INTERSECTION SHEAVES OVER NORMAL SCHEMES

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Abstract. Intersection sheaves are usually defined for a proper flat surjective morphism of Noetherian schemes of relative dimension d and for $d+1$ invertible sheaves on the ambient scheme. In this article, the construction is generalized to the equidimensional proper surjective morphisms over normal separated Noetherian schemes. Applications to the studies on family of effective algebraic cycles and on polarized endomorphisms are also given.

Contents

Introduction

Let $\pi\colon X \to Y$ be a flat proper surjective morphism of Noetherian schemes of relative dimension d. For invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}$ of X, we can associate an invertible sheaf $\mathcal{I}_{X/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1})$ of Y which satisfies suitable conditions similar to those satisfied by the fiber integral of Chern classes:

$$\int_\pi c_1(\mathcal{L}_1) \wedge \cdots \wedge c_1(\mathcal{L}_{d+1}).$$

Especially, if $\pi\colon X \to Y$ is a morphism of algebraic $\Bbbk$-schemes smooth over a field $\Bbbk$, then

$$c_1(\mathcal{I}_{X/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1})) = \pi_* \left(c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_{d+1})\right)$$

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in the Chow group $\mathrm{CH}^1(Y)$, where π_* is the push-forward homomorphism $\mathrm{CH}^{d+1}(X) \to \mathrm{CH}^1(Y)$. In particular, $\mathcal{I}_{X/Y}(\mathcal{L})$ is the norm sheaf of $\mathcal{L}$ in case $d = 0$, and

$$\mathcal{I}_{X/Y}(\mathcal{L}_1, \dots, \mathcal{L}_{d+1}) \simeq \mathcal{I}_{H/Y}(\mathcal{L}_2|_H, \dots, \mathcal{L}_{d+1}|_H)$$

if $\mathcal{L}_1 \simeq \mathcal{O}_X(H)$ for an effective relative Cartier divisor H of X with respect to π. The sheaf $\mathcal{I}_{X/Y}(\mathcal{L}_1, \dots, \mathcal{L}_{d+1})$ is called the *intersection sheaf*, the *intersection bundle*, or the *Deligne pairing* (when $d = 1$). For the Picard groups $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$, we have a homomorphism $\mathrm{Sym}^{d+1} \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ by $\mathcal{I}_{X/Y}$. In [4], Problème 2.1.2, Deligne posed a problem of constructing $\mathcal{I}_{X/Y}$ as a functor $\mathit{PIC}(X)_{\mathrm{is}}^{d+1} \to \mathit{PIC}(Y)_{\mathrm{is}}$ satisfying natural properties on multi-additivity and base change. Here $\mathit{PIC}(X)_{\mathrm{is}}$ denotes the Picard category whose 'objects' are invertible sheaves on X and whose 'morphisms' are isomorphisms of invertible sheaves. The intersection sheaf $\mathcal{I}_{X/Y}$ can be defined a priori as a symmetric difference of $\det \mathbf{R}\,\pi_*(\mathcal{L})$ for invertible sheaves $\mathcal{L}$ (cf. Remark 2.5 below; [5], page 34), but there is a problem of sign related to det. The problem was solved for projective morphisms in [6], [8], [18], and [5] by several methods.

The flatness assumption is important for the functorial properties. In this article, we do not consider the functoriality but the construction of intersection sheaves for non-flat morphisms. More precisely, we shall construct intersection sheaves for proper surjective equi-dimensional morphisms $f\colon X \to Y$ over normal separated Noetherian schemes Y. The following is obtained in Section 3 (cf. Theorems 3.11 and 3.20; Propositions 2.7, 2.15, and 3.17):

Theorem. *Let Y be a normal separated Noetherian integral scheme and $\pi\colon X \to Y$ a projective equi-dimensional surjective morphism of relative dimension d. Let U be a Zariski-open subset of Y such that* $\mathrm{codim}(Y \setminus U) \geq 2$ *and $\pi^{-1}(U) \to U$ is flat. Then the intersection sheaf*

$$\mathcal{I}_{\pi^{-1}(U)/U}(\mathcal{L}_1|_{\pi^{-1}(U)}, \dots, \mathcal{L}_{d+1}|_{\pi^{-1}(U)})$$

defined for invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_{d+1} \in \mathrm{Pic}(X)$, naturally extends to an invertible sheaf $\mathcal{I}_{X/Y}(\mathcal{L}_1, \dots, \mathcal{L}_{d+1})$ of Y. In particular, $\mathcal{I}_{X/Y}$ induces a natural homomorphism $\mathrm{Sym}^{d+1} \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$. *Moreover, it satisfies the following properties*:

(1) *Suppose that, for any i, there exists a surjection $\pi^*\mathcal{G}_i \to \mathcal{L}_i$ for a locally free sheaf $\mathcal{G}_i$ on Y of finite rank. Then there is a surjection*

$$\Phi\colon \mathrm{Sym}^{e_1}(\mathcal{G}_1) \otimes \cdots \otimes \mathrm{Sym}^{e_{d+1}}(\mathcal{G}_{d+1}) \to \mathcal{I}_{X/Y}(\mathcal{L}_1, \dots, \mathcal{L}_{d+1}),$$

where e_i is the intersection number

$$i(\mathcal{L}_1|_F, \dots, \mathcal{L}_{i-1}|_F, \mathcal{L}_{i+1}|_F, \dots, \mathcal{L}_{d+1}|_F)$$

for the generic fiber F of π.

(2) *Let* $g\colon Y' \to Y$ *be a dominant morphism of finite type from another normal separated Noetherian scheme* Y', $f'\colon X' = X \times_Y Y' \to Y'$ *the pullback of* f, *and* $g'\colon X' \to X$ *the pullback of* g. *Then*

$$\mathcal{I}_{X'/Y'}(g'^{*}\mathcal{L}_1, \dots, g'^{*}\mathcal{L}_{d+1}) \simeq g^{*}\,\mathcal{I}_{X/Y}(\mathcal{L}_1, \dots, \mathcal{L}_{d+1}).$$

By the strong assumption on Y, the sheaf $\mathcal{I}_{X/Y}$ is first defined as a reflexive sheaf of rank one, but after certain discussion, it will be shown to be invertible. By the invertibility, we can prove that, for the equi-dimensional morphism $\pi\colon X \to Y$, if X is normal and $\mathbb{Q}$-factorial (i.e., every Weil divisor is $\mathbb{Q}$-Cartier), then so is Y (cf. Theorem 3.15).

The surjection Φ above can be regarded as the homomorphism giving the resultant: For sections $v_i \in \mathrm{H}^0(Y, \mathcal{G}_i)$ and its images $s_i \in \mathrm{H}^0(X, \mathcal{L}_i)$, $\Phi(v_1^{e_1} \otimes \cdots \otimes v^{e_{d+1}})$ is the resultant of sections $s_1, \dots, s_{d+1}$, up to unit. In particular, $\Phi(v_1^{e_1} \otimes \cdots \otimes v^{e_{d+1}})$ does not vanish at a point $y \in Y$ if and only if $\mathrm{div}(s_1) \cap \cdots \cap \mathrm{div}(s_{d+1}) \cap \pi^{-1}(y) = \emptyset$.

The intersection number $i(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_d; \mathcal{F})$ for invertible sheaves $\mathcal{L}_i$ and coherent sheaf $\mathcal{F}$ is defined on projective varieties defined over a field, where $d = \dim \operatorname{Supp} \mathcal{F}$. As an analogy of it, we can define the intersection sheaf $\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \dots, \mathcal{L}_{d+1})$ by replacing X with a coherent sheaf $\mathcal{F}$ with $\dim(\operatorname{Supp} \mathcal{F})/Y = d$. Moreover, we can define $\mathcal{I}_{\mathcal{F}/Y}$ as a homomorphism $\mathrm{Gr}_F^{d+1} K^{\bullet}(X) \to \mathrm{Pic}(Y)$ for the λ-filtration $\{F^i K^{\bullet}(X)\}$ of the Grothendieck K-group $K^{\bullet}(X) = K_0(X)$. In particular, for a locally free sheaf $\mathcal{E}$ on X of rank r and for a Chern polynomial $P = P(x_1, \dots, x_r)$ of weighted degree $d+1$, we have the intersection sheaf $\mathcal{I}_{\mathcal{F}/Y}(P(\mathcal{E})) = \mathcal{I}_{\mathcal{F}/Y}(P(c_1(\mathcal{E}), \dots, c_r(\mathcal{E})))$. A similar result to Theorem above also holds for the intersection sheaves $\mathcal{I}_{\mathcal{F}/Y}(\eta)$ for $\eta \in \mathrm{Gr}_F^{d+1} K^{\bullet}(X)$. Especially, we can prove that if $\mathcal{E}$ is a locally free sheaf of finite rank generated by global sections and if P is numerically positive for ample vector bundles (cf. Definition 2.19) in the sense of [9], then $\mathcal{I}_{\mathcal{F}/Y}(P(\mathcal{E}))$ is also generated by global sections (cf. Proposition 2.21, Corollary 3.18).

Suppose that $f\colon X \to Y$ is an equi-dimensional surjective morphism of normal projective varieties over a field. If the invertible sheaves $\mathcal{L}_i \in \mathrm{Pic}(X)$ are generated by global sections, then so is the intersection sheaf $\mathcal{I}_{X/Y}(\mathcal{L}_1, \dots, \mathcal{L}_{d+1})$. There are similar numerical properties (e.g. ampleness) on the intersection sheaves (cf. Theorems 4.4, 4.7): For example, if $\mathcal{L}_i$ are all ample (resp. nef and big), then so is $\mathcal{I}_{X/Y}(\mathcal{L}_1, \dots, \mathcal{L}_{d+1})$. If $X \subset V \times Y$ for a projective variety V and if $\mathcal{L}_i$ are the pullbacks of very ample invertible sheaves of V by the first projection $X \to V$, then we can show that the intersection sheaf defines the Stein factorization of the morphism $Y \to \mathrm{Chow}(V)$ into the Chow variety of V, which associates a general point $y \in Y$ the algebraic cycle $\mathrm{cyc}(\pi^{-1}(y))$ of V for the fiber $\pi^{-1}(y)$ (cf. Section 4.2). By the property, we have the notion of Chow reduction (cf. Proposition 4.14, Definition 4.15) for a dominant rational map $X \dashrightarrow Y$ of normal projective varieties , and also the notion of special MRC fibration (cf. Theorem 4.18) for uniruled complex projective varieties generalizing the notion of maximal rationally connected (= MRC) fibration (cf. [3], [16], [10]) defined for smooth varieties. The following results on endomorphisms are proved in Theorem 4.19 and Corollary 4.20:

(1) If $f: X \to X$ is a finite surjective morphism of a normal complex uniruled projective variety X, then f descends to an endomorphism $h: Y \to Y$ of the base Y of the special MRC fibration $X \dashrightarrow Y$.

(2) Here, if f is a polarized endomorphism, i.e., $f^*\mathcal{A} \simeq \mathcal{A}^{\otimes q}$ for some $q > 0$ and an ample invertible sheaf $\mathcal{A}$, then the endomorphism h is also polarized.

The motivation of this article is a question by D.-Q. Zhang on a similar result in [25], Proposition 2.2.4 on the endomorphisms of complex normal projective uniruled varieties, where the intersection sheaf is used for proving (2), but the notion of intersection sheaves is defined only for flat morphisms in the paper [25]. The results above solve the question. The results in Section 4.3 are used in a joint paper [19] with D.-Q. Zhang.

It is hopeless to give a similar definition of the intersection sheaves $\mathcal{I}_{X/Y}$ for a proper equi-dimensional surjective non-flat morphism $X \to Y$ over a non-normal base scheme (cf. Remark 3.5). In order to extend the notion of intersection sheaves to the non-normal case, we must add some additional data. For example, [1] treats the intersection sheaves associated to analytic families of cycles parametrized by a reduced complex analytic space, where the definition of the analytic family requires more than the equi-dimensionality.

This article is organized as follows: After preparing basics on K-groups in Section 1, we define and study the intersection sheaves $\mathcal{I}_{\mathcal{F}/Y}$ in Section 2 for Y-flat coherent sheaves $\mathcal{F}$ on X. We use essentially the same argument as in [6], [18], and the description of Chow varieties in [17]. In Section 3, we consider the case where Y is a normal separated Noetherian integral scheme, and prove basic properties including the invertibility and the base change property. We apply these fundamental results obtained in Sections 2 and 3 to projective varieties over a field or the complex number field $\mathbb{C}$ in Section 4. We prove some numerical properties of the intersection sheaves, give a relation to the morphisms into Chow varieties, and finally have some of results on polarized endomorphisms of projective varieties answering the question of D.-Q. Zhang.

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1. Grothendieck K-groups

We recall elementary properties of Grothendieck K-groups (cf. [13], [2]). Let X be a Noetherian scheme. Let $K^\bullet(X)$ (resp. $K_\bullet(X)$) be the Grothendieck group on the locally free sheaves (resp. coherent sheaves) on X. For a locally free (resp. coherent) sheaf $\mathcal{F}$, let $\mathrm{cl}^\bullet(\mathcal{F}) = \mathrm{cl}^\bullet_X(\mathcal{F})$ (resp. $\mathrm{cl}_\bullet(\mathcal{F}) = \mathrm{cl}_{X\bullet}(\mathcal{F})$) denote the corresponding element in $K^\bullet(X)$ (resp. $K_\bullet(X)$). Note that $K^\bullet(X)$ is the K_0 group in the K-theory. The tensor products with locally free sheaves give $K^\bullet(X)$ a ring structure and give $K_\bullet(X)$ a structure of $K^\bullet(X)$-module so that the canonical homomorphism $\boldsymbol{\phi}\colon K^\bullet(X) \to K_\bullet(X)$, which is called the Cartan homomorphism, is $K^\bullet(X)$-linear. Here, $\mathrm{cl}^\bullet(\mathcal{O}_X)$ is the unit element 1 of the ring structure of $K^\bullet(X)$, and $\boldsymbol{\phi}\colon K^\bullet(X) \to K_\bullet(X)$ is regarded as the multiplication map by $\boldsymbol{\phi}(1) = \mathrm{cl}_\bullet(\mathcal{O}_X) \in K_\bullet(X)$. If X is a regular separated Noetherian scheme, then $\boldsymbol{\phi}\colon K^\bullet(X) \to K_\bullet(X)$ is isomorphic by the existence of global locally free resolution for coherent sheaves (cf. [13], Exp. II Proposition 2.2.3 and Exp. II Corollaire 2.2.7.1).

The ring $\mathrm{H}^0(X, \mathbb{Z})$ of locally constant $\mathbb{Z}$-valued functions is a direct summand of $K^\bullet(X)$, in which a projection $\varepsilon\colon K^\bullet(X) \to \mathrm{H}^0(X, \mathbb{Z})$ (called the augmentation map) is given by $\mathrm{cl}^\bullet(\mathcal{E}) \mapsto \operatorname{rank}\mathcal{E}$ for locally free sheaves $\mathcal{E}$. The λ-ring structure of $K^\bullet(X)$ is introduced by setting $\lambda^p(\mathrm{cl}^\bullet(\mathcal{E})) = \mathrm{cl}^\bullet(\bigwedge^p \mathcal{E})$ for locally free sheaves $\mathcal{E}$. The augmentation map ε is a λ-homomorphism with respect to the natural λ-ring structure of $\mathrm{H}^0(X, \mathbb{Z})$. The operator γ^p associated with the λ-ring $K^\bullet(X)$ is defined by $\gamma^p(x) = \lambda^p(x + p - 1)$ for $x \in K^\bullet(X)$. The λ-filtration $\{F^pK^\bullet(X)\}$ of $K^\bullet(X)$ is defined as follows: $F^pK^\bullet(X) = K^\bullet(X)$ for $p \leq 0$, $F^1K^\bullet(X) = \mathrm{Ker}(\varepsilon)$, and $F^pK^\bullet(X)$ for $p \geq 2$ is generated by

$$\gamma^{k_1}(x_1)\gamma^{k_2}(x_2)\cdots\gamma^{k_l}(x_l)$$

with $x_i \in \mathrm{Ker}(\varepsilon)$ and $\sum k_i \geq p$. Then $K^\bullet(X)$ is a filtered ring, i.e., $F^pK^\bullet(X)F^qK^\bullet(X) \subset F^{p+q}K^\bullet(X)$ for p, $q \geq 0$. For $\mathrm{Gr}^i_F K^\bullet(X) = F^iK^\bullet(X)/F^{i+1}K^\bullet(X)$, we have

$$\mathrm{Gr}^0_F(X) \simeq \mathrm{H}^0(X) \quad \text{and} \quad \mathrm{Gr}^1_F(X) \simeq \mathrm{Pic}(X),$$

by [13], Exp. X Théorème 5.3.2, where $\mathrm{Pic}(X)$ denotes the Picard group of X.

On the other hand, $K_\bullet(X)$ also has a natural filtration $\{F^p_{\mathrm{con}}K_\bullet(X)\}$, which is called the *coniview filtration*, is defined as follows (cf. [13] Exp. X Remarque 1.4 and Exp. X Exemple 1.5): $F^p_{\mathrm{con}}K_\bullet(X)$ is generated by $\mathrm{cl}_\bullet(\mathcal{F})$ for coherent sheaves $\mathcal{F}$ with $\operatorname{codim}\operatorname{Supp}\mathcal{F} \geq p$. We have another natural subgroup $F_pK_\bullet(X) \subset K_\bullet(X)$ for $p \geq 0$, which is generated by $\mathrm{cl}_\bullet(\mathcal{F})$ for the coherent sheaves $\mathcal{F}$ with $\dim\operatorname{Supp}\mathcal{F} \leq p$. Note that, $K_\bullet(X) = \bigcup_{p\geq 0} F_pK_\bullet(X)$ does not hold unless $\dim X$ is bounded. If X is of finite type over a field and if X is of pure dimension n, then $F^p_{\mathrm{con}}K_\bullet(X) = F_{n-p}K_\bullet(X)$. The following properties are known (cf. [13], Exp. X Corollaire 1.1.4 and Exp. X Théorème 1.3.2):

- $\operatorname{Gr}^p_{F_{\mathrm{con}}} K_\bullet(X) = F^p_{\mathrm{con}} K_\bullet(X)/F^{p+1}_{\mathrm{con}} K_\bullet(X)$ is generated by $\mathrm{cl}_\bullet(\mathcal{O}_Z)$ for the closed integral subschemes Z of codimension p.
- $\operatorname{Gr}^F_p K_\bullet(X) = F_p K_\bullet(X)/F_{p-1} K_\bullet(X)$ is generated by $\mathrm{cl}_\bullet(\mathcal{O}_Z)$ for the closed integral subschemes Z of dimension p.
- $F^p K^\bullet(X) F^q_{\mathrm{con}} K_\bullet(X) \subset F^{p+q}_{\mathrm{con}} K_\bullet(X)$ and $F^p K^\bullet(X) F_q K_\bullet(X) \subset F_{q-p} K_\bullet(X)$ for any p, $q \geq 0$. In particular, we have $\boldsymbol{\phi}(F^p K^\bullet(X)) \subset F^p_{\mathrm{con}} K_\bullet(X)$, and $\boldsymbol{\phi}(F^p K^\bullet(X)) \subset F_{n-p} K_\bullet(X)$ if $\dim X \leq n$.

Convention. For the sake of simplicity, we write

$$\begin{aligned} F^p(X) &= F^p K^\bullet(X), & F^p_{\mathrm{con}}(X) &= F^p_{\mathrm{con}} K_\bullet(X), & F_p(X) &= F_p K_\bullet(X), \\ G^p(X) &= \operatorname{Gr}^p_F K^\bullet(X), & G^p_{\mathrm{con}}(X) &= \operatorname{Gr}^p_{F_{\mathrm{con}}} K_\bullet(X), & G_p(X) &= \operatorname{Gr}^F_p K_\bullet(X), \\ G^\bullet(X) &= \textstyle\bigoplus_{p\geq 0} G^p(X), & G^\bullet_{\mathrm{con}}(X) &= \textstyle\bigoplus_{p\geq 0} G^p_{\mathrm{con}}(X), & G_\bullet(X) &= \textstyle\bigoplus_{p\geq 0} G_p(X). \end{aligned}$$

Then, $G^\bullet(X)$ is a graded ring; $G^\bullet_{\mathrm{con}}(X)$ and $G_\bullet(X)$ have graded $G^\bullet(X)$-module structures by $G^p(X) \otimes G^q_{\mathrm{con}}(X) \to G^{p+q}_{\mathrm{con}}(X)$ and $G^p(X) \otimes G_q(X) \to G_{q-p}(X)$. We denote by $G(\boldsymbol{\phi})\colon G^p(X) \to G^p_{\mathrm{con}}(X)$ the homomorphism induced from the Cartan homomorphism $\boldsymbol{\phi}\colon K^\bullet(X) \to K_\bullet(X)$.

Remark. Suppose that X is an n-dimensional smooth algebraic variety defined over a filed. Then $\boldsymbol{\phi}\colon K^\bullet(X) \to K_\bullet(X)$ is isomorphic and $F^p_{\mathrm{con}}(X) = F_{n-p}(X) \subset K_\bullet(X)$. Moreover, $K_\bullet(X)$ has a structure of filtered ring by $\boldsymbol{\phi}$ and by $\{F^p_{\mathrm{con}}(X)\}$, i.e., $F^p_{\mathrm{con}}(X) F^q_{\mathrm{con}}(X) \subset F^{p+q}_{\mathrm{con}}(X)$ for any p, $q \geq 0$ (cf. [13], Exp. 0 App. II Théorème 2.11, Corollaire; and Exp. VI Proposition 6.6). Since $\boldsymbol{\phi}(F^p(X)) \subset F^p_{\mathrm{con}}(X)$, $G(\boldsymbol{\phi})\colon G(X) \to G_{\mathrm{con}}(X)$ is a surjective ring homomorphism; however $G(\boldsymbol{\phi})$ is not necessarily isomorphic.

Let $f\colon X \to Y$ be a morphism of Noetherian schemes. Then the λ-ring homomorphism $f^*\colon K^\bullet(Y) \to K^\bullet(X)$ is defined, which maps $\mathrm{cl}^\bullet_Y(\mathcal{E})$ to $\mathrm{cl}^\bullet_X(f^*\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ on Y. Here, $f^* F^p(Y) \subset F^p(X)$, and hence $f^*\colon G^p(Y) \to G^p(X)$ is induced. For a morphism $g\colon Y \to Z$ to another Noetherian scheme Z, we have $(g \circ f)^* = f^* \circ g^*$.

If f is proper, then the group homomorphism $f_*\colon K_\bullet(X) \to K_\bullet(Y)$ is defined, which maps $\mathrm{cl}_{X\bullet}(\mathcal{F})$ to $\sum(-1)^i \mathrm{cl}_{Y\bullet}(\mathrm{R}^i f_* \mathcal{F})$ for a coherent sheaf $\mathcal{F}$. Here, $f_* F_p(X) \subset F_p(Y)$, since $\dim \operatorname{Supp} \mathrm{R}^i f_* \mathcal{F} \leq \dim \operatorname{Supp} \mathcal{F}$ for any i for any coherent sheaf $\mathcal{F}$ on X. In particular, $f\colon G_p(X) \to G_p(Y)$ is induced. For a proper morphism $g\colon Y \to Z$ to another Noetherian scheme Z, we have $(g \circ f)_* = g_* \circ f_*$. We have the following projection formula: If $f\colon X \to Y$ is proper, then

$$f_*(x \cdot f^* y) = f_* x \cdot y \tag{I–1}$$

for any $x \in K_\bullet(X)$ and $y \in K^\bullet(Y)$. This follows from the usual projection formula $\mathrm{R}^i f_*(\mathcal{F} \otimes f^* \mathcal{E}) \simeq \mathrm{R}^i f_* \mathcal{F} \otimes \mathcal{E}$ for coherent sheaves $\mathcal{F}$ on X and locally free sheaves $\mathcal{E}$ on Y.

As a result, we infer that $f_*\colon K_\bullet(X) \to K_\bullet(Y)$ is $K^\bullet(Y)$-linear and $f_*\colon G_\bullet(X) \to G_\bullet(Y)$ is $G^\bullet(Y)$-linear.

Suppose that f is flat. Then $f^*\colon K_\bullet(Y) \to K_\bullet(X)$ is induced (cf. [13], Exp. IV 2.12) which is compatible with $f^*\colon K^\bullet(Y) \to K^\bullet(X)$. Here, $f^*F^p_{\mathrm{con}}(Y) \subset F^p_{\mathrm{con}}(X)$ for any p. If $g\colon Z \to Y$ is a proper morphism, then for the fiber product $W = Z \times_Y X$ and for the natural projections $p_1\colon W \to Z$ and $p_2\colon W \to X$, we have the base change formula

$$f^*g_*(z) = p_{2*}p_1^*(z) \tag{I–2}$$

for $z \in K_\bullet(Z)$ (cf. [13], Exp. IV Proposition 3.1.1). If f is proper, flat, and of relative dimension d, then we have $f_*F^{p+d}_{\mathrm{con}}(X) \subset F^p_{\mathrm{con}}(Y)$ by the formula:

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,f(x)} + \dim \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \Bbbk(f(x)),$$

where $\Bbbk(f(x))$ denotes the residue field of $\mathcal{O}_{Y,f(x)}$. For an open immersion $j\colon U \hookrightarrow X$ and for the closed immersion $i\colon Z \hookrightarrow X$ from the complement $Z = X \setminus U$, we have the following natural exact sequence (cf. [13], Exp. 0 App. II Proposition 2.10):

$$K_\bullet(Z) \xrightarrow{i_*} K_\bullet(X) \xrightarrow{j^*} K_\bullet(U) \to 0. \tag{I–3}$$

An algebraic cycle $Z = \sum n_i Z_i$ of X is a finite linear combination of closed integral subschemes Z_i of X with integral coefficients n_i. If the coefficients n_i are all non-negative, then Z is called effective. If $\dim Z_i = k$ (resp. $\operatorname{codim} Z_i = k$) for all i, then Z is called a cycle of dimension k (resp. of codimension k). The group of algebraic cycles of dimension k (resp. codimension k) is denoted by $\mathcal{Z}_k(X)$ (resp. $\mathcal{Z}^k(X)$).

Definition 1.1. For a coherent sheaf $\mathcal{F}$, we define an effective algebraic cycle by

$$\mathrm{cyc}(\mathcal{F}) := \sum\nolimits_{W \subset \operatorname{Supp}\mathcal{F}} l_W(\mathcal{F})W,$$

where all the irreducible components W of $\operatorname{Supp}\mathcal{F}$ are taken, and $l_W(\mathcal{F})$ denotes the length of the $\mathcal{O}_{X,\eta}$-module $\mathcal{F}_\eta$ for the generic point η of W. If $\dim \operatorname{Supp}\mathcal{F} \le k$, then we set

$$\mathrm{cyc}_k(\mathcal{F}) := \sum\nolimits_{\dim W = k,\, W \subset \operatorname{Supp}\mathcal{F}} l_W(\mathcal{F})W \in \mathcal{Z}_k(X).$$

If $\operatorname{codim} \operatorname{Supp}\mathcal{F} \ge k$, then we set

$$\mathrm{cyc}^k(\mathcal{F}) := \sum\nolimits_{\operatorname{codim} W = k,\, W \subset \operatorname{Supp}\mathcal{F}} l_W(\mathcal{F})W \in \mathcal{Z}^k(X).$$

We write $\mathrm{cyc}(V) = \mathrm{cyc}(\mathcal{O}_V)$ for closed subschemes V.

We have natural homomorphisms $\mathrm{cl}_\bullet\colon \mathcal{Z}^k(X) \to F^k_{\mathrm{con}}(X)$ and $\mathrm{cl}_\bullet\colon \mathcal{Z}_k(X) \to F_k(X)$ defined by $\mathrm{cl}_\bullet(Z) = \sum n_i \,\mathrm{cl}_\bullet(\mathcal{O}_{Z_i})$, where $Z = \sum n_i Z_i$ for closed integral subschemes Z_i and $n_i \in \mathbb{Z}$. Then, $\mathrm{cl}_\bullet(\mathrm{cyc}^k(\mathcal{F})) \equiv \mathrm{cl}_\bullet(\mathcal{F}) \mod F^{k+1}_{\mathrm{con}}(X)$ and $\mathrm{cl}_\bullet(\mathrm{cyc}_k(\mathcal{G})) \equiv \mathrm{cl}_\bullet(\mathcal{G}) \mod F_{k-1}(X)$ for coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ with $\operatorname{codim} \operatorname{Supp}\mathcal{F} = \dim \operatorname{Supp}\mathcal{G} = k$.

Remark. Suppose that X is a smooth quasi-projective variety over a field. The Chow group $\mathrm{CH}_k(X)$ (resp. $\mathrm{CH}^k(X)$) is defined as the quotient group of $\mathscr{Z}_k(X)$ (resp. $\mathscr{Z}^k(X)$) by the rational equivalence relation. Here, $\mathrm{CH}^i(X) \simeq \mathrm{CH}_{n-i}(X)$ for $i \geq 0$, since $\mathscr{Z}^i(X) = \mathscr{Z}_{n-i}(X)$. Then $\mathrm{CH}(X) = \bigoplus_{i=0}^n \mathrm{CH}^i(X)$ has a graded ring structure by the intersection theory, which is called the Chow ring of X. The map $\mathrm{cl}_\bullet \colon \mathscr{Z}^k(X) \to F^k_{\mathrm{con}}(X)$ defined just after Definition 1.1 induces $G(\mathrm{cl}_\bullet) \colon \mathrm{CH}^k(X) \to G^k_{\mathrm{con}}(X)$ and a ring homomorphism $G(\mathrm{cl}_\bullet) \colon \mathrm{CH}(X) \to G_{\mathrm{con}}(X)$.

Definition 1.2 (Chern class)**.** Let X be a Noetherian scheme. For $x \in K^\bullet(X)$, the p-th Chern class of x for $p \geq 0$ in the K-theory is defined to be

$$\boldsymbol{c}^p(x) := \gamma^p(x - \varepsilon(x)) \mod F^{p+1}(X) \in G^p(X),$$

where ε is the augmentation map. For a locally free sheaf $\mathcal{E}$, we write $\boldsymbol{c}^p(\mathcal{E}) = \boldsymbol{c}^p(\mathrm{cl}^\bullet(\mathcal{E}))$.

Remark (cf. [13] Exp. 0 App. II §5). Suppose that X is an n-dimensional smooth quasi-projective variety over a field. Then we have the map of the i-th Chern class $c_i \colon K^\bullet(X) \to \mathrm{CH}^i(X)$ for $0 \leq i \leq n$. The Chern class $c_i(x)$ and the Chern class $\boldsymbol{c}^i(x)$ in the K-theory for $x \in K^\bullet(X)$ is related by

$$G(\mathrm{cl}_\bullet)(c_i(x)) = G(\boldsymbol{\phi})(\boldsymbol{c}^i(x)).$$

Definition 1.3. Let X be a Noetherian scheme. For an invertible sheaf $\mathcal{L}$, we set

$$\boldsymbol{\delta}(\mathcal{L}) = \boldsymbol{\delta}_X(\mathcal{L}) := 1 - \mathrm{cl}^\bullet_X(\mathcal{L}^{-1}) \in F^1(X).$$

Furthermore, for invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_l$ on X, we set

$$\boldsymbol{\delta}(\mathcal{L}_1, \ldots, \mathcal{L}_l) = \boldsymbol{\delta}^{(l)}_X(\mathcal{L}_1, \ldots, \mathcal{L}_l) := \boldsymbol{\delta}(\mathcal{L}_1)\,\boldsymbol{\delta}(\mathcal{L}_2) \cdots \boldsymbol{\delta}(\mathcal{L}_l) \in F^l(X).$$

Remark 1.4.

(1) $\boldsymbol{\delta}(\mathcal{L} \otimes \mathcal{L}') = \boldsymbol{\delta}(\mathcal{L}) + \boldsymbol{\delta}(\mathcal{L}') - \boldsymbol{\delta}(\mathcal{L}, \mathcal{L}')$ for two invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$.

(2) $\boldsymbol{\delta}(\mathcal{L}) \mod F^2(X) = \boldsymbol{c}^1(\mathcal{L}) \in G^1(X)$ for an invertible sheaf $\mathcal{L}$. In fact,

$$\begin{aligned}\boldsymbol{\delta}(\mathcal{L}) - \gamma^1(\mathrm{cl}^\bullet(\mathcal{L}) - 1) &= 1 - \mathrm{cl}^\bullet(\mathcal{L}^{-1}) - (\mathrm{cl}^\bullet(\mathcal{L}) - 1) = (\mathrm{cl}^\bullet(\mathcal{L}) - 1)(\mathrm{cl}^\bullet(\mathcal{L}^{-1}) - 1) \\ &= \gamma^1(\mathrm{cl}^\bullet(\mathcal{L}) - 1)\gamma^1(\mathrm{cl}^\bullet(\mathcal{L}^{-1}) - 1) \in F^2(X).\end{aligned}$$

In particular,

$$\boldsymbol{\delta}(\mathcal{L}_1, \ldots, \mathcal{L}_l) \mod F^{l+1}(X) = \boldsymbol{c}^1(\mathcal{L}_1) \cdots \boldsymbol{c}^1(\mathcal{L}_l) = \boldsymbol{c}^l(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_l).$$

(3) We have the following explicit expression:

$$\begin{aligned}\boldsymbol{\delta}(\mathcal{L}_1, \ldots, \mathcal{L}_l) &= \sum_{k=0}^l (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq l} \mathrm{cl}^\bullet \left(\bigotimes_{j=1}^k \mathcal{L}_{i_j}^{-1} \right) \\ &= 1 - \sum_{i=1}^l \mathrm{cl}^\bullet(\mathcal{L}_i^{-1}) + \cdots + (-1)^l\, \mathrm{cl}^\bullet(\mathcal{L}_1^{-1} \otimes \cdots \otimes \mathcal{L}_l^{-1}).\end{aligned}$$

Remark. The determinant map $\det\colon K^\bullet(X) \to \mathrm{Pic}(X)$ is defined by $\det(\mathrm{cl}^\bullet(\mathcal{E})) = \det\mathcal{E}$ for locally free sheaves $\mathcal{E}$. We note that

$$\det(xy) \simeq \det(x)^{\otimes\varepsilon(y)} \otimes \det(y)^{\otimes\varepsilon(x)}$$

for x, $y \in K^\bullet(X)$. Since det is trivial on $F^2(X)$ by [13], Exp. X Lemma 5.3.4, a homomorphism $G^1(X) \to \mathrm{Pic}(X)$ is induced by det. Its inverse is given by the first Chern class $\boldsymbol{c}^1\colon \mathrm{Pic}(X) \to G^1(X)$.

Definition 1.5. Let $\mathcal{F}$ be a coherent sheaf and $\mathcal{E}$ a coherent locally free sheaf on a Noetherian scheme X. Let σ be a section of $\mathcal{E}$ and let $\sigma^\vee\colon \mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X) \to \mathcal{O}_X$ denote the dual of $\sigma\colon \mathcal{O}_X \to \mathcal{E}$.

(1) The zero subscheme $V(\sigma)$ of the section σ is a closed subscheme defined by $\mathrm{Coker}(\sigma^\vee) = \mathcal{O}_{V(\sigma)}$.

(2) σ is called $\mathcal{F}$-regular, if, for any point $P \in V(\sigma)$ and for a local trivialization $\mathcal{E}_P \simeq \mathcal{O}_P^{\oplus r}$, the germ $\sigma_P \in \mathcal{E}_P$ corresponds to an $\mathcal{F}_P$-regular sequence. In other words, the natural Koszul complex

$$[\cdots \to {\textstyle\bigwedge}^p(\mathcal{E}^\vee) \to {\textstyle\bigwedge}^{p-1}(\mathcal{E}^\vee) \to \cdots \to \mathcal{E}^\vee \xrightarrow{\sigma^\vee} \mathcal{O}_X \to 0]$$

defined by $\sigma^\vee$ induces an exact sequence

$$\text{(I–4)}\quad \cdots \to \mathcal{F} \otimes {\textstyle\bigwedge}^p(\mathcal{E}^\vee) \to \mathcal{F} \otimes {\textstyle\bigwedge}^{p-1}(\mathcal{E}^\vee) \to \cdots \to \mathcal{F} \otimes \mathcal{E}^\vee \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_{V(\sigma)} \to 0.$$

(3) Suppose that $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_l$ for invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_l$ and σ is given by sections σ_i of $\mathcal{L}_i$. Then, we define $V(\sigma_1, \ldots, \sigma_l) := V(\sigma)$. Similarly, $(\sigma_1, \ldots, \sigma_l)$ is called $\mathcal{F}$-regular if so is σ.

Lemma 1.6 (cf. [11], Théorème 2)**.** *Let $\mathcal{E}$ be a locally free sheaf of rank r on X.*

(1) *For the formal power series $\lambda_t(x) := \sum_{p\geq 0} \lambda^p(x)t^p \in K^\bullet(X)[\![t]\!]$ for $x \in K^\bullet(X)$,*

$$\lambda_{-1}(\mathcal{E}) := \lambda_{-1}(\mathrm{cl}^\bullet(\mathcal{E})) = \lambda_t(\mathrm{cl}^\bullet(\mathcal{E}))|_{t=-1}$$

is well-defined as an element of $K^\bullet(X)$, and is equal to $(-1)^r\gamma^r(\mathrm{cl}^\bullet(\mathcal{E}) - r)$. In particular, $\boldsymbol{c}^r(\mathcal{E}) = \lambda_{-1}(\mathcal{E}^\vee) \mod F^{r+1}(X)$.

(2) *Let $\mathcal{F}$ be a coherent sheaf and σ an $\mathcal{F}$-regular section of $\mathcal{E}$. Then,*

$$\text{(I–5)}\qquad \lambda_{-1}(\mathcal{E}^\vee)\,\mathrm{cl}_\bullet(\mathcal{F}) = \mathrm{cl}_\bullet(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)}).$$

In particular,

$$\boldsymbol{c}^r(\mathcal{E})\,\mathrm{cl}_\bullet(\mathcal{F}) \equiv \begin{cases} \mathrm{cl}_\bullet(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)}) \mod F^{k+r+1}_{\mathrm{con}}(X), & \textit{if } \operatorname{codim} \mathcal{F} \geq k, \\ \mathrm{cl}_\bullet(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)}) \mod F_{k-r-1}(X), & \textit{if } \dim \mathcal{F} \leq k. \end{cases}$$

Proof. (1): For $x \in K^\bullet(X)$, formally, we have

$$\lambda_{-1}(x) = \lambda_t(x)|_{t=-1} = \sum\nolimits_{p \geq 0} (-1)^p \lambda^p(x).$$

The formal power series $\gamma_t(x) := \sum_{p\geq 0} \gamma^p(x) t^p$ is related to $\lambda_t(x)$ by $\lambda_t(x) = \gamma_{t/(1+t)}(x)$. By the property: $\lambda^p(\mathrm{cl}^\bullet(\mathcal{E})) = \gamma^p(\mathrm{cl}^\bullet(\mathcal{E}) - r) = 0$ for $p > r$, and by the calculation

$$\lambda_t(x) = \lambda_t(x - r)\lambda_t(r) = \gamma_{t/(1+t)}(x - r)(1 + t)^r = \sum\nolimits_{p\geq 0} \gamma^p(x - r)t^p(1 + t)^{r-p},$$

we have $\lambda_{-1}(\mathcal{E}) = (-1)^r\gamma^r(\mathrm{cl}^\bullet(\mathcal{E}) - r)$. The other formula follows from the equality $\boldsymbol{c}^r(\mathcal{E}) = (-1)^r\, \boldsymbol{c}^r(\mathcal{E}^\vee)$.

(2) is derived from the exact sequence (I–4). □

Remark. If $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$ for invertible sheaves $\mathcal{L}_i$ on X, then, by Remark 1.4,

$$(-1)^r\gamma^r(\mathrm{cl}^\bullet(\mathcal{E}^\vee) - r) = \lambda_{-1}(\mathcal{E}^\vee) = \boldsymbol{\delta}(\mathcal{L}_1, \ldots, \mathcal{L}_r).$$

Definition 1.7 (Intersection number)**.** Assume that X is a scheme proper over $\operatorname{Spec} \Bbbk$ for a field $\Bbbk$. For the the structure morphism $p_X : X \to \operatorname{Spec} \Bbbk$, the composite

$$K_\bullet(X) \xrightarrow{p_{X*}} K_\bullet(\operatorname{Spec} \Bbbk) \simeq \mathrm{H}^0(\operatorname{Spec} \Bbbk, \mathbb{Z}) = \mathbb{Z}$$

maps $\mathrm{cl}_\bullet(\mathcal{F})$ to the Euler characteristic $\chi(X, \mathcal{F})$ for a coherent sheaf $\mathcal{F}$. In particular, it induces the homomorphism $\deg : G_0(X) = F_0(X) \to \mathbb{Z}$, which maps $\mathrm{cl}_\bullet(\mathcal{F})$ for a skyscraper sheaf $\mathcal{F}$ to $\dim \mathrm{H}^0(X, \mathcal{F})$. The intersection number $i(\eta; \xi) \in \mathbb{Z}$ for $\eta \in G^l(X)$ and $\xi \in G_l(X)$ is defined to be the image of the natural homomorphism

$$G^l(X) \otimes G_l(X) \to G_0(X) \xrightarrow{\deg} \mathbb{Z}.$$

If $\eta = \boldsymbol{c}^1(\mathcal{L}_1) \cdots \boldsymbol{c}^1(\mathcal{L}_l)$ for invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_l$, then we write $i(\mathcal{L}_1, \ldots, \mathcal{L}_l; \xi)$ for $i(\eta; \xi)$. For a coherent sheaf $\mathcal{F}$ with $\dim \operatorname{Supp} \mathcal{F} = l$ and a closed subscheme V of dimension l, we write

$$i(\eta; \mathcal{F}) = i(\eta; \mathrm{cl}_\bullet(\mathcal{F})) \quad \text{and} \quad i(\eta; V) = i(\eta; \mathrm{cl}_\bullet(\mathcal{O}_V)).$$

Remark. For $\mathcal{L}_1, \ldots, \mathcal{L}_l \in \operatorname{Pic}(X)$ and $\xi \in G_l(X)$, we have

$$i(\mathcal{L}_1, \ldots, \mathcal{L}_l; \xi) = \deg\left(\boldsymbol{\delta}_X(\mathcal{L}_1, \ldots, \mathcal{L}_l)\xi\right).$$

For a coherent sheaf $\mathcal{F}$ with $\dim \operatorname{Supp} \mathcal{F} = l$, $i(\mathcal{L}_1, \ldots, \mathcal{L}_l; \mathcal{F})$ is just the coefficient of $x_1x_2 \cdots x_l$ of the Snapper polynomial $P_\mathcal{F}(x_1, \ldots, x_l) \in \mathbb{Q}[x_1, \ldots, x_l]$ defined by

$$P_\mathcal{F}(m_1, \ldots, m_l) = \chi(X, \mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{L}_l^{\otimes m_l} \otimes \mathcal{F})$$

for $m_1, \ldots, m_l \in \mathbb{Z}$.

Lemma 1.8. *Let X be a reduced Noetherian scheme and $I = \{X_i\}$ the set of irreducible components of X. Let $\mathrm{rk}_i\colon K_\bullet(X) \to \mathbb{Z}$ be the homomorphism defined by*

$$\mathrm{rk}_i(\mathrm{cl}_\bullet(\mathcal{F})) = l_{X_i}(\mathcal{F}) = \mathrm{length}_{\mathcal{O}_{X,\eta_i}}(\mathcal{F}_{\eta_i})$$

for coherent sheaves $\mathcal{F}$ and for the generic point η_i of X_i. Then $\sum \mathrm{rk}_i\colon K_\bullet(X) \to \bigoplus_I \mathbb{Z}$ induces an isomorphism $G^0_{\mathrm{con}}(X) \simeq \bigoplus_I \mathbb{Z}$.

Proof. If $\mathcal{F}$ is a coherent sheaf with $\operatorname{codim} \operatorname{Supp} \mathcal{F} > 0$, then $\mathrm{rk}_i(\mathcal{F}) = 0$ for any i; Thus $\sum \mathrm{rk}_i$ induces $G^0_{\mathrm{con}}(X) \to \bigoplus_I \mathbb{Z}$. For the surjective homomorphism $\mathcal{Z}^0(X) \to G^0_{\mathrm{con}}(X)$, the composite $\mathcal{Z}^0(X) \to \bigoplus_I \mathbb{Z}$ is just an isomorphism. Thus, $G^0_{\mathrm{con}}(X) \simeq \bigoplus_I \mathbb{Z}$. $\square$

Definition 1.9. Let X be a Noetherian scheme with a morphism $X \to Y$ to an integral scheme Y. For a coherent sheaf $\mathcal{F}$ of X, we denote by $\mathcal{F}_{\mathrm{tor}/Y}$ the unique maximal coherent subsheaf $\mathcal{F}'$ such that $\operatorname{Supp} \mathcal{F}'$ does not dominate Y. We denote by $\mathcal{F}_{\mathrm{t.f.}/Y}$ the quotient sheaf $\mathcal{F}/\mathcal{F}_{\mathrm{tor}/Y}$. In case $X = Y$, then we write $\mathcal{F}_{\mathrm{tor}} = \mathcal{F}_{\mathrm{tor}/Y}$ and $\mathcal{F}_{\mathrm{t.f.}} = \mathcal{F}_{\mathrm{t.f.}/Y}$. The sheaf $\mathcal{F}_{\mathrm{tor}}$ is called the torsion part of $\mathcal{F}$. If $\mathcal{F}_{\mathrm{tor}} = 0$, then $\mathcal{F}$ is called torsion free.

Lemma 1.10. *Suppose that X is a normal separated Noetherian scheme. Then there is an isomorphism $\widehat{\det}\colon G^1_{\mathrm{con}}(X) \xrightarrow{\simeq} \mathrm{Ref}^1(X)$ into the group $\mathrm{Ref}^1(X)$ of reflexive sheaves of rank one on X with the natural commutative diagram*

$$\begin{array}{ccc} G^1(X) & \xrightarrow{\det} & \mathrm{Pic}(X) \\ {\scriptstyle G(\phi)}\downarrow & & \downarrow \\ G^1_{\mathrm{con}}(X) & \xrightarrow{\widehat{\det}} & \mathrm{Ref}^1(X). \end{array}$$

Remark. The group structure of $\mathrm{Ref}^1(X)$ is given by the double-dual ($\vee\vee$) of the tensor product, where $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ and $\mathcal{F}^{\vee\vee} = (\mathcal{F}^\vee)^\vee$ for an $\mathcal{O}_X$-module $\mathcal{F}$. Note that $\mathrm{Ref}^1(X)$ is isomorphic to the Weil divisor class group $\mathrm{CL}(X)$ by $D \mapsto \mathcal{O}_X(D)$ for Weil divisors D. Here $\mathcal{O}_X(D)$ is a subsheaf of the sheaf of germs of rational functions on X defined by

$$\varphi \in \mathrm{H}^0(U, \mathcal{O}_X(D)) \Leftrightarrow \mathrm{div}(\varphi) + D|_U \geq 0$$

for any open subset U, where $\mathrm{div}(\varphi)$ stands for the associated principal divisor.

Proof. We may assume that X is integral. For a coherent sheaf $\mathcal{F}$, we can associate a reflexive sheaf $\mathcal{D}(\mathcal{F})$ of rank one as follows:

- If $\mathcal{F}$ is a torsion sheaf, i.e., $\mathrm{rk}(\mathcal{F}) = l_X(\mathcal{F}) = 0$, then $\mathcal{D}(\mathcal{F}) := \mathcal{O}_X(\mathrm{Div}(\mathcal{F}))$ for the Weil divisor

$$\mathrm{Div}(\mathcal{F}) = \mathrm{cyc}^1(\mathcal{F}) = \textstyle\sum_{\text{prime divisors } \Gamma \subset \operatorname{Supp} \mathcal{F}} l_\Gamma(\mathcal{F})\Gamma.$$

- If $\mathcal{F}$ is torsion free, then
$$\mathcal{D}(\mathcal{F}) := \left(\bigwedge^{\mathrm{rk}(\mathcal{F})} \mathcal{F}\right)^{\vee\vee}.$$
- For a general coherent sheaf $\mathcal{F}$, we define
$$\mathcal{D}(\mathcal{F}) = (\mathcal{D}(\mathcal{F}_{\mathrm{t.f.}}) \otimes \mathcal{D}(\mathcal{F}_{\mathrm{tor}}))^{\vee\vee}.$$

We shall show $\mathcal{D}(\mathcal{F}) \simeq (\mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{H}))^{\vee\vee}$ for any exact sequence $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H} \to 0$ of coherent sheaves: Let $\mathcal{K}$ be the kernel of $\mathcal{F}_{\mathrm{t.f.}} \to \mathcal{H}_{\mathrm{t.f.}}$ and $\mathcal{C}$ the cokernel of $\mathcal{F}_{\mathrm{tor}} \to \mathcal{H}_{\mathrm{tor}}$; then we have an exact sequence
$$0 \to \mathcal{G}_{\mathrm{t.f.}} \to \mathcal{K} \to \mathcal{C} \to 0.$$
Thus, it is enough to show $\mathcal{D}(\mathcal{F}) \simeq (\mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{H}))^{\vee\vee}$ in the case where $\mathcal{H}$ is a torsion sheaf and $\mathcal{F}$ is torsion free. For the generic point η of a prime divisor Γ, $\mathcal{G}_\eta \to \mathcal{F}_\eta$ is written as a homomorphism $h\colon \mathcal{O}_{X,\eta}^{\oplus r} \to \mathcal{O}_{X,\eta}^{\oplus r}$ for $r = \mathrm{rk}\,\mathcal{F} = \mathrm{rk}\,\mathcal{G}$. Thus, $l_\Gamma(\mathcal{H})$ is just the length of $\mathcal{O}_{X,\eta}/\det(h)\mathcal{O}_{X,\eta}$ for the determinant $\det(h)$. Hence, $\mathcal{D}(\mathcal{F}) \simeq (\mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{H}))^{\vee\vee}$.

Therefore, $\mathcal{D}$ gives rise to a homomorphism $K_\bullet(X) \to \mathrm{Ref}^1(X)$, which we write $\widehat{\det}$. Note that $\widehat{\det}$ is zero on $F^2_{\mathrm{con}}(X)$. The homomorphism $\mathrm{cl}_\bullet\colon \mathcal{Z}^1(X) \to G^1_{\mathrm{con}}(X)$ is surjective, and the composite $\mathcal{Z}^1(X) \to \mathrm{Ref}^1(X)$ is the canonical surjection which maps a Weil divisor D to $\mathcal{O}_X(D)$. In order to prove the induced homomorphism $\widehat{\det}\colon G_{n-1}(X) \to \mathrm{Ref}^1(X)$ to be isomorphic, it suffices to show that $\mathrm{cl}_\bullet(Z) = 0 \in G^1_{\mathrm{con}}(X)$ for a divisor Z with $\mathcal{O}_X(Z) \simeq \mathcal{O}_X$. Let $Z = Z_1 - Z_2$ be the decomposition into effective divisors Z_1, Z_2 containing no common prime components. From the equality $\mathrm{cl}_\bullet(\mathcal{O}_{Z_i}) = \mathrm{cl}_\bullet(\mathcal{O}_X) - \mathrm{cl}_\bullet(\mathcal{O}_X(-Z_i))$ for $i = 1$, 2, we have
$$\mathrm{cl}_\bullet(Z) = \mathrm{cl}_\bullet(\mathcal{O}_{Z_1}) - \mathrm{cl}_\bullet(\mathcal{O}_{Z_2}) = \phi(\mathrm{cl}_\bullet(\mathcal{O}_X(-Z_2)) - \mathrm{cl}_\bullet(\mathcal{O}_X(-Z_1))) = 0.$$

Finally, we compare with the other isomorphism $\det\colon G^1(X) \to \mathrm{Pic}(X)$. For $x = \mathrm{cl}^\bullet(\mathcal{E}) - r \in G^1(X)$ for a locally free sheaf $\mathcal{E}$ of rank r, we have $\det(x) = \det(\mathcal{E})$. On the other hand, $\widehat{\det}(\phi(x)) = \mathcal{D}(\mathcal{E}) \simeq \det(\mathcal{E})$. Thus, $\widehat{\det}$ is compatible with det. □

2. Intersection sheaves for flat morphisms

Let $\pi\colon X \to Y$ be a locally projective morphism of Noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. We assume that Y is connected. Thus, the relative dimension

$$d := \dim(\operatorname{Supp}\mathcal{F})/Y = \dim \operatorname{Supp}(\mathcal{F} \otimes \mathcal{O}_{\pi^{-1}(y)})$$

for $y \in Y$ is constant.

Assumption 2.1. For $\pi\colon X \to Y$, we assume that π is a projective morphism and the following (A) or (B) is satisfied:

(A) π is flat.

(B) Y admits an ample invertible sheaf (cf. [12], Définition 4.5.3).

In this section, we shall define the intersection sheaf $\mathcal{I}_{\mathcal{F}/Y}(\eta)$ for $\eta \in G^{d+1}(X)$ under Assumption 2.1. Also we shall study the basic properties of the intersection sheaves.

Note that $\pi_*\mathcal{F}$ is locally free if $\mathrm{R}^p\,\pi_*\mathcal{F} = 0$ for any $p > 0$. Using a π-ample invertible sheaf $\mathcal{A}$, we can show:

Lemma 2.2. *For a locally free sheaf $\mathcal{E}$ of finite rank on X, under* Assumption 2.1*, there is an exact sequence*

$$0 \to \mathcal{E} \to \mathcal{E}_0 \to \cdots \to \mathcal{E}_d \to 0$$

such that $\mathcal{E}_i$ are locally free sheaves of finite rank and $\mathrm{R}^p\,\pi_*(\mathcal{F} \otimes \mathcal{E}_i) = 0$ for any $p > 0$ and any $0 \le i \le d$.

Proof. There exists a positive integer k such that $\mathrm{R}^p\,\pi_*(\mathcal{F} \otimes \mathcal{A}^{\otimes k}) = 0$ for any $p > 0$ and the natural homomorphism

$$\pi^*\pi_*(\mathcal{E}^\vee \otimes \mathcal{A}^{\otimes k}) \to \mathcal{E}^\vee \otimes \mathcal{A}^{\otimes k}$$

is surjective. We shall construct $\mathcal{E}_0$ as follows:

In the case where π is flat, we choose the integer k so that it satisfies also the condition: $\mathrm{R}^p\,\pi_*(\mathcal{E}^\vee \otimes \mathcal{A}^{\otimes k}) = 0$ for any $p > 0$. Then $\pi_*(\mathcal{E}^\vee \otimes \mathcal{A}^{\otimes k})$ is locally free. We set

$$\mathcal{E}_0 := \pi^*\left(\pi_*(\mathcal{E}^\vee \otimes \mathcal{A}^{\otimes k})^\vee\right) \otimes \mathcal{A}^{\otimes k}.$$

Then, $\mathcal{E}$ is regarded as a subbundle of $\mathcal{E}_0$ by the dual of the surjection above, and $\mathrm{R}^p\,\pi_*(\mathcal{F} \otimes \mathcal{E}_0) = 0$ for any $p > 0$.

In the case where Y admits an ample invertible sheaf $\mathcal{H}$, we have a surjection

$$\mathcal{O}_Y^{\oplus N} \to \pi_*(\mathcal{E}^\vee \otimes \mathcal{A}^{\otimes k}) \otimes \mathcal{H}^{\otimes l}$$

for some positive integers l, N. We set

$$\mathcal{E}_0 := \pi^*(\mathcal{H}^{\otimes l})^{\oplus N} \otimes \mathcal{A}^{\otimes k}.$$

Then, $\mathcal{E}$ is a subbundle of $\mathcal{E}_0$ and $\mathrm{R}^p\, \pi_*(\mathcal{F} \otimes \mathcal{E}_0) = 0$ for any $p > 0$.

Considering the same procedure to the quotient bundle $\mathcal{E}_0/\mathcal{E}$, and repeating, we have a long exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}_0 \to \cdots \to \mathcal{E}_{d-1}$$

of locally free sheaves such that $\mathrm{R}^p\, \pi_*(\mathcal{F} \otimes \mathcal{E}_i) = 0$ for any $p > 0$, $0 \le i \le d-1$. We set $\mathcal{E}_d$ to be the cokernel of $\mathcal{E}_{d-2} \to \mathcal{E}_{d-1}$. Then, $\mathcal{E}_d$ is locally free and

$$\mathrm{R}^p\, \pi_*(\mathcal{F} \otimes \mathcal{E}_d) \simeq \mathrm{R}^{p+d}\, \pi_*(\mathcal{F} \otimes \mathcal{E}) = 0$$

for any $p > 0$. Thus, we have an expected exact sequence. □

Therefore, we have a well-defined homomorphism $\pi_*^{\mathcal{F}} \colon K^\bullet(X) \to K^\bullet(Y)$ by

$$\pi_*^{\mathcal{F}}(\mathrm{cl}^\bullet(\mathcal{E})) := \sum (-1)^i \,\mathrm{cl}^\bullet(\pi_*(\mathcal{F} \otimes \mathcal{E}_i))$$

under Assumption 2.1. Here, for any $x \in K^\bullet(X)$, we have

$$\boldsymbol{\phi}(\pi_*^{\mathcal{F}}(x)) = \pi_*(x\, \mathrm{cl}_\bullet(\mathcal{F}))$$

for $\pi_* \colon K_\bullet(X) \to K_\bullet(Y)$ and the Cartan homomorphism $\boldsymbol{\phi}\colon K^\bullet(Y) \to K_\bullet(Y)$.

Remark. Lemma 2.2 shows that $\mathbf{R}\,\pi_*\mathcal{F}$ is a perfect complex even if π is only a locally projective morphism or if Y does not admit ample invertible sheaves. Thus, $\pi_*^{\mathcal{F}}$ is defined as a homomorphism from $K^\bullet(X)$ to the K-group $K^\bullet(Y)_{\mathrm{perf}}$ of the category of $\mathsf{D}(Y)_{\mathrm{perf}}$ of perfect complexes on Y (cf. [13], Exp. IV, Section 2).

Lemma 2.3. *Let $h\colon Y' \to Y$ be a morphism from another Noetherian scheme Y', $X' = X \times_Y Y'$, and let $q_1\colon X' \to X$ and $q_2\colon X' \to Y'$ be natural projections. Then $\mathcal{F}' = q_1^*\mathcal{F}$ is flat over Y', and the equality*

$$q_{2*}^{\mathcal{F}'}(q_1^*x \cdot q_2^*y') = h^*(\pi_*^{\mathcal{F}}(x)) \cdot y'$$

holds in $K^\bullet(Y')_{\mathrm{perf}}$ for $x \in K^\bullet(X)$ and $y' \in K^\bullet(Y')$. If π is flat or if Y and Y' admit ample invertible sheaves, then the same equality holds in $K^\bullet(Y')$.

Proof. We may assume that $y' = 1$ by the projection formula

$$\mathbf{R}\, q_{2*}(\mathcal{F}' \otimes \mathcal{E}' \otimes q_2^*\mathcal{G}') \simeq \mathbf{R}\, q_{2*}(\mathcal{F}' \otimes \mathcal{E}') \otimes \mathcal{G}'$$

for locally free sheaves $\mathcal{E}'$ on X' and $\mathcal{G}'$ on Y'. Since $\mathcal{F}$ is flat over Y, $\mathcal{F}' = q_1^*\mathcal{F}$ is also flat over Y' and is quasi-isomorphic to $\mathbf{L}\, q_1^*\mathcal{F}$. There is a natural base change morphism

$$\Theta\colon\ \mathbf{L}\, h^*\, \mathbf{R}\, \pi_*(\mathcal{F} \otimes \mathcal{E}) \to \mathbf{R}\, q_{2*}(\mathbf{L}\, q_1^*(\mathcal{F} \otimes \mathcal{E})) \simeq_{\mathrm{qis}} \mathbf{R}\, q_{2*}(\mathcal{F}' \otimes q_1^*\mathcal{E})$$

for a locally free sheaf $\mathcal{E}$ on X. It is enough to prove that Θ is a quasi-isomorphism. If $\mathrm{R}^i\,\pi_*(\mathcal{F}\otimes\mathcal{E}) = 0$ for any $i > 0$, then Θ is a quasi-isomorphism by the upper semi-continuity theorem applied to the coherent sheaf $\mathcal{F}\otimes\mathcal{E}$ flat over Y. Hence, by Lemma 2.2, if π is flat or Y and Y' admit ample invertible sheaves, then Θ is a quasi-isomorphism for any locally free sheaf $\mathcal{E}$. There exist open coverings $\{Y_\alpha\}$ of Y and $\{Y'_\alpha\}$ of Y' such that $Y'_\alpha \subset h^{-1}(Y_\alpha)$ and that Y and Y' admits ample invertible sheaves. Thus, Θ restricted to the derived category of Y'_α is a quasi-isomorphism for any α. Hence, Θ itself is also a quasi-isomorphism. □

Definition 2.4. Let $\mathcal{L}_1, \ldots, \mathcal{L}_k$ be invertible sheaves of X, where $k \geq d+1$. Under Assumption 2.1, we define

$$i_{\mathcal{F}/Y}(\mathcal{L}_1,\ldots,\mathcal{L}_k) := \varepsilon\left(\pi_*^{\mathcal{F}}\,\boldsymbol{\delta}_X^{(k)}(\mathcal{L}_1,\ldots,\mathcal{L}_k)\right) \in \mathrm{H}^0(Y,\mathbb{Z}) = \mathbb{Z} \qquad \text{for} \quad k \geq d,$$
$$\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1,\ldots,\mathcal{L}_k) := \det\left(\pi_*^{\mathcal{F}}\,\boldsymbol{\delta}_X^{(k)}(\mathcal{L}_1,\ldots,\mathcal{L}_k)\right) \in \mathrm{Pic}(Y) \qquad \text{for} \quad k \geq d+1,$$

where $\varepsilon\colon K^\bullet(Y) \to \mathrm{H}^0(Y,\mathbb{Z}) = \mathbb{Z}$ is the augmentation map and $\det\colon K^\bullet(Y) \to \mathrm{Pic}(Y)$ is the determinant map. We call $i_{\mathcal{F}/Y}(\underline{\mathcal{L}})$ the relative intersection number and $\mathcal{I}_{\mathcal{F}/Y}(\underline{\mathcal{L}})$ the intersection sheaf for $\underline{\mathcal{L}} = (\mathcal{L}_1,\ldots,\mathcal{L}_k)$. If $\mathcal{F} = \mathcal{O}_X$, then we write $i_{X/Y} = i_{\mathcal{F}/Y}$ and $\mathcal{I}_{X/Y} = \mathcal{I}_{\mathcal{F}/Y}$.

Remark 2.5. By Remark 1.4, we can write

$$\mathcal{I}_{X/Y}(\mathcal{L}_1,\ldots,\mathcal{L}_{d+1}) = \bigotimes\nolimits_{I\subset\{1,\ldots,d+1\}} \left(\det(\mathbf{R}\,\pi_*\mathcal{L}_I^{-1})\right)^{(-1)^{\sharp I}},$$

where $\mathcal{L}_I = \mathcal{L}_{i_1}\otimes\cdots\otimes\mathcal{L}_{i_k}$ for $I = \{i_1,\ldots,i_k\}$ with $\sharp I = k > 0$, and $\mathcal{L}_I = \mathcal{O}_X$ for the empty set $I = \emptyset$. A similar but different formula is written in [5], page 34 (cf. [6], Section IV.1).

Remark. There is also the augmentation map $\varepsilon\colon K^\bullet(Y)_{\mathrm{perf}} \to \mathrm{H}^0(Y,\mathbb{Z})$ and the determinant map $\det\colon K^\bullet(Y)_{\mathrm{perf}} \to \mathrm{Pic}(Y)$, which are lifts of the same maps from $K^\bullet(Y)$, respectively. In fact, ε is defined by ranks of locally free sheaves, and the existence of det is proved by Knudsen–Mumford [15]. Therefore, even if π is only a locally projective morphism and even if Y has no ample invertible sheaves, one can define the relative intersection number $i_{\mathcal{F}/Y}(\underline{\mathcal{L}})$ and the intersection sheaf $\mathcal{I}_{\mathcal{F}/Y}(\underline{\mathcal{L}})$ by using ε and det from $K^\bullet(Y)_{\mathrm{perf}}$.

Lemma 2.6. $\varepsilon(\pi_*^{\mathcal{F}}(x)) = 0$ *for any* $x \in F^{d+1}(X)$. *In particular,* $i_{\mathcal{F}/Y}$ *gives rise to a homomorphism* $G^d(X) \to G^0(Y) \simeq \mathbb{Z}$. *Furthermore,* $i_{\mathcal{F}/Y}(x) = i(x|_{\mathsf{F}};\mathcal{F}\otimes\mathcal{O}_{\mathsf{F}})$ *for any fiber* F *of* π *and for any* $x \in G^d(X)$.

Proof. By Lemma 2.3, $\varepsilon(\pi_*^{\mathcal{F}}(x)) = i(x|_{\mathsf{F}}; \mathcal{F} \otimes \mathcal{O}_{\mathsf{F}})$ for any $x \in F^d(X)$. So, we may assume Y to be $\operatorname{Spec} \Bbbk$ for a field $\Bbbk$. Then $\varepsilon(\pi_*^{\mathcal{F}}(x)) = i(x; \mathcal{F}) = 0$ for $x \in F^{d+1}(X)$, since $x \operatorname{cl}_\bullet(\mathcal{F}) \in F^{d+1}(X)F_d(X) \subset F_{-1}(X) = 0$. □

Proposition 2.7. *Let $\mathcal{L}_1, \ldots, \mathcal{L}_k$ be invertible sheaves on X with surjective homomorphisms $\pi^*\mathcal{G}_i \to \mathcal{L}_i$ for locally free sheaves $\mathcal{G}_i$ of finite rank on Y. If $k \geq d+2$, then $\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_k) \simeq \mathcal{O}_X$. If $k = d+1$, then there is a surjection*

$$\Phi \colon \operatorname{Sym}^{e_1}(\mathcal{G}_1) \otimes \cdots \otimes \operatorname{Sym}^{e_{d+1}}(\mathcal{G}_{d+1}) \to \mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}),$$

where $e_i = i_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{i-1}, \mathcal{L}_{i+1}, \ldots, \mathcal{L}_{d+1})$ for $1 \leq i \leq d+1$.

Proof. Let $q^{(i)} \colon \mathbb{P}_Y^{(i)} := \mathbb{P}_Y(\mathcal{G}_i^\vee) \to Y$ be the projective space bundle associated to $\mathcal{G}_i^\vee$, $\mathbb{P}_X^{(i)} := X \times_Y \mathbb{P}_Y^{(i)}$, and let $p_1^{(i)} \colon \mathbb{P}_X^{(i)} \to X$ and $p_2^{(i)} \colon \mathbb{P}_X^{(i)} \to \mathbb{P}_Y^{(i)}$ be natural projections. For the tautological line bundle $\mathcal{O}(1)$ of $\mathbb{P}_Y^{(i)}$ with respect to $\mathcal{G}_i^\vee$, we have a natural homomorphism

$$p_1^{(i)*}\mathcal{L}_i^\vee \to p_1^{(i)*}\pi^*\mathcal{G}_i^\vee = p_2^{(i)*}q^{(i)*}\mathcal{G}_i^\vee \to p_2^{(i)*}\mathcal{O}(1),$$

and thus a global section $\sigma^{(i)}$ of $p_1^{(i)*}\mathcal{L}_i \otimes p_2^{(i)*}\mathcal{O}(1)$ which defines an effective Cartier divisor $B^{(i)} = \operatorname{div}(\sigma^{(i)})$ on $\mathbb{P}_X^{(i)}$. Then $B^{(i)} \to X$ is a projective space bundle isomorphic to $\mathbb{P}_X(\mathcal{K}_i^\vee)$ for the kernel $\mathcal{K}_i$ of $\pi^*\mathcal{G}_i \to \mathcal{L}_i$. Let $q \colon \mathbb{P}_Y \to Y$ be the fiber product $\mathbb{P}_Y = \mathbb{P}_Y^{(1)} \times \cdots \times \mathbb{P}_Y^{(k)}$ of the projective space bundles, $\mathbb{P}_X := X \times_Y \mathbb{P}_Y \simeq \mathbb{P}_X^{(1)} \times_X \cdots \times_X \mathbb{P}_X^{(k)}$, and let $p_1 \colon \mathbb{P}_X \to X$, $p_2 \colon \mathbb{P}_X \to \mathbb{P}_Y$, and $\pi^{(i)} \colon \mathbb{P}_X \to \mathbb{P}_X^{(i)}$ for $1 \leq i \leq k$ be natural projections. Then

$$V := \bigcap\nolimits_{i=1}^{k} {\pi^{(i)}}^{-1}(B^{(i)}) \simeq B^{(1)} \times_X \cdots \times_X B^{(k)}.$$

The sections $\sigma^{(i)}$ give rise to a global section σ of the locally free sheaf

$$\mathcal{E} = \bigoplus\nolimits_{i=1}^{k} p_1^*\mathcal{L}_i \otimes p_2^*\mathcal{O}(1)^{(i)},$$

where $\mathcal{O}(1)^{(i)}$ is the pullback of $\mathcal{O}(1)$ by $\mathbb{P}_Y \to \mathbb{P}_Y^{(i)}$. Furthermore, V coincides with the zero subscheme $V(\sigma)$ of σ (cf. Definition 1.5). Since V is smooth over X, we infer that σ is a regular section of $\mathcal{E}$. Moreover, σ is $p_1^*\mathcal{F}$-regular, since $V \to X$ is flat. Hence, by Lemma 1.6, we have

$$\operatorname{cl}^\bullet(\mathcal{O}_V) = \lambda_{-1}(\mathcal{E}^\vee) = \boldsymbol{\delta}_{\mathbb{P}_X}^{(k)}\left(p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{O}(1)^{(1)}, \ldots, p_1^*\mathcal{L}_k \otimes p_2^*\mathcal{O}(1)^{(k)}\right).$$

Note that

$$p_{2*}^{p_1^*\mathcal{F}}(p_1^*x \cdot p_2^*y') = p_{2*}^{p_1^*\mathcal{F}}(p_1^*x) \cdot y' = q^*\pi_*^{\mathcal{F}}(x) \cdot y'$$

for $x \in K^\bullet(X)$ and $y' \in K^\bullet(\mathbb{P}_Y)$ by Lemma 2.3. Thus

$$\begin{aligned} &p_2^{p_1^*\mathcal{F}}(\mathrm{cl}^\bullet(\mathcal{O}_V)) - q^*\pi_*^{\mathcal{F}}\,\boldsymbol{\delta}(\mathcal{L}_1, \ldots, \mathcal{L}_k) \\ &\qquad \equiv \textstyle\sum_{i=1}^k q^*\pi_*^{\mathcal{F}}\,\boldsymbol{\delta}_X^{(k-1)}(\mathcal{L}_1, \ldots, \mathcal{L}_{i-1}, \mathcal{L}_{i+1}, \ldots, \mathcal{L}_k) \cdot \boldsymbol{\delta}(\mathcal{O}(1)^{(i)}) \mod F^2(\mathbb{P}_Y) \\ &\qquad \equiv \textstyle\sum_{i=1}^k i_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{i-1}, \mathcal{L}_{i+1}, \ldots, \mathcal{L}_k)\,\boldsymbol{\delta}(\mathcal{O}(1)^{(i)}) \mod F^2(\mathbb{P}_Y). \end{aligned}$$

Since $\det\colon K^\bullet(\mathbb{P}_Y) \to \mathrm{Pic}(\mathbb{P}_Y)$ is trivial on $F^2(\mathbb{P}_Y)$, we have an isomorphism

$$\det p_{2*}^{p_1^*\mathcal{F}}(\mathrm{cl}^\bullet(\mathcal{O}_V)) \simeq q^*\,\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_k) \otimes \textstyle\bigotimes_{i=1}^k (\mathcal{O}(1)^{(i)})^{\otimes e_i} \tag{II–1}$$

for $e_i = i_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{i-1}, \mathcal{L}_{i+1}, \ldots, \mathcal{L}_k)$. Note that

$$\dim(\operatorname{Supp} p_1^*\mathcal{F} \cap V)/Y = \dim(\operatorname{Supp} p_1^*\mathcal{F} \cap V)/\operatorname{Supp}\mathcal{F} + d = \dim \mathbb{P}_Y/Y - k + d.$$

If $k > d$, then $p_2(\operatorname{Supp} p_1^*\mathcal{F} \cap V) \neq \mathbb{P}_Y$ and moreover, $p_2(\operatorname{Supp} p_1^*\mathcal{F} \cap V)$ does not contain any fiber of $q\colon \mathbb{P}_Y \to Y$. Therefore, by the same arguments as in [17], Chapter 5, §§3–4 (cf. [7], [15]), we have an effective Cartier divisor

$$D = D_{\mathcal{F},\underline{\mathcal{L}}} = \mathrm{Div}(p_{2*}(p_1^*\mathcal{F} \otimes \mathcal{O}_V \otimes \mathcal{A}))$$

on $\mathbb{P}_Y$ for an invertible sheaf $\mathcal{A}$ of $\mathbb{P}_X$ such that

$$\mathrm{R}^i\, p_{2*}(p_1^*\mathcal{F} \otimes \mathcal{O}_V \otimes \mathcal{A}) = 0$$

for any $i > 0$. Here, D does not depend on the choice of $\mathcal{A}$, D is a relative Cartier divisor with respect to $q\colon \mathbb{P}_Y \to Y$, i.e., D is flat over Y, and $\operatorname{Supp} D \subset p_2(\operatorname{Supp} p_1^*\mathcal{F} \cap V)$. By construction, we have an isomorphism

$$\det p_{2*}^{p_1^*\mathcal{F}}(\mathrm{cl}^\bullet(\mathcal{O}_V)) \simeq \det(\mathbf{R}\, p_{2*}(p_1^*\mathcal{F} \otimes \mathcal{O}_V)) \simeq \mathcal{O}_{\mathbb{P}_Y}(D).$$

Hence, if $k > d+1$, then $D = 0$ and $i_{\mathcal{F}}(\mathcal{L}_1, \ldots, \mathcal{L}_{i-1}, \mathcal{L}_{i+1}, \ldots, \mathcal{L}_k) = 0$ for any i; thus $\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_k) \simeq \mathcal{O}_Y$ by (II–1). Assume that $k = d+1$. Then (II–1) implies that

$$q^*\,\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}) \otimes \textstyle\bigotimes_{i=1}^{d+1} \left(\mathcal{O}(1)^{(i)}\right)^{\otimes e_i}$$

has a non-zero global section defining the divisor D. The section induces a section of

$$\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}) \otimes \textstyle\bigotimes_{i=1}^{d+1} \mathrm{Sym}^{e_i}(\mathcal{G}_i^\vee),$$

and, equivalently, a homomorphism

$$\Phi\colon \textstyle\bigotimes_{i=1}^{d+1} \mathrm{Sym}^{e_i}(\mathcal{G}_i) \to \mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}).$$

It remains only to show that Φ is surjective. The composition of

$$\textstyle\bigotimes_{i=1}^k \mathcal{O}(-1)^{(i)} \to q^* \bigotimes_{i=1}^{d+1} \mathrm{Sym}^{e_i}(\mathcal{G}_i)$$

and $q^*\Phi$ is an injection between invertible sheaves whose cokernel defines D. In particular, $q^*\Phi$ is surjective outside D. Since D does not contain any fiber of q, we conclude that Φ is surjective. $\square$

Remark 2.8. Let $D = D_{\mathcal{F},\underline{\mathcal{L}}}$ be the effective Cartier divisor of $\mathbb{P}_Y = \mathbb{P}(\mathcal{G}_1^\vee) \times_Y \cdots \times_Y \mathbb{P}(\mathcal{G}_{d+1}^\vee)$ in the proof of Proposition 2.7. In view of the proof, we infer that, for a point $y \in Y$, the effective divisor $D|_{q^{-1}(y)}$ is characterized by the following two conditions:

(1) For $1 \le i \le d$, let $\mathcal{O}(1)^{(i)}$ be the pullback of the tautological invertible sheaf of $\mathbb{P}_Y(\mathcal{G}_i^\vee)$ with respect to $\mathcal{G}_i^\vee$. Then

$$\mathcal{O}_{q^{-1}(y)}(D|_{q^{-1}(y)}) \simeq \bigotimes\nolimits_{i=1}^{d+1} (\mathcal{O}_Y(1)^{(i)})^{\otimes e_i}.$$

(2) Let v_i be a non-zero element of $\mathcal{G}_i \otimes \Bbbk(y)$ for $1 \le i \le d+1$. For $v = (v_1, \ldots, v_{d+1})$, let $[v]$ be a point of $q^{-1}(y)$ corresponding to the surjections $v_i^\vee \colon \mathcal{G}_i^\vee \otimes \Bbbk(y) \to \Bbbk(y)$. Let v_i^X be the global section of $\mathcal{L}_i \otimes \mathcal{O}_{\pi^{-1}(y)}$ defined by $\pi^*\mathcal{G}_i \to \mathcal{L}_i$, and $v^X := (v_1^X, \ldots, v_{d+1}^X)$ as a global section of $(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{d+1}) \otimes \mathcal{O}_{\pi^{-1}(y)}$. Then $[v] \notin \operatorname{Supp} D$ if and only if $V(v^X) \cap \operatorname{Supp} \mathcal{F} = \emptyset$ for the zero subscheme $V(v^X) \subset \pi^{-1}(y)$.

Remark. Assume that $Y = \operatorname{Spec} A$ for a ring A, $X = \mathbb{P}^d_A$, $\mathcal{F} = \mathcal{O}_X$, and $\mathcal{L}_i = \mathcal{O}_{\mathbb{P}^N}(m_i)$ for some $m_i > 0$. Then, for $\mathcal{G}_i = \mathrm{H}^0(X, \mathcal{L}_i) \simeq \operatorname{Sym}^{m_i}(A^{\oplus(d+1)})$, the homomorphism Φ in Proposition 2.7 defines the *resultants*: An element $v_i \in \mathcal{G}_i$ is regarded as a homogeneous polynomial of degree m_i with coefficients in A. Then

$$\Phi(v_1^{e_1} \otimes \cdots \otimes v_{d+1}^{e_{d+1}})$$

is the *resultant* of $v_1, \ldots, v_{d+1}$ up to unit (cf. [5], Section 6.1).

Lemma 2.9 (cf. [6], Section III)**.**

(1) $\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_{\tau(1)}, \ldots, \mathcal{L}_{\tau(d+1)}) \simeq \mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1})$ *for any* $\tau \in \operatorname{Aut}(\{1, \ldots, d+1\}) \simeq \mathfrak{S}_{d+1}$.

(2) *For another invertible sheaf* $\mathcal{L}_1'$*, one has an isomorphism*

$$\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1 \otimes \mathcal{L}_1', \mathcal{L}_2, \ldots, \mathcal{L}_{d+1}) \simeq \mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{d+1}) \otimes \mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1', \mathcal{L}_2, \ldots, \mathcal{L}_{d+1}).$$

(3) *If* σ_1 *is an* $\mathcal{F}$*-regular section of* $\mathcal{L}_1$ *and if* $\mathcal{F} \otimes \mathcal{O}_{B_1}$ *is flat over* Y *for* $B_1 = V(\sigma_1)$*, then*

$$\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}) \simeq \mathcal{I}_{\mathcal{F}\otimes\mathcal{O}_{B_1}/Y}(\mathcal{L}_2|_{B_1}, \ldots, \mathcal{L}_{d+1}|_{B_1}).$$

(4) *If* $d = 0$*, then* $\mathcal{I}_{X/Y}(\mathcal{L})$ *is the norm sheaf of an invertible sheaf* $\mathcal{L}$ *on* X*, i.e.,*

$$\mathcal{I}_{X/Y}(\mathcal{L}) \simeq \det(\pi_*\mathcal{O}_X) \otimes \det(\pi_*\mathcal{L}^{-1})^{-1} \simeq \det(\pi_*\mathcal{L}) \otimes \det(\pi_*\mathcal{O}_X)^{-1}.$$

Proof. (1) follows from Definition 1.3, and (2) from Remark 1.4 and Proposition 2.7.

(3): The section σ_1 induces an exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{L}_1^{-1} \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_{B_1} \to 0.$$

Thus, we have equalities

$$\boldsymbol{\delta}(\mathcal{L}_1)\,\mathrm{cl}_\bullet(\mathcal{F}) = \mathrm{cl}(\mathcal{F} \otimes \mathcal{O}_{B_1}),$$
$$\boldsymbol{\delta}_X(\mathcal{L}_1, \dots, \mathcal{L}_{d+1})\,\mathrm{cl}_\bullet(\mathcal{F}) = i_*\left(\boldsymbol{\delta}_{B_1}(\mathcal{L}_2|_{B_1}, \dots, \mathcal{L}_{d+1}|_{B_1})\,\mathrm{cl}_\bullet(\mathcal{F} \otimes \mathcal{O}_{B_1})\right)$$

in $K_\bullet(X)$ for the closed immersion $i \colon B_1 \subset X$. Hence, the expected isomorphism is derived.

(4) follows from Remark 1.4, (2). □

We recall the following well-known result on Segre classes (cf. [6], Section V):

Lemma 2.10. *Suppose that $X = \mathbb{P}_Y(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ of rank r on Y. Then for the tautological line bundle $\mathcal{O}(1)$ with respect to $\mathcal{E}$, one has*

$$\pi_*(\boldsymbol{\delta}(\mathcal{O}(1))^l) = \begin{cases} 1, & \text{if } 0 \le l < r; \\ \boldsymbol{\delta}(\det \mathcal{E}), & \text{if } l = r. \end{cases}$$

Furthermore, the following equality holds for any $i \ge 0$:

$$\textstyle\sum_{k=0}^{r} \gamma^{r-k}(\mathrm{cl}^\bullet(\mathcal{E}^\vee) - r)\pi_*(\boldsymbol{\delta}(\mathcal{O}(1))^{i+k}) = 0.$$

In particular, for $i \ge 0$,

$$\boldsymbol{s}^i(\mathcal{E}) = \pi_*(\boldsymbol{\delta}(\mathcal{O}(1))^{r+i-1}) \mod F^{i+1}(X) \in G^i(X)$$

can be regarded as the i-th Segre class of $\mathcal{E}$.

Proof. The first assertion follows from

$$\mathrm{R}^p\,\pi_*\mathcal{O}(-l) = \begin{cases} 0, & \text{for } l < r \text{ or } p < r-1; \\ \det \mathcal{E}^\vee, & \text{for } l = r \text{ and } p = r-1, \end{cases}$$

since

$$\boldsymbol{\delta}(\mathcal{O}(1))^l = \textstyle\sum_{i=0}^{l} (-1)^i \binom{l}{i} \mathrm{cl}^\bullet(\mathcal{O}(-i)).$$

Let $\mathcal{G}$ be the cokernel of the natural injection $\mathcal{O}(-1) \to \pi^*\mathcal{E}^\vee$. Then

$$\gamma_t\left(\mathrm{cl}^\bullet(\mathcal{G}) - (r-1)\right) = \gamma_t\left(\mathrm{cl}^\bullet(\pi^*\mathcal{E}^\vee) - r\right)\gamma_t\left(\mathrm{cl}^\bullet(\mathcal{O}(-1)) - 1\right)^{-1}.$$

The left hand side equals the polynomial

$$\lambda_{t/(1-t)}(\mathrm{cl}^\bullet(\mathcal{G}) - (r-1)) = \textstyle\sum_{p=0}^{r-1} \lambda^p(\mathcal{G}) t^p (1-t)^{r-1-p}$$

of degree $r-1$. The right hand side equals

$$\left(\sum_{i=0}^{r}\gamma^i(\pi^*\operatorname{cl}^\bullet(\mathcal{E}^\vee)-r))t^i\right)\left(\sum_{j\geq 0}\boldsymbol{\delta}(\mathcal{O}(1))^j t^j\right),$$

so the coefficient of t^{r+i} for $i\geq 0$ equals

$$\sum_{k=0}^{r}\gamma^{r-k}(\pi^*\operatorname{cl}^\bullet(\mathcal{E}^\vee)-r)\,\boldsymbol{\delta}(\mathcal{O}(1))^{i+k}.$$

By taking π_*, we have the second expected equality. Comparing with the Chern classes $\boldsymbol{c}^p(\mathcal{E}^\vee)=\gamma^p(\operatorname{cl}^\bullet(\mathcal{E}^\vee)-r) \mod F^{p+1}(Y)$, we have

$$\begin{aligned}\sum_{k=0}^{m}\boldsymbol{c}^{m-k}(\mathcal{E}^\vee)\,\boldsymbol{s}^k(\mathcal{E}) &= \sum_{k=0}^{m}\gamma^{m-k}(\operatorname{cl}^\bullet(\mathcal{E}^\vee)-r)\pi_*(\boldsymbol{\delta}(\mathcal{O}(1))^{r+k-1}) \mod F^{m+1}(Y)\\ &= \sum_{k=0}^{r}\gamma^{r-k}(\operatorname{cl}^\bullet(\mathcal{E}^\vee)-r)\pi_*(\boldsymbol{\delta}(\mathcal{O}(1))^{m-1+k}) \mod F^{m+1}(Y)\\ &= 0\end{aligned}$$

for $m>0$. Thus, $\boldsymbol{s}^i(\mathcal{E})$ is the i-th Segre class. □

The following is proved essentially by an argument in [6], Section V.

Proposition 2.11. *If $x\in F^{d+2}(X)$, then $\pi_*^{\mathcal{F}}(x)\in F^2(Y)$. In particular, $\mathcal{I}_{\mathcal{F}/Y}(x):=\det\pi_*^{\mathcal{F}}(x)$ gives rise to a homomorphism $G^{d+1}(X)\to G^1(Y)\simeq \operatorname{Pic}(Y)$.*

Proof. By definition, $F^k(X)$ is generated by elements of the form $\xi=\gamma^{i_1}(x_1)\cdots\gamma^{i_l}(x_l)$ for positive integers i_j with $i_1+\cdots+i_l=k$, where $x_i=\operatorname{cl}(\mathcal{E}_i)-r_i$ for a locally free sheaf $\mathcal{E}_i$ of rank r_i. Thus, $\xi \mod F^{k+1}(X)$ is written as the product $\boldsymbol{c}^{i_1}(\mathcal{E}_1)\cdots\boldsymbol{c}^{i_l}(\mathcal{E}_l)$ of Chern classes. Chern classes are expressed by Segre classes. Thus, $G^k(X)$ is generated by the products $\boldsymbol{s}^{j_1}(\mathcal{E}_1)\cdots\boldsymbol{s}^{j_l}(\mathcal{E}_l)$ of Segre classes. Let r_i be the rank of the locally free sheaf $\mathcal{E}_i$. For the product $p\colon P=\mathbb{P}_X(\mathcal{E}_1)\times_X\cdots\times_X\mathbb{P}_X(\mathcal{E}_l)\to X$ of projective space bundles over X, and for the pullback $\mathcal{O}(1)^{(i)}$ of the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}_X(\mathcal{E}_i)$ by $P\to\mathbb{P}_X(\mathcal{E}_i)$, we have

$$\boldsymbol{s}^{j_1}(\mathcal{E}_1)\cdots\boldsymbol{s}^{j_l}(\mathcal{E}_l)=p_*\left(\boldsymbol{\delta}(\mathcal{O}(1)^{(1)})^{r_1+j_1-1}\,\boldsymbol{\delta}(\mathcal{O}(1)^{(2)})^{r_2+j_2-1}\cdots\boldsymbol{\delta}(\mathcal{O}(1)^{(l)})^{r_l+j_l-1}\right) \mod F^{j_1+\cdots+j_l+1}(X)$$

for $j_1,\ldots,j_l\geq 0$ by Lemma 2.10. If $\sum j_i>d+1$, then $\sum(r_i+j_i-1)>\dim P/Y+1$, and hence,

$$\mathcal{I}_{p^*\mathcal{F}/Y}\left(\boldsymbol{\delta}(\mathcal{O}(1)^{(1)})^{r_1+j_1-1}\cdots\boldsymbol{\delta}(\mathcal{O}(1)^{(l)})^{r_l+j_l-1}\right)=\mathcal{O}_Y,$$

by Proposition 2.7. Therefore, $\pi_*^{\mathcal{F}}(F^{d+2}(X))\subset F^2(Y)$. □

Lemma 2.12. *Let $h\colon Y'\to Y$ be a morphism from a Noetherian scheme Y', $X'=X\times_Y Y'$, and $\mathcal{F}'=q_1^*\mathcal{F}$ for the natural projection $q_1\colon X'\to X$. Assume that Y' admits an ample invertible sheaf when π is not flat. Then one has an isomorphism*

$$h^*\mathcal{I}_{\mathcal{F}/Y}(\eta)\simeq\mathcal{I}_{\mathcal{F}'/Y'}(q_1^*\eta)$$

for any $\eta \in G^{d+1}(X)$*.*

Proof. Proposition 2.11 above and the base change formula

$$q_{2*}^{\mathcal{F}'}(q_1^* x) = h^*(\pi_*^{\mathcal{F}}(x))$$

for $x \in K^\bullet(X)$ shown in Lemma 2.3 imply the equality

$$q_{2*}^{\mathcal{F}'}(q_1^* \eta) = h^*(\pi_*^{\mathcal{F}}(\eta))$$

in $G^1(Y') = \mathrm{Pic}(Y')$. Thus, the required isomorphism is obtained. □

The following Lemma 2.13 and Corollary 2.14 are similar to the projection formulas shown in [6], Proposition IV.2.2 (b), and [18], Propositions 5.2.1 and 5.2.2.

Lemma 2.13. *Let* $\psi\colon Y \to S$ *be a projective surjective flat morphism to a connected Noetherian scheme* S *with the relative dimension* $e = \dim Y/S$*, and* $\mathcal{G}$ *a locally free sheaf on* Y *of finite rank. Suppose that* $\mathcal{F}$ *is flat over* S *and that* S *admits an ample invertible sheaf when* π *is not flat. Then, there exist isomorphisms*

$$\mathcal{I}_{\mathcal{F}\otimes\pi^*\mathcal{G}/S}(\eta\pi^*\theta) \simeq \mathcal{I}_{\mathcal{G}/S}(\boldsymbol{c}^1(\mathcal{I}_{\mathcal{F}/Y}(\eta))\theta) \quad \textit{and} \quad \mathcal{I}_{\mathcal{F}\otimes\pi^*\mathcal{G}/S}(\eta'\pi^*\theta') \simeq \mathcal{I}_{\mathcal{G}/S}(\theta')^{\otimes i_{\mathcal{F}/Y}(\eta')}$$

for $\eta \in G^{d+1}(X)$*,* $\eta' \in G^d(X)$*,* $\theta \in G^e(Y)$*, and* $\theta' \in G^{e+1}(Y)$*.*

Proof. The assertion follows from the projection formula

$$\pi_*^{\mathcal{F}\otimes\pi^*\mathcal{G}}(x\pi^*y) = \psi_*^{\mathcal{G}}(\pi_*^{\mathcal{F}}(x)y)$$

for any $x \in K^\bullet(X)$ and $y \in K^\bullet(Y)$. This is derived from the quasi-isomorphism

$$\mathbf{R}(\psi\circ\pi)_*\left(\mathcal{F}\otimes\pi^*\mathcal{G}\otimes(\mathcal{E}\otimes\pi^*\mathcal{V})\right) \simeq_{\mathrm{qis}} \mathbf{R}\,\psi_*\left((\mathcal{G}\otimes\mathcal{V})\otimes^{\mathrm{L}}\mathbf{R}\,\pi_*(\mathcal{F}\otimes\mathcal{E})\right)$$

for any locally free sheaves $\mathcal{E}$ on X and $\mathcal{V}$ on Y of finite rank. □

Corollary 2.14. *For* $\theta \in G^d(X)$ *and an invertible sheaf* $\mathcal{M}$ *on* Y*, one has an isomorphism*

$$\mathcal{I}_{\mathcal{F}/Y}(\theta\,\boldsymbol{c}^1(\pi^*\mathcal{M})) \simeq \mathcal{M}^{\otimes i_{\mathcal{F}/Y}(\theta)}.$$

Proof. Apply the second isomorphism in Lemma 2.13 to $\theta \in G^d(X)$ and $\boldsymbol{c}^1(\mathcal{M}) \in G^1(Y)$ in the case where ψ is the identity map of Y. □

Proposition 2.15. *Let* $\mathcal{E}$ *be a locally free sheaf on* X *of rank* $d+1$ *admitting a surjection* $\pi^*\mathcal{G} \to \mathcal{E}$ *for a locally free sheaf* $\mathcal{G}$ *on* Y *of finite rank. Then* $i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E})) \geq 0$ *and there is a natural surjection*

$$\Phi\colon\ \mathrm{Sym}^{i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}(\mathcal{G}) \to \mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E})).$$

Proof. We shall prove by essentially the same argument as in Proposition 2.7. Let $q\colon \mathbb{P} := \mathbb{P}_Y(\mathcal{G}^\vee) \to Y$ be the projective space bundle and $\mathcal{O}(1)$ the tautological invertible sheaf associated with $\mathcal{G}^\vee$. Let $\mathbb{P}_X$ be the fiber product $X \times_Y \mathbb{P}$, and let $p_1\colon \mathbb{P}_X \to X$ and $p_2\colon \mathbb{P}_X \to \mathbb{P}$ be the natural projections. Pulling back the natural injection $\mathcal{O}(-1) \to q^*\mathcal{G}$ to $\mathbb{P}_X$, we can consider the composite

$$p_2^*\mathcal{O}(-1) \to p_2^*q^*\mathcal{G} = p_1^*\pi^*\mathcal{G} \to p_1^*\mathcal{E}$$

and hence a section σ of $p_1^*\mathcal{E} \otimes p_2^*\mathcal{O}(1)$. The zero subscheme $V = V(\sigma)$ is isomorphic to $\mathbb{P}_X(\mathcal{K}^\vee)$ for the kernel $\mathcal{K}$ of $\pi^*\mathcal{G} \to \mathcal{E}$. Since $V \to X$ is smooth, the section σ is regular and furthermore $p_1^*\mathcal{F}$-regular. Thus,

$$\mathrm{cl}^\bullet(\mathcal{O}_V) = \lambda_{-1}\left(p_1^*\mathcal{E}^\vee \otimes p_2^*\mathcal{O}(-1)\right) = (-1)^{d+1}\gamma^{d+1}\left(\mathrm{cl}^\bullet(p_1^*\mathcal{E}^\vee \otimes p_2^*\mathcal{O}(-1)) - (d+1)\right)$$

and $\mathrm{cl}_\bullet(\mathcal{F} \otimes \mathcal{O}_V) = \mathrm{cl}_\bullet(\mathcal{F})\,\mathrm{cl}^\bullet(\mathcal{O}_V)$ by Lemma 1.6.

Claim 2.16.

$$\det p_{2*}^{p_1^*\mathcal{F}}(\mathrm{cl}^\bullet(\mathcal{O}_V)) \simeq q^*\,\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E})) \otimes \mathcal{O}(1)^{\otimes i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}.$$

Proof. We set $l = \mathrm{cl}^\bullet(\mathcal{O}(-1)) \in K^\bullet(\mathbb{P})$, $y = \boldsymbol{\delta}(\mathcal{O}(1)) = 1 - l$, and $x = \mathrm{cl}^\bullet(\mathcal{E}^\vee) \in K^\bullet(X)$. Then

$$\begin{aligned}
\lambda_{-1}(p_1^*x \cdot p_2^*l) &= \textstyle\sum_{k\geq 0}(-1)^k p_1^*(\lambda^k(x)) \cdot (p_2^*l)^k = \sum_{k\geq 0} p_1^*(\lambda^k(x)) \cdot p_2^*(y-1)^k \\
&= \textstyle\sum_{0\leq j\leq k\leq d+1}(-1)^{k-j}\binom{k}{j} p_1^*(\lambda^k(x)) \cdot p_2^*y^j \\
&= \textstyle\sum_{j=0}^{d+1} p_1^*\left(\sum_{k=j}^{d+1}(-1)^{k-j}\binom{k}{j}\lambda^k(x)\right) \cdot p_2^*y^j.
\end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned}
p_{2*}^{p_1^*\mathcal{F}}(\mathrm{cl}^\bullet(\mathcal{O}_V)) &= \textstyle\sum_{j=0}^{d+1} q^*\pi_*^{\mathcal{F}}\left(\sum_{k=j}^{d+1}(-1)^{k-j}\binom{k}{j}\lambda^k(x)\right) \cdot y^j \\
&\equiv q^*\pi_*^{\mathcal{F}}(\lambda_{-1}(x)) + q^*\pi_*^{\mathcal{F}}\left(\textstyle\sum_{k=1}^{d+1}(-1)^{k-1}k\lambda^k(x)\right) \cdot y \mod F^2(\mathbb{P}).
\end{aligned}$$

Hence, Claim 2.16 follows from the equality:

$$\text{(II–2)}\qquad \varepsilon\left(\pi_*^{\mathcal{F}}\left(\textstyle\sum_{k=1}^{d+1}(-1)^{k-1}k\lambda^k(\mathrm{cl}^\bullet(\mathcal{E}^\vee))\right)\right) = \varepsilon(\pi_*^{\mathcal{F}}\,\boldsymbol{c}^d(\mathcal{E})).$$

We shall show (II–2) as follows: Comparing the coefficients of t^d on the both side of the equality

$$\gamma_t(x - (d+1)) = \textstyle\sum_{k=0}^{d+1}\lambda^k(x)t^k(1-t)^{d+1-k},$$

we have

$$\begin{aligned}\gamma^d(x-(d+1)) &= \textstyle\sum_{k=0}^{d}(-1)^{d-k}(d+1-k)\lambda^k(x)\\ &= (-1)^d(d+1)\textstyle\sum_{k=0}^{d+1}(-1)^k\lambda^k(x) - (-1)^d\textstyle\sum_{k=1}^{d+1}(-1)^k k\lambda^k(x)\\ &= (-1)^d(d+1)\lambda_{-1}(x) + (-1)^d\left(\textstyle\sum_{k=1}^{d+1}(-1)^{k-1}k\lambda^k(x)\right).\end{aligned}$$

Here $\varepsilon(\pi_*^{\mathcal{F}}\lambda_{-1}(x)) = \varepsilon(\pi_*^{\mathcal{F}}\,\boldsymbol{c}^{d+1}(\mathcal{E}^\vee)) = 0$ by Lemmas 1.6 and 2.6. Thus, we have the equality (II–2) by

$$\begin{aligned}\varepsilon(\pi_*^{\mathcal{F}}\,\boldsymbol{c}^d(\mathcal{E})) &= (-1)^d\varepsilon(\pi_*^{\mathcal{F}}\,\boldsymbol{c}^d(\mathcal{E}^\vee))\\ &= (-1)^d\varepsilon(\pi_*^{\mathcal{F}}\gamma^d(x-(d+1))) = \varepsilon\left(\pi_*^{\mathcal{F}}\left(\textstyle\sum_{k=1}^{d+1}(-1)^{k-1}k\lambda^k(x)\right)\right).\end{aligned}$$ □

Proof of Proposition 2.15 *continued.* We infer that $p_2(\operatorname{Supp} p_1^*\mathcal{F}\cap V)$ does not contain any fiber of $q\colon \mathbb{P}\to Y$ by $\dim(\operatorname{Supp} p_1^*\mathcal{F}\cap V)/Y = N-1 = \dim\mathbb{P}/Y - 1$. Thus, by arguments in [17], Chapter 5, §§3–4, we have an effective relative Cartier divisor $D = D_{\mathcal{F},\mathcal{E}}$ on $\mathbb{P}$ with respect to q such that $D\subset p_2(\operatorname{Supp} p_1^*\mathcal{F}\cap V)$ and

$$\det p_{2*}^{p_1^*\mathcal{F}}(\mathrm{cl}^\bullet(\mathcal{O}_V)) = \mathcal{O}_{\mathbb{P}}(D).$$

By Claim 2.16, we have a global section of

$$q^*\,\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))\otimes\mathcal{O}(1)^{\otimes i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}.$$

Restricting it to a fiber of q, we infer that $i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))\geq 0$. The global section gives a surjection

$$\Phi\colon\ \mathrm{Sym}^{i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}\,\mathcal{G}\to\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E})).$$

by the same argument as in the proof of Proposition 2.7. □

Remark 2.17 (cf. Remark 2.8). Let $D = D_{\mathcal{F},\mathcal{E}}$ be the effective relative Cartier divisor of $\mathbb{P} = \mathbb{P}_Y(\mathcal{G}^\vee)$ in the proof of Proposition 2.15. From the proof, we infer that, for a point $y\in Y$, the effective divisor $D|_{q^{-1}(y)}$ of the fiber $q^{-1}(y)$ of $q\colon\mathbb{P}\to Y$ is characterized by the following two conditions:

(1) For the tautological invertible sheaf $\mathcal{O}(1)$ of the projective space $q^{-1}(y)$, one has

$$\mathcal{O}_{q^{-1}(y)}(D|_{q^{-1}(y)})\simeq\mathcal{O}(1)^{\otimes i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}.$$

(2) Let v be a non-zero element of $\mathcal{G}\otimes\Bbbk(y)$. Let $[v]$ be a point of $q^{-1}(y)$ corresponding to the surjection $v^\vee\colon\mathcal{G}^\vee\otimes\Bbbk(y)\to\Bbbk(y)$. Let v^X be the global section of $\mathcal{E}\otimes\mathcal{O}_{\pi^{-1}(y)}$ defined by $\pi^*\mathcal{G}\to\mathcal{E}$. Then $[v]\notin\operatorname{Supp} D$ if and only if $V(v^X)\cap\operatorname{Supp}\mathcal{F}=\emptyset$ for the zero subscheme $V(v^X)\subset\pi^{-1}(y)$.

Lemma 2.18. *If $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{d+1}$ for invertible sheaves $\mathcal{L}_i$ and if $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_{d+1}$ for locally free sheaves $\mathcal{G}_i$ of finite rank with surjections $\pi^*\mathcal{G}_i \to \mathcal{L}_i$, then the natural surjection*

$$\operatorname{Sym}^e(\mathcal{G}) \to \operatorname{Sym}^{e_1}(\mathcal{G}_1) \otimes \cdots \otimes \operatorname{Sym}^{e_l}(\mathcal{G}_l)$$

to a component is compatible with the surjections Φ in Propositions 2.7 *and* 2.15, *where*

$$e_i = i_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{i-1}, \mathcal{L}_{i+1}, \ldots, \mathcal{L}_{d+1}) \quad \text{and} \quad e = i_{\mathcal{F}/Y}(\mathbf{c}^d(\mathcal{E})) = \sum\nolimits_{i=1}^{d+1} e_i.$$

Proof. Let $\mathcal{V}$ be the locally free sheaf $\bigoplus_{i=1}^{d+1} \mathcal{O}(1)^{(i)}$ on $\mathbb{P}_Y = \mathbb{P}_Y(\mathcal{G}_1^\vee) \times_Y \cdots \times_Y \mathbb{P}_Y(\mathcal{G}_{d+1}^\vee)$, where $\mathcal{O}(1)^{(i)}$ is the pullback of the tautological invertible sheaf by $\mathbb{P}_Y \to \mathbb{P}_Y(\mathcal{G}_i^\vee)$. Then there is a birational morphism $\mu\colon \mathbb{P}(\mathcal{V}) \to \mathbb{P}_Y(\mathcal{G}^\vee)$ for the projective space bundle $\varpi\colon \mathbb{P}(\mathcal{V}) \to \mathbb{P}_Y$ such that the tautological invertible sheaf of $\mathbb{P}(\mathcal{V})$ associated to $\mathcal{V}$ is just the pullback of the tautological invertible sheaf of $\mathbb{P}(\mathcal{G}^\vee)$ by μ. Let $\Gamma_i \subset \mathbb{P}(\mathcal{V})$ be the projective subbundle associated with the quotient locally free sheaf $\mathcal{V}/\mathcal{O}(1)^{(i)}$ for $1 \le i \le d+1$. Then Γ_i is a Cartier divisor such that

$$\mathcal{O}(\Gamma_i) \otimes \varpi^*\mathcal{O}(1)^{(i)} \simeq \mu^*\mathcal{O}(1) \quad \text{and} \quad \mu(\Gamma_i) = \mathbb{P}_Y(\mathcal{G}^\vee/\mathcal{G}_i^\vee) \subset \mathbb{P}_Y(\mathcal{G}^\vee).$$

For a point $y \in Y$, let $v = (v_1, \ldots, v_{d+1})$ be a non-zero element of $\mathcal{G} \otimes \Bbbk(y)$, where $v_i \in \mathcal{G}_i \otimes \Bbbk(y)$. Then $[v] \in \mathbb{P}_Y(\mathcal{G}^\vee) \times_Y y$ is not contained in $\mu(\Gamma_i)$ if and only if $v_i \neq 0$. Let $D_0 = D_{\mathcal{F},\underline{\mathcal{L}}}$ be the effective relative Cartier divisor on $\mathbb{P}_Y$ defining Φ in the proof of Proposition 2.7. Let $D_1 = D_{\mathcal{F},\mathcal{E}}$ be the effective relative Cartier divisor on $\mathbb{P}_Y(\mathcal{G}^\vee)$ defining Φ in the proof of Proposition 2.15. Then,

$$\varpi^*D_0 + \sum\nolimits_{i=1}^{d+1} e_i\Gamma_i \sim \mu^*D_1$$

and $\mu_*(\varpi^*D_0) = D_1$ over $\mathbb{P}_Y(\mathcal{G}^\vee) \setminus \bigcup_{i=1}^{d+1} \mu(\Gamma_i)$, by Remarks 2.8 and 2.17. Hence,

$$\varpi^*D_0 + \sum\nolimits_{i=1}^{d+1} e_i\Gamma_i = \mu^*D_1$$

since the invertible sheaves $\mathcal{O}(1)^{(i)}$ are linearly independent in $\operatorname{Pic}(\mathbb{P}_Y)$. The push-forward on Y of the natural injection

$$\varpi^*\left(\bigotimes\nolimits_{i=1}^{d+1} (\mathcal{O}(1)^{(i)})^{\otimes e_i}\right) \hookrightarrow \mu^*\mathcal{O}(1)^{\otimes e}$$

is just the injection

$$\bigotimes\nolimits_{i=1}^{d+1} \operatorname{Sym}^{e_i}(\mathcal{G}_i^\vee) \hookrightarrow \operatorname{Sym}^e(\mathcal{G}^\vee).$$

Hence, two Φ are related by

$$\Phi\colon \operatorname{Sym}^e(\mathcal{G}) \to \bigotimes\nolimits_{i=1}^{d+1} \operatorname{Sym}^{e_i}(\mathcal{G}_i) \xrightarrow{\Phi} \mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}).$$

□

Definition 2.19 ([9])**.** Let $P = P(x_1, x_2, \dots, x_r) \in \mathbb{Z}[x_1, \dots, x_r]$ be a weighted homogeneous polynomial with the weight of x_i being i. If

$$i(P(\boldsymbol{c}^1(\mathcal{E}), \dots, \boldsymbol{c}^r(\mathcal{E})); X) = \int_X P(\boldsymbol{c}^1(\mathcal{E}), \dots, \boldsymbol{c}^r(\mathcal{E})) > 0$$

for any projective variety X defined over a field and for any ample vector bundle $\mathcal{E}$ of rank r on X, then P is called *numerically positive* for ample vector bundles.

Fact 2.20. If $P \in \mathbb{Z}[x_1, \dots, x_r]$ is a weighted homogeneous polynomial of weight x_i being i, then P is expressed uniquely as $\sum_\lambda a_\lambda P_\lambda$ for the Schur polynomial P_λ associated with the 'partition' λ and $a_\lambda \in \mathbb{Z}$. Fulton–Lazarsfeld [9] showed that P is numerically positive for ample vector bundles if and only if $P \neq 0$ and $a_\lambda \geq 0$.

Proposition 2.21. *Let $P \in \mathbb{Z}[x_1, \dots, x_r]$ be a numerically positive polynomial of degree $d + 1$ for ample vector bundles. Let $\mathcal{E}$ be a locally free sheaf on X of rank r generated by global sections, and $\mathcal{F}$ a coherent sheaf on X flat over Y. Then $\mathcal{I}_{\mathcal{F}/Y}(P(\mathcal{E})) := \mathcal{I}_{\mathcal{F}/Y}(P(\boldsymbol{c}^1(\mathcal{E}), \dots, \boldsymbol{c}^r(\mathcal{E})))$ is generated by global sections.*

Proof. We may assume that P is a Schur polynomial P_λ. By [14] and [9], there is a smooth projective morphism $q \colon W \to X$ and a locally free sheaf $\mathcal{H}$ on W of rank N such that $\mathcal{H}$ is generated by global sections and

$$q_* \, \boldsymbol{c}^N(\mathcal{H}) = P(\boldsymbol{c}^1(\mathcal{E}), \dots, \boldsymbol{c}^r(\mathcal{E})) \in G^{d+1}(X).$$

Thus, $\mathcal{I}_{\mathcal{F}/Y}(P(\mathcal{E})) \simeq \mathcal{I}_{q^*\mathcal{F}/Y}(\boldsymbol{c}^N(\mathcal{H}))$, which is globally generated by Proposition 2.15. □

3. Intersection sheaves over normal base schemes

We shall define the intersection sheaves for non-flat equi-dimensional locally projective morphisms defined only over normal separated Noetherian schemes. Let us fix a normal separated Noetherian scheme Y. For the sake of simplicity, we assume that Y is integral. Let $\pi\colon X \to Y$ be a proper surjective morphism from a Noetherian scheme X. We fix a non-negative integer d.

Definition 3.1. Let $\mathcal{V}_\pi^{(d)}(X)$ be the set of closed integral subschemes Z of X such that $\dim(Z \cap \pi^{-1}(y)) \le d$ for any point $y \in Y$ with $\dim \mathcal{O}_{Y,y} \le 1$. We define $K_\pi^{(d)}(X)$ to be the subgroup of $K_\bullet(X)$ generated by the images of $K_\bullet(Z) \to K_\bullet(X)$ for all the closed integral subschemes $Z \in \mathcal{V}_\pi^{(d)}(X)$. We also define $\mathrm{Coh}_\pi^{(d)}(X)$ to be the set of coherent sheaves $\mathcal{F}$ on X such that any irreducible component of $\operatorname{Supp} \mathcal{F}$ belongs to $\mathcal{V}_\pi^{(d)}(X)$.

Note that, for a closed integral subscheme Z, if $\pi(Z) = Y$ and if $\dim(Z \cap \pi^{-1}(*)) \le d$ for the generic point $*$ of Y, then $Z \in \mathcal{V}_\pi^{(d)}(X)$.

Lemma 3.2. *If $\xi \in K_\pi^{(d)}(X)$, then*

$$\pi_*(F^{d+1}(X)\xi) \subset F^1_{\mathrm{con}}(Y) \quad \textit{and} \quad \pi_*(F^{d+2}(X)\xi) \subset F^2_{\mathrm{con}}(Y)$$

for the push-forward homomorphism $\pi_*\colon K_\bullet(X) \to K_\bullet(Y)$.

Proof. Replacing X with a closed subscheme in $\mathcal{V}_\pi^{(d)}(X)$, we may assume that X is integral. Since $F^{d+i}(X)\xi \subset F^{d+i}_{\mathrm{con}}(X)$, it suffices to show $\pi_* F^{d+i}_{\mathrm{con}}(X) \subset F^i_{\mathrm{con}}(Y)$ for $i = 1$, 2. This is derived from the assertion for $i = 1$, 2 that $\operatorname{codim} \pi(Z) \ge i$ for any integral closed subscheme Z of X with $\operatorname{codim} Z \ge d + i$. Let x be the generic point of Z. Then $\dim \mathcal{O}_{X,x} \ge d + i$. For $y = f(x)$, if $\dim \mathcal{O}_{Y,y} \le 1$, then $\dim \mathcal{O}_{Y,y} \ge i$ by

$$\dim \mathcal{O}_{X,x} \le \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X,x} \otimes \Bbbk(y) \le \dim \mathcal{O}_{Y,y} + d.$$

Thus, the assertion is verified for $i = 1$, 2. $\square$

Definition 3.3. Let ξ be an element of $K_\pi^{(d)}(X)$. By Lemma 3.2, one can define

$$i_{\xi/Y}(\theta) := \mathrm{rk}_Y\left(\pi_*(\theta\xi)\right) \in \mathbb{Z} \quad \text{and} \quad \mathcal{I}_{\xi/Y}(\eta) := \widehat{\det}\left(\pi_*(\eta\xi)\right) \in \mathrm{Ref}^1(Y)$$

for $\theta \in G^d(X)$ and $\eta \in G^{d+1}(X)$. The $i_{\xi/Y}(\theta)$ is called the relative intersection number and $\mathcal{I}_{\xi/Y}(\eta)$ is called the intersection sheaf.

Convention.

(1) If $\theta = \boldsymbol{\delta}_X(\mathcal{L}_1, \dots, \mathcal{L}_d) \mod F^{d+1}(X)$ and $\eta = \boldsymbol{\delta}_X(\mathcal{L}_1, \dots, \mathcal{L}_{d+1}) \mod F^{d+1}(X)$ for invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_{d+1}$ on X, then we write

$$i_{\xi/Y}(\theta) = i_{\xi/Y}(\mathcal{L}_1, \dots, \mathcal{L}_d), \quad \mathcal{I}_{\xi/Y}(\eta) = \mathcal{I}_{\xi/Y}(\mathcal{L}_1, \dots, \mathcal{L}_{d+1}).$$

(2) If $\xi = \mathrm{cl}_\bullet(\mathcal{F})$ for a coherent sheaf $\mathcal{F}$ belonging to $\mathrm{Coh}_\pi^{(d)}(X)$, then $\xi \in K_\pi^{(d)}(X)$, and we write $i_{\mathcal{F}/Y}(\cdot) = i_{\xi/Y}(\cdot)$ and $\mathcal{I}_{\mathcal{F}/Y}(\cdot) = \mathcal{I}_{\xi/Y}(\cdot)$.

(3) If $\xi = \mathrm{cl}_\bullet(V)$ for a closed subscheme V whose irreducible components all belong to $\mathcal{V}_\pi^{(d)}(X)$, then $i_{\xi/Y}$ and $\mathcal{I}_{\xi/Y}$ are written by $i_{V/Y}$ and $\mathcal{I}_{V/Y}$, respectively. Similarly, if $\xi = \mathrm{cl}_\bullet(Z)$ for an algebraic cycle Z whose irreducible components all belong to $\mathcal{V}_\pi^{(d)}(X)$, then $i_{\xi/Y}$ and $\mathcal{I}_{\xi/Y}$ are written by $i_{Z/Y}$ and $\mathcal{I}_{Z/Y}$, respectively.

Remark. For a closed immersion $\iota\colon X \hookrightarrow X'$ into another proper Y-scheme X', and for $\theta' \in G^d(X')$, $\eta' \in G^{d+1}(X')$, we have

$$i_{\xi/Y}(\theta'|_X) = i_{\iota_*(\xi)/Y}(\theta') \quad \text{and} \quad \mathcal{I}_{\xi/Y}(\eta'|_X) = \mathcal{I}_{\iota_*(\xi)/Y}(\eta').$$

Thus, the definitions of $i_{\mathcal{F}/Y}$, $i_{V/Y}$, $i_{Z/Y}$, $\mathcal{I}_{\mathcal{F}/Y}$, $\mathcal{I}_{B/Y}$, and $\mathcal{I}_{Z/Y}$ above cause no confusion.

Example 3.4. Assume that $\mathcal{O}_X \in \mathrm{Coh}_\pi^{(d)}(X)$ and $d = 0$; in other words, $\pi\colon X \to Y$ is generically finite. Then $\mathcal{I}_{X/Y}(\mathcal{L})$ for an invertible sheaf $\mathcal{L}$ on X is nothing but the reflexive sheaf

$$\left(\widehat{\det}\,\pi_*\mathcal{L} \otimes \widehat{\det}(\pi_*\mathcal{O}_X)^\vee\right)^{\vee\vee} \simeq \left(\widehat{\det}(\pi_*\mathcal{O}_X) \otimes \widehat{\det}(\pi_*\mathcal{L}^{-1})^\vee\right)^{\vee\vee}.$$

If X is normal and $\mathcal{L} = \mathcal{O}_X(D)$ for a Cartier divisor D, then $\mathcal{I}_{X/Y}(\mathcal{L}) \simeq \mathcal{O}_Y(\pi_*D)$. In fact, we have an isomorphism

$$\left(\widehat{\det}(\pi_*\mathcal{O}_X) \otimes \widehat{\det}(\pi_*\mathcal{O}_X(-\Delta))^\vee\right)^{\vee\vee} \simeq \mathcal{O}_Y(\pi_*\Delta)$$

for an effective Weil divisor Δ of X, and applying it to effective Weil divisors D_1, D_2 with $D = D_1 - D_2$, we have the isomorphism above (cf. Remark 1.4).

Remark 3.5. In this Section 3, we assume that the base scheme Y to be normal. If Y is only a separated integral scheme, then the intersection sheaves $\mathcal{I}_{X/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1})$ are not naturally defined for an equi-dimensional morphism $\pi\colon X \to Y$ of relative dimension d and invertible sheaves $\mathcal{L}_i$ on X. For example, we consider the following situation: Let Y be a nodal rational cubic plane curve defined over $\mathbb{C}$ and $\pi\colon X \to Y$ the normalization. Let $P \in X$ be a point not lying over the node of Y. One can consider the push-forward $\pi_*(P)$ as a divisor of Y. So, the intersection sheaf $\mathcal{I}_{X/Y}(\mathcal{O}(1))$ for the invertible sheaf $\mathcal{O}(1)$ on $X \simeq \mathbb{P}^1$ is expected to be the invertible sheaf $\mathcal{O}_Y(\pi_*P)$. However, if $P' \in X$ is not lying over the node, then $\pi_*(P)$ is linearly equivalent to $\pi_*(P')$ only when $P = P'$. Hence, we have no natural definition of $\mathcal{I}_{X/Y}(\mathcal{O}(1))$.

Lemma 3.6. *Let $\mathcal{F}$ be a coherent sheaf belonging to* $\mathrm{Coh}_\pi^{(d)}(X)$. *Then*

$$\mathcal{I}_{\mathcal{F}/Y}(\eta) \simeq \mathcal{I}_{(\mathcal{F}_{\mathrm{t.f.}/Y})/Y}(\eta)$$

for any $\eta \in G^{d+1}(X)$ (cf. Definition 1.9).

Proof. Since $\mathrm{cl}_\bullet(\mathcal{F}) = \mathrm{cl}_\bullet(\mathcal{F}_{\mathrm{tor}/Y}) + \mathrm{cl}_\bullet(\mathcal{F}_{\mathrm{t.f.}/Y})$, it is enough to show that

$$\pi_*(\eta\,\mathrm{cl}_\bullet(\mathcal{F}_{\mathrm{tor}/Y})) \in F^2_{\mathrm{con}}(Y)$$

for any $\eta \in G^{d+1}(X)$. This follows from an argument in the proof of Lemma 3.2. In fact, $\mathcal{F}_{\mathrm{tor}/Y}$ is defined on $\pi^{-1}W = X \times_Y W$ for a proper closed subscheme $W \subset Y$ and

$$\pi_*(F^{d+1}(X)\,\mathrm{cl}_\bullet(\mathcal{F}_{\mathrm{tor}/Y})) \subset \mathrm{Image}(F^1_{\mathrm{con}}(W) \to F^2_{\mathrm{con}}(Y)). \qquad \square$$

Remark 3.7. In order to study the intersection sheaf $\mathcal{I}_{\mathcal{F}/Y}(\eta)$ for $\mathcal{F} \in \mathrm{Coh}^{(d)}_\pi(X)$, we may assume that $\mathcal{F}_{\mathrm{tor}/Y} = 0$ by Lemma 3.6. Thus, we may remove the irreducible components of X which do not dominate Y, i.e., we may replace $\mathcal{O}_X$ with $(\mathcal{O}_X)_{\mathrm{t.f.}/Y}$. Hence, we may assume that there is an open subset $U \subset Y$ with $\mathrm{codim}(Y \setminus U) \geq 2$ such that π and $\mathcal{F}$ are flat over U. In particular, we have

$$\begin{aligned} i_{\mathcal{F}/Y}(\theta) &= i_{\mathcal{F}_{\mathrm{t.f.}/Y}|_{\pi^{-1}(U)}/U}\left(\theta|_{\pi^{-1}(U)}\right) = i(\theta; \mathcal{F}_{\mathrm{t.f.}/Y} \otimes \mathcal{O}_{\pi^{-1}(y)}), \\ \mathcal{I}_{\mathcal{F}/Y}(\eta) &\simeq j_*\mathcal{I}_{\mathcal{F}_{\mathrm{t.f.}/Y}|_{\pi^{-1}(U)}/U}\left(\eta|_{\pi^{-1}(U)}\right) \end{aligned}$$

for $y \in U$, $\theta \in G^d(X)$, $\eta \in G^{d+1}(X)$, and for the open immersion $j\colon U \subset Y$.

Remark 3.8. If π is projective, $\mathcal{F}$ is flat over Y, and Assumption 2.1 is satisfied, then $\mathcal{I}_{\mathcal{F}/Y}(\eta)$ defined in Definition 3.3 coincides with the intersection sheaf $\mathcal{I}_{\mathcal{F}/Y}(\eta)$ treated in Section 2. In fact, this is derived from that

$$\boldsymbol{\phi}(\pi^{\mathcal{F}}_*(x)) = \pi_*(x\,\mathrm{cl}_\bullet(\mathcal{F})) \quad \text{and} \quad \widehat{\det}\,\boldsymbol{\phi}(y) = \det(y)$$

for any $x \in K^\bullet(X)$ and $y \in K^\bullet(Y)$. Even if we assume only that π is locally projective and $\mathcal{F}$ is flat over Y, we can define $\mathcal{I}_{\mathcal{F}/Y}(x)$ for $x \in F^{d+1}(X)$ by

$$\mathcal{I}_{\mathcal{F}/Y} = \det \circ \pi^{\mathcal{F}}_*\colon K^\bullet(X) \to K^\bullet(Y)_{\mathrm{perf}} \to \mathrm{Pic}(Y).$$

Then $\mathcal{I}_{\mathcal{F}/Y}(x)$ is also isomorphic to the intersection sheaf $\mathcal{I}_{\mathcal{F}/Y}(\mathrm{cl}^\bullet(x))$ defined in Definition 3.3 by similar formulas

$$\boldsymbol{\phi}_{\mathrm{perf}}(\pi^{\mathcal{F}}_*(x)) = \pi_*(x\,\mathrm{cl}_\bullet(\mathcal{F})) \quad \text{and} \quad \widehat{\det}(\boldsymbol{\phi}_{\mathrm{perf}}(y)) = \det(y)$$

for the Cartan homomorphism $\boldsymbol{\phi}_{\mathrm{perf}}\colon K^\bullet(Y)_{\mathrm{perf}} \to K_\bullet(Y)$ and $y \in K^\bullet(Y)_{\mathrm{perf}}$. The latter formula follows from Lemma 1.10 and an argument in [15], Chapter II (cf. [17], Chapter 5, §3).

Lemma 3.9. *Let $\tau\colon Y' \to Y$ be a dominant morphism from another normal separated Noetherian integral scheme Y' such that* $\mathrm{codim}\,\tau^{-1}(B) \geq 2$ *for any closed set $B \subset Y$ of* $\mathrm{codim}\,B \geq 2$. *Let X' be the fiber product $X \times_Y Y'$, and let $p_1\colon X' \to X$ and $p_2\colon X' \to Y'$*

be the natural projections. For a coherent sheaf $\mathcal{F}$ of X belonging to $\mathrm{Coh}^{(d)}_{\pi}(X)$ and for $\eta \in G^{d+1}(X)$, one has an isomorphism

$$\mathcal{I}_{p_1^*\mathcal{F}/Y'}(p_1^*\eta) \simeq \left(\tau^* \mathcal{I}_{\mathcal{F}/Y}(\eta)\right)^{\vee\vee}.$$

Proof. We may replace Y with a Zariski open subset U such that $\operatorname{codim}(Y \setminus U) \geq 2$, since the isomorphism of the reflexive sheaves follows from that on $\tau^{-1}(U)$. Thus, we may assume that Y is regular and τ is flat. Applying the flat base change formula (I–2): $\tau^*\pi_*(x) = p_{2*}p_1^*(x)$ for $x = \eta\,\mathrm{cl}_\bullet(\mathcal{F})$, we have the expected isomorphism, since $p_1^*\,\mathrm{cl}_\bullet(\mathcal{F}) = \mathrm{cl}_\bullet(p_1^*\mathcal{F})$. □

The following corresponds to Corollary 2.14:

Lemma 3.10. *For $\xi \in K^{(d)}_{\pi}(X)$, $\theta \in G^d(X)$ and an invertible sheaf $\mathcal{M}$ on Y, one has an isomorphism*

$$\mathcal{I}_{\xi/Y}(\theta\, \boldsymbol{c}^1(\pi^*\mathcal{M})) \simeq \mathcal{M}^{\otimes i_{\xi/Y}(\theta)}.$$

Proof. We set $x = \theta\xi$ and $y = \boldsymbol{c}^1(\mathcal{M})$. Then the assertion is derived from $\pi_*(x) \bmod F^1_{\mathrm{con}}(Y) = i_{\xi/Y}(\theta)$ and from the projection formula (I–1): $\pi_*(x\pi^*y) = \pi_*(x)y$. □

We shall show that the intersection sheaf $\mathcal{I}_{\mathcal{F}/Y}(\eta)$ is invertible under certain conditions. The following is one of such results:

Theorem 3.11. *Let $\pi\colon X \to Y$ be a proper surjective morphism onto a normal separated Noetherian scheme Y. Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim(\operatorname{Supp}\mathcal{F} \cap \pi^{-1}(y)) \leq d$ for any $y \in Y$.*

1. *If $\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}$ are invertible sheaves of X such that $\pi^*\pi_*\mathcal{L}_i \to \mathcal{L}_i$ is surjective for any i, then $\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1})$ is an invertible sheaf.*
2. *If π is locally projective, then $\mathcal{I}_{\mathcal{F}/Y}(\eta)$ is invertible for any $\eta \in G^{d+1}(X)$.*

The proof is given after Lemmas 3.12 and 3.14.

Lemma 3.12. *Let $\mathcal{F}$ be a coherent sheaf on X belonging to $\mathrm{Coh}^{(d)}_{\pi}(X)$ and let $\mathcal{E}$ be a locally free sheaf on X of rank $d+1$. Let σ be an $\mathcal{F}$-regular section of $\mathcal{E}$. Then, $\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E})) \simeq \mathcal{O}_Y(D)$ for the codimension one part D of the effective algebraic cycle $\pi_*\operatorname{cyc}(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)})$. Moreover,*

$$\mathrm{cl}_\bullet(D) \equiv \mathrm{cl}_\bullet(\pi_*(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)})) \equiv \pi_*\,\mathrm{cl}_\bullet(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)}) \mod F^2_{\mathrm{con}}(Y).$$

Proof. By Lemma 1.6, we have

$$\boldsymbol{c}^{d+1}(\mathcal{E})\,\mathrm{cl}_\bullet(\mathcal{F}) \equiv \mathrm{cl}_\bullet(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)}) \mod F^{d+2}_{\mathrm{con}}(X).$$

Since $\mathrm{cl}_\bullet(\mathrm{cyc}(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)})) = \mathrm{cl}_\bullet(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)})$ and since $\mathrm{cyc}(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)})$ does not dominate Y, we have

$$\mathrm{cl}_\bullet(D) \equiv \pi_* \,\mathrm{cl}_\bullet(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)}) \mod F^2_{\mathrm{con}}(Y),$$

equivalently, $\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E})) \simeq \mathcal{O}_Y(D)$. Since $\operatorname{codim} \operatorname{Supp} \mathrm{R}^i \pi_*(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)}) \geq 2$ for $i > 0$, we have

$$\mathrm{cl}_\bullet(\pi_*(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)})) \equiv \pi_* \,\mathrm{cl}_\bullet(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)}) \mod F^2_{\mathrm{con}}(Y). \qquad \square$$

Remark 3.13. In the situation of Lemma 3.12, if $\pi^{-1}(y) \cap \operatorname{Supp}(\mathcal{F} \otimes \mathcal{O}_{V(\sigma)}) = \emptyset$ for a point $y \in Y$, then $\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))$ is invertible at y, since $y \notin \operatorname{Supp} D$.

Lemma 3.14. *Let V be a Noetherian scheme over a Noetherian local ring A and $\mathcal{L}$ an invertible sheaf of V generated by finitely many global sections $\sigma_0, \ldots, \sigma_N$. Suppose that the residue field $\Bbbk(A) = A/\mathfrak{m}_A$ is an infinite field. For coherent sheaves $\mathcal{F}_1, \ldots, \mathcal{F}_k$ of V, there exists a global section σ of $\mathcal{L}$ such that $\sigma \in \sum_{i=0}^N A\sigma_i \subset \mathrm{H}^0(V, \mathcal{L})$ and σ is $\mathcal{F}_i$-regular for any $1 \leq i \leq k$.*

Proof. For $1 \leq i \leq k$, let J_i be the set of point $\eta \in V$ with $\operatorname{depth}(\mathcal{F}_{i,\eta}) = 0$; in other words, J_i is the set of associated primes of $\mathcal{F}_i$. Let $W(\eta)$ be the closure of $\{\eta\}$ for $\eta \in J_i$. Then, a global section σ of $\mathcal{L}$ is $\mathcal{F}_i$-regular if and only if $\sigma|_{W(\eta)} \neq 0$ as a section of $\mathcal{L}|_{W(\eta)}$ for any $\eta \in J_i$. Let $\{W_1, \ldots, W_l\}$ be the set $\{W(\eta) \mid \eta \in \bigcup J_i\}$.

By the finite global sections $\sigma_0, \ldots, \sigma_N$, we have a morphism $\psi \colon V \to \mathbb{P}^N_A$ such that $\psi^*\mathcal{O}(1) \simeq \mathcal{L}$. It is enough to find an element $\sigma \in R^N_A := \mathrm{H}^0(\mathbb{P}^N_A, \mathcal{O}(1))$ such that $\{\sigma = 0\}$ does not contain $\psi(W_j)$ for any $1 \leq j \leq l$.

We may replace A by the residue field $\Bbbk(A)$. In fact, if we find a global section $\bar{\sigma} \in R^N(\Bbbk(A)) = \mathrm{H}^0(\mathbb{P}^N_{\Bbbk(A)}, \mathcal{O}(1)) \simeq R^N(A) \otimes_A \Bbbk(A)$ which does not vanish along $\psi(W_j)$ for any j, then a lift $\sigma \in R^N(A)$ of $\bar{\sigma}$ also does not vanish along $\psi(W_j)$. Thus, we assume A to be a field $\Bbbk$.

Let $L_j \subset R^N(\Bbbk)$ be the vector subspace consisting of elements vanishing along $\psi(W_j)$. Then L_j is a proper subspace. Since $\Bbbk$ is infinite, we can find an expected element σ in $R^N(\Bbbk) \setminus \bigcup L_j$. $\square$

We shall prove Theorem 3.11.

Proof of Theorem 3.11. (1): By a flat base change (cf. Lemma 3.9), we may assume that $Y = \operatorname{Spec} A$ for a local ring A. If the residue field $\Bbbk(A)$ is finite, then we replace A with the localization $B = A[x]_\mathfrak{m}$ of the polynomial ring $A[x]$ at the maximal ideal $\mathfrak{m} = \mathfrak{m}_A[x] + xA[x]$. Then $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat and the residue field $\Bbbk(B) = \Bbbk(A)(x)$ is infinite. Thus, we may assume that $\Bbbk(A)$ is infinite.

Now, $\mathcal{L}_i$ are all generated by global sections. Applying Lemma 3.14 successively, for the closed point $y \in Y$, we can find global sections $\sigma_i \in \mathrm{H}^0(X, \mathcal{L}_i)$ such that

$(\sigma_1, \ldots, \sigma_{d+1})$ is $\mathcal{F}$-regular and $\mathrm{Supp}(V \cap \pi^{-1}(y)) = \emptyset$ for $V = V(\sigma_1, \ldots, \sigma_{d+1})$. Therefore, $\mathcal{I}_{\mathcal{F}/Y}(\mathcal{L}_1, \ldots, \mathcal{L}_{d+1})$ is invertible at y by Lemma 3.12 and Remark 3.13.

(2): An element of $G^{d+1}(X)$ is expressed as a homogeneous polynomial of Chern classes of degree $d+1$ for suitable locally free sheaves of finite rank. Since the Chern classes are represented by Segre classes, it suffices to consider the case where $\eta = \boldsymbol{s}^{l_1}(\mathcal{E}_1) \cdots \boldsymbol{s}^{l_k}(\mathcal{E}_k)$ for the Segre classes $\boldsymbol{s}^l(\mathcal{E}_j)$ of locally free sheaves $\mathcal{E}_j$. Let $p \colon \mathbb{P} \to X$ be the fiber product of all $\mathbb{P}(\mathcal{E}_j^\vee)$ over X, and let $\mathcal{L}_j$ the pullback of the tautological invertible sheaf $\mathcal{O}(1)$ with respect to $\mathcal{E}_j$ for any j. Then

$$\eta = p_*(\boldsymbol{c}^1(\mathcal{L}_1)^{N_1+l_1} \cdots \boldsymbol{c}^1(\mathcal{L}_k)^{N_k+l_k})$$

for $N_j = \operatorname{rank} \mathcal{E}_j - 1$ by Lemma 2.10. Therefore,

$$\mathcal{I}_{\mathcal{F}/Y}(\eta) = \mathcal{I}_{p^*\mathcal{F}/Y}(\boldsymbol{c}^1(\mathcal{L}_1)^{N_1+l_1} \cdots \boldsymbol{c}^1(\mathcal{L}_k)^{N_k+l_k}).$$

Thus, we are reduced to the case where $\eta = \boldsymbol{c}^1(\mathcal{M}_1) \cdots \boldsymbol{c}^1(\mathcal{M}_{d+1})$ for $\mathcal{M}_1, \ldots, \mathcal{M}_{d+1} \in \mathrm{Pic}(X)$.

As in the proof of (1), we can localize Y. Hence, we may assume that X admits a relatively very ample invertible sheaf with respect to π. Thus, by the linearity of $\mathcal{I}_{\mathcal{F}/Y}$, we may assume that $\mathcal{M}_i$ are all relatively very ample. Then the assertion follows from (1). $\square$

As an application of Theorem 3.11, we have:

Theorem 3.15. *Let $\pi \colon X \to Y$ be an equi-dimensional locally projective surjective morphism between normal separated Noetherian integral schemes. If X is $\mathbb{Q}$-factorial, then so is Y.*

Proof. Let E be a prime divisor of Y. We shall show that some positive multiple of E is Cartier. Thus, we may assume π to be projective by localizing Y. Let $\mathcal{A}$ be a π-ample invertible sheaf on X and set $\theta = \boldsymbol{c}^1(\mathcal{A})^d \in G^d(X)$ for $d = \dim X/Y$. Then $i_{X/Y}(\theta) > 0$. Since π is equi-dimensional, there exists uniquely an effective Weil divisor D on X such that $\pi^*E = D$ on $\pi^{-1}(U)$ for an open subset $U \subset Y$ with $\mathrm{codim}(Y \setminus U) \geq 2$. By assumption, kD is Cartier for some $k > 0$. Thus, $\mathcal{I}_{X/Y}(\theta\, \boldsymbol{c}^1(\mathcal{O}_X(kD)))$ is an invertible sheaf by Theorem 3.11. On the other hand,

$$\mathcal{I}_{X/Y}(\theta\, \boldsymbol{c}^1(\mathcal{O}_X(kD)))|_U \simeq \mathcal{O}_Y(i_{X/Y}(\theta)kE)|_U,$$

by Lemma 3.10. Hence, $i_{X/Y}(\theta)kE$ is Cartier. $\square$

The following is a result analogous to Lemma 2.13.

Lemma 3.16. *Let $\psi\colon Y \to S$ be a proper surjective morphism to a normal separated Noetherian integral scheme S of relative dimension $e = \dim Y/S$, and $\mathcal{G}$ a torsion free coherent sheaf on Y. Assume that*

- *π, ψ, and $\psi \circ \pi$ are locally projective morphisms, and*
- *$\dim(\operatorname{Supp}\mathcal{F} \cap \pi^{-1}(y)) \le d$ and $\dim(\operatorname{Supp}\mathcal{F} \cap \pi^{-1}\psi^{-1}(s)) \le d + e$ for any $y \in Y$ and $s \in S$.*

Then there exist isomorphisms

$$\mathcal{I}_{\mathcal{F}\otimes\pi^*\mathcal{G}/S}(\eta\pi^*\theta) \simeq \mathcal{I}_{\mathcal{G}/S}(\boldsymbol{c}^1(\mathcal{I}_{\mathcal{F}/Y}(\eta))\theta) \quad \text{and} \quad \mathcal{I}_{\mathcal{F}\otimes\pi^*\mathcal{G}/S}(\eta'\pi^*\theta') \simeq \mathcal{I}_{\mathcal{G}/S}(\theta')^{\otimes i_{\mathcal{F}/Y}(\eta')}$$

for $\eta \in G^{d+1}(X)$, $\eta' \in G^d(X)$, $\theta \in G^e(Y)$, and $\theta' \in G^{e+1}(Y)$.

Proof. Let U be an open subset of Y such that $\operatorname{codim}(Y \setminus U) \ge 2$ and that $\mathcal{G}$ is locally free on U. Then, every irreducible component of $Y \setminus U$ belongs to $\mathcal{V}_\psi^{e+1}(Y)$, and

$$\pi_*(x\,\mathrm{cl}_\bullet(\mathcal{F} \otimes \pi^*\mathcal{G}))|_U = \pi_*(x\,\mathrm{cl}_\bullet(\mathcal{F}))|_U\,\mathrm{cl}^\bullet(\mathcal{G}|_U)$$

for any $x \in K^\bullet(X)$. If $x \in F^{d+1}(X)$ represents η, then, $\boldsymbol{c}^1(\mathcal{I}_{\mathcal{F}/Y}(\eta)) = \pi_*(x\,\mathrm{cl}_\bullet(\mathcal{F}))$ mod $F^2_{\mathrm{con}}(Y)$, since $\mathcal{I}_{\mathcal{F}/Y}(\eta)$ is invertible by Theorem 3.11. If $x \in F^d(X)$ represents η', then $i_{\mathcal{F}/Y}(\eta') = \varepsilon(\pi_*(x\,\mathrm{cl}_\bullet(\mathcal{F})))$. Thus,

$$\psi_*\pi_*(\eta\pi^*\theta\,\mathrm{cl}_\bullet(\mathcal{F} \otimes \pi^*\mathcal{G})) = \psi_*(\boldsymbol{c}^1(\mathcal{I}_{\mathcal{F}/Y}(\eta))\theta\,\mathrm{cl}_\bullet(\mathcal{G})) = \boldsymbol{c}^1(\mathcal{I}_{\mathcal{G}/S}(\boldsymbol{c}^1(\mathcal{I}_{\mathcal{F}/Y}(\eta)\theta))),$$
$$\psi_*\pi_*(\eta'\pi^*\theta'\,\mathrm{cl}_\bullet(\mathcal{F} \otimes \pi^*\mathcal{G})) = i_{\mathcal{F}/Y}(\eta')\psi_*(\theta'\,\mathrm{cl}_\bullet(\mathcal{G})) = i_{\mathcal{F}/Y}(\eta')\,\boldsymbol{c}^1(\mathcal{I}_{\mathcal{G}/S}(\theta')).$$

These equalities induce the expected isomorphisms. □

The following is a generalization of Theorem 3.11, (1). This is proved by an argument analogous to Propositions 2.7 and 2.15 in Section 2. In particular, the proof is independent of that of Theorem 3.11.

Proposition 3.17. *Let $\mathcal{G}$ be a locally free sheaf on Y of rank $N+1$ and $\mathcal{E}$ be a locally free sheaf on X of rank $d+1$ admitting a surjection $\pi^*\mathcal{G} \to \mathcal{E}$. Let $q\colon \mathbb{P} = \mathbb{P}(\mathcal{G}^\vee) \to Y$ be the projective space bundle, $\mathcal{O}(1)$ the tautological invertible sheaf on $\mathbb{P}$ with respect to $\mathcal{G}^\vee$, and let $p_1\colon \mathbb{P}_X \to X$ and $p_2\colon \mathbb{P}_X \to \mathbb{P}$ be the natural projections from $\mathbb{P}_X = X \times_Y \mathbb{P}$. Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim(\pi^{-1}(y) \cap \operatorname{Supp}\mathcal{F}) \le d$ for any point $y \in Y$. Then $\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))$ is invertible and $i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E})) \ge 0$. Moreover, there exist an effective relative Cartier divisor D on $\mathbb{P}$ with respect to $q\colon \mathbb{P} \to Y$, an isomorphism*

$$\text{(III–1)} \qquad \mathcal{O}_{\mathbb{P}}(D) \simeq q^*(\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))) \otimes \mathcal{O}(1)^{i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}$$

and a surjection

$$\Phi\colon \operatorname{Sym}^{i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}\mathcal{G} \to \mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E})).$$

Proof. By replacing X with a closed subscheme, we may assume that $\operatorname{Supp}\mathcal{F} = X$. Thus, $\dim \pi^{-1}(y) \leq d$ for any $y \in Y$. From a natural injection $\mathcal{O}(-1) \to q^*\mathcal{G}$, considering the composition

$$p_2^*\mathcal{O}(-1) \to p_2^*q^*\mathcal{G} = p_1^*\pi^*\mathcal{G} \to p_1^*\mathcal{E},$$

we have a global section σ of $p_1^*\mathcal{E} \otimes p_2^*\mathcal{O}(1)$. Then $V(\sigma)$ is isomorphic to $V = \mathbb{P}_X(\mathcal{K}^\vee)$ for the kernel $\mathcal{K}$ of $\pi^*\mathcal{G} \to \mathcal{E}$. Since $V \to X$ is smooth, the closed immersion $V \subset \mathbb{P}_X$ is locally of complete intersection. Thus, the section σ is $\mathcal{O}_{\mathbb{P}_X}$-regular, and furthermore it is $p_1^*\mathcal{F}$-regular, since V is flat over X. Thus, we have

$$\boldsymbol{c}^{d+1}(p_1^*\mathcal{E} \otimes p_2^*\mathcal{O}(1)) \equiv \mathrm{cl}^\bullet(\mathcal{O}_V) \mod F^{d+2}(\mathbb{P}_X),$$
$$\boldsymbol{c}^{d+1}(p_1^*\mathcal{E} \otimes p_2^*\mathcal{O}(1))\,\mathrm{cl}_\bullet(p_1^*\mathcal{F}) \equiv \mathrm{cl}_\bullet(p_1^*\mathcal{F} \otimes \mathcal{O}_V) \mod F^{d+2}_{\mathrm{con}}(\mathbb{P}_X)$$

by Lemma 1.6. Taking p_{2*}, we have the following equality in $G^1(\mathbb{P})$:

$$\begin{aligned} p_{2*}(\mathrm{cl}_\bullet(p_1^*\mathcal{F} \otimes \mathcal{O}_V)) &= p_{2*}\left(\boldsymbol{c}^{d+1}(p_1^*\mathcal{E} \otimes p_2^*\mathcal{O}(1)) \cdot \mathrm{cl}_\bullet(p_1^*\mathcal{F})\right) \\ &= p_{2*}p_1^*(\boldsymbol{c}^{d+1}(\mathcal{E})\,\mathrm{cl}_\bullet(\mathcal{F})) + p_{2*}p_1^*(\boldsymbol{c}^{d}(\mathcal{E})\,\mathrm{cl}_\bullet(\mathcal{F})) \cdot \boldsymbol{c}^1(\mathcal{O}(1)) \\ &= q^*\pi_*(\boldsymbol{c}^{d+1}(\mathcal{E})\,\mathrm{cl}_\bullet(\mathcal{F})) + q^*\pi_*(\boldsymbol{c}^{d}(\mathcal{E})\,\mathrm{cl}_\bullet(\mathcal{F})) \cdot \boldsymbol{c}^1(\mathcal{O}(1)). \end{aligned}$$

Since q is flat, we have

$$\widehat{\det}(q^*\pi_*(\boldsymbol{c}^{d+1}(\mathcal{E})\,\mathrm{cl}_\bullet(\mathcal{F}))) \simeq q^*\,\widehat{\det}(\pi_*(\boldsymbol{c}^{d+1}(\mathcal{E})\,\mathrm{cl}_\bullet(\mathcal{F}))) \simeq q^*\,\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E})),$$
$$q^*(\pi_*(\boldsymbol{c}^{d}(\mathcal{E})\,\mathrm{cl}_\bullet(\mathcal{F}))) \equiv i_{\mathcal{F}/Y}(\boldsymbol{c}^{d}(\mathcal{E}))\,\mathrm{cl}_\bullet(\mathcal{O}_{\mathbb{P}}) \mod F^1_{\mathrm{con}}(\mathbb{P}).$$

Here, we use the fact that the pullback of a reflexive sheaf by a flat morphism is also reflexive. Therefore,

$$\widehat{\det}\,(p_{2*}(\mathrm{cl}_\bullet(p_1^*\mathcal{F} \otimes \mathcal{O}_V))) \simeq q^*\,\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{F})) \otimes \mathcal{O}(1)^{i_{\mathcal{F}/Y}(\boldsymbol{c}^{d}(\mathcal{E}))}.$$

Let D be the codimension one part of $p_{2*}\,\mathrm{cyc}(p_1^*\mathcal{F} \otimes \mathcal{O}_V)$. Then $\operatorname{Supp} D = p_2(V)$ and

$$\mathcal{O}_{\mathbb{P}}(D) \simeq \widehat{\det}\,(p_{2*}(\mathrm{cl}_\bullet(p_1^*\mathcal{F} \otimes \mathcal{O}_V)))\,.$$

In particular, the isomorphism (III–1) is derived. For an arbitrary point $y \in Y$, $\operatorname{Supp} D = p_2(V)$ does not contain the fiber $q^{-1}(y)$, since

$$\dim p_2(V) \cap q^{-1}(y) \leq \dim(V \cap p_1^{-1}\pi^{-1}(y)) \leq N - 1.$$

Hence, $q^*\,\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))$ is invertible at a point of $q^{-1}(y) \setminus \operatorname{Supp} D$ by (III–1) and Remark 3.13. Thus, $\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))$ is invertible at y and D is a relative Cartier divisor with respect to q. Moreover,

$$i_{\mathcal{F}/Y}(\boldsymbol{c}^{d}(\mathcal{E})) = \deg \mathcal{O}_{\mathbb{P}}(D)|_{q^{-1}(y)} \geq 0.$$

The effective divisor D defines a global section of

$$q_*\left(q^*\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))\otimes\mathcal{O}(1)^{i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}\right)=\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))\otimes\operatorname{Sym}^{i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}\mathcal{G}^\vee$$

and the expected homomorphism Φ by the natural pairing $\operatorname{Sym}^l(\mathcal{G})\otimes\operatorname{Sym}^l(\mathcal{G}^\vee)\to\mathcal{O}_Y$. The surjectivity of Φ follows from the same argument as in the proof of Proposition 2.7. □

Remark. If $\mathcal{F}$ is flat over Y, then, by construction, the surjection Φ in Proposition 3.17 is isomorphic to the surjection Φ in Proposition 2.15.

By Proposition 3.17 and by the argument of Proposition 2.21, we have:

Corollary 3.18. *Let $\mathcal{F}$ be a coherent sheaf with $\dim(\operatorname{Supp}\mathcal{F}\cap\pi^{-1}(y))\le d$ for any $y\in Y$, and let $\mathcal{E}$ be a locally free sheaf on X of rank r generated by global sections. If $r=d+1$, then $\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^r(\mathcal{E}))$ is an invertible sheaf generated by global sections. Moreover, if $P(x_1,\dots,x_r)$ is a numerically positive polynomial of degree $d+1$ for ample vector bundles, then $\mathcal{I}_{\mathcal{F}/Y}(P(\boldsymbol{c}^1(\mathcal{E}),\dots,\boldsymbol{c}^r(\mathcal{E})))$ is also an invertible sheaf generated by global sections.*

Proposition 3.19 below gives a base change property related to the flattening of $\mathcal{F}$. In particular, we have another proof of Theorem 3.11, (2) and Proposition 3.17 in the case of projective morphisms π. The proof uses results in Section 2.

Proposition 3.19. *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim(\operatorname{Supp}\mathcal{F}\cap\pi^{-1}(y))\le d$ for any point $y\in Y$. Let $\tau\colon Y'\to Y$ be proper surjective morphism from a Noetherian integral scheme Y', X' the fiber product $X\times_Y Y'$, and let $p_1\colon X'\to X$ and $p_2\colon X'\to Y'$ be the natural projections. Assume that*

- *$\pi\colon X\to Y$ is a projective morphism,*
- *$\mathcal{F}':=(p_1^*\mathcal{F})_{\mathrm{t.f.}/Y}$ is flat over Y',*
- *Y and Y' admit ample invertible sheaves when π is not flat.*

Then, the following assertions hold for any $\eta\in G^{d+1}(X)$:

(1) *For any closed irreducible curve C contained in a fiber of τ,*

$$\mathcal{I}_{\mathcal{F}'/Y'}(p_1^*\eta)|_C\simeq\mathcal{O}_C.$$

(2) *Assume that $\eta=\boldsymbol{c}^{d+1}(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ of rank $d+1$ on X with a surjection $\pi^*\mathcal{G}\to\mathcal{E}$ for a locally free sheaf $\mathcal{G}$ of finite rank on Y. Then $\mathcal{I}_{\mathcal{F}/Y}(\eta)$ is invertible and the surjection*

$$\Phi'\colon\operatorname{Sym}^{i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}(\tau^*\mathcal{G})\to\mathcal{I}_{\mathcal{F}'/Y'}(p_1^*\boldsymbol{c}^{d+1}(\mathcal{E}))$$

on Y' appearing in Proposition 2.15 *descends to a surjection*

$$\Phi\colon \operatorname{Sym}^{i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}(\mathcal{G}) \to \mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E})).$$

(3) $\mathcal{I}_{\mathcal{F}/Y}(\eta)$ *is an invertible sheaf with an isomorphism*

$$\mathcal{I}_{\mathcal{F}'/Y'}(p_1^*\eta) \simeq \tau^*\mathcal{I}_{\mathcal{F}/Y}(\eta).$$

Proof. (1): Let W be the scheme theoretic inverse image $p_2^{-1}(C) = X' \times_{Y'} C$. Then,

$$\mathcal{I}_{\mathcal{F}'\otimes\mathcal{O}_W/C}(p_1^*\eta|_W) \simeq \mathcal{I}_{\mathcal{F}'/Y'}(p_1^*\eta) \otimes \mathcal{O}_C$$

by Lemma 2.12. Here, $p_1^*\eta|_W = 0 \in G^{d+1}(W)$ by $p_1(W) \subset \pi^{-1}(y)$ and $\dim(\pi^{-1}(y) \cap \operatorname{Supp}\mathcal{F}) \leq d$ for $\{y\} = \tau(C)$. Thus, the intersection sheaf on C is trivial by Proposition 2.11.

(2) The surjection Φ' defines a morphism

$$\varphi\colon Y' \to \mathbb{P}_Y(\operatorname{Sym}^{i_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))}\mathcal{G})$$

over Y so that $\mathcal{I}_{\mathcal{F}'/Y'}(\boldsymbol{c}^{d+1}(\mathcal{E})) \simeq \varphi^*\mathcal{O}(1)$ for the tautological invertible sheaf $\mathcal{O}(1)$. Then $\varphi(Y') \to Y$ is a finite morphism by (1). By Remark 3.7, we may assume that $\mathcal{F}$ is flat over an open subset $U \subset Y$ with $\operatorname{codim}(Y \setminus U) \geq 2$. Then $\mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))$ is invertible on U and there is a surjection

$$\Phi_U\colon \operatorname{Sym}^{i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}(\mathcal{G})|_U \to \mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))|_U.$$

We infer that $\tau^*(\Phi_U)$ and $\Phi'|_{\tau^{-1}(U)}$ are isomorphic to each other by the proof of Proposition 2.15 and Remark 2.17. Therefore, $\varphi(Y') \to Y$ is an isomorphism over U. Since Y is normal and $\varphi(Y')$ is integral, we have $\varphi(Y') \simeq Y$. Hence, Φ' descends to a surjection

$$\Phi\colon \operatorname{Sym}^{i_{\mathcal{F}/Y}(\boldsymbol{c}^d(\mathcal{E}))}(\mathcal{G}) \to \mathcal{M}$$

to an invertible sheaf $\mathcal{M}$ with $\mathcal{M}|_U \simeq \mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))$. Thus, $\mathcal{M} \simeq \mathcal{I}_{\mathcal{F}/Y}(\boldsymbol{c}^{d+1}(\mathcal{E}))$.

(3): As in an argument in Proposition 2.11 or Theorem 3.11, (2), we may assume that $\eta = \boldsymbol{c}^1(\mathcal{L}_1)\cdots\boldsymbol{c}^1(\mathcal{L}_{d+1})$ for π-ample invertible sheaves $\mathcal{L}_i$ such that $\pi^*\pi_*\mathcal{L}_i \to \mathcal{L}_i$ is surjective and $\mathrm{R}^p\pi_*\mathcal{L}_i = 0$ for any $p > 0$. If π is flat, then $\pi_*\mathcal{L}_i$ are locally free. If not, then Y admits an ample invertible sheaf, hence there exist surjections $\mathcal{G}_i \to \pi_*\mathcal{L}_i$ from locally free sheaves $\mathcal{G}_i$ of finite rank. Therefore, the assertion follows from (2) above. □

The base change properties in Lemma 3.9 and Proposition 3.19 are generalized to:

Theorem 3.20. *Let $\pi\colon X \to Y$ be a locally projective surjective morphism over a normal separated Noetherian integral scheme Y, and $\mathcal{F}$ a coherent sheaf on X with* $\dim(\operatorname{Supp}\mathcal{F} \cap \pi^{-1}(y)) \leq d$ *for any point $y \in Y$. Let $\tau\colon Y' \to Y$ be a dominant morphism of finite type from another separated Noetherian integral scheme Y'. Let X' be the fiber product*

$X \times_Y Y'$, $p_1 \colon X' \to X$, $p_2 \colon X' \to Y'$ *be the natural projections, and let* $\mathcal{F}'$ *be the sheaf* $(p_1^*\mathcal{F})_{\mathrm{t.f.}/Y'}$. *Then*

$$\mathcal{I}_{\mathcal{F}'/Y'}(p_1^*\eta) \simeq \tau^* \mathcal{I}_{\mathcal{F}/Y}(\eta)$$

for any for any $\eta \in G^{d+1}(X)$ *provided that one of the following conditions is satisfied*:

(1) $\mathcal{F}'$ *is flat over* Y', τ *is proper surjective with* $\mathcal{O}_Y \simeq \tau_*\mathcal{O}_{Y'}$, *and* p_2 *satisfies* Assumption 2.1.

(2) Y' *is normal.*

Proof. First, we treat the case (1). We may assume that $\mathcal{F}$ is flat over an open subset $U \subset Y$ with $\operatorname{codim}(Y \setminus U) \geq 2$. Then

$$\mathcal{I}_{\mathcal{F}'/Y'}(p_1^*\eta)|_{\tau^{-1}(U)} \simeq \tau^*(\mathcal{I}_{\mathcal{F}/Y}(\eta)|_U)$$

by Lemma 2.3 and Remark 3.8; in other words, the invertible sheaf

$$\mathcal{N} := \mathcal{I}_{\mathcal{F}'/Y'}(p_1^*\eta) \otimes \tau^* \mathcal{I}_{\mathcal{F}/Y}(\eta)^{-1}$$

is trivial on $\tau^{-1}(U)$. By Proposition 3.19 and by $\mathcal{O}_Y \simeq \tau_*\mathcal{O}_{Y'}$, we infer that $\mathcal{M} := \tau_*\mathcal{N}$ is an invertible sheaf on Y and $\mathcal{N} \simeq \tau^*\mathcal{M}$. Here, $\mathcal{M}$ is trivial on U. Hence, $\mathcal{M} \simeq \mathcal{O}_Y$ since Y is normal and $\operatorname{codim}(Y \setminus U) \geq 2$. Therefore, $\mathcal{N} \simeq \mathcal{O}_{Y'}$, and the expected base change formula is obtained.

Second, we consider the case (2). We may replace Y' with an open subset whose complement has codimension greater than one. Hence, we may assume that $\mathcal{F}'$ is flat over Y' also in case (2). By Nagata's completion theorem, Y' is realized as an open subset of an integral scheme $\overline{Y'}$ proper over Y. Let $Y'' \to \overline{Y'}$ be a flattening over $\overline{Y'}$ of the pullback of $\mathcal{F}$ in $X \times_Y \overline{Y'}$. Then, it is enough to show the same base change formula for $X \times_Y Y'' \to Y''$. Precisely speaking, we are reduced to prove Lemma 3.21 below. In fact, the sheaf $\mathcal{I}_{\mathcal{F}'/Y'}(p_1^*x)$ is isomorphic to $\mathcal{I}_{\mathcal{F}'/Y'}(p_1^*\eta)$ if Y' is normal (cf. Remark 3.8). □

Lemma 3.21. *Under the same situation of* Theorem 3.20, *instead of the conditions* (1), (2), *assume that*

(3) p_2 *is locally projective,* τ *is proper surjective, and* $\mathcal{F}'$ *is flat over* Y'.

Let $x \in F^{d+1}(X)$ *be an element representing* η *and set*

$$\mathcal{I}_{\mathcal{F}'/Y'}(p_1^*x) := \det p_{2*}^{\mathcal{F}'}(p_1^*x)$$

for the homomorphisms $p_{2*}^{\mathcal{F}'} \colon K^\bullet(X') \to K^\bullet(Y')_{\mathrm{perf}}$ *and* $\det \colon K^\bullet(Y')_{\mathrm{perf}} \to \operatorname{Pic}(Y')$. *Then, there is a finite birational morphism* $\nu \colon Y'_1 \to Y'$ *from a separated Noetherian integral scheme* Y'_1 *such that*

$$\nu^* \left(\mathcal{I}_{\mathcal{F}'/Y'}(p_1^*x)\right) \simeq \nu^*\tau^* \mathcal{I}_{\mathcal{F}/Y}(\eta).$$

Proof. Let $U \subset Y$ be an open subset $U \subset Y$ such that $\mathcal{F}_{\mathrm{t.f.}/Y}$ is flat over U and $\operatorname{codim}(Y \setminus U) \geq 2$. Then,

$$\mathcal{I}_{\mathcal{F}'/Y'}(p_1^* x)|_{\tau^{-1}(U)} \simeq \tau^*(\mathcal{I}_{\mathcal{F}/Y}(\eta)|_U)$$

by Lemma 2.3 (cf. Remark 3.8). We set

$$\mathcal{N} := \mathcal{I}_{\mathcal{F}'/Y'}(p_1^* x) \otimes \tau^* \mathcal{I}_{\mathcal{F}/Y}(\eta)^{-1}.$$

Then $\mathcal{N}$ has a rational section s which is nowhere vanishing on $\tau^{-1}(U)$. By Proposition 3.19, there is an open covering $\{Y_\alpha\}$ of Y such that $\mathcal{N}$ is trivial over $\tau^{-1}(Y_\alpha)$. Let $Y' \to Y_0 \to Y$ be the Stein factorization of τ. Then, for the morphism $\tau_0 \colon Y' \to Y_0$, $\mathcal{M} := \tau_{0*}\mathcal{N}$ is invertible and $\mathcal{N} \simeq \tau_0^* \mathcal{M}$. Thus, s descends to a rational section s_0 of $\mathcal{M}$ which is nowhere vanishing on $h^{-1}(U)$ for the finite morphism $h \colon Y_0 \to Y$. Let $\mu \colon Y_1 \to Y_0$ be the normalization. Then μ is a finite morphism since $h_* \mu_* \mathcal{O}_{Y_1}$ is the double-dual of $h_* \mathcal{O}_{Y_0}$. Hence, $\mu^*(s_0)$ is a nowhere vanishing section of $\mu^* \mathcal{M}$, since Y_1 is normal and $\operatorname{codim}(Y_1 \setminus \mu^{-1} h^{-1}(U)) \geq 2$. Let Y_1' be the unique integral closed subscheme of $Y_1 \times_{Y_0} Y'$ which dominates Y'. Then $\nu^* \mathcal{N}$ is trivial for the finite birational morphism $\nu \colon Y_1' \to Y'$. □

The following gives a base change property by not necessarily dominant morphisms from normal schemes.

Proposition 3.22. *Let $\pi \colon X \to Y$ be a projective surjective morphism over a normal separated Noetherian integral scheme Y, and $\mathcal{F}$ a coherent sheaf on X with* $\dim(\operatorname{Supp} \mathcal{F} \cap \pi^{-1}(y)) = d$ *for any point $y \in Y$. Let $\nu \colon B \to Y$ be a morphism from a normal separated Noetherian integral scheme B and $\mu \colon X \times_Y B \to X$ the induced morphism. Then there exist a coherent sheaf $\widehat{\mathcal{F}}_B$ on $X \times_Y B$ and a positive integer e such that* $\operatorname{Supp} \widehat{\mathcal{F}}_B \subset \mu^{-1}(\operatorname{Supp} \mathcal{F})$, $\dim(\operatorname{Supp} \widehat{\mathcal{F}}_B \cap (X \times_Y \{b\})) = d$ *for any $b \in B$, and*

$$\nu^* \mathcal{I}_{\mathcal{F}/Y}(\eta)^{\otimes e} \simeq \mathcal{I}_{\widehat{\mathcal{F}}_B/B}(\mu^* \eta)$$

for any $\eta \in G^{d+1}(X)$.

Remark. If $\mathcal{F}$ is flat over Y, then one can take $e = 1$ and $\widehat{\mathcal{F}}_B = \nu^* \mathcal{F}$, by Lemma 2.12.

Proof of Proposition 3.22. Let $\tau \colon Y' \to Y$ be a projective birational morphism from an integral scheme Y' which gives a flattening of $\mathcal{F}/Y$. For the fiber product $X' = X \times_Y Y'$, let $p_1 \colon X' \to X$ and $p_2 \colon X' \to Y'$ be the natural projections. Here, $\mathcal{F}' = (p_1^* \mathcal{F})_{\mathrm{t.f.}/Y'}$ is flat over Y'. There is a closed integral subscheme $B' \subset Y' \times_Y B$ such that $\tau_B := \tau \times_Y \mathrm{id}_B \colon B' \to B$ is surjective and generically finite. We set ν' to be the morphism $B' \subset Y' \times_Y B \to Y'$. Let W and W' be the fiber products $X \times_Y B$ and $X' \times_{Y'} B'$, respectively. Let $\pi_B = \pi \times_Y \mathrm{id}_B \colon W \to B$, $q_1 \colon W' \to W$, $q_2 \colon W' \to B'$, and $\mu' \colon W' \to X'$

be the induced morphisms. Note that the induced morphism $W \to X$ is μ. For the sheaf $\mathcal{F}'_{W'} := \mu'^{*}\mathcal{F}'$, we have

$$\mathcal{I}_{\mathcal{F}'_{W'}/B'}(\mu'^{*}p_1^{*}\eta) \simeq \nu'^{*}\mathcal{I}_{\mathcal{F}'/Y'}(p_1^{*}\eta) \simeq \tau_B^{*}\nu^{*}\mathcal{I}_{\mathcal{F}/Y}(\eta)$$

for any $\eta \in G^{d+1}(X)$ by Lemma 2.12 and Theorem 3.20. On the other hand,

$$\boldsymbol{c}^1(\mathcal{I}_{\mathcal{F}'_{W'}/B'}(\mu'^{*}p_1^{*}\eta)) = q_{2*}^{\mathcal{F}'_{W'}}(\mu'^{*}p_1^{*}\eta) \in G^1(B').$$

Since B is normal, τ_B is a finite morphism over an open subset of B whose complement has codimension greater than one. Hence, $\tau_{B*}F^2_{\mathrm{con}}(B') \subset F^2_{\mathrm{con}}(B)$ for the homomorphism $\tau_{B*}\colon K_\bullet(B') \to K_\bullet(B)$. Therefore, the natural homomorphism $\nu'_*\colon G^1(B') \to G^1_{\mathrm{con}}(B)$ is induced and the following equalities in $G^1_{\mathrm{con}}(B)$ make sense:

$$\tau_{B*}q_{2*}^{\mathcal{F}'_{W'}}(\mu'^{*}p_1^{*}\eta) = \pi_{B*}q_{1*}((\mu'^{*}p_1^{*}\eta)\,\mathrm{cl}_\bullet(\mathcal{F}'_{W'})) = \pi_{B*}((\mu^{*}\eta)\,\mathrm{cl}_\bullet(\mathbf{R}\,q_{1*}\,\mathrm{cl}_\bullet(\mathcal{F}'_{W'})).$$

Now, $\mathrm{Supp}(\mathrm{R}^i\,q_{1*}(\mathcal{F}' \otimes \mathcal{O}_{W'}))$ does not dominate B for any $i > 0$. Hence, for the sheaf $\widehat{\mathcal{F}}_B := q_{1*}\mathcal{F}'_{W'}$ on W, we have

$$\boldsymbol{c}^1(\nu^{*}\mathcal{I}_{\mathcal{F}/Y}(\eta))\tau_{B*}(\mathrm{cl}_\bullet(\mathcal{O}_{B'})) = \tau_{B*}\,\boldsymbol{c}^1(\mathcal{I}_{\mathcal{F}'_{W'}/B'}(\mu'^{*}p_1^{*}\eta)) = \pi_{B*}((\mu^{*}\eta)\,\mathrm{cl}_\bullet(\widehat{\mathcal{F}}_B)) = \pi_{B*}^{\widehat{\mathcal{F}}_B}(\mu^{*}\eta)$$

in $G^1_{\mathrm{con}}(B)$. Therefore,

$$\nu^{*}\mathcal{I}_{\mathcal{F}/Y}(\eta)^{\otimes e} \simeq \mathcal{I}_{\widehat{\mathcal{F}}_B/B}(\mu^{*}\eta)$$

by $\tau_{B*}(\mathrm{cl}_\bullet(\mathcal{O}_{B'})) = e\,\mathrm{cl}_\bullet(\mathcal{O}_B) \in G^0_{\mathrm{con}}(B)$ for the rank e of $\tau_{B*}\mathcal{O}_{B'}$. The remaining assertions follow from $\mathrm{Supp}\,\mathcal{F}'_{W'} \subset \mu'^{-1}p_1^{-1}(\mathrm{Supp}\,\mathcal{F})$ and $\dim(\mathrm{Supp}\,\mathcal{F}'_{W'})/B' = d$. □

The following is an application of Theorem 3.20:

Proposition 3.23. *Let $\pi\colon X \to Y$ be a locally projective surjective morphism of $\Bbbk$-schemes for a field $\Bbbk$ such that Y is a normal separated Noetherian integral scheme, and let $\mathcal{F}$ be a coherent sheaf on X with $\dim(\mathrm{Supp}\,\mathcal{F} \cap \pi^{-1}(y)) \le d$ for any point $y \in Y$. Let T be a normal separated Noetherian integral $\Bbbk$-scheme, $\mathcal{F}_T$ the sheaf $p_1^{*}\mathcal{F}$ for the first projection $p_1\colon X \times_{\Bbbk} T \to X$, and $\tilde{\eta}$ an element of $G^{d+1}(X \times_{\Bbbk} T)$. For a $\Bbbk$-valued point $t \in T(\Bbbk)$, put $X_t := X \times_{\Bbbk} \{t\}$, $Y_t := Y \times_{\Bbbk} \{t\}$, $\mathcal{F}_t := \mathcal{F}_T \otimes \mathcal{O}_{X_t}$, and $\eta_t := \tilde{\eta}|_{X_t} \in G^{d+1}(X_t)$. Then,*

$$\mathcal{I}_{\mathcal{F}_T/Y\times_{\Bbbk}T}(\tilde{\eta})|_{Y_t} \simeq \mathcal{I}_{\mathcal{F}_t/Y_t}(\eta_t).$$

Proof. By taking a flattening of $\mathcal{F}/Y$, and by Theorem 3.20, we may assume that $\mathcal{F}$ is flat over Y. Then the assertion follows from Lemma 2.3. □

4. Intersection sheaves for varieties over a field

In what follows, we shall work in the category of $\Bbbk$-schemes for a fixed field $\Bbbk$. A variety (over $\Bbbk$) is by definition an integral separated scheme of finite type over $\operatorname{Spec}\Bbbk$. We shall study the intersections sheaves for morphisms $X \to Y$ of normal projective varieties. In Section 4.1, we study some numerical properties of $\mathcal{I}_{X/Y}$. In Section 4.2, for a family Z of effective algebraic cycles of pure dimension of X parametrized by Y (hence, Z is a cycle of $X \times Y$) and for an ample invertible sheaf $\mathcal{A}$ of X, we show that the intersection sheaf $\mathcal{I}_{Z/Y}(p_1^*\mathcal{A}, \ldots, p_1^*\mathcal{A})$ is just the pullback of an ample invertible sheaf by the morphism to the Chow variety of X determined by Z/Y. An application to the study of endomorphisms of complex projective normal varieties is given in Section 4.3.

4.1. **Numerical properties of intersection sheaves.** Let $\pi\colon X \to Y$ be a proper surjective equi-dimensional morphism from a projective variety X to a normal variety Y. Note that Y is proper over $\operatorname{Spec}\Bbbk$. We set $d = \dim X/Y$ and $m = \dim Y$. Then the intersection sheaf $\mathcal{I}_{X/Y}(\eta)$ for $\eta \in G^{d+1}(X)$ is also defined as $\pi_* \boldsymbol{\phi}(\eta) \mod F^2_{\mathrm{con}}(Y) = F_{m-2}(Y)$ for $\boldsymbol{\phi}\colon G^{d+1}(X) \to G^{d+1}_{\mathrm{con}}(X) = G_{m-1}(X)$ and $\pi_*\colon G_{m-1}(X) \to G_{m-1}(Y)$.

The base space Y is projective by the following, which is an analogue of [22], Théorème 2 on Kähler spaces:

Theorem 4.1. *Let $\pi\colon X \to Y$ be a proper surjective morphism from a projective scheme to a normal algebraic variety defined over a field. Suppose that π is equi-dimensional. Then Y is projective.*

Proof. Let $\mathcal{A}$ be a very ample invertible of X. We set $\eta = \boldsymbol{c}^1(\mathcal{A})^{d+1} \in G^{d+1}(X)$ for $d = \dim X/Y$. Then $\mathcal{I}_{X/Y}(\eta) = \mathcal{I}_{X/Y}(\mathcal{A}, \ldots, \mathcal{A})$ is an invertible sheaf generated by global sections by Corollary 3.18 (cf. Lemma 2.18). We shall show that $\mathcal{I}_{X/Y}(\eta)$ is ample. For this, it is enough to prove that the intersection number $\mathcal{I}_{X/Y}(\eta)C$ is positive for any irreducible curve $C \subset Y$. Let $\nu\colon B \to C$ be the normalization and $\mu\colon X \times_Y B \to X$ the induced finite morphism. Then there exist a coherent sheaf $\widehat{\mathcal{F}}_B$ on $X \times_Y B$ and a positive integer e such that $\dim \operatorname{Supp} \widehat{\mathcal{F}}_B = d+1$ and

$$\nu^* \mathcal{I}_{X/Y}(\eta)^{\otimes e} \simeq \mathcal{I}_{\widehat{\mathcal{F}}_B/B}(\mu^*\eta)$$

by Proposition 3.22. Hence,

$$e\,\mathcal{I}_{X/Y}(\eta)C = \deg \mathcal{I}_{\widehat{\mathcal{F}}_B/B}(\mu^*\eta) = i(\mu^*\eta; \widehat{\mathcal{F}}_B) = i(\mu^*\mathcal{A}, \ldots, \mu^*\mathcal{A}; \widehat{\mathcal{F}}_B) > 0. \qquad \square$$

In order to calculate the intersection sheaves $\mathcal{I}_{X/Y}(\eta)$, we may replace X with the normalization. In fact, we have:

Lemma 4.2. *Let $\nu\colon \widehat{X} \to X$ be a finite surjective morphism from another projective variety $\widehat{X}$. Then*

$$\mathcal{I}_{\widehat{X}/Y}(\nu^*\eta) \simeq \mathcal{I}_{X/Y}(\eta)^{\otimes e}$$

for any $\eta \in G^{d+1}(X)$ and for the degree $e = \deg(f)$.

Proof. Since e is the rank of $\nu_*\mathcal{O}_{\widehat{X}}$, we have $\nu_* \operatorname{cl}_\bullet(\mathcal{O}_{\widehat{X}}) = e \operatorname{cl}_\bullet(\mathcal{O}_X)$ in $G^0_{\mathrm{con}}(X)$, which proves the formula. □

In Lemma 4.3 and Theorem 4.4 below, we shall give sufficient conditions for an intersection sheaf $\mathcal{I}_{X/Y}(\eta)$ to be ample or nef.

Lemma 4.3. *Let η be an element of $G^{d+1}(X)$ such that $i(\eta; W) \geq 0$ for any closed irreducible subset $W \subset \operatorname{Supp}\mathcal{F}$ of dimension $d+1$. Then $\mathcal{I}_{X/Y}(\eta)$ is nef.*

Proof. For an irreducible curve C of Y, let $B \to C$ be the normalization. Then, by Proposition 3.22, there exist positive integer e and a coherent sheaf $\widehat{\mathcal{F}}_B$ of $X \times_Y B$ such that

$$e\,\mathcal{I}_{X/Y}(\eta)C = i(\mu^*\eta; \widehat{\mathcal{F}}_B) = i(\eta; \mu_*\widehat{\mathcal{F}}_B) \geq 0$$

for the induced finite morphism $\mu\colon X \times_Y B \to X$. □

Theorem 4.4. *Let $\pi\colon X \to Y$ be an equi-dimensional proper surjective morphism of normal projective varieties defined over a field. Let θ be an element of $G^d(X)$ for $d = \dim X/Y$. For an invertible sheaf $\mathcal{L}$ of X, the intersection sheaf $\mathcal{M} := \mathcal{I}_{X/Y}(\theta\, \boldsymbol{c}^1(\mathcal{L}))$ satisfies the following properties*:

1. *If $\mathcal{L}$ is algebraically equivalent to zero, then so is $\mathcal{M}$.*
2. *If $\mathcal{L}$ is numerically trivial, then so is $\mathcal{M}$.*

Assume that $i(\theta; W) \geq 0$ for any closed subscheme $W \subset X$ with $\dim W = d$. Then the following properties are also satisfied:

3. *If $\mathcal{L}$ is nef, then so is $\mathcal{M}$.*
4. *If $\mathcal{L}$ is ample and if $i_{X/Y}(\theta) > 0$, then $\mathcal{M}$ is ample.*

Proof. Let $\overline{\Bbbk}$ be the algebraic closure of the base field $\Bbbk$. Let $\overline{Y}$ be the normalization of a closed integral subscheme of $Y \otimes_{\Bbbk} \overline{\Bbbk}$ which dominates Y. Then, by Lemma 3.9, it suffices to show the assertion for the pullback $X \times_Y \overline{Y} \to \overline{Y}$ of π. Hence, we may assume from the beginning that $\Bbbk$ is algebraically closed.

(1) follows from Proposition 3.23.

(2): $i(\theta\, \boldsymbol{c}^1(\mathcal{L}); W') = 0$ for any closed subscheme $W' \subset X$ of dimension $d+1$, since $\mathcal{L}$ is numerically trivial. Hence, $\mathcal{M}C = 0$ for any irreducible curve $C \subset Y$ by the proof of Lemma 4.3.

(3): By Lemma 4.3, it suffices to show

(IV–1) $$i(\theta\, \boldsymbol{c}^1(\mathcal{L}); W') \geq 0$$

for any closed subvariety $W' \subset X$ of dimension $d+1$. If $\mathcal{L}$ is very ample, then $\dim W' \cap A = d$ for a general member $A \in |\mathcal{L}|$ (cf. Lemma 3.14). Hence,

$$i(\theta\, \boldsymbol{c}^1(\mathcal{L}); W') = i(\theta; W' \cap A) \geq 0.$$

Thus, (IV–1) holds if $\mathcal{L}$ is ample. Even if $\mathcal{L}$ is only nef, $\mathcal{L}^{\otimes N} \otimes \mathcal{A}$ is ample for any ample invertible sheaf $\mathcal{A}$ of X and for any $N > 0$. Thus

$$0 \leq i(\theta\, \boldsymbol{c}^1(\mathcal{L}^{\otimes N} \otimes A); W') = N i(\theta\, \boldsymbol{c}^1(\mathcal{L}); W') + i(\theta\, \boldsymbol{c}^1(\mathcal{A}); W')$$

for any $N > 0$. Hence, (IV–1) holds for any nef invertible sheaf $\mathcal{L}$.

(4): Let $\mathcal{A}$ be an ample invertible sheaf on Y. Then $\mathcal{L}^{\otimes b} \otimes \pi^*\mathcal{A}^{-1}$ is ample for some $b > 0$. Thus,

$$\mathcal{I}_{X/Y}(\theta\, \boldsymbol{c}^1(\mathcal{L}^{\otimes b} \otimes \pi^*\mathcal{A}^{-1})) \simeq \mathcal{M}^{\otimes b} \otimes \mathcal{I}_{X/Y}(\theta\, \boldsymbol{c}^1(\pi^*\mathcal{A}))^{-1} \simeq \mathcal{M}^{\otimes b} \otimes \mathcal{A}^{\otimes(-i_{X/Y}(\theta))}$$

is nef by (3) (cf. Lemma 3.10). Hence, $\mathcal{M}$ is ample. □

In Lemma 4.5 and Theorem 4.7 below, we shall give sufficient conditions for an intersection sheaf $\mathcal{I}_{X/Y}(\eta)$ to be effective, big, or pseudo-effective.

Lemma 4.5. *Let $B \subset Y$ be a closed subset of* $\operatorname{codim}(B) \geq 2$ *and Z an effective algebraic cycle of $X \setminus \pi^{-1}(B)$ of codimension d such that any irreducible component of Z dominates $Y \setminus B$. Let $\theta \in G^d(X)$ be an element such that*

$$\boldsymbol{\phi}(\theta|_{X\setminus\pi^{-1}(B)}) = \mathrm{cl}_\bullet(Z) \quad \in K_\bullet(X \setminus \pi^{-1}(B)).$$

If D is an effective Cartier divisor on X which does not contain any irreducible component of Z, then the intersection sheaf $\mathcal{I}_{X/Y}(\theta\, \boldsymbol{c}^1(\mathcal{O}_X(D)))$ has a non-zero global section.

Proof. Let $Z = \sum n_i Z_i$ be the irreducible decomposition. Then

$$\mathcal{I}_{X/Y}(\theta\, \boldsymbol{c}^1(\mathcal{L}))|_{Y\setminus B} \simeq \bigotimes \mathcal{I}_{Z_i/Y\setminus B}(\mathcal{L}|_{Z_i})^{\otimes n_i}.$$

If $\mathcal{M}$ is an invertible sheaf on Y, then $\mathrm{H}^0(Y, \mathcal{M}) \simeq \mathrm{H}^0(Y \setminus B', \mathcal{M})$ for a closed subset B' with $\operatorname{codim}(B') \geq 2$. Thus, by replacing Y with $Y \setminus B'$ for certain closed subset $B' \supset B$ with $\operatorname{codim}(B') \geq 2$, and X with Z, we are reduced to prove the existence of a non-zero global section of $\mathcal{M} = \mathcal{I}_{X/Y}(\mathcal{O}_X(D))$ for a finite surjective morphism $\pi\colon X \to Y$ of not necessarily projective varieties, where Y is normal, and for an effective Cartier divisor D on X. We may also assume that X is normal by Lemma 4.2. Then, the assertion follows from the property that the push-forward π_*D is effective and from the isomorphism $\mathcal{O}_Y(\pi_*D) \simeq \mathcal{I}_{X/Y}(\mathcal{O}_X(D))$ (cf. Example 3.4). □

Definition 4.6 (cf. [23])**.** Let $\mathcal{N}$ be an invertible sheaf on a normal projective variety X defined over an algebraically closed field, and $W \subset X$ a closed subset. If the following condition is satisfied, then $\mathcal{N}$ is called *weakly positive outside* W:

- For an ample invertible sheaf $\mathcal{A}$ on X, an arbitrary point $x \in X \setminus W$, and for any positive rational number ε, there exist a positive integer m with $m\varepsilon \in \mathbb{Z}$ and an effective divisor D such that $\mathcal{O}_X(D) \simeq \mathcal{N}^{\otimes m} \otimes \mathcal{A}^{\otimes m\varepsilon}$ and $x \notin \operatorname{Supp} D$.

Theorem 4.7. *Let* $\pi\colon X \to Y$ *be an equi-dimensional proper surjective morphism of normal projective varieties defined over an algebraically closed field with* $d = \dim X/Y$*. Let* $\mathcal{N}_1, \ldots, \mathcal{N}_d$ *be invertible sheaves on* X *which are weakly positive outside* $\pi^{-1}(B)$ *for a closed subset* $B \subset Y$ *of* $\operatorname{codim}(B) \geq 2$*. For an invertible sheaf* $\mathcal{L}$ *of* X*, the intersection sheaf* $\mathcal{M} := \mathcal{I}_{X/Y}(\mathcal{N}_1, \ldots, \mathcal{N}_d, \mathcal{L})$ *satisfies the following properties*:

(1) *If* $\mathcal{L}$ *is pseudo-effective, then so is* $\mathcal{M}$*.*
(2) *If* $\mathcal{L}$ *is big and if* $i_{X/Y}(\mathcal{N}_1, \ldots, \mathcal{N}_d) > 0$*, then* $\mathcal{M}$ *is big.*
(3) *If all the* $\mathcal{N}_i$ *and* $\mathcal{L}$ *are nef and big, then* $\mathcal{M}$ *is also nef and big.*

Proof. (1): Let $\mathcal{A}$ be an ample invertible sheaf on X and ε a positive rational number. Then there is an effective divisor Δ such that $\mathcal{O}_X(\Delta) \simeq \mathcal{L}^{\otimes l} \otimes \mathcal{A}^{\otimes l\varepsilon}$ for some $l > 0$ with $l\varepsilon \in \mathbb{Z}$. By the weak positivity, there exist also positive integers $m_1, \ldots, m_d$ and effective divisors $D_1, \ldots, D_d$ such that

- $m_i\varepsilon \in \mathbb{Z}$ and $\mathcal{O}_X(D_i) \simeq \mathcal{N}_i^{\otimes m_i} \otimes \mathcal{A}^{m_i\varepsilon}$ for any $1 \leq i \leq d$,
- $\operatorname{codim}(V \cap \Delta \cap \pi^{-1}(Y \setminus B)) = d + 1$ for the intersection $V = D_1 \cap \cdots \cap D_d$, and
- $V \cap \pi^{-1}(Y \setminus B') \to Y \setminus B'$ is a finite surjective morphism for a closed subset $B' \supset B$ with $\operatorname{codim}(B') \geq 2$.

Hence, $\mathcal{I}_{X/Y}(\mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_d), \mathcal{O}_X(\Delta))$ has a non-zero global section by Lemma 4.5. Taking the limit $\varepsilon \to 0$, we infer that $\mathcal{M}$ is pseudo-effective.

(2): $\mathcal{L}^{\otimes b} \otimes \pi^*\mathcal{A}^{-1}$ has a non-zero global section for an ample invertible sheaf $\mathcal{A}$ and a positive integer b. Hence, by the same argument as in the proof of Theorem 4.4, (4), we infer that $\mathcal{M}^{\otimes b} \otimes \mathcal{A}^{\otimes(-k)}$ is pseudo-effective for $k = i_{X/Y}(\mathcal{N}_1, \ldots, \mathcal{N}_d)$ by (1). Thus, $\mathcal{M}$ is big.

(3) is a consequence of (2) above and Theorem 4.4, (3). □

4.2. **Morphisms into Chow varieties.** Let X be a projective variety, Y a normal variety, and let $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ be the natural projections. Let us fix a non-negative integer d.

Definition 4.8. Let $Z = \sum n_i Z_i$ be an effective algebraic cycle of $X \times Y$, where $n_i > 0$ and Z_i is a closed integral subscheme of $X \times Y$. The cycle Z is called a *family of* *effective algebraic cycles of X of dimension d parametrized by Y* if $p_2\colon Z_i \to Y$ is an equi-dimensional surjective morphism of relative dimension d for any i. We denote by $\operatorname{Supp} Z$ the reduced scheme $\bigcup_i Z_i$.

Let $Z = \sum n_i Z_i$ be a family of effective algebraic cycles of X of dimension d parametrized by Y. For a point $y \in Y$, $Z_i \times_Y y$ is a closed subscheme of $X \otimes_{\Bbbk} \Bbbk(y)$ of pure dimension d, where $\Bbbk(y)$ denotes the residue field of $\mathcal{O}_{Y,y}$. Thus, for the associated cycles $\operatorname{cyc}(Z_i \times_Y y)$, we can define the algebraic cycle $Z(y)$ to be $\sum n_i \operatorname{cyc}(Z_i \times_Y y)$.

Let $\tau\colon Y' \to Y$ be a surjective morphism from a normal variety Y'. Then one can consider the pullback $\tau^* Z$ as follows: Let $\{Z'_{i,j}\}$ be the set of irreducible components of $Z_i \times_Y Y'$ such that $Z'_{i,j}$ dominates Y'. Let $l_{i,j}$ be the length of $Z_i \times_Y Y'$ along $Z'_{i,j}$, i.e.,

$$l_{i,j} = l_{Z'_{i,j}}(\mathcal{O}_{Z_i \times_Y Y'}).$$

We set $\tau^* Z$ to be the cycle $\sum_{i,j} n_i l_{i,j} Z'_{i,j}$. Then, $\tau^* Z$ is a family of effective algebraic cycles of X of dimension d parametrized by Y'. By Lemma 3.6 and Theorem 3.20, we have:

Lemma 4.9. *For any $\eta \in G^{d+1}(X)$, there is an isomorphism*

$$\mathcal{I}_{\tau^* Z/Y'}(p_1^* \eta) \simeq \tau^* \mathcal{I}_{Z/Y}(p_1^* \eta).$$

We shall show the following:

Theorem 4.10. *Let X be a projective variety, Y a normal projective variety, and Z a family of algebraic cycles of X of dimension d parametrized by Y. Then, there exist a proper surjective morphism $\varphi\colon Y \to T$ into a normal projective variety T with connected fibers and a family Z_T of algebraic cycle of X of dimension d parametrized by T such that*

(1) $Z = \varphi^* Z_T$,

(2) $\varphi^* \mathcal{I}_{Z_T/T}(p_1^* \eta) \simeq \mathcal{I}_{Z/Y}(p_1^* \eta)$ *for any* $\eta \in G^{d+1}(X)$,

(3) $\mathcal{I}_{Z_T/T}(p_1^* \mathcal{A}_1, \ldots, p_1^* \mathcal{A}_{d+1})$ *is ample for any ample invertible sheaves $\mathcal{A}_i$ on X,*

where p_1 denotes the first projection $X \times Y \to X$ or $X \times T \to X$.

The proof is given after Lemmas 4.11 and 4.12.

Lemma 4.11. *Let $B \subset Y$ be a connected closed algebraic subset and $F \subset X$ the image $p_1(\operatorname{Supp} Z \cap (X \times B))$. Suppose that* $\dim F = d$. *Then* $\operatorname{Supp} Z \cap (X \times B) = F \times B$ *as an algebraic subset of $X \times Y$.*

Proof. We write $S = \operatorname{Supp} Z$. By construction, there is a natural inclusion $S \cap (X \times B) \subset F \times B$. In order to show the converse inclusion, we may assume B to be irreducible since B is connected. Furthermore, we can reduce to the case where $Z_i \to Y$ is flat for any i

as follows: We can find a birational morphism $Y' \to Y$ from a normal projective variety Y' which gives a flattening of $Z_i \to Y$ for any i. Let Z'_i be the irreducible component of $Z_i \times_Y Y'$ flat over Y'. Then $Z'_i \to Z_i$ is surjective since it is birational. Let S' be the union $\bigcup_i Z'_i$. Then $S' \cap (X \times B') \subset F' \times B'$ for $B' = B \times_Y Y'$ and for the image $F' \subset X$ of $S' \cap (X \times B')$ by the first projection $X \times Y' \to X$. Here, $F = F'$ since $S' \to S$ is surjective. Thus, if $S' \cap (X \times B') = F \times B'$, then we have $S \cap (X \times B) = F \times B$ by considering the image by $X \times Y' \to X \times Y$.

Therefore, we may assume that B is irreducible and $Z_i \to Y$ is flat for any i. Let $\{V_{i,j}\}$ be the set of irreducible components of $Z_i \cap (X \times B)$. Then $p_2(V_{i,j}) = B$ and $\dim V_{i,j} = \dim B + d$, since $V_{i,j} \to B$ is flat at the generic point of $V_{i,j}$. Let $F_{i,j}$ be the image $p_1(V_{i,j})$. Then the natural inclusion $V_{i,j} \subset F_{i,j} \times B$ is just the equality, since the both sides are irreducible subvarieties of $X \times Y$ with the same dimension. Therefore, $Z_i \cap (X \times B) = F_i \times B$ for the union $F_i = \bigcup_j F_{i,j}$, and finally, $S \cap (X \times B) = F \times B$ by $F = \bigcup F_i$. □

Let $\mathcal{A}_1, \ldots, \mathcal{A}_d$ be very ample invertible sheaves on X. Then we can consider the intersection sheaf $\mathcal{M} := \mathcal{I}_{Z/Y}(p_1^*\mathcal{A}_1, \ldots, p_1^*\mathcal{A}_d)$. Here, $\mathcal{M}$ is generated by global sections by Corollary 3.18. Let $\varphi\colon Y \to T$ be the Stein factorization of the morphism

$$\Phi_{|\mathcal{M}|}\colon Y \to |\mathcal{M}|^\vee = \mathbb{P}(\mathrm{H}^0(Y, \mathcal{M})) = \mathrm{Proj}(\mathrm{Sym}\,\mathrm{H}^0(Y, \mathcal{M}))$$

associated with the linear system $|\mathcal{M}|$. In other words, φ is the canonical morphism

$$Y \to T = \mathrm{Proj} \bigoplus\nolimits_{l \geq 0} \mathrm{H}^0(Y, \mathcal{M}^{\otimes l}).$$

Lemma 4.12. *For an integral closed subscheme $B \subset Y$, $\varphi(B)$ is a point if and only if $\mathrm{Supp}\, Z \cap (X \times B) = F \times B$ for a closed subset $F \subset X$. In particular, the morphism φ does not depend on the choices of very ample invertible sheaves $\mathcal{A}_i$.*

Proof. By Lemma 4.11, the latter condition is equivalent to $\dim p_1(\mathrm{Supp}\, Z \cap (X \times B)) = d$. Let $\tau\colon Y' \to Y$ be a projective birational morphism from a normal projective variety Y' which gives a flattening of $Z_i \to Y$ for any i. Then $\tau^*\mathcal{M} \simeq \mathcal{I}_{\tau^*Z/Y'}(p_1^*\eta)$ by Lemma 4.9. Thus $\varphi \circ \tau$ is associated with the family τ^*Z of algebraic cycles parametrized by Y'. Hence, we can replace Y with Y' in order to show the assertion. Therefore, we assume from the beginning that $Z_i \to Y$ is flat for any i. Then,

$$\mathcal{M}|_B \simeq \mathcal{I}_{Z \times_Y B/B}(p_1^*\eta)$$

by Lemma 2.12.

If $\dim p_1(\mathrm{Supp}\, Z \cap (X \times B)) = d$, then $p_1^*\eta\, \mathrm{cl}_\bullet(Z) = 0 \in G^{d+1}_{\mathrm{con}}(Z \times_Y B)$; hence, $\mathcal{M}|_B$ is numerically equivalent to zero, and $\varphi(B)$ is a point.

If $\dim p_1(\operatorname{Supp} Z \cap (X \times B)) \geq d+1$, then there is an irreducible curve $C \subset B$ such that $\dim(\operatorname{Supp} Z \cap (X \times C)) = \dim p_{1*}(\operatorname{Supp} Z \cap (X \times C)) = d+1$ by Lemma 4.11; hence

$$\deg \mathcal{M}|_C = i(p_1^*\eta; Z \times_Y C) > 0,$$

which implies that $\varphi(B)$ is not a point. □

We are ready to prove Theorem 4.10:

Proof of Theorem 4.10. Let $Z_{T,i} \subset X \times T$ be the image of Z_i by $\mathrm{id}_X \times \varphi\colon X \times Y \to X \times T$, and let Z_T be the cycle $\sum n_i Z_{T,i}$. Then the natural inclusion $Z_i \subset Z_{T,i} \times_T Y$ is an equality of algebraic sets by Lemma 4.11. Hence the second projection $p_2\colon Z_{T,i} \to T$ is a surjective equi-dimensional morphism of relative dimension d; thus Z_T is a family of algebraic cycles of X of dimension d parametrized by T. Let $U \subset Y$ be an open dense subset such that $\varphi\colon Y \to T$ is flat along U. Then $Z_{T,i} \times_T U$ is reduced. Hence, the inclusion $Z_i \subset Z_{T,i} \times_T Y$ is an isomorphism over U. Thus, the equality $\varphi^* Z_T = Z$ in (1) follows. The isomorphism (2) follows from (1) and Lemma 4.9. The condition (3) follows from (2) and Lemma 4.12. □

Remark 4.13. We fix a closed immersion $X \hookrightarrow \mathbb{P}^n$ into an n-dimensional projective space $\mathbb{P}^n$ and set $\mathcal{A} = \mathcal{O}(1)|_X$. Let R_n be the vector space $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1))$. We set $\theta = \boldsymbol{c}^1(\mathcal{A})^d \in G^d(X)$ and $\eta = \boldsymbol{c}^1(\mathcal{A})^{d+1} \in G^{d+1}(X)$. Furthermore, we set

$$e := i_{Z/Y}(p_1^*\theta) = i_{Z/Y}(p_1^*\mathcal{A}, \dots, p_1^*\mathcal{A}) = i_{Z/Y}(p_1^*\mathcal{O}(1), \dots, p_1^*\mathcal{O}(1)),$$
$$\mathcal{M} := \mathcal{I}_{Z/Y}(p_1^*\eta) = \mathcal{I}_{Z/Y}(p_1^*\mathcal{A}, \dots, p_1^*\mathcal{A}) = \mathcal{I}_{Z/Y}(p_1^*\mathcal{O}(1), \dots, p_1^*\mathcal{O}(1)).$$

Then by Propositions 2.7 and 3.17 (cf. Lemma 2.18), we have a natural surjection

$$\bigotimes^{d+1} \mathrm{Sym}^e(R_n) \otimes_{\Bbbk} \mathcal{O}_Y \to \mathcal{M}.$$

By construction, the associated morphism

$$\psi\colon Y \to \mathbb{P}\left(\bigotimes^{d+1} \mathrm{Sym}^e(R_n)\right)$$

is just the morphism to the Chow variety $\mathrm{Chow}_{d,e}(X) \subset \mathrm{Chow}_{d,e}(\mathbb{P}^n)$ of d-dimensional algebraic cycles of degree e corresponding to $y \mapsto Z(y)$. Therefore, $\varphi\colon Y \to T$ is just the Stein factorization of ψ.

Proposition 4.14. *Let $\pi\colon X \dashrightarrow Y$ be a dominant rational map from a projective variety X to a normal projective variety Y. Then there exist a normal projective variety T and a birational map $\mu\colon Y \dashrightarrow T$ satisfying the following conditions*:

1. *Let $\Gamma_T \subset X \times T$ be the graph of the composite $\mu \circ \pi\colon X \dashrightarrow Y \dashrightarrow T$. Then $\Gamma_T \to T$ is equi-dimensional.*

(2) *Let* $\mu' \colon Y \dashrightarrow T'$ *be a birational map to another normal projective variety* T' *such that* $\Gamma_{T'} \to T'$ *is equi-dimensional. Then there exists a birational morphism* $\nu \colon T' \to T$ *with* $\mu = \nu \circ \mu'$.

Proof. There exist many birational maps $\mu' \colon Y \dashrightarrow T'$ to normal projective varieties T' such that the graph $\Gamma_{T'} \subset X \times T'$ of $\mu' \circ \pi \colon X \dashrightarrow T'$ induces an equi-dimensional morphism $\Gamma_{T'} \to T'$. For example, a flattening of π creates such a rational map. For the rational map $\mu' \colon Y \dashrightarrow T'$, let us consider the intersection sheaf

$$\mathcal{M}' = \mathcal{I}_{\Gamma_{T'}/T'}(p_1^*\eta)$$

for $\eta = \boldsymbol{c}^1(\mathcal{A})^{d+1} \in G^{d+1}(X)$ for an ample invertible sheaf $\mathcal{A}$ on X. Then $\mathcal{M}'$ is nef and big by Theorem 4.4, since $p_1 \colon \Gamma_{T'} \to X$ is birational. By Theorem 4.10, there is a birational morphism $\varphi \colon T' \to T$ to a normal projective variety T such that $\Gamma_T \to T$ is equi-dimensional for the graph $\Gamma_T \subset X \times T$ of $X \dashrightarrow T' \to T$, and $\mathcal{M}' \simeq \varphi^*\mathcal{M}$ for the ample invertible sheaf

$$\mathcal{M} := \mathcal{I}_{\Gamma_T/T}(p_1^*\eta).$$

We shall show that $X \dashrightarrow T$ and $\mathcal{M}$ are independent for the choice of $X \dashrightarrow T'$. Let $\mu'' \colon Y \dashrightarrow T''$ be another birational map such that $\Gamma_{T''} \to T''$ is equi-dimensional for the graph $\Gamma_{T''} \subset X \times T''$ of $\mu'' \circ \pi$. By replacing T'' by the normalization of the graph of the birational map $T' \dashrightarrow T''$, we may assume that there is a birational morphism $\tau \colon T'' \to T'$ with $\mu' = \tau \circ \mu''$. Then,

$$\mathcal{M}'' := \mathcal{I}_{\Gamma_{T''}/T''}(p_1^*\eta) \simeq \tau^*\mathcal{M}'$$

by Lemma 4.9. Thus, the composite $\varphi \circ \tau \colon T'' \to T' \to T$ is defined only by the invertible sheaf $\mathcal{M}''$. Therefore, T and the birational map $Y \dashrightarrow T$ are uniquely determined up to isomorphism. □

Definition 4.15. The rational map $X \dashrightarrow T$ in Proposition 4.14 is called the Chow reduction of $X \dashrightarrow Y$.

4.3. **Endomorphisms of complex normal projective varieties.** In this section, we assume $\Bbbk$ to be the complex number field $\mathbb{C}$. We shall study surjective endomorphisms $f \colon X \to X$ of a normal projective variety X.

Lemma 4.16. *Let* $\pi \colon X \to Y$, $\pi' \colon X' \to Y'$, $\tau \colon Y' \to Y$, *and* $\tau' \colon X' \to X$ *be surjective morphisms for projective varieties* X, X', Y, *and* Y' *such that*

(1) $\pi \circ \tau' = \tau \circ \pi'$,
(2) Y *and* Y' *are normal,*
(3) π *is equi-dimensional of relative dimension* d,

(4) *for an open dense subset $U' \subset Y'$, the induced morphism $\pi'^{-1}(U') \to X \times_Y U'$ is a finite surjective morphism of degree e.*

Then for any $\eta \in G^{d+1}(X)$, one has an isomorphism

$$\mathcal{I}_{X'/Y'}(\tau'^{*}\eta) \simeq \tau^{*}\,\mathcal{I}_{X/Y}(\eta)^{\otimes e}.$$

Proof. Let $f\colon X' \to X \times_Y Y'$ be the induced morphism, and let $p_1\colon X \times_Y Y' \to X$ and $p_2\colon X \times_Y Y' \to Y'$ be the natural projections. Then

$$f_{*}\operatorname{cl}_{\bullet}(\mathcal{O}_{X'}) - e\operatorname{cl}_{\bullet}(\mathcal{O}_{X\times_Y Y'}) \in \operatorname{Image}(K_{\bullet}(W) \to K_{\bullet}(X \times_Y Y'))$$

for a closed subscheme W with $p_2(W) \cap U = \emptyset$. By the proof of Lemma 3.6 and by Theorem 3.20, we have

$$\mathcal{I}_{X'/Y'}(\tau'^{*}\eta) \simeq \mathcal{I}_{X\times_Y Y'/Y'}(p_1^{*}\eta)^{\otimes e} \simeq \tau^{*}\,\mathcal{I}_{X/Y}(\eta)^{\otimes e}. \qquad \square$$

Proposition 4.17. *Let $f\colon X \to X$ be a surjective endomorphism of a normal projective variety X. Let $\pi\colon X \to Y$ be an equi-dimensional surjective morphism of relative dimension d to a normal projective variety Y such that π has connected fibers and that $\pi \circ f = h \circ \pi$ for a surjective endomorphism $h\colon Y \to Y$. Let $\mathcal{A}$ be a nef and big invertible sheaf on X and let $\mathcal{M}$ be the intersection sheaf $\mathcal{I}_{X/Y}(\mathbf{c}^1(\mathcal{A})^{d+1}) = \mathcal{I}_{X/Y}(\mathcal{A}, \ldots, \mathcal{A})$.*

(1) *If $f^{*}\mathcal{A} \simeq \mathcal{A}^{\otimes q}$ for an integer q, then $h^{*}\mathcal{M}$ is $\mathbb{Q}$-linearly equivalent to $\mathcal{M}^{\otimes q}$.*
(2) *If $f^{*}\mathcal{A}$ is numerically equivalent to $\mathcal{A}^{\otimes q}$ for an integer q, then $h^{*}\mathcal{M}$ is numerically equivalent to $\mathcal{M}^{\otimes q}$.*

Proof. Note that f and h are finite morphisms. In fact, f^{*} induces an automorphism of $\mathrm{NS}(X) \otimes \mathbb{Q}$ for the Néron–Severi group $\mathrm{NS}(X)$; thus an ample divisor of X is $\mathbb{Q}$-linearly equivalent to f^{*} of a Cartier divisor. The induced morphism $(f, \pi)\colon X \to X \times_{Y,h} Y$ is a finite surjective morphism, since π has connected fibers. Thus, $\deg f = e \deg h$ for the mapping degree e of (f, π). Therefore, we can apply Lemma 4.16, and obtain an isomorphism

$$\mathcal{I}_{X/Y}(f^{*}\eta) \simeq h^{*}\,\mathcal{I}_{X/Y}(\eta)^{\otimes e} \tag{IV–2}$$

for any $\eta \in G^{d+1}(X)$. Let us consider the case where $\eta = \mathbf{c}^1(\mathcal{A})^{d+1}$. Then $\mathcal{M} = \mathcal{I}_{X/Y}(\eta)$ is nef and big by Theorem 4.4. In both cases (1) and (2), $f^{*}\mathcal{A}$ is numerically equivalent to $\mathcal{A}^{\otimes q}$. Thus, $\mathcal{I}_{X/Y}(f^{*}\eta)$ is numerically equivalent to $\mathcal{M}^{\otimes q^{d+1}}$ by Theorem 4.4, and $\deg f = q^{d+m}$ for $m = \dim Y$. Hence, $\mathcal{M}^{\otimes q^{d+1}}$ is numerically equivalent to $h^{*}\mathcal{M}^{\otimes e}$ by (IV–2), which implies that $\deg h = q_1^{m}$ for $q_1 = q^{d+1}e^{-1}$. Furthermore, $e = q^{d}$ and $\deg h = q^{m}$ by $\deg f = e \deg h$. In particular, $h^{*}\mathcal{M}$ is numerically equivalent to $\mathcal{M}^{\otimes q}$. In case (1), from (IV–2), we have

$$(h^{*}\mathcal{M} \otimes \mathcal{M}^{\otimes(-q)})^{\otimes e} \simeq \mathcal{O}_Y. \qquad \square$$

Theorem 4.18. *Let X be a complex projective variety. Then there exists a rational map $\pi\colon X \dashrightarrow Y$ into a normal projective variety Y such that*

(1) *Y is not uniruled,*
(2) *The second projection $p_2\colon \Gamma_Y \to Y$ for the graph $\Gamma_Y \subset X \times Y$ of π is equi-dimensional,*
(3) *a general fiber of $\Gamma_Y \to Y$ is rationally connected,*
(4) *π is a Chow reduction.*

Moreover, $\pi\colon X \dashrightarrow Y$ is uniquely determined up to isomorphism.

We call the rational map $\pi\colon X \dashrightarrow Y$ the *special MRC fibration*.

Proof. Let $M \to X$ be a resolution of singularities. Then we have a rational map $f\colon M \dashrightarrow S$ called a maximal rationally connected fibration (MRC fibration, for short) satisfying the following conditions (cf. [3], [16], [10]):

- S is a non-singular non-uniruled variety
- f is holomorphic along $f^{-1}(U)$ for an open dense subset $U \subset S$,
- a general fiber of f is a maximal rationally connected submanifold of M.

Moreover, the maximal rationally connected fibration is unique up to birational equivalence, i.e., if $\mu\colon M' \dashrightarrow M$ is a birational map from a non-singular projective variety M' and $f'\colon M' \dashrightarrow S'$ is a maximal rationally connected fibration of M', then $f \circ \mu = \nu \circ f'$ for a birational map $\nu\colon S' \dashrightarrow S$. Let $\pi\colon X \dashrightarrow Y$ be the Chow reduction of the rational map $X \dashrightarrow M \dashrightarrow S$. Then π is uniquely determined up to isomorphism and satisfies the required conditions. □

Theorem 4.19. *Let $f\colon X \to X$ be a surjective endomorphism of a normal complex projective variety X. Let $\pi\colon X \dashrightarrow Y$ be the special MRC fibration. Then there is an endomorphism $h\colon Y \to Y$ such that $\pi \circ f = h \circ \pi$.*

Proof. Let $X \dashrightarrow Y_1 \to Y$ be the Stein factorization of the composite $\pi \circ f\colon X \dashrightarrow Y$; we set $\pi_1\colon X \dashrightarrow Y_1$ and $\tau\colon Y_1 \to Y$. Then a general fiber of π_1 is rationally connected, and the graph $\Gamma_{Y_1} \subset X \times Y_1$ induces an equi-dimensional morphism $p_2\colon \Gamma_{Y_1} \to Y_1$. Thus, there is a birational morphism $\varphi\colon Y_1 \to Y$ such that $\pi = \varphi \circ \pi_1$ by Proposition 4.14 and by Theorem 4.18. Let $\eta = \boldsymbol{c}^1(\mathcal{A})^{d+1} \in G^{d+1}(X)$ for an ample invertible sheaf $\mathcal{A}$ of X and for $d = \dim X/Y$. Since $\Gamma_{Y_1} \to \Gamma_Y \times_Y Y_1$ is a finite surjective morphism over an open dense subset of Y_1, by applying Lemma 4.16, we have

$$\mathcal{I}_{\Gamma_{Y_1}/Y_1}(p_1^*\eta) \simeq \tau^* \mathcal{I}_{\Gamma_Y/Y}(p_1^*\eta)^{\otimes b}$$

for some $b > 0$, where p_1 denotes the first projection $\Gamma_{Y_1} \to X$ or $\Gamma_Y \to X$. On the other hand, by a property of the Chow reduction (cf. Theorem 4.10), we have

$$\mathcal{I}_{\Gamma_{Y_1}/Y_1}(p_1^*\eta) \simeq \varphi^* \mathcal{I}_{\Gamma_Y/Y}(p_1^*\eta).$$

Hence, φ is a finite morphism, since $\mathcal{I}_{\Gamma_Y/Y}(p_1^*\eta)$ is ample on Y and τ is finite. Therefore, $\varphi\colon Y_1 \simeq Y$, and the endomorphism $h = \tau \circ \varphi^{-1}\colon Y \to Y$ satisfies $\pi \circ f = h \circ \pi$. □

Remark. In Theorem 4.19, if f is étale, then h is induced from the push-forward morphism $\mathrm{Chow}(X) \to \mathrm{Chow}(X)$ given by $Z \mapsto f_*Z$. However, if f is not étale, h is not necessarily induced from the push-forward morphism.

Corollary 4.20. *Let X be a normal complex projective variety admitting a surjective endomorphism $f\colon X \to X$ such that $f^*\mathcal{H} \simeq \mathcal{H}^{\otimes q}$ for a nef and big invertible sheaf $\mathcal{H}$ and a positive integer q. Let $\pi\colon X \cdots\to Y$ be the special MRC fibration. Then there exist an endomorphism $h\colon Y \to Y$ and a nef and big invertible sheaf $\mathcal{M}$ on Y such that $\pi\circ f = h\circ\pi$ and $h^*\mathcal{M} \simeq \mathcal{M}^{\otimes q}$. Here, if $\mathcal{H}$ is ample, then one can take $\mathcal{M}$ to be ample.*

Proof. We have h by Theorem 4.19. The intersection sheaf $\mathcal{M}' = \mathcal{I}_{\Gamma_Y/Y}(p_1^*\,\boldsymbol{c}^1(\mathcal{H})^{d+1})$ is nef and big by Theorem 4.4. If $\mathcal{H}$ is ample, then so is $\mathcal{M}'$, since π is a Chow reduction (cf. Theorem 4.10). Then a suitable power $\mathcal{M}$ of $\mathcal{M}'$ satisfies the required condition by Proposition 4.17. □

Remark. An assertion similar to Corollary 4.20 is proved in [25], Proposition 2.2.4. However, the proof there is sketchy. For example, it uses the intersection sheaves, which are defined only for flat morphisms in [25], but there are no explanation how to reduce to flat morphisms.

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