The competition numbers of complete tripartite graphs

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Abstract

For a graph G, it is known to be a hard problem to compute the competition number $k(G)$ of the graph G in general. In this paper, we give an explicit formula for the competition numbers of complete tripartite graphs.

Keywords: competition graph, competition number, complete tripartite graph

1. Introduction and Main Result

Cohen [1] introduced the notion of a competition graph in connection with a problem in ecology in 1968 (also see [2]). The *competition graph* $C(D)$ of a digraph $D = (V, A)$ is an undirected graph $G = (V, E)$ which has the same vertex set V and has an edge between distinct two vertices $x, y \in V$ if there exists a vertex $a \in V$ such that $(x, a), (y, a) \in A$.

Roberts [5] observed that, for any graph, the graph with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The minimum number of such isolated vertices was called the *competition number* of the graph G and was denoted by $k(G)$. It is difficult to compute the competition number of a graph in general as Opsut [4] has shown that the computation of the competition number of a graph is an NP-hard problem.

But, for a graph in some special classes, it is easy to obtain the competition number of the graph. The following are some of known results for competition numbers.

- If G is a chordal graph which has no isolated vertices, then $k(G) = 1$.
- If G is a triangle-free connected graph, then $k(G) = |E(G)| - |V(G)| + 2$.

As corollaries of these results, we have

- $k(K_n) = 1, \quad k(K_{n,n}) = n^2 - 2n + 2, \quad k(K_{n_1,n_2}) = (n_1 - 1)(n_2 - 1) + 1.$

Competition graphs and the competition numbers of graphs are closely related to edge clique covers and the edge clique cover numbers of the graphs. A *clique* of a graph G is an empty set or a subset of $V(G)$ such that its induced subgraph of G is a complete graph. A clique consisting of 3 vertices is called a *triangle*. An *edge clique cover* (or an *ECC* for short) of a graph G is a family of cliques of G such that each edge of G is contained in some clique in the family. The minimum size of a edge clique cover of G is called the *edge clique cover number* (or the *ECC number* for short) of the graph G, and is denoted by $\theta_e(G)$.

Opsut [4] showed that, for any graph G, the competition number satisfies an inequality $\theta_e(G) - |V(G)| + 2 \leq k(G) \leq \theta_e(G)$. Dutton and Brigham [3] showed that a graph G is a competition graph of some digraph if and only if $\theta_e(G) \leq |V(G)|$, and also characterized the competition graphs of acyclic digraphs by using ECCs as follows.

($*$) A graph G is the competition graph of an acyclic digraph if and only if there exist an ordering $v_1, ..., v_n$ of the vertices of G and an edge clique cover $\{S_1, ..., S_n\}$ of G such that $v_i \in S_j \Rightarrow i < j$.

For other applications of ECCs, see [6].

In this paper, we give an explicit formula for the competition numbers $k(K_{n,n,n})$ of complete tripartite graphs $K_{n,n,n}$. The following is our main result which will be proven in the following section:

Theorem 1. *For $n \geq 2$, the competition number of the complete tripartite graph $K_{n,n,n}$ is given by the following:*

$$k(K_{n,n,n}) = n^2 - 3n + 4. \tag{1.1}$$

2. Proof of Theorem 1

Let $K_{n,n,n}$ $(n \geq 2)$ be a complete tripartite graph on 3 disjoint sets $A := \{a_1, ..., a_n\}$, $B := \{b_1, ..., b_n\}$, and $C := \{c_1, ..., c_n\}$.

Put $\triangle(i,j,l) := \{a_i, b_j, c_l\}$ for $1 \leq i,j,l \leq n$. Then $\triangle(i,j,l)$ are triangles of $K_{n,n,n}$. Note that there are n^3 triangles. Let $\mathcal{F} := \{\triangle(i,j,l) \mid l = i+j-1, 1 \leq i,j \leq n\}$, where $i+j-1$ are reduced to modulo n. Note that $|\mathcal{F}| = n^2$.

Lemma 2. *The family $\mathcal{F}$ is an edge clique cover of $K_{n,n,n}$ of minimum size. In particular, $\theta_e(K_{n,n,n}) = n^2$.*

Proof. Take any edge a_ib_j between A and B, then both a_i and b_j are in $\triangle(i,j,l) \in \mathcal{F}$, where $l = i+j-1$ (mod n). Take any edge a_ic_l between A and C, then both a_i and c_l are in $\triangle(i,j,l) \in \mathcal{F}$, where $j = l-i+1$ (mod n). Take any edge b_jc_l between B and C, then both b_j and c_l are in $\triangle(i,j,l) \in \mathcal{F}$, where $i = l-j+1$ (mod n). Thus the family $\mathcal{F}$ is an ECC of $K_{n,n,n}$.

Since all maximal cliques of $K_{n,n,n}$ have size 3, we may assume that an ECC of $K_{n,n,n}$ of minimum size consists of triangles. Since $|E(K_{n,n,n})| = 3n^2$ and that a triangle has 3 edges, any ECC of $K_{n,n,n}$ has size at least n^2, i.e. $\theta_e(K_{n,n,n}) \geq n^2$. Since $|\mathcal{F}| = n^2$, we conclude $\theta_e(K_{n,n,n}) = n^2$ and thus we have that $\mathcal{F}$ is an ECC of $K_{n,n,n}$ of minimum size. □

Lemma 3. *If two triangles $\triangle, \triangle' \in \mathcal{F}$ are distinct, then $|\triangle \cap \triangle'| \leq 1$.*

Proof. By the definition of $\triangle(i,j,l)$, once two of i, j, l are given, the remaining one is uniquely determined. □

Lemma 4. *We can label the vertices of $K_{n,n,n}$ as $v_1, \ldots, v_{3n}$, and choose trianlges $\triangle_1, ..., \triangle_{3n-3} \in \mathcal{F}$ so that*

$$\triangle_1 \cup ... \cup \triangle_i \subseteq \{v_1, ..., v_{i+3}\} \tag{2.1}$$

for $1 \leq i \leq 3n-3$.

Proof. We label the vertices of $K_{n,n,n}$ as $v_1, v_2, \ldots, v_{3n}$ in the following order:

$$a_1, b_1, c_1, a_2, b_n, c_n, a_n, b_2, c_2, a_{n-1}, b_{n-1}, c_{n-1}, a_{n-2}, b_{n-2}, c_{n-2}, ..., a_3, b_3, c_3 \qquad (2.2)$$

More precisely, we put $v_1, ..., v_9$ as above, and $v_{3s+7} = a_{n-s}$, $v_{3s+8} = b_{n-s}$, $v_{3s+9} = c_{n-s}$ for $1 \leq s \leq n-3$. Now choose triangles from $\mathcal{F}$ and label them as follows.

$$\begin{array}{lll}
\triangle_1 = \{a_1, b_1, c_1\}, & \triangle_2 = \{a_2, b_n, c_1\}, & \triangle_3 = \{a_1, b_n, c_n\}, \\
\triangle_4 = \{a_n, b_1, c_n\}, & \triangle_5 = \{a_n, b_2, c_1\}, & \triangle_6 = \{a_1, b_2, c_2\}, \\
\triangle_7 = \{a_{n-1}, b_2, c_n\}, & \triangle_8 = \{a_2, b_{n-1}, c_n\}, & \triangle_9 = \{a_1, b_{n-1}, c_{n-1}\}, \\
\quad\vdots & \quad\vdots & \quad\vdots \\
\triangle_{3s+4} = \{a_{n-s}, b_2, c_{n-s+1}\}, & \triangle_{3s+5} = \{a_2, b_{n-s}, c_{n-s+1}\}, & \triangle_{3s+6} = \{a_1, b_{n-s}, c_{n-s}\}, \\
\qquad = \{v_{3s+7}, v_6, v_{3s+6}\}, & \qquad = \{v_4, v_{3s+8}, v_{3s+6}\}, & \qquad = \{v_1, v_{3s+8}, v_{3s+9}\}, \\
\quad\vdots & \quad\vdots & \quad\vdots \\
\triangle_{3n-5} = \{a_3, b_2, c_4\}, & \triangle_{3n-4} = \{a_2, b_3, c_4\}, & \triangle_{3n-3} = \{a_1, b_3, c_3\},
\end{array}$$

where $1 \leq s \leq n-3$. Note that $\triangle_i$ are all distinct. Now, we will see that (2.1) holds. For $i = 1, ..., 6$, we can easily check that (2.1) holds. For $i = 7, ..., 3n-3$, it can easily be seen that the vertex of maximum index in $\triangle_i$ has index at most $i+3$. Thus, we conclude $\triangle_1 \cup ... \cup \triangle_i \subseteq \{v_1, ..., v_{i+3}\}$ for $1 \leq i \leq 3n-3$. Hence the lemma holds. □

Now we are ready to prove our main theorem.

Proof of Theorem 1. First, we will show $k(K_{n,n,n}) \geq n^2 - 3n + 4$. Let $k = k(K_{n,n,n})$ for convenience. Then the graph $G := K_{n,n,n} \cup I_k$ is the competition graph of some acyclic digraph D, where I_k denotes a set of k isolated vertices. Then, by $(*)$, we can label the vertices of G as $v_1, \ldots, v_{3n+k}$ so that there exists an ECC $\{S_1, ..., S_{3n+k}\}$ of G satisfying $v_i \in S_j \Rightarrow i < j$. That is, $S_j \subseteq \{v_1, ..., v_{j-1}\}$. Since any edge of G is contained in a triangle and any maximal clique of G has size 3, we may assume that any nonempty clique S_i is a triangle. Therefore we may assume that $S_1 = S_2 = S_3 = \emptyset$. Since $S_4 \subseteq \{v_1, v_2, v_3\}$ and $S_5 \subseteq \{v_1, v_2, v_3, v_4\}$, we may assume that $S_4 = S_5$ by Lemma 3 if they are not empty. Thus the family $\{S_5, S_6, ..., S_{3n+k}\}$ is also an ECC of G, and so we have $\theta_e(G) \leq 3n + k - 4$. However, we know from Lemma 2 that $\theta_e(G) = \theta_e(K_{n,n,n} \cup I_k) = \theta_e(K_{n,n,n}) = n^2$. Hence we have $n^2 \leq 3n + k - 4$, i.e. $k(K_{n,n,n}) = k \geq n^2 - 3n + 4$.

Now we show that $k(K_{n,n,n}) \leq n^2 - 3n + 4$. By Lemma 4, there exist a labeling $v_1, ..., v_{3n}$ of the vertices of $K_{n,n,n}$, and triangles $\triangle_1, ..., \triangle_{3n-3} \in \mathcal{F}$ such that $\triangle_1 \cup ... \cup \triangle_i \subseteq \{v_1, ..., v_{i+3}\}$ for $1 \leq i \leq 3n-3$. Since $|\mathcal{F}| = n^2$, there are $n^2 - 3n + 3$ triangles in $\mathcal{F} \setminus \{\triangle_1, ..., \triangle_{3n-3}\}$. Label those triangles as $T_1, T_2, \ldots, T_{n^2-3n+3}$. Now, we define a digraph D as follows.

$$\begin{aligned} V(D) &= \{v_1, ..., v_{3n}\} \cup \{z_0, z_1, ..., z_{n^2-3n+3}\}, \\ A(D) &= \bigcup_{i=1}^{3n-4} \{(x, v_{i+4}) \mid x \in \triangle_i\} \\ &\quad \cup \{(x, z_0) \mid x \in \triangle_{3n-3}\} \\ &\quad \cup \bigcup_{i=1}^{n^2-3n+3} \{(x, z_i) \mid x \in T_i\}. \end{aligned}$$

Then this digraph D is acyclic. For, vertex z_i has no outgoing arcs for each $i = 0, ..., n^2 - 3n + 3$ and $(v_i, v_j) \in A(D) \Rightarrow i < j$. Since every clique in the ECC $\mathcal{F}$ has a common out-neighbor in D, $E(K_{n,n,n}) \subset E(C(D))$. On the other hand, the in-neighborhood of each vertex in D is either empty or a clique in $\mathcal{F}$, it is true that $E(K_{n,n,n}) \supset E(C(D))$. Thus $C(D) = K_{n,n,n} \cup \{z_0, z_1, ..., z_{n^2-3n+3}\}$. Hence we have $k(K_{n,n,n}) \leq n^2 - 3n + 4$.

Therefore, $k(K_{n,n,n}) = n^2 - 3n + 4$ holds. $\square$

3. Concluding Remarks

In this paper, we compute the competition numbers of complete tripartite graphs on the vertex sets of the same size. We present the following problems for further study:

- What is the competition number of a complete tripartite graphs K_{n_1,n_2,n_3} on the vertex sets of different size?
- What is the competition number of the complete tetrapartite graphs $K_{n,n,n,n}$ (on the vertex sets of the same size)?
- More generally, what is the competition number of a complete multipartite graph $K_{n_1,n_2,...,n_m}$?

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