0 & x^4 & 0 & z & 0 & y'y'' \\ -x & 0 & y & 0 & z & 0 \\ 0 & 0 & x^3 & y & 0 & -z \end{pmatrix} .$$

• $W = (2,4,7;14)$, $A_W = (2,2,2,4)$. For the polynomial $f_W = xyy'y'' + z^2$, $y' := y - x^2$, $y'' := y - \lambda x^2$, $\lambda \neq 0, 1$,

$$V_0 : q_0 = q_1 = \begin{pmatrix} z & xyy'' & -x^2yy'' & 0 \\ y & -z & 0 & -x^2yy'' \\ x & 0 & -z & -xyy'' \\ 0 & x & -y & z \end{pmatrix} ,$$

$$V_{1,2} : q_0 = q_1 = \begin{pmatrix} z & xy'y'' \\ y & -z \end{pmatrix} , \quad V_{2,2} : q_0 = q_1 = \begin{pmatrix} z & xyy'' \\ y' & -z \end{pmatrix} ,$$

$$V_{3,2} : q_0 = q_1 = \begin{pmatrix} z & xyy' \\ y'' & -z \end{pmatrix} , \quad V_{4,4} : q_0 = q_1 = \begin{pmatrix} z & yy'y'' \\ x & -z \end{pmatrix} ,$$

$$V_{\bar{1}} : q_0 = q_1 = \begin{pmatrix} x & -y^2 & 0 & 0 & x^2yy'' & 0 \\ -xy'' & -z & x^2y'' & 0 & 0 & 0 \\ 0 & -xy & z & 0 & xyy'' & 0 \\ 0 & x^3 & 0 & z & -x^3y'' & xy'y'' \\ -x & 0 & y & 0 & -z & 0 \\ 0 & 0 & x^2 & y & 0 & -z \end{pmatrix}.$$

• $W = (2,4,5;12)$, $A_W = (2,2,2,5)$. For the polynomial $f_W = yy'y'' + xz^2$, $y' := y - x^2$, $y'' := y - \lambda x^2$, $\lambda \neq 0, 1$,

$$V_0 : q_0 = \begin{pmatrix} xz & yy'' & -xyy'' & 0 \\ y & -z & 0 & -xyy'' \\ x & 0 & -z & -yy'' \\ 0 & x & -y & xz \end{pmatrix}, \quad q_1 = \begin{pmatrix} z & yy'' & -xyy'' & 0 \\ y & -xz & 0 & -xyy'' \\ x & 0 & -xz & -yy'' \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,2} : q_0 = \begin{pmatrix} xz & y'y'' \\ y & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & y'y'' \\ y & -xz \end{pmatrix},$$

$$V_{2,2} : q_0 = \begin{pmatrix} xz & yy'' \\ y' & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & yy'' \\ y' & -xz \end{pmatrix},$$

$$V_{3,2} : q_0 = \begin{pmatrix} xz & yy' \\ y'' & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & yy' \\ y'' & -xz \end{pmatrix},$$

$$V_{4,5} : q_0 = \begin{pmatrix} y'y'' & z^2 \\ x & -y \end{pmatrix}, \qquad q_1 = \begin{pmatrix} y & z^2 \\ x & -y'y'' \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = \begin{pmatrix} yy'' & yz & -xyy'' & -z^2 & 0 & 0 & 0 \\ xz & -y^2 & 0 & yz & 0 & xyy'' & 0 \\ xy'' & xz & -x^2y'' & y'y'' & 0 & 0 & 0 \\ 0 & -xy & xz & xz & 0 & yy'' & 0 \\ 0 & -x^3 & 0 & 0 & xz & x^2y'' & y'y'' \\ -x & 0 & y & 0 & 0 & -z & 0 \\ 0 & 0 & -x^2 & -x^2 & y & 0 & -z \end{pmatrix},$$

$$q_1 = \begin{pmatrix} y & z & 0 & 0 & 0 & xyy'' & 0 \\ 0 & -y'' & z & xy'' & 0 & 0 & 0 \\ x & 0 & 0 & z & 0 & yy'' & 0 \\ -x & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & x^2 & 0 & z & x^2y' & y'y'' \\ 0 & -x & 0 & y & 0 & -xz & 0 \\ 0 & 0 & 0 & -x^2 & y & 0 & -xz \end{pmatrix}.$$

• $W=(2,3,6;12)$, $A_W=(2,2,3,3)$. For the polynomial $f_W=YY'+z^2$, $Y:=y^2-x^3$, $Y':=y^2-\lambda x^3$, $\lambda\neq 0,1$,

$$V_0: q_0=q_1=\begin{pmatrix} z & yY' & -x^2Y' & 0 \\ y & -z & 0 & -x^2Y' \\ x & 0 & -z & -yY' \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,2}: q_0=\begin{pmatrix} (z+\sqrt{-\lambda}x^3) & y(Y'-x^3) \\ y & -(z-\sqrt{-\lambda}x^3) \end{pmatrix}, \qquad q_1=\begin{pmatrix} (z-\sqrt{-\lambda}x^3) & y(Y'-x^3) \\ y & -(z+\sqrt{-\lambda}x^3) \end{pmatrix},$$

$$V_{2,2}: q_0=\begin{pmatrix} (z-\sqrt{-\lambda}x^3) & y(Y'-x^3) \\ y & -(z+\sqrt{-\lambda}x^3) \end{pmatrix}, \qquad q_1=\begin{pmatrix} (z+\sqrt{-\lambda}x^3) & y(Y'-x^3) \\ y & -(z-\sqrt{-\lambda}x^3) \end{pmatrix},$$

$$V_{3,3}: q_0=\begin{pmatrix} z+\sqrt{-1}y^2 & -x^2(\lambda y^2+Y') \\ x & -(z-\sqrt{-1}y^2) \end{pmatrix}, \qquad q_1=\begin{pmatrix} z-\sqrt{-1}y^2 & -x^2(\lambda y^2+Y') \\ x & -(z+\sqrt{-1}y^2) \end{pmatrix},$$

$$V_{4,3}: q_0=\begin{pmatrix} z-\sqrt{-1}y^2 & -x^2(\lambda y^2+Y') \\ x & -(z+\sqrt{-1}y^2) \end{pmatrix}, \qquad q_1=\begin{pmatrix} z+\sqrt{-1}y^2 & -x^2(\lambda y^2+Y') \\ x & -(z-\sqrt{-1}y^2) \end{pmatrix},$$

$$V_{\bar{1}}: q_0=\begin{pmatrix} z & Y' & -x^2y & \sqrt{-\lambda}x^2y & 0 & x^2Y' \\ y^2 & -z & \sqrt{-\lambda}x^2y & 0 & -x^2Y' & 0 \\ 0 & 0 & -(z-\sqrt{-\lambda}x^3) & y^2 & 0 & 0 \\ 0 & 0 & Y'-x^3 & z+\sqrt{-\lambda}x^3 & 0 & 0 \\ 0 & x & -y & 0 & z & y^2 \\ -x & 0 & 0 & y & Y' & -z \end{pmatrix},$$

$$q_1=\begin{pmatrix} z & Y' & -x^2y & -\sqrt{-\lambda}x^2y & 0 & x^2Y' \\ y^2 & -z & -\sqrt{-\lambda}x^2y & 0 & -x^2Y' & 0 \\ 0 & 0 & -(z+\sqrt{-\lambda}x^3) & y^2 & 0 & 0 \\ 0 & 0 & Y'-x^3 & z-\sqrt{-\lambda}x^3 & 0 & 0 \\ 0 & x & -y & 0 & z & y^2 \\ -x & 0 & 0 & y & Y' & -z \end{pmatrix}.$$

• $W=(2,3,4;10)$, $A_W=(2,2,3,4)$. For the polynomial $f_W=xz'z''+y^2z$, $z':=z-x^2$, $z'':=z-\lambda x^2$, $\lambda\neq 0,1$,

$$V_0: q_0=\begin{pmatrix} xz'' & yz & -x^2z'' & 0 \\ y & -z & 0 & -x^2z'' \\ x & 0 & -z & -yz \\ 0 & x & -y & xz'' \end{pmatrix}, \quad q_1=\begin{pmatrix} z & yz & -x^2z'' & 0 \\ y & -xz'' & 0 & -x^2z'' \\ x & 0 & -xz'' & -yz \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,2}: q_0=\begin{pmatrix} xz'' & yz \\ y & -z' \end{pmatrix}, \qquad q_1=\begin{pmatrix} z' & yz \\ y & -xz'' \end{pmatrix},$$

$$V_{2,2} : q_0 = \begin{pmatrix} xz' & yz \\ y & -z'' \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z'' & yz \\ y & -xz' \end{pmatrix},$$

$$V_{3,3} : q_0 = \begin{pmatrix} y^2 & z'z'' \\ x & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & z'z'' \\ x & -y^2 \end{pmatrix},$$

$$V_{4,4} : q_0 = \begin{pmatrix} yz & z'z'' \\ x & -y \end{pmatrix}, \qquad q_1 = \begin{pmatrix} y & z'z'' \\ x & -yz \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = \begin{pmatrix} yz & yz & -\lambda x^2 z' & -zz' & 0 & 0 & 0 \\ xz'' & -y^2 & (\lambda+1)x^2 y & yz' & 0 & x^2 z'' & 0 \\ xz & xz'' & 0 & yz & 0 & \lambda x^2 z & 0 \\ 0 & -xy & xz' & xz' & 0 & yz & 0 \\ 0 & 0 & xz & xz & xz'' & 0 & yz \\ -x & 0 & y & 0 & 0 & -z & 0 \\ 0 & -x & 0 & 0 & y & z & -z' \end{pmatrix},$$

$$q_1 \begin{pmatrix} y & z & 0 & 0 & 0 & x^2 z'' & 0 \\ 0 & -z & z' & 0 & 0 & (\lambda-1)x^2 z & 0 \\ x & 0 & 0 & z & 0 & yz & 0 \\ -x & 0 & y & -\lambda x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z & z' & 0 & yz \\ 0 & -x & 0 & y & 0 & -xz'' & 0 \\ 0 & 0 & -x & 0 & y & -xz & -xz'' \end{pmatrix}.$$

• $W = (2,3,3;9)$, $A_W = (2,3,3,3)$. For the polynomial $f_W = x^3 y + zz'z''$, $z' := z - y$, $z'' := z - \lambda y$, $\lambda \neq 0, 1$,

$$V_0 : q_0 = \begin{pmatrix} zz'' & -zz'' & x^2 y & 0 \\ y & -z & 0 & x^2 y \\ x & 0 & -z & zz'' \\ 0 & x & -y & zz'' \end{pmatrix}, \quad q_1 = \begin{pmatrix} z & -zz'' & x^2 y & 0 \\ y & -zz'' & 0 & x^2 y \\ x & 0 & -zz'' & zz'' \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,2} : q_0 = \begin{pmatrix} x^3 & z'z'' \\ z & -y \end{pmatrix}, \qquad q_1 = \begin{pmatrix} y & z'z'' \\ z & -x^3 \end{pmatrix},$$

$$V_{2,3} : q_0 = \begin{pmatrix} z'z'' & x^2 y \\ x & -z \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z & x^2 y \\ x & -z'z'' \end{pmatrix},$$

$$V_{3,3} : q_0 = \begin{pmatrix} zz'' & x^2 y \\ x & -z' \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z' & x^2 y \\ x & -zz'' \end{pmatrix},$$

$$V_{4,3} : q_0 = \begin{pmatrix} zz' & x^2 y \\ x & -z'' \end{pmatrix}, \qquad q_1 = \begin{pmatrix} z'' & x^2 y \\ x & -zz' \end{pmatrix},$$

$$V_{\bar{1}}: q_0 = \begin{pmatrix} zz'' & x^3 - zz'' & \lambda zz'' & x^2z'' & 0 & 0 & 0 \\ -yz & z^2 & -\lambda zz'' & -x^2y & 0 & 0 & 0 \\ -y^2 & yz & -yz + zz'' & 0 & -x^2y & 0 & 0 \\ -xy & xz & -xz & zz'' & -zz'' & 0 & 0 \\ -xz'' & xz'' & \lambda xz'' & 0 & 0 & x^3 & z'z'' \\ 0 & 0 & -x & y & -z & 0 & 0 \\ 0 & -x & 0 & -z'' & 0 & z & -y \end{pmatrix},$$

$$q_1 = \begin{pmatrix} z & z & -\lambda z & -x^2 & 0 & x^2z & 0 \\ y & z'' & 0 & 0 & 0 & 0 & 0 \\ 0 & -y & z & 0 & 0 & -x^2y & 0 \\ 0 & -x & 0 & z & 0 & -zz'' & 0 \\ 0 & 0 & -x & y & 0 & -zz'' & 0 \\ 0 & 0 & 0 & z'' & y & -(z'')^2 & z'z'' \\ -x & 0 & 0 & 0 & z & 0 & -x^3 \end{pmatrix}.$$

• $W = (2, 2, 5; 10)$, $A_W = (2, 2, 2, 2, 2)$. For the polynomial $f_W = -xyy_1y_2y_3 + z^2$, $y_1 := y - x$, $y_2 := y - \lambda_1 x$, $y_3 := y - \lambda_2 x$, $\lambda_1, \lambda_2 \neq 0, 1$, $\lambda_1 \neq \lambda_2$,

$$V_0 : q_0 = q_1 = \begin{pmatrix} z & -xyy_2y_3 & xyy_2y_3 & 0 \\ y & -z & 0 & xyy_2y_3 \\ x & 0 & -z & xyy_2y_3 \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,2} : q_0 = q_1 = \begin{pmatrix} z & -yy_1y_2y_3 \\ x & -z \end{pmatrix}, \quad V_{2,2} : q_0 = q_1 = \begin{pmatrix} z & -xy_1y_2y_3 \\ y & -z \end{pmatrix},$$

$$V_{3,2} : q_0 = q_1 = \begin{pmatrix} z & xyy_2y_3 \\ y_1 & -z \end{pmatrix}, \quad V_{4,2} : q_0 = q_1 = \begin{pmatrix} z & -xyy_1y_3 \\ y_2 & -z \end{pmatrix},$$

$$V_{5,2} : q_0 = q_1 = \begin{pmatrix} z & -xyy_1y_2 \\ y_3 & -z \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = q_1 = \begin{pmatrix} z & -xy_2y_3 & 0 & 0 & 0 & 0 \\ yy_1 & -z & 0 & 0 & 0 & 0 \\ xy_1 & 0 & -z & 0 & 0 & xy_1y_2y_3 \\ yy_2 & 0 & 0 & z & xyy_2y_3 & 0 \\ 0 & -y_2 & 0 & -y_1 & -z & 0 \\ 0 & x & -y & 0 & 0 & z \end{pmatrix}.$$

• $W = (2,2,3;8)$, $A_W = (2,2,2,2,3)$. For the polynomial $f_W = yy_1y_2y_3 + xz^2$, $y_1 := y - x$, $y_2 := y - \lambda_1 x$, $y_3 := y - \lambda_2 x$ and $\lambda := \lambda_1$, $\lambda_1 \neq 0,1$, $\lambda_1 \neq \lambda_2$,

$$V_0 : q_0 = \begin{pmatrix} xz & yy_2y_3 & -yy_2y_3 & 0 \\ y & -z & 0 & -yy_2y_3 \\ x & 0 & -z & -yy_2y_3 \\ 0 & x & -y & xz \end{pmatrix}, \quad q_1 = \begin{pmatrix} z & yy_2y_3 & -yy_2y_3 & 0 \\ y & -xz & 0 & -yy_2y_3 \\ x & 0 & -xz & -yy_2y_3 \\ 0 & x & -y & z \end{pmatrix},$$

$$V_{1,2} : q_0 = \begin{pmatrix} xz & y_1y_2y_3 \\ y & -z \end{pmatrix}, \quad q_1 = \begin{pmatrix} z & y_1y_2y_3 \\ y & -xz \end{pmatrix},$$

$$V_{2,2} : q_0 = \begin{pmatrix} xz & yy_2y_3 \\ y_1 & -z \end{pmatrix}, \quad q_1 = \begin{pmatrix} z & yy_2y_3 \\ y_1 & -xz \end{pmatrix},$$

$$V_{3,2} : q_0 = \begin{pmatrix} xz & yy_1y_3 \\ y_2 & -z \end{pmatrix}, \quad q_1 = \begin{pmatrix} z & yy_1y_3 \\ y_2 & -xz \end{pmatrix},$$

$$V_{4,2} : q_0 = \begin{pmatrix} xz & yy_1y_2 \\ y_3 & -z \end{pmatrix}, \quad q_1 = \begin{pmatrix} z & yy_1y_2 \\ y_3 & -xz \end{pmatrix},$$

$$V_{5,3} : q_0 = \begin{pmatrix} z^2 & y_1y_2y_3 \\ y & -x \end{pmatrix}, \quad q_1 = \begin{pmatrix} x & y_1y_2y_3 \\ y & -z^2 \end{pmatrix},$$

$$V_{\bar{1}} : q_0 = \begin{pmatrix} xz & y_1y_2y_3 & xy_2y_3 & -xy_2y_3 & 0 & x^2y_3z & 0 \\ -yz & z^2 & -yy_2y_3 & yy_2y_3 & 0 & -yy_3z & 0 \\ -y^2 & yz & xz & 0 & 0 & -\lambda xyy_3 & 0 \\ -xy & xz & 0 & xz & 0 & (y_1y_2 - \lambda x^2)y_3 & 0 \\ -x^2 & 0 & 0 & 0 & xz & (y_2 - x)xy_3 & y_1y_2y_3 \\ 0 & 0 & -x & y & 0 & -z & 0 \\ 0 & -x & 0 & -x & y & 0 & -z \end{pmatrix},$$

$$q_1 = \begin{pmatrix} z & 0 & (x - y_2)y_3 & -\lambda xy_3 & 0 & xy_3z & 0 \\ y & x & 0 & 0 & 0 & 0 & 0 \\ 0 & -y & z & 0 & 0 & yy_2y_3 & 0 \\ 0 & -x & 0 & z & 0 & yy_2y_3 & 0 \\ x & 0 & 0 & 0 & z & xy_2y_3 & y_1y_2y_3 \\ 0 & 0 & -x & y & 0 & -xz & 0 \\ 0 & 0 & 0 & -x & y & 0 & -xz \end{pmatrix}.$$

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Research Institute for Mathematical Sciences, Kyoto University, 606-8502, Japan
E-mail address: `kajiura@kurims.kyoto-u.ac.jp`

Research Institute for Mathematical Sciences, Kyoto University, 606-8502, Japan
E-mail address: `saito@kurims.kyoto-u.ac.jp`

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka Osaka, 560-0043, Japan
E-mail address: `takahashi@math.sci.osaka-u.ac.jp`