DEGENERATION OF THE STRANGE DUALITY MAP FOR SYMPLECTIC BUNDLES

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1. Introduction

Global sections of the line bundles on a moduli space of vector bundles (or, more generally, principal G-bundles) are called generalized theta functions. The dimension of the vector spaces of generalized theta functions is given by the celebrated Verlinde formula. If you compute several Verlinde numbers, you will find that some of them coincide unexpectedly. Behind the coincidence, there is often geometric meaning.

Let $SU(r)$ be the moduli space of rank r bundles with trivial determinant on a smooth projective curve C of genus g, and let $\mathcal{L}$ be the ample generator of $\mathrm{Pic}SU(r)$. Let $U(n)$ be the moduli space of rank n bundles of slope $g-1$ on C, and let $U(n) \supset \Theta$ be the locus of $E \in U(n)$ such that $\mathrm{H}^0(C, E) \neq 0$. Then by the Verlinde formula, you see that the dimensions of the vector spaces $\mathrm{H}^0(SU(r), \mathcal{L}^{\otimes n})$ and $\mathrm{H}^0(U(n), \mathcal{O}(\Theta)^{\otimes r})$ are equal. There is a strange duality map (cf. [B1])

$$\mathrm{H}^0(SU(r), \mathcal{L}^{\otimes n})^* \to \mathrm{H}^0(U(n), \mathcal{O}(\Theta)^{\otimes r}),$$

and Belkale [Bel], Marian and Oprea [MO] have proved that it is an isomorphism.

Beauville [B3] formulated a strange duality for symplectic bundles as follows. Let $M_{2r}(C; L)$ be the moduli space of rank $2r$ vector bundles E with a non-degenerate alternate bilinear form $E \otimes E \to L$, where L is a line bundle on C. Let

$$\tau : M_{2r}(C; \mathcal{O}_C) \times M_{2s}(C; \omega_C) \to N_{4rs}(C; \omega_C)$$

be the tensor product map, where $N_{4rs}(C; \omega_C)$ is the moduli space of rank $4rs$ vector bundles F with a non-degenerate symmetric bilinear form $F \otimes F \to \omega_C$. If $\mathcal{P}$ is the pfaffian line bundle on $N_{4rs}(C; \omega_C)$, then

$$\tau^* \mathcal{P} \simeq \Xi_{2r}^{\otimes s} \boxtimes \Xi_{2s}^{\otimes r},$$

where Ξ_{2r} and Ξ_{2s} are the ample generators of the Picard groups of $M_{2r}(C; \mathcal{O}_C)$ and $M_{2s}(C; \omega_C)$. The pfaffian line bundle has a canonical section called the pfaffian divisor, which gives rise to the duality map

$$\mathrm{H}^0\left(M_{2r}(C; \mathcal{O}_C), \Xi_{2r}^{\otimes s}\right)^* \to \mathrm{H}^0\left(M_{2s}(C; \omega_C), \Xi_{2s}^{\otimes r}\right). \tag{1.1}$$

The equality of the dimensions of these two vector spaces has been proved by Oxbury and Wilson [O-W]. The strange duality conjecture for symplectic bundles claims that (1.1) is an isomorphism.

The purpose of this paper is to describe how the map (1.1) degenerates as the curve C degenerates to a nodal curve C_0.

Suppose that C_0 is an irreducible nodal curve with only one singular point Q. Let $\mathfrak{n} : \widetilde{C_0} \to C_0$ be the normalization, and set $\{Q_1, Q_2\} := \mathfrak{n}^{-1}(Q)$. Let $\widetilde{M_{2r}} := M_{2r}(\widetilde{C_0}, \{Q_1, Q_2\}; \mathcal{O}_C)$ and $\widetilde{M_{2s}} := M_{2s}(\widetilde{C_0}, \{Q_1, Q_2\}; \omega_C(Q_1 + Q_2))$ be the moduli of parabolic symplectic bundles on the pointed curve $(\widetilde{C_0}; Q_1, Q_2)$ (See Definition 4.1.1).

By the factorization theorem, there are natural isomorphisms

$$\mathrm{H}^0\left(M_{2r}(C_0,\mathcal{O}),\Xi^{\otimes s}\right)\simeq\bigoplus_{\Lambda=(s\geq\lambda_1\geq\cdots\geq\lambda_s\geq 0)}\mathrm{H}^0\left(\widetilde{M_{2r}},\Xi^{(s;\Lambda,\Lambda)}\right),\tag{1.2}$$

$$\mathrm{H}^0\left(M_{2s}(C_0,\omega_{C_0}),\Xi^{\otimes r}\right)\simeq\bigoplus_{N=(r\geq\mu_1\geq\cdots\geq\mu_s\geq 0)}\mathrm{H}^0\left(\widetilde{M_{2s}},\Xi^{(r;N,N)}\right),\tag{1.3}$$

where $\Xi^{(s;\Lambda,\Lambda)}$ and $\Xi^{(r;N,N)}$ are certain line bundles on the moduli of parabolic symplectic bundles (See Definition 4.1.2).

Consider the composite of morphisms

$$\begin{aligned}&\bigoplus_{\Lambda}\mathrm{H}^0\left(\widetilde{M_{2r}},\Xi^{(s;\Lambda,\Lambda)}\right)^*\xrightarrow{\text{dual of (1.2)}}\mathrm{H}^0\left(M_{2r}(C_0,\mathcal{O}),\Xi^{\otimes s}\right)^*\\&\xrightarrow{\text{strange duality map}}\mathrm{H}^0\left(M_{2s}(C_0,\omega_{C_0}),\Xi^{\otimes r}\right)\xrightarrow{(1.3)}\bigoplus_{N}\mathrm{H}^0\left(\widetilde{M_{2s}},\Xi^{(r;N,N)}\right)\end{aligned}\tag{1.4}$$

Note that there is a one-to-one correspondence between the set of Λ's and N's; $\Lambda=(s\geq\lambda_1\geq\cdots\geq\lambda_s\geq 0)$ and $N=(r\geq\mu_1\geq\cdots\geq\mu_s\geq 0)$ correspond if and only if

$$\{\lambda_1+r,\lambda_2+r-1,\dots\lambda_r+1\}\cup\{\mu_1+s,\mu_2+s-1,\dots,\mu_s+1\}=\{1,2,\dots,r+s\}.$$

When Λ and N correspond, we can define a strange duality map (See Section 4.2)

$$\mathrm{H}^0\left(\widetilde{M_{2r}},\Xi^{(s;\Lambda,\Lambda)}\right)^*\to\mathrm{H}^0\left(\widetilde{M_{2s}},\Xi^{(r;N,N)}\right).\tag{1.5}$$

The main theorem (Theorem 4.3.1) says that the composed morphism (1.4) is a direct sum of strange duality maps (1.5) for parabolic symplectic bundles on $(\widetilde{C_0};Q_1,Q_2)$. This implies the following corollary:

Corollary 1.0.1. *If the strange duality for parabolic bundles holds true for $\mathbb{P}^1$ with three points, then it holds true for generic pointed curves.*

The organization of this paper is as follows. In Section 2 we introduce notation about Young diagrams and symplectic flag varieties. In Section 3 we collect some results of representation theory. The key ingredient of the proof of the main theorem is the Howe's skew $(\mathrm{Sp}_{2r},\mathrm{Sp}_{2s})$-duality. In Section 3.3 we recall the Howe's skew $(\mathrm{Sp}_{2r},\mathrm{Sp}_{2s})$-duality, and reformulate it in a geometric setting. In Section 3.4 we consider a key commutative diagram (3.10), and prove an important proposition (Proposition 3.4.2), which is essential to the proof of the main theorem. In Section 4.1 we introduce the moduli of parabolic symplectic bundles. In Section 4.2 we define the strange duality map for parabolic symplectic bundles. In Section 4.3 we state the main theorem. Section 5 is devoted to its proof. As evidence in support of the strange duality conjecture for parabolic symplectic bundles, in Section 6 we prove that the source and the target of the strange duality map for parabolic symplectic bundles have the same dimension.

Notation and Convention. • We shall use moduli stacks of bundles, not coarse moduli spaces.

• Let S be a scheme and $*$ be an object (such as a sheaf, a scheme, a morphism etc.) over S. For an S-scheme T, we denote by $(*)_T$ or $*_T$ the base-change of $*$ by $T\to S$.

2. Preliminaries

2.1. **Young diagrams.** We gather here the terminology on Young diagrams used in this paper.

For positive integers r and s, a Young diagram Λ is said to be *of type* $\leq(r,s)$ if the number of rows of Λ is less than or equal to r and that of columns of Λ is less than or equal to s.

By associating to a non-increasing sequence $s \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0$ of non-negative integers the Young diagram whose i-th row has λ_i boxes, we obtain a one-to-one correspondence between the set of all r-term non-increasing sequences $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ of non-negative integers with $\lambda_1 \leq s$ and the set of all Young diagrams of type $\leq(r,s)$. By this correspondence, we use the terms "Young diagram" and "non-increasing sequence of integers" interchangeably.

For a Young diagram $\Lambda = (\lambda_1 \geq \cdots \geq \lambda_r)$ of type $\leq(r,s)$, we denote by $\widetilde{\Lambda}$ the Young diagram of type $\leq(s,r)$ that is obtained from Λ by interchanging rows and columns. For example, if Λ is the Young diagram $(4,2,1)$ of type $\leq(3,4)$, then $\widetilde{\Lambda}$ is the Young diagram $(3,2,1,1)$ of type $\leq(4,3)$.

For a Young diagram $\Lambda = (\lambda_1 \geq \cdots \geq \lambda_r)$ of type $\leq(r,s)$, we denote by ${}^c\Lambda$ the Young diagram $(s-\lambda_r \geq s-\lambda_{r-1} \geq \cdots \geq s-\lambda_1)$ of type $\leq(r,s)$. The Young diagram Λ^* of type $\leq(s,r)$ is defined to be $\widetilde{{}^c\Lambda}$. It is easy to see that if $\Lambda = (\lambda_1 \geq \cdots \geq \lambda_r)$ and $\Lambda^* = (\mu_1 \geq \cdots \geq \mu_s)$, then

$$\{\lambda_1+r, \lambda_2+r-1, \ldots \lambda_r+1\} \cup \{\mu_1+s, \mu_2+s-1, \ldots, \mu_s+1\} = \{1,2,\ldots,r+s\}.$$

For a Young diagram Λ, we denote by $|\Lambda|$ the number of boxes in Λ.

2.2. **Symplectic flag varieties.** Let S be a scheme, $\mathcal{P}$ a line bundle on S, $\mathcal{E}$ a vector bundle of rank $2r$ on S, and $\pi : \mathcal{E} \otimes \mathcal{E} \to \mathcal{P}$ a non-degenerate alternate bilinear form. *A full flag of $\mathcal{E}$ by isotropic subbundles* means a filtration by isotropic subbundles $\mathcal{E} \supset \mathcal{E}_r \supset \cdots \supset \mathcal{E}_1 \supset \mathcal{E}_0 = 0$ with $\operatorname{rank} \mathcal{E}_i = i$. (Here by "isotropic" we mean that the restriction of π to $\mathcal{E}_i \otimes \mathcal{E}_i$ is zero.)

Let $\mathbf{Fl}(\mathcal{E}) \to S$ be the flag variety parametrizing full flags of $\mathcal{E}$ by isotropic subbundles. Let

$$(\mathcal{E})_{\mathbf{Fl}(\mathcal{E})} \supset \mathcal{E}_r \supset \mathcal{E}_{r-1} \supset \cdots \supset \mathcal{E}_1 \supset \mathcal{E}_0 = 0$$

be the universal full flag by isotropic bundles. Given a tuple of integers $\overrightarrow{q} = (q_1, \ldots, q_r)$, we denote by $\mathcal{O}_{\mathbf{Fl}(\mathcal{E})}(\overrightarrow{q})$ (or simply $\mathcal{O}(\overrightarrow{q})$) the line bundle $\bigotimes_{i=1}^r \left(\mathcal{E}_{i-1}^\perp/\mathcal{E}_i^\perp\right)^{\otimes q_i}$ on $\mathbf{Fl}(\mathcal{E})$.

Let $\mathrm{Sp}(\mathcal{E})$ be the group scheme over S, which parametrizes symplectic automorphisms of $\mathcal{E}$. If $\mathcal{E} \supset \mathcal{E}_r \supset \ldots \mathcal{E}_1 \supset \mathcal{E}_0 = 0$ is a full flag by isotropic subbundles and $\alpha : \mathcal{E} \to \mathcal{E}$ is a symplectic automorphism, then $\mathcal{E} \supset \alpha(\mathcal{E}_r) \supset \cdots \supset \alpha(\mathcal{E}_1) \supset \alpha(\mathcal{E}_0) = 0$ is again a full flag by isotropic subbundles. This gives rises to a left action of $\mathrm{Sp}(\mathcal{E})$ on $\mathbf{Fl}(\mathcal{E})$. The action lifts to the action of each filter $\mathcal{E}_i$ of the universal full flag by isotropic subbundles. Hence the vector bundle $pr_* \mathcal{O}_{\mathbf{Fl}(\mathcal{E})}(\overrightarrow{q})$ on S becomes a (left) $\mathrm{Sp}(\mathcal{E})$-module, where $pr : \mathbf{Fl}(\mathcal{E}) \to S$ is the projection. The following proposition is well-known.

Proposition 2.2.1. *Assume that $S = \operatorname{Spec} k$ with k an algebraically closed field of characteristic zero. The k-vector space $\mathrm{H}^0(\mathbf{Fl}(\mathcal{E}), \mathcal{O}(\overrightarrow{q}))$ is non-zero if and only if $q_1 \geq \cdots \geq q_r \geq 0$. By the correspondence*

$$(q_1, \ldots, q_r) \leftrightarrow \mathrm{H}^0(\mathbf{Fl}(\mathcal{E}), \mathcal{O}(\overrightarrow{q})),$$

there is a one-to-one correspondence between the set of all finite dimensional irreducible representations of the symplectic group $\mathrm{Sp}(\mathcal{E})$ and the set of all $\overrightarrow{q} = (q_1, \ldots, q_r)$ with $q_1 \geq \cdots \geq q_r \geq 0$.

If $\Lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ is a Young diagram having r rows, the finite dimensional irreducible representation of the symplectic group $\mathrm{Sp}(\mathcal{E})(= \mathrm{Sp}_{2r})$ corresponding to Λ by the above correspondence is denoted by ρ_{2r}^{Λ}.

For later use, it would be convenient to prepare here numbering of the filters of a full flag by isotropic subbundles with respect to a Young diagram.

Notation 2.2.2. Let $\Lambda = (s \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ be a Young diagram of type $\leq(r,s)$. Given a full flag of $\mathcal{E}$ by isotropic subbundles

$$\mathcal{E}_\bullet : \mathcal{E} \supset \mathcal{E}_r \supset \cdots \supset \mathcal{E}_1 \supset \mathcal{E}_0 = 0,$$

we put $\mathbb{F}_i^\Lambda(\mathcal{E}_\bullet) := \mathcal{E}_l$ for $s + l - \lambda_l \leq i < s + l + 1 - \lambda_{l+1}$ for $0 \leq i \leq r + s$.

2.3. **A compactification of the symplectic group.** Let S be a scheme, P a line bundle on S, $\mathcal{E}$ and $\mathcal{F}$ locally free $\mathcal{O}_S$-modules of rank $2r$, $\langle -, - \rangle_{\mathcal{E}} : \mathcal{E} \otimes \mathcal{E} \to \mathcal{P}$ and $\langle -, - \rangle_{\mathcal{F}} : \mathcal{F} \otimes \mathcal{F} \to \mathcal{P}$ non-degenerate alternate bilinear forms. We define the non-degenerate alternate bilinear form $\langle -, - \rangle_{\mathcal{E} \oplus \mathcal{F}} : (\mathcal{E} \oplus \mathcal{F}) \otimes (\mathcal{E} \oplus \mathcal{F}) \to \mathcal{P}$ by $\langle (e,f), (e',f') \rangle_{\mathcal{E}\oplus\mathcal{F}} := \langle e, e' \rangle_{\mathcal{E}} - \langle f, f' \rangle_{\mathcal{F}}$. Let $LGr(\mathcal{E} \oplus \mathcal{F})$ be the symplectic Grassmannian parametrizing rank $2r$ isotropic subbundles of $\mathcal{E} \oplus \mathcal{F}$.

Giving a symplectic isomorphism $\mathcal{E} \xrightarrow{\alpha} \mathcal{F}$ is equivalent to giving a rank $2r$ isotropic subbundle $\mathcal{H} \subset \mathcal{E} \oplus \mathcal{F}$ which projects isomorphically to both $\mathcal{E}$ and $\mathcal{F}$ (Consider the graph of α). Therefore $LGr(\mathcal{E} \oplus \mathcal{F})$ is a compactification of $Sp(\mathcal{E}, \mathcal{F})$, the S-scheme parametrizing symplectic isomorphisms from $\mathcal{E}$ to $\mathcal{F}$.

Let $0 \to \mathcal{U} \to (\mathcal{E} \oplus \mathcal{F})_{LGr} \to \mathcal{Q} \to 0$ be the universal sequence on $LGr := LGr(\mathcal{E} \oplus \mathcal{F})$. The action of $Sp(\mathcal{E}) \times_S Sp(\mathcal{F})$ on LGr lifts naturally to an action of the universal quotient bundle $\mathcal{Q}$. So the vector bundle $pr_*(\det \mathcal{Q})^{\otimes n}$ on S becomes a (left) $Sp(\mathcal{E}) \times_S Sp(\mathcal{F})$-module, where $pr : LGr \to S$ is the projection. In [A, Corollary 6.5], we described how this $\mathrm{Sp}(\mathcal{E}) \times_S \mathrm{Sp}(\mathcal{F})$-module decomposes into irreducible modules.

Proposition 2.3.1. *Let n be a positive integer. There is a canonical isomorphism*

$$pr_*(\det \mathcal{Q})^{\otimes n} \simeq \bigoplus_{\overrightarrow{q}=(q_1,\ldots,q_r)} pr_*\left(\mathcal{O}_{\mathbf{Fl}(\mathcal{E})}(\overrightarrow{q}) \boxtimes \mathcal{O}_{\mathbf{Fl}(\mathcal{F})}(\overrightarrow{q})\right) \otimes \mathcal{P}^{\otimes(nr-|\overrightarrow{q}|)} \tag{2.1}$$

of $\mathrm{Sp}(\mathcal{E}) \times_S \mathrm{Sp}(\mathcal{F})$-modules, where $\overrightarrow{q}$ runs through all tuples of integers with $n \geq q_1 \geq \cdots \geq q_r \geq 0$. Here "pr" on the left-hand side is $LGr \to S$ and "pr" on the right-hand side is $\mathbf{Fl}(\mathcal{E}) \times_S \mathbf{Fl}(\mathcal{F}) \to S$.

3. Representation theoretic results

In this section we collect some representation theoretic results. Although the results are used later in the paper in a relative setting (that is, for vector bundles on a scheme or a stack), for simplicity of notation we state and prove propositions for vector spaces in this section. We fix an algebraically closed field k of characteristic zero.

3.1. **Orthogonal Grassmannian.** Let $(V, (-,-)_V)$ be a $2n$-dimensional k-vector space with a non-degenerate symmetric bilinear form. Let $\mathbf{OGr}_n(V)$ be the orthogonal Grassmannian parametrizing isotropic subspaces of V of dimension n. Then $\mathbf{OGr}_n(V)$ has two connected components $\mathbf{OGr}_n^+(V)$ and $\mathbf{OGr}_{2n}^-(V)$; U and $U' \in \mathbf{OGr}_n(V)$ lie in the same connected component if and only if $\dim U \cap U'$ is even.

On $\mathbf{OGr}_n(V)$, there is a short exact sequence

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_{\mathbf{OGr}_n(V)} \to \mathcal{Q} \to 0$$

given by the universal subbundle $\mathcal{U}$ and the universal quotient bundle $\mathcal{Q}$. There is a unique square root of the line bundle $\det \mathcal{Q}$, which we denote by $(\det \mathcal{Q})^{\otimes \frac{1}{2}}$. The action of $\mathrm{SO}(V)$ on $\mathbf{OGr}_n(V)$ naturally lifts to an action on the universal quotient bundle $\mathcal{Q}$. Hence $\mathrm{SO}(V)$ acts on the line bundle $\det \mathcal{Q}$. But $\mathrm{SO}(V)$ does not act on $(\det \mathcal{Q})^{\otimes \frac{1}{2}}$. Instead the spin group $\mathrm{Spin}(V)$ acts on $(\det \mathcal{Q})^{\otimes \frac{1}{2}}$. So the vector space

$$\mathrm{H}^0\left(\mathbf{OGr}_n(V), (\det \mathcal{Q})^{\otimes \frac{1}{2}}\right) = \mathrm{H}^0\left(\mathbf{OGr}_n^+(V), (\det \mathcal{Q})^{\otimes \frac{1}{2}}\right) \oplus \mathrm{H}^0\left(\mathbf{OGr}_n^-(V), (\det \mathcal{Q})^{\otimes \frac{1}{2}}\right)$$

is a representation of $\mathrm{Spin}(V)$, which is called the spin representation. The direct summands are irreducilble representations of $\mathrm{Spin}(V)$, which are called the half-spin representations. Each of them has dimension 2^{n-1} (cf. [FH, §20]).

Notation 3.1.1. For $\bullet \in \{+, -\}$ and $[U \subset V] \in \mathbf{OGr}_n^\bullet(V)$, the closed subset

$$\mathbf{OGr}_n^\bullet(V) \supset \{W \subset V \,|W \cap U \neq 0\}$$

with the reduced scheme structure is a divisor of $\mathbf{OGr}_n^\bullet(V)$. This divisor is the zero-divisor of some section of $(\det \mathcal{Q})^{\otimes \frac{1}{2}}$, which we denote by σ_U (It is determined up to scalar).

3.1.1. Let $(V_1, (-,-)_{V_1})$ and $(V_2, (-,-)_{V_2})$ be $2n$-dimensional k-vector spaces with a non-degenerate symmetric bilinear form. We endow the direct sum $V_1 \oplus V_2$ with the non-degenerate symmetric bilinear form $(-,-)_{V_1 \oplus V_2}$ given by $(v_1 + v_2, v_1' + v_2')_{V_1 \oplus V_2} := (v_1, v_1')_{V_1} - (v_2, v_2')_{V_2}$. If $U_i \in \mathbf{OGr}_n(V_i)$ $(i = 1, 2)$, then we have $U_1 \oplus U_2 \in \mathbf{OGr}_{2n}(V_1 \oplus V_2)$. So we have a morphism

$$\begin{aligned} j : \mathbf{OGr}_n(V_1) \times \mathbf{OGr}_n(V_2) &\to \mathbf{OGr}_{2n}(V_1 \oplus V_2) \\ (U_1, U_2) &\mapsto U_1 \oplus U_2. \end{aligned}$$

We name the connected components of $\mathbf{OGr}_n(V_i)$ and $\mathbf{OGr}_{2n}(V_1 \oplus V_2)$ so that

$$j\left(\mathbf{OGr}_n^+(V_1) \times \mathbf{OGr}_n^+(V_2)\right) \subset \mathbf{OGr}_{2n}^+(V_1 \oplus V_2).$$

Then we have

$$j\left(\mathbf{OGr}_n^-(V_1) \times \mathbf{OGr}_n^-(V_2)\right) \subset \mathbf{OGr}_{2n}^+(V_1 \oplus V_2).$$

Lemma 3.1.2. *Let $\mathcal{U}$ be the universal quotient bundle on $\mathbf{OGr}_{2n}(V_1 \oplus V_2)$. Then the morphim*

$$\begin{aligned} \mathrm{H}^0\left(\mathbf{OGr}_{2n}^+(V_1 \oplus V_2), (\det \mathcal{Q})^{\otimes \frac{1}{2}}\right) \xrightarrow{j^*} &\mathrm{H}^0\left(\mathbf{OGr}_n^+(V_1) \times \mathbf{OGr}_n^+(V_2), j^*(\det \mathcal{Q})^{\otimes \frac{1}{2}}\right) \\ &\oplus \mathrm{H}^0\left(\mathbf{OGr}_n^-(V_1) \times \mathbf{OGr}_n^-(V_2), j^*(\det \mathcal{Q})^{\otimes \frac{1}{2}}\right) \end{aligned}$$

is an isomorphism.

Proof. The source of j^* is a half-spin representation of Spin_{4n}, hence it has dimension 2^{2n-1}. If $\mathcal{Q}_i$ denotes the universal quotient bundle on $\mathbf{OGr}(V_i)$, then $j^*(\det \mathcal{Q})^{\otimes \frac{1}{2}} \simeq (\det \mathcal{Q}_1)^{\otimes \frac{1}{2}} \boxtimes (\det \mathcal{Q}_2)^{\otimes \frac{1}{2}}$. So each direct summand of the target of j^* is a tensor product of half-spin representations of Spin_{2n}, hence it has dimension 2^{2n-2}. Thus the source and the target of j^* have the same dimension.

Let us prove the surjectivity of j^*. For $(U_1 \subset V_2, U_2 \subset V_2) \in \mathbf{OGr}_n^+(V_1) \times \mathbf{OGr}_n^+(V_2)$, the restriction of the section $\sigma_{U_1 \oplus U_2} \in (\det \mathcal{Q})^{\otimes \frac{1}{2}}$ (cf. Notation 3.1.1) to $\mathbf{OGr}_n^+(V_1) \times \mathbf{OGr}_n^+(V_2)$ is $\sigma_{U_1} \boxtimes \sigma_{U_2}$, and its restriction to $\mathbf{OGr}_n^-(V_1) \times \mathbf{OGr}_n^-(V_2)$ is zero. This implies that $\mathrm{Im} j^*$ contains the first direct summand because it is an irreducible $\mathrm{Spin}_{2n} \times \mathrm{Spin}_{2n}$-module. Likewise $\mathrm{Im} j^*$ contains the second direct summand. $\square$

3.2. **The morphism μ_Λ.** Let $(E, (-,-)_E)$ and $(G, (-,-)_G)$ be k-vector spaces with a non-degenerate alternate bilinear form of dimension $2r$ and $2s$ respectively. We endow the tensor product $E \otimes G$ with the non-degenerate symmetric bilinear form $(-,-)_{E\otimes G}$ given by $(e \otimes g, e' \otimes g')_{E\otimes G} := (e, e')_E \cdot (g, g')_G$. Let $\mathbf{OGr}_{2rs}(E \otimes G)$ be the orthogonal Grassmannian parametrizing isotropic subspaces of $E \otimes G$ of dimension $2rs$. We name the connected components of $\mathbf{OGr}_{2rs}(E \otimes G)$ such that $\mathbf{OGr}^+_{2rs}(E \otimes G) \ni E \otimes U$ for an s-dimensional isotropic subspace U of G.

Let Λ be a Young diagram of type $\leq(r, s)$. For full flags by isotropic subspaces

$$E_\bullet : E \supset E_r \supset \dots E_1 \supset E_0 = 0 \quad \text{and} \quad G_\bullet : G \supset G_s \supset \dots G_1 \supset G_0 = 0,$$

we put

$$\mu_\Lambda(E_\bullet, G_\bullet) := \sum_{i=0}^{r+s} \left(\mathbb{F}^\Lambda_i(E_\bullet)^\perp \otimes \mathbb{F}^{\Lambda^*}_i(G_\bullet) + \mathbb{F}^\Lambda_i(E_\bullet) \otimes \mathbb{F}^{\Lambda^*}_i(G_\bullet)^\perp \right) \subset E \otimes G,$$

where we used Notation 2.2.2. You can easily check that $\mu_\Lambda(E_\bullet, G_\bullet)$ is a $2rs$-dimensional isotropic subspace of $E \otimes G$. So associating $\mu_\Lambda(E_\bullet, G_\bullet)$ to $(E_\bullet, G_\bullet)$, we obtain a morphism

$$\mu_\Lambda : \mathbf{Fl}(E) \times \mathbf{Fl}(G) \to \mathbf{OGr}_{2rs}(E \otimes G).$$

Lemma 3.2.1. *We have* $\mathrm{Im}\mu_\Lambda \subset \mathbf{OGr}^+_{2rs}(E \otimes G)$ *if* $|\Lambda|$ *is even, and* $\mathrm{Im}\mu_\Lambda \subset \mathbf{OGr}^-_{2rs}(E \otimes G)$ *if* $|\Lambda|$ *is odd.*

Proof. Let $(\mathbf{e}_1, \dots, \mathbf{e}_{2r})$ and $(\mathbf{g}_1, \dots, \mathbf{g}_{2s})$ be symplectic bases of E and G respectively, i.e., $(\mathbf{e}_i, \mathbf{e}_{r+j})_E = \delta_{i,j}$ $(1 \leq i, j \leq r)$ and $(\mathbf{g}_i, \mathbf{g}_{s+j})_G = \delta_{i,j}$ $(1 \leq i, j \leq s)$. Let $G \supset U$ be the s-dimensional isotropic subspace $\langle \mathbf{g}_1, \dots, \mathbf{g}_s \rangle$. If $E_\bullet$ and $G_\bullet$ are full flags by isotropic subspaces such that

$$E_i = \langle \mathbf{e}_1, \dots, \mathbf{e}_i \rangle \quad \text{and} \quad G_j = \langle \mathbf{g}_1, \dots, \mathbf{g}_j \rangle,$$

then $\dim((E \otimes U) \cap \mu_\Lambda(E_\bullet, G_\bullet)) = 2rs - |\Lambda|$. □

3.3. **Howe's skew $(\mathrm{Sp}_{2r}, \mathrm{Sp}_{2s})$-duality.** We retain the notation in Subsection 3.2. Let $c : \mathrm{Spin}(E \otimes G) \to \mathrm{SO}(E \otimes G)$ be the canonical covering map. For symplectic automorphisms $\alpha : E \to E$ and $\beta : G \to G$, the tensor product $\alpha \otimes \beta : E \otimes G \to E \otimes G$ is an element of $\mathrm{SO}(E \otimes G)$. This defines a morphism $t : \mathrm{Sp}(E) \times \mathrm{Sp}(G) \to \mathrm{SO}(E \otimes G)$ of algebraic groups. Since the symplectic group is symply-connected, there is a unique morphism $\widetilde{t} : \mathrm{Sp}(E) \times \mathrm{Sp}(G) \to \mathrm{Spin}(E \otimes G)$ of algebraic groups such that $c \circ \widetilde{t} = t$. By the morphism $\widetilde{t}$, we can regard a $\mathrm{Spin}(E \otimes G)$-module as a $\mathrm{Sp}(E) \times \mathrm{Sp}(G)$-module. Howe's skew $(\mathrm{Sp}_{2r}, \mathrm{Sp}_{2s})$-duality ([H, Theorem 3.8.9.3]) describes how the spin-representation of $\mathrm{Spin}(E \otimes G)$ decomposes into irreducible $\mathrm{Sp}(E) \times \mathrm{Sp}(G)$-modules:

Theorem 3.3.1. *Let $\mathfrak{S}$ be the spin-representation of* $\mathrm{Spin}(E \otimes G)$ *(cf. Subsection 3.1). Then there is an isomorphism*

$$\mathfrak{S} \simeq \sum_\Lambda \rho^\Lambda_{2r} \otimes \rho^{\Lambda^*}_{2s} \tag{3.1}$$

of $\mathrm{Sp}(E) \times \mathrm{Sp}(G)$*-modules, where Λ runs through all Young diagrams of type* $\leq(r, s)$.

The following geometric form of Howe's skew $(\mathrm{Sp}_{2r}, \mathrm{Sp}_{2s})$-duality will be useful in the sequel.

Corollary 3.3.2. *Let $\mathcal{Q}$ be the universal quotient bundle on $\mathbf{OGr}_{2rs}(E\otimes G)$. The morphisms induced by μ_Λ*

(3.2)
$$\mathrm{H}^0\left(\mathbf{OGr}^+_{2rs}(E\otimes G), (\det \mathcal{Q})^{\otimes\frac{1}{2}}\right) \xrightarrow{\oplus\mu_\Lambda^*} \bigoplus_{\Lambda:|\Lambda|\ even} \mathrm{H}^0\left(\mathbf{Fl}(E)\times\mathbf{Fl}(G), \mu_\Lambda^*\left(\det\mathcal{Q}\right)^{\otimes\frac{1}{2}}\right),$$

(3.3)
$$\mathrm{H}^0\left(\mathbf{OGr}^-_{2rs}(E\otimes G), (\det \mathcal{Q})^{\otimes\frac{1}{2}}\right) \xrightarrow{\oplus\mu_\Lambda^*} \bigoplus_{\Lambda:|\Lambda|\ odd} \mathrm{H}^0\left(\mathbf{Fl}(E)\times\mathbf{Fl}(G), \mu_\Lambda^*\left(\det\mathcal{Q}\right)^{\otimes\frac{1}{2}}\right)$$

are isomorphisms, where Λ runs through all Young diagrams of type $\leq(r,s)$ with $|\Lambda|$ even in (3.2), and with $|\Lambda|$ odd in (3.3).

Proof. We shall prove that the direct sum of (3.2) and (3.3)
(3.4)
$$\mathrm{H}^0\left(\mathbf{OGr}_{2rs}(E\otimes G), (\det \mathcal{Q})^{\otimes\frac{1}{2}}\right) \xrightarrow{\oplus\mu_\Lambda^*} \bigoplus_{\Lambda \text{ of type } \leq(r,s)} \mathrm{H}^0\left(\mathbf{Fl}(E)\times\mathbf{Fl}(G), \mu_\Lambda^*\left(\det\mathcal{Q}\right)^{\otimes\frac{1}{2}}\right)$$

is an isomorphism. Since the source and the target of (3.4) have the same dimension by Theorem 3.3.1, we have only to prove the surjectivity of (3.4). By the lemma below, the direct summands of the target of the morphism (3.4) are distinct irreducible $\mathrm{Sp}(E)\times\mathrm{Sp}(F)$-modules. Therefore it suffices to prove that for each Young diagram Λ of type $\leq(r,s)$, the morphism

$$\mathrm{H}^0\left(\mathbf{OGr}_{2rs}(E\otimes G), (\det \mathcal{Q})^{\otimes\frac{1}{2}}\right) \xrightarrow{\mu_\Lambda^*} \mathrm{H}^0\left(\mathbf{Fl}(E)\times\mathbf{Fl}(G), \mu_\Lambda^*\left(\det\mathcal{Q}\right)^{\otimes\frac{1}{2}}\right) \tag{3.5}$$

is non-zero. Let $(\mathbf{e}_1,\dots,\mathbf{e}_{2r})$ and $(\mathbf{g}_1,\dots,\mathbf{g}_{2s})$ be symplectic bases of E and G (i.e. $(\mathbf{e}_i,\mathbf{e}_{r+j})=\delta_{ij}$ and $(\mathbf{g}_i,\mathbf{g}_{s+j})=\delta_{ij}$). Let

$$E_\bullet : E\supset E_r\supset\dots E_1\supset E_0=0 \quad\text{and}\quad G_\bullet : G\supset G_s\supset\dots G_1\supset G_0=0$$

be the full flags by isotropic subspaces given by $E_i=\langle\mathbf{e}_1,\dots,\mathbf{e}_i\rangle$ and $G_j=\langle\mathbf{g}_1,\dots,\mathbf{g}_j\rangle$. Let

$$E'_\bullet : E\supset E'_r\supset\dots E'_1\supset E'_0=0 \quad\text{and}\quad G'_\bullet : G\supset G'_s\supset\dots G'_1\supset G'_0=0$$

be the full flags by isotropic subspaces given by $E'_i=\langle\mathbf{e}_{r+1},\dots,\mathbf{e}_{r+i}\rangle$ and $G'_j=\langle\mathbf{g}_{s+1},\dots,\mathbf{g}_{s+j}\rangle$. Then you can easily check that $\mu_\Lambda(E_\bullet,G_\bullet)\cap\mu_\Lambda(E'_\bullet,G'_\bullet)=0$. This means that if we put $V:=\mu_\Lambda(E'_\bullet,G'_\bullet)\subset E\otimes G$, then the section $\mu_\Lambda^*(\sigma_V)$ of $\mu_\Lambda^*(\det\mathcal{Q})^{\otimes\frac{1}{2}}$ does not vanish at the point $(E_\bullet,G_\bullet)\in\mathbf{Fl}(E)\times\mathbf{Fl}(G)$. Hence the morphism (3.5) is non-zero. $\square$

Lemma 3.3.3. *For a Young diagram Λ of type $\leq(r,s)$, there is a canonical isomorphism*

$$\mu_\Lambda^*\left(\det\mathcal{Q}\right)\simeq\left(\mathcal{O}_{\mathbf{Fl}(E)}(\Lambda)\boxtimes\mathcal{O}_{\mathbf{Fl}(G)}(\Lambda^*)\right)^{\otimes 2} \tag{3.6}$$

of line bundles on $\mathbf{Fl}(E)\times\mathbf{Fl}(G)$.

Proof. Put $(\lambda_1\geq\dots\geq\lambda_r):=\Lambda$ and $(\mu_1\geq\dots\geq\mu_s):=\Lambda^*$. For full flags by isotropic subspaces

$$E_\bullet : E\supset E_r\supset\dots E_1\supset E_0=0 \quad\text{and}\quad G_\bullet : G\supset G_s\supset\dots G_1\supset G_0=0,$$

there are natural isomorphisms

$$\det\left(\frac{E\otimes G}{\mu_\Lambda(E_\bullet,G_\bullet)}\right)\simeq\bigotimes_{i=1}^{r+s}\det\left(\frac{\mathbb{F}_i^\Lambda(E_\bullet)}{\mathbb{F}_{i-1}^\Lambda(E_\bullet)}\otimes\frac{G}{\mathbb{F}_i^{\Lambda^*}(G_\bullet)^\perp}\right)\otimes\bigotimes_{i=1}^{r+s}\det\left(\frac{\mathbb{F}_{i-1}^\Lambda(E_\bullet)^\perp}{\mathbb{F}_i^\Lambda(E_\bullet)^\perp}\otimes\frac{G}{\mathbb{F}_i^{\Lambda^*}(G)}\right)$$

$$\simeq\bigotimes_{i=1}^{r+s}\left\{\left(\frac{\mathbb{F}_i^\Lambda(E_\bullet)}{\mathbb{F}_{i-1}^\Lambda(E_\bullet)}\right)^{\otimes\dim\mathbb{F}_i^{\Lambda^*}(G_\bullet)}\otimes\left(\frac{\mathbb{F}_{i-1}^\Lambda(E_\bullet)^\perp}{\mathbb{F}_i^\Lambda(E_\bullet)^\perp}\right)^{\otimes 2s-\dim\mathbb{F}_i^{\Lambda^*}(G_\bullet)}\right.$$

$$\left.\otimes\left(\det\left(\frac{G}{\mathbb{F}_i^\Lambda(G_\bullet)^\perp}\right)\otimes\det\left(\frac{G}{\mathbb{F}_i^{\Lambda^*}(G_\bullet)}\right)\right)^{\otimes\dim\mathbb{F}_i^\Lambda(E_\bullet)-\dim\mathbb{F}_{i-1}^\Lambda(E_\bullet)}\right\}$$

$$\simeq\bigotimes_{i=1}^{r+s}\left\{\left(\frac{\mathbb{F}_{i-1}^\Lambda(E_\bullet)^\perp}{\mathbb{F}_i^\Lambda(E_\bullet)^\perp}\right)^{\otimes 2s-2\dim\mathbb{F}_i^{\Lambda^*}(G_\bullet)}\otimes\left(\frac{G}{\mathbb{F}_i^\Lambda(G_\bullet)^\perp}\right)^{\otimes 2(\dim\mathbb{F}_i^\Lambda(E_\bullet)-\dim\mathbb{F}_{i-1}^\Lambda(E_\bullet))}\right\}$$

$$\simeq\left\{\bigotimes_{i=1}^{r}\left(\frac{E_{i-1}^\perp}{E_i^\perp}\right)^{\lambda_i}\otimes\bigotimes_{j=1}^{s}\left(\frac{G_{j-1}^\perp}{G_j^\perp}\right)^{\mu_j}\right\}^{\otimes 2}.$$

Hence we have the isomorphism (3.6). □

3.4. **The multiplication map** m. Let $(E_i,(-,-)_{E_i})$ and $(G_i,(-,-)_{G_i})$ $(i=1,2)$ be vector spaces with a non-degenerate alternate bilinear form. For symplectic isomorphisms $\alpha:E_1\to E_2$ and $\beta:G_1\to G_2$, the tensor product $\alpha\otimes\beta:E_1\otimes G_1\to E_2\otimes G_2$ preserves the symmetric bilinear forms of $E_1\otimes G_1$ and $E_2\otimes G_2$. We denote by $\mathrm{SO}(E_1\otimes G_1,E_2\otimes G_2)$ the connected component of $\mathrm{O}(E_1\otimes G_1,E_2\otimes G_2)$ $(:=\{\gamma:E_1\otimes G_1\to E_2\otimes G_2|\gamma$ preserves the symmetric bilinear forms$\})$ containing $\alpha\otimes\beta$. Let $\mathbf{OGr}_{4rs}^+((E_1\otimes G_1)\oplus(E_2\otimes G_2))$ be the connected component of $\mathbf{OGr}_{4rs}((E_1\otimes G_1)\oplus(E_2\otimes G_2))$ containing $W_1\oplus W_2$ for $W_i\in\mathbf{OGr}_{2rs}^+(E_i\otimes G_i)$.

For $\gamma\in\mathrm{SO}(E_1\otimes G_1,E_2\otimes G_2)$, the graph of γ

$$\Gamma_\gamma:=\{(x,\gamma(x))\,|x\in E_1\otimes G_1\}\subset(E_1\otimes G_1)\oplus(E_2\otimes G_2)$$

determines a point of $\mathbf{OGr}_{4rs}^+((E_1\otimes G_1)\oplus(E_2\otimes G_2))$. So we have a morphism $\mathrm{SO}(E_1\otimes G_1,E_2\otimes G_2)\to\mathbf{OGr}_{4rs}^+((E_1\otimes G_1)\oplus(E_2\otimes G_2))$, which is an open immersion.

By associating $\Gamma_{\alpha\otimes\beta}$ to (α,β), we have a morphism

$$m:\mathrm{Sp}(E_1,E_2)\times\mathrm{Sp}(G_1,G_2)\to\mathbf{OGr}_{4rs}^+((E_1\otimes G_1)\oplus(E_2\otimes G_2)).\tag{3.7}$$

Let $(LGr(E_1\oplus E_2)\times LGr(G_1\oplus G_2))^\circ$ be the open subset

$$(LGr(E_1\oplus E_2)\times\mathrm{Sp}(G_1,G_2))\cup(\mathrm{Sp}(E_1,E_2)\times LGr(G_1\oplus G_2))$$

of $LGr(E_1\oplus E_2)\times LGr(G_1\oplus G_2)$.

We claim that m extends as a morphism

$$\widetilde{m}:(LGr(E_1\oplus E_2)\times LGr(G_1\oplus G_2))^\circ\to\mathbf{OGr}_{4rs}^+((E_1\otimes G_1)\oplus(E_2\otimes G_2)).\tag{3.8}$$

In fact, for isotropic subspaces $U\subset E_1\oplus E_2$ and $V\subset G_1\oplus G_2$ of dimension $2r$ and $2s$ respectively, the morphism

$$U\otimes V\to(E_1\otimes G_1)\oplus(E_2\otimes G_2)$$

is injective if either $(U\subset E_1\oplus E_2)\in Sp(E_1,E_2)$ or $(V\subset G_1\oplus G_2)\in Sp(G_1,G_2)$.

Hence we have a morphism $\tilde{m}$, which is an extension of m.

Lemma 3.4.1. *Let $\mathcal{Q}$ be the universal quotient bundle on $\mathbf{OGr}_{4rs}^+((E_1\otimes G_1)\oplus(E_2\otimes G_2))$. Let $\mathcal{Q}_E$ and $\mathcal{Q}_G$ be the universal quotient bundles on $LGr(E_1\oplus E_2)$ and $LGr(G_1\oplus$*

G_2) respectively. Then there is an isomorphism

$$\widetilde{m}^* (\det \mathcal{Q})^{\otimes \frac{1}{2}} \simeq (\det \mathcal{Q}_E)^{\otimes s} \boxtimes (\det \mathcal{Q}_G)^{\otimes r}\big|_{(LGr(E_1 \oplus E_2) \times LGr(G_1 \oplus G_2))^{\circ}} \tag{3.9}$$

of line bundles on $(LGr(E_1 \oplus E_2) \times LGr(G_1 \oplus G_2))^{\circ}$.

Proof. Let

$$U \subset E_1 \oplus E_2 \quad \text{and} V \subset G_1 \oplus G_2$$

be isotropic subspaces of dimension $2r$ and $2s$ respectively such that the morphism $U \otimes V \to (E_1 \otimes G_1) \oplus (E_2 \otimes G_2)$ is injective. Then there are natural isomorphisms

$$\begin{aligned} &\det \frac{(E_1 \otimes G_1) \oplus (E_2 \otimes G_2)}{U \otimes V} \simeq \det (U \otimes V)^{\vee} \\ &\simeq \left\{ (\det U)^{\otimes 2s} \otimes (\det V)^{\otimes 2r} \right\}^{\vee} \simeq \det \left(\frac{E_1 \oplus E_2}{U} \right)^{\otimes 2s} \otimes \det \left(\frac{G_1 \oplus G_2}{V} \right)^{\otimes 2r}. \end{aligned}$$

Hence we have a natural isomorphism

$$\widetilde{m}^* (\det \mathcal{Q}) \simeq (\det \mathcal{Q}_E)^{\otimes 2s} \boxtimes (\det \mathcal{Q}_G)^{\otimes 2r}\big|_{(LGr(E_1 \oplus E_2) \times LGr(G_1 \oplus G_2))^{\circ}}.$$

By taking the square root, we obtain (3.9). □

Let $\mathcal{Q}_i$ be the universal quotient bundle on $\mathbf{OGr}_{2rs}(E_i \otimes G_i)$.

We have the following commutative diagram (the orthogonal Grassmannians $\mathbf{OGr}^+_{4rs}((E_1 \otimes G_1) \oplus (E_2 \otimes G_2))$, $\mathbf{OGr}^+_{2rs}(E_1 \otimes G_1) \times \mathbf{OGr}^+_{2rs}(E_2 \otimes G_2)$ and $\mathbf{OGr}^-_{2rs}(E_1 \otimes G_1) \times \mathbf{OGr}^-_{2rs}(E_2 \otimes G_2)$ are abbreviated to $\mathbf{OGr}^+_{4rs}$, $\mathbf{OGr}^+_{2rs}$ and $\mathbf{OGr}^-_{2rs}$ respectively):

(3.10)

$$\begin{tikzcd}[column sep=large]
\begin{array}{c} \mathrm{H}^0 \left(LGr(E_1 \oplus E_2), (\det \mathcal{Q}_E)^{\otimes s} \right) \\ \otimes \\ \mathrm{H}^0 \left(LGr(G_1 \oplus G_2), (\det \mathcal{Q}_G)^{\otimes r} \right) \end{array} \arrow[d, "\chi", "\simeq"'] & \mathrm{H}^0 \left(\mathbf{OGr}^+_{4rs}, (\det \mathcal{Q})^{\otimes \frac{1}{2}} \right) \arrow[l, "\widetilde{m}^*"'] \arrow[dd, "(\clubsuit)", "\simeq"'] \\
\begin{array}{c} \bigoplus_M \mathrm{H}^0 \left(\mathbf{Fl}(E_1) \times \mathbf{Fl}(E_2), \mathcal{O}(M) \boxtimes \mathcal{O}(M) \right) \\ \otimes \\ \bigoplus_N \mathrm{H}^0 \left(\mathbf{Fl}(G_1) \times \mathbf{Fl}(G_2), \mathcal{O}(N) \boxtimes \mathcal{O}(N) \right) \end{array} & \\
\begin{array}{c} \bigoplus_{|\Lambda_1|,|\Lambda_2|:\text{even}} \left\{ \begin{array}{c} \mathrm{H}^0 \left(\mathbf{Fl}(E_1) \times \mathbf{Fl}(G_1), \mathcal{O}(\Lambda_1) \boxtimes \mathcal{O}(\Lambda_1^*) \right) \\ \otimes \\ \mathrm{H}^0 \left(\mathbf{Fl}(E_2) \times \mathbf{Fl}(G_2), \mathcal{O}(\Lambda_2) \boxtimes \mathcal{O}(\Lambda_2^*) \right) \end{array} \right\} \\ \oplus \\ \bigoplus_{|\Lambda_1|,|\Lambda_2|:\text{odd}} \left\{ \begin{array}{c} \mathrm{H}^0 \left(\mathbf{Fl}(E_1) \times \mathbf{Fl}(G_1), \mathcal{O}(\Lambda_1) \boxtimes \mathcal{O}(\Lambda_1^*) \right) \\ \otimes \\ \mathrm{H}^0 \left(\mathbf{Fl}(E_2) \times \mathbf{Fl}(G_2), \mathcal{O}(\Lambda_2) \boxtimes \mathcal{O}(\Lambda_2^*) \right) \end{array} \right\} \end{array} \arrow[u, "\phi"] & \begin{array}{c} \mathrm{H}^0 \left(\mathbf{OGr}^+_{2rs}, (\det \mathcal{Q}_1)^{\frac{1}{2}} \boxtimes (\det \mathcal{Q}_2)^{\frac{1}{2}} \right) \\ \otimes \\ \mathrm{H}^0 \left(\mathbf{OGr}^-_{2rs}, (\det \mathcal{Q}_1)^{\frac{1}{2}} \boxtimes (\det \mathcal{Q}_2)^{\frac{1}{2}} \right) \end{array} \arrow[l, "\simeq"', "(\spadesuit)"]
\end{tikzcd}$$

where M and Λ_i run through Young diagrams of type $\leq (r, s)$, and N runs through those of type $\leq (s, r)$. Here $(\clubsuit)$ is the isomorphism in Lemma 3.1.2; the isomorphism χ is a tensor product of the isomorphisms in Proposition 2.3.1; the isomorphism$(\spadesuit)$ is the one in Corollary 3.3.2; and the morphism ϕ is defined such that the above diagram is commutative. Note that all the morphisms are $\mathrm{Sp}(E_1) \times \mathrm{Sp}(E_2) \times \mathrm{Sp}(G_1) \times \mathrm{Sp}(G_2)$-equivariant.

The source of the morphism ϕ is a direct sum of

(3.11)

$$\mathrm{H}^0 \left(\mathbf{Fl}(E_1), \mathcal{O}(\Lambda_1) \right) \otimes \mathrm{H}^0 \left(\mathbf{Fl}(E_2), \mathcal{O}(\Lambda_2) \right) \otimes \mathrm{H}^0 \left(\mathbf{Fl}(G_1), \mathcal{O}(\Lambda_1^*) \right) \otimes \mathrm{H}^0 \left(\mathbf{Fl}(G_1), \mathcal{O}(\Lambda_2^*) \right),$$

such that $|\Lambda_1| \equiv |\Lambda_2| (\mathrm{mod}\ 2)$, and the target of ϕ is a direct sum of

(3.12)
$$\mathrm{H}^0\left(\mathbf{Fl}(E_1), \mathcal{O}(M)\right) \otimes \mathrm{H}^0\left(\mathbf{Fl}(E_2), \mathcal{O}(M)\right) \otimes \mathrm{H}^0\left(\mathbf{Fl}(G_1), \mathcal{O}(N)\right) \otimes \mathrm{H}^0\left(\mathbf{Fl}(G_2), \mathcal{O}(N)\right).$$

We express ϕ as $\left(\phi_{(M,N),(\Lambda_1,\Lambda_2)}\right)$ in a matrix form, where $\phi_{(M,N),(\Lambda_1,\Lambda_2)}$ is a morphism from the direct summand (3.11) to the direct summand (3.12).

Proposition 3.4.2. *The morphism $\phi_{(M,N),(\Lambda_1,\Lambda_2)}$ is zero unless $\Lambda_1 = \Lambda_2 = M = N^*$. If $\Lambda_1 = \Lambda_2 = M = N^*$, then the morphism is a non-zero scalar multiplication.*

Proof. The $\mathrm{Sp}(E_1) \times \mathrm{Sp}(E_2) \times \mathrm{Sp}(G_1) \times \mathrm{Sp}(G_2)$-modules (3.11) and (3.12) are irreducible, and they are isomorphic if and only if $\Lambda_1 = \Lambda_2 = M = N^*$. Therefore $\phi_{(M,N),(\Lambda_1,\Lambda_2)}$ is zero unless $\Lambda_1 = \Lambda_2 = M = N^*$. When $\Lambda_1 = \Lambda_2 = M = N^*$, it is a scalar multiplication by the Schur's Lemma. It remains to be proved that the scalar is non-zero. For this, it suffices to prove that the composed morphism $\pi_{M,M^*} \circ \chi \circ \widetilde{m}^*$ is non-zero, where π_{M,M^*} is the projection

$$\bigoplus_{M,N} \mathrm{H}^0\left(\mathbf{Fl}(E_1) \times \mathbf{Fl}(E_2), \mathcal{O}(M) \boxtimes \mathcal{O}(M)\right) \otimes \mathrm{H}^0\left(\mathbf{Fl}(G_1) \times \mathbf{Fl}(G_2), \mathcal{O}(N) \boxtimes \mathcal{O}(N)\right)$$
$$\to \mathrm{H}^0\left(\mathbf{Fl}(E_1) \times \mathbf{Fl}(E_2), \mathcal{O}(M) \boxtimes \mathcal{O}(M)\right) \otimes \mathrm{H}^0\left(\mathbf{Fl}(G_1) \times \mathbf{Fl}(G_2), \mathcal{O}(M^*) \boxtimes \mathcal{O}(M^*)\right).$$

This has been proved in [A, §8]. ∎

3.4.1. *The relative version of Proposition 3.4.2.* For brevity of notation we have stated the results in the absolute case, that is, for vector spaces. Later in this paper, however, we shall use Proposition 3.4.2 in a relative setting, that is, for vector bundles. In the relative setting we have to be a little careful because we cannot choose canonically a square root $(\det \mathcal{Q})^{\otimes \frac{1}{2}}$ of the determinant of the universal quotient bundle on an orthogonal Grassmannian. Here we formulate the relative version of Proposition 3.4.2.

This time we let $(E_i, (-,-)_{E_i})$ and $(G_i, (-,-)_{G_i})$ $(i = 1, 2)$ be *vector bundles* with a non-degenerate alternate bilinear forms on an algebraic stack S, which is over an algebraically closed field of characteristic zero. We assume that the Picard group of S is torsion-free.

By straghtforward generalization, we can consider the morphism j in Section 3.1.1, the morphism μ_Λ in Section 3.2 and the morphism $\tilde{m}$ in Section 3.4 in this relative situation. We denote them as

$$\begin{aligned} j :& \mathbf{OGr}^+_{2rs} \sqcup \mathbf{OGr}^-_{2rs} \to \mathbf{OGr}^+_{4rs}, \\ \mu^{(i)}_\Lambda :& \mathbf{Fl}(E_i) \times_S \mathbf{Fl}(G_i) \to \mathbf{OGr}_{2rs}(E_i \otimes G_i), \\ \widetilde{m} :& \left(LGr(E_1 \oplus E_2) \times_S LGr(G_1 \oplus G_2)\right)^\circ \to \mathbf{OGr}^+_{4rs}. \end{aligned}$$

(We retain the abbreviation $\mathbf{OGr}^+_{4rs} := \mathbf{OGr}^+_{4rs}\left((E_1 \otimes G_1) \oplus (E_2 \otimes G_2)\right)$ and $\mathbf{OGr}^\bullet_{2rs} := \mathbf{OGr}^\bullet_{2rs}(E_1 \otimes G_1) \times_S \mathbf{OGr}^\bullet_{2rs}(E_2 \otimes G_2)$ $(\bullet \in \{+,-\})$.)

Let $f : \mathbf{OGr}^+_{4rs} \to S$ and $f^\bullet : \mathbf{OGr}^\bullet_{2rs} \to S$ $(\bullet \in \{+,-\})$ be projections.

Let $\mathcal{L}$ be a line bundle on $\mathbf{OGr}^+_{4rs}$ such that there exist a line bundle $\mathcal{A}$ on S and an isomorphism $\mathcal{L}^{\otimes 2} \simeq (\det \mathcal{Q}) \otimes f^*\mathcal{A}^{\otimes 2}$ of line bundles on $\mathbf{OGr}^+_{4rs}$.

Considering Lemma 3.4.1 relatively, we have an isomorphism

(3.13)
$$\widetilde{m}^*\mathcal{L}^{\otimes 2} \simeq \left\{(\det \mathcal{Q}_E)^{\otimes s} \boxtimes (\det \mathcal{Q}_G)^{\otimes r}\right\}^{\otimes 2} \otimes d^*\mathcal{A}^{\otimes 2}\Big|_{(LGr(E_1 \oplus E_2) \times_S LGr(G_1 \oplus G_2))^\circ}$$

of line bundles on $\left(LGr(E_1 \oplus E_2) \times_S LGr(G_1 \oplus G_2)\right)^\circ$, where d is the projection $LGr(E_1 \oplus E_2) \times_S LGr(G_1 \oplus G_2) \to S$. The torsion-freeness of the Picard group of S

implies the torsion-freeness of the Picard group of $LGr(E_1 \oplus E_2) \times_S LGr(G_1 \oplus G_2)$. So we can take the square root of (3.13):

$$\widetilde{m}^* \mathcal{L} \simeq \left\{ (\det \mathcal{Q}_E)^{\otimes s} \boxtimes (\det \mathcal{Q}_G)^{\otimes r} \right\} \otimes d^* \mathcal{A}|_{(LGr(E_1 \oplus E_2) \times_S LGr(G_1 \oplus G_2))^\circ} . \tag{3.14}$$

Let $b_i : \mathbf{Fl}(E_i) \to S$ and $c_i : \mathbf{Fl}(G_i) \to S$ be projections. Combining Proposition 2.3.1 and the isomorphism (3.14), we obtain an isomorphism

$$d_* \widetilde{m}^* \mathcal{L} \simeq \bigoplus_{M,N} b_{1*} \mathcal{O}_{\mathbf{Fl}(E_1)}(M) \otimes b_{2*} \mathcal{O}_{\mathbf{Fl}(E_2)}(M) \otimes c_{1*} \mathcal{O}_{\mathbf{Fl}(G_1)}(N) \otimes c_{2*} \mathcal{O}_{\mathbf{Fl}(G_2)}(N) \otimes \mathcal{A}, \tag{3.15}$$

where M, N run through all Young diagrams of type $\leq (r, s)$.

Considering Lemma 3.1.2 relatively, we have an isomorphism

$$f_* \mathcal{L} \simeq f_*^+ (j|_{\mathbf{OGr}^+_{2rs}})^* \mathcal{L} \oplus f_*^- (j|_{\mathbf{OGr}^-_{2rs}})^* \mathcal{L} \tag{3.16}$$

of vector bundles on S.

If Λ_1 and Λ_2 are Young diagrams of type $\leq (r, s)$ with both $|\Lambda_1|$ and $|\Lambda_2|$ even, then by Lemma 3.3.3 there is an isomorphism

$$\begin{aligned} &(\mu^{(1)}_{\Lambda_1} \otimes \mu^{(2)}_{\Lambda_2})^* (j|_{\mathbf{OGr}^+_{2rs}})^* \mathcal{L}^{\otimes 2} \\ &\simeq \left(\mathcal{O}_{\mathbf{Fl}(E_1)}(\Lambda_1) \boxtimes \mathcal{O}_{\mathbf{Fl}(G_1)}(\Lambda_1^*) \boxtimes \mathcal{O}_{\mathbf{Fl}(E_2)}(\Lambda_2) \boxtimes \mathcal{O}_{\mathbf{Fl}(G_2)}(\Lambda_2^*) \right)^{\otimes 2} \otimes pr^* \mathcal{A}^{\otimes 2}, \end{aligned}$$

where $pr : \mathbf{Fl}(E_1) \times_S \mathbf{Fl}(G_1) \times_S \mathbf{Fl}(E_2) \times_S \mathbf{Fl}(G_2) \to S$ is the projection. Again by the torsion-freeness of the Picard group, we can take the square root of this:

$$\begin{aligned} &(\mu^{(1)}_{\Lambda_1} \otimes \mu^{(2)}_{\Lambda_2})^* (j|_{\mathbf{OGr}^+_{2rs}})^* \mathcal{L} \\ &\simeq \left(\mathcal{O}_{\mathbf{Fl}(E_1)}(\Lambda_1) \boxtimes \mathcal{O}_{\mathbf{Fl}(G_1)}(\Lambda_1^*) \boxtimes \mathcal{O}_{\mathbf{Fl}(E_2)}(\Lambda_2) \boxtimes \mathcal{O}_{\mathbf{Fl}(G_2)}(\Lambda_2^*) \right) \otimes pr^* \mathcal{A}. \end{aligned} \tag{3.17}$$

Likewise, if both $|\Lambda_1|$ and $|\Lambda_2|$ are odd, we have an isomorphism

$$\begin{aligned} &(\mu^{(1)}_{\Lambda_1} \otimes \mu^{(2)}_{\Lambda_2})^* (j|_{\mathbf{OGr}^-_{2rs}})^* \mathcal{L} \\ &\simeq \left(\mathcal{O}_{\mathbf{Fl}(E_1)}(\Lambda_1) \boxtimes \mathcal{O}_{\mathbf{Fl}(G_1)}(\Lambda_1^*) \boxtimes \mathcal{O}_{\mathbf{Fl}(E_2)}(\Lambda_2) \boxtimes \mathcal{O}_{\mathbf{Fl}(G_2)}(\Lambda_2^*) \right) \otimes pr^* \mathcal{A}. \end{aligned} \tag{3.18}$$

From Corollary 3.3.2 and the isomorphisms (3.17) and (3.18), we obtain isomorphisms

$$\begin{aligned} &f_*^+ (j|_{\mathbf{OGr}^+_{2rs}})^* \mathcal{L} \\ &\simeq \bigoplus_{|\Lambda_1|, |\Lambda_2| : \text{even}} b_{1*} \mathcal{O}_{\mathbf{Fl}(E_1)}(\Lambda_1) \otimes c_{1*} \mathcal{O}_{\mathbf{Fl}(G_1)}(\Lambda_1^*) \otimes b_{2*} \mathcal{O}_{\mathbf{Fl}(E_2)}(\Lambda_2) \otimes c_{2*} \mathcal{O}_{\mathbf{Fl}(G_2)}(\Lambda_2^*) \otimes \mathcal{A} \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} &f_*^- (j|_{\mathbf{OGr}^-_{2rs}})^* \mathcal{L} \\ &\simeq \bigoplus_{|\Lambda_1|, |\Lambda_2| : \text{odd}} b_{1*} \mathcal{O}_{\mathbf{Fl}(E_1)}(\Lambda_1) \otimes c_{1*} \mathcal{O}_{\mathbf{Fl}(G_1)}(\Lambda_1^*) \otimes b_{2*} \mathcal{O}_{\mathbf{Fl}(E_2)}(\Lambda_2) \otimes c_{2*} \mathcal{O}_{\mathbf{Fl}(G_2)}(\Lambda_2^*) \otimes \mathcal{A}. \end{aligned}$$

Now considering the commutative diagram (3.10) for vector bundles, we obtain the following commutative diagram (***):

$$
\begin{tikzcd}
d_*\tilde{m}^*\mathcal{L} \arrow[d, "\simeq"', "(3.15)"] & & f_*\mathcal{L} \arrow[ll, "\tilde{m}^*"'] \arrow[dd, "\simeq"', "(3.16)"] \\
\bigoplus_{M,N} \left\{ \begin{array}{c} b_{1*}\mathcal{O}_{\mathbf{Fl}(E_1)}(M) \otimes b_{2*}\mathcal{O}_{\mathbf{Fl}(E_2)}(M) \\ \otimes c_{1*}\mathcal{O}_{\mathbf{Fl}(G_1)}(N) \otimes c_{2*}\mathcal{O}_{\mathbf{Fl}(G_2)}(N) \otimes \mathcal{A} \end{array} \right\} & & \\
\begin{array}{c} \bigoplus_{|\Lambda_1|,|\Lambda_2|:\text{even}} \left\{ \begin{array}{c} b_{1*}\mathcal{O}_{\mathbf{Fl}(E_1)}(\Lambda_1) \otimes c_{1*}\mathcal{O}_{\mathbf{Fl}(G_1)}(\Lambda_1^*) \\ \otimes b_{2*}\mathcal{O}_{\mathbf{Fl}(E_2)}(\Lambda_2) \otimes c_{2*}\mathcal{O}_{\mathbf{Fl}(G_2)}(\Lambda_2^*) \otimes \mathcal{A} \end{array} \right\} \\ \bigoplus \\ \bigoplus_{|\Lambda_1|,|\Lambda_2|:\text{odd}} \left\{ \begin{array}{c} b_{1*}\mathcal{O}_{\mathbf{Fl}(E_1)}(\Lambda_1) \otimes c_{1*}\mathcal{O}_{\mathbf{Fl}(G_1)}(\Lambda_1^*) \\ \otimes b_{2*}\mathcal{O}_{\mathbf{Fl}(E_2)}(\Lambda_2) \otimes c_{2*}\mathcal{O}_{\mathbf{Fl}(G_2)}(\Lambda_2^*) \otimes \mathcal{A} \end{array} \right\} \end{array} \arrow[u, "\phi"'] & & \begin{array}{c} f_*^+(j|_{\mathbf{OGr}^+_{2rs}})^*\mathcal{L} \\ \bigoplus \\ f_*^-(j|_{\mathbf{OGr}^-_{2rs}})^*\mathcal{L}. \end{array} \arrow[ll, "(3.19)"', "\simeq"]
\end{tikzcd}
$$

As in (3.10), the morphism ϕ in the above is defined to make the above diagram commute. We express ϕ as $\left(\phi_{(M,N),(\Lambda_1,\Lambda_2)}\right)$ in a matrix form. Note that the target and source of the morphism ϕ have natural $Sp(E_1)\times_S Sp(E_2)\times_S Sp(G_1)\times_S Sp(G_2)$-action. The following is the relative version of Proposition 3.4.2, which we state as a corollary of Proposition 3.4.2.

Corollary 3.4.3. *The morphism ϕ is $Sp(E_1) \times_S Sp(E_2) \times_S Sp(G_1) \times_S Sp(G_2)$-equivariant. The morphism $\phi_{(M,N),(\Lambda_1,\Lambda_2)}$ is zero unless $\Lambda_1 = \Lambda_2 = M = N^*$. If $\Lambda_1 = \Lambda_2 = M = N^*$, then the morphism is a multiplication map by a nowhere-vanishing function on S.*

Proof. We have only to check the corollary locally on S. But locally, the situation is just the base-change of the absolute case. $\square$

4. Main Theorem

4.1. **The moduli stack of parabolic symplectic bundles.** In this section we shall define a moduli stack of parabolic symplectic bundles, and introduce notation for line bundles on the moduli stack. We shall work over an algebraically closed field k of characteristic zero.

Let C be a connected projective nodal curve of arithmetic genus g, $P^{(1)}, \dots, P^{(m)}$ be distinct smooth points of C, and L a line bundle on C. Put $\overrightarrow{P} := (P^{(1)}, \dots, P^{(m)})$.

Definition 4.1.1. We define the moduli stack $\overline{M}_{2r}(C, \overrightarrow{P}; L)$ as follows. For an affine k-scheme T, an object of the groupoid $\overline{M}_{2r}(C, \overrightarrow{P}; L)(T)$ is the following data:

- a T-flat coherent $\mathcal{O}_{C\times T}$-module $\mathcal{E}$ whose restriction to every geometric fiber $C \times \operatorname{Spec} \overline{k(t)}$ $(t \in T)$ is a rank $2r$ torsion-free sheaf,
- a non-degenerate alternate bilinear form $\mathcal{G} \otimes \mathcal{G} \to pr_C^* L$,
- for every point $P^{(j)}$ $(1 \le j \le m)$, a full flag of $\mathcal{E}$ by isotropic subbundles

$$\mathcal{E}_\bullet^{(j)} : \mathcal{E}^{(j)} \supset \mathcal{E}_r^{(j)} \supset \cdots \supset \mathcal{E}_1^{(j)} \supset \mathcal{E}_0^{(j)} = 0,$$

where $\mathcal{E}^{(j)} = \mathcal{E}|_{P^{(j)}\times T}$.

Isomorphisms of the groupoid $\overline{M}_{2r}(C, \overrightarrow{P}; L)(T)$ are defined obviously.

An object of $\overline{M}_{2r}(C,\overrightarrow{P};L)(T)$ is called *a symplectic parabolic bundle on C parametrized by* T, and an object of $\overline{M}_{2r}(C,\overrightarrow{P};L)(\operatorname{Spec} k)$ is simply called *a symplectic parabolic bundle on* C.

(2) The substack $M_{2r}(C,\overrightarrow{P};L)$ of $\overline{M}_{2r}(C,\overrightarrow{P};L)$ is defined such that an object of $\overline{M}_{2r}(C,\overrightarrow{P};L)(T)$ is in $M_{2r}(C,\overrightarrow{P};L)(T)$ if and only if the $\mathcal{O}_{C\times T}$-module $\mathcal{E}$ above is locally free.

Let

$$\left(\mathcal{E}^{univ},\mathcal{E}^{univ}\otimes\mathcal{E}^{univ}\to pr_C^*L,\mathcal{E}_\bullet^{univ(j)}\ (1\le j\le m)\right)$$

be the universal object of the moduli stack $\overline{M}(C,\overrightarrow{P};L)$.

Definition 4.1.2. Let n be an integer. Let each point $P^{(j)}$ $(1\le j\le m)$ be given a tuple of integers $\Lambda^{(j)}=(\lambda_1^{(j)},\dots,\lambda_r^{(j)})$, and put $\overrightarrow{\Lambda}:=(\Lambda^{(1)},\dots,\Lambda^{(m)})$. We denote by $\Xi^{(n;\overrightarrow{\Lambda})}_{\overline{M}_{2r}(C,\overrightarrow{P};L)}$, or simply $\Xi^{(n;\overrightarrow{\Lambda})}$, the line bundle

$$\left(\det\mathbb{R}pr_*\mathcal{E}^{univ}\right)^{\otimes(-n)}\otimes\bigotimes_{j=1}^m\bigotimes_{i=1}^r\left(\frac{\mathcal{E}_{i-1}^{univ(j)\perp}}{\mathcal{E}_i^{univ(j)\perp}}\right)^{\otimes\lambda_i^{(j)}}$$

on $\overline{M}_{2r}(C,\overrightarrow{P};L)$, where pr is the projection $C\times\overline{M}_{2r}(C,\overrightarrow{P};L)\to\overline{M}_{2r}(C,\overrightarrow{P};L)$.

For later use, we introduce notation for orthogonal bundles as well.

Definition 4.1.3. *An orthogonal sheaf with values in* L *on* C is a torsion-free sheaf F on C together with a non-degenerate symmetric bilinear form $F\otimes F\to L$. If F is a vector bundle, it is called *an orthogonal bundle with values in* L *on* C.

We denote by $\overline{N}_{2t}(C;L)$ the moduli stack of rank $2t$ orthogonal sheaves with values in L on C. The open substack of $\overline{N}_{2t}(C;L)$ consisting of orthogonal bundles is denoted by $N_{2t}(C;L)$.

Consider the special case where $L=\omega_C$. The moduli stack $\overline{N}_{2t}(C;\omega_C)$ is a disjoint union of the open and closed substacks $\overline{N}^+_{2t}(C;\omega_C)$ and $\overline{N}^-_{2t}(C;\omega_C)$. Here an orthogonal sheaf F with values in ω_C lies in the component $\overline{N}^+_{2t}(C;\omega_C)$ if and only if $\dim\mathrm{H}^0(C,F)$ is even.

If $\mathcal{F}^{univ}$ is the universal orhogonal sheaf on $C\times\overline{N}_{2t}(C;\omega_C)$, then the line bundle $\mathcal{D}:=\left(\det\mathbb{R}pr_*\mathcal{F}^{univ}\right)^\vee$ on $\overline{N}_{2t}(C;\omega_C)$ is called the determinant bundle, where $pr:C\times\overline{N}_{2t}(C;\omega_C)\to\overline{N}_{2t}(C;\omega_C)$ is the projection. The determinant line bundle $\mathcal{D}$ has a canonical square root $\mathcal{P}$, the pfaffian bundle (cf. [L-S, Proposition 7.9]). Moreover the pfaffian bundle $\mathcal{P}$ has a canonical section Θ called the pfaffian divisor whose square $\Theta^{\otimes 2}$ is the canonical section of the determinant bundle (cf. [L-S, Section 7.10]).

4.2. **Strange duality for parabolic symplectic bundles.** In this subsection we formulate the strange duality for parabolic symplectic bundles.

Let C and $P^{(1)},\dots,P^{(m)}$ be as in Section 4.1. Assume that each point $P^{(j)}$ $(1\le j\le m)$ is given a Young diagram $\Lambda^{(j)}$ of type $\le(r,s)$.

For a rank $2r$ parabolic symplectic bundle

$$\mathbb{E}:=\left(E,E\otimes E\to\mathcal{O}_C,E_\bullet^{(j)}:E^{(j)}\supset E_r^{(j)}\supset\cdots\supset E_1^{(j)}\supset E_0^{(j)}=0\ (1\le j\le m)\right)$$

and a rank $2s$ parabolic symplectic bundle

$$\mathbb{G}:=\left(G,G\otimes G\to\omega_C(\overrightarrow{P}),G_\bullet^{(j)}:G^{(j)}\supset G_s^{(j)}\supset\cdots\supset G_1^{(j)}\supset G_0^{(j)}=0\ (1\le j\le m)\right),$$

let K be the kernel of the morphism

$$E \otimes G \to \bigoplus_{j=1}^{m} \frac{E^{(j)} \otimes G^{(j)}}{\mu_{\Lambda^{(j)}}(E^{(j)}_\bullet, G^{(j)}_\bullet)}, \tag{4.1}$$

where the vector space $\left(E^{(j)} \otimes G^{(j)}\right)/\mu_{\Lambda^{(j)}}(E^{(j)}_\bullet, G^{(j)}_\bullet)$ is considered to be a skyscraper sheaf at $P^{(j)}$. (Recall that $E^{(j)} := E|_{P^{(j)}}$ and $G^{(j)} := G|_{P^{(j)}}$, and see Section 3.2 for the definition of μ_Λ .)

The alternate bilinear forms of E and G determine a symmetric bilinear form $(E \otimes G) \otimes (E \otimes G) \to \omega(\overrightarrow{P})$ of $E \otimes G$. You can check easily that the restriction to K of this symmetric bilinear form gives rise to a symmetric bilinear form $K \otimes K \to \omega_C$. Since $\deg K = 4rs(g-1)$, it is non-degenerate. Thus K is an orthogonal bundle with values in ω_C on C. We define the morphism

$$\tau_{(C;\overrightarrow{\Lambda})} : M_{2r}(C, \overrightarrow{P}; \mathcal{O}_C) \times M_{2s}(C, \overrightarrow{P}; \omega_C(\overrightarrow{P})) \to N_{4rs}(C; \omega_C)$$

by $(\mathbb{E}, \mathbb{G}) \mapsto K$.

Lemma 4.2.1. *If $\sum_{j=1}^m |\Lambda^{(j)}|$ is even, then $\mathrm{Im}\tau_{(C;\overrightarrow{\Lambda})} \subset N^+_{4rs}(C;\omega_C)$. If $\sum_{j=1}^m |\Lambda^{(j)}|$ is odd, then $\mathrm{Im}\tau_{(C;\overrightarrow{\Lambda})} \subset N^-_{4rs}(C;\omega_C)$.*

Proof. Let $E = \bigoplus_{i=1}^{2r} \mathcal{O}\mathbf{e}_i$ and $G = \bigoplus_{l=1}^{s} \mathcal{O}\mathbf{g}_l \oplus \bigoplus_{l=s+1}^{2s} \omega_C(\overrightarrow{P})\mathbf{g}_l$. Give E and G the non-degenerate alternate bilinear forms given by the matrices

$$\left(\begin{array}{c|c} & 1_r \\ \hline -1_r & \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|c} & 1_s \\ \hline -1_s & \end{array}\right).$$

For $1 \le j \le m$, let $E^{(j)}_\bullet$ and $G^{(j)}_\bullet$ be the full flags by isotropic subspaces of $E^{(j)}(:= E|_{P^{(j)}})$ and $G^{(j)}(:= G|_{P^{(j)}})$ such that $E^{(j)}_a = \langle \mathbf{e}_1, \dots, \mathbf{e}_a \rangle$ $(0 \le a \le r)$ and $G^{(j)}_b = \langle \mathbf{g}_1, \dots, \mathbf{g}_b \rangle$ $(0 \le b \le s)$. For these parabolic symplectic bundles E and G, the kernel K of the morphism (4.1) is

$$\bigoplus_{1\le r\le r,\ 1\le l\le s} \mathcal{O}\mathbf{e}_i \otimes \mathbf{g}_l \oplus \bigoplus_{1\le r\le r,\ 1\le l\le s} \omega_C\left(\sum_{j \text{ s.t. } l\le\lambda^{(j)}_i} P^{(j)}\right)\mathbf{e}_i \otimes \mathbf{g}_{2s+1-l}$$
$$\oplus \bigoplus_{1\le r\le r,\ 1\le l\le s} \mathcal{O}\left(-\sum_{j \text{ s.t. } l\le\lambda^{(j)}_i} P^{(j)}\right)\mathbf{e}_{r+i} \otimes \mathbf{g}_{s+1-l} \oplus \bigoplus_{1\le r\le r,\ 1\le l\le s} \omega_C\mathbf{e}_{r+i} \otimes \mathbf{g}_{2s+1-l}.$$

We have

$$
\begin{aligned}
h^0(K) = rs + \sum_{1\le i\le r,\ 1\le l\le s} & \left\{ h^0\left(\omega_C\left(\sum_{j \text{ s.t. } l\le\lambda_i^{(j)}} P^{(j)}\right)\right)\right. \\
& \left. + h^0\left(\mathcal{O}\left(-\sum_{j \text{ s.t. } l\le\lambda_i^{(j)}} P^{(j)}\right)\right)\right\} + rsg \\
& \equiv rs(1-g) + \sum_{1\le i\le r,\ 1\le l\le s} \chi\left(\mathcal{O}\left(-\sum_{j \text{ s.t. } l\le\lambda_i^{(j)}} P^{(j)}\right)\right) \pmod 2 \\
& = rs(1-g) + \sum_{1\le i\le r,\ 1\le l\le s} \left(-\sharp\left\{j \,\middle|\, l\le\lambda_i^{(j)}\right\} + 1 - g\right) \\
& = 2rs(1-g) - \sum_{j=1}^{m} \sharp\left\{(i,l) \,\middle|\, l\le\lambda_i^{(j)}\right\} = 2rs(1-g) - \sum_{j=1}^{m} |\Lambda^{(m)}| \\
& \equiv \sum_{j=1}^{m} |\Lambda^{(m)}| \pmod 2.
\end{aligned}
$$

Since the moduli stack of symplectic bundles is connected, this proves the lemma. □

Lemma 4.2.2. *Let $\mathcal{P}$ be the pfaffian bundle on $N_{4rs}(C;\omega_C)$. Then we have an isomorphism*

$$
\tau^*_{(C;\overrightarrow{\Lambda})}\mathcal{P} \simeq \Xi^{(s;\overrightarrow{\Lambda})}_{M_{2r}(C,\overrightarrow{P};\mathcal{O}_C)} \boxtimes \Xi^{(r;\overrightarrow{\Lambda^*})}_{M_{2s}(C,\overrightarrow{P};\omega_C(\overrightarrow{P}))} \tag{4.2}
$$

of line bundles on $M_{2r}(C,\overrightarrow{P};\mathcal{O}_C)\times M_{2s}(C,\overrightarrow{P};\omega_C(\overrightarrow{P}))$, where $\overrightarrow{\Lambda^*} = (\Lambda^{(1)*},\dots,\Lambda^{(m)*})$.

Proof. Put $(\mu_1^{(j)}\ge\cdots\ge\mu_s^{(j)}) := \Lambda^{(j)*}$. Fix a point $\mathbb{E} = (E, E\otimes E\to\mathcal{O}_C, E_\bullet^{(j)}) \in M_{2r}(C,\overrightarrow{P};\mathcal{O}_C)$ such that $E = \mathcal{O}^{\oplus 2r}$. For $\mathbb{G} = (G, G\otimes G\to\omega_C(\overrightarrow{P}), G_\bullet^{(j)}) \in M_{2s}(C,\overrightarrow{P};\omega_C(\overrightarrow{P}))$, if K is the kernel of the morphism (4.1), then there are canonical isomorphims

$$
\begin{aligned}
\det\mathbb{R}pr_*(K)^\vee &\simeq \det\mathbb{R}pr_*(E\otimes G)^\vee \otimes \bigotimes_{j=1}^{m} \det\frac{E^{(j)}\otimes G^{(j)}}{\mu_{\Lambda^{(j)}}(E_\bullet^{(j)}, G_\bullet^{(j)})} \\
&\simeq \det\mathbb{R}pr_*(G)^{\otimes(-2r)} \otimes \sum_{j=1}^{m} |\Lambda^{(m)}| \bigotimes_{j=1}^{m} \bigotimes_{i=}^{s} \left(\frac{G_{i-1}^{(j)\perp}}{G_i^{(j)\perp}}\right)^{\otimes 2\mu_i^{(j)}}.
\end{aligned}
$$

Here the latter isomorphism follows fram the isomorphism $E\otimes G\simeq G^{\oplus 2r}$ and Lemma 3.3.3. Therefore we have an isomorphism

$$
\tau^*_{(C;\overrightarrow{\Lambda})}\mathcal{D}\Big|_{\{\mathbb{E}\}\times M_{2s}(C,\overrightarrow{P};\omega_C(\overrightarrow{P}))} \simeq \Xi^{(r;\overrightarrow{\Lambda^*})\otimes 2}_{M_{2s}(C,\overrightarrow{P};\omega_C(\overrightarrow{P}))}, \tag{4.3}
$$

where $\mathcal{D}$ is the determinant line bundle on $N_{4rs}(C;\omega_C)$. Similarly for a fixed point $\mathbb{G}\in M_{2s}(C,\overrightarrow{P};\omega_C(\overrightarrow{P}))$, we have an isomorphism

$$
\tau^*_{(C;\overrightarrow{\Lambda})}\mathcal{D}\Big|_{M_{2r}(C,\overrightarrow{P};\mathcal{O}_C)\times\{\mathbb{G}\}} \simeq \Xi^{(s;\overrightarrow{\Lambda})\otimes 2}_{M_{2r}(C,\overrightarrow{P};\mathcal{O}_C)}. \tag{4.4}
$$

Case (1). C is smooth. In this case, in order to complete the proof, we use the following claim.

Claim 4.2.2.1. If $\mathsf{X}_1, \dots, \mathsf{X}_a$ are moduli stacks of symplectic bundles on a smooth curve, Then $\mathrm{Pic}(\prod_{i=1}^a \mathsf{X}_i)$ is a rank a free abelian group generated by (the pull-backs of) the determinant line bundles on $\mathsf{X}_1, \dots, \mathsf{X}_a$.

Proof of Claim 4.2.2.1. If $a = 1$, then this is proved in [K-N-R] and [L-S]. Their argument was as follows: There is a quotient morphism $\pi : \mathsf{Q} \to \mathsf{X}$ from an infinite Grassmannian to a moduli stack X of symplectic bundles and $\pi^* : \mathrm{Pic}(\mathsf{X}) \to \mathrm{Pic}(\mathsf{Q})$ is injective. The infinite Grassmannian is an inductive limit $\varinjlim \mathsf{Q}_{\mathfrak{w}}$ of projective varieties such that $\mathrm{Pic}(\mathsf{Q}_{\mathfrak{w}}) \xrightarrow{\sim} \mathrm{Pic}(\mathsf{Q}_{\mathfrak{v}}) \simeq \mathbb{Z}$ for $\mathfrak{v} \le \mathfrak{w}$. This implies that $\mathrm{Pic}(\mathsf{Q}) \simeq \mathbb{Z}$. The pull-back of the determinant line bundle on X by π is a generator of $\mathrm{Pic}(\mathsf{Q})$. Hence $\mathrm{Pic}(\mathsf{X})$ is a rank one free abelian group generated by the determinat line bundle.

For $a \ge 1$, this argument applies as well if we verify that $\mathrm{Pic}(\prod_{i=1}^a \mathsf{Q}_i) \simeq \prod_{i=1}^a \mathrm{Pic}(\mathsf{Q}_i)$. But this holds true because $\prod_{i=1}^a \mathsf{Q}_i \simeq \varinjlim (\mathsf{Q}_{1,\mathfrak{w}_1} \times \cdots \times \mathsf{Q}_{a,\mathfrak{w}_a})$ and $\mathrm{Pic}(\prod_{i=1}^a \mathsf{Q}_{i,\mathfrak{w}_i}) \simeq \prod_{i=1}^a \mathrm{Pic}(\mathsf{Q}_{i,\mathfrak{w}_i}) \simeq \mathbb{Z}^a$ ($\mathsf{Q}_{i,\mathfrak{w}_i}$ being a projective variety with $\mathrm{Pic}(\mathsf{Q}_{i,\mathfrak{w}_i}) \simeq \mathbb{Z}$).

This is the end of the proof of Claim 4.2.2.1. $\square$

By the above claim, we have

$$\mathrm{Pic}(M_{2r}(C, \overrightarrow{P}; \mathcal{O}_C) \times M_{2s}(C, \overrightarrow{P}; \omega_C(\overrightarrow{P})) \simeq \mathrm{Pic}(M_{2r}(C, \overrightarrow{P}; \mathcal{O}_C)) \times \mathrm{Pic}(M_{2s}(C, \overrightarrow{P}; \omega_C(\overrightarrow{P})).$$

From (4.3) and (4.4), we obtain an isomorphism

$$\tau^*_{(C;\Lambda)} \mathcal{D} \simeq \left(\Xi^{(s;\overrightarrow{\Lambda})}_{M_{2r}(C,\overrightarrow{P};\mathcal{O}_C)} \boxtimes \Xi^{(r;\overrightarrow{\Lambda^*})}_{M_{2s}(C,\overrightarrow{P};\omega_C(\overrightarrow{P}))} \right)^{\otimes 2}.$$

By the torsion-freeness of the Picard group, we can take the square root of this, and we obtain the isomorphism (4.2).

Case (2). C: general nodal curve.

In this case we need more argument. We omit the proof, but just mention that when C is an irreducible nodal curve with only one singular point and $\overrightarrow{P} = \emptyset$, you can find a complete proof in the next section: Restrict (5.6) to $Sp_{\mathcal{E}} \times Sp_{\mathcal{G}}$.

This is the end of the proof of Lemma 4.2.2. $\square$

If Θ is the canonical section of the pfaffian bundle $\mathcal{P}$ (cf. Section 4.1), then $\tau^*_{(C;\overrightarrow{\Lambda})}\Theta$ induces the duality map

(4.5)
$$\mathrm{H}^0\left(M_{2r}(C, \overrightarrow{P}; \mathcal{O}_C), \Xi^{(s;\overrightarrow{\Lambda})}_{M_{2r}(C,\overrightarrow{P};\mathcal{O}_C)}\right)^* \to \mathrm{H}^0\left(M_{2s}(C, \overrightarrow{P}; \omega_C(\overrightarrow{P})), \Xi^{(r;\overrightarrow{\Lambda^*})}_{M_{2s}(C,\overrightarrow{P};\omega_C(\overrightarrow{P}))}\right)$$

of vector spaces of global sections.

The following is the strange duality for parabolic symplectic bundles.

Conjecture 4.1. *The morphism (4.5) is an isomorphism.*

Remark 4.2.3. The (-1)-multiplication is an automorphism of parabolic symplectic bundles. It induces the multiplication by $(-1)^{\sum|\Lambda^{(j)}|}$ on the fibers of the line bundles $\Xi^{(s;\overrightarrow{\Lambda})}_{M_{2r}(C,\overrightarrow{P};\mathcal{O}_C)}$ and $\Xi^{(r;\overrightarrow{\Lambda^*})}_{M_{2s}(C,\overrightarrow{P};\omega_C(\overrightarrow{P}))}$. Thus if $\sum_{j=1}^m |\Lambda^{(j)}|$ is odd, then the vector spaces $\mathrm{H}^0\left(M_{2r}(C, \overrightarrow{P}; \mathcal{O}_C), \Xi^{(s;\overrightarrow{\Lambda})}\right)$ and $\mathrm{H}^0\left(M_{2s}(C, \overrightarrow{P}; \omega_C(\overrightarrow{P})), \Xi^{(r;\overrightarrow{\Lambda^*})}\right)$ are zero. So the conjecture is trivially true.

4.3. **Compatibility with factorization.** The main theorem (Theorem 4.3.1) of this paper claims that the morphism (4.5) is "compatible with the factorzation morphism". Let us make this meaning precise.

For simplicity of nontation, we assume that C is an *irreducible* nodal curve *with only one singular point* Q. Let $\mathfrak{n} : \widetilde{C} \to C$ be the normalization, and put $\{Q^{(1)}, Q^{(2)}\} := \mathfrak{n}^{-1}(Q)$.

For brevity, we abbreviate $M_{2r}(C, \overrightarrow{P}; \mathcal{O}_C)$, $M_{2s}(C, \overrightarrow{P}; \omega_C(\overrightarrow{P}))$, $M_{2r}(\widetilde{C}, \{Q^{(1)}, Q^{(2)}\} \cup \overrightarrow{P}; \mathcal{O}_C)$ and $M_{2s}(\widetilde{C}, \{Q^{(1)}, Q^{(2)}\} \cup \overrightarrow{P}; \omega_C(Q^{(1)} + Q^{(2)} + \overrightarrow{P}))$ to M_{2r}, M_{2s}, $\widetilde{M}_{2r}$ and $\widetilde{M}_{2s}$ respectively.

The factorization theorem ([A, Theorem 7.3]) says that there is a canonical isomorphism

$$\mathrm{H}^0\left(M_{2r}, \Xi^{(s;\overrightarrow{\Lambda})}\right) \simeq \bigoplus_{M=(\mu_1 \geq \cdots \geq \mu_r)} \mathrm{H}^0\left(\widetilde{M}_{2r}, \Xi^{(s;M,M,\overrightarrow{\Lambda})}\right), \tag{4.6}$$

where M runs through all Young diagrams of type $\leq(r, s)$.

Likewise, there is a canonical isomorphism

$$\mathrm{H}^0\left(M_{2s}, \Xi^{(r;\overrightarrow{\Lambda^*})}\right) \simeq \bigoplus_{N=(\nu_1 \geq \cdots \geq \nu_s)} \mathrm{H}^0\left(\widetilde{M}_{2s}, \Xi^{(r;N,N,\overrightarrow{\Lambda^*})}\right) \otimes_k \left(\omega_C(\overrightarrow{P})\Big|_Q\right)^{\otimes(rs-|N|)}, \tag{4.7}$$

where N runs through all Young diagrams of type $\leq(s, r)$. We fix an isomorphism of vector spaces $\omega_C(\overrightarrow{P})|_Q \simeq k$, and ignore the term $(\omega_C(\overrightarrow{P})|_Q)$ from now on.

Let

$$\tau^*_{(C;\overrightarrow{\Lambda})} : \mathrm{H}^0\left(N_{4rs}(C; \omega_C), \mathcal{P}\right) \to \mathrm{H}^0\left(M_{2r}, \Xi^{(s;\overrightarrow{\Lambda})}\right) \otimes \mathrm{H}^0\left(M_{2s}, \Xi^{(r;\overrightarrow{\Lambda^*})}\right) \tag{4.8}$$

be the morphism between vector spaces of global sections induced by $\tau_{(C;\overrightarrow{\Lambda})}$. Composing (4.8) with the tensor product of the morphisms (4.6) and (4.7), we obtain the morphism

$$\Phi : \mathrm{H}^0\left(N_{4rs}(C; \omega_C), \mathcal{P}\right) \to \bigoplus_{M,N} \mathrm{H}^0\left(\widetilde{M}_{2r}, \Xi^{(s;M,M,\overrightarrow{\Lambda})}\right) \otimes \mathrm{H}^0\left(\widetilde{M}_{2s}, \Xi^{(r;N,N,\overrightarrow{\Lambda^*})}\right). \tag{4.9}$$

Now we come to the main theorem of this paper.

Theorem 4.3.1. *Let Θ be the canonical section of the pfaffian bundle $\mathcal{P}$ on $N_{4rs}(C; \omega_C)$. Then for $\Phi(\Theta) = (\Phi(\Theta)_{M,N})_{M,N}$, the following holds:*

If $M^* \neq N$, then $\Phi(\Theta)_{M,N} = 0$. If $M^* = N$, then (up to non-zero scalar) $\Phi(\Theta)_{M,N}$ is the image of the canonical section $\widetilde{\Theta}$ of the pfaffian bundle $\widetilde{\mathcal{P}}$ on $N_{4rs}(\widetilde{C}; \omega_{\widetilde{C}})$ by the morphism

$$\tau^*_{(\widetilde{C};M,M,\overrightarrow{\Lambda})} : \mathrm{H}^0\left(N_{4rs}(\widetilde{C}; \omega_{\widetilde{C}}), \widetilde{\mathcal{P}}\right) \to \mathrm{H}^0\left(\widetilde{M}_{2r}, \Xi^{(s;M,M,\overrightarrow{\Lambda})}\right) \otimes \mathrm{H}^0\left(\widetilde{M}_{2s}, \Xi^{(r;M^*,M^*,\overrightarrow{\Lambda^*})}\right).$$

Remark 4.3.2. If $\sum_{j=1}^m |\Lambda^{(j)}|$ is odd, the target of Φ is zero, thus the above theorem becomes trivial.

Theorem 4.3.1 is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} H^0\left(M_{2r}, \Xi^{(s;\overrightarrow{\Lambda})}\right)^* \arrow[r] \arrow[d, "\simeq"'] & H^0\left(M_{2s}, \Xi^{(r;\overrightarrow{\Lambda^*})}\right) \arrow[d, "\simeq"] \\ \bigoplus_M H^0\left(\widetilde{M}_{2r}, \Xi^{(s;M,M,\overrightarrow{\Lambda})}\right)^* \arrow[r] & \bigoplus_N H^0\left(\widetilde{M}_{2s}, \Xi^{(r;N,N,\overrightarrow{\Lambda^*})}\right), \end{tikzcd} \tag{4.10}$$

where the upper horizontal arrow is the strange duality map induced by $\tau^*_{(C;\overrightarrow{\Lambda})}\Theta$; the lower horizontal arrow is the direct sum of the strange duality maps induced by $\tau^*_{(C;M,M,\overrightarrow{\Lambda})}\widetilde{\Theta}$; the right vertical arrow is (4.7); and the left vertical arrow is the dual of (4.6).

Therefore, if the strange duality for parabolic symplectic bundles on $(\widetilde{C};\overrightarrow{P}\cup\{Q^{(1)},Q^{(2)}\})$ holds true, then so does it for those on $(C;\overrightarrow{P})$.

By degeneration argument (we need not only irreducible degeneration but also reducible one explained in the next subsection), Theorem 4.3.1 implies the following corollary.

Corollary 4.3.3. *If the strange duality for parabolic symplectic bundles holds true for $\mathbb{P}^1$ with three points, then it holds true for generic pointed curves.*

4.4. **The reducible case.** Here we indicate how to modify the theorem when C is reducible.

For simplicity of notation, we treat the case when C is a union of two smooth irreducible components C_1 and C_2 intersecting at only one point Q. Moreover we assume that $\overrightarrow{P}=\emptyset$ for simplicity. The points of $\mathfrak{n}^{-1}(Q)=\{Q^{(1)},Q^{(2)}\}$ are named such that $Q^{(i)}\in C_i$.

Put

$$\begin{aligned} M_{2r} &:= M_{2r}(C,\mathcal{O}_C), & M_{2s} &:= M_{2s}(C,\omega_C),\\ \widetilde{M}^{(i)}_{2r} &:= M_{2r}(C_i,Q^{(i)};\mathcal{O}_{C_i}), & \widetilde{M}^{(i)}_{2s} &:= M_{2s}(C_i,Q^{(i)};\omega_{C_i}(Q^{(i)})). \end{aligned}$$

By the factorization theorem ([A, Theorem???]) for a reducible curve, we have

$$\mathrm{H}^0(M_{2r},\Xi^{(s)})\simeq\bigoplus_M \mathrm{H}^0\left(\widetilde{M}^{(1)}_{2r},\Xi^{(s,M)}\right)\otimes\mathrm{H}^0\left(\widetilde{M}^{(2)}_{2r},\Xi^{(s,M)}\right)\tag{4.11}$$

and

$$\mathrm{H}^0(M_{2s},\Xi^{(r)})\simeq\bigoplus_N \mathrm{H}^0\left(\widetilde{M}^{(1)}_{2s},\Xi^{(r,N)}\right)\otimes\mathrm{H}^0\left(\widetilde{M}^{(2)}_{2s},\Xi^{(r,N)}\right),\tag{4.12}$$

where M and N run through all Young diagrams of type $\le(r,s)$ and $\le(s,r)$ respectively.

Composing τ^*_C with the tensor product of (4.11) and (4.12), we obtain

$$\Phi:\mathrm{H}^0(N_{4rs}(C;\omega_C),\mathcal{P})\to\bigoplus_{M,N}\begin{pmatrix}\mathrm{H}^0(\widetilde{M}^{(1)}_{2r},\Xi^{(s,M)})\\ \otimes\\ \mathrm{H}^0(\widetilde{M}^{(2)}_{2r},\Xi^{(s,M)})\end{pmatrix}\otimes\begin{pmatrix}\mathrm{H}^0(\widetilde{M}^{(1)}_{2s},\Xi^{(r,N)})\\ \otimes\\ \mathrm{H}^0(\widetilde{M}^{(2)}_{2s},\Xi^{(r,N)})\end{pmatrix}.$$

Put $(\Phi(\Theta)_{M,N})_{M,N}:=\Phi(\Theta)$.

Then the counterpart of Theorem 4.3.1 is:

If $M^*\neq N$, then $\Phi(\Theta)_{M,N}=0$. If $M^*=N$, then (up to non-zero scalar) $\Phi(\Theta)_{M,N}$ is the image of $\Theta_1\otimes\Theta_2$ by the morphism $\tau^*_{(C_1;M)}\otimes\tau^*_{(C_2;M)}$

$$\begin{matrix}\mathrm{H}^0(N_{4rs}(C_1,\omega_{C_1}),\mathcal{P}_1)\\ \otimes\\ \mathrm{H}^0(N_{4rs}(C_2,\omega_{C_2}),\mathcal{P}_2)\end{matrix}\to\begin{matrix}\mathrm{H}^0(\widetilde{M}^{(1)}_{2r},\Xi^{(s,M)})\otimes\mathrm{H}^0(\widetilde{M}^{(1)}_{2s},\Xi^{(r,M^*)})\\ \otimes\\ \mathrm{H}^0(\widetilde{M}^{(2)}_{2r},\Xi^{(s,M)})\otimes\mathrm{H}^0(\widetilde{M}^{(2)}_{2s},\Xi^{(r,M^*)})\end{matrix},$$

where Θ_i is the canonical section of the pfaffian bundle $\mathcal{P}_i$ on $N_{4rs}(C_i,\omega_{C_i})$.

5. Proof of the main theorem

In this section we give a proof of Theorem 4.3.1.

For simplicity of notation, we assume that $\overrightarrow{P}=\emptyset$. Recall the abbreviation:

$$\begin{aligned}
&M_{2r}:=M_{2r}(C;\mathcal{O}_C), && \widetilde{M}_{2r}:=M_{2r}(\widetilde{C},\{Q^{(1)},Q^{(2)}\};\mathcal{O}_C),\\
&M_{2s}:=M_{2s}(C;\omega_C), && \widetilde{M}_{2s}:=M_{2s}(\widetilde{C},\{Q^{(1)},Q^{(2)}\};\omega_{\widetilde{C}}(Q^{(1)}+Q^{(2)})).
\end{aligned}$$

For short, the moduli stacks $M_{2r}(\widetilde{C};\mathcal{O}_{\widetilde{C}})$, $M_{2s}(\widetilde{C};\omega_{\widetilde{C}}(Q^{(1)}+Q^{(2)}))$ and $N_{4rs}\left(\widetilde{C},\omega_{\widetilde{C}}(Q^{(1)}+Q^{(2)})\right)$ are written as $M^{\natural}_{2r}$, $M^{\natural}_{2s}$ and $N^{\natural}_{4rs}$. (Don't confuse $M_{2r}(\widetilde{C};\mathcal{O}_{\widetilde{C}})$ and $M_{2r}(\widetilde{C},\{Q^{(1)},Q^{(2)}\};\mathcal{O}_{\widetilde{C}})$. The former is a moduli stack of (ordinary) symplectic bundles, and the latter is a moduli stack of *parabolic* symplectic bundles.)

Let $h:M^{\natural}_{2r}\times M^{\natural}_{2s}\to N^{\natural}_{4rs}$ be the tensor product morphism.

We also abbreviate $N_{4rs}(C;\omega_C)$, $\overline{N}_{4rs}(C;\omega_C)$ and $N_{4rs}(\widetilde{C};\omega_{\widetilde{C}})$ to N_{4rs}, $\overline{N}_{4rs}$ and $\widetilde{N}_{4rs}$ respectively.

Let $\mathcal{E}$, $\mathcal{G}$ be the universal symplectic bundles on $\widetilde{C}\times M^{\natural}_{2r}$ and $\widetilde{C}\times M^{\natural}_{2s}$. Let $\mathcal{H}$ be the universal orthogonal bundle on $\widetilde{C}\times N^{\natural}_{4rs}$. Put $\mathcal{E}^{(i)}:=\mathcal{E}|_{Q^{(i)}\times M^{\natural}_{2r}}$, $\mathcal{G}^{(i)}:=\mathcal{G}|_{Q^{(i)}\times M^{\natural}_{2s}}$ and $\mathcal{H}^{(i)}:=\mathcal{H}|_{Q^{(i)}\times N^{\natural}_{4rs}}$ We denote by $\overline{\mathcal{E}}^{(i)}$ and $\overline{\mathcal{G}}^{(i)}$ the pull-backs of $\mathcal{E}^{(i)}$ and $\mathcal{G}^{(i)}$ to $M^{\natural}_{2r}\times M^{\natural}_{2s}$ $(i=1,2)$.

We introduce the following abbreviation:

$$\begin{aligned}
Sp_{\mathcal{E}}&:=Sp(\mathcal{E}^{(1)},\mathcal{E}^{(2)}), & Sp_{\mathcal{G}}&:=Sp(\mathcal{G}^{(1)},\mathcal{G}^{(2)}),\\
LGr_{\mathcal{E}}&:=LGr(\mathcal{E}^{(1)}\oplus\mathcal{E}^{(2)}), & LGr_{\mathcal{G}}&:=LGr(\mathcal{G}^{(1)}\oplus\mathcal{G}^{(2)}),\\
SO&:=SO(\overline{\mathcal{E}}^{(1)}\otimes\overline{\mathcal{G}}^{(1)},\overline{\mathcal{E}}^{(2)}\otimes\overline{\mathcal{G}}^{(2)}), & \mathbf{OGr}^{+}_{4rs}&:=\mathbf{OGr}^{+}_{4rs}\left((\overline{\mathcal{E}}^{(1)}\otimes\overline{\mathcal{G}}^{(1)})\oplus(\overline{\mathcal{E}}^{(2)}\otimes\overline{\mathcal{G}}^{(2)})\right),\\
O''&:=O(\overline{\mathcal{E}}^{(1)}\otimes\overline{\mathcal{G}}^{(1)},\overline{\mathcal{E}}^{(2)}\otimes\overline{\mathcal{G}}^{(2)}), & \mathbf{OGr}''_{4rs}&:=\mathbf{OGr}_{4rs}\left((\overline{\mathcal{E}}^{(1)}\otimes\overline{\mathcal{G}}^{(1)})\oplus(\overline{\mathcal{E}}^{(2)}\otimes\overline{\mathcal{G}}^{(2)})\right),\\
O'&:=O\left(h^*\mathcal{H}^{(1)},h^*\mathcal{H}^{(2)}\right), & \mathbf{OGr}'_{4rs}&:=\mathbf{OGr}_{4rs}\left(h^*\mathcal{H}^{(1)}\oplus h^*\mathcal{H}^{(2)}\right),\\
O&:=O\left(\mathcal{H}^{(1)},\mathcal{H}^{(2)}\right), & \mathbf{OGr}_{4rs}&:=\mathbf{OGr}_{4rs}\left(\mathcal{H}^{(1)}\oplus\mathcal{H}^{(2)}\right).
\end{aligned}$$

We have the following commutative diagram:

(5.1)

$$\begin{tikzcd}
M_{2r}\times M_{2s} \arrow[d, "\cong"] \arrow[drrrrr, bend left, "\tau_C"] & & & & & \\
Sp_{\mathcal{E}}\times Sp_{\mathcal{G}} \arrow[r, "m"] \arrow[d, hook] & SO \arrow[r, hook] \arrow[d, hook] & O'' \arrow[r, "\simeq"] \arrow[d, hook] & O' \arrow[r] \arrow[d, hook] & O \arrow[r] \arrow[d, hook] & N_{4rs} \arrow[d, hook] \\
(LGr_{\mathcal{E}}\times LGr_{\mathcal{G}})^{\circ} \arrow[r, "\tilde{m}"] \arrow[d, hook] & \mathbf{OGr}^{+}_{4rs} \arrow[r, hook] \arrow[dd, "f"] & \mathbf{OGr}''_{4rs} \arrow[r, "\simeq"] \arrow[ddl] & \mathbf{OGr}'_{4rs} \arrow[r] \arrow[ddll] & \mathbf{OGr}_{4rs} \arrow[r, "w"] \arrow[dd, "q"] & \overline{N}_{4rs} \\
LGr_{\mathcal{E}}\times LGr_{\mathcal{G}} \arrow[dr, "d"] & & & & & \\
 & M^{\natural}_{2r}\times M^{\natural}_{2s} \arrow[rrr, "h"] & & & N^{\natural}_{4rs} &
\end{tikzcd}$$

where the morphisms d, f and q are natural projections to $M^{\natural}_{2r}\times M^{\natural}_{2s}$, and m and $\widetilde{m}$ are those explained in Section 3.4. The morphism $w:\mathbf{OGr}_{4rs}\to\overline{N}_{4rs}$ is constructed as follows:

Let H be a rank $4rs$ orthogonal bundle with values in $\omega_{\widetilde{C}}\left(Q^{(1)}+Q^{(2)}\right)$ on $\widetilde{C}$, and V an isotropic subspace of dimension $4rs$ of $H|_{Q^{(1)}}\oplus H|_{Q^{(2)}}$. The morphism w

associates to (H,V) the kernel of the mophism

$$\mathfrak{n}_* H \to \frac{H|_{Q^{(1)}} \oplus H|_{Q^{(2)}}}{V},$$

where $(H|_{Q^{(1)}} \oplus H|_{Q^{(2)}})/V$ is a skyscraper sheaf at Q.

We denote by n the composed morphism $\mathbf{OGr}^+_{4rs} \to \overline{N}_{4rs}$. The natural projections $LGr_{\mathcal{E}} \to M^\natural_{2r}$ and $LGr_{\mathcal{G}} \to M^\natural_{2s}$ are denoted by d' and d'' respectively. (So $d = d' \times d''$.)

Lemma 5.0.1. *There is an isomrphism*

$$n^*\mathcal{D} \simeq (\det \mathcal{Q}) \otimes f^* \left(\Xi^{(s)}_{M^\natural_{2r}} \boxtimes \Xi^{(r)}_{M^\natural_{2s}} \right)^{\otimes 2}, \tag{5.2}$$

where $\mathcal{D}$ is the determinant bundle on $\overline{N}_{4rs}$ and $\mathcal{Q}$ is the universal quotient bundle on $\mathbf{OGr}^+_{4rs}$.

Proof. Let $\mathcal{R}$ be the universal quotient bundle on $\mathbf{OGr}_{4rs}$. Let p_1, p_2 and p_3 be the projections $\widetilde{C} \times M^\natural_{2r} \to M^\natural_{2r}$, $\widetilde{C} \times M^\natural_{2s} \to M^\natural_{2s}$ and $\widetilde{C} \times N^\natural_{4rs} \to N^\natural_{4rs}$.

Claim 5.0.1.1. We have

$$h^*(\det \mathbb{R}p_{3*}\mathcal{H}) \simeq (\det \mathbb{R}p_{1*}\mathcal{E})^{\otimes 2s} \boxtimes (\det \mathbb{R}p_{2*}\mathcal{G})^{\otimes 2r}. \tag{5.3}$$

Proof of Claim 5.0.1.1. By Claim 4.2.2.1, line bundles on $M^\natural_{2r} \times M^\natural_{2s}$ are tensor products of line bundles on the factors. But for fixed $E \in M^\natural_{2r}$ and $G \in M^\natural_{2s}$, the isomorphisms

$$h^*(\det \mathbb{R}p_{3*}\mathcal{H})|_{\{E\}\times M^\natural_{2s}} \simeq (\det \mathbb{R}p_{2*}\mathcal{G})^{\otimes 2r},$$
$$h^*(\det \mathbb{R}p_{3*}\mathcal{H})|_{M^\natural_{2r}\times \{G\}} \simeq (\det \mathbb{R}p_{1*}\mathcal{E})^{\otimes 2s}$$

are easy to check (cf. the proof Lemma 4.2.2). This completes the proof of Claim 5.0.1.1. □

By the definition of w, we have a natural isomorphism

$$w^*\mathcal{D} \simeq q^*(\det \mathbb{R}p_3\mathcal{H})^\vee \otimes \det \mathcal{R}. \tag{5.4}$$

The pull-back of $\det \mathcal{R}$ to $\mathbf{OGr}^+_{4rs}$ is clearly isomorphic to $\det \mathcal{Q}$. Thus by pulling back the isomorphism (5.4) to $\mathbf{OGr}^+_{4rs}$ and using (5.3), we obtain (5.2). □

Let $\mathcal{Q}_{\mathcal{E}}$ and $\mathcal{Q}_{\mathcal{G}}$ be the universal quotient bundles on $LGr_{\mathcal{E}}$ and $LGr_{\mathcal{G}}$ respectively.

Then by Lemma 3.4.1, we have an isomorphism of line bundles on $(LGr_{\mathcal{E}} \times LGr_{\mathcal{G}})^\circ$

$$\widetilde{m}^*(\det \mathcal{Q}) \simeq \left\{ (\det \mathcal{Q}_{\mathcal{E}})^{\otimes s} \boxtimes (\det \mathcal{Q}_{\mathcal{G}})^{\otimes r} \right\}^{\otimes 2} \Big|_{(LGr_{\mathcal{E}} \times LGr_{\mathcal{G}})^\circ}. \tag{5.5}$$

Put $\mathcal{L} := n^*\mathcal{P}$, where $\mathcal{P}$ is the pfaffian bundle. Then from the isomorphism (5.2) and (5.5), by taking the square root (this is possible because $LGr_{\mathcal{E}} \times LGr_{\mathcal{G}} \to M^\natural_{2r} \times M^\natural_{2s}$ is a (product of) flag-variety bundle, hence $\mathrm{Pic}(LGr_{\mathcal{E}} \times LGr_{\mathcal{G}})^\circ = \mathrm{Pic}(LGr_{\mathcal{E}} \times LGr_{\mathcal{G}})$ is an free abelian group, in particular torsion-free), we have an isomorphism

$$\widetilde{m}^*\mathcal{L} \simeq \left\{ (\det \mathcal{Q}_{\mathcal{E}})^{\otimes s} \boxtimes (\det \mathcal{Q}_{\mathcal{G}})^{\otimes r} \right\} \otimes d^* \left(\Xi^{(s)}_{M^\natural_{2r}} \boxtimes \Xi^{(r)}_{M^\natural_{2s}} \right) \Big|_{(LGr_{\mathcal{E}} \times LGr_{\mathcal{G}})^\circ}. \tag{5.6}$$

Corresponding to the commutative diagram (5.1) of stacks, we have the following commutative diagram of vector spaces of global sections:

$$
\begin{tikzcd}
\mathrm{H}^0\left(M_{2r},\Xi^{(s)}_{M_{2r}}\right)\otimes \mathrm{H}^0\left(M_{2s},\Xi^{(r)}_{M_{2s}}\right) & \mathrm{H}^0(N_{4rs},\mathcal{P}) \arrow[l, "\beta(:=\tau_C^*)"'] \\
 & \mathrm{H}^0(\overline{N}_{4rs},\mathcal{P}) \arrow[u, "\text{restr.}"'] \arrow[d, "\alpha(:=n^*)"] \\
\begin{array}{c}\mathrm{H}^0\left(LGr_{\mathcal{E}}, d'^*\Xi^{(s)}_{M^\natural_{2r}}\otimes(\det\mathcal{Q}_{\mathcal{E}})^{\otimes s}\right)\\ \otimes \\ \mathrm{H}^0\left(LGr_{\mathcal{G}}, d''^*\Xi^{(s)}_{M^\natural_{2s}}\otimes(\det\mathcal{Q}_{\mathcal{G}})^{\otimes r}\right)\end{array} \arrow[uu, "\gamma:\text{restr.}"] & \mathrm{H}^0\left(\mathbf{OGr}^+_{4rs},\mathcal{L}\right), \arrow[l, "\tilde{m}^*"']
\end{tikzcd}
\tag{5.7}
$$

where the map γ is bijective because of [A, Lemma 7.4].

Let $b_i:\mathbf{Fl}(\overline{\mathcal{E}}^{(i)})\to M^\natural_{2r}\times M^\natural_{2s}$, $c_i:\mathbf{Fl}(\overline{\mathcal{G}}^{(i)})\to M^\natural_{2s}\times M^\natural_{2s}$ be projections. Put

$$
\mathbf{OGr}^\bullet_{2rs}:=\mathbf{OGr}^\bullet_{2rs}\left(\overline{\mathcal{E}}^{(1)}\otimes\overline{\mathcal{G}}^{(1)}\right)\times_{M^\natural_{2r}\times M^\natural_{2s}}\mathbf{OGr}^\bullet_{2rs}\left(\overline{\mathcal{E}}^{(2)}\otimes\overline{\mathcal{G}}^{(2)}\right),
$$

and let $f^\bullet:\mathbf{OGr}^\bullet_{2rs}\to M^\natural_{2r}\times M^\natural_{2s}$ be the projection ($\bullet\in\{-,+\}$).

We have the morphism $j:\mathbf{OGr}^+_{2rs}\sqcup\mathbf{OGr}^-_{2rs}\to\mathbf{OGr}^+_{4rs}$ (see Section 3.1.1).

As in Section 3.4.1, we have the following diagram of vector bundles on $M^\natural_{2r}\times M^\natural_{2s}$:

$$
\begin{tikzcd}
d_*\tilde{m}^*\mathcal{L} \arrow[d, "\epsilon", "\simeq"'] & f_*\mathcal{L} \arrow[l, "\zeta(:=\tilde{m}^*)"'] \arrow[dd, "\delta", "\simeq"'] \\
\bigoplus_{M,N}\left\{\begin{array}{c} b_{1*}\mathcal{O}(M)\otimes b_{2*}\mathcal{O}(M)\otimes c_{1*}\mathcal{O}(N)\\ \otimes c_{2*}\mathcal{O}(N)\otimes\left(\Xi^{(s)}_{M^\natural_{2r}}\boxtimes\Xi^{(r)}_{M^\natural_{2s}}\right)\end{array}\right\} & \\
\begin{array}{c}\bigoplus_{|\Lambda_1|,|\Lambda_2|:\text{even}}\left\{\begin{array}{c} b_{1*}\mathcal{O}(\Lambda_1)\otimes c_{1*}\mathcal{O}(\Lambda_1^*)\otimes b_{2*}\mathcal{O}(\Lambda_2)\\ \otimes c_{2*}\mathcal{O}(\Lambda_2^*)\otimes\left(\Xi^{(s)}_{M^\natural_{2r}}\boxtimes\Xi^{(r)}_{M^\natural_{2s}}\right)\end{array}\right\}\\ \bigoplus \\ \bigoplus_{|\Lambda_1|,|\Lambda_2|:\text{odd}}\left\{\begin{array}{c} b_{1*}\mathcal{O}(\Lambda_1)\otimes c_{1*}\mathcal{O}(\Lambda_1^*)\otimes b_{2*}\mathcal{O}(\Lambda_2)\\ \otimes c_{2*}\mathcal{O}(\Lambda_2^*)\otimes\left(\Xi^{(s)}_{M^\natural_{2r}}\boxtimes\Xi^{(r)}_{M^\natural_{2s}}\right)\end{array}\right\}\end{array} \arrow[u, "\phi"] & \begin{array}{c} f^+_*(j|_{\mathbf{OGr}^+_{2rs}})^*\mathcal{L}\\ \bigoplus \\ f^-_*(j|_{\mathbf{OGr}^-_{2rs}})^*\mathcal{L}.\end{array} \arrow[l, "\eta", "\simeq"']
\end{tikzcd}
\tag{5.8}
$$

If we take global sections, then (using projection formula) from ϵ we obtain the isomorphism

$$
\begin{aligned}
\epsilon^\ddagger:\mathrm{H}^0\left(LGr_{\mathcal{E}}, d'^*\Xi^{(s)}_{M^\natural_{2r}}\otimes(\det\mathcal{Q}_{\mathcal{E}})^{\otimes s}\right)&\otimes\mathrm{H}^0\left(LGr_{\mathcal{G}}, d''^*\Xi^{(r)}_{M^\natural_{2s}}\otimes(\det\mathcal{Q}_{\mathcal{G}})^{\otimes r}\right)\\
&\to\bigoplus_{M,N}\mathrm{H}^0\left(\widetilde{M}_{2r},\Xi^{(s;M,M)}\right)\otimes\mathrm{H}^0\left(\widetilde{M}_{2s},\Xi^{(r;N,N)}\right).
\end{aligned}
\tag{5.9}
$$

Here we used the isomorphisms

$$
\mathbf{Fl}(\mathcal{E}^{(1)})\times_{M^\natural_{2r}}\mathbf{Fl}(\mathcal{E}^{(2)})\simeq\widetilde{M}_{2r},\qquad \mathbf{Fl}(\mathcal{G}^{(1)})\times_{M^\natural_{2s}}\mathbf{Fl}(\mathcal{G}^{(2)})\simeq\widetilde{M}_{2s}.
$$

The morphism Φ in (4.9) is nothing but the composed morphism $\epsilon^\ddagger \circ (\gamma^{-1}) \circ \beta$ (recall that we are assuming $\overrightarrow{\Lambda} = \emptyset$ for simplicity). Thus by the commutativity of the diagram (5.7), we have $\Phi(\Theta) = (\epsilon^\ddagger \circ \widetilde{m}^* \circ \alpha)(\Theta)$, where Θ is the canonical section of the pfaffian bundle $\mathcal{P}$.

Now we shall analyze the composed morphism $\epsilon^\ddagger \circ \widetilde{m}^* \circ \alpha$. In the same way as we obtained the morphism $\epsilon^\ddagger$ from ϵ, we obtain from δ, η and ϕ in the diagram (5.8) the morphisms

$$\begin{aligned}
&\mathrm{H}^0\left(\mathbf{OGr}^+_{4rs}, \mathcal{L}\right) \\
&\xrightarrow{\delta^\ddagger} \mathrm{H}^0\left(\mathbf{OGr}^+_{2rs}, \left(j|_{\mathbf{OGr}^+_{2rs}}\right)^* \mathcal{L}\right) \oplus \mathrm{H}^0\left(\mathbf{OGr}^-_{2rs}, \left(j|_{\mathbf{OGr}^+_{2rs}}\right)^* \mathcal{L}\right) \\
&\xrightarrow{\eta^\ddagger} \bigoplus_{|\Lambda_1|,|\Lambda_2|:\text{ even}} \mathrm{H}^0\left(\widetilde{M}_{2r}, \Xi^{(s;\Lambda_1,\Lambda_2)}_{\widetilde{M}_{2r}}\right) \otimes \mathrm{H}^0\left(\widetilde{M}_{2s}, \Xi^{(r;\Lambda_1^*,\Lambda_2^*)}_{\widetilde{M}_{2s}}\right) \\
&\qquad \oplus \bigoplus_{|\Lambda_1|,|\Lambda_2|:\text{ odd}} \mathrm{H}^0\left(\widetilde{M}_{2r}, \Xi^{(s;\Lambda_1,\Lambda_2)}_{\widetilde{M}_{2r}}\right) \otimes \mathrm{H}^0\left(\widetilde{M}_{2s}, \Xi^{(r;\Lambda_1^*,\Lambda_2^*)}_{\widetilde{M}_{2s}}\right) \\
&\xrightarrow{\phi^\ddagger} \bigoplus_{M,N} \mathrm{H}^0\left(\widetilde{M}_{2r}, \Xi^{(s;M,M)}\right) \otimes \mathrm{H}^0\left(\widetilde{M}_{2s}, \Xi^{(r;N,N)}\right),
\end{aligned}$$

by taking global sections (and using projection formula). We put $(\delta^{\ddagger+}, \delta^{\ddagger-}) := \delta^\ddagger$.

Since the morphism $\widetilde{m}^*$ in (5.7) is also obtained from ζ in (5.8) by taking global section, we have $\epsilon^\ddagger \circ \widetilde{m}^* = \phi^\ddagger \circ \eta^\ddagger \circ \delta^\ddagger$ by the commutativity of the diagram (5.8). Therefore $\Phi(\Theta) = (\phi^\ddagger \circ \eta^\ddagger \circ \delta^\ddagger \circ \alpha)(\Theta)$.

The following two lemmas complete the proof of Theorem 4.3.1.

Lemma 5.0.2. *The (Λ_1, Λ_2)-th component of $(\eta^\ddagger \circ \delta^\ddagger \circ \alpha)(\Theta)$ is equal to $\tau^*_{(\widetilde{C};\Lambda_1,\Lambda_2)}\widetilde{\Theta}$ (up to non-zero scalar), where $\widetilde{\Theta}$ is the canonical section of the pfaffian line bundle $\widetilde{\mathcal{P}}$ on $\widetilde{N}_{4rs}$ (see (4.8) for $\tau^*_{(\widetilde{C};\Lambda_1,\Lambda_2)}$).*

Lemma 5.0.3. *If we express $\phi^\ddagger$ as $\left(\phi^\ddagger_{(M,N)(\Lambda_1,\Lambda_2)}\right)$ in a matrix form, we have $\phi^\ddagger_{(M,N)(\Lambda_1,\Lambda_2)} = 0$ unless $\Lambda_1 = \Lambda_2 = M = N^*$. If $\Lambda_1 = \Lambda_2 = M = N^*$, then $\phi^\ddagger_{(M,N)(\Lambda_1,\Lambda_2)}$ is a non-zero scalar multiplication.*

Proof of Lemma 5.0.2 . Let H be a rank $4rs$ orthogonal bundle with values in $\omega_{\widetilde{C}}(Q^{(1)} + Q^{(2)})$ on $\widetilde{C}$, and V_1, V_2 be $2rs$-dimensional isotropic linear subspaces of $H|_{Q^{(1)}}$ and $H|_{Q^{(2)}}$ respectively. Let K be the kernel of the morphism $H \to \oplus_{i=1,2}(H|_{Q^{(i)}}/V_i)$, where $(H|_{Q^{(i)}}/V_i)$ is a skyscraper sheaf at $Q^{(i)}$. Then you can easily check that K is an orthogonal bundle with values in $\omega_{\widetilde{C}}$. This defines a morphism

$$\nu : \mathbf{OGr}_{2rs}(\mathcal{H}^{(1)}) \times_{N^\natural_{4rs}} \mathbf{OGr}_{2rs}(\mathcal{H}^{(2)}) \to \widetilde{N}_{4rs}.$$

By the canonical isomorphism $h^*\mathcal{H}^{(i)} \simeq \overline{\mathcal{E}}^{(i)} \otimes \overline{\mathcal{G}}^{(i)}$, we have a natural isomorphism

$$\xi^\bullet : \mathbf{OGr}^\bullet_{2rs} \to \mathbf{OGr}_{2rs}(\mathcal{H}^{(1)}) \times_{N^\natural_{4rs}} \mathbf{OGr}_{2rs}(\mathcal{H}^{(2)}),$$

where $\bullet \in \{-, +\}$.

Claim 5.0.3.1. For $\bullet \in \{-, +\}$, we have an isomorphism of line bundles on $\mathbf{OGr}^\bullet_{2rs}$

$$(\nu \circ \xi^\bullet)^* \widetilde{\mathcal{P}} \simeq \left(n \circ \left(j|_{\mathbf{OGr}^\bullet_{2rs}}\right)\right)^* \mathcal{P}. \tag{5.10}$$

Moreover, by this isomrphism, the global section $(\nu \circ \xi^\bullet)^*\widetilde{\Theta}$ of the left-hand side corresponds to $\left(n \circ \left(j|_{\mathbf{OGr}^\bullet_{2rs}}\right)\right)^* \Theta$ of the right-hand side.

Proof of Claim 5.0.3.1. For an orthogonal bundle $\mathbb{F} = (F, F \otimes F \to \omega_{\widetilde{C}}) \in \widetilde{N}_{4rs}$,

$$(\mathfrak{n}_* F, \mathfrak{n}_* F \otimes \mathfrak{n}_* F \to \mathfrak{n}_* \omega_{\widetilde{C}} \xrightarrow{\text{trace}} \omega_C)$$

is an orthogonal sheaf with values in ω_C on C.

This defines a morphism $t : \widetilde{N}_{4rs} \to \overline{N}_{4rs}$. Then $t \circ \nu \circ \xi^\bullet = n \circ (j|_{\mathbf{OGr}^\bullet_{2rs}})$. From this the claim follows. □

By Claim 5.0.3.1, we have $(\eta^\ddagger \circ \delta^\ddagger \circ \alpha)(\Theta) = \eta^\ddagger \left((\nu \circ \xi^+)^* \widetilde{\Theta}, (\nu \circ \xi^-)^* \widetilde{\Theta}\right)$.

The composite of morphisms

$$\begin{aligned}
&\widetilde{M}_{2r} \times \widetilde{M}_{2s} \simeq \left(\mathbf{Fl}(\mathcal{E}^{(1)}) \times_{M^\natural_{2r}} \mathbf{Fl}(\mathcal{E}^{(2)})\right) \times \left(\mathbf{Fl}(\mathcal{G}^{(1)}) \times_{M^\natural_{2s}} \mathbf{Fl}(\mathcal{G}^{(2)})\right) \\
&\simeq \left(\mathbf{Fl}(\overline{\mathcal{E}}^{(1)}) \times_{M^\natural_{2r} \times M^\natural_{2s}} \mathbf{Fl}(\overline{\mathcal{E}}^{(2)})\right) \times_{M^\natural_{2r} \times M^\natural_{2s}} \left(\mathbf{Fl}(\overline{\mathcal{G}}^{(1)}) \times_{M^\natural_{2r} \times M^\natural_{2s}} \mathbf{Fl}(\overline{\mathcal{G}}^{(2)})\right) \\
&\simeq \left(\mathbf{Fl}(\overline{\mathcal{E}}^{(1)}) \times_{M^\natural_{2r} \times M^\natural_{2s}} \mathbf{Fl}(\overline{\mathcal{G}}^{(1)})\right) \times_{M^\natural_{2r} \times M^\natural_{2s}} \left(\mathbf{Fl}(\overline{\mathcal{E}}^{(2)}) \times_{M^\natural_{2r} \times M^\natural_{2s}} \mathbf{Fl}(\overline{\mathcal{G}}^{(2)})\right) \\
&\xrightarrow{\mu_{\Lambda_1} \times \mu_{\Lambda_2}} \mathbf{OGr}^\bullet_{2rs} \xrightarrow{\nu \circ \xi} \widetilde{N}_{4rs}
\end{aligned}$$

is, by construction, nothing but the morphism $\tau_{(\widetilde{C};\Lambda_1,\Lambda_2)}$ (cf. Section 4.2), where $\bullet = +$ or $-$ depending on the parity of $|\Lambda_i|$.

Therefore (Λ_1, Λ_2)-component of $\eta^\ddagger \left((\nu \circ \xi^+)^* \widetilde{\Theta}, (\nu \circ \xi^-)^* \widetilde{\Theta}\right)$ is $\tau^*_{(\widetilde{C};\Lambda_1,\Lambda_2)} \widetilde{\Theta}$. This completes the proof of Lemma 5.0.2. □

Proof of Lemma 5.0.3. By Proposition 3.4.2, we have $\phi^\ddagger_{(M,N)(\Lambda_1,\Lambda_2)} = 0$ unless $\Lambda_1 = \Lambda_2 = M = N^*$. If $\Lambda_1 = \Lambda_2 = M = N^*$, then again by Proposition 3.4.2 we know that $\phi^\ddagger_{(M,N)(\Lambda_1,\Lambda_2)}$ is a multiplication map by a nowhere vanishing function on $M^\natural_{2r} \times M^\natural_{2s}$. But $\mathrm{H}^0\left(M^\natural_{2r} \times M^\natural_{2s}, \mathcal{O}\right) \xleftarrow{\sim} k$. This completes the proof of Lemma 5.0.3. □

6. Equality of Verlinde numbers

The goal of this section is to prove the following theorem.

Theorem 6.0.4. *The source and the target of the strange duality map (4.5) have the same dimension.*

6.1. The Verlinde formula.

We use the following usual notations for Lie algebras.

- $\mathfrak{g}$ is a simple Lie algebra, and $\mathfrak{h}$ is a fixed Cartan subalgebra.
- G and T are the corresponding simple, simply-connected Lie group ant its maximal torus.
- $\mathfrak{h}^* \supset R(\mathfrak{g}, \mathfrak{h})$ is the root symtem, and we fix a basis $\{\alpha_1, \dots, \alpha_n\}$.
- Q is the root lattice, and Q_{long} is the sublattice generated by the long roots.
- $\mathfrak{h}^* \supset P$ is the weight lattice, and P^+ is the set of dominant weights.
- $(-,-)$ is the normalized Killing form (i.e. $(H_\beta, H_\beta) = 2$ for long roots β). By this we identify $\mathfrak{h}$ and $\mathfrak{h}^*$.
- θ is the highest root of $R(\mathfrak{g}, \mathfrak{h})$, and ρ is the half-sum of the positive roots.
- For $l \in \mathbb{N}$, $P_l := \{\lambda \in P_+ | \lambda(H_\theta) \leq l\}$.
- $h := (\rho, \theta) + 1$.

Let C be a connected smooth projective curve of genus g over an algebraically closed field of characteristic zero. Let $\overrightarrow{P} = (P^{(1)}, \dots, P^{(m)})$ be a set of closed points of C. Fix $l \in \mathbb{N}$. Suppose that each point $P^{(j)}$ is labeled by $\Lambda^{(j)} \in P_l$. Put $\overrightarrow{\Lambda} := (\Lambda^{(1)}, \dots, \Lambda^{(m)})$. Denote by $V_C(\overrightarrow{P}, \overrightarrow{\Lambda}, l)$ the conformal block associated to the data $(C, \overrightarrow{P}, \overrightarrow{\Lambda}, l)$ (cf. [T-U-Y]).

The following is the celebrated Verlinde formula (cf. [B2]).

Theorem 6.1.1. *The dimension of the vector space $V_C(\overrightarrow{P}, \overrightarrow{\Lambda}, l)$ is*
(6.1)
$$\left\{(l+h)^{\operatorname{rank}\mathfrak{g}} |P/Q_{long}|\right\}^{g-1} \sum_{\mu \in P_l} \operatorname{Tr}_{V_{\overrightarrow{\Lambda}}} \left(\exp 2\pi\sqrt{-1}\frac{\mu+\rho}{l+h}\right) \prod_{\alpha>0} \left|2\sin\pi\frac{(\alpha,\mu+\rho)}{l+h}\right|^{2-2g},$$
where $V_{\overrightarrow{\Lambda}}$ is the tensor product $\prod_{q=1}^m V_{\Lambda^{(q)}}$ of the irreducible representations of $\mathfrak{g}$ corresponding to $\Lambda^{(q)}$.

6.2. Sp_{2n} **case.** In this section we shall write down the formula (6.1) more explicitly for the symplectic group. As in [O-W], for a positive integer p and a finite set $U = \{u_1, \dots, u_r\}$ of rational numbers, we set

$$\Delta_p(U) := \prod_{1\le i<j\le r} \left(4\sin\left(\frac{(u_i-u_j)\pi}{p}\right)\sin\left(\frac{(u_i+u_j)\pi}{p}\right)\right)^2 \prod_{i=1}^r 4\sin^2(2u_i/p).$$

Let $\mathfrak{g} := \mathfrak{sp}_{2n}$, and $\mathfrak{h}$ the diagonal Cartan subalgebra. Fix the basis $\{\, H_i := E_{i,i} - E_{n+1,n+1} |\, 1 \le i \le n\}$ of the vector space $\mathfrak{h}$, and let $\{L_i\} \subset \mathfrak{h}^*$ be the dual basis. As a basis of the root system, we choose $\{L_1 - L_2, \dots, L_{n-1} - L_n, 2L_n\}$. Then $\theta = 2L_1$, $\rho = nL_1 + (n-1)L_2 + \dots + L_n$ and $h = n+1$. We have $|P/Q_{long}| = 2^n$. For $l \in \mathbb{N}$,

$$P_l = \left\{ \sum_{i=1}^n a_i L_i \,\middle|\, l \ge a_1 \ge a_2 \ge \dots \ge a_n \ge 0 \right\}.$$

Thus P_l can be regarded as the set of Young diagrams of type $\le(n,l)$. If we write $\mu + \rho = u_1L_1 + \dots + u_nL_n$, then we have

$$\left\{(l+h)^{\operatorname{rank}\mathfrak{g}} |P/Q_{long}|\right\}^{g-1} \prod_{\alpha>0} \left|2\sin\pi\frac{(\alpha,\mu+\rho)}{l+h}\right|^{2-2g} = \left(\frac{2(n+l+1)^n}{\Delta_{2(n+l+1)}(\{u_1,\dots,u_n\})}\right)^{g-1}$$

(see [O-W, page 2700] for details). By the Weyl character formula (cf. [FH, §24.2]), we have

$$\operatorname{Tr}_{V_{\overrightarrow{\Lambda}}} \left(\exp 2\pi\sqrt{-1}\frac{\mu+\rho}{l+h}\right) = \prod_{q=1}^m \frac{\det\left(\zeta^{u_j(\lambda_i^{(q)}+n+1-i)} - \zeta^{-u_j(\lambda_i^{(q)}+n+1-i)}\right)}{\det\left(\zeta^{u_j(n+1-i)} - \zeta^{-u_j(n+1-i)}\right)},$$

where $\Lambda^{(q)} = (\lambda_1^{(q)} \ge \dots \ge \lambda_n^{(q)})$ and $\zeta = \exp(\pi\sqrt{-1}/(n+l+1))$. Thus, in the symplectic group case, the Verlinde number (6.1) is equal to

$$\sum_{n+l\ge u_1>\dots>u_n\ge 1} \left(\frac{(2(n+l+1))^n}{\Delta_{2(n+l+1)}(\{u_1,\dots,u_n\})}\right)^{g-1} \prod_{q=1}^m \frac{\det\left(\zeta^{u_j(\lambda_i^{(q)}+n+1-i)} - \zeta^{-u_j(\lambda_i^{(q)}+n+1-i)}\right)}{\det\left(\zeta^{u_j(n+1-i)} - \zeta^{-u_j(n+1-i)}\right)}.$$

6.3. **Proof of the theorem.** Before starting the proof of Theorem 6.0.4, we prepare lemmas on matrices.

If $A = (a_{i,j})$ is an $N \times N$ matrix, and $S = (s_1, \dots, s_n)$ and $T = (t_1, \dots, t_n)$ are two sequences of n distinct integers $\{1, \dots, N\}$, we denote by $A_{S,T}$ the $n \times n$ matrix whose (i,j)-entry is a_{s_i,t_j}.

In the rest of this section we set $\zeta := \exp(\pi\sqrt{-1}/(r+s+1))$. Let W be the $(r+s)\times(r+s)$ symmetric matrix whose (i,j)-entry is $\zeta^{ij} - \zeta^{-ij}$.

Lemma 6.3.1. *(1) $W^2 = -2(r+s+1)I_{r+s}$.*

(2) If $(u_1, \dots, u_r)$ is a sequence of r distinct integers from $\{1, \dots, r+s\}$, we have

$$\det W_{(r,r-1,\dots,1)(u_1,\dots,u_r)} = (-1)^{\sum_{i=1}^r (u_i+1)} \det W_{(s+1,s+2,\dots,s+r)(u_1,\dots,u_r)}.$$

Proof. (1) For $l \in \mathbb{N}$ with $l \not\equiv 0 \pmod{2(r+s+1)}$,

$$\sum_{i=1}^{r+s} \zeta^{li} = \frac{1-(\zeta^l)^{r+s+1}}{1-\zeta^l} - 1 = \frac{\zeta^l - (-1)^l}{1-\zeta^l} = \begin{cases} -1 & \text{if } l \text{ is even ,} \\ \frac{1+\zeta^l}{1-\zeta^l} & \text{if } l \text{ is odd.} \end{cases} \tag{6.2}$$

If $l \equiv 0 \pmod{2(r+s+1)}$, then $\sum_{i=1}^{r+s} \zeta^{li} = r+s$.

The (a,b)-entry of the matrix W^2 is

$$\begin{aligned} \sum_{l=1}^{r+s} (\zeta^{ai} - \zeta^{-ai})(\zeta^{ib} - \zeta^{-ib}) &= \sum_{l=1}^{r+s} \left(\zeta^{(a+b)i} - \zeta^{(a-b)i} + \zeta^{-(a+b)i} - \zeta^{-(a-b)i} \right) \\ &\overset{(6.2)}{=} \begin{cases} -2(r+s+1)\delta_{a,b} & \text{if } a+b \text{ is even,} \\ 0 & \text{if } a+b \text{ is odd.} \end{cases} \end{aligned}$$

(2) This follows from the equality

$$\zeta^{(s+r+1-i)j} - \zeta^{-(s+r+1-i)j} = (-1)^{j+1}(\zeta^{ij} - \zeta^{-ij}).$$

□

In the proof of Theorem 6.0.4, we shall use the following lemma ([FH, Lemma A.42]).

Lemma 6.3.2. *Let A and B be $N \times N$ matrices such that $AB = c \cdot I_N$. Let (S, S') and (T, T') be permutations of the sequence $(1, \dots, N)$, where S and T consist of n integers, S' and T' of $N-n$. Then*

$$c^{N-n} \cdot \det A_{S,T} = \operatorname{sgn}(S, S') \cdot \operatorname{sgn}(T, T') \cdot \det A \cdot \det B_{T',S'}.$$

Now we start the proof of Theorem 6.0.4.

Proof of Theorem 6.0.4. We may assume that C is smooth, for the dimension of the vector spaces of the source and the target of the strange duality map (4.5) follows the fusion rule. If $\sum_{q=1}^m |\Lambda^{(q)}|$ is odd, then the source and the target of (4.5) are zero. Thus we assume that $\sum_{q=1}^m |\Lambda^{(q)}|$ is even.

Moreover we may assume m (the number of points) is even. In fact, if m is odd, we take an extra point $P^{(m+1)} \in C \setminus \{P^{(1)}, \dots, P^{(m)}\}$ and label it by the empty Young diagram $\Lambda^{(m+1)} = (0, \dots, 0)$. Then clearly

$$\mathrm{H}^0 \left(M_{2r}(C, \overrightarrow{P}; \mathcal{O}_C), \Xi^{(s; \overrightarrow{\Lambda})} \right) \simeq \mathrm{H}^0 \left(M_{2r}(C, \overrightarrow{P} + P^{(m+1)}; \mathcal{O}_C), \Xi^{(s; \overrightarrow{\Lambda} + \Lambda^{(m+1)})} \right).$$

Claim 6.3.2.1. There is an isomorphism

$$\mathrm{H}^0 \left(M_{2s}(C, \overrightarrow{P}; \omega_C(\overrightarrow{P})), \Xi^{(r; \overrightarrow{\Lambda^*})} \right) \simeq \mathrm{H}^0 \left(M_{2s}(C, \overrightarrow{P} + P^{(m+1)}; \omega_C(\overrightarrow{P} + P^{(m+1)})), \Xi^{(r; \overrightarrow{\Lambda^*} + \Lambda^{(m+1)*})} \right). \tag{6.3}$$

Proof of Claim 6.3.2.1. If

$$\mathbb{G} = \left(G, G \otimes G \to \omega_C(\overrightarrow{P} + P^{(m+1)}), G^{(q)} \supset G_s^{(q)} \supset \cdots \supset G_0^{(q)} = 0 \ (0 \le q \le m+1) \right)$$

is a point of $M_{2s}(C, \overrightarrow{P} + P^{(m+1)}; \omega_C(\overrightarrow{P} + P^{(m+1)}))$, then put

$$G' = \operatorname{Ker} \left(G \to G^{(m+1)} / G_s^{(m+1)} \right),$$

where $G^{(m+1)}/G_s^{(m+1)}$ is a sky-scraper sheaf at $P^{(m+1)}$. By assoicating to $\mathbb{G}$ the point

$$\left(G', G' \otimes G' \to \omega_C(\overrightarrow{P}), G'^{(q)} \supset G_s^{(q)} \supset \cdots \supset G_0^{(q)} = 0 \ (0 \le q \le m) \right),$$

of $M_{2s}(C, \overrightarrow{P}; \omega_C(\overrightarrow{P}))$, we have a morphism

$$f : M_{2s}(C, \overrightarrow{P} + P^{(m+1)}; \omega_C(\overrightarrow{P} + P^{(m+1)})) \to M_{2s}(C, \overrightarrow{P}; \omega_C(\overrightarrow{P})).$$

By construction, we have $f^*\Xi^{(r;\overrightarrow{\Lambda^*})} \simeq \Xi^{(r;\overrightarrow{\Lambda^*}+\Lambda^{(m+1)*})}$. Since f is a flag-variety bundle, we have the isomorphism (6.3). This completes the proof of Claim 6.3.2.1.

□

In the rest of the proof, we assume that m is even.

By [L-S], there is an isomorphism of vector spaces

$$\mathrm{H}^0\left(M_{2r}(C,\overrightarrow{P};\mathcal{O}_C),\Xi^{(s;\overrightarrow{\Lambda})}_{M_{2r}(C,\overrightarrow{P};\mathcal{O}_C)}\right)\simeq V_C(\overrightarrow{P},\overrightarrow{\Lambda},s).$$

Fix a line bundle L with $L^{\otimes 2}\simeq\omega_C(\overrightarrow{P})$. By applying $\otimes L^{-1}$, we have an isomorphism $M_{2s}(C,\overrightarrow{P};\omega_C(\overrightarrow{P}))\simeq M_{2s}(C,\overrightarrow{P};\mathcal{O}_C)$, so we obtain isomophisms

$$\mathrm{H}^0\left(M_{2s}(C,\overrightarrow{P};\omega_C(\overrightarrow{P})),\Xi^{(r;\overrightarrow{\Lambda^*})}\right)\simeq\mathrm{H}^0\left(M_{2s}(C,\overrightarrow{P};\mathcal{O}_C),\Xi^{(r;\overrightarrow{\Lambda^*})}\right)\simeq V_C(\overrightarrow{P},\overrightarrow{\Lambda^*},r).$$

Thus what we should prove is the equality of the two numbers

(6.4)
$$\sum_{r+s\geq u_1>\cdots>u_r\geq 1}\left(\frac{(2(r+s+1))^r}{\Delta_{2(r+s+1)}(\{u_1,\ldots,u_r\})}\right)^{g-1}\prod_{q=1}^{m}\frac{\det W_{(\alpha_1^{(q)},\ldots,\alpha_r^{(q)})(u_1,\ldots,u_r)}}{\det W_{(r,r-1,\ldots,1)(u_1,\ldots,u_r)}},$$

and

(6.5)
$$\sum_{r+s\geq v_1>\cdots>v_s\geq 1}\left(\frac{(2(r+s+1))^s}{\Delta_{2(r+s+1)}(\{v_1,\ldots,v_s\})}\right)^{g-1}\prod_{q=1}^{m}\frac{\det W_{(\beta_1^{(q)},\ldots,\beta_s^{(q)})(v_1,\ldots,v_s)}}{\det W_{(s,s-1,\ldots,1)(v_1,\ldots,v_s)}},$$

where $\alpha_i^{(q)}:=\lambda_i^{(q)}+r+1-i$ and $\beta_j^{(q)}:=\nu_j^{(q)}+s+1-j$ for $(\lambda_1^{(q)}\geq\cdots\geq\lambda_r^{(q)})=\Lambda^{(q)}$ and $(\nu_1^{(q)}\geq\cdots\geq\nu_s^{(q)}):=\Lambda^{(q)*}$.

The mapping $\{u_1,\ldots,u_r\}\mapsto\{1,\ldots,r+s\}\setminus\{u_1,\ldots,u_r\}$ gives a one-to-one correspondence between the index sets of the summations (6.4) and (6.5). We shall prove that if

$$\{v_1,\ldots,v_s\}=\{1,\ldots,r+s\}\setminus\{u_1,\ldots,u_r\},\tag{6.6}$$

then the terms in the above summations indexed by $(u_1,\ldots,u_r)$ and $(v_1,\ldots,v_s)$ are equal.

If (6.6) holds, then it follows from [O-W, Corollary 1.6], that

$$\frac{(2(r+s+1))^r}{\Delta_{2(r+s+1)}(\{u_1,\ldots,u_r\})}=\frac{(2(r+s+1))^s}{\Delta_{2(r+s+1)}(\{v_1,\ldots,v_s\})}.$$

It remains to show that in the case of (6.6),

$$\prod_{q=1}^{m}\frac{\det W_{(\alpha_1^{(q)},\ldots,\alpha_r^{(q)})(u_1,\ldots,u_r)}}{\det W_{(r,r-1,\ldots,1)(u_1,\ldots,u_r)}}=\prod_{q=1}^{m}\frac{\det W_{(\beta_1^{(q)},\ldots,\beta_s^{(q)})(v_1,\ldots,v_s)}}{\det W_{(s,s-1,\ldots,1)(v_1,\ldots,v_s)}}.\tag{6.7}$$

Note that $\{\beta_1^{(q)},\ldots,\beta_s^{(q)}\}=\{1,\ldots,r+s\}\setminus\{\alpha_1^{(q)},\ldots,\alpha_r^{(q)}\}$. Applying Lemma 6.3.2 as $A=W$ and $B={}^tW(=W)$, we obtain

$$\begin{aligned}&(-2(r+s+1))^s\det W_{(\alpha_1^{(q)},\ldots,\alpha_r^{(q)})(u_1,\ldots,u_r)}\\&=\operatorname{sgn}(\alpha_1^{(q)},\ldots,\alpha_r^{(q)},\beta_1^{(q)},\ldots,\beta_s^{(q)})\operatorname{sgn}(u_1,\ldots,u_r,v_1,\ldots,v_s)\det W\det W_{(\beta_1^{(q)},\ldots,\beta_s^{(q)})(v_1,\ldots,v_s)}\end{aligned}$$

and

$$\begin{aligned}&(-2(r+s+1))^s\det W_{(s+1,\ldots,s+r)(u_1,\ldots,u_r)}\\&=\operatorname{sgn}(s+1,\ldots,s+r,s,\ldots,1)\operatorname{sgn}(u_1,\ldots,u_r,v_1,\ldots,v_s)\det W\det W_{(s,\ldots,1)(v_1,\ldots,v_s)}.\end{aligned}$$

Taking the division on each side of these equations, we obtain

$$\frac{\det W_{(\alpha_1^{(q)},\dots,\alpha_r^{(q)})(u_1,\dots,u_r)}}{\det W_{(s+1,\dots,s+r)(u_1,\dots,u_r)}} = \frac{\operatorname{sgn}(\alpha_1^{(q)},\dots,\alpha_r^{(q)},\beta_1^{(q)},\dots,\beta_s^{(q)})\det W_{(\beta_1^{(q)},\dots,\beta_s^{(q)})(v_1,\dots,v_s)}}{\operatorname{sgn}(s+1,\dots,s+r,s,\dots,1)\det W_{(s,\dots,1)(v_1,\dots,v_s)}}. \tag{6.8}$$

Using Lemma 6.3.1 (2), the left-hand side of (6.8) is equal to

$$(-1)^{\sum_{i=1}^r (u_i+1)} \frac{\det W_{(\alpha_1^{(q)},\dots,\alpha_r^{(q)})(u_1,\dots,u_r)}}{\det W_{(r,r-1,\dots,1)(u_1,\dots,u_r)}}.$$

Taking the product on each side of (6.8) for $1 \le q \le m$, we obtain

$$\begin{aligned}&\prod_{q=1}^m \frac{\det W_{(\alpha_1^{(q)},\dots,\alpha_r^{(q)})(u_1,\dots,u_r)}}{\det W_{(r,r-1,\dots,1)(u_1,\dots,u_r)}} \\ &= \left\{\prod_{q=1}^m \operatorname{sgn}(\alpha_1^{(q)},\dots,\alpha_r^{(q)},\beta_1^{(q)},\dots,\beta_s^{(q)})\right\}\prod_{q=1}^m \frac{\det W_{(\beta_1^{(q)},\dots,\beta_s^{(q)})(v_1,\dots,v_s)}}{\det W_{(s,\dots,1)(v_1,\dots,v_s)}},\end{aligned}$$

here we used the assumption that m is even.

You can check easily that

$$\operatorname{sgn}(\alpha_1^{(q)},\dots,\alpha_r^{(q)},\beta_1^{(q)},\dots,\beta_s^{(q)}) = (-1)^{\frac{r(r-1)}{2}+\frac{s(s-1)}{2}+|\Lambda^{(q)}|}.$$

Hence

$$\prod_{q=1}^m \operatorname{sgn}(\alpha_1^{(q)},\dots,\alpha_r^{(q)},\beta_1^{(q)},\dots,\beta_s^{(q)}) = (-1)^{\sum_{q=1}^m |\Lambda^{(q)}|} = 1,$$

where the last equality follows from the assumption that $\sum_{q=1}^m |\Lambda^{(q)}|$ is even.

This completes the proof of Theorem 6.0.4. $\square$

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