Metric packing for $K_3 + K_3$

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Abstract

In this paper, we consider the metric packing problem for the commodity graph of disjoint two triangles $K_3 + K_3$, which is dual to the multiflow feasibility problem for the commodity graph $K_3 + K_3$. We prove Karzanov's conjecture concerning quarter-integral packings by certain bipartite metrics.

1 Introduction and main result

A *metric* μ on a finite set V is a function $V \times V \to \mathbf{R}$ satisfying $\mu(i,i) = 0$, $\mu(i,j) = \mu(j,i) \geq 0$, and the triangle inequalities $\mu(i,j) + \mu(j,k) \geq \mu(i,k)$ for $i,j,k \in V$. Throughout in this paper, a graph means an undirected graph. Let $G = (V,E)$ be a graph. For a nonnegative edge length function $l : E \to \mathbf{R}_+$, let $d_{G,l}$ denote the graph metric on V induced by (G,l), i.e., $d_{G,l}(i,j)$ is the shortest path length between i and j in G with respect to edge length l. Let d_G denote the metric on V by G with unit edge length.

Let $H = (S,R)$ be another graph on $S \subseteq V$, called a *commodity graph*. A finite set of metrics $\mathcal{M}$ on V together with its nonnegative weight $\lambda : \mathcal{M} \to \mathbf{R}_+$ is called a *fractional H-packing* for (G,l) if it satisfies

$$\begin{aligned} l(ij) &\geq \sum_{\mu \in \mathcal{M}} \lambda(\mu)\mu(i,j) \quad (ij \in E), \\ d_{G,l}(s,t) &= \sum_{\mu \in \mathcal{M}} \lambda(\mu)\mu(s,t) \quad (st \in R). \end{aligned} \tag{1.1}$$

If λ is integral, then it is called an *integral H-packing* for (G,l).

A classical theorem in the network flow theory says that if H consists of a single edge and l is integral, there is an integral H-packing by *cut metrics*. Here a metric d is called a cut metric if there is a set $X \subseteq V$ such that $d(i,j) = 1$ if $|X \cap \{i,j\}| = 1$ and $d(i,j) = 0$ otherwise. This is a *polar* theorem to the famous Ford-Fulkerson's maxflow-mincut theorem [9]. As is well-known, fractional H-packing problems are polar to the multiflow feasibility problems with commodity graph H; see [21, Chapter 70]. The *multiflow feasibility problem* is: given a capacity $c : E \to \mathbf{R}_+$ and a demand $q : R \to \mathbf{R}_+$, find flows f_{st} $(st \in F)$ from s to t of value $q(st)$ such that for each $e \in E$ the total flow through e does not exceed $c(e)$, or establish that no such a flow exists.

For a finite set of metrics $\mathcal{M}$ on V, an obvious necessary condition for multiflow feasibility

$$\sum_{ij \in E} c(ij)\mu(i,j) \geq \sum_{st \in R} q(st)\mu(s,t) \quad (\mu \in \mathcal{M}) \tag{1.2}$$

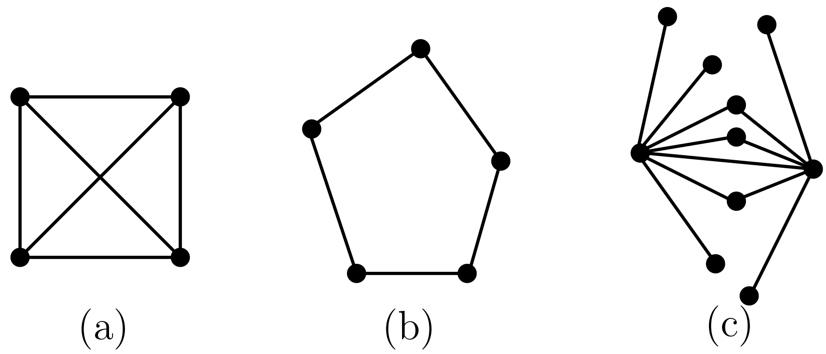


Figure 1: (a) K_4, (b) C_5, and (c) the union of two stars

is also sufficient if and only if for any nonnegative length function $l : E \to \mathbf{R}_+$ there exists a fractional H-packing for (G, l) by $\mathcal{M}$. This is a simple consequence of the linear programming duality.

Papernov [19] has characterized the class of commodity graphs with property that the *cut condition*, i.e., (1.2) by taking $\mathcal{M}$ as cut metrics, is sufficient for multiflow feasibility. He has shown that if H is K_4, C_5, or the union of two stars, then the cut condition is sufficient, where K_n is the complete graph on n vertices, C_m is a cycle on m vertices, and a *star* is a graph all of whose edge have a common vertex; see Figure 1. By polarity, there exists a fractional H-packing by cut metrics in this case.

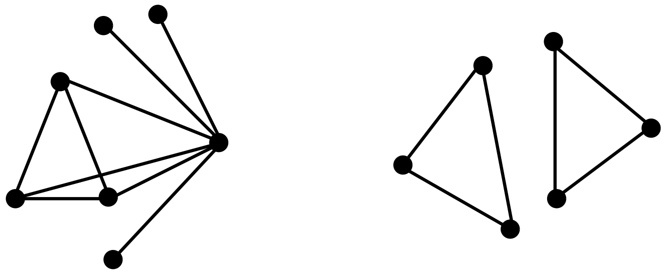
Karzanov [12] has strengthened this result to a half-integral version. Here the length function l on G is said to be *cyclically even* if l is integral and $\sum_{e \in C} l(e)$ is even for any cycle C in G.

Theorem 1.1 ([12]). *Let G be a graph with cyclically even edge length l and H a commodity graph. If H is K_4, C_5, or the union of two stars, then there exists an integral H-packing for (G, l) by cut metrics.*

If H violates the condition of Theorem 1.1, the cut condition is not sufficient for the existence of feasible multiflows, and therefore an H-packing by cut metrics does not exist in general. Karzanov [13] has studied the multiflow feasibility problems for a five-vertex commodity graph, and shown that the $K_{2,3}$-*metric condition* is sufficient. Here, for a graph Γ on X, a metric μ on V is called a Γ-*metric* if there is a map $\phi : V \to X$ such that $\mu(i, j) = d_\Gamma(\phi(i), \phi(j))$ for $i, j \in V$. $K_{n,m}$ denotes the complete bipartite graph with parts of n and m vertices. In particular. a cut metric is nothing but a K_2-metric. The Γ-*metric condition* is (1.2) by taking $\mathcal{M}$ as the set of Γ-metrics. By this result, there is a fractional H-packing by cut metrics and $K_{2,3}$-metrics for a five-vertex commodity graph H. Again Karzanov [15] has strengthened it to:

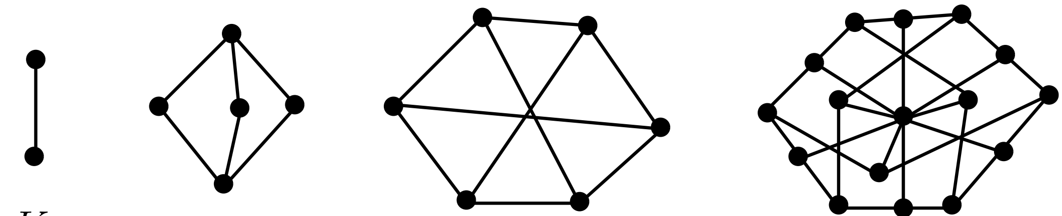
Theorem 1.2 ([15]). *Let G be a graph with cyclically even edge length l, and H a commodity graph. If H has at most five vertices, or is the union of K_3 and a star, then there exists an integral H-packing for (G, l) by cut metrics and $K_{2,3}$-metrics.*

It is natural to ask: what is the class of commodity graphs H with the property that there exists a *finite* set of graphs $\mathcal{G}$ admitting an H-packing for any graph (G, l) by Γ-metrics over $\Gamma \in \mathcal{G}$? It is known that if H has a matching of three edges $K_2 + K_2 + K_2$, there is no such a finite set of graphs $\mathcal{G}$ [15, Section 3]. Therefore, one can expect such fractional or integral H-packings by finite types of metrics only for the class of commodity graphs H without $K_2 + K_2 + K_2$.



(d) (e)

Figure 2: (d) the union of K_3 and a star, and (e) $K_3 + K_3$



K_2 $K_{2,3}$ $K_{3,3}$ $\Gamma_{3,3}$

Figure 3: K_2, $K_{2,3}$, $K_{3,3}$, and $\Gamma_{3,3}$

By direct case-by-case analysis, the commodity graphs H without $K_2 + K_2 + K_2$ are classified into the following:

(1) H has at most five vertices,

(2) H is the union of two stars,

(3) H is the union of K_3 and a star, or

(4) $H = K_3 + K_3$, i.e., the sum of disjoint two triangles.

Theorems 1.1 and 1.2 above solve the first three cases (1-3). For the remaining last case (4), Karzanov [14] has shown that there exists a fractional H-packing by $\Gamma_{3,3}$-metrics. Here $\Gamma_{3,3}$ is the graph of 16 vertices and 27 edges obtained by subdividing each edge of $K_{3,3}$ and connecting each subdivided point to one new point; see Figure 3. In [15, Section 3], Karzanov conjectured that if $H = K_3 + K_3$ and l is cyclically even, there is an integral H-packing for (G, l) by $(1/2)\Gamma_{3,3}$-metrics.

Our main result solves this conjecture affirmatively in a strong form, and also completes the problem of the half or quarter integral H-packing by finite types of metrics.

Theorem 1.3. *Let G be a graph with cyclically even edge length l, and H a commodity graph. If $H = K_3 + K_3$, then there exists an integral H-packing by cut metrics, $K_{2,3}$-metrics, $K_{3,3}$-metrics, and $\Gamma_{3,3}$-metrics.*

Note that cut metrics, $K_{2,3}$-metrics, and $K_{3,3}$-metrics are submetrics of the half of $\Gamma_{3,3}$-metrics. In particular, this achieves an integral H-packing by *integral* metrics. It will turn out that a $K_{3,3}$-metric appears at most once in H-packing (1.1) and its coefficient equals 1. In a sense, a $K_{3,3}$-metric summand is a half-integral *residue* of an integral H-packing by $\Gamma_{3,3}$-metrics.

Our approach to Theorem 1.3 is based on Chepoi's striking proof [5] to Karzanov's half-integral cut and $K_{2,3}$-metric packing results above (Theorems 1.1 and 1.2) using the *tight span* of a metric space, which has been introduced independently by Isbell [11], Dress [8], and Chrobak and Larmore [6]. Since Chepoi's argument relies heavily on the classification result of tight spans of five-point metrics [8], it cannot be applied to six-vertex commodity graph $H = K_3 + K_3$. To overcome this difficulty, we introduce the concept of *H-minimal metrics* that decreases the dimension of tight spans, and develop a certain decomposition theory of two-dimensional tight spans. Our approach is free from the classification result, and gives a geometrical interpretation to the questions why cut, $K_{2,3}$, $K_{3,3}$, and $\Gamma_{3,3}$-metrics arise, and why commodity graph H having $K_2 + K_2 + K_2$ cannot be packed by finite types of metrics.

This paper is organized as follows. In Section 2, we introduce fundamental concepts related to tight spans, and describe how an H-packing problem reduces to a problem of decomposing tight spans. In Section 3, we develop a decomposition theory for two-dimensional tight spans, and prove our main theorem. In Section 4, we give several remarks including a description of an $O(n^2)$ algorithm for an integral $K_3 + K_3$-packing.

Notation. We use the following notation. Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of real and nonnegative real, respectively. Let $\mathbf{Z}$ be the set of integers. The set of functions from a set X to $\mathbf{R}$ is denoted by $\mathbf{R}^X$. For $p, q \in \mathbf{R}^X$, the closed segment between p and q is denoted by $[p, q]$. For $p, q \in \mathbf{R}^X$, $p \leq q$ means $p(i) \leq q(i)$ for each $i \in X$. The *characteristic vector* $\chi_S \in \mathbf{R}^X$ of $S \subseteq X$ is defined as: $\chi_S(i) = 1$ for $i \in S$ and $\chi_S(i) = 0$ for $i \notin S$. We simply denote $\chi_{\{i\}}$ by χ_i, which is the i-th unit vector. For a graph $G = (V, E)$, the edge between $i, j \in V$ is denoted by ij or ji. ii means a loop. For a graph G an subgraph G' of G is called an *isometric subgraph* if $d_G = d_{G'}$ holds on vertices of G'. A *stable set* S of G is a subset of vertices such that there is no edge both of whose endpoints belong to S. For a subset S of vertices in G, the *neighbor* $N(S)$ of S is the set of vertices adjacent to S and not in S. A *partition* of undirected graph G is a partition of vertices such that each part is a stable set. In particular, if there is a bipartition, G is called *bipartite*. G is called a *complete multipartite graph* if G has a partition such that each pair of vertices in different parts has an edge. We often identify a metric space (S, μ) with metric μ. We shall regard a metric as an edge length on the complete graph. A metric is called a *cyclically even* if it is cyclically even as an edge length on the complete graph. We use the standard terminology of polytope theory such as *faces*, *extreme points*, *polyhedral complex*, and so on; see [22].

2 Preliminaries

Main purposes of this section are to introduce fundamental concepts concerning tight spans, and to describe how an H-packing problem reduces to the problem of decomposing tight spans.

Let μ be a metric on a finite set S. We define two polyhedral sets $P(S, \mu)$ and $T(S, \mu)$ as

$$P(S,\mu) \;=\; \{p \in \mathbf{R}^S \mid p(i) + p(j) \geq \mu(i,j) \ (i, j \in V)\}, \tag{2.1}$$

$$T(S,\mu) \;=\; \text{the set of minimal elements of } P(S,\mu). \tag{2.2}$$

$T(S, \mu)$ is called the *tight span* of μ [11, 8, 6]. We immediately see the following characterization of $T(S, \mu)$.

Lemma 2.1. *For $p \in P(S, \mu)$, the following conditions are equivalent:*

(1) *$p \in T(S, \mu)$.*

(2) *for $i \in S$, there is $j \in S$ such that $p(i) + p(j) = \mu(i, j)$.*

(3) *p is contained by a bounded face of $P(S, \mu)$.*

Therefore, $T(S, \mu)$ is the union of bounded faces of $P(S, \mu)$, and thus is compact. For $i \in S$, let μ_i be a vector in $\mathbf{R}^S$ defined by

$$\mu_i(j) = \mu(i, j) \quad (j \in S). \tag{2.3}$$

Namely, μ_i is the i-th column vector of the distance matrix μ.

Lemma 2.2. *μ_i has the following properties:*

(1) *$\{\mu_i\} = T(S, \mu) \cap \{p \in \mathbf{R}^S \mid p(i) = 0\}$ for $i \in S$.*

(2) *$\|\mu_i - \mu_j\|_\infty = \mu(i, j)$ for $i, j \in S$.*

Proof. (1). Take $p \in T(S, \mu)$ with $p(i) = 0$. Then we have $p(j) \geq \mu(i, j)$ for $j \in S$. For $k \in S$, by Lemma 2.1 (2), there is $j \in S$ such that $p(k)+p(j) = \mu(k, j) \leq \mu(k, i)+\mu(i, j) \leq p(k) + p(j)$. Therefore, $p(k) = \mu(k, i)$.

(2). $\mu(i, j) = |\mu_i(i) - \mu_j(i)| \leq \|\mu_i - \mu_j\|_\infty$. Conversely, by the triangle inequality, we have $\mu(i, j) \geq |\mu(i, k) - \mu(j, k)| = |\mu_i(k) - \mu_j(k)|$ for $k \in S$. □

In particular, (S, μ) is isometrically embedded into $(T(X, d), l_\infty)$ by (2). Next we introduce a lattice (a discrete subgroup) in $\mathbf{R}^S$ that behaves nicely with the cyclically evenness. Let L be a lattice in $\mathbf{R}^S$ defined as

$$L = \{p \in \mathbf{R}^S \mid p(i) + p(j) = 0 \mod 2 \ (i, j \in S)\}. \tag{2.4}$$

Namely, L is the set of vectors all of whose components have the same parity. In other words, L is the union of even integer vectors and odd integer vectors.

Lemma 2.3. *If μ is cyclically even, then we have*

$$\mu_i - \mu_j \in L \quad (i, j \in S). \tag{2.5}$$

Proof. By the cyclically evenness, we have

$$\begin{aligned}
(\mu_i - \mu_j)(k) + (\mu_i - \mu_j)(l) &= \mu(i, k) - \mu(j, k) + \mu(i, l) - \mu(j, l) \\
&= \mu(i, k) + \mu(k, j) + \mu(j, l) + \mu(l, j) \mod 2 \\
&= 0 \mod 2.
\end{aligned} \tag{2.6}$$

□

Motivated by this fact, let A_μ be an affine lattice defined by $\mu_i + L$ for $i \in S$. As was suggested in [5], the following *discrete nonexpansive retraction* plays an important role in H-packing problems. Here we give it in a more precise form than that given in [5, Section 2].

Proposition 2.4. *Suppose that μ is cyclically even. For a finite subset U in $P(S, \mu) \cap A_\mu$, there is a map $\phi : U \to T(S, \mu) \cap A_\mu$ such that*

(1) *$\phi(p) = p$ if $p \in U \cap T(S, \mu)$, and*

(2) $\|\phi(p)-\phi(q)\|_\infty \leq \|p-q\|_\infty$ *for* $p,q\in U$.

Proof. In this proof, we simply denote $\|p-q\|_\infty$ by $\|p,q\|$. Note that for $p,q\in A_\mu$ we have $\|p,q\| = p(i)-q(i) \mod 2$ for $i\in S$.

Let $U=\{p_1,p_2,\ldots,p_{n_0},p_{n_0+1},p_{n_0+2},\ldots,p_{n_0+n}\}$ with $\{p_1,p_2,\ldots,p_{n_0}\}\subseteq T(S,\mu)\cap A_\mu$ and $\{p_{n_0+1},p_{n_0+2},\ldots,p_{n_0+n}\}\subseteq P(S,\mu)\cap A_\mu\setminus T(S,\mu)$. If $0\leq n_0\leq 1$ or $n=0$, the existence of such a map ϕ is obvious. So we may assume that $n_0\geq 2$ and $n\geq 1$.

We first define $\phi(p_j)=p_j$ for $1\leq j\leq n_0$. Next we construct $\phi(p_{n_0+i})$ for $1\leq i\leq n$ incrementally. Suppose that we already know $\phi(p_i)$ $(1\leq i\leq n_0+k-1)$ satisfying above (1), (2) for $\{p_1,\ldots,p_{n_0+k-1}\}$, and the condition that $p_i-\phi(p_i)$ is an even vector. Consider the set

$$B_k=\bigcap_{1\leq j<n_0+k}\{p\in\mathbf{R}^S \mid \|\phi(p_j),p\|\leq \|p_j,p_{n_0+k}\|\}. \tag{2.7}$$

Then B_k is nonempty. This follows from the facts that B_k is the intersection of cubes (l_∞-ball), the collection of cubes has Helly property, and $\|p_i,p_{n_0+k}\|+\|p_{n_0+k},p_j\|\geq \|p_i,p_j\|\geq\|\phi(p_i),\phi(p_j)\|$ implies that each pair of those l_∞-balls intersects.

Our goal is to find a point $\phi(p_{n_0+k})$ in $B_k\cap T(S,\mu)\cap A_\mu$ with $p_{n_0+k}-\phi(p_{n_0+k})$ even. B_k is also cube. The maximal element p^* of B_k is given by

$$p^*(l)=\min_{1\leq j<n_0+k}\{\phi(p_j)(l)+\|p_j,p_{n_0+k}\|\}\quad (l\in S). \tag{2.8}$$

Then $p^*\in P(S,\mu)\cap A_\mu$ holds. Indeed, we have

$$\begin{aligned} p^*(l)+p^*(m) &= \phi(p_i)(l)+\|p_i,p_{n_0+k}\|+\phi(p_j)(m)+\|p_j,p_{n_0+k}\| \\ &\geq \mu(l,m)-\phi(p_i)(m)+\phi(p_j)(m)+\|p_i,p_j\| \\ &\geq \mu(l,m). \end{aligned} \tag{2.9}$$

Therefore $p^*\in P(S,\mu)$. To see $p^*\in A_\mu$, we have

$$\begin{aligned} & (\mu_{i'}+p^*)(l)+(\mu_{i'}+p^*)(m) \\ = {} & \mu_{i'}(l)+\phi(p_i)(l)+\|p_i,p_{n_0+k}\|+\mu_{i'}(m)+\phi(p_j)(m)+\|p_j,p_{n_0+k}\| \\ = {} & \phi(p_i)(m)+\|p_i,p_{n_0+k}\|+\phi(p_j)(m)+\|p_j,p_{n_0+k}\| \mod 2 \\ = {} & \phi(p_i)(m)+p_i(m)-p_{n_0+k}(m)+\phi(p_j)(m)+p_j(m)-p_{n_0+k}(m) \mod 2 \\ = {} & 0 \mod 2, \end{aligned} \tag{2.10}$$

where we use the property that $p_i-\phi(p_i)$ is an even vector. Similarly, one can show that all vertices of cube B_j lie on A_μ, and that p^*-p_i is an even vector, The minimal element p_* of B_j is given by

$$p_*(l)=\max_{1\leq j<n_0+k}\{\phi(p_j)(l)-\|p_j,p_{n_0+k}\|\}\quad (l\in S). \tag{2.11}$$

For each $l\in S$, there are $1\leq i,j<n_0+k$ and $m\in S$ such that

$$\begin{aligned} p_*(l)+p_*(m) &= \phi(p_i)(l)+\phi(p_j)(m)-\|p_i,p_{n_0+k}\|-\|p_{n_0+k},p_j\| \\ &= \mu(l,m)-\phi(p_i)(m)+\phi(p_j)(m)-\|p_i,p_{n_0+k}\|-\|p_{n_0+k},p_j\| \\ &\leq \mu(l,m). \end{aligned} \tag{2.12}$$

Thus $p_*\in T(S,\mu)$ or $p_*\notin P(S,\mu)$. From this and the fact that $p(l)+p(m)-\mu(l,m)$ is even, we can construct a desired $\phi(p_{n_0+k})\in T(S,\mu)\cap A_\mu\cap B_k$ by decreasing p^* toward p_* by using steps $\{-2\chi_i\}_{i\in S}$. $\square$

This property reduces an H-packing problem to the problem of decomposing the finite metric $(T(S,\mu)\cap A_\mu, l_\infty)$ [5]. However, to apply this approach to the case $H = K_3+K_3$, we need one more step.

For a graph $H = (S,R)$, a metric μ on S is called an *H-minimal metric* if there is no metric $\mu'(\neq \mu)$ on S such that $\mu' \leq \mu$ and $\mu'(i,j) = \mu(i,j)$ for $ij \in R$.

Lemma 2.5. *For a cyclically even metric μ on S and a graph $H = (S,R)$, there is a cyclically even H-minimal metric of μ^* with $\mu^* \leq \mu$ and $A_\mu = A_{\mu^*}$.*

The proof needs a characterization of H-minimal metrics. For $i \in S$, the set $[i]_\mu$ is defined by $\{i' \in S \mid \mu(i,i') = 0\}$. By triangle inequality, we have $[i]_\mu = [j]_\mu$ or $[i]_\mu \cap [j]_\mu = \emptyset$. Moreover, $\mu(i',j') = \mu(i,j)$ holds for $i' \in [i]_\mu$ and $j' \in [j]_\mu$.

Lemma 2.6. *Let $H = (S,R)$ be a graph and μ a metric on S. Then μ is H-minimal if and only if for each $i,j \in S$ with $\mu(i,j) > 0$,*

(1) *there is $kl \in R$ with $k \in [i]_\mu$ and $l \in [j]_\mu$, or*

(2) *there is $k \in S \setminus [i]_\mu \cup [j]_\mu$ such that $\mu(i,j)+\mu(j,k) = \mu(i,k)$ or $\mu(i,j)+\mu(i,k) = \mu(j,k)$.*

Proof. We first show the only-if part. Suppose that there is $i,j \in S$ with $\mu(i,j) > 0$ not satisfying both (1) and (2). Then, for small $\epsilon > 0$, $\mu' : S \times S \to \mathbf{R}_+$ defined by

$$\mu'(k,l) = \begin{cases} \mu(k,l) - \epsilon & \text{if } \{[k]_\mu, [l]_\mu\} = \{[i]_\mu, [j]_\mu\}, \\ \mu(k,l) & \text{otherwise,} \end{cases} \qquad (k,l \in S) \tag{2.13}$$

is also a metric satisfying $\mu' \leq \mu$ and $\mu'(s,t) = \mu(s,t)$ for $st \in R$. Then μ is not H-minimal.

We show the if part. Suppose that μ is not H-minimal. Then there is a metric $\mu'(\neq \mu)$ with $\mu' \leq \mu$ and $\mu'(s,t) = \mu(s,t)$ for $st \in R$. There are $i,j \in S$ with $\mu'(i,j) < \mu(i,j)$. We take such i,j with $\mu(i,j)$ maximum. Clearly ij does not satisfy (1). If ij satisfies (2) for some k, then we have $\mu(i,k) > \mu(i,j)$ and $\mu'(i,k) < \mu(i,k)$ or $\mu(j,k) > \mu(i,j)$ and $\mu'(j,k) < \mu(j,k)$. Both cases contradict the maximality of $\mu(i,j)$. □

Proof of Lemma 2.5. By the cyclically evenness, we have $\mu(i,j)+\mu(j,k)-\mu(i,k) \in 2\mathbf{Z}$. Therefore, if μ is not H-minimal, then there are $i,j \in S$ violating (1) and (2) in Lemma 2.6. Define $\mu' : V \times V \to \mathbf{R}$ by

$$\mu'(k,l) = \begin{cases} \mu(k,l) - 2 & \text{if } \{[k]_\mu, [l]_\mu\} = \{[i]_\mu, [j]_\mu\}, \\ \mu(k,l) & \text{otherwise,} \end{cases} \qquad (k,l \in S). \tag{2.14}$$

Then μ' is a cyclically metric with $\mu' \leq \mu$ and $\mu'(s,t) = \mu(s,t)$ for $st \in R$. By construction, $A_\mu = A_{\mu'}$ holds. Repeating this process to μ', we obtain a required cyclically even H-minimal metric μ^*. □

The following decomposition theorem is our central subject to prove the main theorem (Theorem 1.3). The proof is given in the next section.

Theorem 2.7. *Suppose that $H = (S,R) = K_3 + K_3$. Let μ be a cyclically even H-minimal metric on S. Then the finite metric space $(T(S,\mu)\cap A_\mu, l_\infty)$ is decomposed into the sum of cut metrics, $K_{2,3}$-metrics, $K_{3,3}$-metrics, and $\Gamma_{3,3}$-metrics with integral coefficients*

Now using this, we can derive our main theorem (Theorem 1.3) as follows. Let $G = (V, E)$ be a connected graph, and let $H = (S, R)$ with $S \subseteq V$ be $K_3 + K_3$. Let l be a cyclically even edge length function on E. Then, clearly, the graph metric $d_{G,l}$ is a cyclically even metric on V. Let μ be the restriction of $d_{G,l}$ to S. By Lemma 2.5, we can take a cyclically even H-minimal metric μ^* with $\mu^* \leq \mu$ and $A_\mu = A_{\mu^*}$. Consider $P(S, \mu^*)$ and $T(S, \mu^*)$. For $k \in V$, we define a vector $p^k \in \mathbf{R}^S$ as:

$$p_k = \mu^*_k \quad (k \in S) \tag{2.15}$$

and

$$p_k(j) = d_{G,l}(k, j) \quad (j \in S, k \in V \setminus S). \tag{2.16}$$

Then we have

$$p_k \in P(S, \mu^*) \cap A_{\mu^*} \quad (k \in V). \tag{2.17}$$

Indeed, we have $p_k = \mu^*_k \in T(S, \mu^*) \cap A_{\mu^*}$ for $k \in S$ and

$$\begin{aligned} p_k(i) + p_k(j) &= d_{G,l}(k, i) + d_{G,l}(k, j) \\ &\geq d_{G,l}(i, j) = \mu(i, j) \geq \mu^*(i, j) \quad (k \in V \setminus S). \end{aligned} \tag{2.18}$$

Therefore $p^k \in P(S, \mu^*)$ for $k \in V \setminus S$. By the cyclically evenness of $d_{G,l}$ and the construction of μ^*, we have $p^k \in A_\mu = A_{\mu^*}$. Then we have

$$\begin{aligned} l(ij) &\geq d_{G,l}(i, j) \geq \|p_i - p_j\|_\infty \quad (ij \in E), \\ d_{G,l}(i, j) &= \mu^*(i, j) = \|p_i - p_j\|_\infty \quad (ij \in R). \end{aligned} \tag{2.19}$$

Let $U = \{p_i\} \subseteq P(S, \mu^*) \cap A_{\mu^*}$. Take a nonexpansive retraction $\phi : U \to T(S, \mu^*) \cap A_{\mu^*}$ in Proposition 2.4. Then we obtain

$$\begin{aligned} l(ij) &\geq \|\phi(p_i) - \phi(p_j)\|_\infty \quad (ij \in E), \\ d_{G,l}(i, j) &= \|\phi(p_i) - \phi(p_j)\|_\infty \quad (ij \in R). \end{aligned} \tag{2.20}$$

Therefore, the decomposition of $(T(S, \mu^*) \cap A_{\mu^*}, l_\infty)$ in Theorem 2.7 yields a required integral H-packing.

3 A decomposition theory for two-dimensional tight spans

The goal of this section is to develop a decomposition theory for two-dimensional tight spans to prove Theorem 2.7. Let (S, μ) be a finite metric space. We further suppose that μ is cyclically even.

The first task is to represent finite metric $(T(S, \mu) \cap A_\mu, l_\infty)$ as the graph metric of a graph obtained by the lattice L. Let $\tilde{\Gamma}_\mu$ be an infinite graph on the vertices $P(S, \mu) \cap A_\mu$ obtained by connecting $p, q \in P(S, \mu) \cap A_\mu$ if $\|p - q\|_\infty = 1$.

Lemma 3.1. *We have*

$$d_{\tilde{\Gamma}_\mu}(p, q) = \|p - q\|_\infty \quad (p, q \in P(S, \mu) \cap A_\mu). \tag{3.1}$$

Proof. ($\geq$) is obvious. We show the converse by constructing a path from p to q with length $\|p - q\|_\infty$. For $p, q \in P(S, \mu) \cap A_\mu$, let U be the set $\{i \in S \mid q(i) < p(i)\}$. Clearly, $p' := p - \chi_U + \chi_{S \setminus U}$ is in $P(S, \mu) \cap A_\mu$. If $p(i) \neq q(i)$ for all $i \in S$, then $\|p - q\|_\infty = 1 + \|p' - q\|_\infty$. If $p(i) = q(i)$ for some $i \in S$, then, by $p - q \in L$, we have $\|p - q\|_\infty \geq 2$, and therefore $\|p - q\|_\infty = 1 + \|p' - q\|_\infty$. Repeating this process to p' and q, we obtain a desired path. $\square$

Let $\mathit{\Gamma}_\mu$ be the subgraphs of $\tilde{\mathit{\Gamma}}_\mu$ induced by $T(S,\mu) \cap A_\mu$. Then $\mathit{\Gamma}_\mu$ is an isometric subgraph of $\tilde{\mathit{\Gamma}}_\mu$. Indeed, for $p, q \in T(S,\mu) \cap A_\mu$, consider the image of a shortest path joining p and q in $\tilde{\mathit{\Gamma}}_\mu$ by a nonexpansive retraction in Proposition 2.4. Then this is a shortest path in $\mathit{\Gamma}_\mu$. In particular, $(T(S,\mu) \cap A_\mu, l_\infty)$ coincides with the graph metric of $\mathit{\Gamma}_\mu$. The decomposability of the graph metric $d_{\mathit{\Gamma}_\mu}$ is our central interest.

It will turn out that two-dimensionality of $T(S,\mu)$ is crucial for $d_{\mathit{\Gamma}_\mu}$ to have a nice decomposability property. To study the dimension of $T(S,\mu)$, we introduce a graph $K(p)$ associated with a point $p \in P(S,\mu)$, which is a fundamental tool to investigate $T(S,\mu)$ [8]. For $p \in P(S,\mu)$, we define the graph $K(p) = (S, E(p))$ as $ij \in E(p) \Leftrightarrow p(i) + p(j) = \mu(i,j)$. Namely, $K(p)$ represents the information of facets of $P(S,\mu)$ containing p. In particular, $p \in T(S,\mu)$ if and only if $K(p)$ has no isolated vertices. Let $F(p)$ be the face of $T(S,\mu)$ containing p as its relative interior. For a face F of $T(S,\mu)$, we denote the corresponding graph by K_F, i.e., $K_F := K(p)$ for a relative interior point $p \in F$. The dimension of $F(p)$ is characterized in a graphical term of $K(p)$.

Lemma 3.2 ([8]). *For $p \in T(S,\mu)$, we have*

$$\dim F(p) = \textit{the number of bipartite components of } K(p). \tag{3.2}$$

Sketch of proof. $\dim F(p)$ is given by the rank of the matrix whose columns are $\{\chi_i + \chi_j \mid ij \in E(p)\}$. The rank of a 0-1 matrix each of whose column has at most two 1's can be characterized in a graphical way as in (3.2). □

It turns out in the proof of the next proposition that the graph $K(p)$ and the commodity graph H are closely related; see Section 4.1 for further discussion. This was a motivation to introduce the concept of H-minimal metrics.

Proposition 3.3. *Let $H = (S,R)$ be a graph and μ an H-minimal metric on S. If H has no n-matching $(n \geq 2)$, then the tight span $T(S,\mu)$ is at most $(n-1)$-dimensional.*

Proof. First we note the following property of a point in the tight span:

(*) For $p \in T(S,\mu)$ and $i, j \in S$, we have $p(i) + \mu(i,j) \geq p(j)$.

Indeed, if $p(j) > p(i) + \mu(i,j)$, then we have $p(j) + p(k) > p(i) + \mu(i,j) + p(k) \geq \mu(i,k) + \mu(i,j) \geq \mu(j,k)$ for any k. This contradicts Lemma 2.1 (2).

Suppose $T(S,\mu)$ is at least n-dimensional. There is a point $p \in T(S,\mu^*)$ such that $K(p)$ has at least n (bipartite) connected components. It suffices to show that each component has at least one edge of H.

Take an edge $ij \in E(p)$ from some component. Then $\mu(i,j) > 0$ must hold. Indeed, suppose $\mu(i,j) = 0$. Then we have $p(i) = p(j) = 0$, and thus $p = \mu_i = \mu_j$. Then $p(i) + p(k) = \mu(i,k)$ and $ik \in E(p)$ holds for $k \in S$. This implies that $K(p)$ is connected. A contradiction.

For $j' \in [j]_\mu$, we have $p(i) + p(j') \leq p(i) + \mu(j,j') + p(j) = \mu(i,j) = \mu(i,j')$ by (*). This implies $ij' \in E(p)$, and consequently $i'j' \in E(p)$ for $i' \in [i]_\mu, j' \in [j]_\mu$. If there is $st \in R$ with $s \in [i]_\mu, t \in [j]_\mu$, then $st \in E(p)$ and we are done.

Suppose not. By Lemma 2.6 (2), there is $k \in S \setminus [i]_\mu \cup [j]_\mu$ such that $\mu(i,j) + \mu(j,k) = \mu(i,k)$ or $\mu(i,j) + \mu(i,l) = \mu(j,l)$. We may assume the former case (by exchanging the role of i, j if necessary). By (*), we have

$$p(i) + p(k) \leq p(i) + \mu(j,k) + p(j) = \mu(i,j) + \mu(j,k) = \mu(i,k). \tag{3.3}$$

Therefore $ik \in E(p)$.

Repeat this process to ik. Since $\mu(i,k) > \mu(i,j)$, after finitely many step we find an edge of H in this components. Therefore, there is at least one edge of H in each component. Thus H has an n-matching. □

In the case where H has no three-matching, $T(S,\mu)$ is at most two dimensional. To concentrate on this case, we assume that $T(S,\mu)$ is at most two-dimensional in the sequel. Our next task is to investigate how the graph Γ_μ is drawn in $T(S,\mu)$. In particular, we will determine the connected components of

$$T(S,\mu) \setminus \bigcup \{[p,q] \mid p,q \in T(S,\mu) \cap A_\mu, \|p-q\|_\infty = 1\}. \tag{3.4}$$

In the subsequent arguments, the following moving process in $T(S,\mu)$ is important.

Lemma 3.4. *Let $p \in T(S,\mu)$ and a maximal stable set U in $K(p)$. For small $\epsilon > 0$ the point $p + \epsilon(-\chi_U + \chi_{S\setminus U})$ is in $T(S,\mu)$. In addition, if $p \in T(S,\mu) \cap A_\mu$, then $p - \chi_U + \chi_{S\setminus U} \in T(S,\mu) \cap A_\mu$.*

Proof. The maximum step $\max\{\epsilon \geq 0 \mid p + \epsilon(-\chi_U + \chi_{S\setminus U}) \in P(S,\mu)\}$ is given by

$$\min_{i,j \in U} (p(i) + p(j) - \mu(i,j))/2. \tag{3.5}$$

Therefore, if U is stable, then (3.5) is positive. Furthermore, by maximality, $K(p + \epsilon(-\chi_U + \chi_{S\setminus U}))$ has no isolated point. This means $p + \epsilon(-\chi_U + \chi_{S\setminus U}) \in T(S,\mu)$. In addition, if $p \in T(S,\mu) \cap A_\mu$, then (3.5) is positive integral by cyclically evenness and the definition of A_μ. $\square$

The first application of this lemma is:

Lemma 3.5. *For $p,q \in T(S,\mu) \cap A_\mu$, if $\|p-q\|_\infty = 1$, then $[p,q] \subseteq T(S,\mu)$.*

Proof. We show that the set $X = \{i \in S \mid p(i) - q(i) = 1\}$ is a nonempty maximal stable set in $K(p)$. If X is empty, then $p < q$, and this contradicts the minimality of q. Thus both X and $S \setminus X$ are nonempty. If there is $i,j \in S$ with $ij \in E(p)$, then $1 = p(i) - q(i) = \mu(i,j) - p(j) - q(i) \leq q(j) - p(j) = -1$. This is a contradiction. Therefore X is stable in $K(p)$. Suppose that X is not a maximal stable set. Then there is $j \in S \setminus X$ such that j is not incident to X. Since $q = p - \chi_S + \chi_{X\setminus S}$, the vertex j is isolated in $K(q)$. This is a contradiction to $q \in T(S,\mu)$. $\square$

The second application reveals the structure of graph $K(p)$.

Lemma 3.6. *For $p \in T(S,\mu)$, graph $K(p)$ has at most two connected components. In addition, if $K(p)$ has two connected components, then $K(p)$ has no loops and both components are complete multipartite.*

Proof. Let U be a maximal stable set of $K(p)$. Then $p' := p + \epsilon(-\chi_U + \chi_{S\setminus U})$ is in $T(S,\mu)$ for small $\epsilon > 0$. If $K(p)$ has at least three components, then $K(p')$ has at three nonbipartite component. This is a contradiction. Suppose that $K(p)$ has two components K^1, K^2. If $K(p)$ has a vertex i with loop ii, then $p(i) = 0$. This implies that $p = \mu_i$ by Lemma 2.2. Then i is adjacent to all vertices. This contradicts the fact that $K(p)$ has two connected components. Suppose that K^1 has two intersecting maximal stable sets U, U'. For small $\epsilon > 0$, $p' := p + \epsilon(-\chi_U + \chi_{N(U)})$ is in $T(S,\mu)$. Then $K(p')$ has three components since $U \cap U'$ is adjacent only to $N(U) \setminus U'$, $U \setminus U'$ is adjacent only to $N(U) \cap U'$, and both U and $N(U)$ are stable in $K(p')$. Therefore, maximal stable sets in K^1 is pairwise disjoint, and this implies that K^1 is complete multipartite. $\square$

Then, from Lemmas 3.2 and 3.6, we see:

(1) $F(p)$ is an extreme point of $T(S,\mu)$ if and only if

(1-1) $K(p)$ is connected nonbipartite or

(1-2) $K(p)$ consists of two nonbipartite complete multipartite components.

(2) $F(p)$ is an edge of $T(S,\mu)$ if and only if

(2-1) $K(p)$ is connected bipartite or

(2-2) $K(p)$ consists of one complete bipartite component and one nonbipartite complete multipartite component.

(3) $F(p)$ is a two-dimensional face of $T(S,\mu)$ if and only if $K(p)$ consists of two complete bipartite components.

In particular, there are two types of edges and extreme points. An edge e of $T(S,\mu)$ is called an l_1-*edge* if $K(p)$ for a relative interior point p in e is a connected bipartite. Other edge e is called an l_∞-*edge*. An extreme point p of $T(S,\mu)$ is called a *core* if $K(p)$ has two nonbipartite components. These concepts have been introduced in [10]. Relationship among l_1-edges, l_∞-edges, cores, and the graph Γ_μ is important for us.

Lemma 3.7. *Let p be an extreme point of $T(S,\mu)$. Then p is integral. In addition, if p is not a core, then $p \in A_\mu$.*

Proof. For $i \in S$, then there is a nonbipartite component contain i. Let C be an odd cycle of this component. We order vertices in C cyclically as $(j_0, j_1, \dots, j_{m-1})$. Then $p(j_0)$ is given by $(\sum_{k=0}^{m-1}(-1)^k\mu(j_k, j_{k+1}))/2$, where the index is taken by modulo m. By cyclically evenness, $p(j_0)$ is integral. There is a path from i to j_0 in $K(p)$. Substituting the relation $p(i') + p(i'') = \mu(i', i'')$ along this path, we obtain $p(i)$ which is integral.

Next we suppose that p is not a core. Fix an odd cycle C in $K(p)$ ordered cyclically as above. For any $i, j \in S$, there are paths connecting from C to i and j, respectively. By calculation, $p(i) + p(j)$ is given by $\sum_{e \in P} \pm\mu(e)$ for some (possibly nonsimple) path joining i and j. Take $k \in S$, then we have

$$(\mu_k - p)(i) + (\mu_k - p)(j) = \mu(k,i) + \mu(k,j) + \sum_{e \in P} \pm\mu(e). \tag{3.6}$$

The right hand side is the sum of μ along some (possibly nonsimple) cycle in the complete graph on S, and thus even. □

For an l_1-edge e, if the corresponding bipartite graph K_e has a bipartition (A,B), then the direction of e is parallel to $\chi_A - \chi_B \in \{1,-1\}^S$. Furthermore, we easily see that neither of the endpoint of e is core. Therefore each l_1-edge e is a series of edges in Γ_μ.

Next we study the shape of a two-dimensional face. We simply call a two-dimensional face *a 2-face*. By calculation, we have the following; see [10] for details.

Lemma 3.8. *Let F be a 2-face of $T(S,\mu)$. Let K^1 and K^2 be two bipartite components of K_F with bipartitions (A_1,B_1) and (A_2,B_2), respectively. For $i \in A_1$ and $j \in A_2$, the projection map $(\cdot)|_{\{i,j\}} : \mathbf{R}^S \to \mathbf{R}^{\{i,j\}}$ is an isometry between (F, l_∞), and $(F|_{\{i,j\}}, l_\infty)$. Moreover $F|_{\{i,j\}}$ is represented as*

$$F|_{\{i,j\}} = \left\{ (p(i), p(j)) \in \mathbf{R}^2 \;\middle|\; \begin{array}{ll} a \le p(i) \le a', & c \le p(i)+p(j) \le c', \\ b \le p(j) \le b', & d \le p(i)-p(j) \le d' \end{array} \right\} \tag{3.7}$$

for $a, a', b, b', c, c', d, d' \in \mathbf{Z}$.

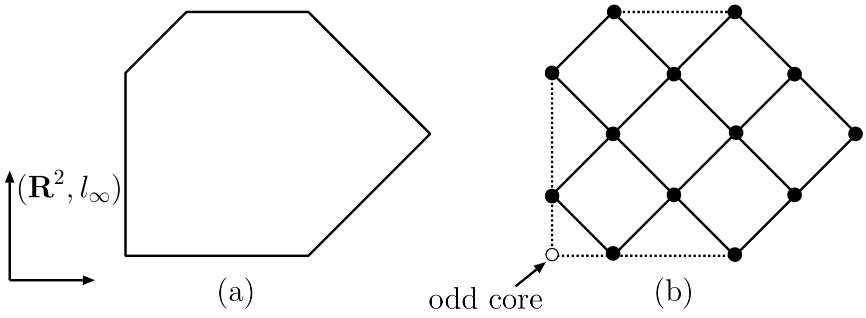


Figure 4: (a) a 2-face and (b) decomposing the 2-face by Γ_μ

Sketch of proof. A point q in F is represented by $p + \alpha(\chi_{A_1} - \chi_{B_1}) + \beta(\chi_{A_2} - \chi_{B_2})$ for some $\alpha, \beta \in \mathbf{R}$ and $p \in F$. From this, we easily see that the projection is an isometry. The coordinate $p(k)$ for $k \in A_1 \cup B_1$ is obtained by substituting relations $p(i') + p(i'') = \mu(i', i'')$ along a path in $K(p)$ connecting i and k. From this, we see the linear inequality description of $F|_{\{i,j\}}$ (3.7). $\square$

Therefore, a 2-face is isomorphic to a polygon in the l_∞-plane $\mathbf{R}^2$ whose edges are parallel to $\chi_1 - \chi_2$ or $\chi_1 + \chi_2$; see Figure 4 (a). Then l_∞-edges in F are parallel to the coordinate axes in the l_∞-plane. As is well-known, the l_∞-plane is isomorphic to the l_1-plane by the map $(x_1, x_2) \mapsto ((x_1+x_2)/2, (x_1-x_2)/2)$; see [7, p. 31]. Then l_1-edges in F are parallel to the coordinate axes in the l_1-plane. In particular, $T(S, \mu)$ is obtained by gluing such polygons along *the same type of edges.*

Next we study the local structure around a core. In general, an edge vector of $T(S, \mu)$ is parallel to $\chi_A - \chi_B$ for some disjoint nonempty subsets $A, B \subseteq S$; consider the orthogonal space of vectors $\{\chi_i + \chi_j \mid ij \in E(p)\}$ having codimension 1. By combining Lemmas 3.4 and 3.2, we see that the edge e adjacent to an extreme point p is given by $[p, p + \alpha(-\chi_A + \chi_{N(A)})]$ for a maximal stable set in some connected component of $K(p)$ and a positive integer α. For a core p, if two complete multipartite components in $K(p)$ have partitions $\{A_1, A_2, \ldots, A_m\}$ and $\{B_1, B_2, \ldots, B_n\}$ $(n, m \geq 3)$, then we say that p *has the type* $(A_1, A_2, \ldots, A_m; B_1, B_2, \ldots, B_n)$. We denote edges adjacent to p parallel to $-\chi_{A_i} + \chi_{\cup_{k \neq i} A_k}$ and $-\chi_{B_j} + \chi_{\cup_{k \neq j} B_k}$ by $e(p, A_i)$ and $e(p, B_j)$, respectively. By the above argument and a routine verification, we have:

Lemma 3.9. *Let p be a core of type $(A_1, A_2, \ldots, A_m; B_1, B_2, \ldots, B_n)$.*

(1) *For an edge e of $T(S, \mu)$, e is adjacent to p if and only if e is $e(p, A_i)$ or $e(p, B_j)$ for some i, j.*

(2) *For a pair of edges e', e'' of $T(S, \mu)$ adjacent to p, both e' and e'' are contained by the common 2-face if and only if (e', e'') is $(e(p, A_i), e(p, B_i))$ or $(e(p, B_i), e(p, A_i))$ for i, j.*

We call a core p *even* if $p \in A_\mu$, and *odd* if $p \notin A_\mu$.

Lemma 3.10. *Let p be an odd core of type $(A_1, A_2, \ldots, A_m; B_1, B_2, \ldots, B_n)$. Then $p - \chi_{A_i} + \chi_{\cup_{k \neq i} A_k}$ and $p - \chi_{B_j} + \chi_{\cup_{k \neq j} B_k}$ are contained in $T(S, \mu) \cap A_\mu$ for i, j.*

Proof. By the argument similar to the proof of Lemma 3.7, $(\mu_l - p)(i) + (\mu_l - p)(j)$ is even for $i, j \in \cup_k A_k$ or $i, j \in \cup_k B_k$. Therefore, $(\mu_l - p)(i) + (\mu_l - p)(j)$ is odd for $i \in \cup_k A_k$ and $j \in \cup_k B_k$. $\square$

Summarizing these arguments, a 2-face in $T(S,\mu)$ is decomposed by $\mathit{\Gamma}_\mu$ as in Figure 4 (b), where the black points are vertices of $\mathit{\Gamma}_\mu$, the white point is an odd core, the broken lines represent l_∞-edges, and other black lines are edges of $\mathit{\Gamma}_\mu$.

Lemma 3.11. *Let e be an l_∞-edge of $T(S,\mu)$. Then there are at least three 2-faces containing e.*

Proof. Let p be a relative interior point in e. Then $K(p)$ consists of one bipartite graph and one nonbipartite graph K. The graph K is complete multipartite with partition $\{A_1, A_2, \ldots, A_m\}$ $(m \geq 3)$. For each A_i, a point $p' := p + \epsilon(-\chi_{A_i} + \chi_{\cup_{k\neq i} A_k})$ for small $\epsilon > 0$ in $T(S,\mu)$, and $K(p')$ has two bipartite components. □

Let us return back to the original problem to determine the closure of connected components of the set

$$T(S,\mu) \setminus \bigcup \{[p,q] \mid p,q \in T(S,\mu) \cap A_\mu, \|p-q\|_\infty = 1\}. \tag{3.8}$$

Recall that the graph $\mathit{\Gamma}_\mu$ decomposes each 2-face of $T(S,\mu)$ as in Figure 4 and $T(S,\mu)$ is obtained by gluing polygons along the same type of edges.

There are unit squares with its edges parallel to $\chi_{A_1\cup A_2} - \chi_{B_1\cup B_2}$ and $\chi_{B_1\cup A_2} - \chi_{A_1\cup B_2}$ for some four-partition $\{A_1, A_2, B_1, B_2\}$ of S. We call it a *square*. Suppose there is an l_∞-edge e having two points of A_μ. Then take two points $p, q \in e$ with $\|p-q\|_\infty = 2$. By Lemma 3.11, there are $m(\geq 3)$ 2-faces of $T(S,\mu)$. Therefore, the closure of component containing (the relative interior of) e is a folder obtained by gluing m right-angled isosceles triangles along their long edge. The graph of boundary edges, which is a subgraph of $\mathit{\Gamma}_\mu$, is the complete bipartite graph $K_{2,m}$. We call it a *$K_{2,m}$-folder.* Suppose that there is an odd core p of type $(A_1, \ldots, A_m, B_1, \ldots, B_n)$ $(m, n \geq 3)$. The closure of the component containing p is the union of triangles whose vertices p, $p - \chi_{A_i} + \chi_{\cup_{k\neq i} A_k}$, and $p - \chi_{B_j} + \chi_{\cup_{k\neq j} B_k}$ over all pairs (i, j) of $1 \leq i \leq m$ and $1 \leq i \leq n$. Its boundary graph, which is an isometric subgraph of $\mathit{\Gamma}_\mu$ induced by $\{p - \chi_{A_i} + \chi_{\cup_{k\neq i} A_k}\}_{i=1}^m \cup \{p - \chi_{B_j} + \chi_{\cup_{k\neq j} B_k}\}_{j=1}^n$, is the complete bipartite graph $K_{n,m}$. We call it a *$K_{n,m}$-folder.* In other words, a $K_{n,m}$-folder is isomorphic to the complex of the join of $K_{n,m}$ and one point. Therefore, the graph $\mathit{\Gamma}_\mu$ decomposes $T(S,\mu)$ into squares, $K_{2,l}$-folders, and $K_{n,m}$-folders. See Figure 5 (a), (b), and (c). Such a folder decomposition of a two-dimensional tight span has already been obtained by [16, 17, 4] via different approach; see Section 4.3 for further discussion. A new point here is a relation among the lattice L, odd cores, and l_1/l_∞-edges.

Remark 3.12. An even core p of type $(A_1, \ldots, A_m, B_1, \ldots, B_n)(m, n \geq 3)$ is contained in n $K_{2,m}$-folders and m $K_{2,n}$-folders. The union of their boundary graphs is $\mathit{\Gamma}_{n,m}$; see Figure 6. Here $\mathit{\Gamma}_{n,m}$ is the graph obtained by subdividing $K_{n,m}$ and connecting each subdivided point to one new point. This will turns out to be a reason why $\mathit{\Gamma}_{3,3}$-metrics appear in the $K_3 + K_3$-packing.

Next we discuss the decomposability property of the graph metric of $\mathit{\Gamma}_\mu$. In fact, this is a special case of the decomposition of a modular graph into its orbit graphs, which is discussed in [3, 18]. We use some of terminology in [18, Section 2] with slight modification.

Two edge e, e' in $\mathit{\Gamma}_\mu$ are called *mates* if there is a rectangle containing e, e' as its parallel edges, or there is a $K_{2,l}$-folder or a $K_{n,m}$-folder containing e, e' as its edges. Two edge e, e' in $\mathit{\Gamma}_\mu$ are said to be *projective* if there is a sequence $e = e_1, e_2, \ldots, e_k = e'$ such that e_i and e_{i+1} are mates. The projectivity is an equivalence relation on edges of $\mathit{\Gamma}_\mu$. An equivalence class is called an *orbit.* For an orbit o, the *orbit graph* $\mathit{\Gamma}_\mu^o$ of $\mathit{\Gamma}_\mu$ is

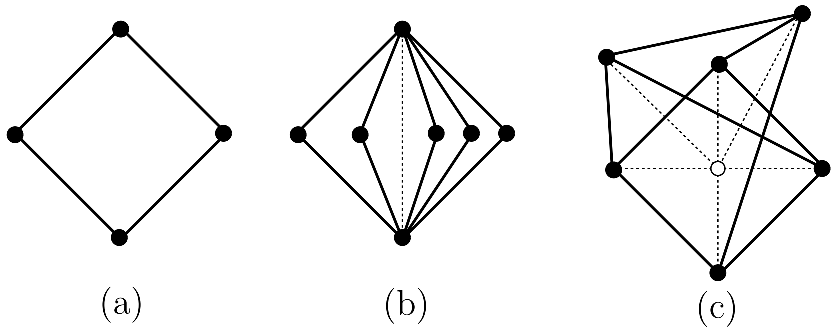


Figure 5: (a) square, (b) $K_{2,5}$-folder, and (c) $K_{3,3}$-folder

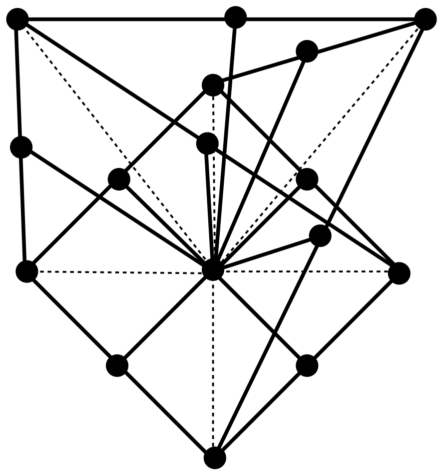


Figure 6: The folder structure around an even core

the graph obtained by contracting all edges not in o and then identifying parallel edges appeared. By the contraction, we define a map ϕ^o from vertices of Γ_μ to vertices of Γ_μ^o in a natural way. Then the graph metric d_{Γ_μ} is decomposed into the graph metrics of the orbit graphs as follows:

Proposition 3.13. *Let O be the set of all orbits of Γ_μ. Then we have:*

$$d_{\Gamma_\mu}(p,q) = \sum_{o\in O} d_{\Gamma_\mu^o}(\phi^o(p),\phi^o(q)) \quad (p,q \in T(S,\mu)\cap A_\mu). \tag{3.9}$$

We will derive Theorem 2.7 from this decomposition principle. In fact, this is a special case of a more general results of modular graphs [2, 18]. Here we give a self-contained proof suitable to our geometric setting. A key is the following Jordan-Hölder type theorem.

Lemma 3.14. *For $p,q \in T(S,\mu)\cap A_\mu$, let $P = (p = p_0, p_1, \ldots, p_k = q)$ and $P' = (p = p'_0, p'_1, \ldots, p'_k = q)$ be two shortest paths joining p and q. For any orbit o, we have*

$$\sum_{\substack{0\le i<k\\ p_{i+1}p_i\in o}} p_{i+1}-p_i = \sum_{\substack{0\le j<k\\ p'_{i+1}p'_i\in o}} p'_{j+1}-p'_j. \tag{3.10}$$

Proof. We may assume $p \neq q$. We use two lemmas. The first is:

(*1) $S_q^p = \{i\in S \mid \|p-q\|_\infty = p(i)-q(i)\}$ is a nonempty stable set in $K(p)$.

Indeed, if $\|p-q\|_\infty = q(i)-p(i)$, then we have $q(i)-p(i) = \mu(j,i)-q(j)-p(i) \le p(j)-q(j)$ for some $j \in S$. Therefore S_q^p is nonempty. Moreover, if there is $i,j \in S_q^p$ with $ij \in E(p)$, then we have $p(i)-q(i) = \mu(i,j)-q(j)-q(i) = q(j)-p(j)$, and this implies $p(i)-q(i) = 0$. A contradiction.

The second is:

(*2) $N_p(S_q^p) = S_p^q$, where $N_p(\cdot)$ is the neighbor operator in $K(p)$.

By the argument above, we have $N_p(S_q^p) \subseteq S_p^q$. Take $i \in S_p^q$, there is $j \in S_q^p$ with $ij \in E(q)$. Then we have $p(i)+p(j) = (p(i)-q(i)) + q(i) + q(j) + (p(j)-q(j)) = q(i)+q(j) = \mu(i,j)$. Therefore, we have $ij \in E(p)$ and $N_p(S_q^p) = S_p^q$.

Now let us start the proof. We use induction on $\|p-q\|_\infty = k$. Let $P = (p = p_0, p_1, \ldots, p_k = q)$ and $P' = (p = p'_0, p'_1, \ldots, p'_k = q)$ be two shortest paths joining p and q. It is obvious for $k = 1$. We may assume that $p_1 \neq p'_1$. Therefore, $\|p_1 - p'_1\|_\infty = 2$. It suffices to show the existence of $p^* \in T(S,\mu)\cap A_\mu$ such that $\|q-p^*\|_\infty = k-2$, $\|p_1-p^*\|_\infty = \|p_2-p^*\|_\infty = 1$, and p,p_1,p'_1,p^* are contained by some square, $K_{2,l}$-folder, or $K_{n,m}$-folder. Indeed, let $P^* = (p^*, p_3^*, \ldots, p_k^* = q)$ be a shortest path joining p^* and q. Then both $P_1^* = (p_1, p^*, p_3^*, \ldots, p_k^* = q)$ and $P_2^* = (p'_1, p^*, p_3^*, \ldots, p_m^* = q)$ are shortest paths. Then apply induction.

Let $p_1 - p = -\chi_A + \chi_B$ and $p'_1 - p = -\chi_{A'} + \chi_{B'}$ for some partitions $\{A,B\}, \{A',B'\}$ of S with all A,B,A',B' nonempty. Since both A and A' are maximal stable sets in $K(p)$, there is no edge between $A\cap A'$ and $A'\cap B$. By these facts and (*1-2), we have $S_q^p \subseteq A\cap A'$ and $S_p^q = N_p(S_q^p) \subseteq N_p(A\cap A') \subseteq B\cap B'$.

(Case 1). Suppose $N_p(A\cap A') = B\cap B'$. In this case, for small $\epsilon > 0$, the vector $p+\epsilon(-\chi_{A\cap A'} + \chi_{N_p(A\cap A')})$ is in $T(S,\mu)$. Therefore both pp_1 and pp_2 are boundary edges of a square or a $K_{2,l}$-folder. In the former case, the point p^* diagonal to p is a desired point. Consider the latter case. The point $\tilde{p} = p - \chi_{A\cap A'} + \chi_{N_p(A\cap A')}$ lies on the l_∞-edges of the $K_{2,l}$-folder. Therefore $K(\tilde{p})$ consists of one complete multipartite component $\tilde{K}$

and one complete bipartite component corresponding to $(A \cap B', A' \cap B)$. Then there is a maximal stable set $\tilde{S}$ in $\tilde{K}$ such that $S_q^p \subseteq S_q^{\tilde{p}} \subseteq \tilde{S} \subseteq A \cap A'$. Thus the point $p^* = \tilde{p} - \chi_{\tilde{S}} + \chi_{N_{\tilde{p}}(\tilde{S})}$ is a desired one.

(Case 2). Suppose $B \cap B' \setminus N_p(A \cap A') \neq \emptyset$. Then any $l \in B \cap B' \setminus N_p(A \cap A')$ must be adjacent to both $A \cap B'$ and $A' \cap B$. For small $\epsilon > 0$, the graph $K(p + \epsilon(-\chi_{A \cap A'} + \chi_{N_p(A \cap A')}))$ consists of one (complete) bipartite component and one nonbipartite (complete multipartite) component. Therefore, the point $p + \epsilon(-\chi_{A \cap A'} + \chi_{N_p(A \cap A')})$ lies on an l_∞-edge. Thus, both pp_1 and pp_2 are boundary edges of a $K_{2,l}$-folder or a $K_{n,m}$-folder. In the former case, the point $p^* = p + 2(-\chi_{A \cap A'} + \chi_{N_p(A \cap A')})$ is a desired one. Suppose the latter case. The point $\tilde{p} := p - \chi_{A \cap A'} + \chi_{N_p(A \cap A')}$ must be an odd core. $K(\tilde{p})$ has two complete multipartite components. Let $\tilde{K}$ be a complete multipartite component containing $A \cap A'$. Then there is a maximal stable set $\tilde{S}$ in $\tilde{K}$ such that $S_q^p \subseteq S_q^{\tilde{p}} \subseteq \tilde{S} \subseteq A \cap A'$. The point $p^* := \tilde{p} - \chi_{\tilde{S}} + \chi_{N_{\tilde{p}}(\tilde{S})}$ is a desired one. $\Box$

The next lemma says that Γ_μ is a modular graph. Here, a graph $G = (V, E)$ is said to be *modular* if for any triple $k_1, k_2, k_3 \in V$ there is k^*, called *a median*, such that $d_G(k_i, k_j) = d_G(k_i, k^*) + d_G(k^*, k_j)$ for $1 \leq i < j \leq 3$.

Lemma 3.15. *Γ_μ is a modular graph.*

Proof. For any triple $p_1, p_2, p_3 \in T(S, \mu) \cap A_\mu$, we define $r_i = (\|p_i - p_j\|_\infty + \|p_i - p_k\|_\infty - \|p_j - p_k\|_\infty)/2$ for $\{i, j, k\} = \{1, 2, 3\}$. By cyclically evenness, all r_1, r_2, r_3 are integer, and $r_i + r_j = \|p_i - p_j\|_\infty$ holds for $1 \leq i < j \leq 3$. Consider the intersection of l_∞-balls

$$B = \bigcap_{1 \leq i \leq 3} \{p \in \mathbf{R}^S \mid \|p_i - p\|_\infty \leq r_i\}. \tag{3.11}$$

By the same argument in the proof of Proposition 2.4, one can show that $B \cap T(S, \mu) \cap A_\mu$ is nonempty. Any point in $B \cap T(S, \mu) \cap A_\mu$ is a median of p_1, p_2, p_3. $\Box$

See Section 4.3 for further discussion about the modularity of Γ_μ. Fix an arbitrary point $p^* \in T(S, \mu) \cap A_\mu$. For an orbit o, we define a map $\phi^o : T(S, \mu) \cap A_\mu \to A_\mu$.

$$\phi^o(p) = \sum_{\substack{0 \leq i < k \\ p_{i+1} p_i \in o}} p_{i+1} - p_i, \tag{3.12}$$

where $(p^* = p_0, p_1, \ldots, p_k = p)$ is a shortest path joining p^* and p. This map is well-defined by Lemma 3.14. By modularity (Lemma 3.15), we have

$$\phi^o(q) - \phi^o(p) = \sum_{\substack{0 \leq i < k \\ q_{i+1} q_i \in o}} q_{i+1} - q_i, \tag{3.13}$$

where $(p = q_0, q_1, \ldots, q_k = q)$ is a shortest path joining p and q. To see this, take a median r of the triple p^*, p, q. Let P_{p^*r}, P_{pr}, and P_{qr} be shortest p^*r, pr, and qr-paths, respectively. Then $P_{p^*r} \cup P_{pr}$, $P_{p^*r} \cup P_{qr}$, and $P_{pr} \cup P_{qr}$ are shortest p^*p, p^*q, and pq-paths, respectively. Substitute the definition of ϕ^o (3.12) into LHS of (3.13) by using shortest paths $P_{p^*r} \cup P_{pr}$ and $P_{p^*r} \cup P_{qr}$. By cancellation, we obtain RHS of (3.13) with respect to the shortest path $P_{pr} \cup P_{qr}$. In particular, for $p, q \in T(S, \mu) \cap A_\mu$ with $\|p - q\|_\infty = 1$, we have

$$\phi^o(p) - \phi^o(q) = \begin{cases} p - q & \text{if } pq \in o, \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

Therefore, the graph of $\phi^o(T(S,\mu) \cap A_\mu)$ obtained by connecting pairs of points having the unit l_∞-distance is isomorphic to the orbit graph Γ_μ^o. Let O be the set of all orbits of Γ_μ. Then we have

$$\|p - q\|_\infty = \sum_{o \in O} \|\phi^o(p) - \phi^o(q)\|_\infty \quad (p, q \in T(S,\mu) \cap A_\mu). \tag{3.15}$$

The image of a shortest path joining p and q by ϕ^o yields a shortest path joining $\phi^o(p)$ and $\phi^o(q)$. Therefore $\|\phi^o(p) - \phi^o(q)\|_\infty$ must equal $d_{\Gamma_\mu^o}(\phi^o(p), \phi^o(q))$. Hence, we complete the proof of Proposition 3.13.

Remark 3.16. We define another map $\psi^o : T(S,\mu) \cap A_\mu \to A_\mu$ by $\psi^o(p) + \phi^o(p) = p$. Corresponding to (3.14), for $p, q \in T(S,\mu) \cap A_\mu$ with $\|p - q\|_\infty = 1$, we have

$$\psi^o(p) - \psi^o(q) = \begin{cases} 0 & \text{if } pq \in o, \\ p - q & \text{otherwise.} \end{cases} \tag{3.16}$$

Then ψ^o has the following geometrical interpretation. For an orbit o, consider the union B_o of $K_{2,l}$-folders and $K_{n,m}$-folders whose boundary edges are in o, and squares at least one of whose parallel pairs of boundary edges is o. Delete the relative interior of B_o from $T(S,\mu)$, and glue the resulting polyhedral set along the boundary of B_0 by translation. This map ψ^o achieves such a gluing translation. This idea will be used by an packing algorithm presented in Section 4.4; also see Figure 10.

Proof of Theorem 2.7. Let us start the proof of Theorem 2.7. Let $H = (S, R)$ be the commodity graph of disjoint two triangles $K_3 + K_3$. We suppose $S = \{1, 2, 3, 1', 2', 3'\}$ and $R = \{12, 23, 31, 1'2', 2'3', 3'1'\}$; see Figure 7 (a). Let μ be a cyclically even H-minimal metric on six-point set S. Our goal is to show that possible orbit graphs of Γ_μ are K_2, $K_{2,3}$, $K_{3,3}$, and isometric subgraphs of $\Gamma_{3,3}$. Then the orbit graph decomposition (Proposition 3.13) yields a desired decomposition.

Suppose that there is no l_∞-edge in $T(S,\mu)$. $T(S,\mu)$ is decomposed into squares. Therefore, each orbit consists of parallel edges, and its orbit graph is K_2. In this case, d_{Γ_μ} is an integral sum of cut metrics, and we obtain an integral H-packing by cut metrics; also see Section 4.1 for further discussion.

We concentrate on the case where $T(S,\mu)$ has l_∞-edges. Recall that there is a $K_{2,l}$-folder or a $K_{m,n}$-folder around an l_∞-edges. We determine possible l, m, n.

Lemma 3.17. *Let e be an l_∞-edge of $T(S,\mu)$ Then the graph K_e is classified into:*

(case 1) *K_e equals H minus one edge, or*

(case 2) *K_e is the disjoint sum of one edge in H and K_4 minus one edge containing one triangle of H.*

Proof. K_e consists of one complete bipartite component and one complete multipartite component K. We show that for any different parts in K there is an edge of H joining them. Take a relative interior point p of e. Then $K_e = K(p)$. Take an edge $ij \in E(p)$ in K. If there is an edge $st \in R$ with $s \in [i]_\mu, t \in [j]_\mu$, then, by the argument in the proof of Lemma 2.6, s and t must belong to parts containing i and j, respectively, and st is a required edge.

Suppose not. By Lemma 2.6 (2), there is $k \in S$ such that $\mu(i,j) + \mu(j,k) = \mu(i,k)$ with $\mu(j,k) > 0$. By the same argument in the proof of Proposition 3.3, we have $ik \in E(p)$. If i, j, k are contained by different parts in K, then $ij, jk, ki \in E(p)$, $p(j) =$

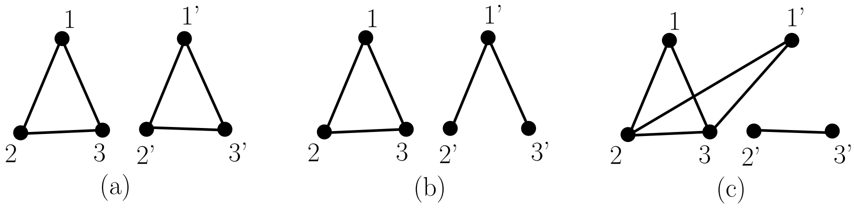


Figure 7: (a) $H = K_3 + K_3$, (b) the case 1, and (c) the case 2

$(\mu(j,k) + \mu(j,i) - \mu(i,k))/2 = 0$, and therefore $p = \mu_j$ (by Lemma 2.2 (2)). Thus j is adjacent to all vertices S. This contradicts the fact that K_e consists of two components. Therefore, j, k are contained by the same parts, i.e., $jk \notin E(p)$. By repeating this process, we can find a required edge of H.

Since the number of vertices of K is three or four, K_e must be (case 1) or (case 2). □

See Figure 7 (b), (c) for examples of the above two cases. By the same argument in the above proof, we have:

Lemma 3.18. *If there exists a core p of $T(S,\mu)$, then $K(p) = H$, and therefore p is a unique core.*

Therefore, the connected component of (3.8) consists of squares, $K_{2,3}$-folders, and one $K_{3,3}$-folder (if an odd core exists).

To investigate the orbit graph, we need to *chase* the orbit started from some edge. For the chase, we use the following lemma that characterizes a *boundary* l_1-edge (corresponding to the case $k = 1$).

Lemma 3.19. *Let e be an l_1-edge of $T(S,\mu)$, Then e is contained by exactly k 2-faces if and only if K_e has exactly $k+2$ maximal stable sets.*

Proof. K_e is connected bipartite. Let (A,B) be the partition of K_e. Both A and B are maximal stable. Suppose there is another maximal stable set C. Take $p \in e$ in the relative interior. Then $p^{C,\epsilon} := p + \epsilon(-\chi_C + \chi_{N(C)})$ is in $T(S,\mu)$ for small $\epsilon > 0$, and $K(p^{C,\epsilon})$ has two bipartite components. Therefore 2-face $F(p')$ contains e. Conversely, if there is 2-face F' containing e, there exists a maximal stable set $C(\neq A,B)$ in $K(p)$ such that $p^{C,\epsilon}$ is in F. For distinct maximal stable sets $C', C''(\neq A,B)$ in $K(p)$, the graphs $K(p^{C',\epsilon})$ and $K(p^{C',\epsilon})$ are distinct, and hence $F(p^{C',\epsilon})$ and $F(p^{C'',\epsilon})$ are distinct. Thus we have done. □

Suppose that there exists an odd core p in $T(S,\mu)$. We show that the orbit graph containing $K_{3,3}$-folder around p is $K_{3,3}$. To see this, we chase the orbit started from a boundary edge of this $K_{3,3}$-folder. The type of p is given by $(\{1\},\{2\},\{3\};\{1'\},\{2'\},\{3'\})$. Let $F_{11'}$ be the 2-face containing edges $e(p,\{1\})$ and $e(p,\{1'\})$, where we use the notation of Lemma 3.9. The graph $K_{F_{11'}}$ is H minus two edges $\{23, 2'3'\}$. Since the graph K_e for an edge e of $F_{11'}$ contains $K_{F_{11'}}$ as a subgraph, it follows from Lemma 3.17 that F has no l_∞-edges except $e(p,\{1\})$ and $e(p,\{1'\})$. Therefore, the orbit started from an edge $[p + \chi_1 - \chi_{23}, p + \chi_{1'} - \chi_{2'3'}]$ hits the edge e of F having direction $\chi_{12'3'} - \chi_{1'23}$, where we simply denote $\chi_{\{1,2',3'\}}$ by $\chi_{12'3'}$. The graph K_e has an edge $11'$. By Lemma 3.19, this edge e is a boundary l_1-edge in $T(S,\mu)$. Hence the orbit started from $[p+\chi_1-\chi_{23}, p+\chi_{1'}-\chi_{2'3'}]$ escapes into the boundary of $T(S,\mu)$; see Figure 8 (a).

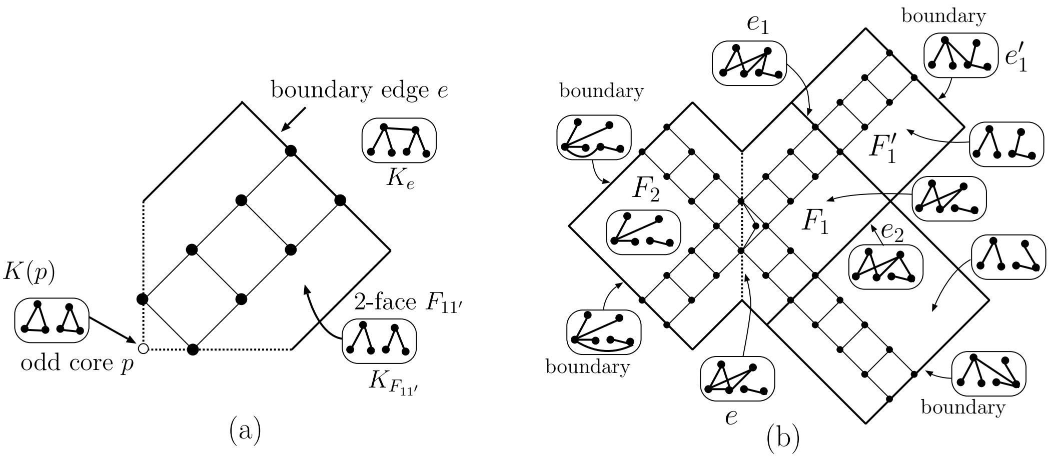


Figure 8: The orbits started from (a) $K_{3,3}$-folder and (b) $K_{2,3}$-folder of (case 2)

The same holds for any $i \in \{1, 2, 3\}$ and $j \in \{1', 2', 3'\}$. Therefore, the orbit started from this $K_{3,3}$-folder does not meet other $K_{2,3}$-folders, and its orbit graph is $K_{3,3}$.

Next we consider the orbit of a $K_{2,3}$-folder containing l_∞-edge e of (case 2) in Lemma 3.17. We may assume that the graph e is $(S, \{12, 23, 31, 1'2, 1'3, 2'3'\})$; see Figure 7 (c). There are three 2-faces F_1, F_2, F_3 containing e. Then, the edges of the graphs K_{F_1}, K_{F_2}, K_{F_3} are given by $\{12, 13, 1'2, 1'3, 2'3'\}$, $\{12, 1'2, 23, 2'3'\}$, and $\{13, 23, 1'2, 2'3'\}$, respectively. For F_1, F_2, F_3, the common edge e is a unique l_∞-edge. In F_2, the orbit started from this $K_{2,3}$-folder hits two edges e_1 and e_2 of F_2. Then K_{e_1} has an edge $22'$ and K_{e_2} has $23'$. Both e_1 and e_2 are in the boundary of $T(S, \mu)$. The same holds for F_3. For F_1, the orbit hits two edges e_1 and e_2 of F_1 such that K_{e_1} has $12'$ or $1'2'$ (or both), and K_{e_2} has $13'$ or $1'3'$ (or both). If K_{e_1} has both $12'$ and $1'2'$, then e_1 is in the boundary of $T(S, \mu)$. Suppose K_{e_1} has only $22'$. Then the edge e_1 is contained by one more 2-face F'_1 with $K_{F'_1} = (S, \{13, 23, 1'2', 2'3'\})$. The orbit hits an edge e'_1 of F'_1. This edge e'_1 is in the boundary of $T(S, \mu)$; see Figure 8 (b). The case where K_{e_1} has only $12'$ does not occur since $K_{F'_1}$ for another 2-face F'_1 containing e_1 has two components one of which has no edge of H which contradicts the proof of Proposition 3.3. For e_2, the argument is the same. Consequently, the orbit started from $K_{2,3}$-folder containing l_∞-edge e of (case 2) does not meet other $K_{2,3}$-folders, and its orbit graph is $K_{2,3}$.

Finally, we consider a $K_{2,3}$-folder $\mathcal{F}_{1'}$ containing l_∞-edge $e_{1'}$ of (case 1). We may assume that $K_{e_{1'}}$ is H minus one edge $2'3'$. Then five vertices of this $K_{2,3}$-folder $\mathcal{F}_{1'}$ are given by $p_{1'}$, $p_{1'} - 2\chi_{1'} + 2\chi_{2'3'}$, $p_{1'} - \chi_{11'} + \chi_{232'3'}$, $p_{1'} - \chi_{21'} + \chi_{132'3'}$, and $p_{1'} - \chi_{31'} + \chi_{122'3'}$ for some $p_{1'} \in e_{1'} \cap A_\mu$.

The edge e_1 is contained by three 2-faces $F_{11'}, F_{21'}, F_{31'}$ whose graphs $K_{F_{11'}}$, $K_{F_{21'}}$, and $K_{F_{21'}}$ are H minus $\{23, 2'3'\}$, H minus $\{13, 2'3'\}$, and H minus $\{13, 1'3'\}$, respectively. Each of 2-faces $F_{11'}, F_{21'}, F_{31'}$ has at most two l_∞-edges. The orbit started from $[p_{1'} - \chi_{11'} + \chi_{231'2'}, p_{1'} - 2\chi_{1'} + 2\chi_{2'3'}]$ through 2-face $F_{11'}$ with direction $-\chi_{11'} + \chi_{232'3'}$ escapes into a boundary edge of $T(S, \mu)$ by the argument same as above. However, the orbit started from $[p_{1'}, p_{1'} - \chi_{231'} + \chi_{12'3'}]$ with direction $-\chi_{12'3'} + \chi_{231'}$ may meet an l_∞-edge. Suppose that this orbit meets an l_1-edge. Then this edge is in the boundary of $T(S, \mu)$ or there is another 2-face F'. For the latter case, the orbit further goes through F' and escapes into the boundary; see Figure 9 (c-2). Suppose this orbit meets an l_∞-edge e_1. Then the graph K_{e_1} is H minus one edge 23, and therefore meets a $K_{2,3}$-folder denoted by $\mathcal{F}_1$. Then five vertices of this $K_{2,3}$-folder $\mathcal{F}_1$ are given by p_1, $p_1 - 2\chi_1 + 2\chi_{23}$, $p_1 - \chi_{11'} + \chi_{232'3'}$, $p_1 - \chi_{12'} + \chi_{231'3'}$, and $p_1 - \chi_{13'} + \chi_{231'2'}$

Similarly, this $K_{2,3}$-folder $\mathcal{F}_1$ meets three 2-faces $F_{11'}$, $F_{12'}$, and $F_{13'}$ whose graphs $K_{F_{11'}}$, $K_{F_{21'}}$, and $K_{F_{31'}}$ are H minus $\{23, 2'3'\}$, H minus $\{13, 2'3'\}$, and H minus $\{12, 2'3'\}$, respectively. Again, the orbits started from $[p_1 - \chi_{11'} + \chi_{232'3'}, p_1 - 2\chi_1 + 2\chi_{23}]$, $[p_1 - \chi_{12'} + \chi_{231'3'}, p_1 - 2\chi_1 + 2\chi_{23}]$, and $[p_1 - \chi_{13'} + \chi_{231'2'}, p_1 - 2\chi_1 + 2\chi_{23}]$ escape into the boundary of $T(S, \mu)$. In the 2-face $F_{12'}$, again the orbit started from $[p_1, p_1 - \chi_{12'}\chi_{231'3'}]$ may meet an l_∞-edge $e_{2'}$ whose graph $K_{e_{2'}}$is H minus one edge $1'3'$, and therefore meets a $K_{2,3}$-folder denoted by $\mathcal{F}_{2'}$. Then five vertices of this $K_{2,3}$-folder $\mathcal{F}_1$ are given by $p_{2'}$, $p_{2'} - 2\chi_{2'} + 2\chi_{1'3'}$, $p_{2'} - \chi_{11'} + \chi_{231'3'}$, $p_{2'} - \chi_{22'} + \chi_{131'3'}$, and $p_{2'} - \chi_{32'} + \chi_{121'3'}$. Similarly, this $K_{2,3}$-folder $\mathcal{F}_1$ meets three 2-faces $F_{12'}$, $F_{22'}$, and $F_{32'}$ whose graphs $K_{F_{12'}}$, $K_{F_{22'}}$, and $K_{F_{32'}}$ are H minus $\{23, 1'3'\}$, H minus $\{13, 1'3'\}$, and H minus $\{12, 1'3'\}$, respectively.

Again the orbit started from edges adjacent to $p_{2'} - 2\chi_{2'} + 2\chi_{1'3'}$ escapes into the boundary. In the 2-face $F_{22'}$, the orbit started from $[p_{2'}, p_{2'} - \chi_{22'} + \chi_{131'3'}]$ may meet an l_∞-edge e_2 whose $K_{e_{2'}}$ is H minus one edge 23, and therefore meets a $K_{2,3}$-folder, which is denoted by $\mathcal{F}_2$. This $K_{2,3}$-folder $\mathcal{F}_2$ meets three 2-faces $F_{21'}$, $F_{22'}$, and $F_{23'}$ whose graphs $K_{F_{21'}}$, $K_{F_{22'}}$, and $K_{F_{23'}}$ are H minus $\{13, 1'2'\}$, H minus $\{13, 2'3'\}$, and H minus $\{13, 1'3'\}$, respectively. Therefore, $\mathcal{F}_{1'}$ and $\mathcal{F}_2$ shares the common 2-face $F_{21'}$. Project four 2-faces $F_{11'}$, $F_{12'}$, $F_{22'}$, $F_{21'}$ by the restriction map $(\cdot)|_{\{1,1'\}} : \mathbf{R}^S \to \mathbf{R}^{\{1,1'\}}$. By Lemma 3.8, this is an injection, and we obtain a tiling by these four 2-faces in the plane. Then edges $e_{1'}$ and $e_{2'}$ have the same 1-th coordinate, and edges e_1 and e_2 have the same $1'$-th coordinate in the plane $\mathbf{R}^{\{1,1'\}}$. Indeed, since $\{p(i) + p(j) = \mu(i, j), 1 \le i < j \le 3\}$ has full rank, the value $p(1)$ is constant in $e_{1'} \cup e_{2'}$. Therefore, in $F_{21'}$, the orbit started from $[p_2, p_2 - \chi_{21'} + \chi_{132'3'}]$ returns back to the first $K_{2,3}$-folder $\mathcal{F}_{1'}$; see Figure 9 (c-1).

Summarizing these arguments, this orbit meets a subset of six $K_{2,3}$-folders $\{\mathcal{F}_j\}_{j\in S}$. Suppose that this orbit meets all six $K_{2,3}$-folders. Translate six $K_{2,3}$-folders so that the points $\{p_j\}_{j\in S}$ coincides. The resulting polyhedral complex is nothing but Figure 6. Thus, the corresponding orbit graph is $\Gamma_{3,3}$. Suppose that some of orbits escapes into the boundary instead of meeting other $K_{2,3}$-folders $\mathcal{F}_j$. The resulting polyhedral complex consists of a proper subset of $K_{2,3}$-folders $\{\mathcal{F}_j\}_{j\in S}$ and squares. A square appears as in the case of Figure 9 (c-2). Namely, the orbit started from $[p_{1'}, p_{1'} - \chi_{11'} + \chi_{232'3'}]$ hits an l_1-edge in $F_{11'}$ with direction $-\chi_{11'} + \chi_{232'3'}$, goes through the adjacent 2-face F', and escapes into the boundary. The orbit started from $[p_{1'}, p_{1'} - \chi_{21'} + \chi_{131'2'}]$ goes thorough 2-faces $F_{21'}$, $F_{22'}$, $F_{1'2}$, and F', crosses the above orbit in F' and escapes into the boundary. The metric space obtained by gluing these three $K_{2,3}$-folders $\mathcal{F}_{1'}, \mathcal{F}_2, \mathcal{F}_{2'}$ and one square is a submetric of the metric space obtained by gluing four $K_{2,3}$-folders $\mathcal{F}_{1'}, \mathcal{F}_2, \mathcal{F}_{2'}, \mathcal{F}_1$.

Consequently, the orbit graph is an isometric subgraph of $\Gamma_{3,3}$. We complete the proof of Theorem 2.7.

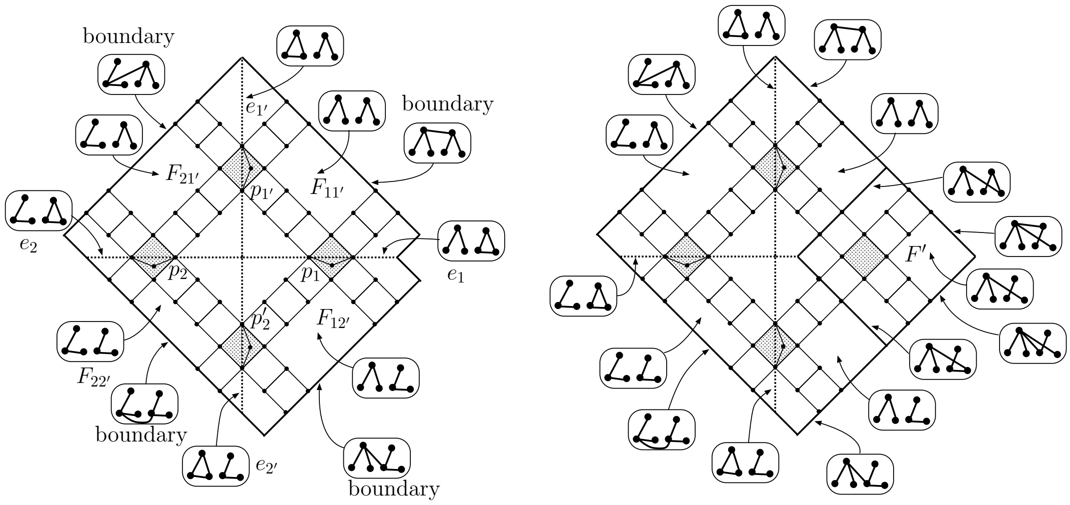
4 Remarks

In this section, we give several remarks.

4.1 H-packing by cut and $K_{2,3}$-metrics

Recall Proposition 3.3 that for a commodity graph H without n-matching, the tight span of an arbitrary H-minimal metric is at most $(n-1)$-dimensional. So it would be valuable to point out a further connection between the commodity graph H and the tight spans of H-minimal metrics.

The graphs K_4, C_5, and the union of two stars are exactly graphs having no $K_2 + K_3$



(c-1) (c-2)

Figure 9: The orbits started from $K_{2,3}$-folder of (case 1)

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and three-matching $K_2 + K_2 + K_2$ [20]; also see [21, Theorem 72.1]. How does this condition reflect the tight span of an H-minimal metric ? The answer is:

Proposition 4.1. *Let $H = (S, R)$ be the graph having no $K_2 + K_3$ and $K_2 + K_2 + K_3$, and let μ be an H-minimal metric on S. Then $T(S, \mu)$ has no l_∞-edges. Consequently, every orbit graph of Γ_μ is K_2.*

Proof. Similar to the proof of Lemma 3.17. □

From this, we obtain Karzanov's half-integral cut packing theorem (Theorem 1.1). This is a tight-span-interpretation of the shape of a commodity graph H admitting cut packing.

The next ask is: what happens for the case that H has at most five vertices or is the union of K_3 and a star ? In this case, the tight span of an H-minimal metric has no core; the proof is similar to Lemma 3.17. Therefore, $K_{n,m}$-folders for $n, m \geq 3$ do not appear. However, l_∞-edge e may exist. Then K_e consists of one bipartite component and one complete multipartite component having three parts; the proof is again similar to Lemma 3.17. Therefore, the tight span is the union of squares and $K_{2,3}$-folders. By chasing the orbit of $K_{2,3}$-folders as in the proof of Theorem 2.7, one can show that the orbit graph is $K_{2,3}$. From this, we obtain Karzanov's half-integral $K_{2,3}$-metric packing theorem (Theorem 1.2).

4.2 The case $\dim T(S, \mu) \geq 3$

In this subsection, we explain that metric spaces $(T(S, \mu) \cap A_\mu, l_\infty)$ arose from three dimensional tight spans cannot be decomposed into finite types of metrics.

Let Γ_L be the graph of L obtained by connecting a pair of points having the unit l_∞-distance. If L is in the plane, then Γ_L is a grid graph, and every submetric of d_{Γ_L} can be decomposed into cut metrics. On the other hand, if the lattice L in three dimensional space, there are infinitely many extreme submetrics in d_{Γ_L}, where a metric is called *extreme* if it is in an extreme ray of the metric cone.

For example, consider the subgraph $\Gamma_{Q_k \cap L}$ of Γ_L induced by the lattice points $Q_k \cap L$ in the affine 3-cube

$$Q_k = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid 0 \leq x_i + x_j \leq 2k \ (1 \leq i < j \leq 3)\} \tag{4.1}$$

for a positive integer k. One can show that $\Gamma_{Q_k \cap L}$ is an isometric subgraph of Γ_L, and the corresponding graph metric $d_{\Gamma_{Q_k \cap L}}$ is extreme. Indeed, the points $Q_1 \cap L$ is given by eight points

$$\{(0,0,0), (-1,1,1), (1,-1,1), (1,1,-1), (2,0,0), (0,2,0), (0,0,2), (1,1,1)\}, \tag{4.2}$$

and therefore the graph $\Gamma_{Q_1 \cap L}$ is the cube plus one diagonal edge; this graph appears in [13, Fig 4 (b)]. It is extreme by Avis' criterion [1]. Since Q_k is obtained by piling Q_1's, by Avis' criterion again, $d_{Q_k \cap L}$ is also extreme. Moreover one can show that for each k there is a metric μ with $\dim T(S, \mu) \geq 3$ such that Γ_μ has $Q_k \cap L$ as an isometric subgraph. Therefore, the graph metric d_{Γ_μ} arose from three dimensional tight spans cannot be decomposed into finite types of metrics. Consequently, the commodity graph H having $K_2 + K_2 + K_2$ cannot be packed by finite types of metrics.

4.3 The folder decomposition, modular closures, and $T(S, \mu) \cap A_\mu$

The folder decomposition of tight spans has already been obtained by Karzanov [17] via the method of modular closures. Here, we explain some relation among the folder decomposition, modular closures, and the points $T(S, \mu) \cap A_\mu$. We need some terminology related to modular metrics. The *least generating graph* (LG-graph) of a metric (S, μ) is the graph on vertices S obtained by connecting a pair $i, j \in S$ if there is no $k \in S \setminus \{i, j\}$ such that $\mu(i, j) = \mu(i, k) + \mu(k, j)$. A metric μ is called *modular* if for any triple $k_1, k_2, k_3 \in S$ there is $k^* \in S$ such that $\mu(k_i, k_j) = \mu(k_i, k^*) + \mu(k^*, k_j)$ for $1 \leq i < j \leq 3$. A *modular closure* $(V, \tilde{\mu})$ of a metric (S, μ) is a certain minimal modular metric containing μ as a submetric. It is constructed by the following process. Initially, set $V := S$ and $\tilde{\mu} := \mu$. Choose a triple $s_0, s_1, s_2 \in V$ without a median, add a new point s^* to V and define the (unique) distances from s^* to the s_k's by

$$\tilde{\mu}(s^*, s_k) = (\tilde{\mu}(s_k, s_i) + \tilde{\mu}(s_k, s_j) - \tilde{\mu}(s_i, s_j))/2 \tag{4.3}$$

for $\{i, j, k\} = \{0, 1, 2\}$. Then define distances from s^* to other points $V \setminus \{s_0, s_1, s_2\} = \{s_3, s_4, \ldots, s_n\}$ by

$$\tilde{\mu}(s^*, s_k) = \max_{0 \leq i < k} \{\tilde{\mu}(s_k, s_i) - \tilde{\mu}(s^*, s_i)\} \quad (3 \leq k \leq n). \tag{4.4}$$

Repeat this procedure for another medianless triple in the current $(V, \tilde{\mu})$ until there is no medianless triple, i.e., $\tilde{\mu}$ is modular. Note that a modular closure $\tilde{\mu}$ depends on the choice of a medianless triple $\{s_0, s_1, s_2\}$ and the order of $V \setminus \{s_0, s_1, s_2\}$. Karzanov [17] has shown that $T(S, \mu)$ is two dimensional if and only if the LG-graph of a modular closure of metric μ is hereditary modular having no $K_{3,3}^-$ as an isometric subgraph, where a graph is called *hereditary modular* if all isometric subgraphs are modular, and $K_{3,3}^-$ is $K_{3,3}$ minus one edge. It is known that a graph is hereditary modular if and only if it is bipartite, and has no isometric k-cycle for $k \geq 6$ [3]. Furthermore, Karzanov [17] has shown that if $\dim T(S, \mu) \leq 2$, then $T(S, \mu)$ is obtained by filling *folders* appropriately into isometric subgraphs $K_{n,m}$ $(n, m \geq 2)$ of the LG-graph of a modular closure of μ as in Figure 5. Interestingly, a modular closure of μ is unique if $\dim T(S, \mu) \leq 2$. However, the uniqueness of a modular closure for a general metric μ is not known [17, p.239 (ii)].

In a sense, our approach to obtain the folder decomposition is opposite to this modular closure approach. In fact, one can show that if $\dim T(S,\mu) \leq 2$, then Γ_μ is hereditary modular without $K_{3,3}^-$. Moreover, a modular closure of a cyclically even metric μ is a submetric of $(T(S,\mu) \cap A_\mu, l_\infty)$. Indeed, consider a vector $p^* \in \mathbf{R}^S$ defined as $p^*(i) = \tilde{\mu}(i, s^*)$ for $i \in S$ by (4.3) and (4.4). Then p^* is in $T(S,\mu) \cap A_\mu$, and $\|p^* - \mu_i\|_\infty = \tilde{\mu}(i, s^*)$ holds for $i \in S$. Consequently, a modular closure $(V, \tilde{\mu})$ is a *tight extension* of (S,μ), i.e, there is no metric $\tilde{\mu}'(\neq \tilde{\mu})$ on V such that $\tilde{\mu}' \leq \tilde{\mu}$ and $\tilde{\mu}' = \mu$ on S; see [8]. Then the restriction map $(\cdot)|_S : T(V, \tilde{\mu}) \to \mathbf{R}^S$ is an isomorphism between $T(V,\tilde{\mu})$ and $T(S,\mu)$ [8, Theorem 3 (vii)], and $(\{\tilde{\mu}_i\}_{i \in V})|_S$ is a subset of $T(S,\mu) \cap A_\mu$. (In particular, the modular closure construction terminates if μ is rational.)

However we still do not know the exact relationship between modular closures and $T(S,\mu) \cap A_\mu$. We leave this issue to a future research topic.

4.4 An $O(n^2)$ algorithm for $K_3 + K_3$-packings

The proof of the main theorem is constructive, and therefore yields a strongly polynomial time for $K_3 + K_3$-packing problems by careful modifications. Here we give an $O(n^2)$ algorithm, where n is the cardinality of vertices of graph $G = (V, E)$. The essential idea is the same as Chepoi's $O(n^2)$ algorithm for cut and $K_{2,3}$-metric packings [5].

Let $G = (V, E)$ be a graph, $H = (S, R) = K_3 + K_3$ a commodity graph on $S \subseteq V$, and l a cyclically even length function. Note that the algorithm presented below works for any commodity graph H without $K_2 + K_2 + K_2$. An algorithm of H-packing of (G, l) by $\Gamma_{3,3}$-metrics is the following:

(s1) Calculate $d_{G,l}(i,j)$ for $(i,j) \in S \times V$. Let μ be the restriction of $d_{G,l}$ to S.

(s2) Take a cyclically even H-minimal metric μ^* on S such that $\mu^* \leq \mu$ and $A_\mu = A_{\mu^*}$.

(s3) Construct $T(S, \mu^*)$.

(s4) Define vectors $U := \{p_k\}_{k \in V} \subseteq P(S, \mu^*)$ by (2.15) and (2.16). Take a nonexpansive retraction $\phi : U \to T(S,\mu^*) \cap A_{\mu^*}$ in Proposition 2.4.

(s5) Decompose finite metric $(\phi(U), l_\infty)$ into $\Gamma_{3,3}$-metrics.

(s1) can be done in $O(n \log n)$ time by Dijkstra algorithm, and both (s2) and (s3) can be done in $O(1)$ time (assuming the size of H fixed). For (s4), the proof of Proposition 2.4 gives an $O(n^2)$ algorithm.

Let us analyze the complexity of (s5). The size of the graph Γ_{μ^*} is not polynomially bounded by $\log(\sum_{i,j} \mu(i,j))$. Therefore, a naive approach to retain Γ_μ does not work. Instead, we chase a kind of *virtual* orbits in $T(S,\mu^*)$ to identify orbit graphs of Γ_μ.

First, we consider the simplest case where $T(S,\mu^*)$ has no l_∞-edges. Take an arbitrary l_1-edge $e = [p, q]$. Take an endpoint p and a point $p^\epsilon := p + \epsilon(q - p)$ for small $\epsilon > 0$. Draw two lines from p and p^ϵ with l_1-direction orthogonal to e until escaping into the boundary of $T(S,\mu)$. Increase ϵ until the line started from p^ϵ meets a point in $\phi(U)$ or $p^\epsilon = q$. Note that such ϵ is integral. Consider the strip sandwiched by two lines. The relative interior of this strip has no points in $\phi(U)$. Delete the relative interior of this strip from $T(S,\mu)$. Then the resulting set consists of two connected components, which yields the bipartition of $\phi(U)$ and a cut metric summand of the H-packing with integral coefficient ϵ.

Next glue this polyhedral set along the boundary of this deleted strip by translating two components together with $\phi(U)$. Such a translation is indeed possible by the

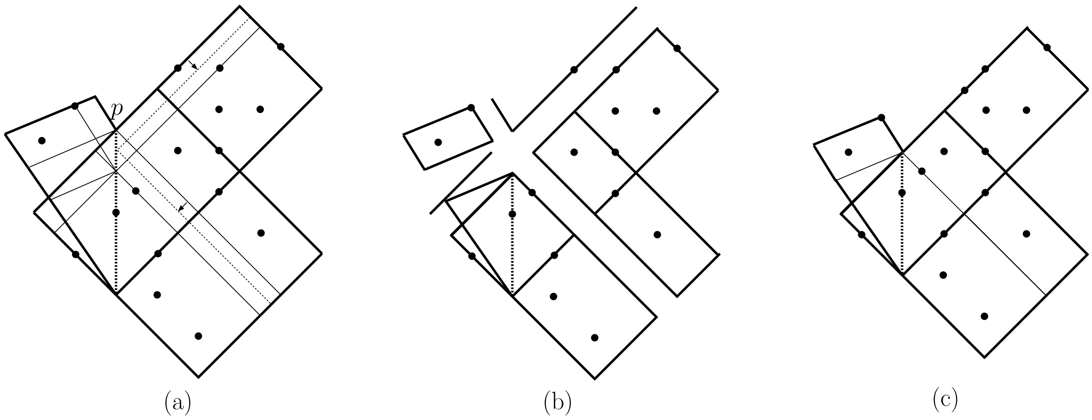

Figure 10: (a) the strip generated by $K_{2,3}$-folder, (b) deleting the strip, and (c) gluing the components

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argument in Remark 3.16. This can be done in $O(n)$ time. Then we obtain a two-dimensional polyhedral set smaller than $T(S, \mu^*)$. Repeat the same process to this set until it becomes one point. The number of the strip-deletion steps will be analysed later.

Second, we consider the case where $T(S, \mu^*)$ has l_∞-edges and has no odd core. The idea is the same as above. Take an l_∞-edge $e = [p, q]$. Take an endpoint p of e and a point $p^\epsilon := p + \epsilon(q - p)$ for small $\epsilon > 0$. Draw lines having l_1-direction started from p and p^ϵ as in Figure 10 (a). Increase ϵ until lines started from p^ϵ meets a point in $\phi(U)$ or $p^\epsilon = q$. Note that such ϵ is an even integer. Then delete the strip sandwiched by these lines (Figure 10 (b)). The resulting components yields a partition of $\phi(U)$, and we obtain a summand of H-packing, which is a $K_{2,3}$-metric or a submetric of a $\mathit{\Gamma}_{3,3}$-metric. Gluing these components along the strip (Figure 10 (b)). Repeat this process until all l_∞-edge vanish. The remaining arguments reduces to the first case.

Finally, we consider the case where $T(S, \mu^*)$ has an odd core p. Note that this odd core is a unique core of $T(S, \mu^*)$ (Lemma 3.18). Consider the $K_{3,3}$-folder containing p, delete the strip generated by this $K_{3,3}$-folder; recall Figure 8 (a). The resulting set consists of six connected components, which gives a unique $K_{3,3}$-metric summand. Gluing these connected components, the remaining argument reduces to the second case.

The number of the strip-deletion steps is bounded by $O(n)$ times. Indeed, consider all lines having a l_1-direction started from $\phi(U)$ and extreme points in $T(S, \mu)$ The number of such lines is $O(n)$. Each strip-deletion step decreases number of such lines. Consequently, we can conclude that (s5) can be done in $O(n^2)$ time, and that a desired integral H-packing can be obtained by $O(n^2)$ time.

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