EFFECTIVE IITAKA FIBRATIONS

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Abstract. For every n-dimensional projective manifold X of Kodaira dimension 2 we show that $\Phi_{|MK_X|}$ is birational to an Iitaka fibration for a *computable* positive integer $M = M(b, B_{n-2})$, where $b > 0$ is minimal with $|bK_F| \neq \emptyset$ for a general fibre F of an Iitaka fibration of X, and where B_{n-2} is the Betti number of a smooth model of the canonical $\mathbb{Z}/(b)$-cover of the $(n-2)$-fold F. In particular, M is a universal constant if the dimension $n \leq 4$.

Building up on the work of H. Tsuji, C.D. Hacon and J. McKernan in [HM] and independently S. Takayama in [Ta] have shown the existence of a constant r_n such that $\Phi_{|mK_X|}$ is a birational map for every $m \geq r_n$ and for every complex projective n-fold X of general type.

If the Kodaira dimension $\kappa = \kappa(X) < n$, consider an Iitaka fibration $f : X \to Y$, i.e. a rational map onto a projective manifold Y of dimension κ with a connected general fibre F of Kodaira dimension zero. We define the *index* b of F to be

$$b = \min\{b' > 0 \mid |b'K_F| \neq \emptyset\},$$

and $B_{n-\kappa}$ to be the $(n-\kappa)$-th Betti number of a nonsingular model of the $\mathbb{Z}/(b)$-cover of F, obtained by taking the b-th root out of the unique member in $|bK_F|$, or as we will say, the *middle Betti number* of the *canonical covering* of F.

Question 0.1. *Is there a constant $M := M(n, b, B_{n-\kappa})$ such that $\Phi_{|MK_X|}$ is (birational to) an Iitaka fibration $f : X \to Y$ for all projective n-folds X of Kodaira dimension κ?*

Assume that for all $s \leq n$ there exists an effective constant $a(s)$ such that for every projective s-fold V one has $|a(s)K_V| \neq \emptyset$ and such that the dimension of $|a(s)K_V|$ is at least one if $\kappa(V) > 0$. Then J. Kollár gives in [Ko86, Th 4.6] a formula for the constant M in 0.1 in terms of $a(s)$ and n.

Question 0.1 has been answered in the affirmative by Fujino-Mori [FM] for $\kappa = 1$. In this note we show that the answer is also affirmative for $\kappa = 2$. In the proof we do not

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assume the existence of minimal models in dimension n. However, the existence of good log minimal models of surfaces will be used in an essential way.

Theorem 0.2. *Let X be an n-dimensional projective manifold of Kodaira dimension 2 with Iitaka fibration $f : X \to Y$. Then there exists a computable positive integer M depending only on the index b of a general fibre F of f and on the middle Betti number B_{n-2} of the canonical covering of F, such that $\Phi_{|MK_X|}$ is birational to f.*

For F an elliptic curve one has $b = 1$ and $B_1 = 2$. For surfaces F of Kodaira dimension zero, the index b divides 12 and there is an upper bound 22 for the middle Betti number B_2 of the smooth minimal model of the canonical covering of F. Hence for $n \leq 4$ the constant M in Theorem 0.2 can be chosen to be independent of X, or as we will say, as a universal constant.

When $\dim X = 3$ and $\kappa(X) = 0, 1$, or 3, one finds similar results in [Mo, §10], [FM, Corollary 6.2], [CC, Th 1.1], [HM] and [Ta]. So we can state:

Corollary 0.3. *There is a computable universal constant M_3 such that $\Phi_{|M_3K_X|}$ is an Iitaka fibration for every 3-dimensional projective manifold X.*

We remark that when $\dim X = 3$ and $\kappa(X) = 2$, Kollár [Ko94, (7.7)] has already shown that there exists a universal constant M' such that $H^0(X, mK_X) \neq 0$ for all $m \geq M'$, under the additional assumption that the Iitaka fibration is non-isotrivial. A direct proof of Corollary 0.3, using the existence of good minimal models, will be given at the end of Section 4. [1]

0.4. Conventions.

We adopt the conventions of Hartshorne's book, of [KMM] and and of [KM]. However, if D is a $\mathbb{Q}$-divisor on X we will often write $H^0(X, D)$ or $H^0(X, \mathcal{O}_X(D))$ instead of $H^0(X, \mathcal{O}_X(\lfloor D \rfloor))$, and write $|D|$ instead of $|\lfloor D \rfloor|$. By abuse of notations we will not distinguish line bundles and linear equivalence classes of divisors.

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[1] After a first version of this article was submitted to the arXiv-server, we learned that the Corollary 0.3 has been obtained indepentently by Adam T. Ringler in [Ri], using different arguments.

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1. Some auxiliary results

Set-up 1.1. Let X be a complex n-fold of Kodaira dimension κ. We will consider an Iitaka fibration $f : X \to Y$ of X with Y nonsingular, and F will denote a general fibre of f. Replacing X by some nonsingular blowup, as in [Vi83, §3] or [FM, §2, §4], one may assume that $f : X \to Y$ is a morphism, that the discriminant of f is contained in a normal crossing divisor of Y and that each effective divisor E in X, with $\mathrm{codim}_Y(f(E)) \geq 2$, is exceptional for some morphism $X \to X'$ with X' nonsingular. In particular, for all $i \geq 1$ and for all such divisors E one has

$$H^0(X, \mathcal{O}_X(ibK_X)) = H^0(X, \mathcal{O}_X(ibK_X + E)) = H^0(Y, \mathcal{O}_Y(ibK_Y) \otimes f_*\mathcal{O}_X(ibK_{X/Y})^{\vee\vee}),$$

where b denotes again the index of F and where $f_*\mathcal{O}_X(ibK_{X/Y})^{\vee\vee}$ is the invertible sheaf, obtained as the reflexive hull of $f_*\mathcal{O}_X(ibK_{X/Y})$. By [FM, Corollary 2.5] one can define the semistable part of $f_*\mathcal{O}_X(ibK_{X/Y})$ as a $\mathbb{Q}$-Cartier divisor $iL^{ss}_{X/Y}$, compatible with base change, such that $\mathcal{O}_Y(iL^{ss}_{X/Y}) \subset f_*\mathcal{O}_X(ibK_{X/Y})^{\vee\vee}$ for i sufficiently divisible, and such that both sheaves coincide if $f : X \to Y$ is semistable in codimension one. In particular, $L^{ss}_{X/Y}$ is nef. We will write

$$L_Y = \frac{1}{b} L^{ss}_{X/Y} \quad \text{and} \quad D_Y = \sum_P \frac{s_P}{b} P$$

for the $\mathbb{Q}$-divisors with $\mathcal{O}_Y(ib(L_Y + D_Y)) = f_*\mathcal{O}_X(ibK_{X/Y})^{\vee\vee}$. We remark that D_Y is supported in the discriminant locus of f and $b(L_Y + D_Y)$ is only a $\mathbb{Q}$-divisor; see [FM, Proposition 2.2].

Let $B_{n-\kappa}$ be the middle Betti number of the canonical covering of F, and

$$N = N(B_{n-\kappa}) = \mathrm{lcm}\{m \in \mathbb{Z}_{>0} \,|\, \varphi(m) \leq B_{n-\kappa}\}.$$

By [FM, Theorem 3.1], $NbL_Y = NL^{ss}_{X/Y}$ is an integral Cartier divisor. By [FM, Proposition 2.8], if $s_P \neq 0$, there exist $u_P, v_P \in \mathbb{Z}_{>0}$ with $0 < v_P \leq bN$ such that

$$0 < \frac{s_P}{b} = \frac{bNu_P - v_P}{bNu_P} < 1.$$

So all the non-zero coefficients of D_Y are contained in

$$\mathrm{A}(b, N) := \{\frac{bNu - v}{bNu} \,|\, u, v \in \mathbb{Z}_{>0};\, 0 < v \leq bN\} \setminus \{0\}.$$

Lemma 1.2. *In the Set-up* 1.1, the following holds true.

(1) The set $\mathrm{A}(b,N)$ is a DCC set in the sense of [AM, §2], and one has
$$\frac{1}{Nb} \leq \operatorname{Inf} \mathrm{A}(b,N).$$
(2) (Y, D_Y) is klt.
(3) The $\mathbb{Q}$-divisor $K_Y + D_Y + L_Y$ is big.
(4) For every $m \in \mathbb{Z}_{>0}$, the map $\Phi_{|mbK_X|}$ is birational to the Iitaka fibration f if and only if $|mb(K_Y + D_Y + L_Y)|$ gives rise to a birational map.
(5) NbL_Y is an integral Cartier divisor.
(6) If $s+1 \in \mathbb{Z}_{>0}$ is divisible by Nb, then $(s+1)D_Y \geq \lceil sD_Y \rceil$.

Proof. Part (1) is obvious and (5) was mentioned already in Set-up 1.1. For (2), we remark that D_Y, as part of the discriminant locus, is a simple normal crossing divisor and that $s_P/b \in (0,1)$. The parts (3) and (4) are obvious, since for all $i \geq 1$
$$H^0(X,\, ibK_X) = H^0(Y,\, ib(K_Y + L_Y + D_Y)).$$
Finally (6) is a consequence of the description of the coefficients of D_Y as elements of $\mathrm{A}(b,N)$. In fact, since all $\beta = \frac{bNu-v}{bNu} \in \mathrm{A}(b,N)$ are larger than or equal to $1 - \frac{1}{u}$ and since $(s+1)\beta \in \frac{1}{u} \cdot \mathbb{Z}$, one finds that $(s+1)\beta \geq \lceil s\beta \rceil$. □

Remark 1.3. If the general fibre F of the Iitaka fibration is a good minimal model, and hence if $\omega_F^b \cong \mathcal{O}_F$, choose an ample invertible sheaf $\mathcal{A}$ on X. Writing h for the Hilbert polynomial of $\mathcal{A}|_F$ one obtains a morphism from the complement Y_0 of the discriminant locus to the moduli scheme M_h of polarized minimal models with Hilbert polynomial h. In [Vi06] we constructed a compactification $\overline{M}_h$ of M_h and a nef $\mathbb{Q}$-Cartier divisor λ on $\overline{M}_h$, which on each curve meeting M_h corresponds to the semistable part. Moreover, λ is ample with respect to M_h. In different terms, there is an effective $\mathbb{Q}$-Cartier divisor Γ on $\overline{M}_h$, supported in $\overline{M}_h \setminus M_h$, such that $\alpha\lambda - \Gamma$ is ample for all $\alpha \geq 1$.

If we choose Y such that $Y_0 \to M_h$ extends to a morphism $\varphi : Y \to \overline{M}_h$, one finds that $L_Y = \varphi^*(\lambda)$, and hence that $\alpha L_Y - \varphi^*(\Gamma)$ is semi-ample. For some constant $C > 0$, depending only on h, both CL_Y and $C(L_Y - \varphi^*(\Gamma))$ are divisors, and choosing C large enough $C(\alpha L_Y - \varphi^*(\Gamma))$ will have lots of global sections for all $\alpha \in \mathbb{Z}_{>0}$ (provided that L_Y is not numerically trivial). Perhaps this allows to answer Question 0.1 in the affirmative, assuming the existence of good minimal models. However, the constant M will depend on the Hilbert polynomial h.

Remark 1.4. Gianluca Pacienza [Pa] recently gave an affirmative answer to Question 0.1 for $\kappa = n-2$, or more precisely if the general fibre F of the Iitaka fibration has a good minimal model, assuming that Y is non-uniruled and that the morphism $Y_0 \to M_h$

in Remark 1.3 is generically finite over its image. Note that the last assumption implies that L_Y is big.

2. Log minimal models of surfaces and pseudo-effectivity

From now on we will restrict ourselves to the case $\kappa = 2$.

Remark 2.1. As we will see in proving Theorem 0.2, the constant $M(b, B_{n-2})$ (later written as $M(b, N)$) can be computed using invariants $\beta(\mathrm{A})$ and $\epsilon(\mathrm{A})$ of the DCC set $\mathrm{A} = \mathrm{A}(b, N)$ (see [AM, Th 4.12] and [Ko94, Complement 5.7.4], or [La, Th 5.4]).

Lemma 2.2. *There is a birational morphism $\sigma : Y \to W$ such that the following holds true.*

(1) *$K_W + D_W + L_W$ is ample and $K_Y + D_Y + L_Y = \sigma^*(K_W + D_W + L_W) + E_\sigma$. Here $D_W := \sigma_* D_Y$, $L_W := \sigma_* L_Y$; $E_\sigma \geq 0$ is an effective σ-exceptional divisor.*
(2) *(W, D_W) is* klt.
(3) *Suppose that L_S is big for $S = Y$ or for $S = W$. Then $L_S \sim_{\mathbb{Q}} L'_S$ with $(S, D_S + L'_S)$* klt.

Proof. As in the proof of [FM, Th 5.2], the nefness of L_Y and the bigness of $K_Y + D_Y + L_Y$ allows to find some $a \in \mathbb{Q}_{>0}$ and a klt-pair (Y, Δ_Y) such that

$$K_Y + L_Y + D_Y \sim_{\mathbb{Q}} a(K_Y + \Delta_Y).$$

Construction 2.3. Starting with the klt-pair (Y, Δ_Y), the log minimal model program (LMMP) for surfaces provides us with a sequence $\gamma : Y \to Z$ of contractions of $(K_Y + \Delta_Y)$-negative extremal rays. If $K_Y + \Delta_Y$ is big or more generally pseudo-effective, γ will be birational. Writing $\Delta_Z = \gamma_* \Delta_Y$, the $\mathbb{Q}$-divisor $E := K_Y + \Delta_Y - \gamma^*(K_Z + \Delta_Z)$ is effective and γ-exceptional, the $\mathbb{Q}$-divisor $K_Z + \Delta_Z$ is nef and (Z, Δ_Z) is klt.

By the abundance theorem for klt log surfaces (see [Ko+], for example), there is a morphism with connected fibres $\psi : Z \to W$ such that $K_Z + \Delta_Z$ is the pullback of an ample $\mathbb{Q}$-divisor H on W.

If $K_Y + \Delta_Y$ is big, as we assumed in Lemma 2.2, $\sigma = \psi \circ \gamma : Y \to W$ is birational and the ample $\mathbb{Q}$-divisor on W is $H = K_W + \Delta_W$. So $K_Z + \Delta_Z = \psi^*(K_W + \Delta_W)$ and hence the divisor E in Construction 2.3 is $K_Y + \Delta_Y - \sigma^*(K_W + \Delta_W)$. In particular, (W, Δ_W) is again klt. We find

$$K_Y + L_Y + D_Y \sim_{\mathbb{Q}} a\sigma^*(K_W + \Delta_W) + aE =$$
$$\sigma^*\sigma_*(K_Y + L_Y + D_Y) + aE = \sigma^*(K_W + L_W + D_W) + aE.$$

(1) is true by Lemma 1.2 (3). Part (2) follows from (1) and Lemma 1.2 (2).

For part (3), we refere to [KM, Proposition 2.61, Corollary 3.5]. □

The answer to Question 0.1 is quite easy when $K_Y + D_Y$ is big, and especially when $L_Y \equiv 0$.

Lemma 2.4. *Suppose that $K_Y + D_Y$ is big. Then there is a constant $M = M(b, N)$ such that for all $s \geq M(b, N)$ with $s+1$ divisible by Nb, the linear system $|(s+1)(K_Y+D_Y+L_Y)|$ defines a birational map. In particular, $\Phi_{|MK_X|}$ is an Iitaka fibration for some $M = M(b, N)$ depending only on the set $\mathrm{A}(b, N)$, and hence only on b and $N = N(B_{n-2})$.*

Proof. Applying Construction 2.3 to (Y, D_Y), we get a birational morphism $\eta : Y \to Z$ such that $K_Z + D_Z = \eta_*(K_Y + D_Y)$ is ample and $E_\eta := K_Y + D_Y - \eta^*(K_Z + D_Z)$ is an η-exceptional effective $\mathbb{Q}$-divisor.

Note that the coefficients of $D_Z = \eta_* D_Y$ still belong to the same DCC set $\mathrm{A}(b, N)$. The Remark (3) on page 60 of [La] allows to apply [La, Th 3.2]. As in [La, Th 5.3] one finds a constant $M(b, N)$, depending only on the set $\mathrm{A}(b, N)$, such that the linear system

$$|K_Y + \lceil s\eta^*(K_Z + D_Z)\rceil|$$

gives rise to a birational map for every $s \geq M(b, N)$. The same [La, Th 3.2] applies to

$$|K_Y + \lceil s(K_Y + D_Y) + (s+1)L_Y\rceil|,$$

since $(s+1)L_Y$ is pseudo-effective and hence the boundary divisor of the above adjoint linear system has nef part larger than $s\eta^*(K_Z + D_Z)$.

Assume further that Nb divides $s + 1$. Then by Lemma 1.2

$$K_Y + \lceil sK_Y + sD_Y + (s+1)L_Y\rceil \leq (s+1)(K_Y + D_Y + L_Y).$$

This implies the first part of Lemma 2.4. Now the second part follows from the first part using Lemma 1.2 (4). □

Lemma 2.5. *Suppose that L_Y is big and $K_Y + D_Y + eL_Y$ is pseudo-effective for some $e \in [0, 1)$. Then there is a constant $M = M(b, N, e)$ such that $\Phi_{|MK_X|}$ is an Iitaka fibration.*

Proof. Consider the Zariski decompositions

$$K_Y + D_Y + eL_Y = P_e + N_e \quad \text{and} \quad K_Y + D_Y + L_Y = P_Y + N_Y.$$

Then $P_Y \geq (1-e)L_Y + P_e$. Since NbL_Y is an integral divisor

$$P_Y^2 \geq (1-e)^2 L_Y^2 \geq \frac{(1-e)^2}{(Nb)^2}.$$

For a very general curve C_t, we have

$$P_Y.C_t \geq (1-e)L_Y.C_t \geq \frac{1-e}{Nb}.$$

Assume that $s(1-e) > 4Nb$. Applying [La, Th 3.2] one finds that the adjoint linear system $|K_Y + \lceil s(K_Y + D_Y + L_Y) + L_Y \rceil|$ (whose boundary divisor has the nef part larger than sP_Y) gives rise to a birational map. Assume further that Nb divides $(s+1)$. The lemma follows from the observation that the latter system is included in the following (see Lemma 1.2):

$$|(s+1)(K_Y + D_Y + L_Y)|.$$

□

The most difficult part of the proof of Theorem 0.2 is the one where $K_Y + D_Y$ is not pseudo-effective, and hence where L_Y is not numerically trivial.

Proposition 2.6. *Suppose that $K_Y + D_Y$ is not pseudo-effective. Then in addition to the birational morphism $\sigma : Y \to W$ constructed in Lemma 2.2 there are a birational morphism $\tau : W \to V$, some $e \in \mathbb{Q} \cap (0,1)$ and effective divisors $E_{\tau\sigma}$, E_τ, E_σ and E_{L_Y} satisfying:*

a) *E_τ is τ-exceptional, E_σ and E_{L_Y} are σ-exceptional, and*

$$E_{\tau\sigma} = E_\sigma + (1-e)E_{L_Y} + \sigma^* E_\tau,$$

b) *Writing $D_W := \sigma_* D_Y, D_V := \tau_* D_W$ etc. one has*

$$\begin{aligned} K_Y + D_Y + eL_Y &= \sigma^*(K_W + D_W + eL_W) + E_\sigma + (1-e)E_{L_Y} \\ &= \sigma^*\tau^*(K_V + D_V + eL_V) + E_{\tau\sigma}. \end{aligned}$$

c) *$\sigma^* L_W = L_Y + E_{L_Y}$, and L_W is nef.*

d) *$e = \min\{e' \mid K_S + D_S + e' L_S$ is pseudo-effective$\}$. Here S can be chosen to be equal to Y, W, or V and the resulting e is independent of this choice.*

e) *(V, D_V) and hence V are* klt*.*

f) *One of the following holds true:*

(1) *$K_V + D_V + eL_V \equiv 0$, the Picard number $\rho(V) = 1$, and V is a* klt *del Pezzo (rational) surface. In particular, $-K_V$ is an ample $\mathbb{Q}$-divisor and V has at most quotient singularities.*

(2) *V is the total space of a $\mathbb{P}^1$-fibration over a curve with general fibre Γ, the Picard number $\rho(V) = 2$, and $K_V + D_V + eL_V \equiv \beta\Gamma$ for some $\beta \in \mathbb{Q}_{>0}$.*

Proof. We start with the morphism $\sigma : Y \to W$ from Lemma 2.2. For $L_W := \sigma_* L_Y$ one has $\sigma^* L_W = L_Y + E_{L_Y}$ where E_{L_Y} is supported in the exceptional locus of σ. Since L_Y is nef, L_W is also nef, and E_{L_Y} is effective. By Lemma 2.2 one finds for all e'

$$K_Y + D_Y + e'L_Y = \sigma^*(K_W + D_W + e'L_W) + E_\sigma + (1 - e')E_{L_Y}. \tag{2.1}$$

So the assertion c) and the first equation in the assertion b) hold true.

Starting from $W_0 = W$ we will construct for some $r \geq 0$ and for $i = 0, \dots, r-1$ a chain of birational morphisms $\tau_i : W_i \to W_{i+1}$, such that W_r satisfies the conditions stated in Proposition 2.6, f) (1) or (2). We will show inductively that the following conditions (c1) - (c5) hold for $i = 1, \dots, r$ and that (c6)-(c8) hold for $i = 1, \dots, r-1$.

(c1) (W_i, D_i) is klt.
(c2) $K_i + D_i$ is not pseudo-effective.
(c3) $K_i + D_i + L_i$ is ample.
(c4) $e_i = \min\{e' \in (0,1) \mid K_i + D_i + e'L_i \text{ is nef }\}$ exists and is rational.
(c5) $1 > e_0 \geq e_1 \geq \dots \geq e_r > 0$.
(c6) $\rho(W_{i+1}) = \rho(W_i) - 1$.
(c7) L_i is nef, and $\tau_i^* L_{i+1} = L_i + E_{L_i}$ for an effective τ_i-exceptional divisor E_{L_i}.
(c8) $K_i + D_i + e_i L_i = \tau_i^*(K_{i+1} + D_{i+1} + e_i L_{i+1})$.

Here $K_i = K_{W_i}$, and D_i or L_i denotes the pushdowns of D_Y or L_Y to W_i. We write $\rho(W_i)$ for the Picard number of W_i.

Claim 2.7. (c3) and (c7) are true for all $i \geq 0$, and (c1) and (c2) hold for $i = 0$.

Proof. Note that τ_i is birational. (c3) and (c7) are true for $i = 0$ and hence they are true for all $i \geq 0$ on surfaces; see Lemma 2.2 and the proof for the assertion c) above. (c1) is also part of Lemma 2.2. For (c2) set $e' = 0$ in the equation (2.1) and use the non-pseudo-effectiveness of $K_Y + D_Y$. □

Claim 2.8.

(i) The conditions (c2) and (c3) for some i imply (c4) with $e_i \in (0,1)$.
(ii) In particular, (c4) and (c5) hold for $i = 0$.

Proof. Knowing (c2) for some i the condition (c3) allows to deduce from [KMM, Th 4-1-1] or [KM, Th 3.5] that there exists a rational number

$$d_i = \max\{d \mid (K_i + D_i + L_i) + d(K_i + D_i) \text{ is nef }\}.$$

Since $K_i + D_i + L_i$ is ample, $d_i > 0$. Then $e_i = 1/(1 + d_i)$. □

Assume now we have found the birational morphisms τ_i for $i < i_0$, that (c1)-(c5) hold for $i = 0, \dots, i_0$ and that (c6)-(c8) hold for $i = 0, \dots, i_0 - 1$.

By [KM, Complement 3.6], the condition (c2) implies the existence of a $K_{i_0} + D_{i_0}$-negative extremal ray R_{i_0}, perpendicular to $K_{i_0} + D_{i_0} + e_{i_0} L_{i_0}$. We choose $\tau_{i_0} : W_{i_0} \to W_{i_0+1}$ to be the contraction of R_{i_0} (i.e., of all the curves proportional to R_{i_0}). In particular, one finds

$$\tau_{i_0}^* \tau_{i_0 *}(K_{i_0} + D_{i_0} + e_{i_0} L_{i_0}) = K_{i_0} + D_{i_0} + e_{i_0} L_{i_0}. \tag{2.2}$$

Suppose that τ_{i_0} is birational. Then for $i = i_0$ the condition (c6) holds. (c8) follows from the equation (2.2).

Knowing (c1)-(c8) for $i = i_0$ it is easy to verify (c1)-(c5) for $i = i_0 + 1$. We remark that (c7) and (c8) for i_0 imply that

$$K_{i_0} + D_{i_0} = \tau_{i_0}^*(K_{i_0+1} + D_{i_0+1}) + e_{i_0} E_{L_{i_0}},$$

so (c1) and (c2) for i_0+1 follow from the corresponding statements for i_0, and hence (c4) for $i_0 + 1$ follows from Claim 2.8.

By the choice of e_{i_0}

$$K_{i_0} + D_{i_0} + e_{i_0} L_{i_0} = \tau_{i_0}^*(K_{i_0+1} + D_{i_0+1} + e_{i_0} L_{i_0+1})$$

is nef. This is possible only if $K_{i_0+1} + D_{i_0+1} + e_{i_0} L_{i_0+1}$ is nef, and hence only if $e_{i_0} \geq e_{i_0+1}$, as claimed in (c5).

If τ_{i_0} is birational, we can continue this process. This way, one obtains birational morphisms $\tau_j : W_j \to W_{j+1}$ $(0 \leq j \leq r)$ satisfying the conditions (c1) - (c8). The condition (c6) implies that $r < \rho(W)$.

If τ_{i_0} is non-birational we set $V = W_{i_0}$ and $e = e_{i_0}$ in Proposition 2.6. The assertions a) and the second half of b) follow from (c5), (c7) and (c8), whereas e) is the same as (c1). It remains to verify d) and f).

Case (1). If the image of τ_{i_0} is a point, we claim that in Proposition 2.6 f) we are in Case (1) there. By the construction, $\rho(V) = 1$.

Recall that the singularities of a klt surface are just quotient singularities. Since L_Y and hence $L_V = \tau_* \sigma_* L_Y$ can not be numerically trivial, it must be a positive multiple of the generator of the Neron-Severi group of V. So the definition of e implies that $K_V + D_V + eL_V \equiv 0$. By [GZ, Lemma 1.3] a klt surface with $-K$ ample is rational.

Case (2). We claim that the second case in Proposition 2.6 f) occurs if τ_{i_0} has a curve W_{i_0+1} as its image. Let Γ denote a general fibre of τ_{i_0}.

For $V = W_{i_0}$ one finds $\rho(V) = 1 + \rho(W_{i_0+1}) = 2$. Our Γ generates the extremal ray R_{i_0} giving rise to the contraction τ_{i_0}. So every fibre of $V \to W_{i_0+1}$ is irreducible (also because

$\rho(V) = 2$). Since the nef divisor $K_V + D_V + eL_V$ is perpendicular to R_{i_0} and hence to the nef divisor Γ, one finds that $K_V + D_V + eL_V \equiv \beta\Gamma$ for some $\beta > 0$.

Since $K_V + D_V + L_V \equiv (1-e)L_V + \beta\Gamma$ is ample, we have $\Gamma.L_V > 0$. Now $0 = \Gamma.\beta\Gamma = \Gamma.(K_V + D_V + eL_V) > \Gamma.K_V$ and hence $\Gamma \cong \mathbb{P}^1$.

We still have to characterize e as the pseudo-effective threshold as claimed in the assertion d) of Proposition 2.6.

Clearly, when $S = V$, our $K_S + D_S + eL_S \equiv \beta\Gamma$ (setting $\beta = 0$ and Γ to be any ample divisor, in Case (1)) is pseudo-effective, so by the assertion b) of Proposition 2.6 the same is true when $S = Y$ or $S = W$.

Conversely, suppose that $K_S + D_S + e'L_S$ is pseudo-effective for some e' and some $S \in \{Y, W, V\}$. Then the same holds for $S = V$ by considering the pushdown.

For $S = V$ we can write this divisor as $\beta\Gamma + (e'-e)L_V$. Thus $0 \le \Gamma.(\beta\Gamma + (e'-e)L_V) = (e'-e)\Gamma.L_V$. Since $K_V + D_V + L_V \equiv \beta\Gamma + (1-e)L_V$ is ample, we have $\Gamma.L_V > 0$ in both Cases (1) and (2), and hence $e' \ge e$. □

The next two Lemmata give a universal upper bound for the threshold e in Proposition 2.6.

Lemma 2.9. *In the situation considered in Proposition* 2.6 f), Case (1), there is a constant $e(b, N) < 1$, depending only on b and N, such that the threshold $e \le e(b, N)$.

Proof. Let $\pi : \widetilde{V} \to V$ be a minimal resolution. So one has a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & W \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\tau} \\ \widetilde{V} & \xrightarrow{\pi} & V. \end{array}$$

As usual, when there is a birational morphism $Y \to S$ we will write D_S and L_S for the direct images of D_Y and L_Y, respectively.

Write $\pi^* K_V = K_{\widetilde{V}} + J$ with J an effective and π-exceptional $\mathbb{Q}$-divisor. Note that (V, D_V) and hence V and $(\widetilde{V}, J)$ are klt. Since $H_{Nb} := NbL_{\widetilde{V}}$ is a nef line bundle with $H_{Nb} - (K_{\widetilde{V}} + J)$ nef and big, [Ka, Th 3.1] tells us that $|2H_{Nb}|$ is base point free.

So $L_{\widetilde{V}}$ is $\mathbb{Q}$-linearly equivalent to $L'_{\widetilde{V}} := H_{2Nb}/2Nb$ for a smooth divisor $H_{2Nb} \in |2H_{Nb}|$ intersecting $D_{\widetilde{V}}$ transversely and away from the fundamental point of the inverse of the birational morphism $\xi : Y \to \widetilde{V}$.

For $L'_Y := \xi^* L'_{\widetilde{V}}$, the pair $(Y, D_Y + L'_Y)$ is klt. Write $\xi^* L_{\widetilde{V}} = L_Y + E$ with $E \ge 0$ ξ-exceptional. Then $K_Y + D_Y + L'_Y \sim_{\mathbb{Q}} (K_Y + D_Y + L_Y) + E$ is big.

So we are allowed to apply the first part of Lemma 2.4 and we find a constant $M(b, N)$ such that $|(t_0 + 1)(K_Y + D_Y + L'_Y)|$ gives rise to a birational map for all $t_0 \in \mathbb{Z}_{>0}$ with

$2Nb|(t_0+1)$ and $t_0 \geq M(b,N)$. Thus the same holds for $|(t_0+1)(K_V+D_V+L'_V)|$ with $(L_V \sim_{\mathbb{Q}})\ L'_V$ the pushdown of L'_Y.

So $(t_0+1)(K_V+D_V+L_V).\bar{\Gamma} \geq 1$ for any movable curve $\bar{\Gamma}$ on V. On $\widetilde{V}$, we take $\Gamma \cong \mathbb{P}^1$ with $\Gamma^2 = 0$ or 1 (when $\widetilde{V}$ is ruled or $\mathbb{P}^2$) such that $\bar{\Gamma} = \pi(\Gamma)$. Note that $\bar{\Gamma}.K_V = \Gamma.(K_{\widetilde{V}}+J) \geq \Gamma.K_{\widetilde{V}} \geq -3$.

If $e \leq 1/2$ there is nothing to show. Otherwise

$$0 = \bar{\Gamma}.(K_V+D_V+eL_V) \geq -3 + e\bar{\Gamma}.L_V \geq -3 + \frac{1}{2}\bar{\Gamma}.L_V,$$

Then

$$6(1-e) \geq (1-e)\bar{\Gamma}.L_V = \bar{\Gamma}.(K_V+D_V+L_V) \geq \frac{1}{t_0+1}$$

gives an upper bound for e. □

Lemma 2.10. *In Case* (2) *of Proposition* 2.6 f)*, there is a constant $\nu = \nu(N,b)$ (depending only on N,b) such that the threshold e satisfies*

$$e \leq 1 - \frac{1}{4\nu} < 1.$$

Proof. Again it is sufficient to consider the case $e \geq 1/2$. We calculate

$$0 = \Gamma.\beta\Gamma = \Gamma.(K_V+D_V+eL_V) \geq -2 + \frac{1}{2}\Gamma.L_V = -2 + \frac{1}{2}\widetilde{\Gamma}.L_Y.$$

Here the fibre $\widetilde{\Gamma}$ is the pullback on Y of the general fibre Γ on V in Proposition 2.6 f), Case (2). Since $K_Y+D_Y+L_Y$ is big and $N\widetilde{\Gamma}.(K_Y+L_Y) \in \mathbb{Z}_{>0}$, we apply [FM, Prop 6.3], obtain $\nu = \nu(N,b)$ satisfying the following and hence conclude the lemma (noting that $E_{\tau\sigma}$ is contained in fibres):

$$\nu \leq \widetilde{\Gamma}.(K_Y+D_Y+L_Y) = (1-e)\widetilde{\Gamma}.L_Y \leq 4(1-e).$$

□

3. The proof of Theorem 0.2 and Corollary 0.3

When K_Y+D is big, especially when $L_Y \equiv 0$, the statement of Theorem 0.2 has been verified in Lemma 2.4. If L_Y is big the theorem follows from Lemmata 2.5, 2.9 and 2.10. So for Theorem 0.2 it remains to consider the case:

Assumption 3.1. L_Y is not numerically trivial, $\kappa(L_Y) \leq 1$, and $\kappa(K_Y+D_Y) \leq 1$.

In particular, the first condition implies that the nef dimension $n(L_Y)$ is one or two.

The morphism $\sigma : Y \to W$, constructed in Lemma 2.2, factors through a minimal resolution $\widetilde{W}$ of W. So $\sigma = \pi \circ \eta$ for

$$Y \xrightarrow{\ \eta\ } \widetilde{W} \xrightarrow{\ \pi\ } W.$$

Write $\pi^* K_W = K_{\widetilde{W}} + J$ with J an effective π-exceptional $\mathbb{Q}$-divisor. As in the last section $L_{\widetilde{W}},\ L_W,\ D_{\widetilde{W}}$ and D_W denote the direct images of L_Y and D_Y.

Lemma 3.2.

(1) $0 \leq L_Y^2 \leq L_{\widetilde{W}}^2 \leq L_W^2$.

(2) *Suppose that* $L_{\widetilde{W}}^2 = 0$. *Then* $L_Y^2 = 0$, $L_Y = \eta^* L_{\widetilde{W}}$ *and* $L_{\widetilde{W}}.K_{\widetilde{W}} = L_Y.K_Y$.

(3) *If* $n(L_Y) = 2$, *then* $L_Y.K_Y \geq 0$.

(4) *Let e be the threshold from Proposition* 2.6 and let P_Y be the positive and N_Y the negative part in the Zariski decomposition

$$K_Y + D_Y + L_Y = P_Y + N_Y.$$

Then $P_Y - (1-e)\sigma^* L_W$ is pseudo-effective. Furthermore,

$$P_Y^2 \geq (1-e)^2 L_W^2 \geq (1-e)^2 L_{\widetilde{W}}^2. \tag{3.1}$$

Proof. Recall that L_Y is nef. Hence the direct images $L_{\widetilde{W}}$ and L_W have the same property. Then $L_{\widetilde{W}} \leq \pi^* L_W$ and $L_Y \leq \eta^* L_{\widetilde{W}}$ which imply (1) and (2). Part (3) is shown in [Am, Th 0.3].

Using the notations from Lemma 2.2, $P_Y = \sigma^*(K_W + D_W + L_W)$ and $N_Y = E_\sigma$. Moreover $K_Y + D_Y + eL_Y$ is pseudo-effective by the choice of e. Then its σ-pushdown $K_W + D_W + eL_W$ is pseudo-effective as well and one obtains the first part of (4). Since P_Y and L_W are nef, the pseudo-effectivity of $P_Y - (1-e)\sigma^* L_W$ implies (3.1). □

Lemma 3.3. *Assume that L_Y is not numerically trivial, that $\kappa(L_Y) \leq 1$ and that either $L_{\widetilde{W}}^2 > 0$ or $L_Y.K_Y \geq 0$. Then $|MK_X|$ is an Iitaka fibration for some $M = M(b, N, e)$ depending only on b, N, e.*

Proof. Keeping the notations from Lemma 3.2 (4), one has:

Claim 3.4.
$$P_Y^2 \geq \frac{(1-e)^2}{3(Nb)^2}.$$

Proof. Assume first that $L_{\widetilde{W}}^2 > 0$. Since $NbL_{\widetilde{W}}$ is an integral Cartier divisor, $L_{\widetilde{W}}^2 \geq 1/(Nb)^2$ and the claim follows from (3.1) in Lemma 3.2 (4).

Assume next that $L_{\widetilde{W}}^2 = 0$. By Lemma 3.2 (2) this implies that $L_Y = \eta^* L_{\widetilde{W}}$ and that $L_{\widetilde{W}}.K_{\widetilde{W}} = L_Y.K_Y$. By the assumption, this intersection number is non-negative. Thus

$$L_W.K_W = L_{\widetilde{W}}.(K_{\widetilde{W}} + J) \geq L_{\widetilde{W}}.K_{\widetilde{W}} \geq 0.$$

If $L_{\widetilde{W}}.K_{\widetilde{W}}$ is positive, by Lemma 1.2 (5) it has to be larger than or equal to $1/Nb$. Applying Lemma 3.2 (4) one finds

$$P_Y^2 \geq P_Y.(1-e)\eta^* L_W = (1-e)(K_W + D_W + L_W).L_W$$
$$\geq (1-e)L_W.K_W \geq (1-e)L_{\widetilde{W}}.K_{\widetilde{W}} \geq \frac{1-e}{Nb} \geq \frac{(1-e)^2}{3(Nb)^2}.$$

If $L_{\widetilde{W}}.K_{\widetilde{W}} = 0$, consider first the case $L_{\widetilde{W}}.\pi' D_W > 0$, where π' stands for the proper transform. By Lemma 1.2 this intersection number is $\geq 1/(Nb)^2$. As above one obtains

$$P_Y^2 \geq (1-e)L_W.D_W = (1-e)L_{\widetilde{W}}.\pi^* D_W \geq (1-e)L_{\widetilde{W}}.\pi' D_W \geq \frac{1-e}{(Nb)^2} \geq \frac{(1-e)^2}{3(Nb)^2}.$$

It remains the case $L_{\widetilde{W}}^2 = L_{\widetilde{W}}.K_{\widetilde{W}} = L_{\widetilde{W}}.\pi' D_W = 0$. Since $K_Y + D_Y + L_Y$ and its η-pushdown $K_{\widetilde{W}} + D_{\widetilde{W}} + L_{\widetilde{W}}$ are big, $0 < L_{\widetilde{W}}.(K_{\widetilde{W}} + D_{\widetilde{W}} + L_{\widetilde{W}}) = L_{\widetilde{W}}.D_{\widetilde{W}}$. Thus $L_{\widetilde{W}}.D_1 > 0$ for some irreducible curve D_1 in $\mathrm{Supp} D_{\widetilde{W}}$ and in the exceptional locus of π. The second condition implies that $D_1 \cong \mathbb{P}^1$ and that $D_1^2 \leq -2$.

If $D_1^2 = -n$ with $n \geq 3$, then [Z, Lemma 1.7] implies that

$$J \geq \frac{n-2}{n} D_1 \geq \frac{1}{3} D_1, \quad \text{and}$$
$$P_Y^2 \geq (1-e)L_W.K_W = (1-e)L_{\widetilde{W}}.(K_{\widetilde{W}} + J) \geq (1-e)L_{\widetilde{W}}.\frac{1}{3}D_1 \geq \frac{1-e}{3Nb} \geq \frac{(1-e)^2}{3(Nb)^2}.$$

If $D_1^2 = -2$ one can factorize $\pi : \widetilde{W} \to W$ as $\pi_1 : \widetilde{W} \to W_1$ and $W_1 \to W$, where π_1 is the contraction of D_1. Then $\pi_1^* L_{W_1} = L_{\widetilde{W}} + aD_1$ with $a = L_{\widetilde{W}}.D_1/2 \geq 1/2Nb$. Note that $0 = L_{\widetilde{W}}^2 = (\pi_1^* L_{W_1} - aD_1)^2 = L_{W_1}^2 - 2a^2$, so $L_W^2 \geq L_{W_1}^2 = 2a^2 \geq 1/2(Nb)^2$ and

$$P_Y^2 \geq (1-e)L_W^2 \geq \frac{1-e}{2(Nb)^2} \geq \frac{(1-e)^2}{3(Nb)^2}.$$

□

In order to prove Lemma 3.3 consider two general points x_1, x_2 of Y.

If the nef dimension $n(L_Y) = 1$, then by [8aut, Proposition 2.11] L_Y is numerically equivalent to a positive multiple of the general fibre Γ of a morphism from Y to a curve. In this case we assume that x_1, x_2 are not in the same fibre of this fibration.

Thus for a very general curve C_t on Y containing x_1, x_2, one has

$$P_Y.C_t \geq (1-e)L_Y.C_t \geq \frac{1-e}{Nb}.$$

Then the adjoint linear system

$$|K_Y + \lceil s_0(K_Y + D_Y + L_Y) + L_Y \rceil|, \quad \text{for} \quad s_0 = b(\lceil \frac{5Nb}{1-e} \rceil + 1) - 1$$

separates the points x_1, x_2. In fact, the nef part of the divisor

$$\lceil s_0(K_Y + D_Y + L_Y) + L_Y \rceil$$

is larger than s_0P_Y and the inequalities

$$s_0P_Y.C_t \geq s_0(1-e)L_Y.C_t \geq \frac{s_0(1-e)}{Nb} \geq 4 \quad \text{and} \quad (s_0P_Y)^2 \geq s_0^2\frac{(1-e)^2}{3(Nb)^2} > 8$$

allow to apply [La, Th 3.2]. Thus, by Lemma 1.2,

$$h^0(X,(s_0+1)K_X) = h^0(Y,(s_0+1)(K_Y+D_Y+L_Y)) \geq$$
$$h^0(X,K_Y+\lceil s_0(K_Y+D_Y+L_Y)+L_Y\rceil) \geq 2.$$

Now by [Ko86, Th 4.6], $\Phi_{|tK_X|}$ is an Iitaka fibration for $t=(s_0+1)(2M+1)+M$, where M is a constant as in [FM, Corollary 6.2], depending only on $A(b,N)$. □

Recall that Assumption 3.1 implies that $n(L_Y)$ is one or two. In the second case, Lemma 3.2 (3) allows to apply Lemma 3.3. So it remains to consider the case below:

Lemma 3.5. *Assume that $n(L_Y)=1$, $L_{\widetilde{W}}^2=0$ and $L_Y.K_Y<0$. Then Y is a ruled surface over a curve C of genus $q(Y)$ with general fibre $\Sigma\cong\mathbb{P}^1$. The $\mathbb{Q}$-divisor L_Y is $\mathbb{Q}$-linearly equivalent to a positive multiple of Σ, and $|MK_X|$ is an Iitaka fibration for some constant $M=M(b,N,q(Y))$ depending only on $b,N,q(Y)$.*

Proof. By Lemma 3.2, one has $L_Y^2=0$ and $L_Y=\eta^*L_{\widetilde{W}}$. By [8aut, Proposition 2.11] L_Y is numerically equivalent to a positive multiple of the general fibre Σ of the nef reduction $\tau: Y\to C$. Since $L_Y.K_Y<0$, one has $\Sigma.K_Y<0$. Thus $2g(\Sigma)-2=\Sigma.K_Y<0$ and $\Sigma\cong\mathbb{P}^1$. So $\tau:Y\to C$ is a $\mathbb{P}^1$-fibration and Y is a ruled surface with $g(C)=q(Y)$.

So the divisor NbL_Y on the ruled surface $Y\to C$ is numerically equivalent to some $\alpha\Sigma$. Considering the intersection of Σ with a section of $\tau:Y\to C$ one sees that α is an integer. The numerically trivial sheaf $NbL_Y-\alpha\Sigma$ is linearly equivalent to the pullback of a numerically trivial sheaf on a relative minimal model of Y which in turn must be the pullback of a sheaf on C. Hence $NbL_Y\sim\tau^*\Pi$ for some integral divisor Π on C of positive degree. Then $(2g(C)+1)\Pi$ is very ample and $(2g(C)+1)NbL_Y$ is linearly equivalent to the disjoint union H of smooth fibres in general position. For ($L_Y\sim_{\mathbb{Q}}$) $L'_Y=H/(2g(C)+1)Nb$ the pair $(Y,D_Y+L'_Y)$ is klt and $K_Y+D_Y+L'_Y$ is big. The coefficients of $D_Y+L'_Y$ lie in the DCC set $A(b,N)\cup\{1/(2g(C)+1)Nb\}$. Hence by the proof of Lemma 2.4 there is a constant M depending only on b,N and $g(C)$ such that $|(s+1)(K_Y+D_Y+L_Y)|$ defines a birational map for all $s\geq M$ with $s+1$ divisible by $(2g(C)+1)Nb$. Now the lemma follows from Lemma 1.2 (4). □

Lemma 3.6. *Keeping the assumptions made in Lemma* 3.5, either K_Y+D is big or $q(Y)\leq 1$.

Proof. If $K_Y + D_Y$ is not pseudo-effective we can apply Proposition 2.6 f). There, in Case (1) the irregularity is zero. So we only have to consider Case (2). Using the notations introduced there, $K_Y + D_Y + L_Y \equiv (1-e)L_Y + \beta\Gamma + E_{\tau\sigma}$ is big, $\Gamma.L_Y > 0$ and hence, using the notation from Lemma 3.5, $\Gamma.\Sigma > 0$. Further $\Gamma \cong \mathbb{P}^1$. So there are two different $\mathbb{P}^1$-fibrations on Y with fibres Γ and Σ, and Y is rational.

Therefore, we may assume that $K_Y + D_Y$ is pseudo-effective with $\kappa(K_Y + D_Y) \leq 1$. Applying Construction 2.3 to the klt-pair (Y, D_Y), we get morphisms $\gamma : Y \to Z$ and $\psi_Z : Z \to B$ with γ birational, and an ample $\mathbb{Q}$-divisor H on B such that

$$E_\gamma = K_Y + D_Y - \gamma^*\psi_Z^*(H)$$

is an effective γ-exceptional $\mathbb{Q}$-divisor consisting of rational curves. By the assumption

$$\dim B = \kappa(H) = \kappa(K_Y + D_Y) \leq 1.$$

Consider the case $\dim B = 1$. So $\psi = \psi_Z \circ \gamma : Y \to B$ is a family of curves over a curve with general fibre Γ. By abuse of notation Γ will also be considered as the general fibre of ψ_Z. For $\alpha := \deg H$, one has

$$K_Y + D_Y \sim_{\mathbb{Q}} \equiv \alpha\Gamma + E_\gamma \quad \text{and} \quad K_Z + D_Z \equiv \alpha\Gamma.$$

Since E_γ is contained in fibres

$$0 = \Gamma.(\alpha\Gamma + E_\gamma) = \Gamma.(K_Y + D_Y) \geq \Gamma.K_Y$$

and Γ is either $\mathbb{P}^1$ or an elliptic curve.

Since $K_Y + D_Y + L_Y \equiv \alpha\Gamma + E_\gamma + L_Y$ is big, $\Gamma.L_Y > 0$. Using the notations from Lemma 3.5, this implies that $\Gamma.\Sigma > 0$ where Y is ruled over C with general fibre Σ. So Γ dominates the base curve C and $q(Y) = g(C) \leq 1$, as claimed.

In case $\dim B = 0$ one has $K_Z + D_Z \equiv 0$ (indeed, $\sim_{\mathbb{Q}} 0$ by [Ko+]). If $L_Y.E_\gamma = 0$, then $K_Y + D_Y + L_Y \equiv L_Y + E_\gamma$ is the Zariski decomposition and hence

$$2 = \kappa(K_Y + D_Y + L_Y) = \kappa(L_Y) \leq n(Y) = 1,$$

a contradiction.

Thus $L_Y.E_\gamma > 0$ and, using the notations from Lemma 3.5, one finds $\Sigma.E_\gamma > 0$. The divisor E_γ is exceptional for the birational morphism to the klt surface Z, whence all its components are isomorphic to $\mathbb{P}^1$. Since one of them intersects Σ, the base curve C in Lemma 3.5 is dominated by $\mathbb{P}^1$ and hence $g(C) = q(Y) = 0$. □

Proof of Theorem 0.2. As recalled at the beginning of this section it remains to verify the theorem under Assumption 3.1. Then the theorem follows from Lemmata 3.3, 3.5, and 3.6, using Lemmata 2.9 and 2.10. □

Proof of Corollary 0.3. When $\kappa(X) = 0$, one can take M_3 to be the Beauville number as in [Mo, §10]. When $\kappa(X) = 1$, the result is just [FM, Corollary 6.2]. When $\kappa(X) = 3$, we can take $M_3 = 77$ by [CC, Th 1.1] (see also [HM], [Ta]). So the only remaining case is the one where $\kappa(X) = 2$. Here the corollary follows from Theorem 0.2 for $n = 3$, $b = 1$, $B_{n-2} = 2$ and $N = N(B_{n-2}) = 12$. □

One can avoid the use of Theorem 0.2 in the proof of the Corollary 0.3 if one uses the existence of good minimal models in dimension three:

Proof of Corollary 0.3, using the existence of minimal models.
As before all cases are known, except the one where $\kappa(X) = 2$. Assume that X is a good minimal threefold. For some $m \gg 0$ the morphism $\pi : X \to S$ associated with $|mK_X|$ has connected fibres and, by the Abundance theorem for threefolds, $K_X \sim_{\mathbb{Q}} \pi^* G$ for an ample $\mathbb{Q}$-Cartier divisor G.

As in [Na, Proof of Corollary (0.4)] (S, Δ) is klt for the effective $\mathbb{Q}$-divisor

$$\Delta := \frac{1}{12} H' + \sum_j a_j D'_j + \sum_i (1 - \frac{1}{m_i}) \Gamma'_i \quad \text{and} \quad K_X \sim_{\mathbb{Q}} \pi^*(K_S + \Delta).$$

Here H', D'_j, Γ'_i stand for the divisors $\mu_* H$, $\mu_* D_j$, $\mu_* \Gamma_i$ in the notation of [Na]. Moreover

$$a_j \in K_2 := \{\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3}\}.$$

We remark that by [KM, Th 3.5.2] S is the canonical model, denoted by W in Lemma 2.2, and that $\Delta = L_W + D_W$. Note that Δ has coefficients in the DCC set

$$\mathrm{Ell} := \{1 - \frac{1}{m} \,|\, m \in \mathbb{Z}_{\geq 2}\} \cup K_2 \cup \{\frac{1}{12}\}.$$

By [La, Th 5.4] there exists a computable constant M, depending only on the DCC set Ell, such that the adjoint linear system $|K_S + \lceil t(K_S + \Delta) \rceil|$ gives rise to a birational map for all $t \geq M$. This adjoint divisor is smaller than or equal to $(t + 1)(K_S + \Delta)$ provided that $12|t$. So $\Phi_{|(t+1)(K_S+\Delta)|}$ is birational and hence $\Phi_{|(t+1)K_X|}$ an Iitaka fibration (see Lemma 1.2 (4)). □

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