COUNTING ENRIQUES QUOTIENTS OF A $K3$ SURFACE

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1. Introduction

In his paper [2], Keum used the following criterion for a $K3$ surface to cover an Enriques surface.

Theorem 1.1 ([2], Theorem 1)**.** *Let X be an algebraic $K3$ surface. Assume that $(*)$: $l(T_X) + 2 \leq \rho(X)$. Then the following two conditions are equivalent:*

(1) *X admits a fixed-point-free involution.*
(2) *There exists a primitive embedding of T_X into $\Lambda^- = U \oplus U(2) \oplus E_8(2)$ such that the orthogonal complement of T_X in Λ^- contains no vectors of self-intersection -2.*

Here T_X and $\rho(X)$ are respectively the transcendental lattice and the Picard number of X. U and E_8 denote the unique even unimodular lattices of signature $(1, 1)$ and $(0, 8)$ respectively. $l(L)$ of a nondegenerate lattice L is the number of minimal generators of the discriminant group $A_L = L^*/L$ of L. Details on A_L are found in [6].

As Keum remarks, the assumption $(*)$ is needed only for the implication (2)⇒(1) and is satisfied if $\rho(X) \geq 12$. In this note, first we show that in fact the equivalence above holds without the assumption $(*)$.

Theorem 1.2. *Let X be a $K3$ surface. Then (1) and (2) above are equivalent conditions.*

The problem is reduced to classifying the lattices which occur in the case $\rho(X) = 11$. This part of the result was stated without proofs in [8]. The proof contains an important fact on Enriques quotients.

Proposition 1.3. *Let X be a $K3$ surface having a free involution. Then the embedding of T_X into the $K3$ lattice $\Lambda \simeq U^{\oplus 3} \oplus E_8^{\oplus 2}$ is unique up to isomorphism.*

Combining this step and the Torelli theorem for Enriques surfaces [4, 1], we can count the number of non-isomorphic Enriques quotients of given $K3$ surface in the following form. This formula is more sophisticated than that is used in [8] and better in computations, as will be shown in [3].

Supported by JSPS Research Fellowships for Young Scientists.

Theorem 1.4. *Let X be a $K3$ surface. Then there is a one-to-one correspondence between the following sets:*

1. *$\{$Enriques quotients of $X\}/\{$isomorphisms$\}$.*
2. *$\{$Primitive embeddings of T_X into Λ^- whose orthogonal complement doesnot contain vectors of self-intersection $-2\}/\sim$, where we define the equivalence $\sim$ between two embeddings i_1 and i_2 by the existence of the following commutative diagram*

$$\begin{tikzcd} T_X \arrow[r, "\varphi"] \arrow[d, "i_1"'] & T_X \arrow[d, "i_2"] \\ \Lambda^- \arrow[r, "\tilde{\varphi}"'] & \Lambda^- \end{tikzcd}$$

with $\tilde{\varphi}$ being an isometry and φ preserves the Hodge structure.

At last of this note, we will consider nine-dimensional family of Enriques surfaces with one node, and apply the theorem to the covering $K3$ surfaces.

The author expresses his sincere gratitude to Professor Shigeru Mukai, who suggested the formulation as in Theorem 1.4.

2. Proof of the Theorem 1.2

It suffices to show (2)$\Rightarrow$ (1) under the condition $\rho(X) \leq 11$. If $\rho(X) \leq 9$, then $\operatorname{rank} T_X \geq 13$, so there are no embeddings as in condition (2).

Assume $\rho(X) = 10$. The condition (2) implies $T_X = \Lambda^-$ since the rank coincide. Then it is a well-known fact that all embeddings of Λ^- into a $K3$ lattice $\Lambda := U^{\oplus 3} \oplus E_8^{\oplus 2}$ are isomorphic. It follows that the Neron-Severi lattice S_X of X is isometric to $U(2) \oplus E_8(2)$ and X has a unique free involution as in [8, 7].

Now assume $\rho(X) = \operatorname{rank} T_X = 11$. Let K be the orthogonal complement of T_X in Λ^-. K is of rank 1, negative definite and contains no vectors of self-intersection -2, so K is of the form $K \simeq \langle -2N \rangle$ with $N \geq 2$. Consider the unique embedding $\Lambda^- \subset \Lambda$, whose orthogonal complement is the lattice $\Lambda^+ = U(2) \oplus E_8(2)$. T_X is the orthogonal complement of $\Lambda^+ \oplus K$ in the unimodular lattice Λ. We classify the primitive hull P of $\Lambda^+ \oplus K$ in Λ. By [6], the overlattice P corresponds to an isotropic subgroup $\Gamma \subset A_{\Lambda^+} \oplus A_K$, which is a pushout of the sign-reversing isomorphism of subgroups $\Gamma_{\Lambda^+} \subset A_{\Lambda^+}$ and $\Gamma_K \subset A_K$. In our case A_{Λ^+} is 2-elementary and A_K is cyclic. Thus $\#\Gamma$ is either 1 or 2.

When $\#\Gamma = 1$, $P = \Lambda^+ \oplus K \simeq U(2) \oplus E_8(2) \oplus \langle -2N \rangle$. It follows that the discriminant form of T_X is isometric to $u(2)^{\oplus 5} \oplus c(2N)$, where $u(2)$ is the discriminant form of $U(2)$ and $c(2N)$ that of $\langle 2N \rangle$. Then Nikulin's theorem [6] applies and T_X is isometric to $\langle 2N \rangle \oplus U(2) \oplus E_8(2)$. On the other hand, we can also apply Nikulin's theorem to P. It follows that the lattice P is unique in its genus, and the homomorphism $O(P) \to O(q_P)$

is surjective. Thus the embedding of T_X into Λ is unique. Note that this is nothing but Proposition 1.3 in this case. It follows that the Neron-Severi lattice S_X of X is isometric to P. Since S_X has Λ^+ as a primitive sublattice whose orthogonal complement K has no vectors of self-intersection -2, we see that X admits a free involution [8].

Next we treat the case $\#\Gamma = 2$. The argument is similar to the case above. $A_K \simeq c(-2N)$ contains the unique element z_K of order 2, which is the nontrivial element of Γ_K. Necessarily the value $z_K^2 \in \mathbb{Q}/2\mathbb{Z}$ is in $\mathbb{Z}/2\mathbb{Z}$, for otherwise there doesn't exist an adequate subgroup $\Gamma_{\Lambda^+} \simeq \Gamma_K$. This shows that N is even, $N = 2M$. z_K^2 can take two values 0 and 1, and in each case a direct computation shows $q_P = (q_{\Lambda^+} \oplus q_K|_{\Gamma^\perp})/\Gamma \simeq u(2)^{\oplus 4} \oplus c(-2M)$. As in the case above, we can apply Nikulin's theorem to both P and T_X. We obtain that $P \simeq U \oplus E_8(2) \oplus \langle -4M \rangle$, $T_X \simeq U \oplus E_8(2) \oplus \langle 4M \rangle$ and the primitive embedding of T_X into Λ is unique. Thus $S_X \simeq P$ and the same reasoning as in $\#\Gamma = 1$ case shows that there exists a free involution on X.

This completes the proof of Theorem 1.2. □

We derive some consequences from the proof above. First, we have actually classified all the possible Neron-Severi lattices S_X of X having a free involution when $\rho(X) = 11$. This was stated in [8] without proofs:

Proposition 2.1. *Let X be a $K3$ surface with a free involution.*

(1) *If $\rho(X) = 10$, then $S_X \simeq U(2) \oplus E_8(2)$. The transcendental lattice T_X is isomorphic to $U \oplus U(2) \oplus E_8(2)$.*
(2) *If $\rho(X) = 11$, then S_X is isomorphic to either $U(2) \oplus E_8(2) \oplus \langle -2N \rangle$, $(N \geq 2)$ or $U \oplus E_8(2) \oplus \langle -4M \rangle$, $(M \geq 1)$. The transcendental lattice T_X is isomorphic to $U(2) \oplus E_8(2) \oplus \langle 2N \rangle$, $(N \geq 2)$ or $U \oplus E_8(2) \oplus \langle 4M \rangle$, $(M \geq 1)$ respectively.*

Second, the main body of the proof of Proposition 1.3 is already done.

Proof of Proposition 1.3. If $\rho(X) = 10$ or 11, then this was in the proof of Theorem 1.2. Suppose $\rho(X) \geq 12$. Then for all prime numbers p, one has

$$l_p(S_X) = l_p(T_X) \leq \operatorname{rank} T_X \leq \operatorname{rank} S_X - 2.$$

Thus Nikulin's theorem implies that S_X is unique in its genus and the homomorphism $O(S_X) \to O(q_{S_X})$ is surjective. This is enough. □

3. Proof of the Theorem 1.4

First we prepare notation. Let $\overline{\mathcal{D}_0} = (\mathcal{D}(\Lambda^-) - \cup_t H_t)/O(\Lambda^-)$ be the moduli space of Enriques surfaces, where

$$\mathcal{D}(\lambda^-) = \{\mathbb{C}\omega \in \mathbb{P}_{\text{lines}}(\Lambda^- \otimes \mathbb{C}) | \omega^2 = 0, \omega\overline{\omega} > 0\},$$

and H_t is the hyperplane orthogonal to vectors $t \in \Lambda^-$ of self-intersection -2. We denote a point in $\overline{\mathcal{D}_0}$ by $[\omega]$. The Torelli theorem for Enriques surfaces asserts that $\overline{\mathcal{D}_0}$ is a coarse moduli space of Enriques surfaces.

For a point $[\omega] \in \overline{\mathcal{D}_0}$, we can define an integral Hodge structure of weight 2 on Λ by taking $\mathbb{C}\omega \subset \Lambda^- \otimes \mathbb{C} \subset \Lambda \otimes \mathbb{C}$ as the $H^{2,0}$ component.

To make the argument clear, we make use of another set:

$$(3): \quad \left\{ [\omega] \in \overline{\mathcal{D}_0} \;\middle|\; \begin{array}{l} (\Lambda, \mathbb{C}\omega) \text{ and } H^2(X, \mathbb{Z}) \text{ are} \\ \text{isomorphic as polarized integral Hodge structures.} \end{array} \right\}.$$

We describe the correspondence of the sets between (1) and (3), and (3) and (2).

Given Enriques quotient Y of X, we can associate the period of Y (i.e., the corresponding point as coarse moduli) in $\overline{\mathcal{D}_0}$. Conversely for a point $[\omega]$ as in (3), we get an Enriques surface. The Hodge structure of its covering $K3$ surface is exactly $(\Lambda, \mathbb{C}\omega)$. Thus the condition (3) assures that Y is an Enriques quotient of X. Thus the sets (1) and (3) are bijective.

Next, for a point $[\omega]$ as in (3), the transcendental lattice T_X of X corresponds to a sublattice $T \subset \Lambda$ by the Hodge isometry. This sublattice is contained in Λ^-, thus determines an element of the set (2). Conversely suppose given an embedding $T_X \subset \Lambda^-$ as in (2). ω_X determines a point $[\omega_X]$ in $\overline{\mathcal{D}_0}$ by the condition (2). To check the condition (3), we use Proposition 1.3. Thus two embeddings $T_X \subset \Lambda$ and $T_X \subset H^2(X, \mathbb{Z})$ are isomorphic, and the condition (3) is fulfilled. This concludes Theorem 1.4.

4. An Example

Let $(x_0 : x_1, y_0 : y_1)$ be the homogeneous coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$ and $i : (x_0 : x_1, y_0 : y_1) \mapsto (x_1 : x_0, y_1 : y_0)$ an involution. We consider the following linear system L of divisors of bidegree $(4, 4)$:

$$a_0x_0^2x_1^2y_0^2y_1^2 + a_1(x_0x_1^3y_0^2y_1^2 + x_0^3x_1y_0^2y_1^2) + a_2(x_1^4y_0^2y_1^2 + x_0^4y_0^2y_1^2) + a_3(x_0^3x_1y_0y_1^3 + x_0x_1^3y_0^3y_1)$$
$$+a_4(x_0^2x_1^2y_0y_1^3 + x_0^2x_1^2y_0^3y_1) + a_5(x_0x_1^3y_0y_1^3 + x_0^3x_1y_0^3y_1) + a_6(x_1^4y_0y_1^3 + x_0^4y_0^3y_1)$$
$$+a_7(x_0^2x_1^2y_1^4 + x_0^2x_1^2y_0^4) + a_8(x_0x_1^3y_1^4 + x_0^3x_1y_0^4) + a_9(x_1^4y_1^4 + x_0^4y_0^4).$$

Simply, divisors $D \in L$ is characterized by the following conditions inside $|\mathcal{O}(4, 4)|$:

- the bihomogeneous equation of D is invariant under i.

- D has multiplicities at least 2 at both $(0:1,1:0)$ and $(1:0,0:1)$.

The general member of L has exactly two ordinary nodes at $(0:1,1:0)$ and $(1:0,0:1)$ as singularities, and doesn't contain the four fixed points $(1:\pm 1,1:\pm 1)$ of i. According to the general construction [1], the double covering $\overline{X}$ of $\mathbb{P}^1\times\mathbb{P}^1$ branched along such D is a $K3$ surface with two nodes and one of the liftings of i, denoted by ε, is a free involution of $\overline{X}$. Thus we obtain a family of Enriques surfaces $\overline{Y}=\overline{X}/\varepsilon$ with one node. Let X and Y be the minimal desingularizations of $\overline{X}$ and $\overline{Y}$. From now on, we consider the general member X.

Proposition 4.1. *We have* $\mathrm{Pic}(X)\simeq U\oplus E_8(2)\oplus\langle -4\rangle$.

Proof. First we note that X has a natural quasi-polarization of degree 4 given by the pullback of $\mathcal{O}(1,1)$ to X. Here quasi-polarization means a nef line bundle on X. If X and X' are isomorphic as quasi-polarized varieties, the isomorphism is induced from an element φ of $\mathrm{Aut}(\mathbb{P}^3)$ that preserves the defining quadratic equation of $\mathbb{P}^1\times\mathbb{P}^1\subset\mathbb{P}^3$ and takes D to D'. The former condition is reduced to saying $\varphi\in\mathrm{Aut}(\mathbb{P}^1\times\mathbb{P}^1)$ and the latter means that φ stabilizes L. It can be checked that the stabilizer G of L is in fact $G=\langle i,\sigma\rangle\simeq(\mathbb{Z}/2\mathbb{Z})^2$, where $\sigma:(x_0:x_1,y_0:y_1)\mapsto(y_0:y_1,x_0:x_1)$. This shows that our family has dimension nine, and general X has Picard number 11, i.e., $\mathrm{Pic}(X)$ is one of the lattices in Proposition 2.1. Let M (resp. K) be the invariant (resp. anti-invariant) part of the action of ε on $\mathrm{Pic}(X)$. As is known, $M\simeq U(2)\oplus E_8(2)$. We consider the two (-2)-curves E_1 and E_2 on X arising from two nodes on $\overline{X}$. ε exchanges them, so $E_1+E_2\in M$ and $E_1-E_2\in K$ and we see that $K\simeq\langle -4\rangle$. The condition $E_1\in\mathrm{Pic}(X)$ shows $[\mathrm{Pic}(X):M\oplus K]=2$. This is the case of Proposition 2.1, $S_X\simeq U\oplus E_8(2)\oplus\langle -4\rangle$. □

Corollary 4.2. $T_X\simeq U\oplus E_8(2)\oplus\langle 4\rangle$.

Remark 4.3. Since the rank of T_X is an odd number, it follows that only isometries of T_X that preserve $\mathbb{C}\omega_X$ when tensored with $\mathbb{C}$ is $\{\pm\mathrm{id}\}$. For this, we can apply the finiteness of that group and the result of Nikulin [5].

Proposition 4.4. *The conjugacy class of free involutions on X is unique, i.e., Y is the unique Enriques surface whose covering $K3$ surface is X.*

Proof. We use Theorem 1.4. The orthogonal complement of T_X in Λ^- is $\langle -4\rangle$. This follows from Section 2, or we can prove it directly as follows. If we take the orthogonal complement of the unimodular component U in the inclusion $T_X\subset\Lambda^-$, it becomes $\langle 4\rangle\oplus E_8(2)\subset U(2)\oplus E_8(2)$. Here we used the uniqueness of these complements in their genera. Dividing by 2, it is reduced to $\langle 2\rangle\oplus E_8\subset U\oplus E_8$. Since $U\oplus E_8$ is unimodular, we see that the orthogonal complement of this inclusion is $\langle -2\rangle$. Thus $K\simeq\langle -4\rangle$.

The calculation of determinant shows $[\Lambda^- : K \oplus T_X] = 2$. The patching group $\gamma = \Lambda^-/K \oplus T_X \subset A_K \oplus A_{T_X}$ is, as usual, the pushout of an isomorphism of $\Gamma_K \subset A_K$ and $\Gamma_T \subset A_{T_X}$. Obviously Γ_K is unique. On the other hand, Γ_T is also unique: Γ_T contains the unique element of the form $2g$, where g is any element of order 4 in A_{T_X}. This is because every other order 2 element of A_{T_X} has order 4 element in its orthogonal complement. In this case the discriminant form $q_K \oplus q_{T_X}|_{\Gamma^\perp}/\Gamma$ will not be isomorphic to $u(2)^5 = q_{\Lambda^-}$. Hence the patching Γ is unique. By the definition of equivalence $\sim$ in Theorem 1.4, this shows the uniqueness of Enriques quotients of X. $\square$

Remark 4.5. Concerning other lattices of Proposition 2.1, similar number of non-isomorphic Enriques quotients is computed in [8]. This needs a little more computations on finite quadratic forms.

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