Extended quadratic algebra and a model of the equivariant cohomology ring of flag varieties

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Abstract

For the root system of type A we introduce and study a certain extension of the quadratic algebra invented by S. Fomin and the first author, to construct a model for the equivariant cohomology ring of the corresponding flag variety. As an application of our construction we describe a generalization of the equivariant Pieri rule for double Schubert polynomials. For a general finite Coxeter system we construct an extension of the corresponding Nichols-Woronowicz algebra. In the case of finite crystallographic Coxeter systems we present a construction of extended Nichols-Woronowicz algebra model for the equivariant cohomology of the corresponding flag variety.

1 Introduction

In the paper [3] S. Fomin and the first author have introduced and study a model for the cohomology ring of flag varieties of type A as a commutative subalgebra generated by the so-called Dunkl elements in a certain noncommutaive quadratic algebra $\mathcal{E}_n$. An advantage of the approach developed in [3] is that it admits a simple generalization which is suitable for description of the quantun cohomology ring of flag varieties, as well as (quantum) Schubert polynomials. Constructions from the paper [3] have been generalized to other finite root systems by the authors in [6]. One of the main constituents of the above constructions is the Dunkl elements. The basic properties of the Dunkl elements are:

1) they are pairwise commuting;

2) in the so-called Calogero-Moser representation [3, 6] they correspond to the *truncated* (i.e. without differential part) rational Dunkl operators [2];

3) in the crystallographic case they correspond – after applying the so-called Bruhat representation [3, 6]– to the Monk formula in the cohomology ring of the flag variety in question;

4) in the crystallographic case, subtraction-free expressions of Schubert polynomials calculated at the Dunkl elements in the algebra $\widetilde{\mathcal{BE}}(\Sigma)$, if exist, provide a combinatorial rule for describing the Schubert basis structural constants, i.e. the intersection numbers of Schubert classes.

In the case of classical root systems Δ, the first author [4] has defined a certain extension $\widetilde{\mathcal{BE}}(\Delta)$ of the algebra $\mathcal{BE}(\Delta)$ together with a pairwise commuting family of elements, called Dunkl elements, which after applying the Calogero-Moser representation exactly coincide with the rational Dunkl operators. One of the main objective of our paper is to study a commutative subalgebra generated by the Dunkl elements in the extended algebra $\widetilde{\mathcal{BE}}(\Delta)$ in the case of type A root systems. Our main result in this direction is:

Theorem 1.1 (Pieri formula in the algebra $\mathcal{E}_n\langle R\rangle[t]$)

$$e_k(\theta_1^{(n)}, \ldots, \theta_m^{(n)}) =$$

$$\sum_{r \geq 0} (-t)^r \; (2r-1)!! \binom{m-k+2r}{2r} \Big\{ \sum_{S,I,J} X_S \prod_{\substack{a=1 \\ i_a \in I, \; j_a \in J}}^{|I|} [i_a, j_a] \Big\}.$$

See Section 2, Theorem 2.1, for a detailed explanation of conditions on sets I, J and S in the summation, and those on indices $\{i_a, j_a\}_{a=1}^{|I|}$ in the product.

When $t = 0$, Theorem 1.1 describes an analog of Pieri's rule for double Schubert polynomials. An important consequence of Theorem 1.1 states

Theorem 1.2 *In the case $t = 0$, a commutative subalgebra generated by the Dunkl elements in the algebra $\mathcal{E}_n\langle R\rangle$ is canonically isomorphic to the T-equivariant cohomology ring of the type A flag variety Fl_n.*

In Section 3 we construct the Bruhat representation of the algebra $\mathcal{E}_n\langle R\rangle[t]$ and study some properties of the former. The existence of Bruhat's representation of the algebra $\mathcal{E}_n\langle R\rangle[t]$ plays a crucial role in applications to the equivariant Schubert calculus, and constitutes an important step in the proof of Corollary 2.2.

Another objective of our paper is to construct a certain extension of the Nichols-Woronowicz model for the coinvariant algebra of a finite Coxeter group W. Recall that the Nichols-Woronowicz algebra model for the cohomology ring of flag varieties has been invented by Y. Bazlov [1]. In Section 4 we introduce a certain extension $\widetilde{\mathcal{B}}_W$ of the Nichols-Woronowicz algebra $\mathcal{B}_W$ and construct a commutative subalgebra in the extended Nichols-Woronowicz algebra. Our second main result states

Theorem 1.3 *For crystallographic root systems and $t = 0$, the commutative subalgebra of $\widetilde{\mathcal{B}}_W$ in question is isomorphic to the T-equivariant cohomology ring of the corresponding flag variety.*

2 Extension of the quadratic algebra

Definition 2.1 *The algebra $\mathcal{E}_n$ is an associative algebra generated by the symbols $[i,j]$, $1 \le i, j \le n$, $i \ne j$, subject to the relations:*
$(0) : [i,j] = -[j,i]$
$(1) : [i,j]^2 = 0,$
$(2) : [i,j][k,l] = [k,l][i,j]$, *if* $\{i,j\} \cap \{k,l\} = \emptyset$,
$(3) : [i,j][j,k] + [j,k][k,i] + [k,i][i,j] = 0.$

Let us consider the extension $\mathcal{E}_n\langle R\rangle[t]$ of the quadratic algebra $\mathcal{E}_n$ by the polynomial ring $R[t] = \mathbb{Z}[x_1, \dots, x_n][t]$ defined by the commutation relations:
(A): $[i,j]x_k = x_k[i,j]$, for $k \ne i, j$,
(B): $[i,j]x_i = x_j[i,j] + t$, $[i,j]x_j = x_i[i,j] - t$, for $i < j$,
(C): $[i,j]t = t[i,j]$.

Note that the $\mathbb{S}_n$-invariant subalgebra $R^{\mathbb{S}_n}[t]$ of $R[t]$ is contained in the center of the algebra $\mathcal{E}_n\langle R\rangle[t]$.

Definition 2.2 (1) *We define the $R[t]$-algebra $\widetilde{\mathcal{E}}_n[t]$ by*

$$\widetilde{\mathcal{E}}_n[t] = \mathcal{E}_n\langle R\rangle[t] \otimes_{R^{\mathbb{S}_n}} R.$$

More explicitly, $\widetilde{\mathcal{E}}_n[t]$ is an algebra over the polynomial ring $\mathbb{Z}[y_1, \dots, y_n]$ generated by the symbols $[i,j]$, $1 \le i, j \le n$, $i \ne j$, and $x_1, \dots, x_n, t$ satisfying the relations in the definition of the algebra $\mathcal{E}_n\langle R\rangle[t]$, together with the identification $e_i(x_1, \dots, x_n) = e_i(y_1, \dots, y_n)$, for $i = 1, \dots, n$. Denote by $\widetilde{\mathcal{E}}_{n,t_0}$ the specialization of $\widetilde{\mathcal{E}}_n[t]$ at $t = t_0$.
(2) *The Dunkl elements $\theta_i \in \widetilde{\mathcal{E}}_n[t]$, $i = 1, \dots, n$, are defined by the formula*

$$\theta_i = \theta_i^{(n)} = x_i + \sum_{j \ne i} [i,j].$$

Remark 2.1 Note that x_i's do not commute with the Dunkl elements, but only symmetric polynomials in x_i's do. By this reason we need the second copy of $R = \mathbb{Z}[y_1, \ldots, y_n]$, where y_i's assumed to be belong to the center of the algebra $\widetilde{\mathcal{E}}_n[t]$, and $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$ for any symmetric polynomial f.

Lemma 2.1 *The Dunkl elements commutes each other.*

Proof. This follows from the fact that

$$(x_i + x_j)[i,j] = [i,j](x_i + x_j).$$

Let $e_k(x_1, \ldots, x_n)$, $1 \le k \le n$, stand for the elementary symmetric polynomial of degree k in the variables $x_1, \ldots, x_n$. We put by definition, $e_0(x_1, \ldots, x_n) = 1$, and $e_k(x_1, \ldots, x_n) = 0$, if $k < 0$.

Theorem 2.1 (Pieri formula in the algebra $\mathcal{E}_n\langle R\rangle[t]$)

$$e_k(\theta_1^{(n)}, \ldots, \theta_m^{(n)}) =$$

$$\sum_{r \ge 0} (-t)^r \; N(m-k, 2r) \Big\{ \sum_{S,I,J} X_S \prod_{\substack{a=1 \\ i_a \in I,\ j_a \in J}}^{|I|} [i_a, j_a] \Big\},$$

where

$$N(a, 2b) = (2b-1)!! \binom{a+2b}{2b},$$

$X_S := \prod_{s \in S} x_s$; *the second summation runs over triples of sets (S, I, J) such that $S \subset \{1, \ldots, k\}$, I and J are subsets of the same cardinality in the set $\{1, \ldots, n\} \backslash S$, and $|I| + |S| + 2r = k$; the product is taken over pairs $(i_a, j_a)_{a=1}^{|I|}$ such that $1 \le i_a \le m < j_a \le n$ and the indices $i_1, \ldots, i_{|I|}$ are all distinct.*

Proof. Let $\mathcal{A} := \{1, \ldots, m\} \subset \{1, \ldots, n\}$, $d := n - m$ and $j_i := m + i$. Denote by $E_k(\mathcal{A})$ the right-hand side of the formula. It will suffice to prove the recursive formula

$$E_k(\mathcal{A} \cup \{j = j_1\}) = E_k(\mathcal{A}) + E_{k-1}(\mathcal{A})(x_j + \sum_{s \ne j} [j, s]).$$

For a subset $I = \{i_1, \ldots, i_l\} \subset \{1, \ldots, n\}$ and $p \notin I$, we use the symbol

$$\langle\langle I | p \rangle\rangle = \sum_{w \in \mathbb{S}_l} [i_{w(1)}, p] \cdots [i_{w(l)}, p]$$

as defined in [8]. We have the following decompositions:

$$
\begin{aligned}
&E_k(\mathcal{A})\\
&\quad = \sum_{r\geq 0}(-t)^r N(m-k,2r) X_S \sum_{S\subset\mathcal{A}} \sum_{I_1\cdots I_d\subset_{k-2r-|S|}\mathcal{A}\setminus S} \langle\!\langle I_1|j_1\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle\\
&\quad = \sum_{r\geq 0}(-t)^r N(m-k,2r)(A_1^r + A_2^r),\\
&E_k(\mathcal{A}\cup\{j\})\\
&\quad = \sum_{r\geq 0}(-t)^r N(m-k+1,2r) X_S \sum_{S\subset\mathcal{A}} \sum_{I_2\cdots I_d\subset_{k-2r-|S|}\mathcal{A}\cup\{j\}\setminus S} \langle\!\langle I_2|j_2\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle\\
&\quad = \sum_{r\geq 0}(-t)^r N(m-k+1,2r)(B_1^r + B_2^r + B_3^r),\\
&E_{k-1}(\mathcal{A})\sum_{s\neq j}[j,s]\\
&\quad = \sum_{r\geq 0}(-t)^r N(m-k+1,2r) X_S \sum_{S\subset\mathcal{A}} \sum_{I_1\cdots I_d\subset_{k-1-2r-|S|}\mathcal{A}\setminus S} \langle\!\langle I_1|j_1\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle \sum_{s\neq j}[j,s]\\
&\quad = \sum_{r\geq 0}(-t)^r N(m-k+1,2r)(C_1^r + C_2^r + C_3^r + C_4^r),\\
&E_{k-1}(\mathcal{A})x_j\\
&\quad = \sum_{r\geq 0}(-t)^r N(m-k+1,2r) X_S \sum_{S\subset\mathcal{A}} \sum_{I_1\cdots I_d\subset_{k-1-2r-|S|}\mathcal{A}\setminus S} \langle\!\langle I_1|j_1\rangle\!\rangle\cdots\langle\!\langle I_d|j_d\rangle\!\rangle x_j\\
&\quad = \sum_{r\geq 0}(-t)^r N(m-k+1,2r)(D_1^r + D_2^r),
\end{aligned}
$$

where A_i^r, B_i^r, C_i^r, D_i^r are defined as follows.

- A_1^r is the sum of terms with $I_1=\emptyset$; A_2^r is the sum of terms with $I_1\neq\emptyset$.
- B_1^r is the sum of terms with $j\notin S\cup I_2\cup\cdots\cup I_d$; B_2^r is the sum of terms with $j\in I_2\cup\cdots\cup I_d$; B_3^r is the sum of terms with $j\in S$.
- C_1^r is the sum of terms with $s\in\mathcal{A}\setminus(S\cup I_1\cup\cdots\cup I_d)$; C_2^r is the sum of terms with $s\in I_2\cup\cdots\cup I_d\cup\mathcal{A}^c$; C_3^r is the sum of terms with $s\in S$; C_4^r is the sum of terms with $s\in I_1$.
- D_1^r is the sum of terms with $I_1=\emptyset$; D_2^r is the sum of terms with $I_1\neq\emptyset$.

Based on the same arguments used in [8], we can see that $A_1^r = B_1^r$, $A_2^r + C_1^r = 0$, $B_2^r = C_2^r$ and $C_4^r = 0$. It is also easy to see that $B_3^r = D_1^r$. Now we have

$$\begin{aligned}
&E_k(\mathcal{A}) + E_{k-1}(\mathcal{A})(x_j + \sum_{s\neq j}[j,s]) - E_k(\mathcal{A}\cup\{j\}) \\
&= \sum_{r\geq 0}\sum_{S\subset\mathcal{A}}(-t)^r \left(N(m-k,2r)(A_1^r + A_2^r) - N(m-k+1,2r)(B_1^r - C_1^r - C_3^r - D_2^r)\right) \\
&= \sum_{r\geq 1}\sum_{S\subset\mathcal{A}}(-t)^r \left(N(m-k,2r) - N(m-k+1,2r)\right)(A_1^r + A_2^r) \\
&\quad + \sum_{r\geq 0}\sum_{S\subset\mathcal{A}}(-t)^r N(m-k+1,2r)(C_3^r + D_2^r).
\end{aligned}$$

From the commutation relation $[i,j]x_j = x_i[i,j] - t$, we have

$$\begin{aligned}
D_2^r > &= X_S \sum_{\substack{I_1\cdots I_d\subset_{k-1-2r-|S|}\mathcal{A}\setminus S \\ I_1=\{a_1,\ldots,a_{|I_1|}\}}} \sum_{w\in\mathbb{S}_{|I_1|}} x_{a_{w(|I_1|)}}[a_{w(1)},j]\cdots[a_{w(|I_1|)},j]\langle\langle I_2|j_2\rangle\rangle\cdots\langle\langle I_d|j_d\rangle\rangle \\
&\quad -tX_S \sum_{\substack{I_1\cdots I_d\subset_{k-1-2r-|S|}\mathcal{A}\setminus S \\ I_1=\{a_1,\ldots,a_{|I_1|}\}}} \sum_{w\in\mathbb{S}_{|I_1|}} [a_{w(1)},j]\cdots[a_{w(|I_1|-1)},j]\langle\langle I_2|j_2\rangle\rangle\cdots\langle\langle I_d|j_d\rangle\rangle \\
&= \sum_{s\notin S} X_{S\cup\{s\}} \sum_{I_1\cdots I_d\subset_{k-1-2r-(|S|+1)}\mathcal{A}\setminus S\cup\{s\}} \langle\langle I_1|j_1\rangle\rangle[s,j]\langle\langle I_2|j_2\rangle\rangle\cdots\langle\langle I_d|j_d\rangle\rangle \\
&\quad -(m-k+2r+2)tX_S \sum_{I_1\cdots I_d\subset_{k-2-2r-|S|}\mathcal{A}\setminus S} \langle\langle I_1|j_1\rangle\rangle\cdots\langle\langle I_d|j_d\rangle\rangle \\
&= -C_3^r + (-t)(m-k+2r+2)(A_1^{r+1} + A_2^{r+1}).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&(-t)^{r+1}\left(N(m-k,2(r+1)) - N(m-k+1,2(r+1))\right)(A_1^{r+1} + A_2^{r+1}) \\
&= -(-t)^{r+1}(2r+1)!!\frac{(m-k+2r+2)!}{(2r+1)!(m-k+1)!}(A_1^{r+1} + A_2^{r+1}) \\
&= -(-t)^r(2r-1)!!\frac{(m-k+2r+1)!}{(2r)!(m-k+1)!}\cdot(-t)(m-k+2r+2)(A_1^{r+1} + A_2^{r+1}) \\
&= -(-t)^r N(m-k+1,2r)(C_3^r + D_2^r).
\end{aligned}$$

This shows the desired result.

Corollary 2.1 *The list of relations in the algebra* $\widetilde{\mathcal{E}}_n[t]$

$$e_k(\theta_1^{(n)},\ldots,\theta_n^{(n)}) =$$

$$e_k(y_1,\ldots,y_n)+\sum_{r\ge 1}(-t)^r\ (2r-1)!!\binom{n-k+2r}{2r}e_{k-2r}(y_1,\ldots,y_n),\ 1\le k\le n,$$

describes the complete set of relations among the Dunkl elements $\theta_1^{(n)},\ldots,\theta_n^{(n)}$.

Corollary 2.2 *For* $t=0$, *the subalgebra of* $\widetilde{\mathcal{E}}_{n,0}$ *generated by the Dunkl elements* $\theta_1,\ldots,\theta_n$ *over* $H_T(\mathrm{pt})=\mathbb{Z}[y_1,\ldots,y_n]$ *is isomorphic to the* T*-equivariant cohomology ring* $H_T^*(Fl_n)$.

Proof. Let $(0=U_0\subset U_1\subset\cdots\subset U_n)$ be the universal flag over Fl_n. First of all it follows from Corollary 2.1 that the natural map $z_i:=-c_1(U_i/U_{i-1})\mapsto\theta_i$, $y_i\mapsto y_i$ defines a surjective homomorphism

$$\pi: H_T^*(Fl_n)\longrightarrow\mathbb{Z}[y_1,\ldots,y_n][\theta_1,\ldots,\theta_n]\subset\widetilde{\mathcal{E}}_{n,0}.$$

On the other hand, it follows from the definitions that the image of Dunkl's element θ_i in the Bruhat representation (see Section 3) acts according to the rule:

$$\theta_i\ \underline{w}=y_{w(i)}\underline{w}+\sum_{\substack{j>i\\ l(wt_{ij})=l(w)+1}}\underline{wt_{ij}}-\sum_{\substack{j<i\\ l(wt_{ij})=l(w)+1}}\underline{wt_{ij}}.$$

This rule exactly corresponds to the Monk formula for double Schubert polynomials, see e.g. [7, Exercise 2.7.2]. Therefore the element $\underline{\mathrm{id.}}$ generates $\mathbb{Z}[y_1,\ldots,y_n]\langle\mathbb{S}_n\rangle$ over $\mathbb{Z}[y_1,\ldots,y_n][\theta_1,\ldots,\theta_n]$. Since $\mathbb{Z}[y_1,\ldots,y_n]\langle\mathbb{S}_n\rangle$ is isomorphic as a $\mathbb{Z}[y_1,\ldots,y_n]$-module to the quotient $\mathbb{Z}[y_1,\ldots,y_n,z_1,\ldots,z_n]/J_n$, where J_n denotes the ideal generated by the elements $e_k(z_1,\ldots,z_n)-e_k(y_1,\ldots,y_n)$, $1\le k\le n$. This exactly means that the homomorphism π is an isomorphism.

We expect that for general t the subalgebra in $\widetilde{\mathcal{E}}_n[t]$ generated by the Dunkl elements $\theta_1,\ldots,\theta_n$ over $\mathbb{Z}[y_1,\ldots,y_n][t]$ is isomorphic to the $(T\times\mathbf{C}^\times)$-equivariant cohomology ring $H^*_{T\times\mathbf{C}^\times}(Fl_n)$.

Theorem 2.2 *The subalgebra generated by the elements* $g_1:=[1,2], g_2:=[2,3],\ldots,\ g_{n-1}:=[n-1,n]$ *in the algebra* $\mathcal{E}_n\langle R\rangle[t]$ *is isomorphic to the nil degenerate affine Hecke algebra of type* $A_{n-1}^{(1)}$, *i.e. the algebra given by two* *sets of generators $g_1, \ldots, g_{n-1}$ and $x_1, \ldots, x_n$ subject to the set of defining relations:*

$$g_i^2 = 0, \quad g_i g_j = g_j g_i, \ \text{ if } |i-j| > 1, \quad g_i g_j g_i = g_j g_i g_j, \ \text{if } |i-j| = 1,$$

$$x_i \ x_j = x_j \ x_i, \quad x_k \ g_i = g_i \ x_k, \ \text{if } k \neq i, i+1, \ \ g_i x_i - x_{i+1} g_i = t.$$

3 Bruhat representation

Let us recall the definition of the Bruhat representation of the algebra $\mathcal{E}_n$ on the group ring of the symmetric group $\mathbb{Z}\langle \mathbb{S}_n \rangle = \oplus_{w \in \mathbb{S}_n} \mathbb{Z} \cdot \underline{w}$. The operator σ_{ij}, $i < j$, is defined as follows:

$$\sigma_{ij}(\underline{w}) = \begin{cases} \underline{w t_{ij}}, & \text{if } l(wt_{ij}) = l(w) + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then the Bruhat representation of $\mathcal{E}_n$ is defined by $[i,j].\underline{w} := \sigma_{ij}(\underline{w})$.

Now we extend the Bruhat representation to that of the algebra $\mathcal{E}_n\langle R \rangle[t]$ defined on

$$R[t]\langle \mathbb{S}_n \rangle = \oplus_{w \in \mathbb{S}_n} \mathbb{Z}[y_1, \ldots, y_n][t] \cdot \underline{w}.$$

For $f(y) \in \mathbb{Z}[y_1, \ldots, y_n][t]$ and $w \in \mathbb{S}_n$, we define the $\mathbb{Z}[t]$-linear operators $\tilde{\sigma}_{ij}$, $i < j$, and ξ_k as follows:

$$\tilde{\sigma}_{ij}(f(y)\underline{w}) = \begin{cases} t(\partial_{w(i)w(j)} f(y))\underline{w} + f(y)\underline{w t_{ij}}, & \text{if } l(wt_{ij}) = l(w) + 1, \\ t(\partial_{w(i)w(j)} f(y))\underline{w}, & \text{otherwise,} \end{cases}$$

$$\xi_k(f(y)\underline{w}) = (y_{w(k)} f(y))\underline{w}.$$

Proposition 3.1 *The algebra $\mathcal{E}_n\langle R \rangle[t]$ acts $\mathbb{Z}[t]$-linearly on $\mathbb{Z}[y][t]\langle \mathbb{S}_n \rangle$ via $[ij] \mapsto \tilde{\sigma}_{ij}$ and $x_k \mapsto \xi_k$.*

Proof. Let us check the compatibility with the defining relations of the algebra $\widetilde{\mathcal{E}}_n[t]$. We show the compatibility only with the relations (1), (3) and (B). The rest are easy to check.

Let us start with the relation (1). We have

$$\begin{aligned} \tilde{\sigma}_{ij}^2(f(y)\underline{w}) &= \tilde{\sigma}_{ij}\left(t(\partial_{w(i)w(j)} f(y))\underline{w} + f(y)\sigma_{ij}(\underline{w})\right) \\ &= t^2(\partial^2_{w(i)w(j)} f(y))\underline{w} + t(\partial_{w(i)w(j)} f(y))\sigma_{ij}(\underline{w}) \\ &\quad + t(\partial_{w(j)w(i)} f(y))\sigma_{ij}(\underline{w}) + f(y)\sigma_{ij}^2(\underline{w}). \end{aligned}$$

Since $\partial^2_{w(i)w(j)} = 0$, $\sigma_{ij}^2 = 0$ and $\partial_{w(i)w(j)} = -\partial_{w(j)w(i)}$, we get $\tilde{\sigma}_{ij}^2 = 0$.

For the relation (3), we have

$$\begin{aligned}
\tilde{\sigma}_{ij}\tilde{\sigma}_{jk}(f(y)\underline{w}) &= \tilde{\sigma}_{ij}\left(t(\partial_{w(j)w(k)}f(y))\underline{w} + f(y)\sigma_{jk}(\underline{w})\right) \\
&= t^2(\partial_{w(i)w(j)}\partial_{w(j)w(k)}f(y))\underline{w} + t(\partial_{w(j)w(k)}f(y))\sigma_{ij}(\underline{w}) \\
&\quad + t(\partial_{w(i)w(k)}f(y))\sigma_{jk}(\underline{w}) + f(y)\sigma_{ij}\sigma_{jk}(\underline{w}).
\end{aligned}$$

We also obtain $\tilde{\sigma}_{jk}\tilde{\sigma}_{ki}(f(y)\underline{w})$ and $\tilde{\sigma}_{ki}\tilde{\sigma}_{ij}(f(y)\underline{w})$ by the cyclic permutation of i, j, k. The 3-term relations

$$\partial_{w(i)w(j)}\partial_{w(j)w(k)} + \partial_{w(j)w(k)}\partial_{w(k)w(i)} + \partial_{w(k)w(i)}\partial_{w(i)w(j)} = 0$$

and

$$\sigma_{ij}\sigma_{jk} + \sigma_{jk}\sigma_{ki} + \sigma_{ki}\sigma_{ij} = 0$$

show the desired equality

$$\tilde{\sigma}_{ij}\tilde{\sigma}_{jk} + \tilde{\sigma}_{jk}\tilde{\sigma}_{ki} + \tilde{\sigma}_{ki}\tilde{\sigma}_{ij} = 0.$$

Finally, we check the relation (B). We have

$$\begin{aligned}
\tilde{\sigma}_{ij}\xi_i(f(y)\underline{w}) &= \tilde{\sigma}_{ij}(y_{w(i)}f(y)\underline{w}) \\
&= t\partial_{w(i)w(j)}(y_{w(i)}f(y))\underline{w} + (y_{w(i)}f(y))\sigma_{ij}(\underline{w}) \\
&= t(f(y)\underline{w}) + t(y_{w(j)}\partial_{w(i)w(j)}f(y))\underline{w} + y_{wt_{ij}(j)}\sigma_{ij}(\underline{w}) \\
&= \xi_j\tilde{\sigma}_{ij}(f(y)\underline{w}) + t(f(y)\underline{w}).
\end{aligned}$$

Theorem 3.1 *Let $\mathfrak{S}_w(x, y)$ be the double Schubert polynomial corresponding to $w \in \mathbb{S}_n$. Then, we have*

$$\mathfrak{S}_w(\theta, y)(\underline{\mathrm{id.}}) = \underline{w}.$$

Proof. This follows from the Monk formula for the double Schubert polynomials and

$$\begin{aligned}
(\theta_i - y_{w(i)})(\underline{w}) &= \xi_i(\underline{w}) + \sum_{j \neq i}\sigma_{ij}(\underline{w}) - y_{w(i)}\underline{w} \\
&= \sum_{j<i, l(wt_{ij})=l(w)+1}\underline{wt_{ij}} - \sum_{j>i, l(wt_{ij})=l(w)+1}\underline{wt_{ij}}.
\end{aligned}$$

Remark 3.1 Only when $t = 0$, one can extend $\mathbb{Z}[y][t]$-linearly the Bruhat representation of the algebra $\mathcal{E}_n\langle R\rangle[t]$ to that of the algebra $\widetilde{\mathcal{E}}_n[t]$. In fact, Theorem 3.1 describes the multiplicative structure of the $\mathbb{Z}[y]$-subalgebra generated by the Dunkl elements in $\widetilde{\mathcal{E}}_{n,0}$, which is isomorphic to $H_T^*(Fl_n)$. Nevertheless, the property of the double Schubert polynomials in Theorem 3.1 holds for arbitrary t.

4 Quantization

Definition 4.1 *The algebra $\mathcal{E}_n^q$ is a $\mathbb{Z}[q_1, \ldots, q_{n-1}]$-algebra defined by the same generators and relations as in the definition of the algebra $\mathcal{E}_n$ except that the relation* (1) *is replaced by*

$$(1)' \quad [i,j]^2 = \begin{cases} q_i & \text{if } i = j-1, \\ 0, & \text{if } i < j-1. \end{cases}$$

The extension $\mathcal{E}_n^q\langle R\rangle[t]$ of the algebra $\mathcal{E}_n^q$ is also defined by the relations (A), (B) and (C).

The Bruhat representation for $\mathcal{E}_n$ is deformed to the quantum Bruhat representation for $\mathcal{E}_n^q$. We define the quantum Bruhat operator σ_{ij}^q, $i < j$, acting on $\mathbb{Z}[q_1, \ldots, q_{n-1}]\langle \mathbb{S}_n\rangle = \oplus_{w\in\mathbb{S}_n}\mathbb{Z}[q_1, \ldots, q_{n-1}] \cdot \underline{w}$ as follows:

$$\sigma_{ij}^q(\underline{w}) = \begin{cases} q_{ij}\underline{wt_{ij}}, & \text{if } l(wt_{ij}) = l(w) - 2(j-i) + 1, \\ \underline{wt_{ij}}, & \text{if } l(wt_{ij}) = l(w) + 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $f(y) \in \mathbb{Z}[y_1, \ldots, y_n][t]$ and $w \in \mathbb{S}_n$, we define the $\mathbb{Z}[q_1, \ldots, q_{n-1}][t]$-linear operators $\tilde{\sigma}_{ij}^q$ by

$$\tilde{\sigma}_{ij}^q(f(y)\underline{w}) = t(\partial_{w(i)w(j)}f(y))\underline{w} + f(y)\sigma_{ij}^q(\underline{w}).$$

We can check the well-definedness of the quantum extended Bruhat representation $[ij] \mapsto \tilde{\sigma}_{ij}^q$, $x_k \mapsto \xi_k$ of the algebra $\mathcal{E}_n^q\langle R\rangle[t]$ in the same way as the proof of Proposition 3.1.

Theorem 4.1 *Let $\mathfrak{S}_w^q(x,y)$ be the quantum double Schubert polynomial corresponding to $w \in \mathbb{S}_n$. Then, we have*

$$\mathfrak{S}_w^q(\theta, y)(\underline{\mathrm{id.}}) = \underline{w}$$

under the quantum extended Bruhat representation.

Proof. This follows from the quantum Monk formula for the quantum Schubert polynomials.

Corollary 4.1 *The quantum double Schubert polynomials* $\mathfrak{S}_w^q(x,y)$ *are characterized by the conditions:*
(1) $\mathfrak{S}_w^q(x,y)|_{q=0} = \mathfrak{S}_w(x,y)$,
(2) $\mathfrak{S}_w^q(x,y)$ *is a linear combination of polynomials* $\mathfrak{S}_v(x,y)$ *with* $v \leq w$ *over* $\mathbb{Z}[q_1,\ldots,q_{n-1}]$,
(3) $\mathfrak{S}_w^q(\theta,y)(\underline{\mathrm{id.}}) = \underline{w}$.

5 Nichols-Woronowicz model

The model of the equivariant cohomology ring $H_T^*(Fl_n)$ in the algebra $\widetilde{\mathcal{E}}_n$ has a natural interpretation in terms of the Nichols-Woronowicz algebra. The Nichols-Woronowicz approach leads us to the uniform construction for arbitrary root systems.

We denote by $\mathcal{B}_W$ the Nichols-Woronowicz algebra associated to the Yetter-Drinfeld module

$$V = \bigoplus_{\alpha\in\Delta} \mathbb{R}[\alpha]/([\alpha]+[-\alpha])$$

over the finite Coxeter group W of the root system Δ. Let $\mathfrak{h}$ be the reflection representation of W and $R = \mathrm{Sym}\mathfrak{h}^*$ the ring of polynomial functions on $\mathfrak{h}$. Let us consider the extension $\mathcal{B}_W\langle R\rangle[t]$ of the algebra $\mathcal{B}_W$ by the polynomial ring $R[t]$ defined by the commutation relation

$$[\alpha]x = s_\alpha(x)[\alpha] + t(x,\alpha) \quad \text{for } \ x \in \mathfrak{h}^*.$$

Definition 5.1 *We define the* R*-algebra* $\widetilde{\mathcal{B}}_W$ *by*

$$\widetilde{\mathcal{B}}_W = \mathcal{B}_W\langle R\rangle[t] \otimes_{R^W} R.$$

Choose a W-invariant constants $(c_\alpha)_\alpha$. Let us consider a linear map $\mu : \mathfrak{h}^* \to \widetilde{\mathcal{B}}_W$ defined as

$$\mu(x) = x + \sum_{\alpha\in\Delta_+} c_\alpha (x,\alpha)[\alpha]$$

for $x \in \mathfrak{h}^*$.

Proposition 5.1 $[\mu(x), \mu(y)] = 0$, $x, y \in \mathfrak{h}^*$.

The linear map μ extends to a homomorphism of algebras

$$\mu : R \to \mathcal{B}_W\langle R\rangle[t].$$

Denote by $\widetilde{\mu}$ the composite of the homomorphisms

$$R \otimes_{\mathbb{Z}} R \stackrel{\mu\otimes 1}{\to} \mathcal{B}_W\langle R\rangle[t] \otimes_{\mathbb{Z}} R \to \widetilde{\mathcal{B}}_W.$$

Theorem 5.1 *If $t = 0$ and the constants $(c_\alpha)_\alpha$ are generic, the image of the homomorphism $\widetilde{\mu}$ is isomorphic to the algebra $R \otimes_{R^W} R$. In particular, when W is the Weyl group, it is isomorphic to the T-equivariant cohomology ring $H^*_T(G/B)$ of the corresponding flag variety G/B.*

The proof is based on the correspondence between the twisted derivation D_α and the divided difference operator ∂_α. We define the operator D_α as the twisted derivation on $\widetilde{\mathcal{B}}_W$ determined by the conditions:
(1): $D_\alpha(x) = 0$, for $x \in R$,
(2): $D_\alpha([\beta]) = \delta_{\alpha,\beta}$, for $\alpha, \beta \in \Delta_+$,
(3): $D_\alpha(fg) = D_\alpha(f)g + s_\alpha(f)D_\alpha(g)$.
The operator D_α is linear with respect to R on the second component.

Proposition 5.2

$$\cap_{\alpha\in\Delta_+}\mathrm{Ker}(D_\alpha) = R[t] \otimes_{R^W} R$$

Proof. Any element $\omega \in \mathcal{B}_W\langle R\rangle[t]$ can be written as

$$\omega = f_1\varphi_1 + \cdots + f_k\varphi_k,$$

where $f_1, \ldots, f_k \in R[t]$ are linearly independent, and $\varphi_1, \ldots, \varphi_k \in \mathcal{B}_W$. We have

$$D_\alpha(\omega) = s_\alpha(f_1)D_\alpha(\varphi_1) + \cdots + s_\alpha(f_k)D_\alpha(\varphi_k)$$

from the twisted Leibniz rule. If $D_\alpha(\omega) = 0$, we have $D_\alpha(\varphi_1) = \cdots = D_\alpha(\varphi_k) = 0$. Hence, $\omega \in \cap_{\alpha\in\Delta_+}\mathrm{Ker}(D_\alpha)$ implies that $\varphi_i \in \mathcal{B}_W^0 = \mathbb{R}$ for $i = 1, \ldots, k$. This means $\omega \in R[t]$.

Proposition 5.3

$$D_\alpha(\widetilde{\mu}(x)) = c_\alpha\widetilde{\mu}(\partial_\alpha(x))$$

for $x \in R \otimes_{\mathbb{Z}} R$.

Proof. When $x = \beta \otimes 1$, $\beta \in \Delta$, we can check that

$$D_\alpha(\widetilde{\mu}(\beta \otimes 1)) = c_\alpha(\beta, \alpha) = c_\alpha \widetilde{\mu}(\partial_\alpha(\beta)).$$

Hence, we have $D_\alpha(\widetilde{\mu}(x)) = c_\alpha \widetilde{\mu}(\partial_\alpha(x))$ for $x \in \mathfrak{h}^* \otimes R$. On the other hand, the both-hands sides satisfy the same twisted Leibniz rule, so it follows that $D_\alpha(\widetilde{\mu}(x)) = c_\alpha \widetilde{\mu}(\partial_\alpha(x))$ for $x \in R \otimes R$.

(*Proof of Theorem 5.1*) If $x \in R^W \otimes_{\mathbb{Z}} R$, we have $D_\alpha(\widetilde{\mu}(x)) = 0$ for every $\alpha \in \Delta_+$ from Proposition 5.3. This implies from Proposition 5.2 that $\widetilde{\mu}(x) \in R^W \otimes_{R^W} R$. When $t = 0$, $\widetilde{\mu}(x)$ coincides with the element of R which is obtained by replacing all the symbols $[\alpha]$ by zero in $\widetilde{\mu}(x)$. Hence, the homomorphism $\widetilde{\mu}$ factors through $R \otimes_{R^W} R \to \widetilde{\mathcal{B}}_W$. Since a linear basis of the coinvariant algebra of W gives an R^W-basis of R, it is easy to see that $R \otimes_{R^W} R \to \widetilde{\mathcal{B}}_W$ is injective.

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