Braided differential structure on Weyl groups, quadratic algebras and elliptic functions

To the memory of Leonid Vaksman

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Abstract

We discuss a class of generalized divided difference operators which give rise to a representation of Nichols-Woronowicz algebras associated to Weyl groups. For the root system of type A, we also study the condition for the deformations of the Fomin-Kirillov quadratic algebra, which is a quadratic lift of the Nichols-Woronowicz algebra, to admit a representation given by generalized divided difference operators. The relations satisfied by the mutually commuting elements called Dunkl elements in the deformed Fomin-Kirillov algebra are determined. The Dunkl elements correspond to the truncated elliptic Dunkl operators via the representation given by the generalized divided difference operators.

Introduction

The rational Dunkl operators, which were introduced in [5] for any finite Coxeter group, constitute a remarkable family of operators of differential-difference type. The Dunkl operators are defined to be the ones acting on the functions on the reflection representation V of the corresponding Weyl group W. For the root system of type A_{n-1}, the Dunkl operators $D_1, \dots, D_n$ are defined by the formula

$$D_i := \frac{\partial}{\partial x_i} + \sum_{j \neq i} \frac{1 - s_{ij}}{x_i - x_j},$$

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where s_{ij} is the transposition of i and j. They are S_n-invariant and mutually commute. The Dunkl opearotors play an important role in the representation theory and in the study of integrable systems. Here we would like to mention only a remarkable result, due to Dunkl, that the algebra generated by *truncated Dunkl operetors* is isomorphic to the coinvariant algebra of the corresponding finite Coxeter group [5], [2]. A trigonometric generalization of Dunkl operators has been proposed by Cherednik [4], and an elliptic one by Buchstaber, Felder and Veselov [3]. The basic requirement for such generalizations is that the operators to be constructed are bounded to pairwise commute. Another important property of rational Dunkl operators, namely, their W-invariance, may be broken for generalizations.

For a crystallographic irreducible root system R, Buchstaber, Felder and Veselov [3] have determined the conditions on the functions $f_\alpha(z)$, $\alpha \in R$, so that the operators

$$\nabla_\xi = \partial_\xi + \sum_{\alpha \in R_+} (\alpha, \xi) f_\alpha((\alpha, x)) s_\alpha$$

satisfy the commutativity condition $[\nabla_\xi, \nabla_\eta] = 0$ for all $\xi, \eta \in V$. Here, we denote by R_+ the set of positive roots and by s_α the reflection corresponding to a root α. Under the assumption of the W-invariance of ∇_ξ, they proved that the solutions of the functional equation for f_α must be rational unless R is of type B_2. Without the assumption of the W-invariance, some elliptic solutions given by Kronecker's σ-function may appear. If R is of type A_n, such functions exhaust the general solution.

The present paper contains two main results. The first one is concerned about the existence of a representation given by the generalized divided difference operators for the (certain extension of) Nichols-Woronowicz algebra $\mathcal{B}_W$ corresponding to a Weyl group W. Our second main result describes relations among the Dunkl elements in the elliptic extension of the Fomin-Kirillov algebra introduced originally in [7]. In particular, we describe the relations among truncated elliptic Dunkl operators of type A_{n-1}. By analogy with Dunkl's theorem mentioned above, one can consider the algebra generated by truncated elliptic Dunkl operators of type A_{n-1} as an elliptic deformation of the cohomology ring of the flag variety Fl_n. We also prove an elliptic analogue of the Pieri rule in the elliptic extension of Fomin-Kirillov algebra. These results can be considered as further generalizations of those obtained in [6], [15], since the latter correspond to certain degenerations of the elliptic case, see Section 4 for details.

The Nichols-Woronowicz algebra $\mathcal{B}(M)$ is a braided analogue of the symmetric algebra, which is defined for a given braided vector space M. Nichols [14] studied graded bialgebras generated by the primitive elements of degree one. The braided Hopf algebra $\mathcal{B}(M)$ satisfying such a condition was called Nichols algebra by Andruskiewitsch and Schneider [1]. The algebra $\mathcal{B}(M)$ has been constructed also in the theory of the differential forms on quantum groups due to Woronowicz [16]. Woronowicz constructed $\mathcal{B}(M)$ as a braided symmetric (or exterior) algebra based on the construction of his (anti-)symmetrizer. The Nichols-Woronowicz algebra provides a natural framework for the braided differential calculus, which was developed by Majid [11].

In this paper we are interested in the Nichols-Woronowicz algebra associated to a particular kind of braided vector space called Yetter-Drinfeld module. See [2] for more details of general construction of $\mathcal{B}(M)$. In our case, we use a $\mathbb{C}$-vector space M_W spanned by the symbols $[\alpha] = -[-\alpha]$, $\alpha \in R$, with the braiding $\psi : M_W^{\otimes 2} \to M_W^{\otimes 2}$, $[\alpha] \otimes [\beta] \mapsto [s_\alpha(\beta)] \otimes [\alpha]$. The algebra $\mathcal{B}_W = \mathcal{B}(M_W)$ of our interest is defined to be the quotint of the tensor algebra of M_W by the kernel of the braided symmetrizer.

Milinski and Schneider [13] and Majid [12] have pointed out that the algebra $\mathcal{B}_W$ for $W = S_n$ is a quotient of the Fomin-Kirillov quadratic algebra $\mathcal{E}_n$ defined in [6]. The algebra $\mathcal{B}_{S_n}$ is conjectured to be isomorphic to $\mathcal{E}_n$. Fomin and the first author introduced the algebra $\mathcal{E}_n$ to construct a model of the cohomology ring of the flag variety Fl_n. In [2], Bazlov has reformulated their construction of the model of the cohomology ring in terms of the Nichols-Woronowicz algebra $\mathcal{B}_W$, and generalized it to arbitrary finite Coxeter groups. The braided differential operators on the algebra $\mathcal{B}_W$, which were used by Majid [12] for root system of type A, play an essential role in Bazlov's construction. His construction also has an important implication on the representation of $\mathcal{B}_W$, since the braided differential operators act on the coinvariant algebra of W as the divided difference operators $\partial_\alpha = (1 - s_\alpha)/\alpha$, $\alpha \in R$.

In Section 1, we discuss the conditions for the generalized divided difference operators

$$\mathbf{D}_\alpha = f_\alpha((\alpha, \xi)) + g_\alpha((\alpha, \xi)) s_\alpha$$

to give rise a representation of $\mathcal{B}_W$. These conditions are interpreted as functional equations for f_α and g_α. We prove that the operators corresponding to the W-invariant solutions described in [3] define a representation of $\mathcal{B}_W$. Komori [10] studied when the operators $\mathbf{D}_\alpha$ satisfy the Yang-Baxter equation. Since the generators $[\alpha]$ of the algebra $\mathcal{B}_W$ satisfy the Yang-Baxter equation, our operators also correspond to special part of the solutions found in [10].

In order to get a more general class of solutions like elliptic functions, we have to loose part of defining relations of $\mathcal{B}_W$. In Section 2 we introduce a deformed version $\tilde{\mathcal{E}}_n(\psi_{ij})$ of the Fomin-Kirillov quadratic algebra, which is defined for a given family of meromorphic functions $\psi_{ij}(z)$, $1 \le i, j \le n$, $i \neq j$. The algebra $\tilde{\mathcal{E}}_n(\psi_{ij})$ admits the representation by the operators $\mathbf{D}_\alpha$ only when $\psi_{ij}(z)$ is given by the Weierstrass $\wp$-function or its degenerations. In this case, the operator $\mathbf{D}_\alpha$ exactly corresponds to the general solution for A_{n-1}-system obtained in [3, Theorem 16].

Our second main result is the study of relations among the Dunkl elements in the elliptic extension $\tilde{\mathcal{E}}_n(\psi_{ij})$ of the Fomin-Kirilov algebra. The Dunkl elements $\theta_1, \ldots, \theta_n \in \tilde{\mathcal{E}}_n(\psi_{ij})$ are mutually commuting elements defined by $\theta_i = \sum_{j \neq i} [ij]$. The images of the Dunkl elements, via the representation $[\alpha] \mapsto \mathbf{D}_\alpha$, become the so-called truncated (or level zero) elliptic Dunkl operators, cf [3]. It is well-known that the (truncated) rational or trigonometric Dunkl operators can be obtained as certain degenerations of the (truncated) elliptic Dunkl operators. The identities among the Dunkl elements in $\tilde{\mathcal{E}}_n(\psi_{ij})$ are also satisfied by the corresponding truncated elliptic Dunkl operators or their degenerations. In the context of Schubert calculus, the Dunkl elements describe the multiplication by the classes of standard line bundles in the cohomology ring of the flag variety. The formula of the elementary symmetric polynomials in the Dunkl elements in the Fomin-Kirillov algebra reflects the Pieri formula. In Section 3 we give a formula for the deformed elementary symmetric polynomial $E_k(\theta_i \mid i \in I)$ in the algebra $\tilde{\mathcal{E}}_n(\psi_{ij})$.

The algebra $\tilde{\mathcal{E}}_n(\psi_{ij})$ has degenerations to variants of the deformation of the Fomin-Kirillov algebra. In particular, the multiparameter deformation $\mathcal{E}_n^p$ studied in [6] and [15], and the extended quadratic algebra $\tilde{\mathcal{E}}_n\langle R\rangle[t]$ defined in [9] after the specialization $t = 0$ can be regarded as degenerations of $\tilde{\mathcal{E}}_n(\psi_{ij})$. In Section 4 we show that our algebra recovers the Pieri formulas in the corresponding degenerations.

1 Representation of Nichols-Woronowicz algebra

Let us consider the reflection representation V of the Weyl group W. Denote by $R \subset V$ the set of roots for the Weyl group W. Fix R_+ the set of positive roots in R. Let $\{\alpha_1, \dots, \alpha_r\} \subset R_+$ be the set of simple roots. The Weyl group W naturally acts on the space $\mathcal{M} = \mathcal{M}(V_{\mathbb{C}})$ of meromorphic functions on $V_{\mathbb{C}}$. We also denote by $\mathcal{M}_0$ the space of meromorphic functions on $\mathbb{C}$.

We discuss the generalized Calogero-Moser representaion of the Nichols-Woronowicz algebra $\mathcal{B}_W$ for the Weyl group W. The Nichols-Woronowicz algebra $\mathcal{B}_W = \mathcal{B}(M_W)$ is associated to the Yetter-Drinfeld module M_W generated by the symbols $[\alpha]$, $\alpha \in R$. Define the operator $\mathbf{D}_\alpha$, $\alpha \in R$, acting on $\mathcal{M}$ by

$$\mathbf{D}_\alpha = f_\alpha((\alpha, \xi)) + g_\alpha((\alpha, \xi))s_\alpha, \ \ \xi \in V,$$

where s_α is the reflection with respect to α, and $f_\alpha, g_\alpha \in \mathcal{M}_0$. We assume that $f_{-\alpha}(z) = -f_\alpha(-z)$ and $g_{-\alpha}(z) = -g_\alpha(-z)$ so that $\mathbf{D}_{-\alpha} = -\mathbf{D}_\alpha$.

Lemma 1.1 *The divided difference operator $\partial_\alpha = (1 - s_\alpha)/(\alpha, \xi)$ gives a well-defined representation of $\mathcal{B}_W$ on P.*

Proof. From the construction of the model of the coinvariant algebra P_W in [2], we can see that the natural action of the braided differential operator $\overleftarrow{D}_\alpha$ on P_W coincides with the divided difference operator ∂_α. Since $P = P^W \otimes P_W$, we can extend P^W-linearly the action of $\mathcal{B}_W$ on P_W to that on P.

Lemma 1.2 *If $[\alpha] \mapsto \mathbf{D}_\alpha$ defines the representation of $\mathcal{B}_W$, then f_α must be an odd function, and*

$$g_\alpha(z) = f_\alpha(z)\phi_\alpha(z),$$

where $\phi_\alpha(z)\phi_\alpha(-z) = 1$.

Proof. The condition $\mathbf{D}_\alpha^2 = 0$ is equivalent to the equations

$$f_\alpha(z)^2 + g_\alpha(-z)g_\alpha(z) = 0$$

and

$$g_\alpha(z) \cdot (f_\alpha(z) + f_\alpha(-z)) = 0.$$

The second equation shows that f_α is odd. Define the function $\phi_\alpha(z)$ by

$$\phi_\alpha(z) = \frac{g_\alpha(z)}{f_\alpha(z)}.$$

Then the first equation can be written as

$$\phi_\alpha(z)\phi_\alpha(-z) = 1.$$

We take the standard realization of the root systems of type A_n and B_n as follows:

$$R(A_n) = \{ij = \epsilon_i - \epsilon_j \mid 1 \le i, j \le n, i \ne j\},$$
$$R(B_n) = \{ij = \epsilon_i - \epsilon_j, \overline{ij} = \epsilon_i + \epsilon_j, i = \epsilon_i \mid 1 \le i, j \le n, i \ne j\},$$

where $(\epsilon_1, \dots, \epsilon_n)$ is an orthonormal basis of V.

Proposition 1.1 *Suppose that R is not of type A_1 or B_2. If the operators $\mathbf{D}_\alpha$ give a representation of $\mathcal{B}_W$, then $f_\alpha(z) = k_\alpha/z$ and $g_\alpha(z) = \pm k_\alpha e^{\lambda_\alpha z}/z$, where k_α are W-invariant constants and the choice of the signature $\pm$ is independent of roots α. The constants λ_α are obtained as $\lambda_\alpha = \lambda(\alpha^\vee)$ from an element $\lambda \in V^*$. Conversely, the operators $\mathbf{D}_\alpha$ corresponding to the above solutions give the representation of $\mathcal{B}_W$.*

Proof. When R is of type A_2, we have the functional equations

$$f_{12}(x-y)f_{23}(y-z) + f_{23}(y-z)f_{31}(z-x) + f_{31}(z-x)f_{12}(x-y) = 0, \quad (1)$$
$$g_{12}(x-y)g_{23}(x-z) + g_{23}(y-z)g_{31}(y-x) + g_{31}(z-x)g_{12}(z-y) = 0. \quad (2)$$

If f_{12} is regular at the origin, then we have $f_{12}(0) = 0$ since f_{12} is odd. We have $f_{23}(x-z)f_{31}(z-x) = 0$ by putting $x = y$ in the equation (1), and hence f_{12}, f_{23} and f_{13} must be constantly zero. So we may assume f_{12}, f_{23} and f_{13} have a pole at the origin. Now the equation (1) shows

$$f_{31}(z-x)^{-1} + f_{12}(x-y)^{-1} + f_{23}(y-z)^{-1} = 0.$$

Therefore we have

$$f_{12}(x) = f_{23}(x) = f_{13}(x) = \frac{k}{x}$$

for some constant k. From Lemma 1.2, we can write

$$g_{ij}(x) = \frac{k\phi_{ij}(x)}{x},$$

where $\phi_{ij}(x)\phi_{ij}(-x) = 1$. From the results in [3, Theorem 16], we can conclude that

$$g_{ij}(x) = \pm\frac{ke^{\lambda_{ij}x}}{x},$$

where $\lambda_{12}+\lambda_{23}+\lambda_{31}=0$. When R contains B_2 as a subsystem, the argument works well. If R contains the subsystem $\{\pm 12, \pm\overline{12}, \pm 1, \pm 2\}$ of type B_2, we have the functional equations

$$f_{12}(x-y)f_1(x) - f_2(y)f_{12}(x-y) + f_{\overline{12}}(x+y)f_2(y) + f_1(x)f_{\overline{12}}(x+y) = 0, \quad (3)$$
$$g_{12}(x-y)g_1(y) - g_2(y)g_{12}(x+y) + g_{\overline{12}}(x+y)g_2(-x) + g_1(x)g_{\overline{12}}(-x+y) = 0. \quad (4)$$

Since R is not of type A_1 or B_2, R contains a subsystem of type A_2. We may assume that f_{12}, $f_{\overline{12}}$, g_{12} and $g_{\overline{12}}$ are determined from the subsystems of type A_2 in R as follows:

$$f_{12}(x) = f_{\overline{12}}(x) = \frac{k}{x}, \quad g_{12}(x) = \frac{ke^{\lambda_{12}x}}{x}, \quad g_{\overline{12}}(x) = \frac{ke^{\lambda_{\overline{12}}x}}{x}.$$

Then the functional equations (3) and (4) can be written as

$$\left(\frac{1}{x-y} + \frac{1}{x+y}\right) f_1(x) + \left(-\frac{1}{x-y} + \frac{1}{x+y}\right) f_2(y) = 0, \quad (5)$$
$$\frac{e^{\lambda_{12}(x-y)}}{x-y} g_1(y) - \frac{e^{\lambda_{12}(x+y)}}{x+y} g_2(y) + \frac{e^{\lambda_{\overline{12}}(x+y)}}{x+y} g_2(-x) + \frac{e^{\lambda_{\overline{12}}(-x+y)}}{-x+y} g_1(x) = 0. \quad (6)$$

Hence we get

$$f_1(x) = f_2(x) = \frac{k'}{x}$$

from the equation (5). The equation (6) is written as

$$\begin{aligned} &(x+y)e^{\lambda_{12}(x-y)}\frac{\phi_1(y)}{y} - (x-y)e^{\lambda_{12}(x+y)}\frac{\phi_2(y)}{y} \\ &-(x-y)e^{\lambda_{\overline{12}}(x+y)}\frac{\phi_2(-x)}{x} - (x+y)e^{\lambda_{\overline{12}}(-x+y)}\frac{\phi_1(x)}{x} \\ &= 0. \end{aligned} \quad (7)$$

We obtain, by taking the limit $y \to 0$,

$$e^{-\lambda_{\overline{12}}x}\phi_1(x) + e^{\lambda_{\overline{12}}x}\phi_2(-x) = e^{\lambda_{12}x}(2 + x(\phi_1'(0) - \phi_2'(0) - 2\lambda_{12})),$$

and by taking the limit $x \to 0$,

$$e^{-\lambda_{12}y}\phi_1(y) + e^{\lambda_{12}y}\phi_2(y) = e^{\lambda_{\overline{12}}y}(2 + y(\phi_1'(0) + \phi_2'(0) - 2\lambda_{\overline{12}})).$$

After eliminating $\phi_2(y)$ and $\phi_2(-x)$ from the equation (7), we have

$$e^{-(\lambda_{12}+\lambda_{\overline{12}})x}\frac{\phi_1(x)}{x^2}-\frac{1}{x^2}-\frac{\phi_1'(0)-\lambda_{12}-\lambda_{\overline{12}}}{x}$$

$$=e^{-(\lambda_{12}+\lambda_{\overline{12}})y}\frac{\phi_1(y)}{y^2}-\frac{1}{y^2}-\frac{\phi_1'(0)-\lambda_{12}-\lambda_{\overline{12}}}{y}.$$

This means that the both sides must be a constant C. Hence, we have

$$\phi_1(x)=e^{(\lambda_{12}+\lambda_{\overline{12}})x}(1+(\phi_1'(0)-\lambda_{12}-\lambda_{\overline{12}})x+Cx^2).$$

From the condition $\phi_1(x)\phi_1(-x)=1$, we get

$$\phi_1'(0)=\lambda_{12}+\lambda_{\overline{12}},\ \ C=0.$$

Therefore we conclude that

$$g_1(x)=\pm\frac{k'e^{\lambda_1 x}}{x},\ g_2(x)=\pm\frac{k'e^{\lambda_2 x}}{x},$$

where $\lambda_1=\lambda_{12}+\lambda_{\overline{12}}$, $\lambda_2=-\lambda_{12}+\lambda_{\overline{12}}$.

When $R_+=\{\alpha_1,\alpha_1+\alpha_2,2\alpha_1+3\alpha_2,\alpha_1+2\alpha_2,\alpha_1+3\alpha_2,\alpha_2\}$ is of type G_2, we have the quadratic relation in the algebra $\mathcal{B}_W$ as follows:

$$[\alpha_1][\alpha_1+\alpha_2]+[\alpha_1+\alpha_2][2\alpha_1+3\alpha_2]+[2\alpha_1+3\alpha_2][\alpha_1+2\alpha_2]$$

$$+[\alpha_1+2\alpha_2][\alpha_1+3\alpha_2]+[\alpha_1+3\alpha_2][\alpha_2]=[\alpha_2][\alpha_1].$$

This equation shows that the constants $(\lambda_\gamma)_{\gamma\in R_+}$ are subject to the following constraints

$$\lambda_{\alpha_1+\alpha_2}=3\lambda_{\alpha_1}+\lambda_{\alpha_2},\lambda_{2\alpha_1+3\alpha_2}=2\lambda_{\alpha_1}+\lambda_{\alpha_2},\lambda_{\alpha_1+2\alpha_2}=3\lambda_{\alpha_1}+2\lambda_{\alpha_2},\lambda_{\alpha_1+3\alpha_2}=\lambda_{\alpha_1}+\lambda_{\alpha_2}.$$

This means that $\lambda_\gamma=\lambda(\gamma^\vee)$ for some $\lambda\in V^*$.

Consider the multiplication operators

$$\mathbf{e}=e^{\sum_{i=1}^r\lambda_{\alpha_i}\pi_i(\xi)}$$

and

$$\mathbf{e}_+=(\prod_{\beta\in R_+}\beta)e^{\sum_{i=1}^r\lambda_{\alpha_i}\pi_i(\xi)},$$

where π_i is the fundamental dominant weight corresponding to α_i. For the operator $\mathbf{D}_\alpha = k_\alpha(1 - e^{\lambda_\alpha(\alpha,\xi)} s_\alpha)/(\alpha, \xi)$, we have

$$\mathbf{D}_\alpha = k_\alpha \mathbf{e} \circ \partial_\alpha \circ \mathbf{e}^{-1}.$$

For the operator $\mathbf{D}_\alpha = k_\alpha(1 + e^{\lambda_\alpha(\alpha,\xi)} s_\alpha)/(\alpha, \xi)$, we have

$$\mathbf{D}_\alpha = k_\alpha \mathbf{e}_+ \circ \partial_\alpha \circ \mathbf{e}_+^{-1}.$$

Namely, $\mathbf{D}_\alpha$ is conjugate to ∂_α up to a constant k_α. Hence the operators $\mathbf{D}_\alpha$ give rise to a representation of $\mathcal{B}_W$ from Lemma 1.1.

Proposition 1.2 *If R is of type B_2, then $\mathcal{B}_W$ is a 64-dimensional algebra defined by the following relations:*
(i) $[12]^2 = \overline{[12]}^2 = [1]^2 = [2]^2 = 0$,
(ii) $[12]\overline{[12]} = \overline{[12]}[12]$, $[1][2] = [2][1]$,
(iii) $[12][1] - [2][12] + [1]\overline{[12]} + \overline{[12]}[2] = 0$, $[1][12] - [12][2] + \overline{[12]}[1] + [2]\overline{[12]} = 0$,
(iv) $[12][1]\overline{[12]}[1] + \overline{[12]}[1][12][1] + [1][12][1]\overline{[12]} + [1]\overline{[12]}[1][12] = 0$,
(v) $[1][12][1][12] = [12][1][12][1]$.

The relations above were considered in [8] and [13]. The algebra defined by these relations is a finite-dimensional algebra with the Hilbert polynomial $(1+t)^4(1+t^2)^2$. Milinski and Schneider [13] and Bazlov [2] have shown that these relations are also satisfied in the algebra $\mathcal{B}_W$. They also checked that the algebra $\mathcal{B}_W$ has dimension 64. Hence, the relations above exhaust the independent defining relations for the algebra $\mathcal{B}_W$ in B_2 case.

Proposition 1.3 *Let R be of type B_2.*
(i) *The functions f_α must be as follows:*

$$f_1(x) = f_2(x) = \frac{A}{\mathrm{sn}(ax, k)},$$

$$f_{12}(x) = f_{\overline{12}}(x) = \frac{B}{\mathrm{sn}(\varepsilon ax, \tilde{k})},$$

where A, B, a, k are arbitrary constants, and $\tilde{k} = (1-k)/(1+k)$, $\varepsilon = (1+k)/\sqrt{-1}$.
(ii) *If one assumes the W-invariance $w \circ \mathbf{D}_\alpha \circ w^{-1} = \mathbf{D}_{w(\alpha)}$, $w \in W$, then*

$$g_\alpha(x) = \pm f_\alpha(x),$$

where the choice of the signature is independent of α.
(iii) *If the functions* $f_\alpha(z)$ *are chosen as in* (i) *and* $g_\alpha(x) = \pm e^{\lambda(\alpha^\vee)x} f_\alpha(x)$, $\lambda \in V^*$, *then the operators* $\mathbf{D}_\alpha$ *give a representation of* $\mathcal{B}_W$.

Proof. (i) This follows from the 4-term quadratic equations and [3, Theorem 6]. The relation

$$[12][1] - [2][12] + \overline{[12]}[2] + [1]\overline{[12]} = 0$$

implies

$$f_{12}(x-y)f_1(x) - f_2(y)f_{12}(x-y) + f_{\overline{12}}(x+y)f_2(y) + f_1(x)f_{\overline{12}}(x+y) = 0.$$

From the equations

$$(f_{12}(x-y) + f_{\overline{12}}(-x+y))g_1(x) = 0, \; (f_1(y) - f_2(y))g_{12}(x-y) = 0,$$

we have $f_{12} = f_{\overline{12}}$ and $f_1 = f_2$. Hence the functions $f_{12} = f_{\overline{12}}$ and $f_1 = f_2$ are the solutions found in [3] in the invariant case, i.e.,

$$f_1(x) = f_2(x) = \frac{A}{\mathrm{sn}(ax, k)},$$

$$f_{12}(x) = f_{\overline{12}}(x) = \frac{B}{\mathrm{sn}(\varepsilon a x, \tilde{k})}.$$

(ii) The W-invariance shows $g_{12}(z) = g_{\overline{12}}(z)$ and $g_1(z) = g_2(z)$. Moreover, the functions $g_\alpha(z)$ must be odd functions. On the other hand, we may set $g_\alpha(z) = f_\alpha(z)\phi_\alpha(z)$ with $\phi_\alpha(z)\phi_\alpha(-z) = 1$ from Lemma 1.2. Since both of f_α and g_α are odd functions, ϕ_α must be even function. Hence, we have $\phi_\alpha(z) = \pm 1$.
(iii) In this case, we can check that the operators $\mathbf{D}_\alpha$ satisfy all the relations listed in Proposition 1.2 by direct computation.

2 Representation of quadratic algebra

Definition 2.1 For a given family of functions $\varphi_{ij}(z) = -\varphi_{ji}(-z)$, $\psi_{ij}(z) = \psi_{ji}(z) \in \mathcal{M}_0$, $1 \le i, j \le n$, $i \ne j$, the algebra $\tilde{\mathcal{E}}_n(\varphi_{ij}, \psi_{ij})$ is a $\mathbb{C}$-algebra generated by the symbols $\langle ij \rangle$ and functions $f(\xi)$ in $\mathcal{M}$ subject to the relations:
(i) $\langle ij \rangle^2 = \psi_{ij}(x_i - x_j)$,
(ii) $\langle ij \rangle \langle kl \rangle = \langle kl \rangle \langle ij \rangle$ for $\{i, j\} \cap \{k, l\} = \emptyset$,
(iii) $\langle ij \rangle \langle jk \rangle + \langle jk \rangle \langle ki \rangle + \langle ki \rangle \langle ij \rangle = 0$,
(iv) $(\langle ij \rangle - \varphi_{ij}(x_i - x_j)) f(\xi) = f(s_{ij}\xi)(\langle ij \rangle - \varphi_{ij}(x_i - x_j))$.

Remark 2.1 The algebra $\mathcal{M}^{S_n}$ of S_n-invariant functions is contained in the center of $\tilde{\mathcal{E}}_n(\varphi_{ij}, \psi_{ij})$. Hence $\tilde{\mathcal{E}}_n(\varphi_{ij}, \psi_{ij})$ has a structure of the $\mathcal{M}^{S_n}$-algebra.

In this section we consider when the quadratic algebra $\tilde{\mathcal{E}}_n(\varphi_{ij}, \psi_{ij})$ has a generalization of the Calogero-Moser representation. For $\lambda \in \mathbb{C} \setminus \mathbb{Z} + \mathbb{Z}\tau$, define the function $\sigma_\lambda(z) = \sigma_\lambda(z|\tau)$ by the formula

$$\sigma_\lambda(z) = \frac{\vartheta_1(z-\lambda)\vartheta_1'(0)}{\vartheta_1(z)\vartheta_1(-\lambda)},$$

where $\vartheta_1(z)$ is Jacobi's theta function

$$\vartheta_1(z) = -\sum_{n=-\infty}^{+\infty} \exp\left(2\pi\sqrt{-1}\left((z+\frac{1}{2})(n+\frac{1}{2}) + \frac{\tau}{2}(n+\frac{1}{2})^2\right)\right).$$

Proposition 2.1 *The algebra $\tilde{\mathcal{E}}_n(\varphi_{ij}, \psi_{ij})$ has the generalized Calogero-Moser representation if and only if $\varphi_{ij}(z) = a/z$ and the functions ψ_{ij} have one of the following forms:*
(i)

$$\psi_{ij}(z) = \frac{A}{z^2} - K(\wp(bz) - \wp(\lambda_i - \lambda_j)),$$

(ii)

$$\psi_{ij}(z) = \frac{A}{z^2} - K\frac{\sin^2(b(z - \lambda_i + \lambda_j))}{\sin^2(bz)\sin^2(b(\lambda_i - \lambda_j))},$$

(iii)

$$\psi_{ij}(z) = \frac{A-K}{z^2} + \frac{K}{(\lambda_i - \lambda_j)^2}.$$

Here, $A = a^2$, K and b are parameters.

Proof. If the generalized Calogero-Moser representation

$$\langle ij \rangle \mapsto \mathbf{D}_{ij} = f_{ij}(x_i - x_j) + g_{ij}(x_i - x_j)s_{ij}$$

is well-defined for the algebra $\tilde{\mathcal{E}}_n(\varphi_{ij}, \psi_{ij})$, then $\varphi_{ij}(z) = f_{ij}(z)$ must be a rational function a/z as we have seen in the proof of Proposition 1.1. The functions g_{ij} are also determined from [3, Theorem 16]. Hence the operator $\mathbf{D}_{ij}$ must be one of the following:
(i)

$$\mathbf{D}_{ij} = \frac{a}{x_i - x_j} + k\sigma_{\lambda_i - \lambda_j}(b(x_i - x_j))e^{(\alpha_i - \alpha_j)(x_i - x_j)}s_{ij},$$

(ii)

$$\mathbf{D}_{ij} = \frac{a}{x_i - x_j} + k\frac{\sin(b(x_i - x_j - \lambda_i + \lambda_j))}{\sin(b(x_i - x_j))\sin(b(\lambda_i - \lambda_j))} e^{(\alpha_i - \alpha_j)(x_i - x_j)} s_{ij},$$

(iii)

$$\mathbf{D}_{ij} = \frac{a}{x_i - x_j} + k\left(\frac{1}{x_i - x_j} - \frac{1}{\lambda_i - \lambda_j}\right) e^{(\alpha_i - \alpha_j)(x_i - x_j)} s_{ij}.$$

In case (i), we have

$$\psi_{ij}(x_i - x_j) = \mathbf{D}_{ij}^2 = \frac{A}{(x_i - x_j)^2} - K(\wp(b(x_i - x_j)) - \wp(\lambda_i - \lambda_j))$$

with $A = a^2$, $K = k^2$. We also have the desired result in cases (i) and (ii) in a similar way.

Remark 2.2 The trigonometric solution (ii) is obtained from the elliptic solution (i) by taking the limit $\tau \to +\infty\sqrt{-1}$ and replacing λ_i by $b\lambda_i$. The rational solution (iii) is obtained from the trigonometric solution by taking the limit $b \to 0$ after replacing K by Kb^2.

Under the assumption of Proposition 2.1, the functions $\varphi_{ij}(z)$ are determined to be the rational function a/z. In the rest of this paper, we denote just by $\tilde{\mathcal{E}}_n(\psi_{ij})$ the quadratic algebra $\tilde{\mathcal{E}}_n(\varphi_{ij}, \psi_{ij})$ with $\varphi_{ij}(z) = a/z$. If we introduce a new set of generators $[ij] = \langle ij \rangle - a/(x_i - x_j)$, then the algebra $\tilde{\mathcal{E}}_n(\psi_{ij})$ is defined by the following relations:
(i)$'$ $[ij]^2 = \psi_{ij}(x_i - x_j) - A/(x_i - x_j)^2$,
(ii)$'$ $[ij][kl] = [kl][ij]$ for $\{i, j\} \cap \{k, l\} = \emptyset$,
(iii)$'$ $[ij][jk] + [jk][ki] + [ki][ij] = 0$,
(iv)$'$ $[ij]f(\xi) = f(s_{ij}\xi)[ij]$.

3 Subalgebra generated by Dunkl elements

In this section, the functions ψ_{ij} are assumed to be chosen as in Proposition 2.1 (i) with $K = b = 1$ for simplicity.

We define the Dunkl elements θ_i in the algebra $\tilde{\mathcal{E}}_n(\psi_{ij})$ by the formula

$$\theta_i = \sum_{j \neq i} [ij].$$

We can easily see the following from the defining quadratic relations for $\tilde{\mathcal{E}}_n(\psi_{ij})$.

Proposition 3.1 *The Dunkl elements $\theta_1, \ldots, \theta_n$ commute pairwise.*

In the rest of this section, we discuss the structure of the commutative subalgebra generated by the Dunkl elements $\theta_1, \ldots, \theta_n$ over $\mathcal{M}^{S_n}$ in the algebra $\tilde{\mathcal{E}}_n(\psi_{ij})$. We use an abbreviation $x_{ij} := x_i - x_j$, $\lambda_{ij} := \lambda_i - \lambda_j$ in the following.

Lemma 3.1 ([6, Lemma 7.3]) *For distinct $i_1, \ldots, i_k$, one has the following relation in the algebra $\tilde{\mathcal{E}}_n(\psi_{ij})$ for $k \geq 3$.*

$$\sum_{a=1}^{k} [i_a\ i_{a+1}][i_a\ i_{a+2}] \cdots [i_a\ i_k] \cdot [i_a\ i_1][i_a\ i_2] \cdots [i_a\ i_{a-1}] = 0. \tag{8}$$

Proof. The proof is done by induction on k. For $k = 3$, the relation (8) is just the 3-term relation

$$[i_1\ i_2][i_2\ i_3] + [i_2\ i_3][i_3\ i_1] + [i_3\ i_1][i_1\ i_2] = 0.$$

Let $Q_k(i_1, \ldots, i_k)$ denote the left-hand side of the above relation. By using the 3-term relation

$$[i_a\ i_{k-1}][i_a\ i_k] = [i_{k-1}\ i_k][i_a\ i_{k-1}] - [i_a\ i_k][i_{k-1}\ i_k],$$

we get

$$\begin{aligned}
&\sum_{a=1}^{k} [i_a\ i_{a+1}][i_a\ i_{a+2}] \cdots [i_a\ i_k] \cdot [i_a\ i_1][i_a\ i_2] \cdots [i_a\ i_{a-1}] \\
= &\sum_{a=1}^{k-2} [i_a\ i_{a+1}] \cdots [i_a\ i_{k-2}] \cdot \Big([i_{k-1}\ i_k][i_a\ i_{k-1}] - [i_a\ i_k][i_{k-1}\ i_k] \Big) \cdot [i_a\ i_1] \cdots [i_a\ i_{a-1}] \\
&+ [i_{k-1}\ i_k][i_{k-1}\ i_1] \cdots [i_{k-1}\ i_{k-2}] + [i_k\ i_1][i_k\ i_2] \cdots [i_k\ i_{k-1}] \\
= &[i_{k-1}\ i_k] Q_{k-1}(i_1, \ldots, i_{k-1}) - Q_{k-1}(i_1, \ldots, i_{k-2}, i_k)[i_{k-1}\ i_k] = 0.
\end{aligned}$$

Lemma 3.2 *For distinct $i_1, \ldots, i_k, m$, one has the following relation in the algebra $\tilde{\mathcal{E}}_n(\psi_{ij})$ for $k \geq 2$.*

$$(-1)^{k+1} \sum_{a=1}^{k} [i_a\ m][i_{a+1}\ m] \cdots [i_k\ m] \cdot [i_1\ m][i_2\ m] \cdots [i_{a-1}\ m][i_a\ m]$$

$$= \sum_{a=1}^{k} \wp(\lambda_{i_a m}) [i_a\ i_{a+1}][i_a\ i_{a+2}] \cdots [i_a\ i_k] \cdot [i_a\ i_1][i_a\ i_2] \cdots [i_a\ i_{a-1}] \tag{9}$$

Proof. The proof is done by induction on k. For $k = 2$, we have

$$\begin{aligned}
&[i_1\ m][i_2\ m][i_1\ m] + [i_2\ m][i_1\ m][i_2\ m] \\
&\quad = \Big([i_2\ m][i_1\ i_2] - [i_1\ i_2][i_1\ m]\Big)[i_1\ m] + [i_2\ m]\Big([i_2\ m][i_1\ i_2] - [i_1\ i_2][i_1\ m]\Big) \\
&\quad = -[i_1\ i_2](\psi_{i_1\ m}(x_{i_1\ m}) - Ax_{i_1\ m}^{-2}) + (\psi_{i_2\ m}(x_{i_2\ m}) - Ax_{i_2\ m}^{-2})[i_1\ i_2] \\
&\quad = \Big(\psi_{i_2\ m}(x_{i_2\ m}) - \psi_{i_1\ m}(x_{i_2\ m})\Big)[i_1\ i_2] \\
&\quad = \Big(\wp(\lambda_{i_2\ m}) - \wp(\lambda_{i_1\ m})\Big)[i_1\ i_2].
\end{aligned}$$

Let $P_k(i_1, \ldots, i_k; m)$ denote the left-hand side of the relation (9). Here we show only the relation

$$P_k(1, 2, \ldots, k; m) = \sum_{a=1}^{k} \wp(\lambda_{am})[a\ a{+}1][a\ a{+}2] \cdots [a\ k] \cdot [a\ 1][a\ 2] \cdots [a\ a{-}1],$$

since the general relations can be proved in similar manner. By using the quadratic relation $[i_{k-1}\ m][i_k\ m] = [i_k\ m][i_{k-1}\ i_k] - [i_{k-1}\ i_k][i_{k-1}\ m]$ and the assumption of the induction, we obtain

$$\begin{aligned}
&P_k(1, \ldots, k; m) \\
&\quad = [k-1\ k] \cdot P_{k-1}(1, \ldots, k-2, k-1; m) - P_{k-1}(1, \ldots, k-2, k; m) \cdot [k-1\ k] \\
&\quad = [k-1\ k] \cdot \sum_{a=1}^{k-1} \wp(\lambda_{am})[a\ a+1] \cdots [a\ k-1] \cdot [a\ 1] \cdots [a\ a-1] \\
&\qquad - \sum_{a=1}^{k-2} \wp(\lambda_{am})[a\ a+1] \cdots [a\ k-2][a\ k] \cdot [a\ 1] \cdots [a\ a-1] \cdot [k-1\ k] \\
&\qquad - \wp(\lambda_{km})[k\ 1][k\ 2] \cdots [k\ k-2][k-1\ k] \\
&\quad = \sum_{a=1}^{k-2} \wp(\lambda_{am})[a\ a+1] \cdots [a\ k-2]\Big([k-1\ k][a\ k-1] - [a\ k][k-1\ k]\Big)[a\ 1] \cdots [a\ a-1] \\
&\qquad + \wp(\lambda_{k-1\ m})[k-1\ k][k-1\ 1] \cdots [k-1\ k-2] \\
&\qquad + \wp(\lambda_{km})[k\ 1][k\ 2] \cdots [k\ k-2][k\ k-1] \\
&\quad = \sum_{a=1}^{k-2} \wp(\lambda_{am})[a\ a+1] \cdots [a\ k-2]\Big([a\ k-1][a\ k]\Big)[a\ 1] \cdots [a\ a-1] \\
&\qquad + \wp(\lambda_{k-1\ m})[k-1\ k][k-1\ 1] \cdots [k-1\ k-2]
\end{aligned}$$

$$
\begin{aligned}
& + \wp(\lambda_{km})[k\ 1][k\ 2]\cdots[k\ k-2][k\ k-1] \\
= & \sum_{a=1}^{k} \wp(\lambda_{am})[a\ a+1][a\ a+2]\cdots[a\ k]\cdot[a\ 1][a\ 2]\cdots[a\ a-1].
\end{aligned}
$$

Example. $(k=4)$

$$
\begin{aligned}
& [1m][2m][3m][4m][1m] + [2m][3m][4m][1m][2m] \\
& +[3m][4m][1m][2m][3m] + [4m][1m][2m][3m][4m] \\
= & -[1m][2m][34][3m][1m] + [1m][2m][4m][34][1m] \\
& - [2m][34][3m][1m][2m] + [2m][4m][34][1m][2m] \\
& - [34][3m][1m][2m][3m] + [4m][34][1m][2m][3m] \\
& - [4m][1m][2m][34][3m] + [4m][1m][2m][4m][34] \\
= & -[34]\Big([1m][2m][3m][1m] + [2m][3m][1m][2m] + [3m][1m][2m][3m]\Big) \\
& + \Big([1m][2m][4m][1m] + [2m][4m][1m][2m] + [4m][1m][2m][4m]\Big)[34] \\
= & -[34]\Big(\wp(\lambda_{1m})[12][13] + \wp(\lambda_{2m})[23][21] + \wp(\lambda_{3m})[31][32]\Big) \\
& +\Big(\wp(\lambda_{1m})[12][14] + \wp(\lambda_{2m})[24][21] + \wp(\lambda_{4m})[41][42]\Big)[34] \\
= & -\wp(\lambda_{1m})[12]([34][13] - [14][34]) - \wp(\lambda_{2m})([34][23] - [24][34])[21] \\
& -\wp(\lambda_{3m})[34][31][32] - \wp(\lambda_{4m})[41][42][43] \\
= & -\wp(\lambda_{1m})[12][13][14] - \wp(\lambda_{2m})[23][24][21] - \wp(\lambda_{3m})[34][31][32] - \wp(\lambda_{4m})[41][42][43]
\end{aligned}
$$

Remark 3.1 Lemma 3.2 is a deformed version of [6, Lemma 7.2] and [15, Lemma 5.3]. Though the identity (9) looks similar to the one in [15, Lemma 5.3], they are different formulas. In our case, $[ij]^2 = \psi_{ij}(x_{ij}) - Ax_{ij}^{-2}$ is not central, and $[ij]^2 \neq \wp(\lambda_{ij})$.

For a subset $I \subset \{1, \ldots, n\}$ with $\#I = 2k$, define the function $\phi(I) = \phi(x_i | i \in I)$ by the following formula:

$$
\phi(I) := \sum \prod_{i=1}^{k} \wp(x_{a_i b_i}),
$$

where the summension is taken over the choice of pairs (a_i, b_i), $1 \leq i \leq k$, such that $I = \{a_1, \ldots, a_k, b_1, \ldots b_k\}$, $a_1 < \cdots < a_k$ and $a_i < b_i$. We also define the deformed elementary symmetric polynomial $E_k(I) = E_k(X_i \,|\, i \in I)$ by the recursion relations:

$$E_0(I) = 1,\ E_k(I \cup \{j\}) = E_k(I) + E_{k-1}(I)X_j + \sum_{i \in I} \wp(\lambda_{ij}) E_{k-2}(I \setminus \{i\}).$$

Theorem 3.1 *One has the following formula in the algebra* $\tilde{\mathcal{E}}_n(\psi_{ij})$ *:*

$$E_k(\theta_i \mid i \in I) = \sum_{l=0}^{[k/2]} \sum_{I_0 \subset I, \#I_0 = 2l} \phi(I_0) \sum_{(*)} [a_1\ b_1] \cdots [a_{k-2l}\ b_{k-2l}], \tag{10}$$

where $(*)$ *stands for the conditions that* $a_i \in I \setminus I_0$*;* $b_i \notin I$*;* $a_1, \ldots, a_{k-2l}$ *are distinct;* $b_1 \leq \cdots \leq b_{k-2l}$*.*

Corollary 3.1

$$E_k(\theta_1, \ldots, \theta_n) = \begin{cases} \sum_{I_0 \subset I, \#I_0 = k} \phi(I_0) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Proof of Theorem 3.1. Denote by $F_k(I)$ the right-hand side of the fomula (10). For $I \subset \{1, \ldots, n\}$ and $j \notin I$, we will show the recursive formula

$$F_k(I \cup \{j\}) = F_k(I) + \theta_j F_{k-1}(I) + \sum_{i \in I} \wp(\lambda_{ij}) F_{k-2}(I \setminus \{i\}).$$

Let $J = \{j_1 = j, \ldots, j_d\}$ be the set $\{1, \ldots, n\} \setminus I$. For $L = \{l_1, \ldots, l_m\} \subset \{1, \ldots, n\}$ and $r \notin L$, we define

$$\langle\langle L \mid r \rangle\rangle := \sum_{w \in S_m} [l_{w(1)}\ r][l_{w(2)}\ r] \cdots [l_{w(m)}\ r].$$

In order to show the formula above, we use the following decompositions which are similar to those used in [15]. In the following, the symbol $I_1 \cdots I_d \subset_m I$ means that $I_1, \ldots, I_d \subset I$ are disjoint and $\#I_1 + \cdots + \#I_d = m$. Here, some of $I_1, \ldots, I_d$ may be empty. Let us consider the decompositions:

$$\begin{aligned} F_k(I) &= \sum_{l=0}^{[k/2]} \sum_{I_0 \subset_{2l} I} \phi(I_0) \sum_{I_1 \ldots I_d \subset_{k-2l} I \setminus I_0} \langle\langle I_1 \mid j_1 \rangle\rangle \langle\langle I_2 \mid j_2 \rangle\rangle \cdots \langle\langle I_d \mid j_d \rangle\rangle \\ &= A_1 + A_2, \end{aligned}$$

$$
\begin{aligned}
F_k(I\cup\{j\}) &= \sum_{l=0}^{[k/2]} \sum_{I_0'\subset_{2l} I\cup\{j\}} \phi(I_0') \sum_{I_2'\dots I_d'\subset_{k-2l}(I\cup\{j\})\setminus I_0'} \langle\!\langle I_2' \mid j_2\rangle\!\rangle\langle\!\langle I_3' \mid j_3\rangle\!\rangle\cdots\langle\!\langle I_d' \mid j_d\rangle\!\rangle \\
&= B_1+B_2+B_3, \\
\theta_j F_{k-1}(I) &= \sum_{s\neq j}[js] \sum_{l=0}^{[(k-1)/2]} \sum_{I_0''\subset_{2l} I}\phi(I_0'') \sum_{I_1''\dots I_d''\subset_{k-1-2l} I\setminus I_0''} \langle\!\langle I_1'' \mid j_1\rangle\!\rangle\cdots\langle\!\langle I_d'' \mid j_d\rangle\!\rangle \\
&= C_1+C_2+C_3+C_4,
\end{aligned}
$$

where A_1 is the sum of terms with $I_1=\emptyset$; A_2 is the sum of terms with $I_1\neq\emptyset$; B_1 is the sum of the terms with $j\notin I_0\cup I_2'\cup\cdots\cup I_d'$; B_2 is the sum of terms with $j\in I_2'\cup\cdots\cup I_d'$; B_3 is the sum of terms with $j\in I_0''$; C_1 is the sum of terms with $s\in I\setminus(I_0''\cup I_1''\cup\cdots\cup I_d'')$; C_2 is the sum of terms with $s\in I_2''\cup\cdots\cup I_d''\cup J$; C_3 is the sum of terms with $s\in I_0''$; C_4 is the sum of terms with $s\in I_1''$. Then we can see that $A_1=B_1$, $A_2+C_1=0$ and $B_2=C_2$ by the same argument in [15].

Note that the formula in Lemma 3.2 holds only for $k\geq 2$. For any subset $K=\{k_1,\dots,k_m\}$ with $j\notin K$, we have

$$
\begin{aligned}
&\sum_{s\in K}\left([js]\langle\!\langle K \mid j\rangle\!\rangle + \sum_{L\subset K\setminus\{s\}} \wp(\lambda_{js})\langle\!\langle L \mid s\rangle\!\rangle\langle\!\langle K\setminus L\setminus\{s\} \mid j\rangle\!\rangle\right) \\
&= \sum_{s\in K}\Bigg([js][sj]\langle\!\langle K\setminus\{s\} \mid j\rangle\!\rangle + [js]\sum_{w\in S_m, k_{w(1)}\neq s}[k_{w(1)}\, j]\cdots[k_{w(k)}\, j] \\
&\qquad +\wp(\lambda_{js})\langle\!\langle K\setminus\{s\} \mid j\rangle\!\rangle + \sum_{L\subset K\setminus\{s\}, L\neq\emptyset}\wp(\lambda_{js})\langle\!\langle L \mid s\rangle\!\rangle\langle\!\langle K\setminus L\setminus\{s\} \mid j\rangle\!\rangle\Bigg) \\
&= \sum_{s\in K}\wp(x_{js})\langle\!\langle K\setminus\{s\} \mid j\rangle\!\rangle
\end{aligned}
$$

from Lemma 3.2 and $[ij]^2=\psi_{ij}(x_{ij})-Ax_{ij}^{-2}=-(\wp(x_{ij})-\wp(\lambda_{ij}))$. This shows

$$
\begin{aligned}
&C_3+C_4+\sum_{i\in I}\wp(\lambda_{ij})F_{k-2}(I\setminus\{i\}) \\
&= \sum_{l=1}^{[(k-1)/2]}\sum_{I_0''\subset_{2l} I}\sum_{s\in I_0''}\phi(\{j\}\cup I_0''\setminus\{s\})\cdot[js] \sum_{I_1''\cdots I_d''\subset_{k-1-2l} I\setminus I_0''}\langle\!\langle I_1'' \mid j_1\rangle\!\rangle\cdots\langle\!\langle I_d'' \mid j_d\rangle\!\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{l=0}^{[(k-1)/2]} \sum_{I_0''\subset_{2l} I} \phi(I_0'') \sum_{I_1''\cdots I_d''\subset_{k-1-2l} I\setminus I_0''} \sum_{s\in I_1''} \wp(x_{js}) \langle\!\langle I_1''\setminus\{s\}|\, j_1\rangle\!\rangle \cdots \langle\!\langle I_d'' \mid j_d\rangle\!\rangle \\
&= -\sum_{l=1}^{[(k-1)/2]} \sum_{I_0''\subset_{2l} I, j\in I_0''} \phi(I_0'') \sum_{I_1''\cdots I_d''\subset_{k-2l} I\setminus I_0'', I_1''\neq\emptyset} \langle\!\langle I_1'' \mid j_1\rangle\!\rangle \cdots \langle\!\langle I_d'' \mid j_d\rangle\!\rangle \\
&+ \sum_{l=0}^{[(k-1)/2]} \sum_{I_0''\subset_{2l+2} I, j\in I_0''} \phi(I_0'') \sum_{I_1''\cdots I_d''\subset_{k-2l-2} I\setminus I_0''} \langle\!\langle I_1'' \mid j_1\rangle\!\rangle \cdots \langle\!\langle I_d'' \mid j_d\rangle\!\rangle \\
&= \sum_{l=1}^{[k/2]} \sum_{I_0''\subset_{2l} I, j\in I_0''} \phi(I_0'') \sum_{I_2''\cdots I_d''\subset_{k-2l} I\setminus I_0''} \langle\!\langle I_2'' \mid j_2\rangle\!\rangle \cdots \langle\!\langle I_d'' \mid j_d\rangle\!\rangle \\
&= B_3.
\end{aligned}
$$

Example. One has the following formula for $E_3(\theta_1, \theta_2, \theta_3)$ in $\tilde{\mathcal{E}}_5(\psi_{ij})$:

$$\theta_1\theta_2\theta_3 + \wp(\lambda_{23})\theta_1 + \wp(\lambda_{13})\theta_2 + \wp(\lambda_{12})\theta_3 =$$

$$\sum_{(**)}[a_1\ b_1][a_2\ b_2][a_3\ b_3] + \psi_{12}(x_{12})([34]+[35]) + \psi_{13}(x_{13})([24]+[25]) + \psi_{23}(x_{23})([14]+[15]),$$

where $(**)$ stands for the condition that $\{a_1, a_2, a_3\} = \{1, 2, 3\}$; $b_1, b_2, b_3 \in \{4, 5\}$ and $b_1 \leq b_2 \leq b_3$.

4 Degenerations

Some variants of the cohomology ring of the flag variety

$$Fl_n = SL_n(\mathbb{C})/(\text{upper triangular matrices})$$

have the model as the commutative subalgebra in deformations of the quadratic algebra $\mathcal{E}_n$. We see how the deformations of $\mathcal{E}_n$ used for the constructions of the model of the cohomology rings can be recovered as degenerations of our algebra $\tilde{\mathcal{E}}_n(\psi_{ij})$.

Let $T \subset SL_n(\mathbb{C})$ be the torus consisting of the diagonal matrices. We identify the polynomial ring $R = \mathbb{Z}[x_1, \ldots, x_n]$ with the T-equivariant cohomology ring $H_T(\text{pt.})$. The authors introduced the extended quadratic algebra $\tilde{\mathcal{E}}_n\langle R\rangle[t]$ to construct a model of the T-equivariant cohomology ring $H_T(Fl_n)$ in [9]. In case $\psi_{ij}(z) = 0$ for any distinct i and j, the algebra $\tilde{\mathcal{E}}_n(\psi_{ij} = 0)$

is defined by the relations $[ij]^2 = 0$, $[ij][kl] = [kl][ij]$ for $\{i,j\} \cap \{k,l\} = \emptyset$, $[ij][jk] + [jk][ki] + [ki][ij] = 0$ and $[ij]x_i = x_j[ij]$. Since these relations are same as the defining relations for the algebra $\tilde{\mathcal{E}}_n\langle R\rangle[t]|_{t=0}$ introduced in [9], the $\mathbb{C}$-subalgebra of $\tilde{\mathcal{E}}_n(\psi_{ij} = 0)$ generated by $[ij]$'s and $x_1, \ldots, x_n$ is isomorphic to $\tilde{\mathcal{E}}_n\langle R\rangle[t]|_{t=0}$. The subsequent result shows that the elements

$$\theta'_i := x_i + \theta_i = x_i + \sum_{j \neq i}[ij], \quad i = 1, \ldots, n,$$

generate a commutative R-subalgebra of $\tilde{\mathcal{E}}_n(\psi_{ij} = 0) \otimes_{R^{S_n}} R$ which is isomorphic to the T-equivariant cohomology ring $H_T(Fl_n)$.

Proposition 4.1 ([9, Corollary 2.2]) *Let I be a subset of $\{1, \ldots, n\}$. In the algebra $\tilde{\mathcal{E}}_n(\psi_{ij} = 0)$, one has*

$$e_k(\theta'_i \mid i \in I) = \sum_{m=0}^{k} \sum_{I_0 \subset_m I} (\prod_{i \in I_0} x_i) \sum_{(*)} [a_1\, b_1] \cdots [a_{k-m}\, b_{k-m}], \tag{11}$$

where $(*)$ stands for the conditions that $a_i \in I \setminus I_0$; $b_i \notin I$; $a_1, \ldots, a_{k-m}$ are distinct; $b_1 \leq \cdots \leq b_{k-m}$. In particular, one has

$$e_k(\theta'_1, \ldots, \theta'_n) = e_k(x_1, \ldots, x_n), \quad 1 \leq k \leq n.$$

Proof. The idea is similar to the proof of Theorem 3.1. Denote by $F'_k(I)$ the right-hand side of (11). For $j \notin I$, we will show that

$$F'_k(I \cup \{j\}) = F'_k(I) + F'_{k-1}(I)(x_j + \theta_j).$$

We use the same notation as the one used in the proof of Theorem 3.1. Let us consider the decompositions:

$$\begin{aligned}
F'_k(I) &= \sum_{m=0}^{k} \sum_{I_0 \subset_m I} (\prod_{i \in I_0} x_i) \sum_{I_1 \ldots I_d \subset_{k-m} I \setminus I_0} \langle\!\langle I_1 \mid j_1 \rangle\!\rangle \langle\!\langle I_2 \mid j_2 \rangle\!\rangle \cdots \langle\!\langle I_d \mid j_d \rangle\!\rangle \\
&= A'_1 + A'_2, \\
F'_k(I \cup \{j\}) &= \sum_{m=0}^{k} \sum_{I'_0 \subset_m I \cup \{j\}} (\prod_{i \in I'_0} x_i) \sum_{I'_2 \ldots I'_d \subset_{k-m} (I \cup \{j\}) \setminus I'_0} \langle\!\langle I'_2 \mid j_2 \rangle\!\rangle \langle\!\langle I'_3 \mid j_3 \rangle\!\rangle \cdots \langle\!\langle I'_d \mid j_d \rangle\!\rangle \\
&= B'_1 + B'_2 + B'_3, \\
F'_{k-1}(I)\theta_j &= \sum_{m=0}^{k-1} \sum_{I''_0 \subset_m I} (\prod_{i \in I''_0} x_i) \sum_{I''_1 \ldots I''_d \subset_{k-1-m} I \setminus I''_0} \langle\!\langle I''_1 \mid j_1 \rangle\!\rangle \cdots \langle\!\langle I''_d \mid j_d \rangle\!\rangle \sum_{s \neq j} [js] \\
&= C'_1 + C'_2 + C'_3 + C'_4,
\end{aligned}$$

where A'_1 is the sum of terms with $I_1 = \emptyset$; A'_2 is the sum of terms with $I_1 \neq \emptyset$; B'_1 is the sum of the terms with $j \notin I_0 \cup I'_2 \cup \cdots \cup I'_d$; B'_2 is the sum of terms with $j \in I'_2 \cup \cdots \cup I'_d$; B'_3 is the sum of terms with $j \in I''_0$; C'_1 is the sum of terms with $s \in I \setminus (I''_0 \cup I''_1 \cup \cdots \cup I''_d)$; C'_2 is the sum of terms with $s \in I''_2 \cup \cdots \cup I''_d \cup J$; C'_3 is the sum of terms with $s \in I''_0$; C'_4 is the sum of terms with $s \in I''_1$. Moreover, we decompose $F'_{k-1}(I)x_j$ as follows:

$$\begin{aligned} F'_{k-1}(I)x_j &= \sum_{m=0}^{k-1} \sum_{I''_0 \subset_m I} (\prod_{i \in I''_0} x_i) \sum_{I''_1 \ldots I''_d \subset_{k-1-m} I \setminus I''_0} \langle\!\langle I''_1 \mid j_1 \rangle\!\rangle \cdots \langle\!\langle I''_d \mid j_d \rangle\!\rangle x_j \\ &= D'_1 + D'_2, \end{aligned}$$

where D'_1 is the sum of terms with $I''_1 = \emptyset$ and D'_2 is the sum of terms with $I''_1 \neq \emptyset$. As before, we can easily see that $A'_1 = B'_1$, $A'_2 + C'_1 = 0$ and $B'_2 = C'_2$. It is also clear that $B'_3 = D'_1$. Since the relations $[ij]^2 = 0$ are assumed, the degenerate version of the formula (9), which is same as [6, Lemma 7.2], holds in $\tilde{\mathcal{E}}_n(\psi_{ij} = 0)$:

$$\sum_{a=1}^{k} [i_a\ m][i_{a+1}\ m] \cdots [i_k\ m] \cdot [i_1\ m][i_2\ m] \cdots [i_{a-1}\ m][i_a\ m] = 0, \quad \text{for } k \geq 1.$$

This formula implies $C'_4 = 0$. Finally, the following computation completes the proof:

$$\begin{aligned} & C'_3 + D'_2 \\ &= \sum_{m=1}^{k-1} \sum_{I''_0 \subset_m I} (\prod_{i \in I''_0} x_i) \sum_{I''_1 \ldots I''_d \subset_{k-1-m} I \setminus I''_0} \langle\!\langle I''_1 \mid j_1 \rangle\!\rangle \cdots \langle\!\langle I''_d \mid j_d \rangle\!\rangle \sum_{s \in I''_0} [js] \\ &+ \sum_{m=0}^{k-1} \sum_{I''_0 \subset_m I} (\prod_{i \in I''_0} x_i) \sum_{I''_1 \ldots I''_d \subset_{k-1-m} I \setminus I''_0, I''_1 \neq \emptyset} \langle\!\langle I''_1 \mid j_1 \rangle\!\rangle \cdots \langle\!\langle I''_d \mid j_d \rangle\!\rangle x_j \\ &= -\sum_{m=1}^{k-1} \sum_{I''_0 \subset_m I} (\prod_{i \in I''_0} x_i) \sum_{I''_1 \ldots I''_d \subset_{k-1-m} I \setminus I''_0} \sum_{s \in I''_0} \langle\!\langle I''_1 \mid j_1 \rangle\!\rangle [sj] \langle\!\langle I_2 \mid j_2 \rangle\!\rangle \cdots \langle\!\langle I''_d \mid j_d \rangle\!\rangle \\ &+ \sum_{m=0}^{k-1} \sum_{I''_0 \subset_m I} (\prod_{i \in I''_0} x_i) \sum_{I''_1 \ldots I''_d \subset_{k-1-m} I \setminus I''_0, I''_1 \neq \emptyset} \sum_{s \in I''_1} x_s \langle\!\langle I''_1 \setminus \{s\} | j_1 \rangle\!\rangle [sj] \langle\!\langle I_2 | j_2 \rangle\!\rangle \cdots \langle\!\langle I''_d | j_d \rangle\!\rangle \\ &= 0. \end{aligned}$$

Let us consider another kind of degeneration. Consider the elliptic solution obtained in Proposition 2.1 (i). If we put $K = \kappa\delta^2$ and $\lambda_{ij} = \delta\Lambda_{ij}$, then we have

$$\lim_{\delta\to 0} \psi_{ij}(x_{ij}) = Ax_{ij}^{-2} + \kappa\Lambda_{ij}^{-2}.$$

In this situation, the functions $[ij]^2 = \psi_{ij}(x_{ij}) - Ax_{ij}^{-2}$ become central parameters $\kappa\Lambda_{ij}^{-2}$. Then the $\mathbb{C}$-algebra generated by the brackets $[ij]$ in $\tilde{\mathcal{E}}_n(\psi_{ij} = \kappa\Lambda_{ij}^{-2})$ is isomorphic to the multiparameter deformation of $\mathcal{E}_n$ introduced in [6, Section 15], which is denoted by $\mathcal{E}_n^p$ in [15], under the identification of the parameters $p_{ij} = p_{ji} = \kappa\Lambda_{ij}^{-2}$. In this case, the functions $\phi(I)$ are constantly zero, so Theorem 3.1 is reduced to the following:

Proposition 4.2 ([6, Conjecture 15.1], [15, Theorem 3.1]) *Assume that the functions ψ_{ij} are chosen as in Proposition 2.1 (iii) with $K = \kappa\delta^2$, $\lambda_{ij} = \delta\Lambda_{ij}$. In the limit $\delta \to 0$, one has*

$$E_k(\theta_i, i \in I; p) = \sum_{(*)'} [a_1\ b_1] \cdots [a_k\ b_k],$$

where $(*)'$ stands for the conditions that $a_i \in I$; $b_i \notin I$; $a_1, \ldots, a_k$ are distinct; $b_1 \le \cdots \le b_k$.

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