Tight extensions of distance spaces and the dual fractionality of undirected multiflow problems

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Abstract

In this paper, we give a complete characterization of the class of weighted maximum multiflow problems whose *dual* polyhedra have bounded fractionality. This is a common generalization of two fundamental results of Karzanov. The first is a characterization of commodity graphs H for which the dual of maximum multiflow problem with respect to H has bounded fractionality, and the second is a characterization of metrics d on terminals for which the dual of metric-weighed maximum multiflow problem has bounded fractionality. A key ingredient of the present paper is a *non-metric* generalization, due to the present author, of the *tight span*, which was originally introduced for metrics by by Isbell, Dress, and Chrobak and Larmore. A theory of non-metric tight spans provides a unified duality framework to the weighted maximum multiflow problem, and gives a unified interpretation of combinatorial dual solutions of several known minimax theorems in the multiflow theory.

1 Introduction and main results

Let $G = (V, E, c)$ be an undirected graph with nonnegative edge capacity $c : E \to \mathbf{R}_+$, and let $S \subseteq V$ be terminals and μ a nonnegative weight for each pair of S. A path $P \subseteq E$ is called an *S-path* if its endpoints are distinct vertices in S. A *multiflow* (multicommodity flow) is a set $\mathcal{P}$ of S-paths in G together with nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying the capacity constraint $\sum_{P \in \mathcal{P}: e \in P} \lambda(P) \leq c(e)$ for each $e \in E$. The *weighted maximum multiflow problem* with respect to G and (S, μ), denoted by $M(G; S, \mu)$, is formulated as:

$$M(G; S, \mu) \quad \text{Maximize} \sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P) \text{ over all multiflows } (\mathcal{P}, \lambda) \text{ in } G,$$

where $s_P, t_P \in S$ are the endpoints of P. One of the intriguing issues in the multiflow theory is the fractionality of optimal multiflows; see [18] [25, Part VII]. The *fractionality* of (S, μ) is the least positive integer k such that $M(G; S, \mu)$ has a $1/k$-integral optimal flow for *any* graph $G = (V, E, c)$ with $S \subseteq V$ and integral capacity c. If such a k does not exist, the fractionality of (S, μ) is defined to be infinity. The question is:

(F) What is a necessary and sufficient condition for (S, μ) to have bounded fractionality ?

The 0-1 weight cases have a particular combinatorial interest. In this case, 0-1 weight μ can be regarded as a *commodity graph*, and $M(G; S, \mu)$ is the problem of maximizing the total sum of multiflows flowing pairs of terminals specified by $\mu(i,j) = 1$. For example, when S is a 2-set $\{i,j\}$ with $\mu(i,j) = 1$, which corresponds to the single-commodity flow problem, the famous Ford-Fulkerson's maxflow-mincut theorem [12] states that there exists an integral optimal flow. The case where S is a 4-set $\{i,j,k,l\}$ and $\mu(i,j) = \mu(k,l) = 1$ and others are zero corresponds to the two-commodity flow problem. Hu's biflow-mincut theorem [14] says that there exists a half-integral optimal flow. For the case of $\mu(i,j) = 1$ for all distinct $i,j \in S$ (the *free multiflow problem*), Lovász [23] and Cherkassky [7] have shown the existence of half-integral optimal flows. Such results for 0-1 weights μ are further generalized by Karzanov and Lomonosov [21] to a certain class of commodity graphs. In the cases of non 0-1 weights, the so-called *multiflow locking theorem* by Karzanov and Lomonosov [21] states the existence of half-integral optimal flows for a class of cut-decomposable metrics μ. All of those results give sufficient conditions, but a complete answer for (F) is still unknown (even for the 0-1 weight case).

Since $M(G; S, \mu)$ is a linear program, we may think of its dual problem $M^*(G; S, \mu)$, which is given as:

$$
\begin{array}{lll}
M^*(G; S, \mu) & \text{Minimize} & \displaystyle\sum_{e \in E} c(e) l(e) \\
& \text{subject to} & \displaystyle\sum_{e \in P} l(e) \geq \mu(s_P, t_P) \text{ for all } S\text{-paths } P, \\
& & l(e) \geq 0 \quad (e \in E).
\end{array}
$$

Corresponding to the (primal) fractionality above, the *dual fractionality* of (S, μ) for integral μ is the least positive integer k such that $M^*(G; S, \mu)$ has a $1/k$-integral optimal solution with for *any* graph $G = (V, E, c)$ with $S \subseteq V$. Then the dual fractionality problem is:

(F*) What is a necessary and sufficient condition for (S, μ) with integral μ to have bounded dual fractionality ?

As was observed in [17], a necessary condition for bounded dual fractionality is also necessary for bounded primal fractionality. Namely, for fixed (S, μ), if $M(G; S, \mu)$ has a $1/k$-integral optimal flow for any graph G with $S \subseteq V$ and integral capacity c, then $M^*(G; S, \mu)$ has also a $1/k$-integral optimum for any graph G. The converse is not true in general. In particular, the primal fractionality is greater than equal to the dual fractionality.

There are two fundamental results in this direction due to Karzanov. The first is in the 0-1 weight case. For a 0-1 weight μ on S, the *commodity graph* $H_\mu = (S, F_\mu)$ is defined as: $ij \in F_\mu$ if $\mu(i,j) = 1$.

Theorem 1.1 ([17]). *For a 0-1 weight μ on S whose commodity graph H_μ has no isolated vertices, the following two statements hold:*

(1) *If H_μ satisfies:*

  (P) *For any three pairwise intersecting maximal stable sets A, B, C of H_μ we have $A \cap B = B \cap C = C \cap A$,*

then there exists a $1/4$-integer optimal solution to $M^*(G; S, \mu)$ for any graph $G = (V, E, c)$ with $S \subseteq V$.

(2) *If H_μ violates condition* (P)*, then there exists no integer k such that $M^*(G;S,\mu)$ has a $1/k$-integral optimal solution for any graph $G=(V,E,c)$ with $S \subseteq V$.*

This result completely answers (F*) for 0-1 weight cases. Karzanov [18] conjectured that there exists a 1/4-integral optimal flow of $M(G;S,\mu)$ for 0-1 weight μ with the property (P). However, this conjecture is still unsolved.

The second is in the case where μ is a metric on S. By definition, μ is a *metric* on S if μ satisfies $\mu(i,j)=\mu(j,i)\geq 0$, $\mu(i,i)=0$, and the triangle inequality $\mu(i,j)\leq \mu(i,k)+\mu(k,j)$ for all $i,j,k\in S$. In addition, a weight μ is called *cyclically even* if μ is integral and $\mu(i,j)+\mu(j,k)+\mu(k,i)$ is an even integer for all $i,j,k\in S$. For a metric μ on S, the *tight span* $T(S,\mu)$, introduced independently by Isbell [16], Dress [11], and Chrobak and Larmore [8], is defined by the set of minimal elements of the polyhedron

$$P(S,\mu)=\{p\in \mathbf{R}^S \mid p(i)+p(j)\geq \mu(i,j)\ (i,j\in S)\} \tag{1.1}$$

We will explain the tight spans in detail later. Karzanov's result for $M^*(G;S,\mu)$ for a metric μ is the following:

Theorem 1.2 ([20])**.** *For a cyclically even metric μ on S, the following two statements hold:*

(1) *If the dimension of $T(S,\mu)$ is at most 2, then there exists a half-integral optimal solution to $M^*(G;S,\mu)$ for any graph $G=(V,E,c)$ with $S\subseteq V$.*

(2) *If the dimension of $T(S,\mu)$ is greater than 2, there is no integer k such that $M^*(G;S,\mu)$ has a $1/k$-integral optimal solution for any graph $G=(V,E,c)$ with $S\subseteq V$.*

Although (2) is not explicit in [20], it is a consequence of his characterization of primitively finite metrics. Therefore, this theorem completely answers (F*) for the metric-weighted case.

The main result of this paper is to give a complete answer to the problem (F*) by establishing a common generalization of the above two theorems of Karzanov. In fact, Theorem 1.2 holds for non-metric weights, where the polyhedral set $T(S,\mu)$ is defined for non-metric weight μ as in the metric case above. Specifically, our main result is given by the following theorem.

Theorem 1.3. *For a cyclically even weight μ on S, we have:*

(1) *If the dimension of $T(S,\mu)$ is at most 2, there exists a half-integral optimal solution to $M^*(G;S,\mu)$ for any graph $G=(V,E,c)$ with $S\subseteq V$.*

(2) *If the dimension of $T(S,\mu)$ is greater than 2, there is no integer k such that $M^*(G;S,\mu)$ has a $1/k$-integral optimal solution for any graph $G=(V,E,c)$ with $S\subseteq V$.*

It is not so obvious the fact that condition (P) in Theorem 1.1 is equivalent to 2-dimensionality of $T(S,\mu)$ for 0-1 weight μ. We give a direct proof of this fact later.

Our result suggests that we cannot expect a combinatorial min-max theorem in $M(G;S,\mu)$ for fixed (S,μ) with $T(S,\mu)\geq 3$ and any graph G, although we still do not know whether this condition (1) is sufficient for bounded primal fractionality. If it is sufficient, it gives a complete answer for (F).

Overview. The proof of Theorem 1.3 is based on a novel relationship between multiflows and the tight span $T(S,\mu)$ as generalized for non-metric μ. This is the central topic in this paper. A certain duality relationship between multiflows and metrics was pioneered by Onaga and Kakusho [24] and Iri [15] in 70's, and further developed by Lomonosov and Karzanov [22, 17]. In mid 90's, Bandelt, Chepoi, and Karzanov revealed the significance of tight spans in multiflow theory [3, 5, 19, 20]. Our approach to Theorem 1.3 also lies on this line of research developments.

For a metric space (S,μ), a metric space (V,d) is called an *extension* of (S,μ) if $S \subseteq V$ and $\mu(i,j) = d(i,j)$ for all $i,j \in S$. Namely (S,μ) is a submetric of (V,d). It is easy to see that $M^*(G;S,\mu)$ for metric μ is equivalent to the following *minimum extension problem*; see [20, p.240] for example.

$$\text{(MEP) Minimize} \sum_{i,j\in V} c(i,j)d(i,j) \text{ over all extensions } (V,d) \text{ of } (S,\mu),$$

where we extend the capacity c for all pairs of V by defining $c(i,j) = 0$ for $ij \notin E$. A key observation is that an optimum of (MEP) is attained by a *tight* extension because of $c \geq 0$. Here an extension (V,d) of (S,μ) is called *tight* if there is no extension (V,d') of (S,μ) with $d' \leq d$ and $d' \neq d$. Namely, a tight extension is a minimal extension.

The tight extension of metric spaces has been studied independently by Isbell, Dress, and Chrobak and Larmore, and they have shown that for a metric space (X,d) there is an essentially unique *universal* tight extension, called the *tight span*, such that every tight extension of (X,d) is a submetric of the tight span of (X,d). The tight span is realized by the set $T(X,d)$ of minimal elements of $P(X,d) \subseteq \mathbf{R}^X$ defined in (1.1), endowed with the l_∞-metric. The above-mentioned universality property of $(T(X,d),l_\infty)$ can be explained as follows.

Theorem 1.4 ([16, 11, 8])**.** *For a finite metric space (X,d), the space $(T(X,d),l_\infty)$ has the following properties:*

(1) *Let $h = h_{X,d} : X \to \mathbf{R}^X$ be a map defined as*

$$(h(i))(j) = d(i,j) \quad (i,j \in X) \tag{1.2}$$

(i.e., h is the i-th column vector of the distance matrix d). Then we have $h(i) \in T(X,d)$ and $d(i,j) = \|h(i) - h(j)\|_\infty$ for $i,j \in X$, and therefore (X,d) is isometrically embedded into $(T(X,d),l_\infty)$. In particular, $(T(X,d),l_\infty)$ is an extension of (X,d).

(2) *$(T(X,d),l_\infty)$ is a tight extension of (X,d).*

(3) *For any tight extension (Y,d') of (X,d), there uniquely exists a map $\phi : Y \to T(X,d)$ such that $\phi(i) = h(i)$ for $i \in X$ and $d'(i,j) = \|\phi(i) - \phi(j)\|_\infty$ for $i,j \in Y$.*

By this theorem, one can easily see that the minimum extension problem (MEP) is equivalent to the following *continuous location problem* in $T(S,\mu)$ (a variant of the p-median problem, called the *p-facility minisum problem with mutual communication* [26]).

$$\begin{aligned} \text{Minimize} \quad & \sum_{i,j\in V} c(i,j)\|p^i - p^j\|_\infty \\ \text{subject to} \quad & p^i \in T(S,\mu) \quad (i \in V), \\ & p^i = h_{S,\mu}(i) \quad (i \in S). \end{aligned}$$

Therefore, the several nice properties of multiflows and metric extension problems can be characterized in terms of the geometric properties of the space $T(S,\mu)$ [20].

To apply this idea to $M^*(G;S,\mu)$ for general weights μ, we first generalize Theorem 1.4 to a *non-metric distance space* (X,d). Here we call d a *distance* if $d(i,j) = d(j,i) \geq 0$ and $d(i,i) = 0$ for $i,j \in X$. That is, the triangle inequality is not imposed. Namely, a distance is nothing but a nonnegative weight. When we emphasize non-metricity, we call it a *non-metric* distance. To begin with, we generalize the concept of extension of metrics for non-metric distances as follows. For a distance space (X,d), let us call a distance space (Y,d') an *extension* of (X,d) if if it satisfies $X \subseteq Y$, $d'(i,j) = d(i,j)$ for $i,j \in X$, and

$$d'(i,k) + d'(k,j) \geq d'(i,j) \quad (k \in Y \setminus X,\ i,j \in Y). \tag{1.3}$$

This condition prohibits shortcuts using a point in $Y \setminus X$. Just as in the metric case, an extension (Y,d') of (X,d) is called *tight* if there is no extension (Y,d'') of (X,d) with $d'' \leq d'$ and $d'' \neq d'$. To represent non-metric distance spaces in the l_∞-space, we extend the l_∞-metric to the l_∞-distance between subsets of points. The l_∞-distance $\|P,Q\|_\infty$ for two subsets P,Q is defined as

$$\|P,Q\|_\infty = \min\{\|p-q\|_\infty \mid p \in P, q \in Q\}. \tag{1.4}$$

We simply denote $\|P,\{q\}\|_\infty$ by $\|P,q\|_\infty$ As an extension of Theorem 1.4, we obtain the following, where it should be clear that $T(X,d)$ is defined also for a non-metric d as the set of minimal elements of $P(X,d)$.

Theorem 1.5. *For a finite distance space (X,d), the metric space $(T(X,d),l_\infty)$ has the following properties:*

(1) *Let $\eta = \eta_{X,d} : X \to 2^{T(X,d)}$ be a set-valued map defined as*

$$\eta(i) = T(X,d) \cap \{p \in \mathbf{R}^X \mid p(i) = 0\}. \tag{1.5}$$

Then we have $d(i,j) = \|\eta(i),\eta(j)\|_\infty$ for $i,j \in X$, and therefore (X,d) is isometrically embedded into $(2^{T(X,d)},l_\infty)$. In particular, $(\eta(X) \cup T(X,d),l_\infty)$ is an extension of (X,d), where $\eta(X) \cup T(X,d)$ means $\{\eta(i)\}_{i\in X} \cup \{\{p\}\}_{p\in T(X,d)} \subseteq 2^{T(X,d)}$.

(2) *$(\eta(X) \cup T(X,d),l_\infty)$ is a tight extension of (X,d).*

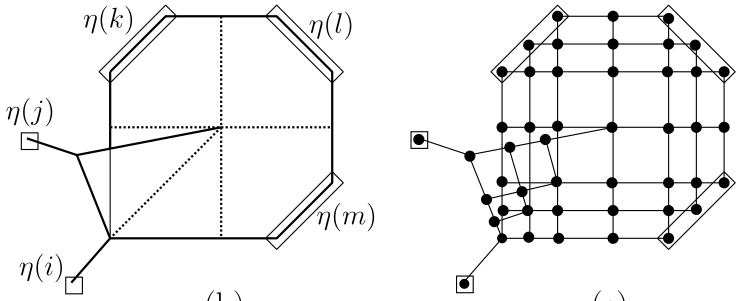
(3) *For any tight extension (Y,d') of (X,d), there uniquely exists a map $\phi : Y \setminus X \to T(X,d)$ such that $d'(i,j) = \|\phi(i),\phi(j)\|_\infty$ and $d'(i,k) = \|\phi(i),\eta(k)\|_\infty$ for $i,j \in Y \setminus X$ and $k \in X$.*

Therefore, the set $T(X,d)$ for non-metric (X,d) is justified to be called the *tight span* of (X,d). Property (1) has already been shown in our previous paper [13, Theorem 2.4]. The essential distinction between Theorems 1.4 and 1.5 is the way of embedding of X to $T(X,d)$. Namely, we represent a non-metric distance as the l_∞-distance among *subsets* in $T(X,d)$. If d is a metric, $\eta(i)$ consists of a single point $h(i)$ (Lemma 3.2). If d violates the triangle inequality, some $\eta(i)$'s are "regions" in $T(X,d)$. Figure 1 (a) and (b) illustrate a distance d on 5-set $X = \{i,j,k,l,m\}$ and its tight span $T(X,d)$, respectively. In the case, $T(X,d)$ is a 2-dimensional polyhedral complex obtained by gluing three pentagons and three triangles along the broken lines. Then $\eta(k)$, $\eta(l)$, and $\eta(m)$ are segments in $T(X,d)$ caused by violations of triangle inequalities, e.g., $2 = d(j,l) + d(l,m) < d(j,m) = 3$.

Non-metric tight spans provide a unified duality framework to the weighted maximum multiflow problems for general nonnegative weights. Problem $M^*(G;S,\mu)$ is equivalent

$$d = \begin{array}{c|ccccc} & i & j & k & l & m \\ \hline i & 0 & 2 & 3 & 4 & 2 \\ j & 2 & 0 & 3 & 3 & 3 \\ k & 3 & 3 & 0 & 1 & 3 \\ l & 4 & 3 & 1 & 0 & 1 \\ m & 2 & 3 & 3 & 1 & 0 \end{array}$$

(a)



(b) (c)

Figure 1: (a) distance d, (b) tight span $T(X, d)$, and (c) $T(X, d) \cap Z$

to a certain minimum extension problem similar to (MEP); see Section 5. Then, by Theorem 1.5, it is further transformed equivalently to

$$\begin{array}{rll} \text{(TSD) Minimize} & \displaystyle\sum_{i,j \in V} c(i,j) \|p^i - p^j\|_\infty & \\ \text{subject to} & p^i \in T(S, \mu) & (i \in V), \\ & p^i \in \eta_{S,\mu}(i) & (i \in S). \end{array}$$

We call it the *tight-span-dual* to the weighted maximum multiflow problem. Here p^i for $i \in S$ is not be fixed to a point $h(i)$ but is constraint to the region $\eta(i)$. In a sense, p^i is a *(vector) potential* at $i \in V$, and $\|p^i - p^j\|_\infty$ is a *potential difference.* In a single-commodity case; X is a 2-set, $T(X, d)$ is a segment, and therefore p^i can be regarded as a scalar potential. Theorem 1.3 (1) follows from the following characterization when the continuous location problem (TSD) becomes a *discrete* location problem in $T(X, d)$.

Theorem 1.6. *For a rational distance μ on a finite set S, the following two statements hold:*

(1) *If the dimension of $T(S, \mu)$ is at most 2. there exists a finite set of points Z in $T(S, \mu)$ such that for any graph $G = (V, E, c)$ with $S \subseteq V$, the optimal solution of* (TSD) *for $(G; S, \mu)$ can be taken from Z, i.e.,* (TSD) *is equivalent to the discrete location problem:*

$$\begin{array}{rll} \text{(TSD-Z) Minimize} & \displaystyle\sum_{i,j \in V} c(i,j) \|p^i - p^j\|_\infty & \\ \text{subject to} & p^i \in T(S, \mu) \cap Z & (i \in V), \\ & p^i \in \eta_{S,\mu}(i) \cap Z & (i \in S). \end{array}$$

(2) *In addition, if μ is cyclically even, we can take Z such that the l_∞-distance among Z is a multiple of* $1/2$.

Figure 1 (c) illustrates the point Z in this theorem as the black points; also see Figure 20 for further examples. In a sense, the above Z can be regarded as *integer points* in $T(S, \mu)$, although Z is not a subset of the ordinary integer points $\mathbf{Z}^S$ in general. Furthermore, solutions of (TSD-Z) provide *combinatorial* dual solutions to $M(G; S, \mu)$ and this gives a unified interpretation of the combinatorial dual of several known minimax theorems in the multiflow theory mentioned above. Indeed, the constraints in (TSD-Z) imply that it is an optimization over certain partitions of V. For example, consider a distance of a 2-set, which corresponds to the single-commodity case Then its tight span is a line segment, and Z can be taken to be its endpoints, and therefore (TSD-Z) is the problem of finding a minimum cut; see [20, p. 241] for a related argument.

An intuitive reason why the 2-dimensionality of $T(X,d)$ implies bounded dual fractionality is the following well-known property of the l_∞-metric; see [9, p. 31].

$$(\mathbf{R}^2, l_\infty) \text{ is isomorphic to } (\mathbf{R}^2, l_1) \text{ by the map } (x_1, x_2) \mapsto (\frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{2}).$$

In fact, $(T(X,d), l_\infty)$ will turn out to be obtained by gluing certain l_∞-spaces (Proposition 6.2). If $\dim T(X,d) \leq 2$, then $T(X,d)$ is a 2-complex of l_1-spaces and therefore has nice decomposability properties.

Karzanov's proof of Theorem 1.2 is based on his elegant characterization of *minimizable graphs* [19], and a number of properties of modular closures and least generating graphs (LG-graphs) of metrics [20]. Here, minimizable graphs are graphs G with property that (MEP) with its graph metric becomes the discrete location problem on G.

Such a graph metric approach does not seem to be extended to the case of non-metric distances. In particular, we do not know an analogue of LG-graphs and modular closures of non-metric distances. Instead, our proof of Theorem 1.6 relies mainly on Theorem 1.5 and the geometry of the space $T(X,d)$.

This paper is organized as follows. We begin with examining elementary properties of distance extensions in Section 2, study geodesic properties of $T(X,d)$ in Section 3, and then give a proof of Theorem 1.5 in Section 4. Our proof is based on Dress' original proof of Theorem 1.4 with a careful treatment of the "partial" triangle inequalities (1.3). In Section 5, we verify that $M^*(G; S, \mu)$ is indeed equivalent to (TSD). In Sections 6, 7, and 8, we further study the geometry of $T(X,d)$ with connection to primitive extensions. In particular, we devise the *billiard construction* to draw a certain grid on $T(X,d)$ as in Figure 1 (c), which is an alternative to the modular closure approach. By combining a simple modification of Karzanov's *orbit splitting method* [20], we give an explicit construction of the minimal Z in Theorem 1.6 (1). Then, we prove Theorem 1.6 (1) in Section 9 and Theorem 1.3 (2) in Section 10. Section 11 is devoted to proving the half-integrality assertion in Theorem 1.6 (2). In Section 12, we verify that condition (P) in Theorem 1.1 is indeed equivalent to 2-dimensionality of the tight span of 0-1 distances, and also give an explicit combinatorial construction of tight spans for 2-dimensional 0-1 distances. Concluding Section 13 gives some remarks.

Notation. We use the following notation. Let $\mathbf{R}_+$ be the set of nonnegative real. Let $\mathbf{Z}$ be the set of integer. The set of functions from a set X to $\mathbf{R}$ is denoted by $\mathbf{R}^X$. For $p, q \in \mathbf{R}^X$, $p \leq q$ means $p(i) \leq q(i)$ for each $i \in X$. For $p \in \mathbf{R}^X$ and $S \subseteq X$, the restriction of p to S is denoted by $p|_S$. Similarly, for a distance d on X and $S \subseteq X$, the restriction of d to S is denoted by $d|_S$. The l_∞-length $\|p - q\|_\infty$ is often simply denoted by $\|p, q\|_\infty$ or $\|p, q\|$. The *characteristic vector* $\chi_S \in \mathbf{R}^X$ of $S \subseteq X$ is defined as: $\chi_S(i) = 1$ for $i \in S$ and $\chi_S(i) = 0$ for $i \notin S$. We simply denote $\chi_{\{i\}}$ by χ_i, which is the i-th unit vector. For an undirected graph $G = (V, E)$, the edge between $i, j \in V$ is denoted by ij or ji. ii means a loop. E_V is the set of (non-loop) edges of the complete graph on vertices V. A *stable set* S of G is a subset of vertices such that there is no edge both of whose endpoints belong to S. A *partition* of undirected graph G is a partition of vertices such that each part is a stable set. In particular, if there is a bipartition, G is called *bipartite*. G is called a *complete multipartite graph* if G has a partition such that each pair of vertices in different parts has an edge. We often regard distance d on X as $d \in \mathbf{R}_+^{E_X}$. We often identify a distance space (X,d) with distance d. We use the standard terminology of polytope theory such as *faces*, *extreme points*, *polyhedral complex* or *subdivision* and so on; see [27].

2 Elementary properties of distance extensions

We begin with some elementary properties of extensions of distance spaces. Let (X, d) be a finite distance space. A distance space (Y, d') is an *extension* of (X, d) if it satisfies $X \subseteq Y$, $d'(i,j) = d(i,j)$ for $i, j \in X$, and

$$d'(i,k) + d'(k,j) \geq d'(i,j) \quad (i,j \in Y, k \in Y \setminus X). \tag{2.1}$$

An extension (Y, d') of (X, d) is *tight* if there is no extension (Y, d'') of (X, d) such that $d'' \leq d'$ and $d'(i,j) < d''(i,j)$ for some $i, j \in Y$. For notational simplicity, we often use the same d for (X, d) and its extension (Y, d).

It should be noted that if (X, d) is a metric, an extension (Y, d) is *not* a metric in general. However, a tight extension of a metric space is always a metric as follows.

Lemma 2.1. *If d is a metric, then any tight extension (Y, d) of (X, d) is a metric.*

Proof. For a distance space (Y, d), the *metric closure* $(Y, \overline{d})$ defined as

$$\overline{d}(i,j) = \inf\{\textstyle\sum_{k=0}^{m-1} d(i_k, i_{k+1}) \mid i = i_0, i_1, i_2, \ldots, i_m = j : \{i_k\}_{k=1}^{m-1} \subseteq Y,\ m \geq 1\}. \tag{2.2}$$

Namely, $\overline{d}$ is the graph metric on the complete graph endowed with the edge length $d(i,j)$ on each edge ij (if Y is finite). Then we have $\overline{d} \leq d$, and $\overline{d}(i,j) < d(i,j)$ for some $i, j \in Y$ if (Y, d) is non-metric. If (Y, d) is an extension of (X, d), then $\overline{d}(i,j) = d(i,j)$ for $i, j \in X$ holds by the inequalities (2.1). Therefore, if an extension (Y, d) is tight, then (Y, d) coincides with $(Y, \overline{d})$, which is a metric. □

Therefore, our theory of tight extensions is compatible to the theory of metric extensions. The following retraction property is due to Dress [11, p.331, (1.9)] (his proof in [11, p.332, remark] does not use the triangle inequality).

Lemma 2.2 ([11])**.** *There is a map $\psi : P(X, d) \to T(X, d)$ such that*

(1) $\|\psi(p), \psi(q)\|_\infty \leq \|p, q\|_\infty$ *for $p, q \in P(X, d)$, and*

(2) $\psi(p) \leq p$ *for $p \in P(X, d)$ (and thus $\psi(p) = p$ for $p \in T(X, d)$).*

In particular, ψ is a non-expansive retraction from $P(X, d)$ to $T(X, d)$.

Sketch of proof. For $p \in P(X, d)$, let p^* be defined as $p^*(i) = \max_{j \in X}\{d(i,j) - p(j)\}$ for $i \in X$. A map $\tau : P(X, d) \to \mathbf{R}^X$ is defined by $\tau(p) = (p + p^*)/2$. Then we have $\tau(p) \in P(X, d)$, $\|\tau(p), \tau(q)\| \leq \|p, q\|$, and $\tau(p) \leq p$ for $p, q \in P(X, d)$. From this, we have a desired retraction $\psi := \lim_{n \to \infty} \tau^n$. □

The following criterion for the tightness is an extension of (the easy part of) [11, Theorem 1].

Lemma 2.3. *Let (Y, d) be an extension of (X, d). If (Y, d) satisfies*

$$d(i,j) = \begin{cases} \max_{k \in X}\{d(j,k) - d(i,k)\} & (i \in Y \setminus X, j \in X), \\ \max_{k,l \in X}\{d(k,l) - d(i,k) - d(j,l)\} & (i,j \in Y \setminus X), \end{cases} \tag{2.3}$$

then (Y, d) is tight.

Proof. Note that ($\geq$) in (2.3) always holds for any extension by definition. Let (Y, d') be another extension of (X, d) with $d' \leq d$. For $i \in Y \setminus X, j \in X$, we have

$$d'(i,j) \leq d(i,j) = \max_{k \in X}\{d(j,k) - d(i,k)\} \leq \max_{k \in X}\{d(j,k) - d'(i,k)\} \leq d'(i,j). \qquad (2.4)$$

For $i, j \in Y \setminus X$, we have

$$\begin{aligned} d'(i,j) &\leq d(i,j) = \max_{k,l \in X}\{d(k,l) - d(i,k) - d(j,l)\} \\ &\leq \max_{k,l \in X}\{d(k,l) - d'(i,k) - d'(j,l)\} \leq d'(i,j). \qquad (2.5) \end{aligned}$$

Therefore we have $d = d'$, and (Y, d) is tight. $\square$

3 The space $T(X, d)$ and its geodesic properties

We define two polyhedral sets $P(X, d), T(X, d) \subseteq \mathbf{R}^X$ as

$$P(X,d) = \{p \in \mathbf{R}^X \mid p(i) + p(j) \geq d(i,j) \ (i,j \in X)\}, \qquad (3.1)$$

$$T(X,d) = \text{the set of minimal elements in } P(X,d). \qquad (3.2)$$

In a sense, $P(X, d)$ and $T(X, d)$ are *the space of one-element extensions* and *the space of one-element tight extensions*, respectively. To see this, consider one-element extension $(X \cup \{k\}, d)$ of (X, d). Then a vector $\{d(i, k)\}_{i \in X}$ satisfies $d(i, k) + d(j, k) \geq d(i, j)$, and hence $d(\cdot, k) \in P(X, d)$. Conversely, $p \in P(X, d)$ determines a one-element extension by $d(i, k) := p(i)$. It will turn out that this space $T(X, d)$ of one-element tight extensions governs all possible tight extensions.

We introduce the undirected graph $K_{X,d}(p) = K(p)$ associated with $p \in P(X, d)$ which is a fundamental tool to investigate the space $T(X, d)$; see [11, Section 3] or [13, Section 3]. For $p \in P(X, d)$, we define the graph $K(p) = (X, E(p))$ by

$$ij \in E(p) \overset{\text{def}}{\Longleftrightarrow} p(i) + p(j) = d(i,j) \quad (i,j \in X). \qquad (3.3)$$

Note that $E(p)$ may contain loop edges, like ii for $i \in X$. The graph $K(p)$ expresses the information of facets of $P(X, d)$ which contain p.

Let $F(p)$ denote the face of $P(X, d)$ that contains p in its relative interior. Then one can easily see that the following characterization of elements of $T(X, d)$; see also [11, 13].

Lemma 3.1. *For $p \in P(X, d)$, the following conditions are equivalent.*

(a) *p is in $T(X, d)$.*

(b) *For any $i \in X$, there is $j \in X$ such that $p(i) + p(j) = d(i, j)$.*

(c) *$K(p)$ has no isolated vertices.*

(d) *$F(p)$ is bounded.*

In particular, $T(X, d)$ is the union of the bounded faces of $P(X, d)$. Recall two embedding maps $h = h_{X,d} : X \to \mathbf{R}^X$ and $\eta = \eta_{X,d} : X \to 2^{T(X,d)}$ defined as

$$\begin{aligned} (h(i))(j) &= d(i,j) \quad (i,j \in X), \qquad (3.4) \\ \eta(i) &= T(X,d) \cap \{p \in \mathbf{R}^X \mid p(i) = 0\}. \qquad (3.5) \end{aligned}$$

Point $h(i)$ and region $\eta(i)$ are related in the following way.

Lemma 3.2. *Let (X,d) be a finite distance space. If an element $k \in X$ satisfies*

$$d(i,k) + d(k,j) \geq d(i,j) \quad (i,j \in X). \tag{3.6}$$

Then we have $\eta(k) = \{h(k)\}$. In this case, in $K(h(k))$, the vertex k is adjacent to all vertices, i.e, $kj \in E(h(k))$ for any $j \in X$.

Proof. Let $p \in \eta(k)$. Then we have $p(i) \geq d(i,k)$ since $p(k) = 0$. By Lemma 3.1 (b), there is $j \in X$ such that $p(i) + p(j) = d(i,j)$. By (3.6), we have $d(i,k) + d(k,j) \leq p(i) + p(j) = d(i,j) \leq d(i,k) + d(k,j)$. Hence we obtain $p(i) = d(i,k)$. □

For $p, q \in T(X,d)$, consider the image of the segment $[p,q] \subseteq P(X,d)$ by the non-expansive retraction in Lemma 2.2. Then it is a geodesic in $T(X,d)$ between p and q. Therefore we have:

Proposition 3.3. *The metric space $(T(X,d), l_\infty)$ is geodesic, i.e., for $p,q \in T(X,d)$ there is a path in $T(X,d)$ connecting p and q with its length $\|p-q\|_\infty$.*

Next we present a useful way of moving a point $p \in T(X,d)$ to another point in $T(X,d)$ using a stable set of $K(p)$. For a subset of vertices S of a graph, the *neighborhood* $N(S)$ of S is the set of vertices which are adjacent to S and are not in S. For a stable set S of $K(p)$ and a sufficiently small $\epsilon > 0$, one can easily see $p + \epsilon(-\chi_S + \chi_{N(S)}) \in P(X,d)$. The following lemma, which concerns about the condition for $p + \epsilon(-\chi_S + \chi_{N(S)}) \in T(X,d)$, is also easily examined by using Lemma 3.1.

Lemma 3.4. *For $p \in T(X,d)$, let S be a stable set in $K(p)$. If each vertex in $N(S \cup N(S))$ is covered by an edge which is not adjacent to $N(S)$, then for a sufficiently small $\epsilon > 0$, a vector $p^{S,\epsilon}$ defined as*

$$p^{S,\epsilon} = p + \epsilon(-e_S + e_{N(S)}) \tag{3.7}$$

is contained by $T(X,d)$. In particular, any maximal stable set in $K(p)$ is such a set.

As applications of Lemma 3.4, we have the further geodesic properties of $T(X,d)$.

Proposition 3.5. *For $p \in T(X,d)$, we have*

$$p(i) = \|p, \eta(i)\|_\infty \quad (i \in X). \tag{3.8}$$

Moreover there is a path in $T(X,d)$ connecting p and $\eta(i)$ with length $p(i)$.

Proof. Since each $q \in \eta(i)$ is $q(i) = 0$ by definition, we have $p(i) \leq \min_{q \in \eta(i)} \|p - q\|_\infty$. We show ($\geq$) by constructing a path from p to $\eta(i)$ with its length $p(i)$. If $p(i) = 0$, then $p \in \eta(i)$ and therefore (3.8) holds. Now we assume $p(i) > 0$. Take a maximal stable set S containing i. Then move $p \to p^{S,\epsilon}$ as much as $p^{S,\epsilon} \in T(X,d)$. Then we have $\|p, p^{S,\epsilon}\|_\infty = \epsilon$. Set $p \leftarrow p^{S,\epsilon}$. Repeat this process until $p(i) = 0$. By finitely many steps, we obtain a desired path with length $p(i)$. □

The following has already been obtained by [13] (in a slightly strong form). We give a proof for completeness.

Proposition 3.6 ([13])**.** *The following holds:*

$$d(i,j) = \|\eta(i), \eta(j)\|_\infty \quad (i,j \in X). \tag{3.9}$$

Moreover there is a path in $T(X,d)$ connecting $\eta(i)$ and $\eta(j)$ with length $d(i,j)$.

Proof. It is easy to see that there is $p \in \eta(i)$ with $ij \in E(p)$; take a minimal $p \in P(X,d)$ with $p(i) = 0$ and $p(j) = d(i,j)$. Now we may assume $d(i,j) > 0$. Take a maximal stable set S containing j. Move $p \to p^{S,\epsilon}$ as much as $p^{S,\epsilon} \in T(X,d)$. Reset $p \leftarrow p^{S,\epsilon}$, and repeat this process to p until $p(j) = 0$. In this process, the vertex j is always in $N(S)$. Therefore, the resulting path from $\eta(i)$ to $\eta(j)$ has the length $d(i,j)$. □

By the above proposition, we obtain the following.

Corollary 3.7. *The distance space $(\eta(X) \cup T(X,d), l_\infty)$ is an extension of (X,d).*

4 Proof of Theorem 1.5

Theorem 1.5 (1) follows from Proposition 3.6. We show (2).

Proof of Theorem 1.5 (2). We show that $(\eta(X) \cup T(X,d), l_\infty)$ satisfies the condition (2.3) of Lemma 2.3. For $q \in T(X,d), j \in X$, we have

$$\|q, \eta(j)\| = q(j) = d(i,j) - q(i) = \|\eta(i), \eta(j)\| - \|q, \eta(i)\|$$

for some $i \in X$, where we use Propositions 3.3 and 3.5 for the first and the last equalities. For $p, q \in T(X,d)$, we have

$$\begin{aligned} \|p,q\| &= \max_{i \in X} |p(i) - q(i)| = p(i^*) - q(i^*) = d(i^*, j^*) - p(j^*) - q(i^*) \\ &= \|\eta(i^*), \eta(j^*)\| - \|p, \eta(j^*)\| - \|q, \eta(i^*)\|, \end{aligned}$$

where we assume $\max_{i \in X} |p(i) - q(i)| = p(i^*) - q(i^*)$ and $p(i^*) = d(i^*, j^*) - p(j^*)$ for some $i^*, j^* \in X$ by Lemma 3.1 (b), and we use Propositions 3.3 and 3.5 for the last equality. □

For the proof of (3), the following lemma is crucial, which corresponds to [11, Theorem 3 (vii)].

Lemma 4.1. *Let (Y,d) be a tight extension of a finite distance space (X,d). The restriction map $(\cdot)|_X : \mathbf{R}^Y \to \mathbf{R}^X$ is a bijection and an isometry between $T(Y,d)$ and $T(X,d)$.*

Proof. Since (Y,d) is tight for (X,d) and $(\eta(Y) \cup T(Y,d), l_\infty)$ is tight for (Y,d), $(\eta(Y) \cup T(Y,d), l_\infty)$ must be tight for (X,d). By the proof of Proposition 3.6, for $i,j \in X$, $\|\eta_{Y,d}(i)|_X, \eta_{Y,d}(j)|_X\| = d(i,j)$ must hold. By Lemma 3.2, we have $h(i) \in T(Y,d)$ for $i \in Y \setminus X$. Therefore, $(\eta_{Y,d}(X)|_X \cup T(Y,d)|_X, l_\infty)$ is an extension of (X,d). By $\|p|_X, q|_X\| \leq \|p,q\|$ for $p,q \in \mathbf{R}^Y$ and the tightness of $(\eta(Y) \cup T(Y,d), l_\infty)$, we have $(T(Y,d)|_X, l_\infty) \simeq (T(Y,d), l_\infty)$. Therefore the restriction map is an isometry and thus an injection. Clearly, $T(Y,d)|_X \subseteq P(X,d)$. By the existence of a non-expansive retraction from $P(X,d)$ to $T(X,d)$ in Lemma 2.2, it must hold $T(Y,d)|_X \subseteq T(X,d)$. To see $T(Y,d)|_X \supseteq T(X,d)$, for $p \in T(X,d)$, take a minimal $\hat{p} \in P(Y,d)$ satisfying $\hat{p}|_X = p$ (by using Zorn's lemma if Y is infinite). Then $\hat{p}$ is also minimal in $P(Y,d)$ and therefore is in $T(Y,d)$. □

We are ready to prove Theorem 1.5 (3).

Proof of Theorem 1.5 (3). Consider $T(Y,d')$ for (Y,d'). Then for $k \in Y \setminus X$, $\eta(k)$ is a single point $h(k)$ by Lemma 3.2. Therefore, by Lemma 4.1, the restriction map $(\cdot)|_X$ induces a desired isometry $\phi : Y \setminus X \to T(X,d)$. Finally, we show the uniqueness of such a map. Now $\phi', \phi'' : Y \setminus X \to T(X,d)$ be such maps. Then, by Proposition 3.5, we have $(\phi'(k))(i) = \|\eta(i), \phi'(k)\| = \|\eta(i), \phi''(k)\| = (\phi''(k))(i)$ for $i \in X, k \in Y \setminus X$. This implies $\phi' = \phi''$. □

5 The tight-span-dual to the weighted maximum multiflow problem

In this section, we show that the dual of the weighted maximum multiflow problem is indeed equivalent to the tight-span-dual (TSD). Let $G = (V, E, c)$ be a graph with nonnegative edge capacity, and μ a distance on $S \subseteq V$. We extend c on every pair of V by $c(i,j) = 0$ for $ij \notin E$.

Proposition 5.1. *Let $\hat{S} = \{\hat{\imath} \mid i \in S\}$ be a disjoint copy of S. We regard μ as a distance on $\hat{S}$. Then we have the following:*

$$\begin{aligned}
&\text{The optimal value of } M(G;S,\mu) \text{ or } M^*(G;S,\mu)\\
&= \min\left\{ \sum_{i,j\in V} c(i,j)d(i,j) \;\middle|\; \begin{array}{l} (\hat{S}\cup V, d) \text{ is an extension of } (\hat{S},\mu), \\ \text{and satisfies } d(i,\hat{\imath}) = 0 \text{ for } i \in S \end{array} \right\}\\
&= \min\left\{ \sum_{i,j\in V} c(i,j)d(i,j) \;\middle|\; \begin{array}{l} (\hat{S}\cup V, d) \text{ is a tight extension of } (\hat{S},\mu), \\ \text{and satisfies } d(i,\hat{\imath}) = 0 \text{ for } i \in S \end{array} \right\}\\
&= \min\left\{ \sum_{i,j\in V} c(i,j)\|p^i - p^j\|_\infty \;\middle|\; p^i \in T(S,\mu)\ (i \in V),\ p^i \in \eta(i)\ (i \in S) \right\}.
\end{aligned}$$

Proof. We show the first equality. Let $(\hat{S} \cup V, d)$ be an extension of $(\hat{S}, \mu)$. Then define $l \in \mathbf{R}_+^E$ as $l(ij) = d(i,j)$ for $ij \in E$. Then, by definition of the extension (2.1) and $d(i,\hat{\imath}) = 0$ for $i \in S$, we have

$$\sum_{e\in P} l(e) \geq d(s_P, t_P) = d(s_P, t_P) + d(s_P, \hat{s_P}) + d(t_P, \hat{t_P}) \geq d(\hat{s_P}, \hat{t_P}) = \mu(s_P, t_P), \quad (5.1)$$

and therefore ($\leq$). Conversely, let $l \in \mathbf{R}_+^E$ be a nonnegative weight on edges satisfying $\sum_{e\in P} l(e) \geq \mu(s_P, t_P)$. Let $d_{G,l}$ be the graph metric of G with edge length l. From this, we define a distance d on $\hat{S} \cup V$ as

$$d(\hat{\imath},\hat{\jmath}) = \mu(\hat{\imath},\hat{\jmath}),\ d(i,\hat{\jmath}) = d_{G,l}(i,j),\ d(i,j) = d_{G,l}(i,j). \quad (5.2)$$

By $\sum_{e\in P} l(e) \geq \mu(s_P,t_P)$ and $d(i,j) \leq l(ij)$ for $ij \in E$, we may replace l by d, and the objective value does not increase. By construction, we have $d(i,\hat{\imath}) = 0$. By $\sum_{e\in P} l(e) \geq \mu(s_P,t_P)$, d is an extension of μ. Therefore we have ($\geq$). The second inequality follows from $c \geq 0$. The third follows from Theorem 1.6 and $d(i,\hat{\imath}) = 0 \Leftrightarrow p^i \in \eta(i)$. □

The proof of Theorem 1.6 (1) is based on the formulation of Proposition 5.1. However, to prove Theorem 1.3 (2), we cannot use this formulation. We shall explain the reason. It is known that $M^*(G : S,\mu)$ is also equivalent to:

$$\text{Minimize } \textstyle\sum_{i,j\in V} c(i,j)d(i,j) \text{ over metric } d \text{ on } V \text{ with } d|_S \geq \mu \quad (5.3)$$

This is a variant of the so-called Onaga-Kakusho-Iri Japanese theorem [24, 15]. If μ is a metric, we may assume $d(i,j) = \mu(i,j)$ for $i,j \in S$, and therefore we obtain (MEP). Then the dual fractionality is the least positive integer k such that for all $V \supseteq S$ the polyhedron

$$\{d\text{: metric on } V \mid d|_S \geq \mu\ \} + \mathbf{R}_+^{E_V} \quad (5.4)$$

is $1/k$-integral. On the other hand, the formulation of Proposition 5.1 is a linear optimization over the face, determined by $d(i,\hat{i}) = 0$, of the *extension polyhedron*

$$\{d\text{: distance on } V \cup \hat{S} \mid (V,d) \text{ is an extension of } (\hat{S},\mu)\ \} + \mathbf{R}_+^{E_{V\cup\hat{S}}}. \tag{5.5}$$

One can easily see that (5.4) is a projection of the face of the extension polyhedron (5.5). Therefore, the fractionality of (5.4) may be better than that of (5.5). Such a phenomenon is caused by non-metricity of μ; in the metric case, this projection is bijection. For example, consider the extension polyhedron (5.5) for the distance μ on 4-set $\{1,2,3,4\}$ defined as: $\mu(1,2) = \mu(3,4) = 1$ and others are zero. Then, by using Theorem 9.1 and Proposition 12.3, one can show that (5.5) for this μ is half-integral. On the other hand, the polyhedron (5.4) is integral; this is a consequence of Hu's biflow-mincut theorem [14]. Therefore, to prove Theorem 1.3 (2), we have to investigate the polyhedron (5.4).

Remark 5.2. The formulation in Proposition 5.1 naturally provides a similar duality framework for the *weighted maximum multiport-multiflow problem*; the terminals are disjoint subsets $\mathcal{S}$ of V, the weight μ is defined on pairs of $\mathcal{S}$, and we pack $\mathcal{S}$-paths (the set of paths whose ends belong to distinct terminals in $\mathcal{S}$) fractionally with maximizing the total sum of weight μ. In this case we simply replace $p^i \in \eta(i)$ $(i \in S)$ by $p^i \in \eta(S)$ $(i \in S \in \mathcal{S})$ in (TSD).

6 The metric structure of faces of $T(X,d)$

The proof of Theorem 1.6, needs further investigation of metric properties of $T(X,d)$. We begin with a characterization of the dimension of the face $F(p)$ for $p \in T(X,d)$ by the graph $K(p)$. Note that $\dim F(p)$ is equal to $|X|$ minus the rank of the matrix whose column vectors are $\{\chi_i + \chi_j \mid ij \in E(p)\}$. Since the rank of a matrix which has two 1's in each column can be graph-theoretically characterized, we have the following; see [11] or [13, Section 3].

Proposition 6.1. *For $p \in T(X,d)$, we have*

$$\dim F(p) = \textit{the number of bipartite components of } K(p), \tag{6.1}$$

where loops are regarded as odd cycles.

Since the metric space $(T(X,d), l_\infty)$ is geodesic (Proposition 3.3), it is obtained by gluing the metric spaces (F, l_∞) of faces F of $T(X,d)$. The next proposition concerns about the shape of (F, l_∞).

Proposition 6.2. *Let F be a k-dimensional face of $T(X,d)$. Then the metric space (F, l_∞) is isomorphic to a polytope Q in the k-dimensional l_∞-space $(\mathbf{R}^k, l_\infty)$ represented as*

$$Q = \left\{ x \in \mathbf{R}^k \;\middle|\; \begin{array}{ll} b_{ij} \leq x_i + x_j \leq b'_{ij} & (1 \leq i \leq j \leq k), \\ c_{ij} \leq x_i - x_j \leq c'_{ij} & (1 \leq i < j \leq k) \end{array} \right\} \tag{6.2}$$

for some $b_{ij}, b'_{ij}, c_{ij}, c'_{ij} \in \mathbf{R}$. Moreover, the isomorphism is induced by the restriction map $(\cdot)|_S : \mathbf{R}^X \to \mathbf{R}^S$ for some $S \subseteq X$ with cardinality k.

Proof. Take any $p^* \in F$ in its relative interior. By Proposition 6.1, the graph $K(p^*)$ has exactly k bipartite components with partitions $(A_1,B_1),(A_2,B_2),\ldots,(A_k,B_k)$. Take $j_i \in A_i$ for $i = 1,\ldots,k$. We show that the restriction map $(\cdot)_{\{j_1,\ldots,j_k\}} : \mathbf{R}^X \to \mathbf{R}^{\{j_1,\ldots,j_k\}}$

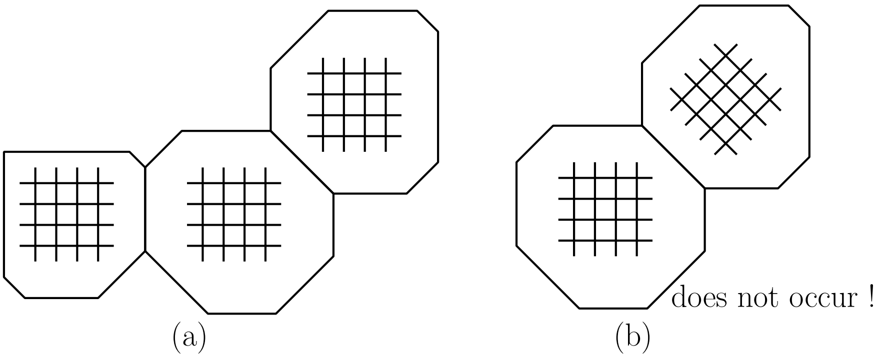


Figure 2: Gluing l_1-octagons

induces an isomorphism between (F, l_∞) and $((F)_{\{j_1,\dots,j_k\}}, l_\infty)$. Take $p, q \in F$ with $p \neq q$. Then both $K(p)$ and $K(q)$ contain $K(p^*)$ as a subgraph. Now we assume $\|p - q\|_\infty = p(i^*) - q(i^*) > 0$. If $i^* \notin \bigcup_{1 \le i \le k} A_i \cup B_i$, then i^* is in some nonbipartite component in $K(p^*)$. Let E be the set of edges of this nonbipartite component. Then, the linear equality system $\{p(i) + p(j) = d(i,j) \ (ij \in E)\}$ has full rank, and therefore its solution is unique. This implies $p(i^*) = q(i^*)$ which contradicts $p(i^*) - q(i^*) > 0$. If $i^* \in A_i \cup B_i$, then there is $i' \in A_i \cup B_i$ with $i^*i' \in E(p)$, and we have $p(i^*) - q(i^*) = d(i^*, i') - p(i') - q(i^*) \le q(i') - p(i')$. Therefore $\|p - q\|_\infty$ is attained also by i'. Since there is a path from i^* to j_i, $\|p - q\|_\infty$ is attained by j_i. This concludes that the restriction map $(\cdot)_{\{j_1,\dots,j_k\}}$ is an isometry.

Next we show that $(F)_{\{j_1,\dots,j_k\}}$ is represented as (6.2). Let $p \in F$. For $j \in A_i$, there is a path from j to j_i. By substituting the equality $p(i') + p(i'') = d(i', i'')$ along the path, the jth component of p can be represented as $p(j) = b + p(j_i)$ for some $b \in \mathbf{R}$ Similarly if $j \in B_i$, we have $p(j) = c - p(i_j)$ for some $c \in \mathbf{R}$. Substitute such relations to the inequalities $p(j') + p(j'') \ge d(j', j'')$. Then we obtain the desired the linear inequality representation of $(F)_{\{j_1,\dots,j_k\}}$. $\square$

The 2-dimensional case is important for us. In this case, Q is (a Minkowski summand of) an octagon in l_∞-plane. We call it an l_∞-*octagon* (though it is a k-gon for $3 \le k \le 8$). Recall that l_∞-plane is l_1-plane. By the map $(x_1, x_2) \mapsto ((x_1 + x_2)/2, (x_1 - x_2)/2)$, we again obtain an octagon in the l_1-plane. We call it an l_1-*octagon*.

Proposition 6.3. *Let F be a 2-dimensional face of $T(X,d)$. Then the metric space (F, l_∞) is isomorphic to an octagon Q in the l_1-plane represented as*

$$Q = \left\{ x \in \mathbf{R}^2 \;\middle|\; \begin{array}{ll} a_1 \le x_1 \le a_1', & b \le x_1 + x_2 \le b', \\ a_2 \le x_2 \le a_2', & c \le x_1 - x_2 \le c' \end{array} \right\} \tag{6.3}$$

for some $a_1, a_1', a_2, a_2', b, b', c, c' \in \mathbf{R}$.

We can draw the l_1-coordinate on 2-dimensional face F, and observe that there are two types of edges of F: edges parallel to an l_1-axis and edges parallel to an l_1-axis. The next proposition says that if $\dim T(X,d) \le 2$, the metric space $(T(X,d), l_\infty)$ is constructed by gluing l_1-octagons along same type of edges; see Figure 2.

Proposition 6.4. *Suppose $\dim T(X,d) \le 2$. Let F, F' be 2-dimensional faces of $T(X,d)$ with $\dim F \cap F' = 1$. The edge $e = F \cap F'$ is parallel to an l_1-axis on F if and only if e is parallel to an l_1-axis on F'.*

Proof. Let F be a 2-dimensional face of $T(X,d)$. Take $p \in F$ in its relative interior. Then $K(p)$ has exactly two bipartite components K_1 and K_2 with bipartitions (A_1, B_1) and (A_2, B_2), respectively. First we show that

(*1) both K_1 and K_2 are complete bipartite graphs.

Suppose that K_1 is not complete bipartite. There are vertices $i \in A_1$ and $j \in B_1$ such that $ij \notin E(p)$. Take a maximal stable set S containing $\{i,j\}$. For small $\epsilon > 0$, $p + \epsilon(-\chi_S + \chi_{N(S)}) \in T(X,d)$ and $K(p + \epsilon(-\chi_S + \chi_{N(S)}))$ has at least three bipartite components. This contradicts $\dim T(X,d) \leq 2$ (Proposition 6.1). Then we observe, from the projection map of Proposition 6.2, that

(*2) the two l_1-axes on F are parallel to $\chi_{A_1 \cup A_2} - \chi_{B_1 \cup B_2}$ and $\chi_{A_1 \cup B_2} - \chi_{B_1 \cup A_2}$, and

(*3) the two l_∞-axes on F are parallel to $\chi_{A_1} - \chi_{B_1}$ and $\chi_{A_2} - \chi_{B_2}$.

Let e be an edge of F. Since e is one-dimensional face of $T(X,d)$ and is a face of F, the graph K_e corresponding to e has $K(p)$ as a subgraph and has exactly one bipartite component K. Then we show:

(*4) K is complete bipartite if and only if e is parallel to an l_∞-axis of F.

We may assume that $p' := p + \epsilon(\chi_{B_1 \cup B_2} - \chi_{A_1 \cup A_2})$ is in the relative interior of the edge e for $\epsilon > 0$. In $K(p')$, there appears an edge $ij \in E(p')$ such that $i, j \in A_1$, $i, j \in A_2$, $i \in A_1, j \in X \setminus \{A_1 \cup A_2 \cup B_1 \cup B_2\}$, $i \in A_2, j \in X \setminus \{A_1 \cup A_2 \cup B_1 \cup B_2\}$, or $i \in A_1, j \in A_2$. The first four cases, $K(p')$ has exactly one complete bipartite graph K_1 or K_2. In this case, the edge e is parallel to an l_∞-axis on F by (*3). The last case, $K(p')$ has exactly one bipartite graph which is not complete. In this case, the edge e is parallel to an l_1-axis on F by (*2). Since the property (*4) is independent of F, we have done. □

Therefore, there are two types of edges in $T(X,d)$. An edge e of $T(X,d)$ is called an l_∞-*edge* if there is a 2-face containing e such that e is parallel to an l_∞-axis of F. Other edge e is called an l_1-*edge*, which is parallel to an l_1-axis of some 2-face containing e, or is a maximal 1-face; there is no 2-face containing e. This definition is well-defined. By the proof of Proposition 6.4, we obtain a characterization of l_1-edges in term of $K(\cdot)$ below, where the *loop component* of $K(\cdot)$ is a connected component all of whose vertices have a loop. Note that vertices having a loop are pairwise adjacent, and therefore the loop component is a complete graph.

Lemma 6.5. *Let e be an edge of 2-dimensional tight span $T(X,d)$ and p a point in the relative interior of e.*

(1) *e is an l_1-edge if and only if $K(p)$ has exactly one bipartite component and has no non-loop non-bipartite components.*

(2) *e is an l_∞-edge if and only if $K(p)$ has exactly one complete bipartite component and exactly one non-loop non-bipartite component.*

7 Constructions of primitive extensions using $T(X,d)$

As is seen in Proposition 5.1 in Section 5, $M^*(G;S,\mu)$ is equivalent to linear optimization of a face of the extension polyhedron (5.5). For a finite distance space (X,d), the extension polyhedron to $Y \supseteq X$ is

$$\{d'\colon \text{distance on } Y \mid d' \text{ is an extension of } d\ \} + \mathbf{R}_+^{E_Y}. \tag{7.1}$$

In this section, we give several constructions of extreme points of this polyhedron using the geometry of $T(X,d)$, which is a basis for a construction of minimal Z in Theorem 1.6 (1).

An extension (Y,d) of (X,d) is called *an extreme extension* if it is an extreme point of the extension polyhedron. We call an extension (Y,d) of (X,d) *positive* if $d(i,j) > 0$ for $i,j \in Y \setminus X$ with $i \neq j$. This condition excludes the case where the distance matrix d has the same columns in $Y \setminus X$. A positive extreme extension is called *a primitive extension.* Clearly, every extreme extension is a tight extension. By Theorem 1.5, for every finite tight extension (Y,d) of (X,d), there uniquely exists the finite multiset $\{p^i\}_{i\in Y\setminus X} \subseteq T(X,d)$ such that (Y,d) is represented as $(\{\eta(i)\}_{i\in X} \cup \{p^i\}_{i\in Y\setminus X}, l_\infty)$. In the case of a positive tight extension (Y,d), the corresponding set $\{p^i\}_{i\in Y\setminus X} \subseteq T(X,d)$ is all distinct. Motivated by this fact, a finite subset $P \subseteq T(X,d)$ is called *a primitive set* if $(\eta(X) \cup P, l_\infty)$ is a primitive extension. We note an obvious lemma.

Lemma 7.1. *Let (Y,d) be a tight extension of (X,d). If there is extension (Y,d') and (Y,d'') such that $d = (1/2)(d'+d'')$, then both extensions (Y,d') and (Y,d'') are tight.*

We are interested in the case where $d = (1/2)(d'+d'')$ implies $d = d' = d''$. For a finite multiset $P = \{p_i\}_{i\in U} \subseteq T(X,d)$ indexed by U, A distance d^P denotes the corresponding tight extension of (X,d) by P. We analyze the case where d^P is decomposed to $(d'+d'')/2$ for some extensions d', d''. By the previous lemma and Theorem 1.5, there uniquely exist $P' = \{p'_i\}_{i\in U} \subseteq T(X,d)$ and $P'' = \{p''_i\}_{i\in U} \subseteq T(X,d)$ such that $d' = d^{P'}$ and $d'' = d^{P''}$. For notational simplicity, for finite multisets $P, P', P'' \subseteq T(X,d)$, $d^P = (d^{P'} + d^{P''})/2$ means that P, P', and P'' are indexed by some finite set U as above. In particular, for $p \in P$, the corresponding elements of P' and P'' are denoted by p' and p'', respectively.

We investigate the relation among P, P' and P'' in terms of geometric properties of $T(X,d)$. For $p,q \in T(X,d)$, we define the *interval* $I[p,q]$ of p,q in $T(X,d)$ as

$$I[p,q] = \{r \in T(X,d) \mid \|p,r\|_\infty + \|r,q\|_\infty = \|p,q\|_\infty\}. \tag{7.2}$$

We can extend this definition to two subsets P,Q in $T(X,d)$ in a natural way.

Lemma 7.2. *Let P be a finite multiset in $T(X,d)$. Suppose that $d^P = (d^{P'} + d^{P''})/2$ for some multisets $P', P \subseteq T(X,d)$. Then we have the following:*

(1) *$p = (p'+p'')/2$ for $p \in P$.*

(2) *If $p \in I[q,r]$ for $p,q,r \in P$, then $p' \in I[q',r']$ and $p'' \in I[q'',r'']$.*

(3) *If $p \in I[\eta(i),\eta(j)]$ for $p \in P, i,j \in X$, then $p', p'' \in I[\eta(i),\eta(j)]$.*

(4) *If $p \in P$ is contained by some face F of $T(X,d)$, then both p' and p'' are also contained by F.*

Proof. (1) follows from Proposition 3.5, i.e.,

$$p(k) = \|p,\eta(k)\| = (\|p',\eta(k)\| + \|p'',\eta(k)\|)/2 = (p'(k) + p''(k))/2. \tag{7.3}$$

We show (2). The condition $p \in I[q,r]$ means the equality of the corresponding triangle inequality, which is a valid inequality of the extension polyhedron (5.5). Therefore, both $d^{P'}$ and $d^{P''}$ must satisfy this equality.

(3) also follows from the similar reason.

(4) follows from (3). Indeed, $p \in I[\eta(i),\eta(j)]$ is equivalent to $p(i) + p(j) = d(i,j)$ by Proposition 3.5. Recall that this equality defines a face of $T(X,d)$. $\square$

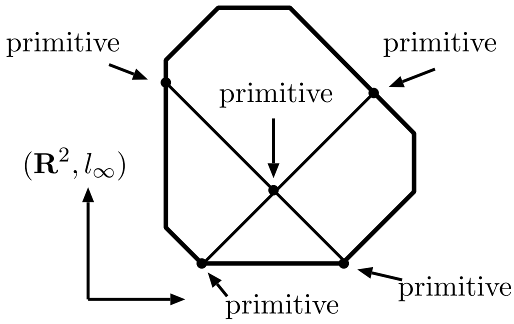


Figure 3: Constructing a primitive set

Consequences of Lemma 7.2 are as follows.

Corollary 7.3. *Any nonempty subset of extreme points of $T(X,d)$ is primitive.*

This is natural since $T(X,d)$ is the space of one-element tight extensions, and its extreme point corresponds to one-element extreme extension.

Corollary 7.4. *Let $P \subseteq T(X,d)$ be a primitive set. Suppose that there are distinct $p,q,r,s \in P$ such that $I[p,q] \cap I[r,s] = \{t\}$ for $t \in T(X,d) \setminus P$. Then $P \cup \{t\}$ is primitive.*

We give one more construction. See also Figure 3.

Lemma 7.5. *Suppose that $\dim T(X,d) \leq 2$. Let $P \subseteq T(X,d)$ be a primitive set. Suppose that there are $p \in P$, a face F of $T(X,d)$ containing p, and a vector $t \in \mathbf{R}^X$ parallel to an l_1-axis of F such that $p + \epsilon t \in F$ for sufficiently small $\epsilon > 0$, Let $p^*(\neq p)$ be an endpoint of the segment $F \cap \{p + \mathbf{R}t\}$. Then $P \cup \{p^*\}$ is primitive.*

Proof. Let e be an edge of F containing p^* as its relative interior. Then we have $\|p^*, p\| = \min_{q \in e} \|q, p\|$. Furthermore p^* is an end of the segment $e \cap \{s \in \mathbf{R}^X \mid \|p,s\| = \|p,p^*\|\}$. Then p^* cannot be decomposed properly into $(p' + p'')/2$ for $p', p'' \in e$ with $\|p, p^*\| = (\|p,p'\| + \|p',p''\|)/2$. $\square$

Corollaries 7.3, 7.4 and Lemma 7.5 gives a successive construction of a primitive set from extreme points of $T(X,d)$, which will be discussed in the next section.

8 Constructions of l_1-grids

The main purpose of this section is to construct a certain kind of l_1-grid on 2-dimensional $T(X,d)$ whose grid-points are primitive. The idea of drawing an l_1-coordinate on 2-dimensional $T(X,d)$ is due to Chepoi [5] for metric tight spans on at most 5-point set. The argument presented here is an extension of Chepoi's idea to general 2-dimensional non-metric tight spans, and connects it to primitive extensions.

Now suppose that $\dim T(X,d) \leq 2$. Recall Propositions 6.3 that $T(X,d)$ can be constructed by gluing l_1-octagons. An *l_1-subdivision* Δ of an l_1-octagon $Q \subseteq \mathbf{R}^2$ is a polyhedral subdivision of Q such that each 2-face of Δ is

(r) a rectangle with edges parallel to l_1-axes of $\mathbf{R}^2$ or

(t) a right-angled isosceles triangle whose two short edges are parallel to l_1-axes of $\mathbf{R}^2$.

We extend this definition to $T(X,d)$. An l_1-*grid* Δ of $T(X,d)$ is a 2-dimensional polyhedral complex satisfying following properties:

(1) The union of the members of Δ is equal to $T(X,d)$.

(2) For each 2-face $F \in T(X,d)$, the restriction $\{C \in \Delta \mid C \subseteq F\}$ induces an l_1-subdivision of an l_1-octagon F.

A vertex of an l_1-grid is called a *grid-point*. The long edge of a triangle is called an l_∞-*edge*, and other edge is called an l_1-*edge*. The graph of l_1-edges behaves nicely as follows.

Proposition 8.1. *Let Δ be an l_1-grid for $T(X,d)$. Then we have the following.*

(1) *For two grid-points p, q in Δ, there is a geodesic between p and q consisting of l_1-edges of Δ.*

(2) *For a grid-point p in Δ. and $i \in X$, there is a grid-point $q \in \eta(i)$ with $\|p,q\|_\infty = \|p,\eta(i)\|_\infty$*

(3) *For $i,j \in X$, there are grid-points $p \in \eta(i), q \in \eta(j)$ with $\|p,q\|_\infty = \|\eta(j),\eta(i)\|_\infty$.*

Proof. (1). Let $L \subseteq T(X,d)$ be a geodesic from p to q. Suppose that L does not lie on l_1-edges of Δ. Then there is a member F in Δ such that L meets a point not in l_1-edges of F. Let F be such first member of Δ. Let p', q' be the endpoint of $L \cap F$. We may assume that p' is a grid-point of Δ and q' is in the boundary of F. Suppose that F is a rectangle. Then we modify L so that p' and q' are connected by a geodesic boundary path in F. Then the resulting path is also geodesic. Suppose that F is a triangle. If q' lies on an l_1-edge (a short edge) of F, then we modify P as above. If q' lies on the long edge of F, then there is a triangle F' in Δ such that F' and F share the long edge by Proposition 6.4. Let $q''(\neq q')$ be the endpoint of $P \cap F'$. Then q'' lies on an l_1-edge of F'. Then we modify L so that p' and q'' are connected by a geodesic boundary path in $F \cup F'$. The modified path is also a geodesic between p and q. Repeat this process, we obtain a desired geodesic consisting of l_1-edges of Δ.

(2). Let $L \subseteq T(X,d)$ be a geodesic from p to $\eta(i)$. Suppose the endpoint $q(\neq p)$ of L is not a grid-point in Δ. Apply the above modification to L starting from p. Let F be the final member in Δ meeting L at a point not in l_1-edges F. Then F must be a triangle, q is on the long edge of F, and the other end p' of $F \cap L$ is the right angled vertex in F. Since the l_∞-length from p' to arbitrary point in the long edge is same, we can take q as one of ends of the long edge of F.

(3). In the proof of Proposition 3.6, we can take $p \in \eta(i)$ as an extreme point of $T(X,d)$; see [13] for detail. Then p is a grid-point of Δ. Apply (2). □

Remark 8.2. Chepoi [6] studied 2-dimensional complex constructed by gluing rectangles and right triangles, and explored some of interesting geodesic and graph-theoretic properties. By using his arguments in [6, Section 7], one can show that the graph of l_1-edges of an l_1-grid of a 2-dimensional tight span is a *hereditary modular graph without induced $K_{3,3}$ and $K_{3,3}^-$*. A hereditary modular graph is just a bipartite graph without isometric k-cycles for $k \geq 6$ [2].

An l_1-grid for $T(X,d)$ may not exist for an irrational distance d; see Remark 8.4. However if d is rational, an l_1-grid always exists. To see this, take an integer k such that such that vertices of $T(X,kd)$ are *even* integral vectors. Project 2-face F of $T(X,kd)$ to 2-dimensional plane as in Proposition 6.2, and transform by $(x_1,x_2) \mapsto$

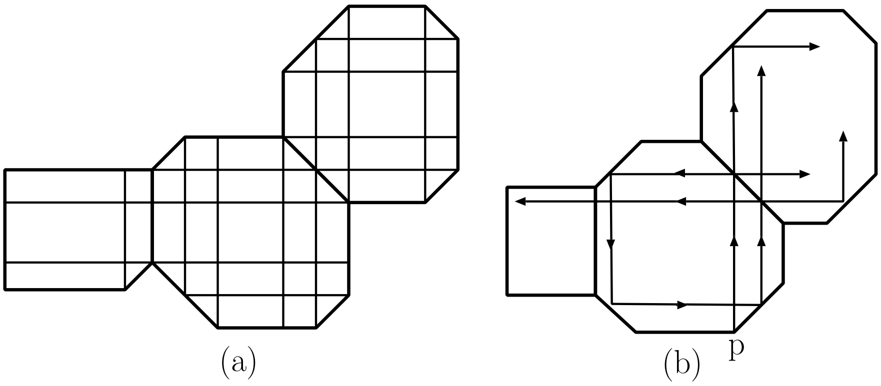


Figure 4: (a) an l_1-grid and (b) the billiard construction

$((x_1+x_2)/2, (x_1-x_2)/2)$, the resulting l_1-octagon is integral since vertices of F are even integral vectors. The lines $\{\mathbf{R}\chi_1 + n\chi_2\}_{n\in\mathbf{Z}} \cup \{m\chi_1 + \mathbf{R}\chi_2\}_{m\in\mathbf{Z}}$ in $\mathbf{Z}^2$ decompose this l_1-octagon into unit squares and right-angled isosceles triangles. Taking the inverse image of this decomposition we can obtain an subdivision of F. We apply this subdivision for all 2-faces of $T(X,d)$. Since two adjacent 2-faces can be put on $\mathbf{R}^2$ as Figure 2 (a) by Proposition 6.4, the union of these subdivision (and maximal 1-faces of $T(X,d)$) forms an l_1-grid of $T(X,d)$. If all l_1-edges of an l_1-grid has the same length $1/k$, we call it the $(1/k)$-*uniform* l_1-grid. In this case, the distance among grid-points are multiple of $1/k$ by Proposition 8.1. As will be seen, the existence of $(1/k)$-uniform l_1-grids guarantees $1/(2k)$-integrality of primitive extensions.

The above construction of an l_1-grid may be redundant since the set of the grid-points is not primitive in general. Here we present a construction of the minimal l_1-grid, called the *billiard construction*, whose grid-points are primitive.

Algorithm: the billiard construction

(b0) $L \leftarrow \emptyset$ and $P \leftarrow$ the set of extreme points of $T(X,d)$.

(b1) Take a point $p \in P$, a 2-face F of $T(X,d)$ containing p, and a feasible l_1-direction t in F such that there is no line in L such that it has a direction t and contains p. If no such p, F, and t exist, then go to (b5).

(b3) Let $l = \{p + \mathbf{R}t\} \cap F$, and let $q(\neq p)$ be the endpoint of l.

(b4) $L \leftarrow L \cup \{l\}$ and $P \leftarrow P \cup \{q\}$.

(b5) Decompose each 2-face of $T(X,d)$ by the lines in L. Δ is the set consisting of resulting rectangles, triangles, and maximal 1-faces of $T(X,d)$ (and their faces).

This algorithm proceeds as the billiard ball. The billiard ball starts from some pocket (vertex) toward an l_1-direction, hits the wall of l_∞-edges in 45-degree angle, and moves toward another l_1-direction, turning 90-degrees around; see Figure 4 (b). The locus of the balls draw an l_1-grid as follows.

Proposition 8.3. *If d is a rational distance, the billiard construction terminates after finite number of steps, and yields an l_1-grid for $T(X,d)$ whose set of grid-points is primitive.*

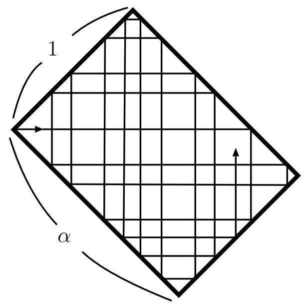


Figure 5: The billiard ball never stops

Proof. It is easy to see that the locus generated by the billiard is a subset of grid-lines of the $(1/k)$-uniform l_1-grid for some integer k. Hence the billiard stops after finite number of steps, and we conclude that Δ is an l_1-grid by the condition of the termination.

To show the latter part, we note the following property of l_1-axes on a 2-face F.

(*) For $p, q \in F$, if $p - q$ is parallel to an l_1-axis, then $I[p, q] = [p, q]$.

Indeed, let K be the graph corresponding to F. Then K has exactly two bipartite components with partitions (A_1, B_1), (A_2, B_2). We may assume that $p - q$ is parallel to $\chi_{A_1 \cup A_2} - \chi_{B_1 \cup B_2}$. Clearly $[p, q] \subseteq I[p, q]$. Take a point $r \in I[p, q]$. Then there is $0 \leq \alpha \leq 1$ such that $r(i) = \alpha p(i) + (1 - \alpha) q(i)$ for all $i \in A_1 \cup A_2 \cup B_1 \cup B_2$. For $j \in X \setminus A_1 \cup A_2 \cup B_1 \cup B_2$, vertex j must have a loop in K, and thus $p(j) = q(j) = 0$. Therefore $r(j) = 0$ must hold since $r \in T(X, d)$ is a minimal element of $P(X, d)$.

The grid-points of Δ are the intersection points of the locus of billiard and the endpoint of lines L. By this fact and property (*), we can apply Corollaries 7.3, 7.4, and Lemma 7.5. Then the grid-points are primitive. □

Remark 8.4. If d is irrational, the billiard may not terminate and therefore $T(X, d)$ has no l_1-grids. For example, consider the distance d on 4-set $\{1, 2, 3, 4\}$ defined by $d(1, 2) = 1$, $d(3, 4) = \alpha$ for irrational positive real α, and others are zero (the case $\alpha = 1$ corresponds the two-commodity flow problem). Then the tight span $T(X, d)$ is the 45-degree rotation of a rectangle in the l_1-plane with the edge length ratio $(1 : \alpha)$. Then the billiard ball started from some pocket never falls other pocket; see Figure 5. In particular, since the intersection points of this locus are primitive, d has infinitely many primitive extensions.

Although this l_1-grid by the billiard is the unique minimal l_1-grid by construction, it is not sufficient to describe all possible primitive sets. Next we explain a simple modification of Karzanov's *orbit splitting method* [20]. The essential distinction is to need to deal with l_∞-edges explicitly.

Two edges e and e' of an l_1-grid Δ are said to be *projective* if there is a sequence of edges $e = e_0, e_1, \ldots, e_m = e'$ such that for $0 \leq i \leq m - 1$ there is a triangle in Δ containing e_i and e_{i+1}, or a rectangle in Δ containing e_i and e_{i+1} as its parallel edges. The projectivity is an equivalence relation on the set of edges of l_1-grid. An equivalence class is called an *orbit*. For an orbit o, *the band* of o is the set consisting of rectangles at least one of whose parallel pairs of edges belongs to o and triangles all of whose edges belong to o.

An l_1-grid is said to be *orientable* if we can orient its edges in such a way that for a rectangle, its parallel edges have the same direction, and for a triangle, its 45-degree angle vertex is a source or a sink; see Figure 6.

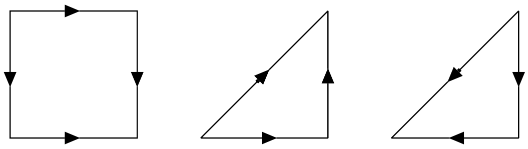


Figure 6: Orientations of a rectangle and a triangle

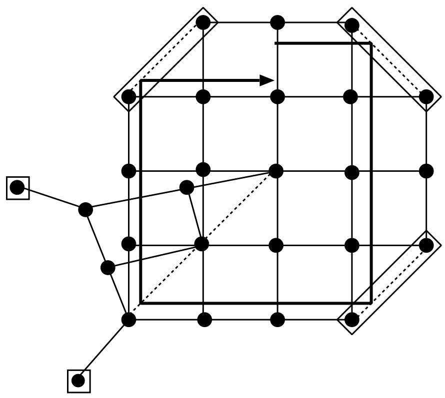


Figure 7: Non-orientable 1/2-uniform grid

It is easy to see that l_1-grid is non-orientable if and only if there is an orbit having a sequence of edges $p_0q_0, p_1q_1, \dots, p_mq_m$ such that $p_m = q_0$, $q_m = p_0$, and for $0 \leq i \leq m-1$ there is a rectangle with edges $\{p_iq_i, p_{i+1}q_{i+1}, p_ip_{i+1}, q_iq_{i+1}\}$ or a triangle with vertices $p_i, p_{i+1}, q_i = q_{i+1}$ or $p_i = p_{i+1}, q_i, q_{i+1}$. Such an orbit is called a *non-orientable orbit*. A characterization of non-orientable orbits using the billiard is also useful. We observe that the billiard ball started from a point p on an edge $e = qr$ of the l_1-grid meets all edges of the orbit of e, and meets such edges only. If this orbit is non-orientable, then the billiard ball started from a point $p = \alpha q + \beta r$ for $\alpha + \beta = 1$, $\alpha, \beta > 0$ returns back to the edge e at the point $\beta r + \alpha q$; see Figure 7. Therefore, the billiard ball goes round this orbit *twice*. Conversely, if this orbit is orientable, the billiard ball goes around this orbit once.

Figure 7 illustrates the 1/2-uniform l_1-grid, generated by the billiard, for the tight span given in Figure 1 (b) in the introduction. This l_1-grid has one non-orientable orbit, and therefore is non-orientable.

Take a point p on the midpoint on arbitrary edge of an orbit o, and start the billiard (b1-b3) from p. In the resulting l_1-grid, if o is non-orientable, then o is transformed into one orientable orbit of the twice size, and if o is orientable, then o is split into two orientable orbits. In particular, the orientability of other orbits are not affected. This operation corresponds to splitting each element of the band of the orbit as in Figure 8, and we can orient this split orbit so that newly added points are sinks. This process is called an *orbit splitting*. Applying orbit splittings to each non-orientable orbit, we have an orientable l_1-grid. Figure 1 (c) is the result of an orbit splitting for Figure 7. By the arguments above, we have:

Proposition 8.5. *The orbit splitting yields an orientable l_1-grid.*

Furthermore, the new grid-points by the orbit splitting for non-orientable orbits after the billiard are also primitive

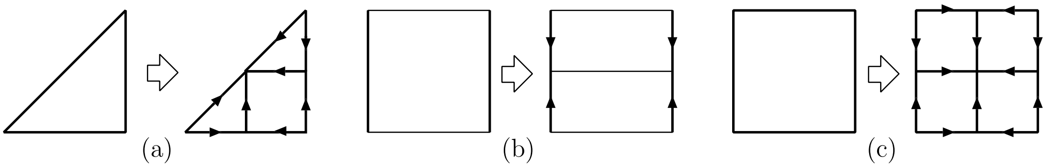


Figure 8: Splitting and orienting a triangle and rectangles

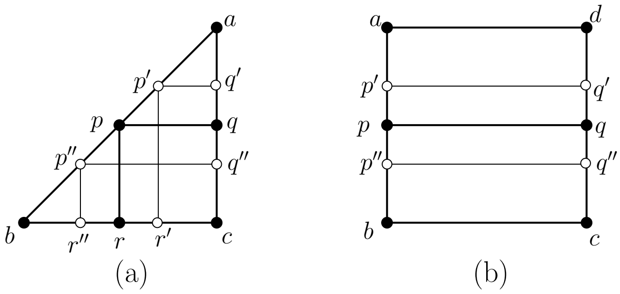


Figure 9: p, q, r in the proof of Proposition 8.6

Proposition 8.6. *The set of grid-points of the l_1-grid generated by the billiard and the orbit splittings to each non-orientable orbit is primitive.*

Proof. Let Δ be the l_1-grid generated by the billiard, and Δ' the l_1-grid obtained by applying the orbit splitting to one non-orientable orbit. It suffices to show that the grid point P of Δ' are primitive. Suppose that $d^P = (d^{P'} + d^{P''})/2$ with $d^{P'} \neq d^{P''}$ for some $P', P'' \subseteq T(X, d)$, where we use the notation of Lemma 7.2. Then there is $p \in P$ such that the corresponding $p' \in P'$ is in the different position to p. We may assume that p is a new point added by the orbit splitting. Therefore there exists an edge $e = ab$ of a non-orientable orbit in Δ' such that p is the midpoint of the edge e. Now we suppose that $p \neq p'$. If e is an l_∞-edge, then p', p'' must be also on the edge e by Lemma 7.2(2),(4). There is a triangle having vertices a, b, c such that the points $q = (b + c)/2$ and $r = (a + c)/2$ are new grid points added by the orbit splitting. Let $q', r' \in P'$ and $q'', r'' \in P''$ be the corresponding points of q, r. We may assume that p' is $\alpha a + \beta b$ for $\alpha > \beta \geq 0$ with $\alpha + \beta = 1$. Then p'' is $\beta a + \alpha b$ by $p = (p' + p'')/2$ (Lemma 7.2 (1)). Moreover, q', q'' are on bc with $q = (q' + q'')/2$, and r', r'' are on ac with $r = (r' + r'')/2$ again by Lemma 7.2. By $p \in I[q, r]$, we have $p' \in I[q', r']$ (Lemma 7.2 (2)). Therefore the possible configuration satisfying $d^P = (d^{P'} + d^{P''})/2$ is exactly the case where q' is $\alpha b + \beta c$ and r' is $\beta a + \alpha c$; see Figure 9 (a). If e is an l_1-edge of a rectangle with edges ab, cd, ac, bd. Then $q = (c + d)/2$ is a new grid point. By the same arguments above. Let $q' \in P'$ be a corresponding point of q. If $p' = \alpha a + \beta b$, then $q' = \beta c + \alpha d$; see Figure 9 (a). Therefore the corresponding points of P' of newly added grid-points of Δ by the orbit splitting are exactly the intersection points by the billiard started from p'. However, by $p \neq p'$, the billiard started from p' goes around the non-orientable orbit of e twice. Thus $p' = p''$. A contradiction. □

We show in the next section that this primitive set P by the billiard and the orbit splitting is in fact finite *universal* primitive set, i.e., every primitive set of $T(X, d)$ is a subset of P.

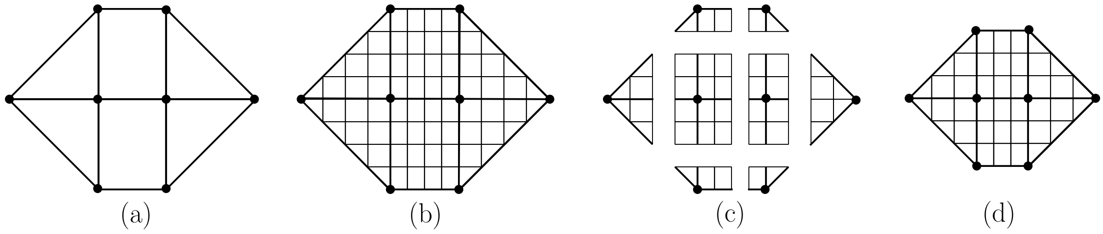


Figure 10: (a) l_1-grid Δ, (b) 1/4-subdivision Δ^4, (c) $T(X,d) \setminus B_i^\circ$, and (d) $(3/4)\Delta^3$

9 Proof of Theorem 1.6 (1)

Suppose that d is a rational distance with $T(X,d) \leq 2$. Theorem 1.6 (1) is immediate from the following, which seems to be a non-metric variant of [20, Lemma 5.1].

Theorem 9.1. *Let Z be the grid-points of the l_1-grid for $T(X,d)$ generated by the billiard and the orbit splittings to each non-orientable orbit. Then every primitive set of $T(X,d)$ is a subset of Z.*

Therefore, every extreme extension of d is represented as d^P for a multiset $P \subseteq Z$. Since the constraint $p^i \in \eta(i)$ in (TSD) or $d(i,\hat{i}) = 0$ defines a face of the extension polyhedron (5.5), Z in Theorem 1.6 (1) can be taken as Z in the above theorem. The primitivity has already shown in Proposition 8.6. For a subset $Z \subseteq T(X,d)$, a tight extension (Y,d) is a *Z-extension* if there is a map $\phi : Y \setminus X \to Z$ such that $d = d^{\phi(Y\setminus X)}$, where we use the notation in Section 7. We prove:

Proposition 9.2. *Let Δ be an orientable l_1-grid for $T(X,d)$ and Z the grid-point of Δ. Every finite tight extension of d can be represented as a convex combination of Z-extensions of d.*

For an integer k, we define the $1/k$-subdivision of Δ as follows. For each rectangle R in Δ, divide it equally into k^2 rectangles congruence to $(1/k)R$ For each triangle T in Δ, divide it into k triangles congruence to $(1/k)T$ and $(k^2-k)/2$ squares obtained by gluing two $(1/k)T$'s along its long edge. Similarly, divide each (maximal) edge in Δ equally into k edges. The resulting l_1-grids, denoted by Δ^k, is called the *$1/k$-subdivision* of Δ; see Figure 10 (b).

Let (Y,d) be a rational tight extension of (X,d). Then there uniquely exists $P = \{P^i \mid i \in Y \setminus X\}$ such that $d = d^P$. Since d' is rational, each P^i is a rational vector. There is an integer k such that that each P^i lies on a grid-point on Δ^k. Let $\mathcal{O} = \{O_1, O_2, \ldots, O_m\}$ be the set of orbits of Δ. By orientability, in Δ^k, each orbit O_i of Δ is split into k-orbits. We take one orbit o_i of Δ^k from each split k-orbits of O_i. For a member T of the band of o_i, the *interior* of T (with respect to o_i) is the subset of T obtained by deleting all of its vertices and its l_1-edges not in o_i. Let B_i° is the union of the interior of members of the band of o_i. Consider $T(X,d) \setminus B_i^\circ$, and glue each connected components along the boundary of B_i°. Then we obtain the polyhedral set congruence to $(k-1)/kT(X,d)$; see Figure 10 (c), (d). Expand this set in factor $k/(k-1)$ so that it becomes the original $T(X,d)$. This contraction/expansion induces a continues map ϕ from $T(X,d) \setminus B_i^\circ$ to $T(X,d)$. We observe that $\phi(P)$ lies on the grid-points of Δ^{k-1}. Since each connected components of $T(X,d) \setminus B^\circ$ has exactly one grid-point of Δ by the orientability of Δ, we can define another map ψ from $T(X,d) \setminus B^\circ$ to Z as: $\psi(p)$ is the unique grid-point of Δ in the connected component containing p. By construction, $d^{\psi(P)}$ is a Z-extension. Then we have:

Lemma 9.3.
$$d^P = \frac{k-1}{k} d^{\phi(P)} + \frac{1}{k} d^{\psi(P)}. \tag{9.1}$$

Proof. Let p, q be grid-points of Δ^k. By Proposition 8.1, there is a geodesic L between p and q consisting of l_1-edges of Δ^k. Then $\phi(l \setminus B_i^\circ)$ is a path between $\phi(p)$ and $\phi(q)$, and its length is $k/(k-1)$-times as longer as the sum of the length of the segments in $L \setminus B_i^\circ$. A sequence of points $\psi(l \setminus B_i^\circ)$ induces a path between $\psi(p)$ and $\psi(q)$ with its length k-times as longer as the sum of the length of the segments in $L \cap B_i$. Therefore, we have
$$\|p, q\| \geq \frac{k-1}{k} \|\phi(p), \phi(q)\| + \frac{1}{k} \|\psi(p), \psi(q)\|. \tag{9.2}$$
Consequently, we have
$$d^P \geq \frac{k-1}{k} d^{\phi(P)} + \frac{1}{k} d^{\psi(P)}. \tag{9.3}$$
The equality must hold by the tightness of d^P. □

Apply the same process to $d^{\phi(P)}$. Recall that $\phi(P)$ lies on the grid-points of Δ^{k-1}. Consequently, we obtain a desired convex combination of Z-extensions.

10 Proof of Theorem 1.3 (2)

The goal of this section is to prove Theorem 1.3 (2). Recall Section 5 that $M^*(G; S, \mu)$ for (S, μ) is equivalent to the linear optimization of over metrics d with $d|_S \geq \mu$. Motivated by this fact, we call a metric (S, d) a *minimal dominant* of (S, μ) if $d \geq \mu$ and there is no metric $d' (\neq d)$ on $d' \geq \mu$ with $d' \leq d$. First we show:

Lemma 10.1. *For a distance (S, μ) with $\dim T(S, \mu) \geq k$, there exists a minimal dominant d of μ such that $\dim T(S, d) \geq k$*

Proof. Let F be a k-dimensional face of $T(S, \mu)$ and p a point of the relative interior of F. By Proposition 6.1, $K(p)$ has exactly k bipartite components with bipartitions $(A_1, B_1), (A_2, B_2), \ldots, (A_k, B_k)$. For small $\epsilon > 0$, points $p_i^\pm := p + \epsilon(\mp \chi_{A_i} \pm \chi_{B_i})$ $(i = 1, \ldots, k)$ are in the relative interior of F. $p_i^\pm$ are vertices of the cross polytope with center p. Then $K(p) = K(p_i^\pm)$. We take edges $u_i^+ u_i^- \in E(p)$ with $u_i^+ \in A_i, u_i^- \in B_i$ for $i = 1, \ldots, k$. By construction of $p_i^\pm$, we have
$$\begin{aligned} \mu(u_i^+, u_i^-) &= p_i^+(u_i^+) + p_i^+(u_i^-) = p_i^+(u_i^+) + 2\epsilon + p_i^-(u_i^-) \\ &= \|\eta(u_i^+), p_i^+\| + \|p_i^+, p_i^-\| + \|p_i^-, \eta(u_i^-)\|. \end{aligned} \tag{10.1}$$
We take $q_i^\pm \in \eta(u_i^\pm)$ with $\|q_i^\pm, p_i^\pm\| = \|\eta(u_i^\pm), p_i^\pm\|$. Then, $\|q_i^+, q_i^-\| = \mu(u_i^+, u_i^-)$ must hold by (10.1). We define a metric μ' on $2k$-set $U := \{u_i^+, u_i^-\}_{i=1}^k$ as $\mu'(u_i^+, u_j^\pm) := \|q_i^+, q_j^\pm\|$. Then $\mu' \geq \mu|_U$ with $\mu'(u_i^+, u_i^-) = \mu(u_i^+, u_i^-)$. Consider $T(U, \mu')$. Then $p|_U$, the restriction of p to U, is also in $P(U, \mu')$, and furthermore we have:

the graph $K_{U,\mu'}(p|_U)$ is exactly k-matching $\{u_i^+ u_i^-\}_{i=1}^k$.

Indeed, $u_i^+ u_i^- \in E_{U,\mu'}(p|_U)$ is obvious. We show $u_i^+ u_j^\pm \notin E_{U,\mu'}(p|_U)$ if $i \neq j$. By construction of $p_i^\pm$, we have
$$\begin{aligned} \mu(u_i^+, u_j^\pm) &\leq \|\eta(u_i^+), p_i^+\| + \|p_i^+, p_j^\pm\| + \|p_j^\pm, \eta(u_j^\pm)\| \\ &= \|\eta(u_i^+), p_i^+\| + \epsilon + \|p_j^\pm, \eta(u_j^\pm)\| \\ &< \|\eta(u_i^+), p_i^+\| + 2\epsilon + \|p_j^\pm, \eta(u_j^\pm)\| \\ &= \|\eta(u_i^+), p\| + \|p, \eta(u_j^\pm)\| = p(u_i^+) + p(u_j^\pm). \end{aligned} \tag{10.2}$$

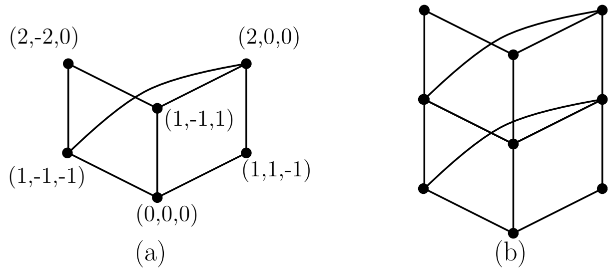


Figure 11: (a) $K_{3,3}^{-}$ and (b) piling two $K_{3,3}^{-}$'s

Therefore, $\dim T(U,\mu') \geq k$ by Proposition 6.1. Let μ'' be a minimal dominant of $\mu|_U$ on U with $\mu'' \leq \mu'$. By $\mu(u_i^+, u_i^-) = \mu'(u_i^+, u_i^-) = \mu''(u_i^+, u_i^-)$, again $p|_U \in P(U,\mu'')$, and $K_{U,\mu''}(p)$ is still k-matching $\{u_i^+ u_i^-\}_{i=1}^k$. Therefore, $\dim T(U,\mu'') \geq k$. We can extend μ'' to a minimal dominant d of μ with $d|_U = \mu''$. Dress' dimension criterion (Theorem 12.1) implies $\dim T(S,d) \geq k$. $\square$

Second we recall the notion of primitive metrics. A metric d on a finite set V is called *extreme* if d lies on an extreme ray of the *metric cone*, which is a polyhedral cone in $\mathbf{R}_+^{E_V}$ defined by the triangle inequalities. An extreme metric d is called *primitive* if $d(i,j) > 0$ for $i,j \in V$ with $i \neq j$; this condition prohibits the situation that the distance matrix corresponding μ has same column (or row) vectors since $d(i,j) = 0$ for distinct i,j implies $d(i,k) = d(j,k)$ for k by the triangle inequalities.

We recall a useful construction of primitive metrics by graphs due to Avis [1]. For an undirected graph $G = (V,E)$, the graph metric d_G on V defined by the shortest path length on vertices, where each edge length has the unit length. A subgraph $G' = (U,F)$ of G is called an *isometric subgraph* if $d_{G'}(i,j) = d_G(i,j)$ for $i,j \in U$. In particular, an *isometric* 4*-cycle* C of G is a 4-cycle $C = \{uv, vw, wz, zu\} \subseteq E$ satisfying $d_G(u,w) = d_G(v,z) = 2$.

Proposition 10.2 ([1], also see [22])**.** *Let $G = (V,E)$ be an undirected graph. Suppose that for each pair of edges $uv, wz \in E$, there is a sequence of isometric 4-cycles $\{\{u_iv_i, u_iu_{i+1}, v_iv_{i+1}, u_{i+1}v_{i+1}\}\}_{i=1}^{m-1}$ such that $u_1v_1 = uv$ and $u_mv_m = wz$. Then the graph metric of d_G is primitive.*

If $\dim T(S,\mu) \geq 3$, we can construct an infinite sequence of tight extensions $(S \cup Z_i, \mu_i)$ with $|Z_{i+1}| > |Z_i|$ for $i = 1,2,3,\ldots$ such that $\mu_i|_{Z_i}$ is a primitive metric, where $|Z_i|$ means the cardinality of Z_i. Take a 3-dimensional face F of $T(S,\mu)$, by Proposition 6.2, F is isomorphic to a (3-dimensional) subspace of $(\mathbf{R}^3, l_\infty)$. Let Z_1 be a 6-element subset of $\mathbf{R}^3$ defined as

$$Z_1 = \{(0,0,0), (1,1,-1), (1,-1,1), (1,-1,-1), (2,0,0), (2,-2,0)\}. \tag{10.3}$$

Then, (Z_1, l_∞) is primitive. Indeed, it is the graph metric of $K_{3,3}^{-}$ (the graph of $K_{3,3}$ minus one edge), which is primitive by Proposition 10.2; see Figure 11 (a). Subsequently, let Z_2 be defined by $(1/2)(Z_1 \cup \{(1,-1,1) + Z_1\})$. Then (Z_2, l_∞) is also primitive since it is (the half of) the graph metric of the graph of piled two $K_{3,3}^{-}$'s; see Figure 11. Therefore, we recursively define Z_k by $1/2(Z_{k-1} \cup \{(1,-1,1) + Z_{k-1}\})$ Then (Z_k, l_∞) is primitive for $k = 1,2,\ldots$. We can take points $P_k \subseteq F$ isomorphic to (a dilation of) Z_k. The corresponding sequence of tight extensions $\mu^{P_k}(k = 1,2,\ldots)$ of (S,μ) is a desired one. Now we are ready to prove Theorem 1.3 (2).

Proof of Theorem 1.3 (2). We show that for any integer $k > 0$, there is $V \subseteq S$ such that the polyhedron (5.4) is not $1/k$-integral. By Lemma 10.1, we can take a minimal dominant d of μ such that $\dim T(S, d) \geq 3$. For an integer l, consider the tight extension d^{P_l} on $V := S \cup P_l$ with respect to some 3-dimensional face F as above. Then d^{P_l} is a minimal element of the polyhedron (5.4). Decompose d^{P_l} into a convex combination $\sum_i \lambda_i d^{P_l^i}$ of extreme points of (5.4), where P_l^i is a (multi)subset in F by Lemma 7.2 (4). By the primitivity of (P_l, l_∞) that is the restriction of d^{P_l} to P_l, there is a summand $d^{P_l^j}$ in the convex combination such that $d^{P_l^j}|_{P_l} = \alpha d^{P_l}|_{P_l}$ for some positive α. Since the face F is bounded, the l_∞-distance among P_l^j is bounded by some positive constant C independent on l. Hence, $d^{P_l^j}$ has an element smaller than C/l. We can take a large l such that $C/l < 1/k$. This implies that the polyhedron (5.4) is not $1/k$-integral. □

Combining Theorem 9.1, we obtain an extension of Karzanov's primitively finiteness result [20].

Corollary 10.3. *A rational distance d on a finite set X has finite primitive extensions if and only if* $\dim T(X, d) \leq 2$.

In particular, the converse of Theorem 1.6 (1) also holds.

11 Proof of the half-integrality

In this section, we prove Theorem 1.6 (2). We begin with the fundamental lemma.

Lemma 11.1. *If d is a cyclically even distance, then the polyhedron $P(X, d)$ is integral.*

Proof. Let p be an extreme point of $T(X, d)$. Then the graph $K(p)$ has no bipartite components. Take a nonbipartite component K. Then there is an odd cycle C in K. We order vertices in C cyclically as $(i_0, i_1, \dots, i_{k-1})$. Then $p(i_0)$ is given by $(\sum_{j=0}^{k-1}(-1)^j d(i_j, i_{j+1}))/2$, where the index is taken by modulo k. By the cyclically evenness, $p(i_0)$ is integral, and thus $p(i_j)$ is integral. Let i' be a arbitrary vertex of K. There is a path connecting i' to C. $p(i')$ is determined by substituting $p(j) + p(j') = d(j, j')$ along this path. Consequently p is integral. □

Now that the 1/4-integrality is easy. Indeed, by the previous lemma, we can take the 1/2-uniform l_1-grid for $T(X, d)$. This 1/2-uniform l_1-grid may be non-orientable, By the orbit splittings, we obtain an orientable 1/4-uniform l_1-grid. Therefore every primitive extension is a multiple of 1/4. In fact, surprisingly, this 1/2-uniform l_1-grid is orientable. The remaining of this section is devoted to proving this fact:

Theorem 11.2. *Suppose that d be a cyclically even distance with* $\dim T(X, d) \leq 2$. *The 1/2-uniform l_1-grid for $T(X, d)$ is orientable.*

As was seen in Figure 7, the fractionality and non-orientability come from l_∞-edges. Motivated by this observation, we introduce the concept " a core" of $T(X, d)$, which is a source of l_∞-edges. An extreme point p of $T(X, d)$ is called a *core* of $T(X, d)$ if $K(p)$ has exactly two non-loop non-bipartite components. Recall that the loop component is a connected component whose all vertices have a loop. We call a vertex having a loop a *loop vertex*. By Lemma 6.5, one can easily see that all edges adjacent to a core p are l_∞-edges. The detailed structure of $K(p)$ is given as follows

Lemma 11.3. *Let p be a core. There is a partition $\{A_1, \dots A_m, B_1, \dots, B_n, C\}$ of X having the following properties:*

(1) *C is the set of loop vertices (C may be empty).*

(2) *The subgraph of $K(p)$ induced by $X \setminus C$ consists of two complete multipartite components with partitions $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_n\}$.*

(3) *If $kl \in E(p)$ for $k \in A_i(B_j)$ and $l \in C$, then for any $k' \in A_i(B_j)$ we have $k'l \in E(p)$.*

Proof. Let K_1 and K_2 be non-loop non-bipartite components of $K(p)$. Let S_1 and S_2 be maximal stable sets of K_1 and K_2 respectively. Then $S := S_1 \cup S_2$ is a maximal stable set of $K(p)$. By Lemma 3.4, $p' = p + \epsilon(-\chi_S + \chi_{N(S)})$ is in $T(X, d)$ for small $\epsilon > 0$. In particular $K(p')$ has exactly two complete bipartite components (by the proof of Proposition 6.4). From this, we easily see the existence of the above partition. □

The subpartition $(A_1, \ldots A_m; B_1, \ldots, B_n)$ is called *the type of p*. $K(p)$ has exactly two non-loop non-bipartite components; the component in $K(p)$ containing $\{A_i\}$ is called the *A-component*, and the component containing $\{B_j\}$ is called the *B-component*. Since loop vertices are pairwise adjacent, at most one non-loop component has a loop. In particular, $\max(m, n) \geq 3$ must holds, and if $m \leq 2$, the A-component must have a loop.

We easily see that for small $\epsilon > 0$, a point $p' := p + \epsilon(-\chi_{A_i} + \chi_{N_p(A_i)})$ is on the edge of $T(X, d)$ adjacent to p, where $N_p(\cdot)$ is the neighborhood operator in the graph $K(p)$. Indeed, $K(p')$ consists of the B-component of $K(p)$, one complete bipartite graph with partition $(A_i, N_p(A_i))$, and the (possibly empty) loop component. Motivated by this fact, the edges adjacent to p with directions $-\chi_{A_i} + \chi_{N_p(A_i)}$ and $-\chi_{B_i} + \chi_{N_p(B_i)}$ are denoted by $e(p, A_i)$ and $e(p, B_j)$, respectively. By the structure of the graph $K(p)$, we see that all edges adjacent to p are such edges, and that $e(p, A_i)$ and $e(p, B_j)$ are contained by the common 2-face, and $e(p, A_i)$ and $e(p, A_j)$ are not contained by the common 2-face if $i \neq j$. Summarizing these facts, the local structure around a core p of $T(X, d)$ is given as follows:

Corollary 11.4. *Let p be a core of type $(A_1, \ldots, A_m; B_1, \ldots, B_n)$. Then we have:*

(1) *e is an edge adjacent to p if and only if e is $e(p, A_i)$ or $e(p, B_j)$ for some i, j.*

(2) *Two edges e', e'' adjacent to p belong to the common 2-face if and only if (e', e'') is $(e(p, A_i), e(p, B_j))$ or $(e(p, B_i), e(p, A_j))$ for some i, j.*

Let Δ be the 1/2-uniform l_1-grid. For a core p, Δ^p is the subcomplex of Δ containing p as a vertex, i.e., Δ^p is the *star* at p of Δ. By the previous corollary, we obtain a combinatorial description of Δ^p.

Corollary 11.5. *Let p be a core of type $(A_1, \ldots, A_m; B_1, \ldots, B_n)$. The complex Δ^p is isomorphic to the join of one point p and the subdivision of the complete bipartite graph $K_{n,m}$.*

See Figure 12 for (a) the complete bipartite graph $K_{3,3}$ and (b) the complex Δ^p obtained by subdividing $K_{3,3}$ and taking the one-point join, where the broken lines represent l_∞-edges.

A key of the proof of Theorem 11.2 is based on the following observation:

If all l_∞-edges have even length, there is the *integral* uniform l_1-grid.

Indeed, an l_∞-octagon all of whose l_∞-edges have even length is an *integral* polygon in the $\mathbf{Z}$-lattice $\{(x_1, x_2) \in \mathbf{Z}^2 \mid x_1 + x_2 \text{ is even }\}$. By the map $(x_1, x_2) \mapsto ((x_1 + x_2)/2, (x_1 - x_2)/2)$, the resulting l_1-octagon is integral in $\mathbf{Z}^2$. Then, we obtain the integral uniform l_1-grid for $T(X, d)$. Consequently, we obtain the orientable 1/2-uniform l_1-grid by the orbit splittings. Related to this, we have:

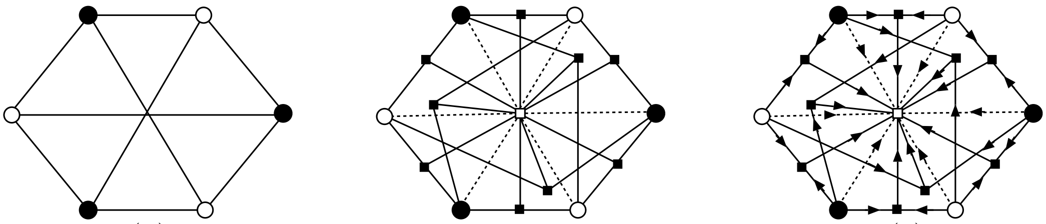


Figure 12: (a) $K_{3,3}$, (b) the complex Δ^p, and (c) an orientation of Δ^p

Lemma 11.6. *Let e be an l_∞-edge. If both endpoints are not cores, then e has even length.*

Proof. Let p be a point in the relative interior of e. By Lemma 6.5, $K(p)$ has one bipartite component K with bipartition (A, B) and one non-bipartite non-loop component $\tilde{K}$. Let q, r be the endpoints of e. Then we may assume $q - r = \|q, r\|(-\chi_A + \chi_B)$. Therefore $\|q, r\| = -q(i) + r(i)$ for $i \in A$. In $K(q)$, there is an edge connecting A and $\tilde{K}$ since q is not a core. Similarly, in $K(r)$, there is an edge connecting B and $\tilde{K}$. Let C be an odd cycle in $\tilde{K}$. Then $q(i)$ is $\sum_{e \in P'} \pm d(e) + (1/2)\sum_{e \in C} \pm d(e)$ for a path P' connecting i and C in $K(q)$, and $r(i)$ is $\sum_{e \in P''} \pm d(e) + (1/2)\sum_{e \in C} \pm d(e)$ for a path P'' connecting i and C in $K(r)$. By calculation, $-q(i) + r(i)$ is $\sum_{e \in \tilde{C}} \pm d(e)$ for cycle $\tilde{C}$ in (X, E_X), and is even by the cyclically evenness. □

In particular, if $T(X, d)$ has no core, then there is the integral uniform l_1-grid. The proof of Theorem 11.2 is completed by showing the existence of a set of cores $\{p_i\}_{i \in I}$ having the following property:

$$\text{Each } l_\infty\text{-edge of } T(X, d) \setminus (\textstyle\bigcup_{i \in I} |\Delta^{p_i}|^\circ) \text{ has even length,}$$

where $|\Delta^{p_i}|^\circ$ is the union of the relative interior of the elements in Δ^{p_i}. Namely we can hollow $T(X, d)$ out of some cores to make the resulting polyhedral set, which is also a complex of l_1-octagons, have the integral uniform l_1-grid Δ^*. Apply the orbit splitting to each orbit of Δ^* and orient it as in Figure 6. Moreover, Δ^{p_i} itself is orientable, and can be oriented as in Figure 12 (c), i.e., orient the graph of Δ^{p_i} so that p_i is the unique sink and vertices adjacent to p_i by l_∞-edges are sources. Then restore each Δ^{p_i} to the original position, which induces an orientation of the 1/2-uniform l_1-grid Δ. Thus we can conclude that the 1/2-uniform l_1-grid Δ is orientable.

Let G_c be the subgraph of 1-skeleton graph of $T(X, d)$ consisting of edges adjacent to some core. Our final goal is to prove the following intriguing properties of G_c.

Proposition 11.7. *G_c has the following properties:*

(1) *For each core p, there is a path in G_c connecting p and some non-core vertex.*

(2) *Let P be a path in G_c connecting a pair of non-core vertices. Then the length (measured by l_∞-metric) of P is even.*

From this properties, the following definition is well-defined:

> A core p is said to be *odd* if a path in G_c connecting p and some non-core vertex has odd length.

The set of odd cores is a desired one. Indeed, if an l_∞-edge e has odd length, then the exactly one of its endpoints is an odd core. On the other hand, if e has even length, then both of its endpoints are odd cores or neither of its endpoints is an odd core.

Now we begin to prove Proposition 11.7. We still need some preliminary arguments. Let p be a core of type $(A_1, \ldots, A_m; B_1, \ldots, B_n)$. We call a move tracing edge $e(p, A_i)$ for some A_i an A-*move*. Similarly, a B-*move* is to trace edge $e(p, B_j)$ for some B_j. If the end $p'(\neq p)$ of $e(p, A_i)$ is a core, then the type of p' is $(N_p(A_i), \tilde{A}_1, \ldots, \tilde{A}_l; B_1, \ldots, B_n)$ for some $\tilde{A}_k \subseteq A_i$ with $1 \leq k \leq l$, and the B-component does not change. Therefore, from the A and B-components of $K(p)$, the A and B-components of $K(p')$ at other core p' in the component of G_c containing p is well-defined. Let $G_c^A(p)$ be the subgraph of G_c which is reachable from p by A-moves. Similarly, we can define the subgraph $G_c^B(p)$ by B-moves. Let $p \to q \to r$ $(p \neq r)$ be A-moves in $G_c^A(p)$. Then we have $q - p = \|q, p\|(-\chi_{A_i} + \chi_{N_p(A_i)})$ and $r - q = \|r, q\|(-\chi_{\tilde{A}_j} + \chi_{N_q(\tilde{A}_i)})$. In particular, $\tilde{A}_j \subseteq A_i$ and $N_q(\tilde{A}_i) \supseteq N_p(A_i)$. At least one of the inclusions is strict. From this we have:

(a1) Both $G_c^A(p)$ and $G_c^B(p)$ are trees.

Let $p_1, p_2, \ldots, p_k$ be a path in $G_c^A(p)$. Then we may assume that $p_{i+1} - p_i$ is represented by $\|p_{i+1}, p_i\|(-\chi_{A^i} + \chi_{N_{p_i}(A^i)})$ for some A_i in the subpartition of the type of p_i. By $A^{i+1} \subseteq A^i$ and $N_{p_{i+1}}(A^{i+1}) \supseteq N_{p_i}(A^i)$, we have $\|p_1, p_k\| = \sum_{i=1}^{k-1} \|p_i, p_{i+1}\|$. Hence we have:

(a2) Both $G_c^A(p)$ and $G_c^B(p)$ are a geodesic subgraph of $T(X, d)$.

Let u, v be non-core vertices in $G_c^A(p)$. Then, we have:

(a3) $\|u, v\|$ is even. By (a2) the length of a path in $G_c^A(p)$ connecting non-core vertices is even.

Indeed, let $u = p_1, p_2, \ldots, p_k = v$ be the unique path in $G_c^A(p)$. We have $p_{i+1} - p_i = \|p_{i+1}, p_i\|(-\chi_{A^i} + \chi_{N_{p_i}(A^i)})$ for $i = 1, \ldots, k-1$. Take $j \in A^k \subseteq A^i$ and $j' \in N_1(A^1) \subseteq N_{p_i}(A^i)$. Then $\|u, v\| = -p_k(j) + p_1(j')$. In $K(p_k)$, there is an edge between A^k and B-component. In $K(p_1)$, there is an edge between A^1 and B-component. By the argument similar to the proof of Lemma 11.6, we can conclude that $\|u, v\|$ is even.

Finally, we easily to see:

(a4) The total length of all l_∞-edges of an integral l_∞-octagon is even.

We are ready to prove Proposition 11.7. (1) is not difficult. Let p be a core of type $(A_1, \ldots, A_m; B_1, \ldots, B_n)$. Consider $G_c^A(p)$ and take a leaf q of the tree $G_c^A(p)$. If q is not a core, then we are done. Suppose that q is a core. Then the type of q is $(A_1'; B_1, \ldots, B_n)$. Consider $G_c^B(q)$. Then every leaf of the tree $G_c^B(q)$ is a non-core vertex. Indeed, if a leaf r of $G_c^B(q)$ is a core, the type of r must be $(A_1'; B_1')$. This is impossible.

To prove (2), we assume:

Assumption: There is an odd length path in G_c connecting non-core vertices.

We simply call it a *violating path*. We call a core p in the path P a *bending point* if P interchanges A-move and B-move at p. By (a3), any violating path has at least one bending point. We first claim:

(a5) There is a violating path having exactly one bending point.

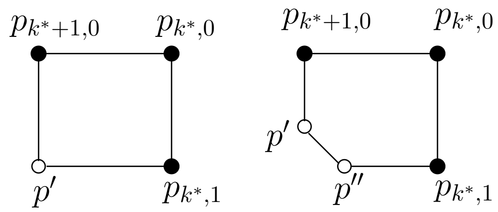


Figure 13: The shapes of $F_{k^*,0}$

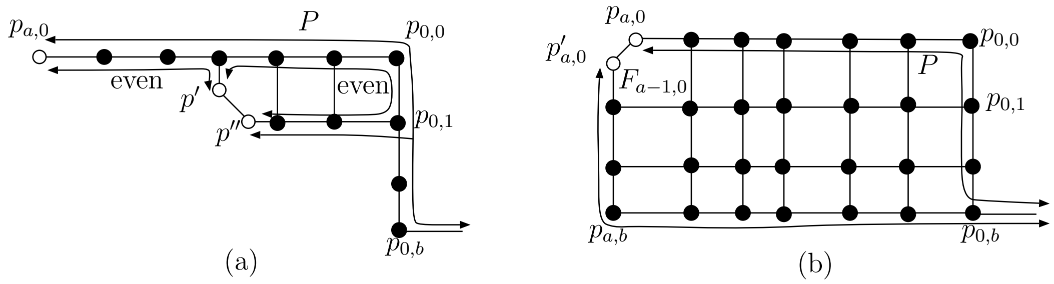


Figure 14: Another violating paths

Suppose that every violating path has at least two bending point. We can take a violating path P with properties that (1) the number of bending points is minimum, (2) the total length is minimum among violating paths with property (1), and (3) the length between the first and the second bending point is minimum among violating paths with properties (1) and (2). By minimality, vertices except endpoints u, v of P are cores. The path P is $u = p_{a,0}, p_{a-1,0}, \ldots, p_{1,0}, p_{0,0}, p_{0,1}, \ldots, p_{0,b}, \ldots, v$, where a, b are positive integers, $p_{0,0}$ is the first bending point, and $p_{0,b}$ is the second bending point. We may assume that $p_{k,0} \to p_{k+1,0}$ is A-move in $G^A(p_{0,0})$, and $p_{0,l} \to p_{0,l+1}$ is B-move in $G^B(p_{0,0})$. We further assume that the types of $p_{k,0}$ and $p_{0,l}$ are $(A^k_1, \ldots, A^k_{m_k}; B^0_1, \ldots, B^0_{n_0})$ and $(A^0_1, \ldots, A^0_{m_0}; B^l_1, \ldots, B^l_{n_l})$, respectively, and edge $p_{k,0}p_{k+1,0}$ is given by $e(p_{k,0}, A^k_{i_k})$ for $k = 0, \ldots, a-1$, and edge $p_{0,l}p_{0,l+1}$ is given by $e(p_{0,l}, B^l_{j_l})$ for $l = 0, \ldots, b-1$. Let $F_{k,0}$ for $0 \leq k \leq a-1$ be the 2-face of $T(X, d)$ containing $e(p_{k,0}, A^k_{i_k})$ and $e(p_{k,0}, B^0_{j_0})$; the existence of such a 2-face is guaranteed by Corollary 11.4. Then we have:

- For $0 \leq k \leq a-2$, $F_{k,0}$ is a rectangle each of whose vertices is a core.

We simply call it a *core-rectangle*. Indeed, suppose that there is k^* such that $F_{k^*,0}$ is not a core-rectangle. We take the smallest k^*. Then $F_{k,0}$ for $0 \leq k \leq k^*-1$ is a core-rectangle $p_{k,0}p_{k+1,0}p_{k+1,1}p_{k,1}$, and $F_{k^*,0}$ is a rectangle $p_{k^*,0}p_{k^*+1,0}p'p_{k^*,1}$ or a pentagon $p_{k^*,0}p_{k^*+1,0}p'p''p_{k^*,1}$, where p', p'' are non-core vertices; see Figure 13, where the black points are cores and the white points are non-core vertices. Then the path $p_{a,0}, p_{a-1,0}, \ldots, p_{k^*+1,0}, p'$ has only one bending point $p_{k^*+1,0}$ and therefore has even length by the assumption; See Figure 14 (a). Combining (a4), the path $p''(\text{or } p'), p_{k^*,1}, , \ldots, p_{0,1}, p_{0,2}, \ldots, p_{0,b}, \ldots v$ is a shorter violating path than P. A contradiction.

- The shape of $F_{a-1,0}$ is one of Figure 15.

Indeed, other possible seven cases listed in Figure 16 are impossible by similar arguments above. The last case contradicts the minimality of the length between the first and the second bending points.

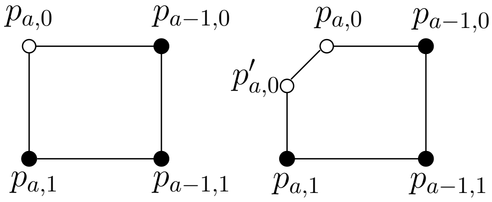


Figure 15: The shapes of $F_{a-1,0}$

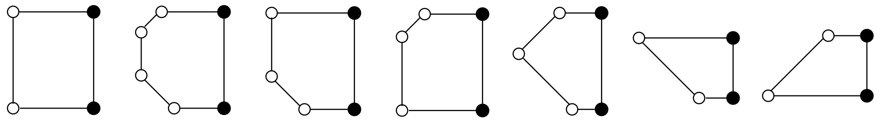


Figure 16: Other possibilities of $F_{a-1,0}$

Let $p_{k,1}$ for $1 \leq k \leq a-1$ be a core adjacent to $p_{k,0}$ by edge $e(p_{k,0}, B^0_{j_0})$ and $p_{a,1}$ a core adjacent to $p_{a,0}$ or $p'_{a,0}$ in $F_{a-1,0}$. As above, let $F_{k,1}$ for $0 \leq k \leq a-1$ be a 2-face of $T(X,d)$ containing edges $e(p_{k,0}, A^k_{i_k})$ and $e(p_{k,1}, B^1_{j_1})$. Then $F_{k,1}$ for $0 \leq k \leq a-1$ is a core-rectangle $p_{k,1}p_{k+1,1}p_{k+1,2}p_{k,2}$ by the similar arguments above. Repeat this process, we can further unfold core-rectangles $F_{k,l} = p_{k,l}p_{k+1,l}p_{k+1,l+1}p_{k,l+1}$ for $1 \leq k \leq a-1$ and $0 \leq l \leq b-1$; see Figure 14 (b). Note that $p_{k,l}p_{k+1,l}$ and $p_{k,l'}p_{k+1,l'}$ are parallel, and $p_{k,l}p_{k,l+1}$ and $p_{k',l}p_{k',l+1}$ are parallel. If the next point v' of $p_{0,b}$ is $p_{1,b}$, then this contradicts the minimality assumptions. Then the path $p_{a,0}$(or $p'_{a,0}$)$, p_{a,1}, p_{a,2}, \ldots, p_{a,b}, p_{a-1,b}, \ldots, p_{0,b}, v' \ldots v$ is also violating and the number of its bending points is less than that of P. A contradiction. Hence we can conclude (a5).

Take a minimal length one-bending violating path $P = p_{a,0}, \ldots, p_{0,0}, p_{0,1}, \ldots, p_{0,b}$ whose $\min(a,b)$ is minimal, where $p_{0,0}$ is a unique bending point. Types of cores $p_{k,0}$ and $p_{0,l}$ are given as above. Then, by the similar arguments above, $F_{k,0}$ and $F_{0,l}$ for $0 \leq k \leq a-1$ and $0 \leq l \leq b-1$ are core-rectangles.

In $K(p_{a,0})$, there is an edge between $A^{a-1}_{i_{a-1}}$ and B-component. If there is an edge connecting $A^{a-1}_{i_{a-1}}$ and $B^0_{j_0}$ in $K(p_{a,0})$, then the shape of $F_{a-1,0}$ is the sixth type of Figure 16. This is impossible by the minimality of P. Therefore, there are two cases: (A1) in $K(p_{a,0})$ there is a vertex in the B-component adjacent to both $A^{a-1}_{i_{a-1}}$ and $B^0_{j_0}$, and (A2) there is no such a vertex, but there is an edge connecting $A^{a-1}_{i_{a-1}}$ and a loop in the B-component. For the case (A1), the shape of $F_{a-1,0}$ is the left in Figure 15. For the case (A2), the shape of $F_{a-1,0}$ the right in Figure 15. Similarly, in $K(p_{0,b})$, there are two cases: (B1) there is a vertex adjacent to both $B^{b-1}_{j_{b-1}}$ and $A^0_{i_0}$ in $K(p_{0,b})$, and (B2) there is no such a vertex, but there is an edge connecting $B^{b-1}_{j_{b-1}}$ and some loop vertex which belongs to A-component.

The cases (A2) and (B2) do not occur simultaneously. Indeed, if (A2) and (B2) occur, then in $K(p_{0,0})$ the both components have a loop. This is impossible since loop vertices are pairwise adjacent. Therefore, it suffices to consider the two cases (A1)-(B1) and (A2)-(B1).

First we treat the case (A1)-(B1). Let $(A^a_1, \ldots, A^a_{m_a}; B^1_1, \ldots, B^1_{n_1})$ be the type of $p_{a,1}$. There is $A^a_{i_a} \subseteq A^{a-1}_{i_{a-1}}$ such that in $K(p'_{a,0})$ there is a vertex v adjacent both $A^a_{i_a}$ and $B^0_{j_0}$. Consider 2-face $F_{a,0}$ containing $p_{a,1}p'_{a,0}$ and $e(p_{a,1}, A^a_{i_a})$. Then the vertex $p'_{a,0}$ must have 45-degree angle in $F_{a,0}$. Indeed, $S := A^a_{i_a} \cup B^0_{j_0}$ is maximal stable in $K(p'_{a,0})$, the subgraph of $K(p'_{a,0})$ consisting of edges between S and $N(S)$ is a connected bipartite graph by the

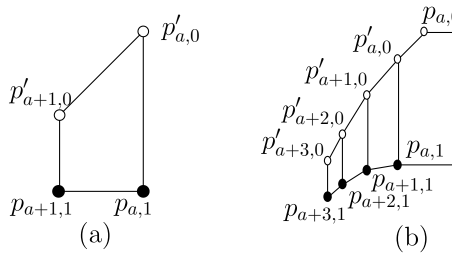


Figure 17: (a) $F_{a,0}$ and (b) unfolding $F_{a,0}, F_{a+1,0}, F_{a+2,0}, F_{a+3,0}$.

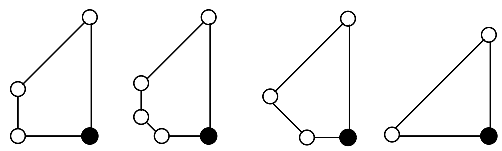


Figure 18: Other possibilities of $F_{a,0}$

existence of v. Therefore $p'_{a,0} + \epsilon(-\chi_S + \chi_{N(S)})$ is on an edge of $F_{a,0}$ (by Lemma 3.4 and Proposition 6.1). The shape of 2-face $F_{a,0}$ is (a) of Figure 17. The other cases listed in Figure 18 are impossible by the minimality of P. Let $(A_1^{a+1}, \ldots, A_{m_{a+1}}^{a+1}; B_1^1, \ldots, B_{n_1}^1)$ be the type of $p_{a+1,1}$. Then, there is $A_{i_{a+1}}^{a+1} \subseteq A_{i_a}^a$ such that in $K(p'_{a+1,0})$, the vertex v is again adjacent to both $A_{i_{a+1}}^{a+1}$ and $B_{j_0}^0$. the shape of 2-face $F_{a,0}$ containing $p_{a,1}p'_{a,0}$ and $e(p_{a,1}, A_{i_a}^a)$ is again (a) of Figure 17. Repeating this process as Figure 17 (b), there is a nonnegative integer a' such that in $K(p_{a+a',1})$ the A-component has a loop. We apply the same process to $p_{0,b}$. Then there is a nonnegative integer b' such that the B-component has a loop vertex in $K(p_{1,b+b'})$. By minimality assumption, we can tile 2-faces $F_{k,l}$ for $1 \le k \le a + a' - 1$, $1 \le l \le b + b' - 1$, which are core-rectangles except $F_{a+a'-1,b+b'-1}$. Then the final 2-face $F_{a+a'-1,b+b'-1}$, which is a pentagon or a rectangle, must contain a non-core vertex. If $F_{a+a'-1,b+b'-1}$ is a pentagon $p_{a+a',b+b'-1}p_{a+a'-1,b+b'-1}p_{a+a'-1,b+b'}p''p'$, neither p' or p'' is core. Suppose that $F_{a+a'-1,b+b'-1}$ is a rectangle. The vertex $p_{a+a',b+b'}$ diagonal to $p_{a+a'-1,b+b'-1}$ is non-core. Indeed, In the B-moves $p_{a+a',1} \to p_{a+a',2} \to \cdots \to p_{a+a',b+b'-1}$, the A-component, which has a loop, is invariant. In $K(p_{a+a',b+b'})$, there is a loop in $B_{j_{b+b'-1}}^{b+b'-1}$. Since loop vertices are adjacent each other, $K(p_{a+a',b+b'})$ has only one component, and thus $p_{a+a',b+b'}$ is not a core.

Project 2-faces $F_{k,l}$ isometrically into 2-dimensional plane by the map $p \mapsto (p(i^*), p(j^*))$ for $i^* \in A_{i_{a+a'-1}}^{a+a'-1}$ and $j^* \in B_{i_{b+b'-1}}^{b+b'-1}$. Then we obtain a tiling of an integral l_∞-octagon; see Figure 19. By (a3), both edges $p'_{a+a',0}p_{a+a',b+b'}$ (or $p'_{a+a',0}p'$) and $p'_{0,b+b'}p_{a+a',b+b'}$ (or $p'_{0,b+b'}p''$) have even length. Therefore by (a4) the length of path P must be even. This is a contradiction to the first assumption.

Next we treat the case (A2)-(B1). Similarly, we unfold $F_{0,b+1}, F_{0,b+2}, \ldots F_{0,b+b'}$ until the B-component has a loop in $K(p_{1,b+b'})$. In $K(p_{a,1})$, the A-component already has a loop. Consider 2-faces $F_{k,l}$ for $0 \le k \le a-1$ and $0 \le l \le b+b'-1$ as above. Then the final $F_{a,b+b'-1}$ has a non-core vertex. The remaining arguments are the same as above.

Now we complete the proof of Proposition 11.7 and therefore Theorem 11.2.

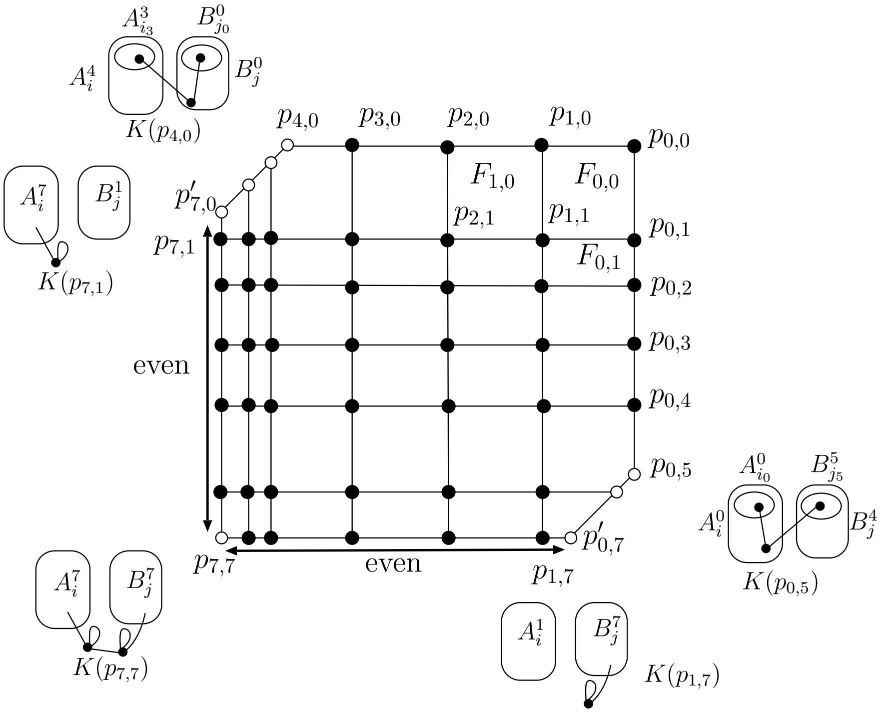


Figure 19: Tiling an l_∞-octagon by $F_{k,l}$

12 0-1 distances

In this section, we verify that the condition (P) in Theorem 1.1 is indeed equivalent to the 2-dimensionality of 0-1 distances, and give an explicit combinatorial construction of the tight span of a 2-dimensional 0-1 distance.

First we present Dress' criterion [11, Theorem 9] of the dimension of tight spans.

Theorem 12.1 ([11], also see [13])**.** *For a distance d on a finite set X and a positive integer n, the following conditions are equivalent.*

(a) $\dim T(X,d) \geq n$.

(b) *There exists a $2n$-element subset $Y \subseteq X$ and a perfect matching $M \subseteq E_Y$ such that χ_M is the unique optimal solution of the following linear programming over the fractional matching polytope:*

$$\begin{aligned} &\text{Maximize} && \sum_{i,j\in Y} \lambda_{ij} d(i,j) \\ &\text{subject to} && \sum_{i,j \in Y} (\chi_i + \chi_j)\lambda_{ij} = \chi_Y, \ \lambda_{ij} \geq 0 \quad (i,j \in Y). \end{aligned} \tag{12.1}$$

(c) *There exists a $2n$-element subset $Y \subseteq X$ and a perfect matching $M \subseteq E_Y$ such that χ_M attains the unique maximum of*

$$\max_{M', C_1,\ldots,C_m} \sum_{ij\in M} d(i,j) + \frac{1}{2}\sum_{k=1}^{m}\sum_{ij\in C_i} d(i,j), \tag{12.2}$$

where the maximum is taken over pairwise vertex disjoint matching M' and odd cycles $C_1,\ldots,C_m (m \geq 0)$.

(d) *There exists a $2n$-element subset $\{i_1, i_{-1}, i_2, i_{-2}, \ldots, i_n, i_{-n}\} \subseteq X$ such that*

$$\sum_{k \in \{\pm 1, \pm 2, \ldots \pm n\}} d(i_k, i_{-k}) > \sum_{k \in \{\pm 1, \pm 2, \ldots \pm n\}} d(i_k, i_{\sigma(k)}) \tag{12.3}$$

holds for any permutation σ of $\{\pm 1, \pm 2, \ldots \pm n\}$ with $\sigma(i) \neq -i$ for any $i \in \{\pm 1, \pm 2, \ldots \pm n\}$.

The equivalence between (a) and (d) is the original form given in [11].

Sketch of proof. (b) $\Leftrightarrow$ (c) is immediate from the characterization of extreme points of the fractional matching polytopes [4]; see also [25, p. 522]. (c) $\Leftrightarrow$ (d) is also immediate from the facts that a permutation can be decomposed into disjoint cyclic permutation, and that even cycle is the vertex-disjoint sum of two matchings. (a) $\Leftrightarrow$ (b) can be shown by using Proposition 6.1 and the complementary slackness condition; see [13, Appendix] for detail. $\square$

Specializing Theorem 12.1 to 0-1 distance d and $n = 3$, we have the following.

Proposition 12.2. *For a 0-1 distance d on X whose H_d has no isolated vertex, the following conditions are equivalent:*

(a) $\dim T(X, d) \leq 2$.

(b) *There is no six-elements subset U such that the induced subgraph $H_d(U)$ of H_d by U has a unique perfect matching and has no vertex disjoint two triangles.*

(P) *For any three distinct pairwise intersecting maximal stable sets A, B, C of H_d, we have $A \cap B = B \cap C = C \cap A$.*

Proof. It is easy to see that the condition (b) is equivalent to the negation of the condition (c) of Theorem 12.1 for 0-1 distances and $n = 3$.

(b) $\Rightarrow$ (P). Suppose that there are three distinct pairwise intersecting maximal stable sets A, B, C of H_d such that $(B \cap C) \setminus A$ is nonempty. Take $i \in (B \cap C) \setminus A$. Since A is a *maximal* stable set, there is $i' \in A \setminus (B \cup C)$ with $ii' \in F_d$

(Case 1). Suppose that $A \cap B \cap C$ is empty. Then both $(A \cap C) \setminus B$ and $(A \cap B) \setminus C$ are nonempty. Take $j \in (A \cap C) \setminus B$ and $k \in (A \cap B) \setminus C$. There are $j' \in B \setminus (A \cup C)$, $k' \in C \setminus (A \cup B)$ with $jj', kk' \in F_d$. Let $U = \{i, i', j, j', k, k'\}$. Then the induced subgraph $H_d(U)$ consists of three edges $\{ii', jj', kk'\}$, which is a unique perfect matching.

(Case 2). Suppose $A \cap B \cap C$ is not empty. Take $j \in B \setminus C$. Then there is $j' \in C \setminus B$ with $jj' \in F_d$. Take $k \in A \cap B \cap C$. By the condition that H_d has no isolated vertex, there is $k' X \setminus (A \cup B \cup C)$ with $kk' \in F_d$. Let $U = \{i, i', j, j', k, k'\}$. Consider the induced subgraph $H_d(U)$ which has a perfect matching $\{ii', jj', kk'\}$. In $H_d(U)$, a vertex k is covered by edge kk' only. Therefore, $H_d(U)$ does not have vertex disjoint two triangles. Moreover, any perfect matching must use edge kk'. A vertex i is not adjacent to j and j'. Therefore $\{ii', jj', kk'\}$ is a unique perfect matching of $H_d(U)$

(P) $\Rightarrow$ (a). Suppose that $\dim T(X, d) \geq 3$. Then there is $p \in T(X, d)$ such that the graph $K(p)$ has three bipartite components. We can take three edges $i_1 i'_1, i_2 i'_2, i_3 i'_3 \in E(p)$ from different bipartite components. Since d is a 0-1 distance, we have $i_k i'_k \in F_d$ for $k = 1, 2, 3$. By $p(i_k) + p(i'_k) = 1$, we may assume that $p(i_k) \geq 1/2 \geq p(i'_k)$ and $p(i_1) \geq p(i_2) \geq p(i_3)$. Consequently we have $p(i'_1) \leq p(i'_2) \leq p(i'_3)$. Since $p(i) + p(j) \leq 1$ implies $ij \notin F_d$, three sets $\{i'_1, i'_2, i'_3\}$, $\{i'_1, i'_2, i_3\}$, and $\{i'_1, i_2\}$ are pairwise intersecting stable sets of $H_d(U)$ violating condition (P). Then we can extend this triple to pairwise intersecting maximal stable sets of H_d violating condition (P). $\square$

Finally, we give an explicit combinatorial construction of $T(X,d)$ for a 2-dimensional 0-1 distance d. Let $\mathcal{A}_d$ be the set of maximal stable sets of H_d and $\mathcal{K}_d$ the set of maximal clique of the intersection graph of $\mathcal{A}_d$.

Proposition 12.3. *Let d be a 2-dimensional 0-1 distance on X whose H_d has no isolated vertices. Let $\{p_S\}_{S\in\mathcal{A}_d}$, $\{p_K\}_{K\in\mathcal{K}_d}$, and p_O be points in $T(X,d)$ defined as*

$$
\begin{aligned}
p_S &= \chi_{X\setminus S} \quad (S \in \mathcal{A}_d), && (12.4)\\
p_K &= (1/2)\chi_{\cup_{S\in K}S\setminus\cap_{S\in K}S} + \chi_{X\setminus\cup_{S\in K}S} \quad (K\in\mathcal{K}_d), && (12.5)\\
p_O &= (1/2)\chi_X. && (12.6)
\end{aligned}
$$

Then we have

$$T(X,d) = \bigcup\{ \text{ convex hull of } \{p_S, p_K, p_O\} \mid S\in K\in\mathcal{K}_d\}. \qquad (12.7)$$

Proof. ($\supseteq$) in (12.7) is straightforward. We show ($\subseteq$). Take a sufficiently generic $p \in T(X,d)$. By the facts $0\le p\le 1$ and that H_d has no isolated vertices, $K(p)$ does not have the loop component. Combining the genericity of p, we can conclude that $K(p)$ is one complete bipartite graph or the (vertex-disjoint) sum of two complete bipartite graphs K_1, K_2. For the first case, let A and B be the two parts of the bipartite graph $K(p)$. Then we have $p(i)=\alpha, p(j)=\beta$ for $i\in A, j\in B$ and α,β with $\alpha+\beta=1$ and $0<\alpha<1/2<\beta<1$. Then A is a maximal stable set of H_d. Therefore $p=(\beta-\alpha)p_A + 2\alpha p_O$.

For the second case, let A_i and B_i be the two parts of the bipartite graph K_i for $i=1,2$. Similarly, $(p(i),p(j),p(k),p(l)) = (\alpha_1,\beta_1,\alpha_2,\beta_2)$ for $(i,j,k,l)\in A_1\times B_1\times A_2\times B_2$ and $1<\alpha_1<\alpha_2<1/2<\beta_2<\beta_1<1$ with $\alpha_1+\beta_1=\alpha_2+\beta_2=1$. Then $A_1\cup A_2$ is a maximal stable set of H_d, and there is no edge in H_d between A_1 and B_2. By condition (P), there is a maximal set K of pairwise intersecting stable sets such that $A_1\cup A_2\in K$, and the union and intersection of members in K are $X\setminus B_1$ and A_1, respectively. By calculation, we have $p = 2\alpha_1 p_O + (\alpha_1+\beta_1-2\alpha_2)p_S + (2\alpha_2-2\alpha_1)p_K$. □

Namely, $T(X,d)$ is the complex of the join of the point p_O and the clique-vertex incidence graph of $\mathcal{A}_d$ and $\mathcal{K}_d$. Figure 20 illustrates the tight spans with their minimal orientable l_1-grids for commodity graphs (a) $H=K_2+K_2$, (b) $H=K_2+K_3$, and (c) $H=K_3+K_3$. Karzanov's original proof [17] of Theorem 1.1 is based on the concept of *frameworks* of graph $G=(V,E,c)$ and commodity graph H_μ, which is a certain subpartition of V. He has shown that $M^*(G;S,\mu)$ is equivalent to discrete optimization over all possible frameworks. In our setting, frameworks can be interpreted as feasible configurations to (TSD-Z) of the 1/4-uniform l_1-grid.

13 Concluding remarks

We interpreted $T(X,d)$ as the space of one-element tight extensions. This interpretation naturally leads us to the space of one-element tight extensions of an *asymmetric distance space* (X,γ), where a asymmetric distance γ is a function $\gamma : X\times X\to\mathbf{R}_+$ satisfying $\gamma(i,i)=0$ for $i\in X$, and $\gamma(i,j)=\gamma(j,i)$ is not imposed. So an asymmetric analogue $T(X,\gamma)$ are given by the set of minimal elements of

$$P(X,\gamma) = \{(p,q)\in\mathbf{R}_+^{X\times X} \mid p(i)+q(j)\ge\gamma(i,j)\ (i,j\in X)\}. \qquad (13.1)$$

This space $T(X,\gamma)$ is the intersection of the nonnegative orthant and the *tropical convex hull* of γ which was introduced by Develin and Sturmfels [10]. The forthcoming paper,

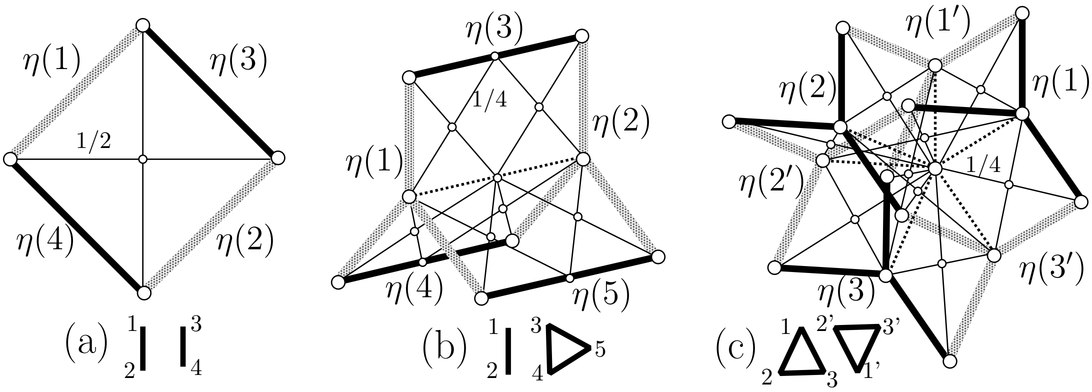


Figure 20: Tight spans for 0-1 distances

jointly with S. Koichi, will develop a parallel theory for this asymmetric tight span $T(X, \gamma)$ and its applications to *directed* multiflow problems.

Apart from the fractionality issues, the design of combinatorial or practical algorithms specialized to general multiflow problems is still a challenging problem. The tight-span-dual problem and the geometry of $T(X, d)$ explored in this paper might give a basis against this challenge.

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