# On the Logarithmic Asymptotics of the Sixth Painlevé Equation – Part I

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**Abstract**

We study the solutions of the sixth Painlevé equation with a logarithmic asymptotic behavior at a critical point. We compute the monodromy group associated to the solutions by the method of monodromy preserving deformations and we characterize the asymptotic behavior in terms of the monodromy itself. This is the first of two papers aimed at the characterization/classification of the logarithmic behaviors, in terms of the monodromy data.

## 1 Introduction

We consider the sixth Painlevé equation:

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right]\frac{dy}{dx}$$
$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right], \qquad \text{(PVI)}.$$

The generic solution has essential singularities and/or branch points in 0,1,$\infty$. It's behavior at these points is called *critical*. Other singularities which may appear are poles and depend on the initial conditions. A solution of (PVI) can be analytically continued to a meromorphic function on the universal covering of $\mathbf{P}^1\backslash\{0,1,\infty\}$. For generic values of the integration constants and of the parameters $\alpha$,$\beta$,$\gamma$,$\delta$, it cannot be expressed via elementary or classical transcendental functions. For this reason, it is called a *Painlevé transcendent*. Solving (PVI) means: i) Determine the critical behavior of the transcendents at the *critical points* $x=0,1,\infty$. Such a behavior must depend on two integration constants. ii) Solve the *connection problem*, namely: find the relation between couples of integration constants at $x=0,1,\infty$.

(PVI) is the isomonodromy deformation equation of a Fuchsian system of differential equations [18]:

$$\frac{d\Psi}{d\lambda}=A(x,\lambda)\ \Psi, \qquad A(x,\lambda):=\left[\frac{A_0(x)}{\lambda}+\frac{A_x(x)}{\lambda-x}+\frac{A_1(x)}{\lambda-1}\right], \quad \lambda\in\mathbf{C}. \tag{1}$$

The $2\times 2$ matrices $A_i(x)$ depend on $x$ in such a way that there exists a fundamental matrix solution $\Psi(\lambda,x)$ such that its monodromy does not change for small deformations of $x$. They also depend on the parameters $\alpha,\beta,\gamma,\delta$ of (PVI) through more elementary parameters $\theta_0,\theta_x,\theta_1,\theta_\infty$, according to the following relations:

$$-A_\infty:=A_0+A_1+A_x=-\frac{\theta_\infty}{2}\sigma_3, \quad \theta_\infty\neq 0. \qquad \text{Eigenvalues }(A_i)=\pm\frac{1}{2}\theta_i, \quad i=0,1,x; \tag{2}$$

$$\alpha=\frac{1}{2}(\theta_\infty-1)^2, \quad -\beta=\frac{1}{2}\theta_0^2, \quad \gamma=\frac{1}{2}\theta_1^2, \quad \left(\frac{1}{2}-\delta\right)=\frac{1}{2}\theta_x^2 \tag{3}$$

Here $\sigma_3:=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ is the Pauli matrix. The condition $\theta_\infty\neq 0$ is not restrictive, because $\theta_\infty=0$ is equivalent to $\theta_\infty=2$. The equations of monodromy preserving deformation (Schlesinger equations), can be written in Hamiltonian form and reduce to (PVI), being the transcendent $y(x)$ the solution $\lambda$ of $A(x,\lambda)_{1,2}=0$. Namely:

$$y(x)=\frac{x\ (A_0)_{12}}{x\ \ [(A_0)_{12}+(A_1)_{12}]-(A_1)_{12}}, \tag{4}$$

The matrices $A_i(x)$, $i=0,x,1$, depend on $y(x)$, $\frac{dy(x)}{dx}$ and $\int y(x)$ through rational functions, which are given in [18] and in subsection 8.2.

This paper, and the second paper, are devoted to the computation of the monodromy group of (1) associated to the solutions with a logarithmic critical behavior, and to the action of the symmetries of (PVI) on the monodromy of (1). They are part of a project to classify the critical behaviors in terms of the monodromy data of the system (1). This project has been the motivation of our papers [11] [12] [14].

In our paper [14], we developed a "constructive" procedure which we called *matching*. It enabled us to compute the leading term of the critical behavior of a transcendent $y(x)$ and the monodromy data of (1) when the matrices $A_i(x)$ are those associated to $y(x)$. Originally, such an approach was suggested by Its and Novokshenov in [15], for the second and third Painlevé equations. The method of Jimbo [17] can be regarded as a matching procedure. This approach was further developed and used by Kapaev, Kitaev, Andreev, and Vartanian (see for example the case of the fifth Painlevé equation, in [2]). Our approach in [14] is new, because we introduced non-fuchsian systems associated to (PVI) in the process of matching. In this way we obtained new asymptotic behaviors. The matching procedure will be reviewed in section 2.

We developed the matching procedure in order to discover new critical behaviors and to classify the critical behaviors themselves in terms of associated monodromy data. Denote by $M_0$, $M_x$, $M_1$ a monodromy representation of (1). The critical behaviors associated to monodromy matrices satisfying the relation $\mathrm{tr}(M_iM_j)\neq\pm 2$, $i\neq j\in\{0,x,1\}$, is known from the work [17]. But when $\mathrm{tr}(M_iM_j)=\pm 2$, we cannot naively extend the procedure of [17]. In addition, many cases corresponding to non generic values of $\alpha,\beta,\gamma,\delta$ are not yet studied. The matching procedure was developed in [14], as a general method to study the cases $\mathrm{tr}(M_iM_j)=\pm 2$ and the non generic cases of $\alpha,\beta,\gamma,\delta$. The logarithmic solutions, some of the Taylor's series solutions and the trigonometric solutions of [14] actually appear when $\mathrm{tr}(M_iM_j)=\pm 2$ for some $i\neq j=0,x,1$.

The values of the traces $\mathrm{tr}(M_0M_x)$, $\mathrm{tr}(M_1M_x)$, $\mathrm{tr}(M_0M_1)$ characterize the critical behaviors at $x=0,1,\infty$ respectively. This is a known fact, which follows from the solution of the connection problem (see also subsection 8.3). For example, in the generic case studied in [17] we find the following behaviors at the critical points [17][10][11][12][13][3][25]:

$$y(x)=\begin{cases} a x^{1-\sigma}(1+O(|x|^{\epsilon})), & x\to 0,\\[2mm] y(x)=1-a^{(1)}(1-x)^{1-\sigma^{(1)}}(1+O(|1-x|^{\epsilon})), & x\to 1,\\[2mm] y(x)=a^{(\infty)}x^{\sigma^{(\infty)}}(1+O(|x|^{-\epsilon})), & x\to\infty,\end{cases}$$

where $\epsilon$ is a small positive number, $a$, $\sigma$, $a^{(1)}$ , $\sigma^{(1)}$, $a^{(\infty)}$ , $\sigma^{(\infty)}$ are complex numbers such that $a$, $a^{(i)}\neq 0$ and $0<\Re\sigma<1$, $0<\Re\sigma^{(1)}<1$, $0<\Re\sigma^{(\infty)}<1$. The connection problem among the three sets of parameters $(a,\sigma)$, $(a^{(1)},\sigma^{(1)})$, $(a^{(\infty)}\sigma^{(\infty)})$ was first solved in [17] and its solution implies that:

$$2\cos(\pi\sigma)=\mathrm{tr}(M_0M_x),\quad 2\cos(\pi\sigma^{(1)})=\mathrm{tr}(M_1M_x),\quad 2\cos(\pi\sigma^{(\infty)})=\mathrm{tr}(M_0M_1);$$

while $a$, $a^{(1)}$, $a^{(\infty)}$ are rational functions of the $\mathrm{tr}(M_iM_j)$'s ($i\neq j=0,x,1$) and depend on the $\theta_\nu$'s ($\nu=0,x,1,\infty$) through trigonometric functions and $\Gamma$-functions rationally combined. In this sense, the three traces determine the critical behavior at the three critical points.

Before we present the result of the paper, it is worth summarizing the results obtained by the matching procedure in [14]. We first consider the point $x=0$. Let $\sigma$ be a complex number defined by:

$$\mathrm{tr}(M_0M_x)=2\cos(\pi\sigma),\qquad 0\leq\Re\sigma\leq 1.$$

The matching procedure yields the following behaviors for $x\to 0$:

$$y(x)\sim\ a\ x^{1-\sigma},\qquad \text{if } \Re\sigma>0; \tag{5}$$

$$y(x)\sim x\left\{iA\ \sin\bigl(i\sigma\ln x+\phi\bigr)+\frac{\theta_0^2-\theta_x^2+\sigma^2}{2\sigma^2}\right\},\quad \text{if } \Re\sigma=0,\quad \sigma\neq 0.$$

In the above formulae, $\sigma$ is one of the integration constants, while $a$, or $\phi$, is the other. $A$ is:

$$A:=\left[\frac{\theta_0^2}{\sigma^2}-\left(\frac{\theta_0^2-\theta_x^2+\sigma^2}{2\sigma^2}\right)^2\right]^{\frac{1}{2}}.$$

As we mentioned, the behavior (5) was first studied in [17]. For special values of $\sigma \neq 0$, the first leading term above is zero and we need to consider the next leading terms:

$$y(x) \sim \frac{\theta_0}{\theta_0+\theta_x}\, x \;\mp\; \frac{r}{\theta_0+\theta_x}\, x^{1+\sigma}, \quad \sigma = \pm(\theta_0+\theta_x) \neq 0,$$

$$y(x) \sim \frac{\theta_0}{\theta_0-\theta_x}\, x \;\mp\; \frac{r}{\theta_0-\theta_x}\, x^{1+\sigma}, \quad \sigma = \pm(\theta_0-\theta_x) \neq 0.$$

When $\sigma = 0$, the matching procedure of [14] yields the logarithmic behaviors:

$$y(x) \sim x\left\{\frac{\theta_x^2-\theta_0^2}{4}\left[\ln x + \frac{4r+2\theta_0}{\theta_0^2-\theta_x^2}\right]^2 + \frac{\theta_0^2}{\theta_0^2-\theta_x^2}\right\}, \qquad \text{if } \theta_0^2 \neq \theta_x^2, \tag{6}$$

$$y(x) \sim x\;(r \;\pm\; \theta_0\; \ln x), \qquad \text{if } \theta_0^2 = \theta_x^2. \tag{7}$$

Here $r$ is an integration constant.

In [14] we also computed all the solutions with Taylor expansions at a critical point. They fall within three equivalent classes (the equivalence relations are Backlund transformations of (PVI)), with representatives characterized by $\sigma = \pm(\theta_1 \pm \theta_\infty), 1, -1$ respectively. To these classes, we must add the singular solutions $y = 0, x, 1$. The associated monodromy groups are characterized by reducible subgroups generated by $M_0M_x$ and $M_1$. Taylor solutions are studied also in [19], by the isomonodromy deformation method; and in [4] [5] [6] [7] by a power geometry technique.

The critical behaviors at $x = 1, \infty$ can be obtained from those at $x = 0$ by the action of some of the Backlund transformations of (PVI). See subsection 8.3.

The monodromy data for the solution (5) are computed in [17][10][11][12][13][3]. The monodromy data for the Taylor expansions are computed in [14] and [19].

In [14] we did not compute the monodromy associated to the logarithmic behaviors, postponing this problem to the present paper and its companion paper in preparation. We are going to show that logarithmic critical behaviors at $x = 0$ are associated to $\mathrm{tr}(M_0M_x) = \pm 2$, at $x = 1$ to $\mathrm{tr}(M_1M_x) = \pm 2$, and at $x = \infty$ to $\mathrm{tr}(M_0M_1) = \pm 2$.

Once the monodromy data are known, the connection problem is solved (see subsection 8.3)

We computed the logarithmic asymptotic behaviors in [14] as a result of the matching procedure (in the framework of the method of monodromy preserving deformations). In [4] [5] [6] [7] [8], A.D.Bruno and I.V.Goryuchkina constructed the asymptotic expansions, including logarithmic ones, by a power geometry technique [9]. By this technique, the authors of [7] claim that they have obtained *all* the critical behaviors for (PVI). The logarithmic asymptotics for real solutions of (PVI) is studied in [24]. Our approach, being based on the method of isomonodromy deformations, allows to solve the connection problem, while the results of [4]– [8] and [24] are local.

## 1.1 Results

In this paper:

1) In Section 3 we justify the project of classifying the transcendents in terms of monodromy data of (1). We establish the necessary and sufficient conditions such that there exist a one to one correspondence between a set of monodromy data of system (1) and a transcendent of (PVI). The result is Proposition 1. The definition of *monodromy data* itself is given in Section 3.

2) We compute the monodromy data associated to the logarithmic solutions (6) in the generic case $\theta_0, \theta_x, \theta_1, \theta_\infty \notin \mathbf{Z}$. The result is Proposition 2, Section 5. In particular, $\mathrm{tr}(M_0M_x) = 2$.

3) In Proposition 3 of Section 6, we compute the monodromy group associated to the solution (7). In particular, $\mathrm{tr}(M_0M_x) = 2$. The parameter $r$ will be computed as a function of the $\theta_\nu$'s, $\nu = 0, x, 1, \infty$ and of $\mathrm{tr}(M_0M_1)$.

4) We consider a non generic case of (6), which occurs when:

$$\theta_x = \theta_1 = 0, \qquad \theta_\infty = 1, \qquad \theta_0 = 2p \neq 0, \quad p \in \mathbf{Z}. \tag{8}$$

Therefore:

$$y(x) \sim \left[1 - p^2 \left(\ln x + \frac{r+p}{p^2}\right)\right], \qquad x \to 0. \tag{9}$$

The monodromy of the associated system (1) is computed in Proposition 4, Section 7. It is important to observe that the monodromy is independent of $r$. This means that the parameter $r$ cannot be determined in terms of the monodromy data. Therefore, (9) is a *one parameter class of solutions (*parameter $r$*) associated to the same monodromy data.* We prove in Proposition 4 that the solution (9) is associated to:

$$\operatorname{tr}(M_0 M_x) = 2, \quad \operatorname{tr}(M_0 M_1) = 2, \quad \operatorname{tr}(M_1 M_x) = -2.$$

This special values of the traces imply that the behavior at $x = \infty$ and $x = 1$ is also logarithmic. $\operatorname{tr}(M_0 M_x) = 2$ is associated to the logarithmic behavior of type $\ln^2 x$ at $x = 0$. $\operatorname{tr}(M_0 M_1) = 2$ is associated to the logarithmic behavior of type $\ln^2(1/x)$ at $x = \infty$. $\operatorname{tr}(M_1 M_x) = -2$ is associated to the logarithmic behavior of type $1/\ln^2(1-x)$ at $x = 1$. Actually a solution (9) has the following behaviors at the three critical points:

$$y(x) \sim \begin{cases} x\left[1 - p^2(\ln x + \rho_0)^2\right], & x \to 0, \\ 1 - p^2\left(\ln \frac{1}{x} + \rho_\infty\right)^2, & x \to \infty, \\ 1 - \frac{1}{p^2(\ln(1-x)+\rho_1)^2}, & x \to 1. \end{cases} \tag{10}$$

where:

$$\rho_0 = \frac{(r+p)}{p^2}, \quad \rho_\infty = \frac{\pi(4\ln 2 - 1 + \rho_0)}{\pi - i(4\ln 2 - 1 + \rho_0)} - 2\ln 2 + 1, \quad \rho_1 = \frac{\pi^2}{4\ln 2 - 1 + \rho_0} - \ln 2 + 1.$$

The behavior at $x = 1$ differs from those at $x = 0, \infty$ for the inverse of $\ln(1-x)$ appears. This is actually due to the fact that $\operatorname{tr}(M_1 M_x) = -2$. We will prove the above behaviors in section 8.4, and in the second paper by a different method.

In general, the logarithmic behaviors of "type (6)" at the critical points are as follows:

$$y(x) \sim x\left\{\frac{\theta_x^2 - \theta_0^2}{4}\left[\ln x + \frac{4r + 2\theta_0}{\theta_0^2 - \theta_x^2}\right]^2 + \frac{\theta_0^2}{\theta_0^2 - \theta_x^2}\right\}, \quad x \to 0. \tag{11}$$

$$y(x) \sim \frac{\theta_0^2}{\theta_0^2 - \theta_1^2} + \frac{\theta_1^2 - \theta_0^2}{4}\left[\ln\frac{1}{x} + \frac{4r + 2\theta_0}{\theta_0^2 - \theta_1^2}\right]^2, \quad x \to \infty. \tag{12}$$

$$y(x) \sim 1 - (1-x)\left\{\frac{\theta_1^2}{\theta_1^2 - \theta_x^2} + \frac{\theta_x^2 - \theta_1^2}{4}\left[\ln(1-x) + \frac{4r + 2\theta_1}{\theta_1^2 - \theta_x^2}\right]^2\right\}, \quad x \to 1. \tag{13}$$

$$y(x) = \frac{4}{[\theta_1^2 - (\theta_\infty - 1)^2]\ln^2 x}\left[1 + \frac{8r + 4\theta_\infty - 4}{\theta_1^2 - (\theta_\infty - 1)^2}\frac{1}{\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \quad x \to 0. \tag{14}$$

$$y(x) = \frac{4\,x}{[(\theta_\infty - 1)^2 - \theta_x^2]\ln^2 x}\left[1 - \frac{8r + 4(\theta_\infty - 1)}{\theta_x^2 - (\theta_\infty - 1)^2}\frac{1}{\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \quad x \to \infty. \tag{15}$$

$$y(x) = 1 + \frac{4}{(\theta_1^2 - \theta_0^2)\ln^2(x-1)}\left[1 - \frac{8r + 4\theta_0}{\theta_0^2 - \theta_1^2}\frac{1}{\ln(x-1)} + O\left(\frac{1}{\ln^2(x-1)}\right)\right], \quad x \to 1. \tag{16}$$

In general, the log-behaviors of "type (7)" are:

$$y(x) \sim x\,(r \ \pm\ \theta_0\ \ln x), \qquad x \to 0, \qquad \theta_0^2 = \theta_x^2. \tag{17}$$

$$y(x) \sim r \pm \theta_0 \ln x, \qquad x \to \infty, \qquad \theta_0^2 = \theta_1^2. \tag{18}$$

$$y(x) \sim 1 - (1-x)\big(r \pm \theta_1 \ln(1-x)\big), \qquad x \to 1, \qquad \theta_1^2 = \theta_x^2. \tag{19}$$

$$y(x) \sim \frac{1}{r \pm (\theta_\infty - 1)\ln x}, \qquad x \to 0, \qquad (\theta_\infty - 1)^2 = \theta_1^2. \tag{20}$$

$$y(x) = \pm \frac{x}{(\theta_\infty - 1)\ln x}\left[1 \mp \frac{r}{(\theta_\infty - 1)\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right]., \qquad x \to \infty, \qquad (\theta_\infty - 1)^2 = \theta_x^2. \tag{21}$$

$$y(x) = 1 \pm \frac{1}{\theta_0 \ln(x-1)}\left[1 \mp \frac{r}{\theta_0 \ln(x-1)} + O\left(\frac{1}{\ln^2(x-1)}\right)\right], \qquad x \to 1, \qquad (\theta_\infty - 1)^2 = \theta_0^2. \tag{22}$$

The above are proved in Section 8, making use of the Backlund transformations of (PVI). The behaviors (11), (17) are associated to $\mathrm{tr}(M_0 M_x) = 2$; (12), (18) are associated to $\mathrm{tr}(M_0 M_1) = 2$; (13), (19) are associated to $\mathrm{tr}(M_1 M_x) = 2$. This fact is proved in Section 8.3. The behaviors (14), (20) are associated to $\mathrm{tr}(M_0 M_x) = -2$; (15), (21) are associated to $\mathrm{tr}(M_0 M_1) = -2$; (16), (22) are associated to $\mathrm{tr}(M_1 M_x) = -2$. This fact is proved in the second paper. We note that generically a solution (6) does not have the logarithmic behavior at $x = 1, \infty$, because the traces $\mathrm{tr}(M_1 M_x)$, $\mathrm{tr}(M_0 M_1)$ are not equal to $\pm 2$. The case (9) is special, in that the log-behavior appears at the three critical points.

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# 2 Matching Procedure

This section is a review of the matching procedure of [14]. We explain how the asymptotic behavior of a transcendent is derived, and how the associated monodromy is computed.

## 2.1 Leading Terms of $y(x)$

We consider $x \to 0$. We divide the $\lambda$-plane into two domains. The "outside" domain is defined for $\lambda$ sufficiently big:

$$|\lambda| \geq |x|^{\delta_{OUT}}, \qquad \delta_{OUT} > 0. \tag{23}$$

Therefore, (1) can be written as:

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_x}{\lambda}\sum_{n=1}^{\infty}\left(\frac{x}{\lambda}\right)^n + \frac{A_1}{\lambda - 1}\right]\Psi. \tag{24}$$

The "inside" domain is defined for $\lambda$ comparable with $x$, namely:

$$|\lambda| \leq |x|^{\delta_{IN}}, \qquad \delta_{IN} > 0. \tag{25}$$

Therefore, $\lambda \to 0$ as $x \to 0$, and we rewrite (1) as:

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} - A_1 \sum_{n=0}^{\infty}\lambda^n\right]\Psi. \tag{26}$$

If the behavior of $A_0(x)$, $A_1(x)$ and $A_x(x)$ is sufficiently good, we expect that the higher order terms in the series of (24) and (26) are small corrections which can be neglected when $x \to 0$. If this is the case, (24) and (26) reduce respectively to:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_x}{\lambda}\sum_{n=1}^{N_{OUT}}\left(\frac{x}{\lambda}\right)^n + \frac{A_1}{\lambda - 1}\right]\Psi_{OUT}, \tag{27}$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} - A_1 \sum_{n=0}^{N_{IN}} \lambda^n \right] \Psi_{IN}, \tag{28}$$

where $N_{IN}$, $N_{OUT}$ are suitable integers. The simplest reduction is to Fuchsian systems:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[ \frac{A_0 + A_x}{\lambda} + \frac{A_1}{\lambda - 1} \right] \Psi_{OUT}, \tag{29}$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} \right] \Psi_{IN}. \tag{30}$$

In [14] we considered reduced non-fuchsian systems for the first time in the literature, where the fuchsian reduction has been privileged. We showed that in some relevant cases it cannot be used, being the non-fuchsian reduction necessary.

Generally speaking, we can parameterize the elements of $A_0 + A_x$ and $A_1$ of (29) in terms of $\theta_1$, the eigenvalues of $A_0 + A_x$ and the eigenvalues $\theta_\infty$ of $A_0 + A_x + A_1$. We also need an additional unknown function of $x$. In the same way, we can explicitly parameterize the elements of $A_0$ and $A_x$ in (30) in terms of $\theta_0$, $\theta_x$, the eigenvalues of $A_0 + A_x$ and another additional unknown function of $x$. Cases when the reductions (27) and (28) are non-fuchsian deserve particular care, as it has been done in [14]. Our purpose is to find the leading terms of the unknown functions when $x \to 0$, in order to determine the critical behavior of $A_0(x)$, $A_1(x)$, $A_x(x)$ and of (4).

The leading term can be obtained as a result of two facts:
i) Systems (27) and (28) are isomonodromic. This imposes constraints on the form of the unknown functions. Typically, one of them must be constant.
ii) Two fundamental matrix solutions $\Psi_{OUT}(\lambda, x)$, $\Psi_{IN}(\lambda, x)$ must match in the region of overlap, provided this is not empty:

$$\Psi_{OUT}(\lambda, x) \sim \Psi_{IN}(\lambda, x), \qquad |x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}, \quad x \to 0 \tag{31}$$

This relation is to be intended in the sense that the leading terms of the local behavior of $\Psi_{OUT}$ and $\Psi_{IN}$ for $x \to 0$ must be equal. This determines a simple relation between the two functions of $x$ appearing in $A_0$, $A_x$, $A_1$, $A_0 + A_x$. (31) also implies that $\delta_{IN} \leq \delta_{OUT}$.

Practically, to fulfill point ii), we match a fundamental solution of (27) for $\lambda \to 0$, with a fundamental solution of the system obtained from (28) by the change of variables $\mu := \lambda/x$, namely with a solution of:

$$\frac{d\Psi_{IN}}{d\mu} = \left[ \frac{A_0}{\mu} + \frac{A_x}{\mu - 1} - xA_1 \sum_{n=0}^{N_{IN}} x^n \mu^n \right] \Psi_{IN}, \qquad \mu := \frac{\lambda}{x}. \tag{32}$$

To summarize, *matching* two fundamental solutions of the reduced *isomonodromic* systems (27) and (28), we obtain the leading term(s), for $x \to 0$, of the entries of the matrices of the original system (1). The procedure is algorithmic, the only assumption being (31).

This method is sometimes called *coalescence of singularities*, because the singularity $\lambda = 0$ and $\lambda = x$ coalesce to produce system (27), while the singularity $\mu = \frac{1}{x}$ and $\mu = \infty$ coalesce to produce system (32). Coalescence of singularities was first used by M. Jimbo in [17] to compute the monodromy matrices of (1) for the class of solutions of (PVI) with leading term $y(x) \sim \ a\ x^{1-\sigma}$, $0 < \Re\sigma < 1$.

## 2.2 Computation of the Monodromy Data

In the "$\lambda$-plane" $\mathbf{C}\backslash\{0, x, 1\}$ we fix a base point $\lambda_0$ and three loops, which are numbered in order 1, 2, 3 according to a counter-clockwise order referred to $\lambda_0$. We choose $0, x, 1$ to be the order $1, 2, 3$. We denote the loops by $\gamma_0$, $\gamma_x$, $\gamma_1$. See figure 1. The monodromy matrices of a fundamental solution $\Psi(\lambda)$ w.r.t. this base of loops are denoted $M_0$, $M_x$, $M_1$. The loop at infinity will be $\gamma_\infty = \gamma_0\gamma_x\gamma_1$, so $M_\infty = M_1 M_x M_0$. As a consequence, the following relation holds:

$$\cos(\pi\theta_0)\mathrm{tr}(M_1 M_x) + \cos(\pi\theta_1)\mathrm{tr}(M_0 M_x) + \cos(\pi\theta_x)\mathrm{tr}(M_1 M_0)$$

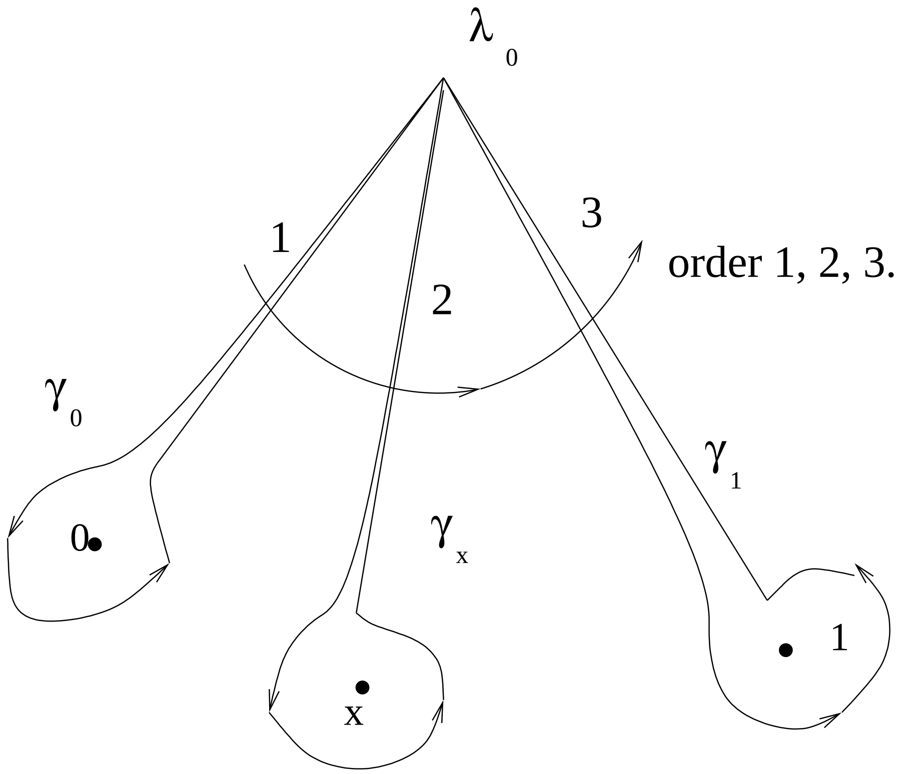


Figure 1: The ordered basis of loops

$$= 2\cos(\pi\theta_\infty) + 4\cos(\pi\theta_1)\cos(\pi\theta_0)\cos(\pi\theta_x).$$

The monodromy matrices are determined by $\mathrm{tr}(M_\nu)$, $\mathrm{tr}(M_\nu M_\mu)$ , $\nu, \mu = 0, x, 1, \infty$ [3].

As a consequence of isomonodromicity, there exists a fundamental solution $\Psi_{OUT}$ of (27) such that

$$M_1^{OUT} = M_1, \qquad M_\infty^{OUT} = M_\infty,$$

where $M_1^{OUT}$ and $M_\infty^{OUT}$ are the monodromy matrices of $\Psi_{OUT}$ at $\lambda = 1, \infty$. Moreover, $M_0^{OUT} = M_x M_0$. There also exists a fundamental solution $\Psi_{IN}$ of (28) such that:

$$M_0^{IN} = M_0, \qquad M_x^{IN} = M_x,$$

where $M_0^{IN}$ and $M_x^{IN}$ are the monodromy matrices of $\Psi_{IN}$ at $\lambda = 0, x$.

The method of coalescence of singularities is useful when the monodromy of the reduced systems (27), (28) can be explicitly computed. This is the case when the reduction is fuchsian (namely (29), (30)), because fuchsian systems with three singular points are equivalent to a Gauss hypergeometric equation (see Appendix 1). For the non-fuchsian reduction, in general we can compute the monodromy when (27), (28) are solvable in terms of special or elementary functions.

In order for this procedure to work, not only $\Psi_{OUT}$ and $\Psi_{IN}$ must match with each other, as in subsection 2.1, but also $\Psi_{OUT}$ must match with a fundamental matrix solution $\Psi$ of (1) in a domain of the $\lambda$ plane, and $\Psi_{IN}$ must match with *the same* $\Psi$ in another domain of the $\lambda$ plane.

The standard choice of $\Psi$ is as follows:

$$\Psi(\lambda) = \begin{cases} \left[I + O\left(\frac{1}{\lambda}\right)\right]\ \lambda^{-\frac{\theta_\infty}{2}\sigma_3}\lambda^{R_\infty}, & \lambda \to \infty; \\ \psi_0(x)\big[I + O(\lambda)\big]\ \lambda^{\frac{\theta_0}{2}\sigma_3}\lambda^{R_0}C_0, & \lambda \to 0; \\ \psi_x(x)\big[I + O(\lambda - x)\big]\ (\lambda - x)^{\frac{\theta_x}{2}\sigma_3}(\lambda - x)^{R_x}C_x, & \lambda \to x; \\ \psi_1(x)\big[I + O(\lambda - 1)\big]\ (\lambda - 1)^{\frac{\theta_1}{2}\sigma_3}(\lambda - 1)^{R_1}C_1, & \lambda \to 1; \end{cases} \tag{33}$$

Here $\psi_0(x)$, $\psi_x(x)$, $\psi_1(x)$ are the diagonalizing matrices of $A_0(x)$, $A_1(x)$, $A_x(x)$ respectively. They are defined by multiplication to the right by arbitrary diagonal matrices, possibly depending on $x$. $C_\nu$, $\nu = \infty, 0, x, 1$, are invertible *connection matrices*, independent of $x$ [18]. Each $R_\nu$, $\nu = \infty, 0, x, 1$, is also independent of $x$, and:

$$R_\nu = 0 \text{ if } \theta_\nu \notin \mathbf{Z}, \qquad R_\nu = \begin{cases} \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, & \text{if } \theta_\nu > 0 \text{ integer} \\ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, & \text{if } \theta_\nu < 0 \text{ integer} \end{cases}$$

If $\theta_i = 0$, $i = 0, x, 1$, then $R_i$ is to be considered the Jordan form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of $A_i$. Note that for the loop $\lambda \mapsto \lambda e^{2\pi i}$, $|\lambda| > \max\{1, |x|\}$, we immediately compute the monodromy at infinity:

$$M_\infty = \exp\{-i\pi\theta_\infty\} \ \exp\{2\pi i R_\infty\}.$$

Let $\Psi_{OUT}$ and $\Psi_{IN}$ be the solutions of (27) and (28) matching as in (31). We explain how they are matched with (33).

**(*) Matching $\Psi \ \leftrightarrow \ \Psi_{OUT}$:**
$\lambda = \infty$ is a fuchsian singularity of (27), with residue $-A_\infty/\lambda$. Therefore, we can always find a fundamental matrix solution with behavior:

$$\Psi_{OUT}^{Match} \ = \left[ I + O\left(\frac{1}{\lambda}\right)\right] \ \lambda^{-\frac{\theta_\infty}{2}\sigma_3}\lambda^{R_\infty}, \quad \lambda \to \infty.$$

This solution matches with $\Psi$. Also $\lambda = 1$ is a fuchsian singularity of (27). Therefore, we have:

$$\Psi_{OUT}^{Match} \ = \psi_1^{OUT}(x)\big[I + O(\lambda - 1)\big] \ (\lambda - 1)^{\frac{\theta_1}{2}\sigma_3}(\lambda - 1)^{R_1} C_1^{OUT}, \quad \lambda \to 1;$$

Here $C_1^{OUT}$ is a suitable connection matrix. $\psi_1^{OUT}(x)$ is the matrix that diagonalizes the leading terms of $A_1(x)$. Therefore, $\psi_1(x) \sim \psi_1^{OUT}(x)$ for $x \to 0$. As a consequence of isomonodromicity, $R_1$ is the same of $\Psi$.

As a consequence of the matching $\Psi \ \leftrightarrow \ \Psi_{OUT}^{Match}$, the monodromy of $\Psi$ at $\lambda = 1$ is:

$$M_1 = {C_1}^{-1} \exp\{i\pi\theta_1\sigma_3\} \exp\{2\pi i R_1\} C_1, \quad \text{with } C_1 \equiv C_1^{OUT}.$$

We finally need an invertible connection matrix $C_{OUT}$ to connect $\Psi_{OUT}^{Match}$ with the solution $\Psi_{OUT}$ appearing in (31). Namely, $\Psi_{OUT}^{Match} = \Psi_{OUT} C_{OUT}$.

**(*) Matching $\Psi \ \leftrightarrow \ \Psi_{IN}$:**

As a consequence of the matching $\Psi \ \leftrightarrow \ \Psi_{OUT}^{Match}$, we have to choose the IN-solution which matches with $\Psi_{OUT}^{Match}$. This is $\Psi_{IN}^{Match} := \Psi_{IN} C_{OUT}$.

Now, $\lambda = 0, x$ are fuchsian singularities of (28). Therefore:

$$\Psi_{IN}^{Match} = \begin{cases} \psi_0^{IN}(x)\big[I + O(\lambda)\big] \ \lambda^{\frac{\theta_0}{2}\sigma_3}\lambda^{R_0} C_0^{IN}, & \lambda \to 0; \\ \psi_x^{IN}(x)\big[I + O(\lambda - x)\big] \ (\lambda - x)^{\frac{\theta_x}{2}\sigma_3}(\lambda - x)^{R_x} C_x^{IN}, & \lambda \to x; \end{cases}$$

The above hold for fixed small $x \neq 0$. Here $C_0^{IN}$ and $C_x^{IN}$ are suitable connection matrices. $\psi_0^{IN}(x)$ and $\psi_x(x)^{IN}$ are diagonalizing matrices of the leading terms of $A_0(x)$ and $A_x(x)$. For $x \to 0$ they match with $\psi_0(x)$ and $\psi_x(x)$ of $\Psi$ in (36). On the other hand, as a consequence of isomonodromicity, the matrices $R_0$ and $R_x$ are the same of $\Psi$. The above $\Psi_{IN}^{Match}$ has the same behavior of $\Psi$ at $\lambda \to 0$ and $\lambda \to x$; moreover, it is an approximation of $\Psi$ for $x$ small. The matrices $C_0^{IN}$, $C_x^{IN}$ are independent of $x$. So, the matching $\Psi \ \leftrightarrow \ \Psi_{IN}$ is realized and the connection matrices $C_0$ and $C_x$ coincide with $C_0^{IN}$, $C_x^{IN}$ respectively. As a result, we obtain the monodromy matrices for $\Psi$:

$$M_0 = {C_0}^{-1} \exp\{i\pi\theta_0\sigma_3\} \exp\{2\pi i R_0\} C_0, \qquad C_0 \equiv C_0^{IN},$$

$$M_x = {C_x}^{-1} \exp\{i\pi\theta_x\sigma_3\} \exp\{2\pi i R_x\} C_x, \qquad C_x \equiv C_x^{IN}.$$

Our reduction is useful if the connection matrices $C_1^{OUT}$, $C_0^{IN}$, $C_x^{IN}$ can be computed explicitly. This is possible for the fuchsian reduced systems (29), (30). For non-fuchsian reduced systems, we discussed the computability in [14].

# 3 Classification in Terms of Monodromy Data

Two conjugated systems:

$$\frac{d\Psi}{d\lambda} = A(x,\lambda)\ \Psi, \quad \frac{d\tilde{\Psi}}{d\lambda} = \tilde{A}(x,\lambda)\ \tilde{\Psi},$$

$$\tilde{\Psi} = W\Psi, \qquad \det(W) \neq 0, \qquad \tilde{A} = WAW^{-1},$$

admit fundamental matrix solutions with the same monodromy matrices (w.r.t. the same basis of loops). The matrix $\tilde{A}(x,\lambda)$ defines the same solution of (PVI) associated to $A(x,\lambda)$ only if the following condition holds:

$$\tilde{A}_0 + \tilde{A}_1 + \tilde{A}_x = -\frac{\theta_\infty}{2}\sigma_3, \qquad \text{where } \tilde{A}_i = WA_iW^{-1}, \quad i = 0, x, 1.$$

Namely, $W\sigma_3W^{-1} = \sigma_3$. This occurs if and only if $W$ is *diagonal.* The transformation of $A(x,\lambda)$ is therefore:

$$WA(x,\lambda)W^{-1} = \begin{pmatrix} A_{11}(x,\lambda) & \frac{w_2}{w_1}A_{12}(x,\lambda) \\ \frac{w_1}{w_2}A_{21}(x,\lambda) & A_{22}(x,\lambda) \end{pmatrix}, \qquad \text{where } W = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}.$$

We conclude that the equation $A_{12}(x,\lambda) = 0$ is the same and then:

*Two conjugate fuchsian systems, satisfying (2) (3), define the same solution of PVI if and only if the conjugation is diagonal.*

Note that $\theta_\infty \neq 0$ is a necessary condition, otherwise any $W$ would be acceptable and then $A_{12}(x,\lambda) = 0$ would not define $y(x)$ uniquely.

The problem of finding a (branch of a) transcendent associated to a monodromy representation is the problem of finding a fuchsian system (1) having the given monodromy. This problem is called *Riemann-Hilbert problem*, or $21^{th}$ *Hilbert problem.* For a given PVI there is a one-to-one correspondence between a monodromy representation and a branch of a transcendent if and only if the Riemann-Hilbert problem has a unique solution $A(x,\lambda)$, defined up to diagonal conjugation.

• **Riemann-Hilbert problem (R.H.)**: find the coefficients $A_i(x)$, $i = 0, x, 1$ from the following *monodromy data*:

a) A fixed order of the poles $0, x, 1$. Namely, we choose a base of loops. Here we choose the order (1,2,3)=(0,$x$,1). See figure 1.

b) The exponents $\theta_0, \theta_x, \theta_1, \theta_\infty$, with $\theta_\infty \neq 0$.

c) Matrices $R_0, R_x, R_1, R_\infty$, such that:

$$R_\nu = 0 \text{ if } \theta_\nu \notin \mathbf{Z}, \qquad R_\nu = \begin{cases} \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, & \text{if } \theta_\nu > 0 \text{ integer} \\ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, & \text{if } \theta_\nu < 0 \text{ integer} \end{cases}$$

$$R_j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{if } \theta_j = 0, \quad j = 0, x, 1.$$

c) three monodromy matrices $M_0, M_x, M_1$ relative to the loops, similar to the matrices $\exp\{i\pi\theta_i\sigma_3\}\exp\{2\pi i R_i\}$, $i = 0, x, 1$, satisfying (for the chosen order of loops $\gamma_0\gamma_x\gamma_1 = \gamma_\infty$):

$$M_1\ M_x\ M_0 = e^{-i\pi\theta_\infty\sigma_3}e^{2\pi i R_\infty}$$

Solving the Riemann-Hilbert problem means that we have to find invertible *connection matrices*, $C_\nu$, $\nu = \infty, 0, x, 1$, such that.

$$C_j^{-1} e^{i\pi\theta_j\sigma_3} e^{2\pi i R_j} C_j = M_j, \qquad j = 0, x, 1; \tag{34}$$

$$C_\infty^{-1} e^{-i\pi\theta_\infty\sigma_3} e^{2\pi i R_\infty} C_\infty = e^{-i\pi\theta_\infty\sigma_3} e^{2\pi i R_\infty}. \tag{35}$$

and a matrix valued meromorphic function $\Psi(x,\lambda)$ such that:

$$\Psi(x,\lambda) = \begin{cases} \left[I + O\left(\frac{1}{\lambda}\right)\right]\ \lambda^{-\frac{\theta_\infty}{2}\sigma_3}\lambda^{R_\infty} C_\infty, & \lambda \to \infty; \\ \psi_0(x)\left[I + O(\lambda)\right]\ \lambda^{\frac{\theta_0}{2}\sigma_3}\lambda^{R_0} C_0, & \lambda \to 0; \\ \psi_x(x)\left[I + O(\lambda - x)\right]\ (\lambda - x)^{\frac{\theta_x}{2}\sigma_3}(\lambda - x)^{R_x} C_x, & \lambda \to x; \\ \psi_1(x)\left[I + O(\lambda - 1)\right]\ (\lambda - 1)^{\frac{\theta_1}{2}\sigma_3}(\lambda - 1)^{R_1} C_1, & \lambda \to 1; \end{cases} \tag{36}$$

Here $\psi_0$, $\psi_x$, $\psi_1$ are invertible matrices depending on $x$. The coefficient of the fuchsian system are then given by

$$A(x;\lambda) := \frac{d\Psi(x,\lambda)}{d\lambda}\Psi(x;\lambda)^{-1}.$$

A $2\times 2$ R.H. is always solvable at a fixed $x$ [1]. As a function of $x$, the solution $A(x;\lambda)$ extends to a meromorphic function on the universal covering of $\bar{\mathbf{C}}\backslash\{0,1,\infty\}$. Now we prove the following fact:

*The R.H. admits diagonally conjugated solutions (fuchsian systems), except when at least one* $\theta_\nu \in \mathbf{Z}\backslash\{0\}$ *and simultaneously* $R_\nu = 0$.

This can be equivalently stated in the form of the following:

**Proposition 1** *There is a one to one correspondence between the* monodromy data $\theta_0, \theta_x, \theta_1$, $R_0, R_x, R_1$, $\theta_\infty \neq 0$, $R_\infty$, $M_0, M_x, M_1$ *(defined up to conjugation), satisfying a), b), c) above, and a (branch of a) transcendent* $y(x)$, *except when at least one* $\theta_\nu \in \mathbf{Z}\backslash\{0\}$ *and simultaneously* $R_\nu = 0$.

To say in other words, the one to one correspondence is realized if and only if one of the following conditions is satisfied:

(1) $\theta_\nu \notin \mathbf{Z}$, for every $\nu = 0, x, 1, \infty$;

(2) if some $\theta_\nu \in \mathbf{Z}$ and $R_\nu \neq 0$, $\theta_\nu \neq 0$

(3) if some $\theta_j = 0$ $(j = 0, x, 1)$ and simultaneously $\theta_\infty \notin \mathbf{Z}$, or $\theta_\infty \in \mathbf{Z}$ and $R_\infty \neq 0$.

Note that for $\theta_j = 0$, $M_j$ can be put in Jordan form $\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$. Therefore Proposition 1 says that:

*There is one to one correspondence except when one of the matrices* $M_i$ $(i = 0, x, 1)$, *or* $M_\infty = M_1 M_x M_0$, *is equal to* $\pm I$.

*Proof:* The proof is based on the observation that a triple of monodromy matrices $M_0$, $M_x$, $M_1$ may be realized by two fuchsian systems which are not conjugated. The crucial point is that the solutions of (34), (35) are not unique. Two sets of particular solutions $C_\nu$ and $\tilde{C}_\nu(\nu = 0, x, 1, \infty)$ give to fuchsian systems:

$$\frac{d\Psi(x,\lambda)}{d\lambda}\Psi(x,\lambda)^{-1} = A(x,\lambda), \qquad \frac{d\tilde{\Psi}(x,\lambda)}{d\lambda}\tilde{\Psi}(x,\lambda)^{-1} = \tilde{A}(x,\lambda).$$

These may be not diagonally conjugated. If this happens, there is no one-to-one correspondence between a set of monodromy data and a solutions of PVI.

We study the structure of the solutions of (34), (35). Equation (35) has the following solutions:

i) If $\theta_\infty \notin \mathbf{Z}$ (and then $R_\infty = 0$),

$$C_\infty = \begin{pmatrix} p_\infty & 0 \\ 0 & q_\infty \end{pmatrix}, \quad p_\infty, q_\infty \in \mathbf{C}\backslash\{0\}$$

ii) If $\theta_\infty \in \mathbf{Z}$ and $R_\infty \neq 0$,

$$C_\infty = \begin{pmatrix} p_\infty & q_\infty \\ 0 & p_\infty \end{pmatrix}, \quad \text{if } R_\infty = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

$$C_\infty = \begin{pmatrix} p_\infty & 0 \\ q_\infty & p_\infty \end{pmatrix}, \quad \text{if } R_\infty = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}.$$

where $p_\infty, q_\infty \in \mathbf{C}$, $p_\infty \neq 0$.

iii) If $\theta_\infty \in \mathbf{Z}$ and $R_\infty = 0$, then $C_\infty$ is *any* invertible matrix.

Equation (34), may have different solutions $C_j$ and $\tilde{C}_j$. Therefore $C_j\tilde{C}_j^{-1}$ is a solution of:

$$\left(C_j\tilde{C}_j^{-1}\right)^{-1} e^{i\pi\theta_j\sigma_3}e^{2\pi i R_j}\; C_j\tilde{C}_j^{-1} = e^{i\pi\theta_j\sigma_3}e^{2\pi i R_j}.$$

i) If $\theta_j \notin \mathbf{Z}$ (and then $R_j = 0$), we have:

$$C_j\tilde{C}_j^{-1} = \begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix}, \qquad a_j, b_j \in \mathbf{C}\backslash\{0\}$$

ii) If $\theta_j \in \mathbf{Z}$ and $R_j \neq 0$, we have:

$$C_j\tilde{C}_j^{-1} = \begin{pmatrix} a_j & b_j \\ 0 & a_j \end{pmatrix}, \quad a_j, b_j \in \mathbf{C}, \;\; a_j \neq 0; \qquad \text{if } R_j = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

$$C_j\tilde{C}_j^{-1} = \begin{pmatrix} a_j & 0 \\ b_j & a_j \end{pmatrix}, \quad a_j, b_j \in \mathbf{C}, \;\; a_j \neq 0; \qquad \text{if } R_j = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}.$$

In particular, for $\theta_j = 0$ , $R_j$ is the Jordan form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

iii) If $\theta_j \in \mathbf{Z}$ and $R_j = 0$, then $C_j\tilde{C}_j^{-1}$ is any invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let $C_\nu$ and $\tilde{C}_\nu$ ($\nu = 0, x, 1, \infty$) be two sets of solutions of (34) (35) and let us denote by $\Psi$ and $\tilde{\Psi}$ the corresponding solutions of the R.H. We observe that:

i) for $\theta_j \notin \mathbf{Z}$ ($j = 0, x, 1$):

$$(\lambda - j)^{\frac{\theta_j}{2}\sigma_3} \begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix} = \begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix} (\lambda - j)^{\frac{\theta_j}{2}\sigma_3}.$$

ii) For $\theta_j \in \mathbf{Z}$ and $R_j \neq 0$:

$$(\lambda - j)^{\frac{\theta_j}{2}\sigma_3}(\lambda - j)^{R_j} \begin{pmatrix} a_j & b_j \\ 0 & a_j \end{pmatrix} = \left[a_j I + (\lambda - j)^{|\theta_j|} \begin{pmatrix} 0 & b_j \\ 0 & 0 \end{pmatrix}\right] (\lambda - j)^{\frac{\theta_j}{2}\sigma_3}(\lambda - j)^{R_j},$$

or

$$(\lambda - j)^{\frac{\theta_j}{2}\sigma_3}(\lambda - j)^{R_j} \begin{pmatrix} a_j & 0 \\ b_j & a_j \end{pmatrix} = \left[a_j I + (\lambda - j)^{|\theta_j|} \begin{pmatrix} 0 & 0 \\ b_j & 0 \end{pmatrix}\right] (\lambda - j)^{\frac{\theta_j}{2}\sigma_3}(\lambda - j)^{R_j},$$

for $R_j$ upper or lower triangular respectively.

iii) For $\theta_j \in \mathbf{Z}$ and $R_j = 0$:

$$(\lambda - j)^{\frac{\theta_j}{2}\sigma_3} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b\lambda^{\theta_j} \\ c\lambda^{-\theta_j} & d \end{pmatrix} \; (\lambda - j)^{\frac{\theta_j}{2}\sigma_3}$$

We conclude that, for $\lambda \to j$:

$$\Psi\tilde{\Psi}^{-1} \sim \begin{cases} \begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix}, & \text{if } \theta_j \notin \mathbf{Z}; \\ \begin{cases} a_j I, \text{ if } \theta_j \neq 0, \\ \begin{pmatrix} a_j & b_j \\ 0 & a_j \end{pmatrix}, \text{ if } \theta_j = 0, \end{cases} & \text{if } \theta_j \in \mathbf{Z},\ R_j \neq 0 \\ \begin{cases} \text{Arbitrary invert. matrix, if } \theta_j = 0, \\ \mathcal{C}\ (\lambda - j)^{-|\theta_j|} \to \infty,\ \text{otherwise}, \end{cases} & \text{if } \theta_j \in \mathbf{Z},\ R_j = 0 \end{cases}$$

The matrix $\mathcal{C}$ above is $\mathcal{C} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ or $\mathcal{C} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$.

Let $C_\infty$ and $\tilde{C}_\infty$ be two solutions of (35).

i) If $\theta_\infty \notin \mathbf{Z}$ (and then $R_\infty = 0$), we have

$$C_\infty \tilde{C}_\infty^{-1} = \begin{pmatrix} a_\infty & 0 \\ 0 & b_\infty \end{pmatrix}, \quad a_\infty, b_\infty \in \mathbf{C}\backslash\{0\}.$$

ii) If $\theta_\infty \in \mathbf{Z}$ and $R_\infty \neq 0$, we have

$$C_\infty \tilde{C}_\infty^{-1} = \begin{pmatrix} a_\infty & b_\infty \\ 0 & a_\infty \end{pmatrix}, \quad a_\infty, b_\infty \in \mathbf{C},\ \ a_\infty \neq 0; \qquad \text{if } R_\infty = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

$$C_\infty \tilde{C}_\infty^{-1} = \begin{pmatrix} a_\infty & 0 \\ b_\infty & a_\infty \end{pmatrix}, \quad a_\infty, b_\infty \in \mathbf{C},\ \ a_\infty \neq 0; \qquad \text{if } R_\infty = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}.$$

iii) If $\theta_\infty \in \mathbf{Z}$ and $R_\infty = 0$, then $C_\infty \tilde{C}_\infty^{-1}$ is *any* invertible matrix.

Therefore, for $\lambda \to \infty$ we have:

$$\Psi\tilde{\Psi}^{-1} \sim \begin{cases} \begin{pmatrix} a_\infty & 0 \\ 0 & b_\infty \end{pmatrix} & \text{if } \theta_\infty \notin \mathbf{Z}; \\ \left(I + O\left(\frac{1}{\lambda}\right)\right)\left(a_\infty I + \frac{b_\infty}{\lambda^{|\theta_\infty|}}\right) \to a_\infty I, & \text{if } \theta_\infty \in \mathbf{Z}\backslash\{0\},\ R_\infty \neq 0 \\ \mathcal{C}_\infty \lambda^{|\theta_\infty|} \to \infty, & \text{if } \theta_\infty \in \mathbf{Z}\backslash\{0\},\ R_\infty = 0 \end{cases}$$

The matrix $\mathcal{C}_\infty$ above is $\mathcal{C}_\infty = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ or $\mathcal{C}_\infty = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$.

From the above result we conclude that $\Psi\tilde{\Psi}^{-1}$ is analytic on $\bar{\mathbf{C}}$ and then it is a constant matrix $W$, except when at least one $\theta_\nu \in \mathbf{Z}\backslash\{0\}$ and simultaneously $R_\nu = 0$. Except for this case, we have:

$$\Psi = W\tilde{\Psi} \quad \Longrightarrow \quad \tilde{A}(x,\lambda) = WA(x,\lambda)W^{-1}.$$

We observe that: $W = \lim_{\lambda\to\infty} \Psi\tilde{\Psi}^{-1}$ (in the cases $\theta_\infty \notin \mathbf{Z}$, or for $\theta_\infty \in \mathbf{Z}$ ($\theta_\infty \neq 0$) and $R_\infty \neq 0$). Therefore $W$ is diagonal.

Proposition 1 is proved. □

# 4 Logarithmic asymptotics (6) and (7)

We consider cases when (1) can be reduced to the fuchsian systems (29) and (30). Let $\sigma$ be a complex number defined, up to sign, by:

$$\text{tr } (M_0 M_x) = 2\cos(\pi\sigma), \qquad |\Re\sigma| \leq 1.$$

In our paper [14], we computed all the asymptotic behaviors for $0 \leq \Re\sigma < 1$, as they can be obtained from the matching procedure when (29) and (30) are fuchsian. Among them, we obtained (6) and (7).

*Note:* For solutions with expansion:

$$y(x) = x(A_1 + B_1 \ln x + C_1 \ln^2 x + D_1 \ln^3 x + ...) + x^2(A_2 + B_2 \ln x + ...) + ..., \qquad x \to 0.$$

only the following cases are possible:

$$y(x) = \begin{cases} \frac{\theta_0}{\theta_0 \pm \theta_x} x + O(x^2) \quad \text{[Taylor expansion]}, \\ x\left(\frac{\theta_0^2 - B_1^2}{\theta_0^2 - \theta_x^2} + B_1 \ln x + \frac{\theta_x^2 - \theta_0^2}{4} \ln^2 x\right) + x^2(...) + ..., \\ x\ (A_1 \pm \theta_0 \ln x) + x^2(...) + ..., \qquad \text{and } \theta_0 = \pm\theta_x. \end{cases} \tag{37}$$

$A_1$ and $B_1$ are parameters. We see that the higher orders in (6) and (7) are $O(x^2 \ln^m x)$, for some integer $m > 0$.

## 4.1 Review of the Derivation of (6) and (7)

Let $x \to 0$. The reduction to the fuchsian systems (29) is possible if in the domain (23) we have:

$$|(A_0 + A_x)_{ij}| \gg \left|(A_x)_{ij}\ \frac{x}{\lambda}\right|, \qquad \text{namely:}\ \ |(A_0 + A_x)_{ij}| \gg \left|(A_x)_{ij}\ x^{1-\delta_{OUT}}\right|. \tag{38}$$

Let us denote with $\hat{A}_i$ the leading term of the matrix $A_i$, $i = 0, x, 1$. We can substitute (29) with:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{\hat{A}_0 + \hat{A}_x}{\lambda} + \frac{\hat{A}_1}{\lambda - 1}\right]\ \Psi_{OUT} \tag{39}$$

**Lemma 1** *If the approximation (29) is possible, then $\hat{A}_0 + \hat{A}_x$ has eigenvalues $\pm\frac{\sigma}{2} \in \mathbf{C}$ independent of $x$, defined (up to sign and addition of an integer) by* $\mathrm{tr}(M_x M_0) = 2\cos(\pi\sigma)$. *Let $r_1 \in \mathbf{C}$, $r_1 \neq 0$. For $\theta_\infty \neq 0$, the leading terms are:*

$$\hat{A}_1 = \begin{pmatrix} \frac{\sigma^2 - \theta_\infty^2 - \theta_1^2}{4\theta_\infty} & -r_1 \\ \frac{[\sigma^2 - (\theta_1 - \theta_\infty)^2][\sigma^2 - (\theta_1 + \theta_\infty)^2]}{16\theta_\infty^2}\ \frac{1}{r_1} & -\frac{\sigma^2 - \theta_\infty^2 - \theta_1^2}{4\theta_\infty} \end{pmatrix}, \tag{40}$$

*and*

$$\hat{A}_0 + \hat{A}_x = \begin{pmatrix} \frac{\theta_1^2 - \sigma^2 - \theta_\infty^2}{4\theta_\infty} & r_1 \\ -\frac{[\sigma^2 - (\theta_1 - \theta_\infty)^2][\sigma^2 - (\theta_1 + \theta_\infty)^2]}{16\theta_\infty^2}\ \frac{1}{r_1} & -\frac{\theta_1^2 - \sigma^2 - \theta_\infty^2}{4\theta_\infty} \end{pmatrix}. \tag{41}$$

*Proof:* Observe that $\mathrm{tr}(\hat{A}_0 + \hat{A}_x) = \mathrm{tr}(A_0 + A_x) = 0$, thus, for any $x$, $\hat{A}_0 + \hat{A}_x$ has eigenvalues of opposite sign, that we denote $\pm\tilde{\sigma}(x)/2$. Then, we recall that $x$ is a monodromy preserving deformation, therefore the monodromy matrices of (39) are independent of $x$. At $\lambda = 0, 1, \infty$ they are:

$$M_0^{OUT} = \begin{cases} M_x M_0 \\ M_0 M_x \end{cases}, \quad M_1^{OUT} = M_1, \quad M_\infty^{OUT} = M_\infty.$$

Thus, $\det(M_0^{OUT}) = 1$, because $\det(M_x)$=$\det(M_0) = 1$. Therefore, there exists a constant matrix $D$ and a complex constant number $\sigma$ such that:

$$D^{-1}\ M_0^{OUT}\ D\ = \begin{cases} \mathrm{diag}(\exp\{-i\pi\sigma\}, \exp\{i\pi\sigma\}), \\ \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}, \text{ or } \begin{pmatrix} \pm 1 & 0 \\ * & \pm 1 \end{pmatrix}, \quad \sigma \in \mathbf{Z} \end{cases}$$

We conclude that $\tilde{\sigma}(x) \equiv \sigma$. We also have $\mathrm{tr}(M_0^{OUT}) = 2\cos(\pi\sigma)$.

Now consider the gauge:

$$\Phi_1 := \lambda^{-\frac{\sigma}{2}}(\lambda-1)^{-\frac{\theta_1}{2}}\ \Psi_{OUT}. \qquad \frac{d\Phi_1}{d\lambda}=\left[\frac{\hat{A}_0+\hat{A}_x-\frac{\sigma}{2}}{\lambda}+\frac{\hat{A}_1-\frac{\theta_1}{2}}{\lambda-1}\right]\ \Phi_1 \tag{42}$$

We can identify $\hat{A}_0+\hat{A}_x-\frac{\sigma}{2}$ and $\hat{A}_1-\frac{\theta_1}{2}$ with $B_0$ and $B_1$ of Proposition 5 in Appendix 1, case (69), with $a=\frac{\theta_\infty}{2}+\frac{\theta_1}{2}+\frac{\sigma}{2}$, $b=-\frac{\theta_\infty}{2}+\frac{\theta_1}{2}+\frac{\sigma}{2}$, $c=\sigma$. $\Box$

In principle, $r_1$ may be a function of $x$. If the monodromy of system (39) depends on $r_1$, then $r_1$ is a constant independent of $x$. This is the case here.

For all the computations which follow, involving system (39) or (42), we note that the hypothesis $\theta_\infty\neq 0$ excludes cases (70), (71) and the Jordan cases (72)–(74).

The reduction to the fuchsian system (30) is possible for $x\to 0$ in the domain (25) if:

$$\left|\frac{(A_0)_{ij}}{\lambda}+\frac{(A_x)_{ij}}{\lambda-x}\right|\gg|(A_1)_{ij}|, \qquad \text{namely: } \left|\frac{(A_0+A_x)_{ij}}{x^{\delta_{IN}}}\right|\gg|(A_1)_{ij}|. \tag{43}$$

We can rewrite (30) using just the leading terms of the matrices:

$$\frac{d\Psi_{IN}}{d\lambda}=\left[\frac{\hat{A}_0}{\lambda}+\frac{\hat{A}_x}{\lambda-x}\right]\Psi_{IN}, \tag{44}$$

Then, we re-scale $\lambda$ and consider the following system:

$$\frac{d\Psi_{IN}}{d\mu}=\left(\frac{\hat{A}_0}{\mu}+\frac{\hat{A}_x}{\mu-1}\right)\Psi_{IN}, \qquad \mu:=\frac{\lambda}{x}$$

We know that there exists a matrix $K_0(x)$ such that:

$$K_0^{-1}(x)\ (\hat{A}_0+\hat{A}_x)\ K_0(x)=\begin{pmatrix}\frac{\sigma}{2} & 0\\ 0 & -\frac{\sigma}{2}\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

Let $\hat{\hat{A}}_i := K_0^{-1}\hat{A}_iK_0$, $i=0,x$. By a gauge transformation, we get the system:

$$\Psi_{IN}=:K_0(x)\ \Psi_0, \qquad \frac{d\Psi_0}{d\mu}=\left[\frac{\hat{\hat{A}}_0}{\mu}+\frac{\hat{\hat{A}}_x}{\mu-1}\right]\Psi_0, \tag{45}$$

*Important Remark* (see [14]): Conditions (38), (43) are satisfied if and only if $|\Re\sigma|<1$, $0<\delta_{IN}\leq\delta_{OUT}<1$.

## 4.2 Matching for $\sigma=0$. Proof of (6) and (7)

We suppose now $\sigma=0$.

### 4.2.1 Case $\theta_0\pm\theta_x\neq 0$. Proof of (6)

**Lemma 2** *Let $r_1\in\mathbf{C}$, $r_1\neq 0$. The matrices of system (39) are:*

$$\hat{A}_1=\begin{pmatrix}-\frac{\theta_\infty{}^2+\theta_1{}^2}{4\theta_\infty} & -r_1\\ \frac{[\theta_1^2-\theta_\infty^2]^2}{16\theta_\infty^2 r_1} & \frac{\theta_\infty{}^2+\theta_1{}^2}{4\theta_\infty}\end{pmatrix}, \quad \hat{A}_0+\hat{A}_x=\begin{pmatrix}\frac{\theta_1^2-\theta_\infty^2}{4\theta_\infty} & r_1\\ -\frac{[\theta_\infty^2-\theta_1^2]^2}{16\theta_\infty^2 r_1} & \frac{\theta_\infty^2-\theta_1^2}{4\theta_\infty}\end{pmatrix}, \qquad \forall r_1\neq 0.$$

*A fundamental matrix solution can be chosen with the following behavior at $\lambda=0$:*

$$\Psi_{OUT}(\lambda)=[G_0+O(\lambda)]\begin{pmatrix}1 & \log\lambda\\ 0 & 1\end{pmatrix}, \qquad G_0=\begin{pmatrix}1 & 0\\ \frac{\theta_\infty{}^2-\theta_1{}^2}{4\theta_\infty\ r_1} & \frac{1}{r_1}\end{pmatrix}.$$

*Proof:* The system (42) is:
$$\frac{d\Phi_1}{d\lambda}=\left[\frac{\hat{A}_0+\hat{A}_x}{\lambda}+\frac{\hat{A}_1-\frac{\theta_1}{2}}{\lambda-1}\right]\ \Phi_1,$$
We identify $\hat{A}_0+\hat{A}_x$ and $\hat{A}_1-\frac{\theta_1}{2}$ with $B_0$ and $B_1$ of proposition 5 in Appendix 1, diagonalizable case (69) (we recall that (70)–(74) never occur when $\theta_\infty\neq 0$) with $a=\frac{\theta_\infty}{2}+\frac{\theta_1}{2}$, $b=-\frac{\theta_\infty}{2}+\frac{\theta_1}{2}$, $c=0$.

The behavior of a fundamental solution is a standard result in the theory of Fuchsian systems. The matrix $G_0$ is defined by ${G_0}^{-1}\left(\hat{A}_0+\hat{A}_x\right)G_0=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. □

**Lemma 3** *Let $r\in\mathbf{C}$. The matrices of system (45) are:*
$$\hat{\hat{A}}_0=\begin{pmatrix}r+\frac{\theta_0}{2} & \frac{4\ r\ (r+\theta_0)}{\theta_x^2-\theta_0^2}\\ \frac{\theta_0^2-\theta_x^2}{4} & -r-\frac{\theta_0}{2}\end{pmatrix},\qquad \hat{\hat{A}}_x=\begin{pmatrix}-r-\frac{\theta_0}{2} & 1-\frac{4\ r\ (r+\theta_0)}{\theta_x^2-\theta_0^2}\\ \frac{\theta_x^2-\theta_0^2}{4} & r+\frac{\theta_0}{2}.\end{pmatrix}. \tag{46}$$
*There exist a fundamental solution of (45) with the following behavior at $\mu=\infty$:*
$$\Psi_0(\mu)=\left[I+O\left(\frac{1}{\mu}\right)\right]\ \begin{pmatrix}1&\log\mu\\0&1\end{pmatrix},\qquad \mu\to\infty.$$
*Proof:* We do a gauge transformation:
$$\Phi_0:=\mu^{-\frac{\theta_0}{2}}(\mu-1)^{-\frac{\theta_x}{2}}\ \Psi_0,\qquad \frac{d\Phi_0}{d\mu}=\left[\frac{\hat{\hat{A}}_0-\frac{\theta_0}{2}}{\mu}+\frac{\hat{\hat{A}}_x-\frac{\theta_x}{2}}{\mu-1}\right]\ \Phi_0. \tag{47}$$
We identify $\hat{\hat{A}}_0-\frac{\theta_0}{2}$, $\hat{\hat{A}}_x-\frac{\theta_x}{2}$ with $B_0$ and $B_1$ in the Appendix 1, Proposition 5, case (72), with parameters $a=\frac{\theta_0}{2}+\frac{\theta_x}{2}$, $c=\theta_0$. In particular,
$$\hat{\hat{A}}_0-\frac{\theta_0}{2}+\hat{\hat{A}}_x-\frac{\theta_x}{2}=\begin{pmatrix}-\frac{\theta_0+\theta_x}{2} & 1\\ 0 & -\frac{\theta_0+\theta_x}{2}\end{pmatrix} \tag{48}$$
Here the values of the parameters satisfy the conditions $a\neq 0$ and $a\neq c$, namely $\theta_0\pm\theta_x\neq 0$. From the matrices (72), we obtain $\hat{\hat{A}}_0=B_0+\theta_0/2$ and $\hat{\hat{A}}_x=B_1+\theta_x/2$. Keeping into account (48), by the standard theory of fuchsian systems we have:
$$\Phi_0(\mu)=\left[I+O\left(\frac{1}{\mu}\right)\right]\ \mu^{-\frac{\theta_0+\theta_x}{2}}\ \begin{pmatrix}1&\log\mu\\0&1\end{pmatrix},\quad \mu\to\infty.$$
This proves the behavior of $\Psi_0(\mu)$. □

If the monodromy of the system (45) depends on $r$, then $r$ is a constant independent of $x$. This is the case here.

The matching condition $\Psi_{OUT}(\lambda)\sim K_0(x)\ \Psi_0\left(\lambda/x\right)$ becomes:
$$K_0(x)\ \begin{pmatrix}1&\log\left(\frac{\lambda}{x}\right)\\0&1\end{pmatrix}\ \sim\ G_0\ \begin{pmatrix}1&\log\lambda\\0&1\end{pmatrix}\quad\Longrightarrow\quad K_0(x)\sim\begin{pmatrix}1&0\\ \frac{\theta_\infty^2-\theta_1^2}{4\ \theta_\infty\ r_1}&\frac{1}{r_1}\end{pmatrix}\ \begin{pmatrix}1&\log x\\0&1\end{pmatrix}.$$
From the above result, together with (46), we compute $\hat{A}_0=K_0\hat{\hat{A}}_0{K_0}^{-1}$, $\hat{A}_1=K_0\hat{\hat{A}}_1{K_0}^{-1}$. For example,
$$\hat{A}_0=G_0\ \begin{pmatrix}r+\frac{\theta_0}{2}+\frac{\theta_0^2-\theta_x^2}{4}\log x & \frac{\theta_x^2-\theta_0^2}{4}\log^2 x\ -2\left(r+\frac{\theta_0}{2}\right)\log x\ +\frac{4\ r(r+\theta_0)}{\theta_x^2-\theta_0^2}\\ \frac{\theta_0^2-\theta_x^2}{4} & \frac{\theta_x^2-\theta_0^2}{4}\log x-\left(r+\frac{\theta_0}{2}\right)\end{pmatrix}\ {G_0}^{-1}.$$
A similar expression holds for $\hat{A}_x$. The leading terms of $y(x)$ are obtained from (4) with matrix entries $(\hat{A}_1)_{12}=-r_1$ and:
$$(\hat{A}_0)_{12}=r_1\ \left[\frac{\theta_x^2-\theta_0^2}{4}\log^2 x-2\left(r+\frac{\theta_0}{2}\right)\log x+\frac{4\ r(r+\theta_0)}{\theta_x^2-\theta_0^2}\right].$$

The result is:

$$y(x) \sim x \left[\frac{\theta_x^2-\theta_0^2}{4}\log^2 x - 2\left(r+\frac{\theta_0}{2}\right)\log x + \frac{4\, r(r+\theta_0)}{\theta_x^2-\theta_0^2}\right] \tag{49}$$

$$= x \left\{\frac{\theta_x^2-\theta_0^2}{4}\log^2 x - 2\left(r+\frac{\theta_0}{2}\right)\log x + \frac{4}{\theta_x^2-\theta_0^2}\left[\left(r+\frac{\theta_0}{2}\right)^2-\frac{\theta_0^2}{4}\right]\right\}.$$

The above is (6).

#### 4.2.2 Case $\theta_0 \pm \theta_x = 0$. Proof of (7)

We consider here the cases (73), (74) of Proposition 5 applied to the system (47).

*Case (73)* is the case $\sigma = 0$, $\theta_0 = -\theta_x$, with $a = 0$, $c = \theta_0$ in the system (47). From Proposition 5 we immediately have:

$$\hat{\hat{A}}_0 = \begin{pmatrix} \frac{\theta_0}{2} & r \\ 0 & -\frac{\theta_0}{2} \end{pmatrix}, \quad \hat{\hat{A}}_x = \begin{pmatrix} \frac{\theta_x}{2} & 1-r \\ 0 & -\frac{\theta_x}{2} \end{pmatrix}.$$

The behavior of $\Psi_0$ and $\Psi_{OUT}$, and the matching are the same of subsection 4.2.1. We obtain the same $K_0(x)$. Therefore:

$$(\hat{A}_0)_{12} = r_1\ (r - \theta_0\ \ln x), \quad (\hat{A}_1)_{12} = -r_1.$$

This gives the leading terms:

$$y(x) \sim x(r - \theta_0\ \ln x) = x(r + \theta_x\ \ln x). \tag{50}$$

In the same way, we treat the other cases. *Case (73)* with $a = c$, is the case $\sigma = 0$, $\theta_0 = \theta_x$. As above, we find $y(x) \sim x(r - \theta_0\ \ln x) = x(r - \theta_x\ \ln x)$. *Case (74)* with $a = 0$, is the case $\sigma = 0$, $\theta_0 = -\theta_x$. We find $y(x) \sim x(r + \theta_0\ \ln x) = x(r - \theta_x\ \ln x)$. *Case (74)* with $a = c$, is the case $\sigma = 0$, $\theta_0 = \theta_x$. We find $y(x) \sim x(r + \theta_0\ \ln x) = x(r + \theta_x\ \ln x)$.

Both (49) and (50) contain more than one term, and in principle only the leading one is certainly correct. To prove that they are all correct, we observe that (49) and (50) can be obtained also by direct substitution of $y(x) = x(A_1 + B_1 \ln x + C_1 \ln^2 x + D_1 \ln^3 x + ...) + x^2(A_2 + B_2 \ln x + ...) + ...$ into (PVI). We can recursively determine the coefficients by identifying the same powers of $x$ and $\ln x$. As a result we obtain only the five cases (37), which include (49) and (50).

The reader can verify that conditions (38), (43) are satisfied.

## 5 Monodromy Data associated to the solution (6)

In this section, we compute the monodromy data for the solution (6) in the generic case $\theta_\nu \notin \mathbf{Z}$ for any $\nu = 0, x, 1, \infty$. We need some notations. Let $\gamma_E$ denote the Euler's constant. Let:

$$\psi_E(x) = \frac{d \ln \Gamma(x)}{dx}, \qquad x \neq 0, -1, -2, -3, ....$$

In particular, $\psi_E(1) = -\gamma_E$.

**Proposition 2** *Let $\theta_0, \theta_x, \theta_1, \theta_\infty \notin \mathbf{Z}$. The monodromy group associated to (6) is generated by:*

$$M_0 = EC_{0\infty}^{(*)}\ \exp\{i\pi\theta_0\sigma_3\}\ \left[EC_{0\infty}^{(*)}\right]^{-1},$$

$$M_x = EC_{0\infty}^{(*)}\ {C_{01}^{(*)}}^{-1}\ \exp\{i\pi\theta_x\sigma_3\}\ C_{01}^{(*)}\ \left[EC_{0\infty}^{(*)}\right]^{-1},$$

$$M_1 = BC_{01}^{-1}\ \exp\{i\pi\theta_1\sigma_3\}\ C_{01}B^{-1}.$$

*The matrices above are:*

$$E = \begin{pmatrix} \frac{4q}{\theta_x^2-\theta_0^2} & \frac{4}{\theta_0^2-\theta_x^2} \\ \frac{4}{\theta_x^2-\theta_0^2} & 0 \end{pmatrix}$$

$$q=-4i\pi\epsilon+$$
$$+\frac{1}{\theta_0^2-\theta_x^2}\left\{4r+2(\theta_0-\theta_x)+(\theta_x^2-\theta_0^2)\left[\psi\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}\right)+\psi\left(\frac{\theta_x}{2}-\frac{\theta_0}{2}+1\right)+2\gamma_E\right]\right\},$$

*where* $\epsilon=\pm1$.

$$C_{0\infty}^{(*)}=\begin{pmatrix}-\dfrac{e^{i\pi\epsilon\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}\right)}\Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}\right)\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}\right)} & -\dfrac{e^{i\pi\epsilon\left(\frac{\theta_x}{2}-\frac{\theta_0}{2}\right)}\Gamma(1-\theta_0)}{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}\right)\Gamma\left(\frac{\theta_x}{2}-\frac{\theta_0}{2}\right)}\\ \dfrac{e^{i\pi\epsilon\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}\right)}\pi\sin\pi\theta_0\ \Gamma(1+\theta_0)}{\sin\pi\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}\right)\sin\pi\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}\right)\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}\right)\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}\right)} & 0\end{pmatrix}$$

$$C_{01}^{(*)}=\begin{pmatrix}\dfrac{\Gamma(-\theta_x)\Gamma(1+\theta_0)}{\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}\right)\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}\right)^2} & \dfrac{\Gamma(-\theta_x)\Gamma(1-\theta_0)}{\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}\right)\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}\right)^2}\\ \dfrac{\Gamma(\theta_x)\Gamma(1+\theta_0)}{\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}\right)\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}\right)^2} & \dfrac{\Gamma(\theta_x)\Gamma(1-\theta_0)}{\left(-\frac{\theta_0}{2}+\frac{\theta_x}{2}\right)\Gamma\left(-\frac{\theta_0}{2}+\frac{\theta_x}{2}\right)^2}\end{pmatrix},$$

$$C_{01}=\begin{pmatrix}\dfrac{\Gamma(-\theta_1)}{\Gamma\left(1-\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)\Gamma\left(\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)} & -\dfrac{\Gamma\left(1+\frac{\theta_1}{2}-\frac{\theta_\infty}{2}\right)\Gamma\left(\frac{\theta_\infty}{2}+\frac{\theta_1}{2}\right)}{\Gamma(1+\theta_1)}\\ \dfrac{\Gamma(\theta_1)}{\Gamma\left(1+\frac{\theta_1}{2}-\frac{\theta_\infty}{2}\right)\Gamma\left(\frac{\theta_\infty}{2}+\frac{\theta_1}{2}\right)} & 0\end{pmatrix},$$

$$B=\begin{pmatrix}1&\omega\\0&1\end{pmatrix},\qquad \omega:=\psi_E\left(\frac{\theta_\infty}{2}+\frac{\theta_1}{2}\right)-\psi_E\left(\frac{\theta_1}{2}-\frac{\theta_\infty}{2}+1\right)+2\gamma_E,$$

*With the above choice, we have:*

$$M_1M_xM_0=C_{OUT}\ \exp\{-i\pi\theta_\infty\sigma_3\}\ C_{OUT}^{-1},$$

*where:*

$$C_{OUT}=BC_{0\infty}^{-1}D^{-1},\qquad D=\begin{pmatrix}1&0\\0&\frac{1-\theta_\infty}{r_1}\end{pmatrix},\qquad C_{0\infty}=\begin{pmatrix}1&-\dfrac{\pi\ e^{-i\frac{\pi}{2}(\theta_1+\theta_\infty)}}{\sin\frac{\pi}{2}(\theta_1+\theta_\infty)}\\ 1&-\dfrac{\pi\ e^{-i\frac{\pi}{2}(\theta_1-\theta_\infty)}}{\sin\frac{\pi}{2}(\theta_1-\theta_\infty)}\end{pmatrix}$$

*We also note that* $\mathrm{tr}(M_0M_x)=2$.

If we compute $\mathrm{tr}(M_0M_1)$ and $\mathrm{tr}(M_1M_x)$ we find two quadratic polynomials of $q$. Then, $q$ can be derived as a function $\mathrm{tr}(M_0M_1)$ and $\mathrm{tr}(M_1M_x)$. In this way we obtain

$$r=r(\theta_0,\theta_x,\theta_1,\theta_\infty,\mathrm{tr}(M_0M_1),\mathrm{tr}(M_1M_x))\tag{51}$$

We omit the long formula which results. Direct computation shows also that $\mathrm{tr}(M_0M_1)$ and $\mathrm{tr}(M_1M_x)$ depend on $\epsilon$ only through $q$. Therefore, different choices of $\epsilon$ just change the branch of (6), because they change $4r/(\theta_0^2-\theta_x^2)$ of $8\pi i$.

## 5.1 Derivation of Proposition 2

The matching $\Psi_{OUT}\leftrightarrow\Psi_{IN}$ has been realized by:

$$\Psi_{OUT}(x,\lambda)=[G_0+O(\lambda)]\ \begin{pmatrix}1&\log\lambda\\0&1\end{pmatrix},\qquad G_0=\begin{pmatrix}1&0\\ \frac{{\theta_\infty}^2-{\theta_1}^2}{4\theta_\infty\ r_1}&\frac{1}{r_1}\end{pmatrix}.$$

$$\Psi_{IN}(x,\lambda)=K_0(x)\Psi_0\left(\frac{\lambda}{x}\right),\qquad \Psi_0(\mu)=\left[I+O\left(\frac{1}{\mu}\right)\right]\ \begin{pmatrix}1&\log\mu\\0&1\end{pmatrix},\quad \mu\to\infty.$$

**MATCHING** $\Psi\leftrightarrow\Psi_{OUT}$**.**

The correct choice of $\Psi_{OUT}^{Match}$ must match with:

$$\Psi=\left[I+O\left(\frac{1}{\lambda}\right)\right]\lambda^{-\frac{\theta_\infty}{2}\sigma_3},\quad \lambda\to\infty.$$

System (42) is (69) of Appendix 1, with:

$$a=\frac{\theta_\infty}{2}+\frac{\theta_1}{2},\quad b=-\frac{\theta_\infty}{2}+\frac{\theta_1}{2},\quad c=0.$$

If we write:

$$\Psi_{OUT}=(\lambda-1)^{\frac{\theta_1}{2}}\begin{pmatrix}\varphi_1 & \varphi_2\\ \xi_1 & \xi_2\end{pmatrix},$$

then $\varphi_1$ and $\varphi_2$ are independent solutions of the hypergeometric equation (75):

$$\lambda(1-\lambda)\,\frac{d^2\varphi}{d\lambda^2}+\big(1+c-(a+[b+1]+1)\,\lambda\big)\,\frac{d\varphi}{d\lambda}-a(b+1)\,\varphi=0,$$

while $\xi_i$ are given by (76):

$$\xi_i=\frac{1}{r}\left[\lambda(1-\lambda)\,\frac{d\varphi_i}{d\lambda}\;-a\left(\lambda+\frac{b-c}{a-b}\right)\,\varphi_i\right],\quad i=1,2.$$

We need a complete set of solutions at $\lambda=0,1,\infty$.

We explain some preliminary facts. Let us consider a Gauss hypergeometric equation in standard form:

$$z\,(1-z)\,\frac{d^2\varphi}{dz^2}+\big[\gamma_0-(\alpha_0+\beta_0+1)\,z\big]\,\frac{d\varphi}{dz}-\alpha_0\beta_0\,\varphi=0 \tag{52}$$

($\alpha_0,\beta_0,\gamma_0$ here are not the coefficients of (PVI)! We are just using the same symbols only here). We refer to the paper by N.E. Norlund [22] in order to choose three sets of two independent solutions which can be easily expanded in series at $z=0,1,\infty$ respectively. Solutions with logarithmic or polynomial behaviors at $z=0$ may occur when $\gamma_0\in\mathbf{Z}$. The role of $\gamma_0$ at $z=1$ and $z=\infty$ is played by $\alpha_0+\beta_0-\gamma_0+1$ and $\alpha_0-\beta_0+1$ respectively. Therefore, solutions with logarithmic or polynomial behaviors at $z=1$ may occur when $\alpha_0+\beta_0-\gamma_0+1\in\mathbf{Z}$, at $z=\infty$ when $\alpha_0-\beta_0+1\in\mathbf{Z}$. Some more words must be said about the choice of independent solutions. We consider the point $z=0$.

For $\gamma_0\not\in\mathbf{Z}$, we choose the following two independent solutions:

$$\varphi_1(z)=F(\alpha_0,\beta_0,\gamma_0;\ z),\qquad \varphi_2(z)=z^{1-\gamma_0}F(\alpha',\beta',\gamma';\ z).$$

Here $F$ is the standard hypergeometric function and $\alpha'=\alpha_0-\gamma_0+1$, $\beta'=\beta_0-\gamma_0+1$, $\gamma'=2-\gamma_0$.

If $\gamma_0=0,-1,-2,...$, then:

$$\varphi_1(z)=f(\alpha_0,\beta_0,\gamma_0;\ z),\quad \varphi_2(z)=z^{1-\gamma_0}F(\alpha',\beta',\gamma';\ z),\qquad \text{if } \alpha_0 \text{ or } \beta_0=0,-1,...,\gamma.$$

$$\varphi_1(z)=z^{1-\gamma_0}\mathcal{G}(\alpha',\beta',\gamma';\ z),\quad \varphi_2(z)=z^{1-\gamma_0}F(\alpha',\beta',\gamma';\ z),\qquad \text{if } \alpha_0 \text{ and } \beta_0\neq 0,-1,...,\gamma.$$

Here $f$ is the truncation of $F$ at the order $z^{-\gamma}$. $\mathcal{G}$ is one of the functions $g$, $g_1$, $g_0$ or $G$ with logarithmic behavior, introduced in [22], section 2. They are listed in Appendix 3.

If $\gamma_0=2,3,...$, then:

$$\varphi_1(z)=F(\alpha_0,\beta_0,\gamma_0;\ z),\quad \varphi_2(z)=z^{1-\gamma_0}f(\alpha',\beta',\gamma';\ z),\qquad \text{if } \alpha_0 \text{ or } \beta_0=1,2,...,\gamma-1.$$

$$\varphi_1(z)=F(\alpha_0,\beta_0,\gamma_0;\ z),\quad \varphi_2(z)=\mathcal{G}(\alpha_0,\beta_0,\gamma_0;\ z),\qquad \text{if } \alpha_0 \text{ and } \beta_0\neq 1,2,...,\gamma-1.$$

If $\gamma_0=1$, then:

$$\varphi_1(z)=F(\alpha_0,\beta_0,\gamma_0;\ z),\quad \varphi_2(z)=\mathcal{G}(\alpha_0,\beta_0,\gamma_0;\ z).$$

The point $z=1$ is treated in the same way, with the substitution:

$$\alpha_0\mapsto\alpha_0,\quad \beta_0\mapsto\beta_0,\quad \gamma_0\mapsto\alpha_0+\beta_0-\gamma_0+1;\quad \varphi\mapsto\varphi,\quad z\mapsto 1-z.$$

The point $z=\infty$ is treated in the same way, with the substitution:

$$\alpha_0\mapsto\alpha_0,\quad \beta_0\mapsto\alpha_0-\gamma_0+1,\quad \gamma_0\mapsto\alpha_0-\beta_0+1;\quad \varphi\mapsto z^{-\alpha_0}\varphi,\quad z\mapsto\frac{1}{z}.$$

In our case:

$$\alpha_0=a=\frac{\theta_\infty}{2}+\frac{\theta_1}{2},\quad \beta_0=b+1=\frac{\theta_1}{2}-\frac{\theta_\infty}{2}+1,\quad \gamma_0=c+1=1,\qquad z=\lambda.$$

Because $\gamma_0 = 1$, we have a logarithmic solution at $\lambda = 0$. As for $\lambda = 1$, $\alpha_0 + \beta_0 - \gamma_0 + 1 = 1 + \theta_1$ and for $\lambda = \infty$, $\alpha_0 - \beta_0 + 1 = \theta_\infty$. We suppose $\theta_1$ and $\theta_\infty \notin \mathbf{Z}$. We choose the following set of independent solutions at $\lambda = 0, 1, \infty$ respectively (the upper label indicates the singularity):

$$\begin{cases} \varphi_1^{(0)} = F(\alpha_0, \beta_0, \gamma_0; \lambda), \\ \varphi_2^{(0)} = g(\alpha_0, \beta_0, \gamma_0; \lambda); \end{cases}$$

$$\begin{cases} \varphi_1^{(1)} = F(\alpha_0, \beta_0, \alpha_0 + \beta_0 - \gamma_0 + 1; 1 - \lambda), \\ \varphi_2^{(1)} = (1-\lambda)^{\gamma_0 - \alpha_0 - \beta_0} F(\gamma_0 - \alpha_0, \gamma_0 - \beta_0, \gamma_0 - \alpha_0 - \beta_0 + 1; 1 - \lambda); \end{cases}$$

$$\begin{cases} \varphi_1^{(\infty)} = \lambda^{-\alpha_0} F(\alpha_0, \alpha_0 - \gamma_0 + 1, 1 + \alpha_0 - \beta_0; \lambda^{-1}), \\ \varphi_2^{(\infty)} = \lambda^{-\beta_0} F(\beta_0, \beta_0 - \gamma_0 + 1, 1 - \alpha_0 + \beta_0; \lambda^{-1}); \end{cases}$$

Let:

$$\Psi_{OUT}^{(i)} = (\lambda - 1)^{\frac{\theta_1}{2}} \begin{pmatrix} \varphi_1^{(i)} & \varphi_2^{(i)} \\ \xi_1^{(i)} & \xi_2^{(i)} \end{pmatrix}, \quad i = 0, 1, \infty.$$

From Norlund, 3.(1) and 3.(2) we get:

$$\Psi_{OUT}^{(0)} = \Psi_{OUT}^{(1)} C_{01}, \quad |\arg \lambda| < \pi, \quad |\arg(1-\lambda)| < \pi,$$

where $C_{01}$ is written in Proposition 2. From Norlund, 10.(1) and 10.(3) we obtain:

$$\Psi_{OUT}^{(0)} = \Psi_{OUT}^{(\infty)} C_{0\infty}, \quad 0 < \arg z < \pi,$$

where $C_{0\infty}$ is written in Proposition 2.

• *Note about the computation:* In order to apply the formulae of Norlund, 10.(1) and 10.(3) we have to transform $g$ into $g_1$, using the formula (see Norlund, formula (24)):

$$g(\alpha, \beta, \gamma; z) = g_1(\alpha, \beta, \gamma; z) - \frac{\pi}{\sin \pi\alpha} e^{i\pi\epsilon\alpha} F(\alpha, \beta, \gamma; z), \tag{53}$$

where $\epsilon$ is an integer introduced as follows. $g(\alpha, \beta, \gamma; z)$ is defined for $|\arg(z)| < \pi$, while $g_1(\alpha, \beta, \gamma; z)$ is defined for $|\arg(-z)| < \pi$. Moreover, $-z = e^{i\epsilon\pi} z$. In $g(\alpha, \beta, \gamma; z)$, $\ln(z)$ is negative for $0 < z < 1$ (namely, $\arg(z) = 0$), while in $g_1(\alpha, \beta, \gamma; z)$, $\ln(-z)$ is negative for $-1 < z < 0$. Namely, for $-1 < z < 0$, we have $\arg(z) = -\pi\epsilon$. Formula (53) holds true for $0 < \arg z < \pi$ when $\epsilon = -1$, and for $-\pi < \arg z < 0$ when $\epsilon = 1$.

In the formulae of Norlund, 10.(1) and 10.(3) it is required that $|\mathrm{agr}(-z)| < \pi$, namely $|\arg(e^{i\epsilon\pi} z)| < \pi$. This limitation must be restricted to $0 < \arg z < \pi$ when $\epsilon = -1$, and for $-\pi < \arg z < 0$ when $\epsilon = 1$ in order to apply (53).

In our computations we have chosen $0 < \arg z < \pi$ (i.e. $\epsilon = -1$), because this is the choice which gives the order $M_1 M_x M_0 = \exp\{-i\pi\theta_\infty\sigma_3\}$. The choice $-\pi < \arg z < 0$ ($\epsilon = 1$) gives $M_x M_1 M_0 = \exp\{-i\pi\theta_\infty\sigma_3\}$.

We expand $\varphi_1^{(0)}$, $\varphi_2^{(0)}$ in series at $\lambda = 0$ and we get:

$$\Psi_{OUT}^{(0)} = G_0 \big[I + O(\lambda)\big] \begin{pmatrix} 1 & \ln\lambda \\ 0 & 1 \end{pmatrix} B e^{i\frac{\pi}{2}\theta_1}, \quad \lambda \to 0,$$

where $B$ is written in Proposition 2. Namely:

$$\Psi_{OUT}^{(0)} = \Psi_{OUT}\ B e^{i\frac{\pi}{2}\theta_1}$$

We expand $\varphi_1^{(\infty)}$, $\varphi_2^{(\infty)}$ in series at $\lambda = \infty$, obtaining:

$$\Psi_{OUT}^{(\infty)} = \left[I + O\left(\frac{1}{\lambda}\right)\right] \lambda^{-\frac{\theta_\infty}{2}\sigma_3}\ D, \quad \lambda \to \infty,$$

where $D$ is written in Proposition 2. Namely,

$$\Psi_{OUT}^{(\infty)} = \Psi_{OUT}^{Match}\ D.$$

Combining the above results we get:

$$\begin{aligned}\Psi_{OUT}^{Match} &= \Psi_{OUT}^{(\infty)} D^{-1} \\ &= \Psi_{OUT}^{(0)} C_{0\infty}^{-1} D^{-1} \\ &= \Psi_{OUT}\ B C_{0\infty}^{-1} D^{-1} e^{i\frac{\pi}{2}\theta_1} \quad \equiv \Psi_{OUT} C_{OUT}.\end{aligned}$$

The matrix $BC_{0\infty}^{-1}D^{-1}e^{i\frac{\pi}{2}\theta_1}$ is $C_{OUT}$. It differs from the matrix $C_{OUT}$ of proposition 2 by the factor $e^{i\frac{\pi}{2}\theta_1}$, which simplifies in the formulae. We also have:

$$\Psi_{OUT}^{Match} = \Psi_{OUT}^{(1)} C_{01} C_{0\infty}^{-1} D^{-1}.$$

Finally, it is an elementary computation to see that

$$\Psi_{OUT}^{(1)} = (\lambda-1)^{\frac{\theta_1}{2}} \begin{pmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} \\ \xi_1^{(1)} & \xi_2^{(1)} \end{pmatrix} \mapsto \Psi_{OUT}^{(1)} e^{i\pi\theta_1\sigma_3}, \qquad \text{when } \lambda-1 \mapsto (\lambda-1)e^{2\pi i}.$$

Thus, a choice for the matrix $M_1$ of (1) is

$$\begin{aligned}M_1 \equiv M_1^{OUT} &= D C_{0\infty} C_{01}^{-1}\ e^{i\pi\theta_1\sigma_3}\ C_{01} C_{0\infty}^{-1} D^{-1}, \\ &= C_{OUT}^{-1} \big[ B C_{01}^{-1}\ e^{i\pi\theta_1\sigma_3}\ C_{01} B^{-1} \big] C_{OUT}.\end{aligned}$$

**MATCHING $\Psi \leftrightarrow \Psi_{IN}$**

The system:

$$\Phi_0 := \mu^{-\frac{\theta_0}{2}} (\mu-1)^{-\frac{\theta_x}{2}}\ \Psi_0, \qquad \frac{d\Phi_0}{d\mu} = \left[ \frac{\hat{\hat{A}}_0 - \frac{\theta_0}{2}}{\mu} + \frac{\hat{\hat{A}}_x - \frac{\theta_x}{2}}{\mu-1} \right]\ \Phi_0.$$

is (72) of Appendix 1, with:

$$a = \frac{\theta_0}{2} + \frac{\theta_x}{2}, \quad c = \theta_0.$$

The equation for $\xi$ is in Gauss hypergeometic form (77):

$$\mu(\mu-1)\frac{d^2\xi}{d\mu^2} + \big(1 + c - 2(a+1)\mu\big)\frac{d\xi}{d\mu} - a(a+1)\xi = 0, \tag{54}$$

while $\varphi$ is given by (78):

$$\varphi(\mu) = \frac{1}{a(a-c)} \left[ \mu(\mu-1)\frac{d\xi}{d\mu} + (a\mu - c - r)\xi \right].$$

In the standard form

$$\mu\ (1-\mu)\ \frac{d^2\xi}{d\mu^2} + \big[\gamma_0 - (\alpha_0+\beta_0+1)\ \mu\big]\ \frac{d\xi}{d\mu} - \alpha_0\beta_0\ \xi = 0, \tag{55}$$

we have:

$$\alpha_0 = a = \frac{\theta_0}{2} + \frac{\theta_x}{2}, \quad \beta_0 = a+1 = \frac{\theta_0}{2} + \frac{\theta_x}{2} + 1, \quad \gamma_0 = c+1 = \theta_0 + 1; \qquad z = \mu.$$

Therefore $\gamma_0 = 1 + \theta_0$, $\alpha_0 + \beta_0 - \gamma_0 + 1 = 1 + \theta_x$, $\alpha_0 - \beta_0 + 1 = 0$, and (54) has no logarithmic solutions at $\mu = 0, 1$ if $\theta_0, \theta_1 \notin \mathbf{Z}$. On the other hand, at $\mu = \infty$ we may have a solution with logarithmic or polynomial behavior.

For $\theta_0, \theta_x \notin \mathbf{Z}$, we choose the following independent solutions at $\mu = 0, 1, \infty$ respectively::

$$\begin{cases} \xi_1^{(0)} = F(\alpha_0, \beta_0, \gamma_0;\ \mu) \\ \xi_2^{(0)} = \mu^{1-\gamma_0} F(\alpha_0 - \gamma_0 + 1, \beta_0 - \gamma_0 + 1, 2 - \gamma_0;\ \mu); \end{cases}$$

$$\begin{cases} \xi_1^{(1)} = F(\alpha_0, \beta_0, \alpha_0+\beta_0-\gamma_0+1;\ 1-\mu) \\ \xi_2^{(1)} = (1-\mu)^{\gamma_0-\alpha_0-\beta_0} F(\gamma_0-\beta_0, \gamma_0-\alpha_0, 1+\gamma_0-\alpha_0-\beta_0;\ 1-\mu); \end{cases}$$

$$\begin{cases} \xi_1^{(\infty)} = \mu^{-\beta_0} g_1(\beta_0, 1-\gamma_0+\beta_0, 1-\alpha_0+\beta_0;\ \mu^{-1}) \\ \xi_2^{(\infty)} = \mu^{-\beta_0} F(\beta_0, 1-\gamma_0+\beta_0, 1-\alpha_0+\beta_0;\ \mu^{-1}); \end{cases}$$

Let us construct three fundamental matrices form the above three sets of independent solutions:

$$\Psi_0^{(i)} := \mu^{\frac{\theta_0}{2}} (\mu-1)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(i)} & \varphi_2^{(i)} \\ \xi_1^{(i)} & \xi_2^{(i)} \end{pmatrix}, \quad i = 0, 1, \infty$$

The connection formulae between solutions at $\mu = 0$ and 1 is a standard one, and can be found in any book on special functions:

$$\Psi_0^{(0)} = \Psi_0^{(1)} C_{01}^{(*)}, \qquad |\arg(\mu)| < \pi, \quad |\arg(1-\mu)| < \pi$$

where $C_{01}^{(*)}$ is given in the statement of Proposition 2. The connection formulae between solutions at $\mu = 0$ and $\mu = \infty$ can be found in Norlund [22], formulae 9.(1) and 9.(5) (case $m = 1$). We get:

$$\Psi_0^{(0)} = \Psi_0^{(\infty)} C_{0\infty}^{(*)}, \qquad |\arg(-\mu)| < \pi,$$

where $C_{0\infty}^{(*)}$ can be read in Proposition 2 and $-\mu = e^{-i\pi\epsilon}\mu$ (when $\mu < 0$, $\arg(\mu) = \pi\epsilon$).

• *Note about the computation:* In order to apply the formulae 9.(1) and 9.(5) of Norlund, we have made use of the formula:

$$g_1(\alpha, \beta, \gamma; z) = g_1(\beta, \alpha, \gamma; z) + \frac{\pi \sin \pi(\beta-\alpha)}{\sin \pi\beta \ \sin \pi\alpha} \ F(\alpha, \beta, \gamma; z).$$

We expand $\xi_1^{(\infty)}$, $\xi_2^{(\infty)}$, $\varphi_1^{(\infty)}$, $\varphi_2^{(\infty)}$ for $\mu \to \infty$. We obtain:

$$\Psi_0^{(\infty)} = \left[ I + \left(\frac{1}{\mu}\right) \right] \begin{pmatrix} 1 & \ln \mu \\ 0 & 1 \end{pmatrix} \ E, \qquad \mu \to \infty$$

where $E$ can be read in Proposition 2. Thus,

$$\Psi_0^{(\infty)} = \Psi_0 \ E,$$

where $\Psi_0$ is the matrix used in the matching $\Psi_{OUT} \leftrightarrow \Psi_{IN}$. Expanding $\xi_1^{(0)}$, $\xi_2^{(0)}$, $\varphi_1^{(0)}$, $\varphi_2^{(0)}$ for $\mu \to 0$ we get:

$$\Psi_0^{(0)} = (\mu-1)^{-\frac{\theta_x}{2}} \begin{pmatrix} \frac{4(\theta_0+r)}{\theta_0^2-\theta_x^2} & \frac{4r}{\theta_0^2-\theta_x^2} \\ 1 & 1 \end{pmatrix} \ [1+O(\mu)] \ \mu^{\frac{\theta_0}{2}\sigma_3}, \qquad \mu \to 0.$$

Expanding $\xi_1^{(1)}$, $\xi_2^{(1)}$, $\varphi_1^{(1)}$, $\varphi_2^{(1)}$ for $\mu \to 1$ we get:

$$\Psi_0^{(1)} = \begin{pmatrix} \frac{2(\theta_0-\theta_x+2r)}{\theta_0^2-\theta_x^2} & \frac{2(\theta_0+\theta_x+2r)}{\theta_0^2-\theta_x^2} \\ 1 & 1 \end{pmatrix} [1+O(1-\mu)](1-\mu)^{\frac{\theta_x}{2}\sigma_3}, \qquad \mu \to 1.$$

The above imply that:

$$\Psi_0^{(0)} \mapsto \Psi_0^{(0)} e^{i\pi\theta_0\sigma_3}, \qquad \text{for } \mu \mapsto \mu e^{2\pi i},$$

$$\Psi_0^{(1)} \mapsto \Psi_0^{(1)} e^{i\pi\theta_x\sigma_3}, \qquad \text{for } \mu-1 \mapsto (\mu-1)e^{2\pi i}.$$

Finally, we observe that:

$$\Psi_{IN}^{Match} = \Psi_{IN} C_{OUT},$$

$$\Psi_{IN} = K_0(x)\Psi_0 = K_0(x) \ \Psi_0^{(\infty)} \ E^{-1} = \begin{cases} K_0(x) \ \Psi_0^{(0)} \ {C_{0\infty}^{(*)}}^{-1} E^{-1} \\ K_0(x) \ \Psi_1^{(0)} \ C_{01}^{(*)} {C_{0\infty}^{(*)}}^{-1} E^{-1} \end{cases}$$

As a result of the matching procedure we get:

$$M_0 \equiv M_0^{IN} = C_{OUT}^{-1}\ \left[E C_{0\infty}^{(*)}\ e^{i\pi\theta_0\sigma_3}\ {C_{0\infty}^{(*)}}^{-1} E^{-1}\right]\ C_{OUT},$$

$$M_x \equiv M_1^{IN} = C_{OUT}^{-1}\ \left[E C_{0\infty}^{(*)} {C_{01}^{(*)}}^{-1}\ e^{i\pi\theta_x\sigma_3}\ C_{01}^{(*)} {C_{0\infty}^{(*)}}^{-1} E^{-1}\right]\ C_{OUT}.$$

□

When we come to the computation of the traces, we find:

$$\mathrm{tr}(M_0M_1) = \mathbf{a}q^2 + (\mathbf{b} - 2\mathbf{a}\omega)q + (\mathbf{c} - \mathbf{b}\omega + \mathbf{a}\omega^2),$$

$$\mathrm{tr}(M_1M_x) = \mathbf{A}q^2 + (\mathbf{B} - 2\mathbf{A}\omega)q + (\mathbf{C} - \mathbf{B}\omega + \mathbf{A}\omega^2),$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ , $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are complicated long trigonometric expressions in sines and cosines of the parameters $\pi\theta_\nu,\ \nu = 0, x, 1, \infty$. We omit to write them. The above form for the system which determines $q$ (and therefore $r$) implies that:

$$q = 2\omega + \big\{ \text{ solution of the system for } \omega = 0\big\}.$$

Moreover:

$$\big\{ \text{ solution of the system for } \omega = 0\big\} = \frac{\mathbf{a}\big(\mathbf{C} - \mathrm{tr}(M_1M_x)\big) - \mathbf{A}\big(\mathbf{c} - \mathrm{tr}(M_0M_1)\big)}{\mathbf{Ab} - \mathbf{aB}}$$

$$\equiv \frac{\mathbf{b}\big(\mathbf{C} - \mathrm{tr}(M_1M_x)\big) - \mathbf{B}\big(\mathbf{c} - \mathrm{tr}(M_0M_1)\big)}{\mathbf{a}\big(\mathbf{C} - \mathrm{tr}(M_1M_x)\big) - \mathbf{A}\big(\mathbf{c} - \mathrm{tr}(M_0M_1)\big)}$$

We omit all the explicit expressions.

# 6 Monodromy Data associated to the Solution (7)

**Proposition 3** **[1].** *The monodromy group associated to the solution (7):*

$$y(x) \sim x(r + \theta_0\ \ln x),$$

*is generated by:*

$$M_0 = E\ \exp\{-i\pi\theta_0\sigma_3\}\ E^{-1}, \qquad M_x = EU^{-1}\ \exp\{i\pi\theta_x\sigma_3\}\ UE^{-1},$$

$$M_1 = BC_{01}^{-1}\ \exp\{i\pi\theta_1\sigma_3\}\ C_{01}B^{-1};$$

*where $B$, $C_{01}$ are given in Proposition 2 and:*

$$E := \begin{pmatrix} e^{-i\frac{\pi}{2}\theta_0} & \frac{r}{\theta_0} - \Psi_E(\theta_0+1) - \gamma_E - i\pi \\ 0 & e^{i\frac{\pi}{2}\theta_0} \end{pmatrix}, \qquad U := \begin{pmatrix} 1 & -\Gamma(\theta_0+1)\Gamma(-\theta_0) \\ 0 & 1 \end{pmatrix}.$$

*Conversely, the parameter $r$ is:*

$$\frac{r}{\theta_0} = -\frac{\pi}{4}\ \frac{\mathrm{tr}(M_0M_1)}{\sin\pi\theta_0 \sin\frac{\pi}{2}(\theta_\infty+\theta_1)\sin\frac{\pi}{2}(\theta_\infty-\theta_1)} + (\Psi_E(\theta_0+1) + i\pi + \gamma_E) +$$

$$+\frac{\pi}{2}\ \frac{\cos\pi(\theta_0+\theta_1)}{\sin\pi\theta_0 \sin\frac{\pi}{2}(\theta_\infty+\theta_1)\sin\frac{\pi}{2}(\theta_\infty-\theta_1)} - \frac{\omega}{2}\ \frac{\left[\cos\pi(\theta_0+\theta_1) - \cos\pi(\theta_0-\theta_1)\right]}{\sin\pi\theta_0\sin\pi\theta_1}. \qquad (56)$$

*$\omega$ is given in Proposition 2.*

**[2].** *The monodromy group and $r$ for the solution (7):*

$$y(x) \sim x(r - \theta_0\ \ln x),$$

*are obtained from the results in* **[1]**, *with the substitution $\theta_0 \mapsto -\theta_0$.*

*Proof:* For the matching $\Psi_{OUT} \leftrightarrow \Psi_{IN}$ and $\Psi \leftrightarrow \Psi_{OUT}$, we proceed as in the proof of Proposition 2.

**MATCHING** $\Psi \leftrightarrow \Psi_{IN}$

Consider the case $\theta_0 = \theta_x$. For this case, the system for $\Phi_0$ can be chosen to be (73) or (74), with $a = c = \theta_0$. Here we refer to system (74). Therefore, a fundamental solution is (see Proposition 6):

$$\Psi_0^{(0)} := \mu^{\frac{\theta_0}{2}} (\mu-1)^{\frac{\theta_0}{2}} \Phi_0 =$$

$$= e^{i\frac{\pi}{2}\theta_0} \begin{pmatrix} \mu^{-\frac{\theta_0}{2}}(1-\mu)^{\frac{\theta_0}{2}} & \frac{\pi}{\theta_0}(1-\mu)^{-\frac{\theta_0}{2}}\mu^{\frac{\theta_0}{2}} - \frac{1}{\theta_0+1}\mu^{\frac{\theta_0}{2}+1}(1-\mu)^{\frac{\theta_0}{2}} \; F(1+\theta_0, 1+\theta_0, 2+\theta_0; \mu) \\ 0 & \mu^{\frac{\theta_0}{2}}(1-\mu)^{-\frac{\theta_0}{2}} \end{pmatrix}.$$

Here, the branch is: $(\mu-1) = e^{i\pi}(1-\mu)$. When $\mu \to \infty$, we write the hypergeometric function as follows, using the connection formula 9.(1) in Norlund [22]:

$$F(1+\theta_0, 1+\theta_0, 2+\theta_0; \mu) = e^{i\pi\theta_0}(\theta_0+1)\mu^{-1-\theta_0} g_1\left(0, 1+\theta_0, 1; \frac{1}{\mu}\right), \quad 0 < \arg\mu < 2\pi.$$

Here, we have used the branch $-\mu = e^{-i\pi}\mu$. The function $g_1$ is:

$$g_1\left(0, 1+\theta_0, 1; \frac{1}{\mu}\right) = \Psi_E(1+\theta_0) + \gamma_E + i\pi - \ln\mu + \sum_{\nu=1}^{\infty} \frac{(1+\theta_0)_\nu}{\nu \; \nu!}\mu^{-\nu}, \qquad \mu \to \infty.$$

From the above, we obtain:

$$\Psi_0^{(0)} = \left[1 + \left(\frac{1}{\mu}\right)\right] \begin{pmatrix} 1 & \ln\mu \\ 0 & 1 \end{pmatrix} E e^{i\frac{\pi}{2}\theta_0} \equiv \Psi_0 \; E e^{i\frac{\pi}{2}\theta_0}.$$

Here, $\Psi_0$ is the matrix used in the matching $\Psi_{OUT} \leftrightarrow \Psi_{IN}$ and $E$ is in the statement of the proposition. When $\mu \to 1$, we use the connection formula:

$$F(1+\theta_0, 1+\theta_0, 2+\theta_0; \mu) =$$

$$= \Gamma(-\theta_0)\Gamma(2+\theta_0)F(1+\theta_0, 1+\theta_0, 1+\theta_0; 1-\mu) + \frac{\Gamma(\theta_0)\Gamma(2+\theta_0)}{\Gamma(1+\theta_0)^2}(1-\mu)^{-\theta_0}F(1,1,1-\theta_0; 1-\mu).$$

Therefore,

$$\Psi_0^{(0)} = e^{i\frac{\pi}{2}\theta_0}(I + O(1-\mu)) \begin{pmatrix} 1 & \frac{r}{\theta_0} - \frac{\Gamma(\theta_0)\Gamma(\theta_0+2)}{(\theta_0+1)\Gamma(\theta_0+1)^2} \\ 0 & 1 \end{pmatrix} (1-\mu)^{\frac{\theta_0}{2}\sigma_3} U, \qquad \mu \to 1.$$

Finally, when $\mu \to 0$,we have:

$$\Psi_0^{(0)} = e^{i\frac{\pi}{2}\theta_0}(1 + O(\mu)) \begin{pmatrix} 1 & r/\theta_0 \\ 0 & 1 \end{pmatrix} \mu^{-\frac{\theta_0}{2}\sigma_3}.$$

Let $C_{OUT}$ be the same matrix introduced in the proof of Proposition 2. We have:

$$\Psi_{IN}^{Match} = \Psi_{IN} C_{OUT} = K_0(x)\Psi_0 C_{OUT} = K_0(x)\Psi_0^{(0)} E^{-1} C_{OUT}.$$

This implies that:

$$M_x = C_{OUT}^{-1} \; E U^{-1} \; \exp\{i\pi\theta_x\sigma_3\} \; U E^{-1} \; C_{OUT},$$
$$M_0 = C_{OUT}^{-1} \; E \; \exp\{-i\pi\theta_0\sigma_3\} \; E^{-1} \; C_{OUT}.$$

The matrix $C_{OUT}$ has been simplified in the statement of the proposition.

The proof for $\theta_0 = -\theta_x$ is analogous (for example, it is the case (73) with $a = 0$, $c = \theta_0$). $\square$

# 7 Monodromy Data for the Non-generic Case (9)

We consider the non-generic case

$$\theta_0 = 2p, \quad p \in \mathbf{Z}, \quad \theta_0 \neq 0, \qquad \theta_1 = \theta_x = 0, \quad \theta_\infty = 1.$$

In this case, the solutions (6) becomes (9). We show here that the solutions (9) are not in one to one correspondence with a set of monodromy data. Namely, to a given set of *monodromy data*, as defined in Proposition 1, there corresponds a one parameter family (9), where $r$ is a free parameter (i.e. $r$ is not a function of the traces of the product of the monodromy matrices).

We miss the one-to-one correspondence because the conditions in Proposition 1 are not realized. Namely, the matrix $R_0$ associated to (9) is:

$$R_0 = 0, \qquad \text{while } \theta_0 \in \mathbf{Z} \text{ and } \theta_0 \neq 0.$$

This fact is contained in the following Proposition.

**Proposition 4** *The monodromy group associated to (9) is generated by:*

$$M_0 = I, \qquad M_x = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 1 - \frac{8i}{\pi} \ln 2 & -\frac{32i}{\pi} (\ln 2)^2 \\ \frac{2i}{\pi} & 1 + \frac{8\ i}{\pi} \ln 2 \end{pmatrix}.$$

*In particular,*

$$\mathrm{tr}(M_0 M_x) = \mathrm{tr}(M_0 M_1) = 2, \qquad \mathrm{tr}(M_1 M_x) = -2$$

*The monodromy is independent of the parameter $r$ in (9).*

*Note:* With the above choice the monodrmy at infinity: $M_1 M_x$ ( or $M_x M_1$) is not in standard Jordan form. Namely:

$$M_\infty^+ = M_1 M_x = \begin{pmatrix} 1 - \frac{8i}{\pi} \ln 2 & -\frac{2i}{\pi} (4 \ln 2 + i\pi)^2 \\ \frac{2i}{\pi} & -3 + \frac{8i}{\pi} \ln 2 \end{pmatrix}, \quad M_\infty^- = M_x M_1 = \begin{pmatrix} -3 - \frac{8i}{\pi} \ln 2 & \frac{2i}{\pi} (4i \ln 2 + \pi)^2 \\ \frac{2i}{\pi} & 1 + \frac{8i}{\pi} \ln 2 \end{pmatrix}$$

They can be put in Jordan form respectively by the following matrices:

$$C_{OUT}^+ = \begin{pmatrix} 1 - \frac{4i}{\pi} \ln 2 & -\frac{16i}{\pi} (\ln 2)^2 r_1 \\ \frac{i}{\pi} & \left(1 + \frac{4i}{\pi} \ln 2\right) r_1 \end{pmatrix}, \qquad C_{OUT}^- = \begin{pmatrix} 1 + \frac{4i}{\pi} \ln 2 & \frac{16i}{\pi} (\ln 2)^2 r_1 \\ -\frac{i}{\pi} & \left(1 - \frac{4i}{\pi} \ln 2\right) r_1 \end{pmatrix}, \qquad r_1 \in \mathbf{C}.$$

We obtain:

$${C_{OUT}^+}^{-1} M_\infty^+ C_{OUT}^+ = \begin{pmatrix} -1 & 2\pi i\ r_1 \\ 0 & -1 \end{pmatrix}, \qquad {C_{OUT}^-}^{-1} M_\infty^- C_{OUT}^- = \begin{pmatrix} -1 & 2\pi i\ r_1 \\ 0 & -1 \end{pmatrix}.$$

On the other hand:

$${C_{OUT}^+}^{-1} M_1 C_{OUT}^+ = {C_{OUT}^-}^{-1} M_1 C_{OUT}^- = \begin{pmatrix} 1 - \frac{8i}{\pi} \ln 2 & -\frac{32i}{\pi} (\ln 2)^2\ r_1 \\ \frac{2i}{\pi\ r_1} & 1 + \frac{8i}{\pi} \ln 2 \end{pmatrix},$$

$${C_{OUT}^+}^{-1} M_x C_{OUT}^+ = \begin{pmatrix} -1 - \frac{8i}{\pi} \ln 2 & \frac{2i}{\pi} (4i \ln 2 + \pi)^2\ r_1 \\ \frac{2i}{\pi\ r_1} & 3 + \frac{8i}{\pi} \ln 2 \end{pmatrix},$$

$${C_{OUT}^-}^{-1} M_x C_{OUT}^- = \begin{pmatrix} 3 - \frac{8i}{\pi} \ln 2 & \frac{2i}{\pi} (4i \ln 2 - \pi)^2\ r_1 \\ \frac{2i}{\pi\ r_1} & -1 + \frac{8i}{\pi} \ln 2 \end{pmatrix}$$

## 7.1 Derivation of Proposition 4

The matching $\Psi_{OUT} \leftrightarrow \Psi_{IN}$ has been realized by

$$\Psi_{OUT}(x,\lambda) = [G_0 + O(\lambda)] \begin{pmatrix} 1 & \log\lambda \\ 0 & 1 \end{pmatrix}, \qquad G_0 = \begin{pmatrix} 1 & 0 \\ \frac{1}{4r_1} & \frac{1}{r_1} \end{pmatrix}.$$

$$\Psi_{IN}(x,\lambda) = K_0(x)\Psi_0\left(\frac{\lambda}{x}\right), \qquad \Psi_0(\mu) = \left[I + O\left(\frac{1}{\mu}\right)\right] \begin{pmatrix} 1 & \log\mu \\ 0 & 1 \end{pmatrix}, \quad \mu \to \infty.$$

**MATCHING $\Psi \leftrightarrow \Psi_{OUT}$.**

The correct choice of $\Psi_{OUT}^{Match}$ must match with:

$$\Psi = \left[I + O\left(\frac{1}{\lambda}\right)\right] \lambda^{-\frac{1}{2}\sigma_3}\lambda^{R_\infty}, \quad R_\infty = \begin{pmatrix} 0 & -r_1 \\ 0 & 0 \end{pmatrix}, \qquad \lambda \to \infty.$$

System (42) is (69) of Appendix 1, with:

$$a = \frac{1}{2}, \quad b = -\frac{1}{2}, \quad c = 0.$$

If we write:

$$\Psi_{OUT} = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \xi_1 & \xi_2 \end{pmatrix},$$

then $\varphi_1$ and $\varphi_2$ are independent solutions of the hypergeometric equation (75):

$$\lambda(1-\lambda)\,\frac{d^2\varphi}{d\lambda^2} + \big(1 + c - (a + [b+1] + 1)\ \lambda\big)\,\frac{d\varphi}{d\lambda} - a(b+1)\ \varphi = 0,$$

and

$$\xi_i = \frac{1}{r}\left[\lambda(1-\lambda)\,\frac{d\varphi_i}{d\lambda}\ - a\left(\lambda + \frac{b-c}{a-b}\right)\ \varphi_i\right], \quad i = 1,2.$$

We need a complete set of solutions at $\lambda = 0, 1, \infty$. In the standard Gauss hypergeometric form (52) we have $\alpha_0 = \beta_0 = 1/2$, $\gamma_0 = 1$. Since $\gamma_0 = 1$, $\alpha_0 + \beta_0 - \gamma_0 + 1 = 1$ and $\alpha_0 - \beta_0 + 1 = 1$, we expect solutions with logarithmic behaviors at $\lambda = 0, 1, \infty$. We choose three sets of independent solutions:

$$\begin{cases} \varphi_1^{(0)} = F(\alpha_0,\beta_0,\gamma_0;\lambda) \equiv F\left(\frac{1}{2},\frac{1}{2},1;\lambda\right), \\ \varphi_1^{(0)} = g(\alpha_0,\beta_0,\gamma_0;\lambda) \equiv g\left(\frac{1}{2},\frac{1}{2},1;\lambda\right); \end{cases}$$

$$\begin{cases} \varphi_1^{(1)} = F(\alpha_0,\beta_0,\alpha_0+\beta_0-\gamma_0+1;1-\lambda) \equiv F\left(\frac{1}{2},\frac{1}{2},1;1-\lambda\right), \\ \varphi_1^{(1)} = g(\alpha_0,\beta_0,\alpha_0+\beta_0-\gamma_0+1;1-\lambda) \equiv g\left(\frac{1}{2},\frac{1}{2},1;1-\lambda\right); \end{cases}$$

$$\begin{cases} \varphi_1^{(\infty)} = \lambda^{-\beta_0}F(\beta_0,\beta_0-\gamma_0+1,\beta_0-\alpha_0+1;\lambda^{-1}) \equiv \lambda^{-\frac{1}{2}}F\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{\lambda}\right), \\ \varphi_1^{(\infty)} = \lambda^{-\beta_0}g(\beta_0,\beta_0-\gamma_0+1,\beta_0-\alpha_0+1;\lambda^{-1}) \equiv \lambda^{-\frac{1}{2}}g\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{\lambda}\right); \end{cases}$$

Let

$$\Psi_{OUT}^{(i)} = \begin{pmatrix} \varphi_1^{(i)} & \varphi_2^{(i)} \\ \xi_1^{(i)} & \xi_2^{(i)} \end{pmatrix},$$

From Norlund, formulae 5.(1) and 5.(2) we get:

$$\Psi_{OUT}^{(0)} = \Psi_{OUT}^{(1)}C_{01}, \qquad C_{01} = \begin{pmatrix} 0 & -\pi \\ -\frac{1}{\pi} & 0 \end{pmatrix}; \qquad |\arg\lambda| < \pi, \quad |\arg(1-\lambda)| < \pi.$$

From Norlund, formulae 12.(1) and 12.(3) we get:

$$\Psi_{OUT}^{(0)} = \Psi_{OUT}^{(\infty)}C_{0\infty}, \qquad C_{0\infty} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{\pi}e^{i\frac{\pi}{2}\epsilon} & 1 \end{pmatrix};$$

$$0 < \arg \lambda < \pi \quad (\epsilon = 1), \qquad -\pi < \arg \lambda < 0 \quad (\epsilon = -1).$$

• *Note on the computation:* In order to apply 12.(1) we need:

$$g_1\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{1}{\lambda}\right) = g\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{1}{\lambda}\right) + \pi e^{i\frac{\pi}{2}\epsilon} F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{1}{\lambda}\right)$$

$$0 < \arg \lambda < \pi \quad (\epsilon = 1), \qquad -\pi < \arg \lambda < 0 \quad (\epsilon = -1).$$

$\epsilon$ appears in the computations when we express: $-\lambda = e^{-i\pi\epsilon}\lambda$.

We expand the solutions for $\lambda \to 0$ and we get:

$$\Psi_{OUT}^{(0)} = G_0(1 + O(\lambda)) \begin{pmatrix} 1 & \ln \lambda \\ 0 & 1 \end{pmatrix} B, \qquad B = \begin{pmatrix} 1 & -4 \ln 2 \\ 0 & 1 \end{pmatrix}, \qquad \lambda \to 0.$$

Namely,

$$\Psi_{OUT}^{(0)} = \Psi_{OUT} B.$$

Then expansion when $\lambda \to \infty$ yields::

$$\Psi_{OUT}^{(\infty)} = \left[ I + O\left(\frac{1}{\lambda}\right) \right] \lambda^{-\frac{1}{2}\sigma_3} \lambda^{R_\infty} D, \qquad \lambda \to \infty;$$

$$D = \begin{pmatrix} 1 & -\ln 16 \\ 0 & \frac{1}{r_1} \end{pmatrix}, \qquad R_\infty = \begin{pmatrix} 0 & -r_1 \\ 0 & 0 \end{pmatrix}.$$

Namely,

$$\Psi_{OUT}^{(\infty)} = \Psi_{OUT}^{Match} D.$$

From the above:

$$\begin{aligned} \Psi_{OUT}^{Match} &= \Psi_{OUT}^{(\infty)} D^{-1} \\ &= \Psi_{OUT}^{(0)} C_{0\infty}^{-1} D^{-1} \\ &\equiv \Psi_{OUT}\, C_{OUT}, \qquad \text{where} \quad C_{OUT} = B C_{0\infty}^{-1} D^{-1}. \end{aligned}$$

It is easy to see that:

$$\Psi_{OUT}^{(1)} \mapsto \Psi_{OUT}^{(1)} \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}, \qquad \text{when } \lambda - 1 \mapsto e^{2\pi i}\, (\lambda - 1).$$

This, together with the connection formulae

$$\begin{aligned} \Psi_{OUT}^{Match} &= \Psi_{OUT}^{(0)} C_{0\infty}^{-1} D^{-1}, \\ &= \Psi_{OUT}^{(1)} C_{01} C_{0\infty}^{-1} D^{-1}, \end{aligned}$$

yields:

$$\begin{aligned} M_1 \equiv M_1^{OUT} &= D C_{0\infty} C_{01}^{-1} \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} C_{01} C_{0\infty}^{-1} D^{-1} \\ &= C_{OUT}^{-1} B C_{01}^{-1} \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} C_{01} B^{-1} C_{OUT}. \end{aligned}$$

We have two choices for $C_{OUT}$, depending on $\epsilon = \pm 1$ in $C_{0\infty}$. These have been called $C_{OUT}^{+}$ and $C_{OUT}^{-}$ in the Note, after Proposition 4.

**MATCHING $\Psi \leftrightarrow \Psi_{IN}$**

The system:

$$\Phi_0 := \mu^{-p}\ \Psi_0, \qquad \frac{d\Phi_0}{d\mu} = \left[ \frac{\hat{\hat{A}}_0 - p}{\mu} + \frac{\hat{\hat{A}}_x}{\mu - 1} \right] \Phi_0.$$

is (72) of Appendix 1, with:

$$a = p, \quad c = 2p.$$

The equation for $\xi$ is in Gauss hypergeometic form (77):

$$\mu(\mu-1)\frac{d^2\xi}{d\mu^2}+\big(1+c-2(a+1)\mu\big)\frac{d\xi}{d\mu} - a(a+1)\xi = 0, \tag{57}$$

$$\varphi(\mu) = \frac{1}{a(a-c)}\left[\mu(\mu-1)\frac{d\xi}{d\mu} + (a\mu - c - r)\xi\right].$$

In the standard form (55), we have:

$$\alpha_0 = p, \quad \beta_0 = 1+p, \quad \gamma_0 = 1+2p; \qquad z = \mu.$$

Therefore $\gamma_0 = 1+2p$, $\alpha_0+\beta_0-\gamma_0+1 = 1$, $\alpha_0-\beta_0+1 = 0$, and (57) may have solutions with logarithmic or polynomial behaviors at $\mu = 0, 1, \infty$.

The choice of three sets of independent solutions requires a distinction of sub cases $p > 0$ and $p < 0$. As before, we denote:

$$\Psi_0^{(i)} = \mu^p \Phi_0^{(i)}, \qquad \Phi_0^{(i)} = \begin{pmatrix} \varphi_1^{(i)} & \varphi_2^{(i)} \\ \xi_1^{(i)} & \xi_2^{(i)} \end{pmatrix}, \quad i = 0, 1, \infty.$$

* *CASE* $p > 0$. We choose:

$$\begin{cases} \xi_1^{(0)} = F(\alpha_0, \beta_0, \gamma_0; \mu), \\ \xi_2^{(0)} = \mu^{1-\gamma_0} f(\alpha_0-\gamma_0+1, \beta_0-\gamma_0+1, 2-\gamma_0; \mu); \end{cases}$$

$$\begin{cases} \xi_1^{(1)} = F(\alpha_0, \beta_0, \alpha_0+\beta_0-\gamma_0+1; 1-\mu), \\ \xi_2^{(1)} = g(\alpha_0, \beta_0, \alpha_0+\beta_0-\gamma_0+1; 1-\mu); \end{cases}$$

$$\begin{cases} \xi_1^{(\infty)} = \mu^{-\beta_0} F(\beta_0, \beta_0-\gamma_0+1, \beta_0-\alpha_0+1; \mu^{-1}), \\ \xi_2^{(\infty)} = \mu^{-\beta_0} g_1(\beta_0-\gamma_0+1, \beta_0, \beta_0-\alpha_0+1; \mu^{-1}); \end{cases}$$

From Norlund, formulae 5.(1), 5.(7) we get:

$$\Psi_0^{(0)} = \Psi_0^{(1)} C_{01}^{(*)}, \quad |\arg(1-\lambda)| < \pi, \qquad C_{01}^{(*)} = \begin{pmatrix} 0 & \frac{p\Gamma(p)^2}{\Gamma(2p)} \\ -\frac{2\Gamma(2p)}{\Gamma(p)^2} & 0 \end{pmatrix}.$$

From Norlund, formulae 12.(1), 12.(6) we get:

$$\Psi_0^{(0)} = \Psi_0^{(\infty)} C_{0\infty}^{(*)}, \quad |\arg(-\mu)| < \pi, \qquad C_{0\infty}^{(*)} = (-1)^{p+1} \begin{pmatrix} 0 & \frac{p^2\Gamma(p)^2}{\Gamma(2p)} \\ \frac{2p\ \Gamma(2p)}{\Gamma(p)^2} & 0 \end{pmatrix},$$

where $-\mu = e^{-i\pi\eta}\mu$, $\eta = \pm 1$.

We compute the behavior of $\varphi_i^{(\infty)}$, $\xi_i^{(\infty)}$ $(i = 1, 2)$ for $\mu \to \infty$. In the computation, $\ln(-1/\mu)$ appears in $g_1$. We write $-1/\mu = e^{i\pi\eta}/\mu$, $\arg\mu = \eta\pi$ when $-\infty < \mu < 0$. The final result (after expanding in series):

$$\Psi_0^{(\infty)}(\mu) = \left[I + O\left(\frac{1}{\mu}\right)\right] \begin{pmatrix} 1 & \ln\mu \\ 0 & 1 \end{pmatrix} E, \quad \mu \to \infty, \qquad E = \begin{pmatrix} p^{-2} & Q_> \, p^{-2} \\ 0 & -p^{-2} \end{pmatrix},$$

$$Q_> = \psi_E(p) + \psi_E(p+1) + 2\gamma_E + i\pi\eta - \frac{p+r}{p^2}.$$

Namely,

$$\Psi_0^{(\infty)} = \Psi_0 E,$$

where $\Psi_0$ is the matrix for the matching $\Psi_{OUT} \leftrightarrow \Psi_{IN}$. Expanding $\varphi_i^{(0)}$, $\xi_i^{(0)}$ for $\mu \to 0$ we get:

$$\Psi_0^{(0)} = \begin{pmatrix} \frac{r+2p}{p^2} & \frac{r}{p^2} \\ 1 & 1 \end{pmatrix} \big[I + O(\mu)\big]\mu^{p\sigma_3}, \quad \mu \to 0.$$

Expanding $\varphi_i^{(1)}$, $\xi_i^{(1)}$ for $\mu \to 1$ we get:

$$\Psi_0^{(1)} = \begin{pmatrix} \frac{p+r}{p^2} & \frac{p+r}{p^2}(\psi_E(p) + \psi_E(p+1) + 2\gamma_E) - \frac{1}{p^2} \\ 1 & \psi_E(p) + \psi_E(p+1) + 2\gamma_E \end{pmatrix} \big[I + O(1-\mu)\big] \begin{pmatrix} 1 & \ln(1-\mu) \\ 0 & 1 \end{pmatrix}, \quad \mu \to 1.$$

* *CASE* $p < 0$. We choose:

$$\begin{cases} \xi_1^{(0)} = f(\alpha_0, \beta_0, \gamma_0; \mu), \\ \xi_2^{(0)} = \mu^{1-\gamma_0} F(\alpha_0 - \gamma_0 + 1, \beta_0 - \gamma_0 + 1, 2 - \gamma_0; \mu); \end{cases}$$

$$\begin{cases} \xi_1^{(1)} = F(\alpha_0, \beta_0, \alpha_0 + \beta_0 - \gamma_0 + 1; 1-\mu), \\ \xi_2^{(1)} = g_0(\alpha_0, \beta_0, \alpha_0 + \beta_0 - \gamma_0 + 1; 1-\mu); \end{cases}$$

$$\begin{cases} \xi_1^{(\infty)} = \mu^{-\beta_0} F(\beta_0, \beta_0 - \gamma_0 + 1, \beta_0 - \alpha_0 + 1; \mu^{-1}), \\ \xi_2^{(\infty)} = \mu^{-\beta_0} g_1(\beta_0, \beta_0 - \gamma_0 + 1, \beta_0 - \alpha_0 + 1; \mu^{-1}); \end{cases}$$

From Norlund, formulae 8.(6), 8.(11) we compute:

$$\Psi_0^{(0)} = \Psi_0^{(1)} C_{01}^{(*)}, \quad |\arg(1-\lambda)| < \pi, \qquad C_{01}^{(*)} = \begin{pmatrix} -\frac{p\Gamma(-p)^2}{\Gamma(-2p)} & 0 \\ 0 & -\frac{2\Gamma(-2p)}{\Gamma(-p^2)} \end{pmatrix}.$$

From Norlund, formulae 13.(1), 13.(6) we compute:

$$\Psi_0^{(0)} = \Psi_0^{(\infty)} C_{0\infty}^{(*)}, \quad |\arg(-\mu)| < \pi, \qquad C_{0\infty}^{(*)} = (1)^{p+1} \begin{pmatrix} \frac{p^2\Gamma(-p)^2}{\Gamma(-2p)} & 0 \\ 0 & -\frac{2p\ \Gamma(-2p)}{\Gamma(-p)^2} \end{pmatrix}.$$

We compute the behavior of $\varphi_i^{(\infty)}$, $\xi_i^{(\infty)}$ $(i = 1, 2)$ for $\mu \to \infty$. In the computation, $\ln(-1/\mu)$ appears in $g_1$. We write $-1/\mu = e^{i\pi\eta}/\mu$, $\arg \mu = \eta\pi$ when $-\infty < \mu < 0$. The final result (expanding in series):

$$\Psi_0^{(\infty)}(\mu) = \left[I + O\left(\frac{1}{\mu}\right)\right] \begin{pmatrix} 1 & \ln\mu \\ 0 & 1 \end{pmatrix} E, \quad \mu \to \infty, \qquad E = \begin{pmatrix} p^{-2} & Q_< \, p^{-2} \\ 0 & -p^{-2} \end{pmatrix},$$

$$Q_< = \psi_E(-p) + \psi_E(-p+1) + 2\gamma_E + i\pi\eta - \frac{p+r}{p^2}.$$

Namely,

$$\Psi_0^{(\infty)} = \Psi_0 E,$$

where $\Psi_0$ is the matrix for the matching $\Psi_{OUT} \leftrightarrow \Psi_{IN}$. Expanding $\varphi_i^{(0)}$, $\xi_i^{(0)}$ for $\mu \to 0$ we get:

$$\Psi_0^{(0)} = \begin{pmatrix} \frac{r+2p}{p^2} & \frac{r}{p^2} \\ 1 & 1 \end{pmatrix} \big[I + O(\mu)\big]\mu^{p\sigma_3}, \quad \mu \to 0.$$

Expanding $\varphi_i^{(1)}$, $\xi_i^{(1)}$ for $\mu \to 1$ we get:

$$\Psi_0^{(1)} = \begin{pmatrix} \frac{p+r}{p^2} & \frac{p+r}{p^2}(\psi_E(-p) + \psi_E(1-p) + 2\gamma_E) - \frac{1}{p^2} \\ 1 & \psi_E(-p) + \psi_E(1-p) + 2\gamma_E \end{pmatrix} \big[I + O(1-\mu)\big] \begin{pmatrix} 1 & \ln(1-\mu) \\ 0 & 1 \end{pmatrix}, \quad \mu \to 1.$$

* Both for $p > 0$ and $p < 0$ we have:

$$\begin{aligned} \Psi_{IN} = K_0(x)\Psi_0 &= K_0(x)\Psi_0^{(\infty)} E^{-1}, \\ &= K_0(x)\Psi_0^{(0)} {C_{0\infty}^{(*)}}^{-1} E^{-1}, \\ &= K_0(x)\Psi_0^{(1)} C_{01}^{(*)} {C_{0\infty}^{(*)}}^{-1} E^{-1}; \end{aligned}$$

together with $\Psi_{IN}^{Match}=\Psi_{IN}C_{OUT}$. We conclude that the monodromy of (1) is:

$$M_0\equiv M_0^{IN}=I,\qquad M_x\equiv M_1^{IN}=C_{OUT}^{-1}\left[EC_{0\infty}^{(*)}{C_{01}^{(*)}}^{-1}\begin{pmatrix}1&2\pi i\\0&1\end{pmatrix}C_{01}^{(*)}{C_{0\infty}^{(*)}}^{-1}E^{-1}\right]C_{OUT}.$$

The connection matrices $E$, $C_{0\infty}^{(*)}$, $C_{01}^{(*)}$ have different form for $p>0$ and for $p<0$. We also have two choices for $C_{OUT}$, depending on $\epsilon=\pm1$ in $C_{0\infty}$. These have been called $C_{OUT}^{+}$ and $C_{OUT}^{-}$ in the comments just after Proposition 4. Multiplying by $C_{OUT}$ and $C_{OUT}^{-1}$ to the left and right respectively we get three generators for the monodromy group:

$$M_0=I,\qquad M_1=BC_{01}^{-1}\begin{pmatrix}1&2\pi i\\0&1\end{pmatrix}C_{01}B^{-1},\qquad M_x=EC_{0\infty}^{(*)}{C_{01}^{(*)}}^{-1}\begin{pmatrix}1&2\pi i\\0&1\end{pmatrix}C_{01}^{(*)}{C_{0\infty}^{(*)}}^{-1}E^{-1}.$$

With this choice, we obtain the matrices of the Proposition 4. We observe that

$$C_{OUT}^{-1}M_1M_xM_0C_{OUT}=\begin{pmatrix}-1&2\pi i\\0&-1\end{pmatrix},\qquad \epsilon=1;$$

$$C_{OUT}^{-1}M_xM_1M_0C_{OUT}=\begin{pmatrix}-1&2\pi i\\0&-1\end{pmatrix},\qquad \epsilon=-1.$$

$$\mathrm{tr}(M_0M_x)=\mathrm{tr}(M_0M_1)=2,\qquad \mathrm{tr}(M_1M_x)=-2.$$

□

# 8 Logarithmic Behaviors at $x=1$ and $x=\infty$ – Symmetries and their Action on the Monodromy Data – Connection Problem

In this section we compute the logarithmic asymptotic behaviors at $x=1,\infty$. This is easily done by applying the action of some Backlund transformations of (PVI) on (6) and (7). They act as birational transformations on $y(x)$ and $x$, and as permutations on the $\theta_\nu$'s, $\nu=0,x,1,\infty$. In order to know the monodromy data which are associated to the solutions of (PVI) obtained from (6) and (7) by the Backlund transformations, we also compute their action on the monodromy data.

The birational transformations are described in [23]; some of them form a representation of the permutation group and are generated by:

$$\sigma^1:\quad \theta_1'=\theta_0,\ \ \theta_0'=\theta_1;\qquad \theta_x'=\theta_x,\ \ \theta_\infty'=\theta_\infty;\qquad y'(x')=1-y(x),\quad x=1-x'.$$

$$\sigma^2:\quad \theta_0'=\theta_\infty-1,\ \ \theta_\infty'=\theta_0+1;\qquad \theta_1'=\theta_1,\ \ \theta_x'=\theta_x;\qquad y'(x')=\frac{1}{y(x)},\quad x=\frac{1}{x'}.$$

$$\sigma^3:\quad \theta_x'=\theta_1,\ \ \theta_1'=\theta_x;\qquad \theta_0'=\theta_0,\ \ \theta_\infty'=\theta_\infty;\qquad y'(x')=\frac{1}{x}y(x),\quad x=\frac{1}{x'}.$$

It is convenient to consider also:

$$\theta_0'=\theta_x,\ \ \theta_x'=\theta_0;\qquad \theta_1'=\theta_1,\ \ \theta_\infty'=\theta_\infty;\qquad y'(x')=\frac{x-y(x)}{x-1},\quad x=\frac{x'}{x'-1};\tag{58}$$

$$\theta_0'=\theta_\infty-1,\ \ \theta_x'=\theta_1,\ \ \theta_1'=\theta_x,\ \ \theta_\infty'=\theta_0+1;\qquad y'(x')=\frac{x}{y(x)},\quad x=x'.\tag{59}$$

$$\theta_x'=\theta_1,\ \ \theta_1'=\theta_\infty-1,\ \ \theta_\infty'=\theta_x+1;\qquad \theta_0'=\theta_0;\qquad y'(x')=\frac{y(x)}{y(x)-x},\quad x=\frac{x'-1}{x'}.\tag{60}$$

The transformantion (58) is the composition $\sigma^1\cdot\sigma^3\cdot\sigma^1$. (59) is $\sigma^2\cdot\sigma^3$. (60) is the composition of $\sigma^2$, (58), (59). For brevity, we will call the Backlund transformations with the name "symmetries".

### 8.1 Action on the Transcendent. Formulae (11)-(16) and (17)-(22)

The symmetry $\sigma^3$, acting on the transcendent (6), gives the behavior:

$$y'(x') \sim \frac{{\theta'_0}^2}{{\theta'_0}^2-{\theta'_1}^2}+\frac{{\theta'_1}^2-{\theta'_0}^2}{2}\left[\ln\frac{1}{x'}+\frac{4r+2\theta'_0}{{\theta'_0}^2-{\theta'_1}^2}\right]^2, \quad x' \to \infty;$$

We prove below that $\sigma^3$ maps $\mathrm{tr}(M_0M_x)$ to $\mathrm{tr}(M'_0M'_1)$, where $M'_\nu$, $\nu=0,x,1,\infty$ are th emonodromy matrices for the system (1) associated to $y'(x')$, with respect to the same basis of loops (see below). Therefore $\mathrm{tr}(M'_0M'_1)=2$.

The symmetry $\sigma^1$, acting on the transcendent (6), gives the behavior:

$$y'(x') \sim 1-(1-x')\left\{\frac{{\theta'_1}^2}{{\theta'_1}^2-{\theta'_x}^2}+\frac{{\theta'_x}^2-{\theta'_1}^2}{4}\left[\ln(1-x')+\frac{4r+2\theta'_1}{{\theta'_1}^2-{\theta'_x}^2}\right]^2\right\}, \quad x' \to 1.$$

As it is proved below, $\sigma^1$ maps $\mathrm{tr}(M_0M_x)$ to $\mathrm{tr}(M'_1M'_x)$ and thus $\mathrm{tr}(M'_1M'_x)=2$.

The action of (59) gives the behavior:

$$y'(x') \sim \frac{1}{\frac{{\theta'_1}^2-(\theta'_\infty-1)^2}{4}\left[\ln x'+\frac{4r+2\theta'_\infty-2}{(\theta'_\infty-1)^2-{\theta'_1}^2}\right]^2+\frac{(\theta'_\infty-1)^2}{(\theta'_\infty-1)^2-{\theta'_1}^2}}, \quad x' \to 0,$$

Namely,

$$y'(x') = \frac{4}{[{\theta'_1}^2-(\theta'_\infty-1)^2]\ln^2 x'}\left[1+\frac{8r+4\theta'_\infty-4}{{\theta'_1}^2-(\theta'_\infty-1)^2}\frac{1}{\ln x'}+O\left(\frac{1}{\ln^2 x'}\right)\right], \quad x' \to 0.$$

The symmetry (60) gives:

$$y'(x') \sim 1+\frac{1}{\frac{{\theta'_1}^2-{\theta'_0}^2}{4}\left[\ln(x'-1)+\frac{4r+2\theta'_0}{{\theta'_0}^2-{\theta'_1}^2}\right]^2+\frac{{\theta'_0}^2}{{\theta'_0}^2-{\theta'_1}^2}}, \quad x' \to 1.$$

Namely:

$$y'(x') = 1+\frac{4}{({\theta'_1}^2-{\theta'_0}^2)\ln^2(x'-1)}\left[1-\frac{8r+4\theta'_0}{{\theta'_0}^2-{\theta'_1}^2}\frac{1}{\ln(x'-1)}+O\left(\frac{1}{\ln^2(x'-1)}\right)\right], \quad x' \to 1$$

The symmetry $\sigma^2$ yields:

$$y'(x') \sim \frac{x'}{\frac{{\theta'_x}^2-(\theta'_\infty-1)^2}{4}\left[\ln\frac{1}{x'}+\frac{4r+2\theta'_\infty-2}{(\theta'_\infty-1)^2-{\theta'_x}^2}\right]^2+\frac{(\theta'_\infty-1)^2}{(\theta'_\infty-1)^2-{\theta'_x}^2}}, \quad x' \to \infty.$$

Namely,

$$y(x) = \frac{4\;x'}{[(\theta'_\infty-1)^2-{\theta'_x}^2]\ln^2 x'}\left[1-\frac{8r+4(\theta'_\infty-1)}{{\theta'_x}^2-(\theta'_\infty-1)^2}\frac{1}{\ln x'}+O\left(\frac{1}{\ln^2 x'}\right)\right], \quad x' \to \infty.$$

We study the action of the symmetries on (7). If we apply $\sigma^1$ we find:

$$y'(x') \sim 1-(1-x')\big(r \pm \theta'_1\ln(1-x')\big), \quad x' \to 1, \quad \theta'_1=\pm\theta'_x.$$

The action of $\sigma^3$ gives:

$$y'(x') \sim r \pm \theta'_0 \ln x', \quad x' \to \infty, \quad \theta'_0=\pm\theta'_1.$$

The action of (59) gives:

$$y'(x') \sim \frac{1}{r \pm (\theta'_\infty - 1)\ln x'}, \qquad x \to 0, \qquad \theta'_\infty - 1 = \pm\theta'_1.$$

The action of (60) gives:

$$y'(x') \sim 1 + \frac{1}{r \pm \theta'_0 \ln\left(\frac{x'-1}{x'}\right)}, \qquad x' \to 1, \qquad \theta'_\infty - 1 = \pm\theta'_0.$$

Namely:

$$y'(x') = 1 \pm \frac{1}{\theta'_0 \ln(x'-1)}\left[1 \mp \frac{r}{\theta'_0 \ln(x'-1)} + O\left(\frac{1}{\ln^2(x'-1)}\right)\right], \qquad x' \to 1, \qquad \theta'_\infty - 1 = \pm\theta'_0.$$

The action of $\sigma^2$ gives:

$$y'(x') \sim \frac{x'}{r \pm (\theta_\infty - 1)\ln x'}, \qquad x' \to \infty, \qquad \theta'_\infty - 1 = \pm\theta'_x.$$

Namely:

$$y'(x') = \pm\frac{x'}{(\theta'_\infty - 1)\ln x'}\left[1 \mp \frac{r}{(\theta'_\infty - 1)\ln x'} + O\left(\frac{1}{\ln^2 x'}\right)\right], \qquad \theta'_\infty - 1 = \pm\theta'_x.$$

When we drop the index $'$ from the above formulae, we get the asymptotic behaviors (11)–(16) and (17)–(22).

## 8.2 Action of $\sigma^1$ and $\sigma^3$ on the Monodromy Data

To compute the action of the symmetries on the monodromy of system (1), it is important that we choose the same base of loops in the $\lambda$-plane that we used to parameterize a transcendent in terms of the monodromy data. Therefore, we consider an ordered base of loops in the "$\lambda$-plane" $\mathbf{C}\backslash\{0, x, 1\}$ as we did in Sub-Section 2.2, figure 1.

Consider the system associated to $y(x)$:

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{A_1}{\lambda - 1}\right]\Psi, \tag{61}$$

The monodromy matrices of a fundamental solution $\Psi(\lambda)$ w.r.t. the chosen base of loops are denoted $M_0$, $M_x$, $M_1$. The loop at infinity will be $\gamma_\infty = \gamma_0\gamma_x\gamma_1$, so $M_\infty = M_1 M_x M_0$. We need to construct the system associated to $y'(x')$:

$$\frac{d\Psi'}{d\lambda'} = \left[\frac{A'_0}{\lambda'} + \frac{A'_{x'}}{\lambda' - x'} + \frac{A'_1}{\lambda' - 1}\right]\Psi', \tag{62}$$

We will determine the relation between (61) and (62), between a fundamental solutions $\Psi(\lambda)$ and a fundamental solution $\Psi'(\lambda')$ and between their respective monodromy matrices $M_0$, $M_x$, $M_1$ and $M'_0$, $M'_{x'}$, $M'_1$. The monodromy $M'_0$, $M'_{x'}$, $M'_1$ are understood to be referred to the order $1, 2, 3 = 0, x', 1$. In order to do this, we will construct $A'_j(x', y'(x'), dy'/dx')$, $j = 0, x', 1$ and we will see how they are related to the matrices $A_j(x, y(x), dy/dx)$.

The explicit formulas to write $A_j(x, y(x), dy/dx)$ can be found at page 443-445 of [18]:

$$(A_0)_{12} = -k\frac{y}{x}, \quad (A_1)_{12} = k\frac{y-1}{x-1}, \quad (A_x)_{12} = -k\frac{y-x}{x(x-1)};$$

$$\frac{d}{dx}\ln k = (\theta_\infty - 1)\frac{y-x}{x(x-1)} \quad \Longrightarrow \quad k(x) = k_0 \exp\left\{(\theta_\infty - 1)\int^x \frac{y(s) - s}{s(s-1)}ds\right\}, \quad k_0 \in \mathbf{C}.$$

$$(A_i)_{11} = z_i + \frac{\theta_i}{2}, \quad i = 0, x, 1.$$

$$z_0 = \frac{y}{x\theta_\infty}\Big\{y(y-1)(y-x)\tilde{z}^2+\big[\theta_1(y-x)+x\theta_x(y-1)-2\kappa_2(y-1)(y-x)\big]\tilde{z}+\kappa_2^2(y-x-1)-\kappa_2(\theta_1+x\theta_x)\Big\},$$

$$z_1 = -\frac{y-1}{(x-1)\theta_\infty}\Big\{y(y-1)(y-x)\tilde{z}^2+\big[(\theta_1+\theta_\infty)(y-x)+x\theta_x(y-1)-2\kappa_2(y-1)(y-x)\big]\tilde{z}+\kappa_2^2(y-x)+$$
$$-\kappa_2(\theta_1+x\theta_x)-\kappa_2(\kappa_2+\theta_\infty)\Big\},$$

$$z_x = \frac{y-x}{x(x-1)\theta_\infty}\Big\{y(y-1)(y-x)\tilde{z}^2+\big[\theta_1(y-x)+x(\theta_x+\theta_\infty)(y-1)-2\kappa_2(y-1)(y-x)\big]\tilde{z}+$$
$$+\kappa_2^2(y-1)-\kappa_2(\theta_1+x\theta_x)-x\kappa_2(\kappa_2+\theta_\infty)\Big\},$$

$$\kappa_2 = -\left\{\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{\theta_1}{2}+\frac{\theta_\infty}{2}\right\}, \quad \tilde{z} = \frac{1}{2}\frac{x(x-1)}{y(y-1)(y-x)}\frac{dy}{dx}-\frac{1}{2}\left\{\frac{1}{y-x}+\frac{\theta_0}{y}+\frac{\theta_x}{y-x}+\frac{\theta_1}{y-1}\right\},$$

$$(A_0)_{21} = \frac{z_0 x}{ky}(z_0+\theta_0), \quad (A_1)_{21} = -\frac{(x-1)z_1}{k(y-1)}(z_1+\theta_1), \quad (A_x)_{21} = \frac{x(x-1)z_x}{k(y-x)}(z_x+\theta_x).$$

We also recall that $(A_0)_{12}/\lambda + (A_x)_{12}/(\lambda-x) + (A_1)_{12}/(\lambda-1) = \frac{k(\lambda-y)}{\lambda(\lambda-1)(\lambda-x)}$.

**Symmetry $\sigma_3$:** We compute the matrices $A_i'$, $i = 0, x', 1$, through the above formulas. By direct computation we find:

$$\tilde{z}' = x\tilde{z}, \quad z_0' = z_0, \quad z_1' = z_x, \quad z_x' = z_1.$$

Therefore we find:

$$A_0' = K^{-1}A_0K, \quad A_1' = K^{-1}A_xK, \quad A_{x\prime}' = K^{-1}A_1K; \qquad K := \begin{pmatrix} \frac{k}{xk'} & 0 \\ 0 & 1 \end{pmatrix}.$$

We also note that $d(\ln k')/dx' = d(\ln k)/dx + (\theta_\infty - 1)/x$, thus: $k' = kx^{\theta_\infty-1}$. Anyway, the specific form of $k/k'$ is not important here. What is important is that the matrix $K$ is *diagonal*. Then we can write

$$\frac{d\Psi'}{d\lambda'} = \left[\frac{A_0'}{\lambda'}+\frac{A_{x'}'}{\lambda'-x'}+\frac{A_1'}{\lambda'-1}\right]\Psi' = K^{-1}\left[\frac{A_0}{\lambda'}+\frac{A_1}{\lambda'-x'}+\frac{A_x}{\lambda'-1}\right]K\Psi',$$

With the change of variables:

$$\lambda' = \frac{\lambda}{x}, \quad x' = \frac{1}{x},$$

we get:

$$\frac{d\Psi'}{d\lambda} = K^{-1}\left[\frac{A_0}{\lambda}+\frac{A_1}{\lambda-1}+\frac{A_x}{\lambda-x}\right]K\Psi'.$$

With the gauge:

$$\Psi = K\Psi',$$

We finally get (61):

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda}+\frac{A_1}{\lambda-1}+\frac{A_x}{\lambda-x}\right]\Psi. \tag{63}$$

It is important to note that the gauge is *diagonal*, a fact that ensures that, for the gauge-transformed system, the solution $\lambda$ of the equation obtained by setting the matrix element $(1,2)$ equal to zero defines the same $y(x)$. We conclude that the systems (61) and (62) are related by a *diagonal* gauge transformation and the exchange of the point $x$ and 1. In other words, we can take as (62) the system:

$$\frac{d\Psi}{d\lambda'} = \left[\frac{A_0}{\lambda'}+\frac{A_1}{\lambda'-x'}+\frac{A_x}{\lambda'-1}\right]\Psi, \tag{64}$$

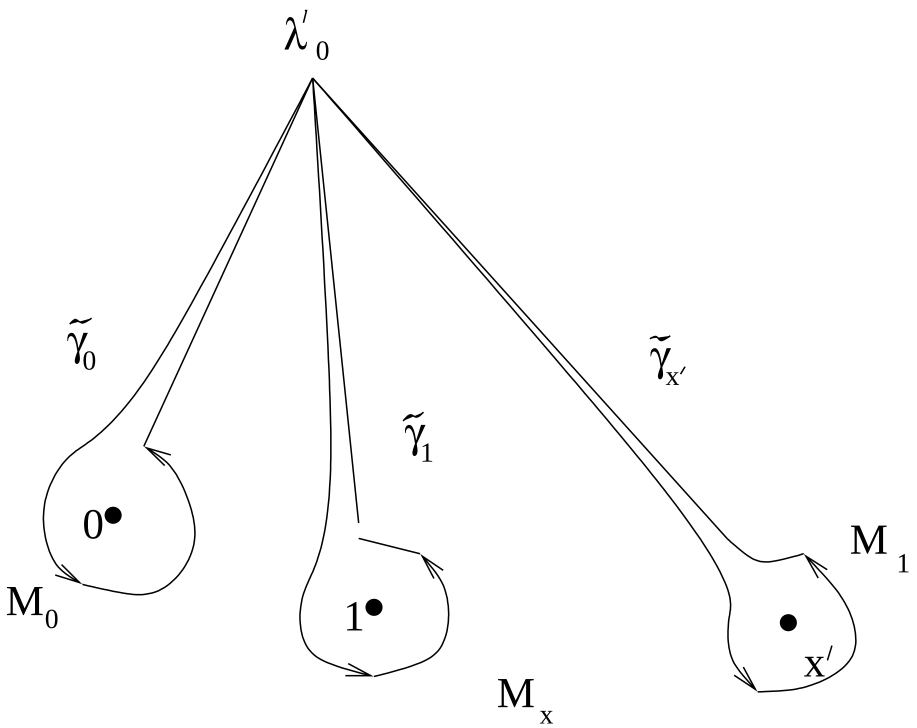


Figure 2:

where $\Psi(\lambda)$ is also a fundamental matrix solution of (61). The equation defining $y'(x')$ is:

$$\left[\frac{A_0}{\lambda'}+\frac{A_1}{\lambda'-x'}+\frac{A_x}{\lambda'-1}\right]_{1,2}=0 \quad\Longrightarrow\quad \lambda'=y'(x'),$$

while:

$$\left[\frac{A_0}{\lambda}+\frac{A_1}{\lambda-1}+\frac{A_x}{\lambda-x}\right]_{1,2}=0 \quad\Longrightarrow\quad \lambda=y(x),$$

Therefore, (62) can be obtained from (61) simply by a change of variables $\lambda'=\lambda/x$, $x=1/x'$. The result is that the points $\lambda=x,1$ are exchanged to $\lambda'=1,x'$.

We compute the monodromy of (64) in terms of the monodromy of (63). For the latter, we have fixed in the beginning of the section a ordered base of loops $\gamma_0$, $\gamma_x$, $\gamma_1$. But for (64), the points $1,x'$ are exchanged. The loops $\tilde{\gamma}_0$, $\tilde{\gamma}_1$ $\tilde{\gamma}_{x'}$ of figure 2 correspond to the order 1,2,3. Their monodromy matrices are:

$$M_{\tilde{\gamma}_0}=M_0, \quad M_{\tilde{\gamma}_1}=M_x, \quad M_{\tilde{\gamma}_{x'}}=M_1.$$

We need a new basis of loops such that the order $1,2,3$ be $0,x',1$. Let us denote these loops $\gamma_0',\gamma_{x'}',\gamma_1'$ of figure 3. For the basis in figure 3 we easily see that:

$$\gamma_0'=\tilde{\gamma}_0, \qquad \gamma_{x'}'=\tilde{\gamma}_1\ \tilde{\gamma}_x\tilde{\gamma}_1^{-1}, \qquad \gamma_1'=\tilde{\gamma}_1.$$

Let $M_0',M_{x'}',M_1'$ be the monodromy matrices for the orderded loops $\gamma_0',\gamma_{x'}',\gamma_1'$. Therefore we have:

$$\begin{aligned} M_0'&=M_{\tilde{\gamma}_0}=M_0,\\ M_{x'}'&=M_{\tilde{\gamma}_1}^{-1}M_{\tilde{\gamma}_{x'}}M_{\tilde{\gamma}_1}\equiv M_x^{-1}M_1M_x,\\ M_1'&=M_{\tilde{\gamma}_1}\equiv M_x.\end{aligned}$$

From the above results we compute the traces:

$$\begin{aligned} \mathrm{tr}(M_0'M_{x'}')=&\ -\mathrm{tr}(M_0M_1)\ -\mathrm{tr}(M_0M_x)\mathrm{tr}(M_1M_x)\ +4\big(\cos(\pi\theta_\infty)\cos(\pi\theta_x)+\cos(\pi\theta_0)\cos(\pi\theta_1)\big),\\ \mathrm{tr}(M_0'M_1')=&\ \mathrm{tr}(M_0M_x),\\ \mathrm{tr}(M_1'M_{x'}')=&\ \mathrm{tr}(M_1M_x).\end{aligned}$$

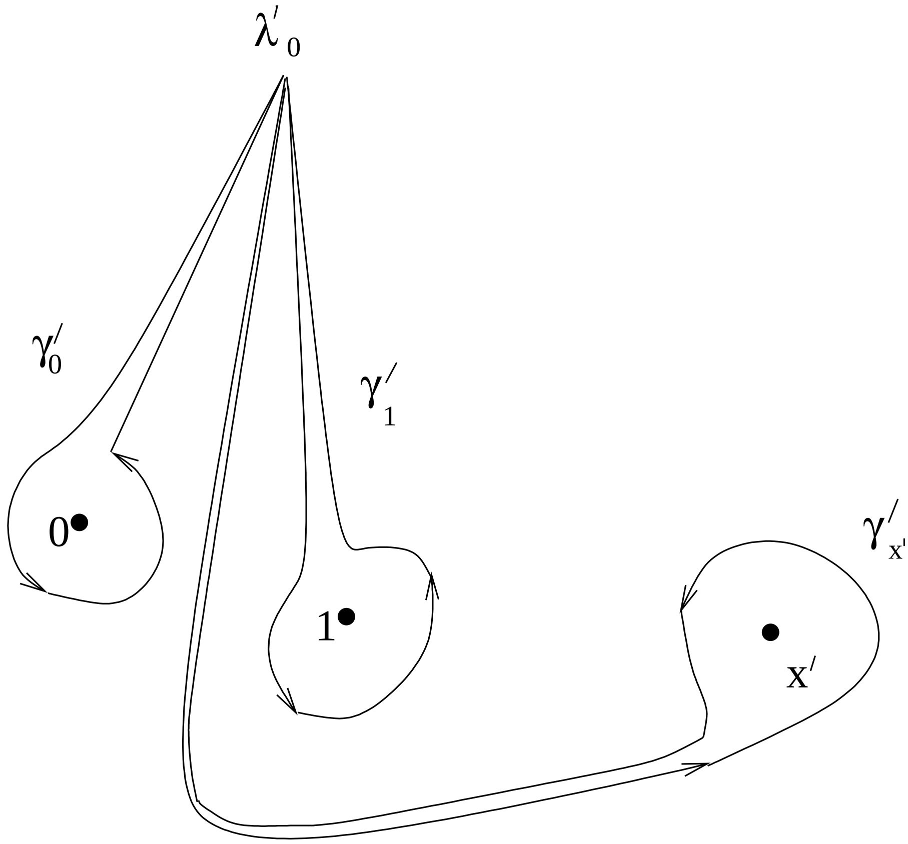


Figure 3:

The above follow from the identity:

$$\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB^{-1}), \quad A, B \ 2 \times 2 \text{ matrices }, \quad \det(B) = 1$$

and from:

$$\mathrm{tr}(M_1 M_x M_0) = e^{i\pi\theta_\infty} + e^{-i\pi\theta_\infty}, \quad \mathrm{tr}(M_i) = e^{i\pi\theta_i} + e^{-i\pi\theta_i}, \quad i = 0, x, 1$$

**Symmetry** $\sigma_1$**:** We repeat the computation $A_0'$, $A_{x'}', A_1'$ as above. As a result we find that the system (62) is – up to diagonal conjugation:

$$\frac{d\Psi}{d\lambda'} = \left[\frac{A_0}{\lambda' - 1} + \frac{A_1}{\lambda'} + \frac{A_x}{\lambda' - x}\right]\Psi, \quad \lambda' = 1 - \lambda, \quad x' = 1 - x, \tag{65}$$

where $\Psi(\lambda)$ is also a fundamental matrix of (61). In other words, (65) can be obtained from (61) by the change of variables $\lambda' = \lambda - 1$, $x = 1 - x'$. The relation between the two systems is simply that the points $\lambda = 0, 1$ are exchanged to $\lambda' = 1, 0$. The base $\gamma_0, \gamma_x\gamma_1$ becomes the basis $\tilde{\gamma}_1, \tilde{\gamma}_{x'}, \tilde{\gamma}_0$, in figure 4. The monodromy matrices are:

$$M_{\tilde{\gamma}_1} = M_0, \quad M_{\tilde{\gamma}_{x'}} = M_x, \quad M_{\tilde{\gamma}_0} = M_1.$$

We introduce the ordered basis $\gamma_0', \gamma_{x'}', \gamma_1'$ of figure 5 and we easily compute:

$$\gamma_0' = \tilde{\gamma}_0, \quad \gamma_{x'}' = \tilde{\gamma}_0^{-1}\tilde{\gamma}_{x'}\tilde{\gamma}_0, \quad \gamma_1' = \tilde{\gamma}_0^{-1}\tilde{\gamma}_{x'}^{-1}\tilde{\gamma}_1\tilde{\gamma}_{x'}\tilde{\gamma}_0.$$

Therefore:

$$M_0' = M_1, \quad M_{x'}' = M_1 M_x M_1^{-1}, \quad M_1' = M_1 M_x M_0 M_x^{-1} M_1^{-1};$$

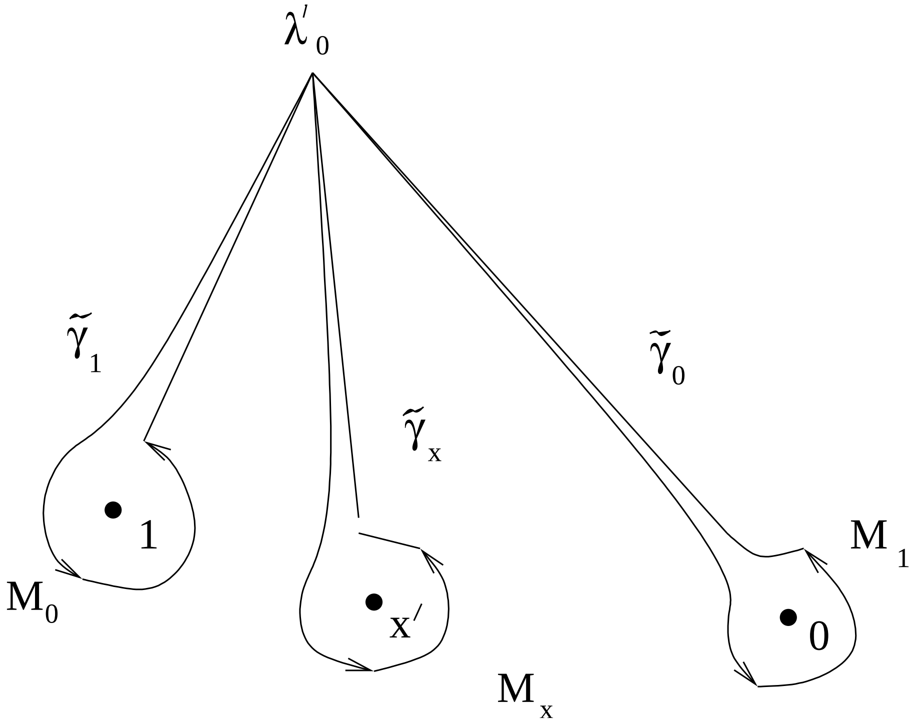


Figure 4:

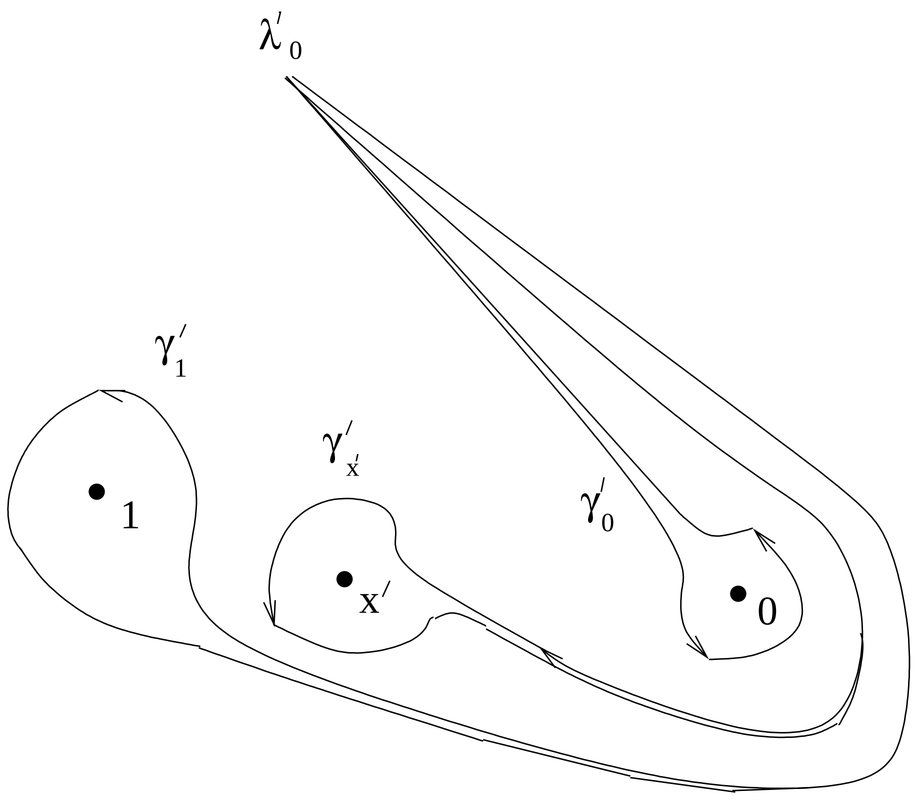


Figure 5:

and:

$$\mathrm{tr}(M_0'M_{x'}') = \quad \mathrm{tr}(M_1M_x),$$

$$\mathrm{tr}(M_0'M_1') = \quad -\mathrm{tr}(M_0M_1) \quad -\mathrm{tr}(M_1M_x)\mathrm{tr}(M_0M_x) \ + \ 4\big(\cos(\pi\theta_\infty)\cos(\pi\theta_x)+\cos(\pi\theta_1)\cos(\pi\theta_0)\big)$$

$$\mathrm{tr}(M_1'M_{x'}') = \quad \mathrm{tr}(M_0M_x).$$

### 8.3 Connection Problem

When we act with a Backlund transformation on $y(x)$ for $x \to 0$, we obtain the asymptotic behavior for $x' \to$ (the image of $x = 0$). $r$ in (6) is expressed in terms of the monodromy data. Let us write the dependence on the monodromy data in a synthetic way as follows:

$$y(x) = y(x; \Theta; \mathbf{TR}_{MM}),$$

where $\Theta = \theta_0, \theta_x, \theta_1, \theta_\infty$; $\mathbf{TR}_{MM} = \mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)$.

When we act with a symmetry on the above transcendent, we get:

$$y'\big(\ x'; \Theta(\Theta'); \mathbf{TR}_{MM}\big(\mathbf{TR}_{M'M'}\big)\ \big).$$

Here $\Theta(\Theta')$ stands for the $\theta_\nu$'s expressed in terms of the $\theta_\nu'$'s, and $\mathbf{TR}_{MM}\big(\mathbf{TR}_{M'M'}\big)$ stands for the traces of the products of the $M_j$'s as functions of the traces of the products of the $M_j'$'s. For example:

For $\sigma_3$:

$$2 \equiv \mathrm{tr}(M_0M_x) = \quad \mathrm{tr}(M_0'M_1'),$$

$$\mathrm{tr}(M_0M_1) = \quad -\mathrm{tr}(M_0'M_{x'}') \quad -\mathrm{tr}(M_0'M_1')\mathrm{tr}(M_1'M_{x'}') \quad +4\big(\cos(\pi\theta_\infty')\cos(\pi\theta_1')+\cos(\pi\theta_0')\cos(\pi\theta_x')\big),$$

$$\mathrm{tr}(M_1M_x) = \quad \mathrm{tr}(M_1'M_{x'}').$$

For $\sigma^1$:

$$2 \equiv \mathrm{tr}(M_0M_x) = \quad \mathrm{tr}(M_1'M_{x'}'),$$

$$\mathrm{tr}(M_0M_1) = \quad -\mathrm{tr}(M_0'M_1') \quad -\mathrm{tr}(M_1'M_{x'}')\mathrm{tr}(M_0'M_{x'}') \ + \ 4\big(\cos(\pi\theta_\infty')\cos(\pi\theta_x')+\cos(\pi\theta_1')\cos(\pi\theta_0')\big)$$

$$\mathrm{tr}(M_1M_x) = \quad \mathrm{tr}(M_0'M_{x'}'),$$

In order to obtain the formulas which express $r$ in terms of the monodromy data for the solutions (12) and (13), (18) and (19), we substitute in (51) of Proposition 2 or in (56) of Proposition (3), the $\theta_\nu$'s as functions of the $\theta_\nu'$'s and the $\mathrm{tr}(M_iM_j)$ as functions of the $\mathrm{tr}(M_i'M_j')$. When this is done, we can drop the index $'$. The above also proves that (12), (18) are associated to $\mathrm{tr}(M_0M_1) = 2$, while (13) , (19) are associated to $\mathrm{tr}(M_1M_x) = 2$.

### 8.4 The case of (9): asymptotic behavior (10)

We apply the above results for the transformation of the traces to the case (9). First of all, we observe that the solutions obtained from the above by the symmetry (59) are:

$$y(x) \sim -\frac{1}{p^2\ln^2 x}\left[1 - 2\frac{p+r}{p^2}\frac{1}{\ln x} + \frac{4p^2+6rp+3r^2}{p^4}\frac{1}{\ln^2 x}\right],$$

with:

$$\theta_0 = \theta_x = \theta_1 = 0, \quad \theta_\infty = 2p+1.$$

These contain the family of *Chazy solutions* studied in [21] (for $\mu = -1/2$ in [21]), namely:

$$y(x) \sim -\frac{1}{\ln^2 x}\left[1 - \frac{2+2r}{\ln x} + \frac{4+6r+3r^2}{\ln^2 x}\right], \qquad \theta_0 = \theta_x = \theta_1 = 0, \quad \theta_\infty = 3 \quad (p = 1).$$

The symmetry $\sigma^3$ transforms (9) into :

$$y'(x') \sim 1 - p^2 \left( \ln \frac{1}{x'} + \frac{r+p}{p^2} \right)^2, \qquad x' \to \infty, \tag{66}$$

$$\big(\mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)\big) \quad \longmapsto \quad \big(\mathrm{tr}(M_0'M_x'), \mathrm{tr}(M_0'M_1'), \mathrm{tr}(M_1'M_x')\big) \equiv (2, 2, -2),$$
$$(\theta_0, \theta_x, \theta_1, \theta_\infty) = (2p, 0, 0, 1) \quad \longmapsto \quad (\theta_0', \theta_x', \theta_1', \theta_\infty') = (2p, 0, 0, 1).$$

Therefore, the transformed solution is again associated to the same monodromy data of (9).

Now we apply (60). We obtain:

$$y'(x') = 1 - \frac{1}{p^2 \left( \ln(1-x) + \frac{r+p}{p^2} \right)^2}, \qquad x' \to 1 \tag{67}$$

$$\big(\mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)\big) \quad \longmapsto \quad \big(\mathrm{tr}(M_0'M_x'), \mathrm{tr}(M_0'M_1'), \mathrm{tr}(M_1'M_x')\big) \equiv (2, 2, -2),$$
$$(\theta_0, \theta_x, \theta_1, \theta_\infty) = (2p, 0, 0, 1) \quad \longmapsto \quad (\theta_0', \theta_x', \theta_1', \theta_\infty') = (2p, 0, 0, 1).$$

The transformation of the traces by the action of (60) will be proved in the second paper. The transformed solution is again associated to the same monodromy data of (9).

Actually, a transcendents (9) has a behaviors (67) at $x = 1$ and a behavior (66) at $x = \infty$. Namely, it is the transcendent (10). The parameters $r$ appearing in (9), (66) and (67) are not the same. Their relation will be determined below.

The rigorous proof of (10) is as follows. For $\theta_0 = \theta_x = \theta_1 = 0$ and $\theta_\infty = 2p + 1$, $p \in \mathbf{Z}$, (PVI) was completely studied in [21]. There are two classes of solutions:

(1) Chazy solutions for any $p \neq 0$. The Chazy solutions for a given $p \neq 0$ can be obtained applying a birational transformation to the Chazy solutions for $p = 1$.

(2) Picard solutions for any $p$. The Picard solutions for a given $p \neq 0$ can be obtained applying a birational transformation to the Picard solutions for $p = 0$.

The symmetry (59) transforms the Chazy solutions of (PVI) with $\theta_0 = \theta_x = \theta_1 = 0$, $\theta_\infty = 2p+1$, $p = 1$, to the solution:

$$y(x) = \frac{8x\ \omega\omega' \big(2(x-1)\omega' + \omega\big)(2x\omega' + \omega)}{\big[(2x\omega' + \omega)^2 - 4x\omega'^2\big]^2}, \tag{68}$$

associated to

$$\theta_0 = 2p, \quad p = 1, \quad \theta_x = \theta_1 = 0, \quad \theta_\infty = 1.$$

Here,

$$\omega = \omega_1 + \nu\omega_2, \quad \nu \in \mathbf{C}, \qquad \omega' = d\omega/dx.$$

The $\omega_i$, $i = 1, 2$ are two independent solutions of the hypergeometric equation $x(x-1)\omega'' + (1 - 2x)\omega' - 1/4\omega = 0$, namely:

$$\omega_1 = F\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x\right), \qquad \omega_2 = g\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x\right).$$

Any other case $p \in \mathbf{Z}$, $p \neq 0$, can be obtained by a birational transformation of (68), as it is already proved in [21] for the Chazy solutions. If we expand (68) for $x \to 0$ we obtain (9), with:

$$\nu = 1/(4\ln 2 - 1 + \rho_0), \qquad \rho_0 \equiv \frac{r+p}{p^2}.$$

Thanks to the representation (68), we can compute the parameters in (10):

$$\rho_\infty = \frac{\pi(4\ln 2 - 1 + \rho_0)}{\pi - i(4\ln 2 - 1 + \rho_0)} - 2\ln 2 + 1, \quad \rho_1 = \frac{\pi^2}{4\ln 2 - 1 + \rho_0} - \ln 2 + 1.$$

This is done by expanding $\omega_1$, $\omega_2$ for $x \to 1$, $x \to \infty$. In order to do this, we use the connection formulae in Norlund [22]. From 5.(1) and 5.(2), we get:

$$\omega_1 = -\frac{1}{\pi} g\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1-x\right), \qquad \omega_1 = -\pi F\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1-x\right);$$

From 12.(1), 12.(3) we get:

$$\omega_1 = \frac{x^{-\frac{1}{2}}}{\pi}\left[\pi F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{1}{x}\right) - ig\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{1}{x}\right)\right],$$

$$\omega_2 = x^{-\frac{1}{2}} g\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{1}{x}\right).$$

It is not possible to compute the relation between $\rho_0$, $\rho_\infty$ and $\rho_1$ by the method of monodromy preserving deformations, due to the lack of one to one correspondence between a solution (the parameter $r$, i.e. $\rho_0$) and the monodromy data.

*Note 1:* The pure braid group (Appendix 2) acts as follows:

$$\beta_i \cdot \beta_i : \ \big(\mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)\big) = (2,2,-2) \ \longmapsto \ (2,2,-2), \qquad i=1,2.$$

It leaves $\big(\mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)\big)$ invariant, thus the log-behaviors at $x=0,1,\infty$ are preserved in the analytic continuation of (10).

*Note 2:* The symmetry $\sigma^1$ transforms:

$$\big(\mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)\big) \mapsto \big(\mathrm{tr}(M_0'M_x'), \mathrm{tr}(M_0'M_1'), \mathrm{tr}(M_1'M_x')\big) \equiv (-2,2,2),$$

$$(\theta_0,\theta_x,\theta_1,\theta_\infty) = (2p,0,0,1) \mapsto (\theta_0',\theta_x',\theta_1',\theta_\infty') = (0,0,2p,1).$$

Therefore, the solution:

$$y'(x') \sim 1-(1-x')\left[-p^2\left(\ln(1-x)+\frac{r+p}{p^2}\right)^2+1\right], \qquad x'\to 1$$

is not associate to the same monodromy data of (9).

# 9 Appendix 1

**Proposition 5** *Let $B_0$, $B_1$ be $2\times 2$ matrices such that*

$$\textit{Eigenvalues } (B_0) = 0, -c, \quad \textit{Eigenvalues } (B_1) = 0, c-a-b.$$

*and $B_0+B_1$ is either diagonalizable:*

$$B_0+B_1 = \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \quad \textit{(it may happen that } a=b\textit{)},$$

*or it is a Jordan form:*

$$B_0+B_1 = \begin{pmatrix} -a & 1 \\ 0 & -a \end{pmatrix}.$$

*Then, $B_0$ and $B_1$ can be computed as in the following cases. Let $r$, $s$ be any complex numbers.*

**1) Diagonalizable case.**
*Case $a\neq b$:*

$$B_0 := \begin{pmatrix} \frac{a(b-c)}{a-b} & r \\ \frac{ab(a-c)(c-b)}{r(a-b)^2} & \frac{b(c-a)}{a-b} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{a(c-a)}{a-b} & -r \\ -(B_0)_{21} & \frac{b(b-c)}{a-b} \end{pmatrix}, \quad r\neq 0 \tag{69}$$

*Case $a=b$. We have two sub-cases:*

$$\textit{If } a=b=c: \quad B_0 = \begin{pmatrix} -c-s & r \\ -\frac{s(c+s)}{r} & s \end{pmatrix}, \quad B_1 = \begin{pmatrix} s & -r \\ \frac{s(c+s)}{r} & -c-s \end{pmatrix}. \tag{70}$$

$$\textit{If } a=b=0: \quad B_0 = \begin{pmatrix} -c-s & r \\ -\frac{s(c+s)}{r} & s \end{pmatrix}, \qquad B_1 = -B_0. \tag{71}$$

*The transpose matrices of all the above cases are also possible.*

**2) Jordan case.**
*For $a \neq 0$ and $a \neq c$ we have:*

$$B_0 = \begin{pmatrix} r & \frac{r(r+c)}{a(a-c)} \\ a(c-a) & -c-r \end{pmatrix}, \quad B_1 = \begin{pmatrix} -a-r & 1-\frac{r(r+c)}{a(a-c)} \\ a(a-c) & c-a+r \end{pmatrix}. \tag{72}$$

*For $a = 0$, or $a = c$, we have two possibilities:*

$$B_0 = \begin{pmatrix} 0 & r \\ 0 & -c \end{pmatrix}, \quad B_1 = \begin{pmatrix} -a & 1-r \\ 0 & -a+c \end{pmatrix}; \tag{73}$$

*or*

$$B_0 = \begin{pmatrix} -c & r \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} c-a & 1-r \\ 0 & -a \end{pmatrix} \tag{74}$$

**Proposition 6** *Let $B_0$ and $B_1$ be as in Proposition 5. The linear system:*

$$\frac{d}{dz}\begin{pmatrix} \varphi \\ \xi \end{pmatrix} = \left[\frac{B_0}{z} + \frac{B_1}{z-1}\right] \begin{pmatrix} \varphi \\ \xi \end{pmatrix}$$

*may be reduced to a Gauss hyper-geometric equation, in the following cases.*

*Diagonalizable case (i.e. from (69) to (71)):*

$$z(1-z)\ \frac{d^2\varphi}{dz^2} + \big(1+c-(a+[b+1]+1)\ z\big)\ \frac{d\varphi}{dz} - a(b+1)\ \varphi = 0. \tag{75}$$

*The component $\xi$ is obtained by the following equalities, according to the different cases of Proposition 5.*

*Cases (69):*

$$\xi = \frac{1}{r}\left[z(1-z)\ \frac{d\varphi}{dz}\ - a\left(z + \frac{b-c}{a-b}\right)\ \varphi\right] \tag{76}$$

*Case (70):*

$$\xi = \frac{1}{r}\left[z(1-z)\ \frac{d\varphi}{dz} + (c+s-c\ z)\ \varphi\right]$$

*Case (71):*

$$\xi = \frac{1}{r}\left[z(1-z)\ \frac{d\varphi}{dz} + (c+s)\ \varphi\right]$$

*Jordan case (72): The equation for $\xi$ is in Gauss hypergeometic form:*

$$z(z-1)\frac{d^2\xi}{dz^2} + \big(1+c-2(a+1)z\big)\frac{d\xi}{dz} - a(a+1)\xi = 0, \tag{77}$$

$$\varphi(z) = \frac{1}{a(a-c)}\left[z(z-1)\frac{d\xi}{dz} + (az-c-r)\xi\right]. \tag{78}$$

*Jordan case (73): The equation for $\xi$:*

$$\frac{d\xi}{dz} = \left(-\frac{c}{z} + \frac{c-a}{z-1}\right)\xi \quad \Longrightarrow \quad \xi(z) = \begin{cases} D\ z^{-c}(1-z)^c, & a = 0; \\ D\ z^{-c}, & a = c; \end{cases} \qquad D \in \mathbf{Z}$$

*The equation for $\varphi$:*

$$\frac{d\varphi}{dz} = \begin{cases} \left[\frac{r}{z} + \frac{1-r}{z-1}\right]\ D\frac{(1-z)^c}{z^c}, & a = 0; \\ -\frac{c}{z-1}\varphi + \left[\frac{r}{z} + \frac{1-r}{z-1}\right]\ \frac{D}{z^c}, & a = c; \end{cases}$$

*The equation for $\varphi$ can be integrated. If $c \notin \mathbf{Z}$ we obtain (by variation of parameters):*

$$\varphi(z) = \begin{cases} E + D\left[-\frac{r}{c}(1-z)^c z^{-c} + \frac{1}{c-1}\ z^{1-c}F(1-c,1-c,2-c;z)\right], & a=0; \\ E(1-z)^{-c} + D\left[-\frac{r}{c}z^{-c} + \frac{1}{c-1}\ z^{1-c}(1-z)^{-c}F(1-c,1-c,2-c;z)\right], & a=c; \end{cases} \quad D, E \in \mathbf{C}$$

*If $c \in \mathbf{Z}$, the solution contains a logarithmic term.*

*Jordan case (74): The equation for $\xi$:*

$$\frac{d\xi}{dz} = -\frac{a}{z-1}\xi \quad \Longrightarrow \quad \xi(z) = \begin{cases} D, & a=0; \\ D\ (1-z)^{-c}, & a=c; \end{cases} \quad D \in \mathbf{C}$$

*The equation for $\varphi$:*

$$\frac{d\varphi}{dz} = \begin{cases} \left(-\frac{c}{z} + \frac{c}{z-1}\right)\varphi + \left(\frac{r}{z} + \frac{1-r}{z-1}\right)D, & a=0; \\ -\frac{c}{z}\varphi + \left(\frac{r}{z} + \frac{1-r}{z-1}\right)\frac{D}{(1-z)^a}, & a=c; \end{cases}$$

*The equation for $\varphi$ can be integrated. If $c \notin \mathbf{Z}$ we obtain (by variation of parameters):*

$$\varphi(z) = \begin{cases} E(1-z)^c z^{-c} + D\left[\frac{r}{c} - \frac{1}{c+1}\ z(1-z)^c F(1+c,1+c,2+c;z)\right], & a=0; \\ Ez^{-c} + D\left[\frac{r}{c}(1-z)^{-c} - \frac{1}{c+1}\ zF(1+c,1+c,2+c;z)\right], & a=c; \end{cases} \quad E, D, \in \mathbf{C}$$

*If $c \in \mathbf{Z}$, the solution contains a logarithmic term.*

# 10 Appendix 2: Action of the Braid Group and Analytic Continuation

The subject of this Appendix is well known. Let us denote a branch of a transcendent, in one to one correspondence with the monodromy data $\theta_0, \theta_x, \theta_1, \theta_\infty$; $\mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)$), with the following notation:

$$y(x; \theta_0, \theta_x, \theta_1, \theta_\infty; \mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)),$$

Its analytic continuation, when $x$ goes around a loop around one of the singular points $x = 0, 1, \infty$, is obtained by an action of the pure braid group on the monodromy data. This means that the new branch is:

$$y(x; \theta_0, \theta_x, \theta_1, \theta_\infty; \mathrm{tr}(M_0^\beta M_x^\beta), \mathrm{tr}(M_0^\beta M_1^\beta), \mathrm{tr}(M_1^\beta M_x^\beta)),$$

where $\beta$ is a pure braid, and $M_j \mapsto M_j^\beta$ is its action.

It is convenient to replace (1) by

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0(u)}{\lambda - u_1} + \frac{A_x(u)}{\lambda - u_2} + \frac{A_1(u)}{\lambda - u_3}\right]\Psi,$$

where we have restored three parameters of isomonodromy deformation $u_1, u_2, u_3$. The ordered basis of loops $\gamma_1, \gamma_2, \gamma_3$ is in figure 6. The monodromy matrices which correspond to the loops are $M_0, M_x, M_1$.

When $x$ goes around a loop around $x = 0$, the monodromy data of the system (1) change by the action of the pure braid $\beta_1 \cdot \beta_1$, where $\beta_1$ is the elementary braid which exchanges $u_1$ and $u_2$, namely which continuously deforms $(u_1, u_2, u_3) \mapsto (u_1', u_2', u_3') := (u_2, u_1, u_3)$. The basis $\gamma_1, \gamma_2, \gamma_3$ is deformed, but it is still denoted by $\gamma_1, \gamma_2, \gamma_3$ in figure 7. The monodromy matrices remain unchanged, because the deformation is monodromy preserving. The monodromy matrices obtained by the action of the braid are the monodromy matrices for:

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0(u')}{\lambda - u_1'} + \frac{A_x(u')}{\lambda - u_2'} + \frac{A_1(u')}{\lambda - u_3'}\right]\Psi,$$

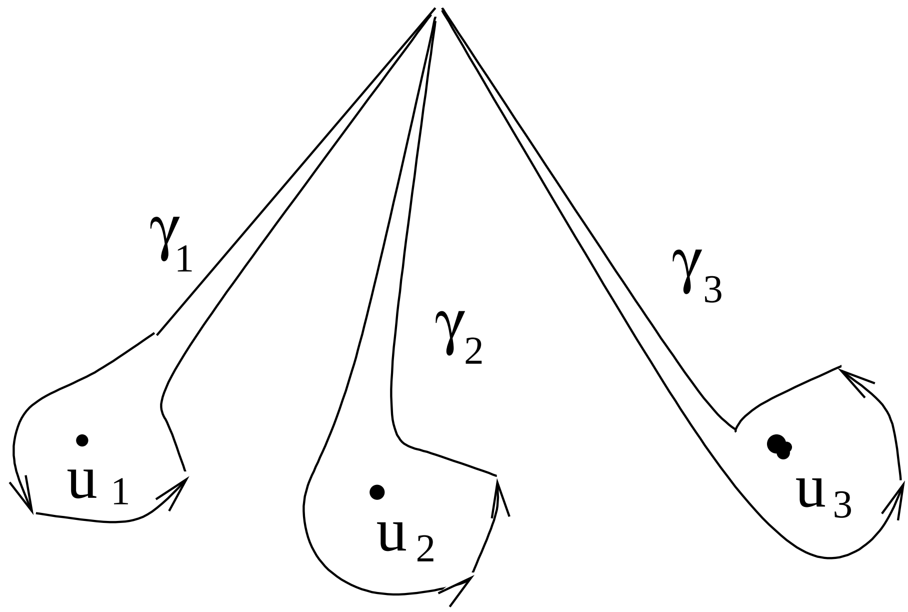

Figure 6:

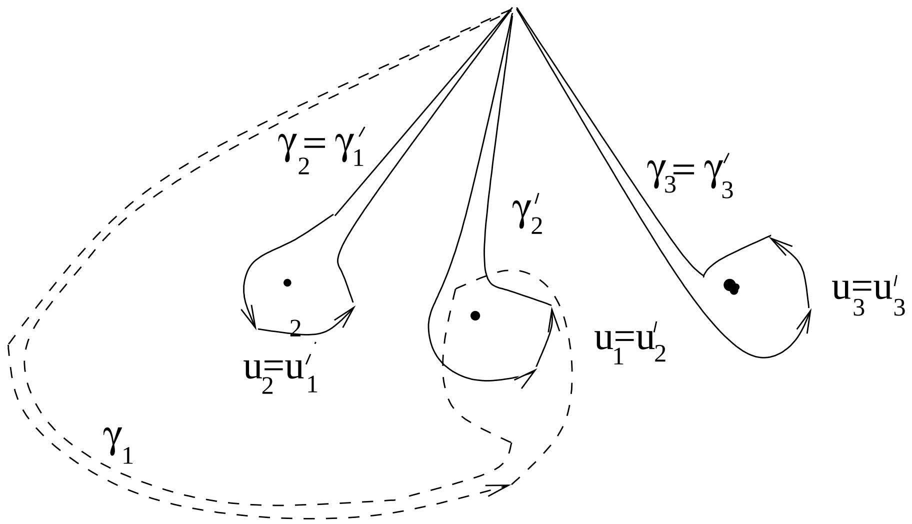

Figure 7:

w.r.t to the basis $\gamma_1', \gamma_2', \gamma_3'$ of figure 7.

We have:

$$\gamma_1' = \gamma_2, \quad \gamma_2' = \gamma_2^{-1}\gamma_1\gamma_2, \quad \gamma_3' = \gamma_3.$$

Therefore:

$$M_0^{\beta_1} = M_x, \quad M_x^{\beta_1} = M_x M_0 M_x^{-1}, \quad M_1^{\beta_1} = M_1.$$

If follows that:

$$M_0^{\beta_1\cdot\beta_1} = M_x M_0 M_x^{-1},$$
$$M_x^{\beta_1\cdot\beta_1} = M_x M_0 M_x M_0^{-1} M_x^{-1},$$
$$M_1^{\beta_1\cdot\beta_1} = M_1;$$

$$\operatorname{tr}(M_0^{\beta_1\cdot\beta_1} M_x^{\beta_1\cdot\beta_1}) = \operatorname{tr}(M_0 M_x)$$

$$\operatorname{tr}(M_0^{\beta_1\cdot\beta_1} M_1^{\beta_1\cdot\beta_1}) = -\operatorname{tr}(M_0M_1) - \operatorname{tr}(M_1M_x)\operatorname{tr}(M_0M_x) + 4\big(\cos(\pi\theta_\infty)\cos(\pi\theta_x) + \cos(\pi\theta_1)\cos(\pi\theta_0)\big),$$

$$\operatorname{tr}(M_1^{\beta_1\cdot\beta_1} M_x^{\beta_1\cdot\beta_1}) = \operatorname{tr}(M_1M_x)\big[\operatorname{tr}(M_0M_x)^2 - 1\big] + \operatorname{tr}(M_0M_x)\operatorname{tr}(M_0M_1) +$$
$$-4\big[\cos(\pi\theta_\infty)\cos(\pi\theta_x) + \cos(\pi\theta_1)\cos(\pi\theta_0)\big]\operatorname{tr}(M_0M_x) + 4\big[\cos(\pi\theta_\infty)\cos(\pi\theta_0) + \cos(\pi\theta_1)\cos(\pi\theta_x)\big].$$

We observe that $\operatorname{tr}(M_0M_x)$ is unchanged. This means that the log-behavior at $x = 0$ is preserved when $x$ goes around a small loop around $x = 0$.

When $x$ goes around a loop around $x = 1$, the monodromy data of the system (1) change by the action of the pure braid $\beta_2 \cdot \beta_2$, where $\beta_2$ is the elementary braid which exchanges $u_2$ and $u_3$, namely which continuously deforms $(u_1, u_2, u_3) \mapsto (u_1', u_2', u_3') := (u_1, u_3, u_2)$. The basis $\gamma_1, \gamma_2, \gamma_3$ is deformed, and we still denote it $\gamma_1, \gamma_2, \gamma_3$ in figure 8. The monodromy matrices remain unchanged. The monodromy matrices obtained by the action of the braid group are the monodromy matrices w.r.t to the basis $\gamma_1', \gamma_2', \gamma_3'$ of figure 8. We have:

$$\gamma_1' = \gamma_1, \quad \gamma_2' = \gamma_3, \quad \gamma_3' = \gamma_3^{-1}\gamma_2\gamma_3.$$

$$M_0^{\beta_2} = M_0, \quad M_x^{\beta_2} = M_1, \quad M_1^{\beta_2} = M_1 M_x M_1^{-1}.$$

Therefore:

$$\operatorname{tr}(M_0^{\beta_2\cdot\beta_2} M_x^{\beta_2\cdot\beta_2}) = -\operatorname{tr}(M_0M_x) - \operatorname{tr}(M_0M_1)\operatorname{tr}(M_1M_x)$$
$$+4\big(\cos(\pi\theta_\infty)\cos(\pi\theta_1) + \cos(\pi\theta_0)\cos(\pi\theta_x)\big),$$

$$\operatorname{tr}(M_0^{\beta_2\cdot\beta_2} M_1^{\beta_2\cdot\beta_2}) = \operatorname{tr}(M_0M_1)\big[\operatorname{tr}(M_1M_x)^2 - 1\big] + \operatorname{tr}(M_0M_x)\operatorname{tr}(M_1M_x) +$$
$$-4\big[\cos(\pi\theta_\infty)\cos(\pi\theta_1) + \cos(\pi\theta_0)\cos(\pi\theta_x)\big]\operatorname{tr}(M_1M_x) + 4\big[\cos(\pi\theta_\infty)\cos(\pi\theta_x) + \cos(\pi\theta_0)\cos(\pi\theta_1)\big],$$

$$\operatorname{tr}(M_1^{\beta_2\cdot\beta_2} M_x^{\beta_2\cdot\beta_2}) = \operatorname{tr}(M_1M_x).$$

We observe that $\operatorname{tr}(M_1M_x)$ is unchanged. This means that the log-behavior at $x = 1$ is preserved when $x$ goes around a small loop around $x = 1$.

Any pure braid can be obtained by the two generators $\beta_1 \cdot \beta_1$, $\beta_2 \cdot \beta_2$ introduced above.

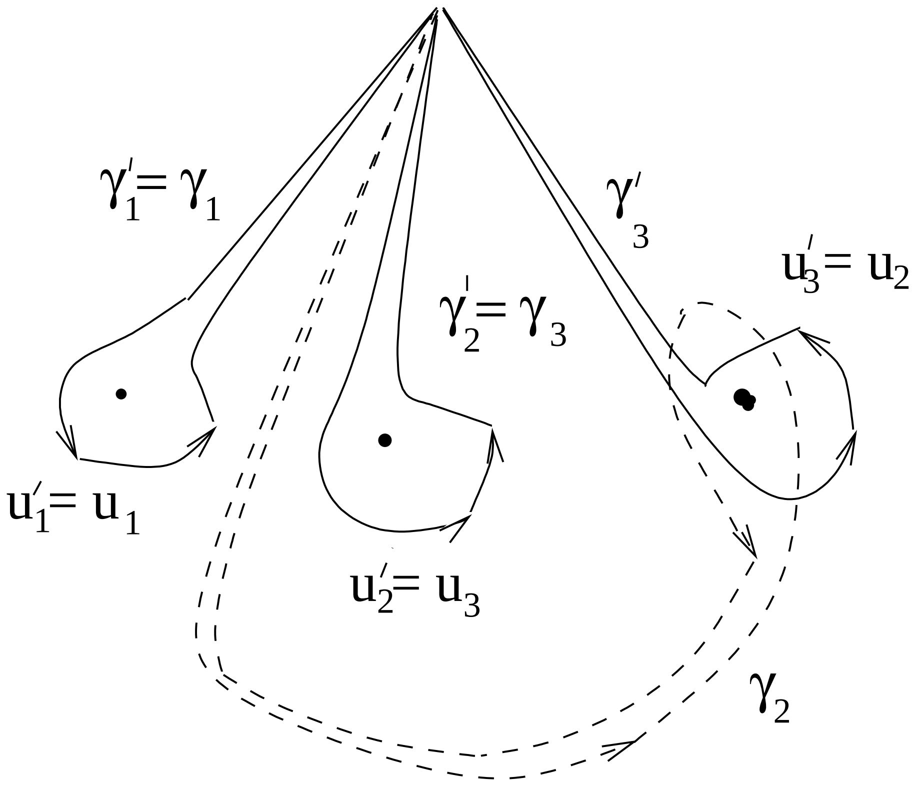


Figure 8:

# 11 Appendix 3: Functions introduced in [22]

$$(a)_n := a(a+1)(a+2)...(a+n-1), \qquad (a)_{-n} := \frac{1}{(a-1)(a-2)(a-3)...(a-n)}.$$

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n.$$

$$G(a,b,c;z) = \sum_{n=1}^{c-1} (-1)^{n-1}(n-1)! \frac{(a)_{-n}(b)_{-n}}{(c)_{-n}} z^{-n} +$$
$$+ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} \big( [\psi_E(a+n) - \psi_E(a) +$$
$$+ \psi_E(b+n) - \psi_E(b) - \psi_E(c+n) + \psi_E(c) - \psi_E(1+n) + \psi_E(1)] + \ln z \big) z^n.$$

$$g(a,b,c;z) = \sum_{n=1}^{c-1} (-1)^{n-1}(n-1)! \frac{(a)_{-n}(b)_{-n}}{(c)_{-n}} z^{-n} +$$
$$+ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} [\psi_E(a+n) + \psi_E(b+n) - \psi_E(c+n) - \psi_E(1+n) + \ln z] z^n.$$

$$g_1(a,b,c;z) = \sum_{n=1}^{c-1} (-1)^{n-1}(n-1)! \frac{(a)_{-n}(b)_{-n}}{(c)_{-n}} z^{-n} +$$
$$+ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} [\psi_E(1-a-n) + \psi_E(b+n) - \psi_E(c+n) - \psi_E(1+n) + \ln z] z^n.$$

$$g_0(a,b,c;z) = \sum_{n=1}^{c-1}(-1)^{n-1}(n-1)!\frac{(a)_{-n}(b)_{-n}}{(c)_{-n}}z^{-n}+$$

$$+\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}[\psi_E(1-a-n)+\psi_E(1-b-n)-\psi_E(c+n)-\psi_E(1+n)+\ln z]z^n.$$

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