The Bender-Wu analysis and the Voros theory. II

by

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Abstract

In our earlier paper ([AKT1]), by interpreting the formal transformation to the Airy equation near a simple turning point as the symbol of a microdifferential operator, we derived the Voros connection formula or, equivalently, the discontinuity function of a Borel transformed WKB solution at its movable singularities. In this paper we extend this approach to the two turning points problem; by constructing the formal transformation which brings a Schrödinger equation with two paired simple turning points that merge (i.e., a merging-turning-points equation or an MTP equation for short) to the Weber equation and by interpreting it as the symbol of a microdifferential operator, we reduce the analysis of an MTP equation to that of the Weber equation. Then, combining this transformation theory with the so-called "Sato's conjecture" for the Weber equation, we obtain the discontinuity function of a Borel transformed WKB solution of an MTP equation at its fixed singularities.

0 Introduction

In our earlier paper [AKT1] we discussed how to understand the pioneering work of Bender and Wu ([BW]) in the framework of exact WKB analysis ([V], [P1]), i.e., WKB analysis based upon the Borel resummation. This is what Silverstone ([S]) also aimed at; the paper [S] clearly explains how the Borel resummation method clarifies several ambiguous points in traditional WKB analysis. An important point of [AKT1] is that the formal transformation used in [S] can be interpreted as the symbol of a microdifferential operator acting on the Borel transformed WKB solutions ([AKT1, Section 2]). In a neighborhood of a simple turning point, this interpretation enabled us to derive the

Voros connection formula from the connection formula for Gauss' hypergeometric functions. But, when two turning points are relevant, we encounter the following troubles in putting the idea into practice.

Problem 1. *To perform the actual computation we use an integral operator that represents the microdifferential operator in question. In the case of two turning points problem we are to analyze the analytic structure of a Borel transformed WKB solution at two singular points whose relative location is fixed* (*the so-called "fixed singularities"*), *and we need to guarantee the existence of a sufficiently large domain of definition of the integral operator for this purpose. In* [AKT1, Section 2], *we studied only "movable singularities" which eventually merge, and troubles of this sort did not arise.*

Problem 2. *In the situation where only one simple turning point is relevant, the Borel transformed WKB solution of the canonical equation* (*the Airy equation*) *can be explicitly written down in terms of hypergeometric functions. When two turning points are relevant such a concrete expression cannot be expected. Hence some other way of describing analytic properties of the Borel transformed WKB solutions of the canonical equation* (*i.e., the Weber equation this time*) *should be found.*

Problem 3. *In the two turning points problem, the canonical equation contains an infinite series* $E(\eta) = \sum_{k\geq 0} E_k \eta^{-k}$ *as the parameter* E *contained in the Weber equation. Hence we have to find the correct analytic meaning of WKB solutions of an equation whose coefficients contain such infinite series. In this paper we use the terminology "*∞*-Weber equation" to designate the Weber equation with an infinite series as its parameter, if such a distinction is necessary.*

Our answers to these problems are as follows.

To cope with Problem 1, we consider a Schrödinger operator that depends on a parameter t (tied up with the energy in most applications) and that has two paired simple turning points which merge to form a double turning point at $t = 0$. Such an operator is called a merging-paired-simple-turning-points operator, or, for short, a merging-turning-points (MTP) operator. To be more concrete, an MTP operator is a Schrödinger operator of the form

$$\frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x}, t) \quad (\eta : \text{a large parameter}) \tag{0.1}$$

which depends on a parameter t, where the potential $Q(\tilde{x}, t)$ satisfies the following conditions:

(0.2) $Q(\tilde{x}, t)$ is holomorphic near the origin $(\tilde{x}, t) = (0, 0)$,

(0.3) $Q(\tilde{x}, 0) = c\tilde{x}^2 + O(\tilde{x}^3)$ (c : a non-zero constant),

(0.4) for each t ($\neq 0$), the equation $Q(\tilde{x}, t) = 0$ in $\tilde{x}$ has two distinct simple roots which merge together at $t = 0$, whereas other roots stay uniformly away from 0 for sufficiently small t.

(The definition shall be made more precise concerning the merging speed of two simple turning points in Section 2. Cf. Definition 2.1 in Section 2.) Then we can construct a transformation that brings the following MTP equation

$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x}, t)\right) \widetilde{\psi} = 0 \tag{0.5}$$

uniformly to the following t-dependent ∞-Weber equation

$$\left(\frac{d^2}{dx^2} - \eta^2 (E(t, \eta) - \frac{1}{4}x^2)\right) \psi = 0. \tag{0.6}$$

The precise meaning of the "uniform transformation" will be given in terms of the transformation of the Borel transformed WKB solution (Section 2, Remark 2.4). Intuitively speaking, we define the uniformity of transformation through the uniformity with respect to t of the domain of definition of the integral operator determined by the transformation. Since the distance of "fixed singularities" of (0.6) tends to 0 as t tends to 0 (Section 4, Remark 4.1), the uniformity guarantees that they are contained in the domain of definition of the integral operator for sufficiently small t. Thus Problem 1 disappears for an MTP operator with t sufficiently small. Before establishing the transformation theorem (Theorem 2.2 and Theorem 2.4) for $t \neq 0$, we first prove the result for $t = 0$ in Section 1. The result plays an important auxiliary role in our later discussions in Section 2, and it is also of its own interest as it gives the transformation theory in the situation where a double turning point is relevant. (Cf. [P1], [DDP2], [P2], [T].) The required transformation theory for an MTP operator with $t \neq 0$ (Theorem 2.2 and Theorem 2.4) is constructed through a perturbation of the transformation found for $t = 0$. In Sections 1 and 2 we concentrate our attention to the formal structure of the transformation, and the estimation of the growth order of the obtained series is separately discussed in Appendices A and B.

In solving Problem 2 we make use of "Sato's conjecture" ([KT1]), whose clear-cut proof has recently been given by Shen and Silverstone ([SS]). (See also [V], which gives a transcendental proof for the parabola potential (versus the inverted-parabola potential used in Sections 2 and 3).) An important consequence of Sato's conjecture is that the discontinuity function (or, more specifically, the alien derivative) of the Borel transformed WKB solution of the Weber equation is an E-independent constant multiple of the Borel transformed WKB solution evaluated at a fixed singularity. (Theorem 3.1; see also [DDP1] and [CNP].) Note that the study of one simple turning point problem given in [AKT1] makes use of the explicit form of the Borel transformed WKB solution of the Airy equation only in analyzing the structure of its discontinuity at a movable singular point. Hence this seemingly somewhat weaker result suffices for the study of the connection problem.

To answer Problem 3 we make full use of the estimation of the coefficients of the series $E(t,\eta)$ (Appendix B); it is a symbol of a microdifferential operator. This observation enables us to employ the same technique as was used in [AKT1] to give an analytic meaning to the formal coordinate transformation in the independent variable of the Schrödinger equation, i.e., x-variable. This time we regard E, together with x, as an auxiliary variable in a resurgent function in η-variable, that is, we interpret

$$\tilde{\psi}(x,\eta) = \psi(x,\eta,E(\eta)) \tag{0.7}$$

as

$$\tilde{\psi}(x,\eta) = \sum_{n\geq 0} \frac{(E_1\eta^{-1} + E_2\eta^{-2} + \cdots)^n}{n!} \frac{\partial^n}{\partial E_0^n} \psi(x,\eta,E_0), \tag{0.8}$$

or

$$\begin{aligned} \tilde{\psi}_B(x,y) \\ = \sum_{n\geq 0} \frac{(E_1(\frac{\partial}{\partial y})^{-1} + E_2(\frac{\partial}{\partial y})^{-2} + \cdots)^n}{n!} \frac{\partial^n}{\partial E_0^n} \psi_B(x,y,E_0). \end{aligned} \tag{0.9}$$

Because of the growth order condition that E_k's satisfy (Appendix B, (B.107)), the infinite series

$$\mathcal{E} =: \sum_{n\geq 0} \frac{1}{n!}(E_1\eta^{-1} + E_2\eta^{-2} + \cdots)^n \theta^n : \tag{0.10}$$

is a well-defined microdifferential operator, where θ stands for the symbol of the operator $\partial/\partial E_0$ and the ideograph : : designates the normal order product of the symbol. (Theorem 4.1; see [A] for the definition of a normal order product; it consistently assigns a microdifferential operator to each symbol.)

Combining all these answers to Problems 1, 2 and 3, we describe in Section 5 how to obtain concrete results from the transformation theory developed in Section 2.

Appendices A and B give detailed proofs of required results on the estimation of coefficients of several series formally constructed in Sections 1 and 2. In particular, we like to call the attention of the reader to Proposition B.1; this result gives another constructive proof of the existence of $(q_0(\tilde{q}), E_0)$ in Theorem 3.1 of [AKT1]. The proof given in [AKT1] was rather geometric and transcendental, while the construction of the corresponding object $(x_0(\tilde{x}, t), E_0(t))$ in this paper is more algebro-analytic. It is noteworthy that the construction scheme for $(x_k^{(j)}(\tilde{x}), E_k^{(j)})$ is uniform with respect to indices j and k and that still their growth orders substantially differ depending on whether j tends to ∞ or k tends to ∞.

In ending this introduction, we express our heartiest thanks to Professor H.J. Silverstone and Professor T. Koike for the stimulating discussions with them. The extended stay of Professor Silverstone at RIMS has given us fresh impetus to attack the two turning points problem again, which we had set aside for quite a while.

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1 Reduction of an MTP equation to the canonical form at $t = 0$

The purpose of this section is to find the canonical form of an MTP equation at $t = 0$. As we emphasized in Introduction, the results in this section (Theorems 1.1, 1.4 and 1.5 below) may be regarded as reduction theorems for a general operator with a double turning point.

Theorem 1.1. *Let $Q(\tilde{x}, t)$ be the potential of an MTP operator* (0.1) *in Introduction. Suppose that there exists an open disk U centered at the origin $\tilde{x} = 0$ for which the following hold:*

(1.1) $$Q(\tilde{x}, 0) \text{ is holomorphic on } U,$$

(1.2) $$Q(\tilde{x}, 0) \neq 0 \text{ on } U - \{0\}.$$

Then we can find an open neighborhood ω of the origin, a sequence $\{E_k^{(0)}\}_{k\geq 0}$ of constants and a sequence $\{x_k^{(0)}(\tilde{x})\}_{k\geq 0}$ of holomorphic functions on ω so that the series $E^{(0)}(\eta) = \sum_{k\geq 0} E_k^{(0)} \eta^{-k}$ and $x^{(0)}(\tilde{x}, \eta) = \sum_{k\geq 0} x_k^{(0)}(\tilde{x})\eta^{-k}$, where η is the large parameter contained in the MTP operator (0.1), *formally satisfy the following relations* (1.3) $\sim$ (1.7) *on ω:*

(1.3)
$$Q(\tilde{x}, 0) = \left(\frac{dx^{(0)}(\tilde{x}, \eta)}{d\tilde{x}}\right)^2 (E^{(0)}(\eta) - \frac{x^{(0)}(\tilde{x}, \eta)^2}{4}) - \frac{\eta^{-2}}{2}\{x^{(0)}(\tilde{x}, \eta); \tilde{x}\},$$

(1.4) $$x_0^{(0)}(0) = 0,$$

(1.5) $$\frac{dx_0^{(0)}}{d\tilde{x}}(0) \neq 0,$$

(1.6) $$E_0^{(0)}, E_{2p+1}^{(0)} = 0 \quad (p = 0, 1, 2, \dots)$$

(1.7) $$x_{2p+1}^{(0)} = 0 \quad (p = 0, 1, 2, \dots).$$

Here, and in what follows, $\{x;\tilde{x}\}$ designates the Schwarzian derivative, i.e.,

$$\left(\frac{d^3x}{d\tilde{x}^3}\Big/\frac{dx}{d\tilde{x}}\right)-\frac{3}{2}\left(\frac{d^2x}{d\tilde{x}^2}\Big/\frac{dx}{d\tilde{x}}\right)^2. \tag{1.8}$$

Proof. Comparing the coefficients of like powers of η in (1.3), we find

$$Q(\tilde{x},0)=\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2(E_0^{(0)}-\frac{1}{4}x_0^{(0)2}), \tag{1.9}$$

$$\begin{aligned}0=2\left(\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_1^{(0)}}{d\tilde{x}}\right)(E_0^{(0)}-\frac{1}{4}x_0^{(0)2})\\+\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2(E_1^{(0)}-\frac{1}{2}x_0^{(0)}x_1^{(0)}),\end{aligned} \tag{1.10}$$

$$\begin{aligned}0=2\left(\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_n^{(0)}}{d\tilde{x}}\right)(E_0^{(0)}-\frac{1}{4}x_0^{(0)2})\\+\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2(E_n^{(0)}-\frac{1}{2}x_0^{(0)}x_n^{(0)})+R_n^{(0)}\qquad(n\geq 2),\end{aligned} \tag{1.11.n}$$

where

(1.12)

$$\begin{aligned}R_n^{(0)}=&\sum_{\substack{k_1+k_2+l=n\\k_1,k_2,l<n}}\frac{dx_{k_1}^{(0)}}{d\tilde{x}}\frac{dx_{k_2}^{(0)}}{d\tilde{x}}E_l^{(0)}\\&-\frac{1}{4}\sum_{\substack{k_1+k_2+l_1+l_2=n\\k_1,k_2,l_1,l_2<n}}\frac{dx_{k_1}^{(0)}}{d\tilde{x}}\frac{dx_{k_2}^{(0)}}{d\tilde{x}}x_{l_1}^{(0)}x_{l_2}^{(0)}\end{aligned}$$

$$+\frac{1}{2}\sum_{k+l+\mu=n-2}\sum_{\mu_1+\cdots+\mu_l=\mu}\frac{d^3x_k^{(0)}}{d\tilde{x}^3}\frac{dx_{\mu_1+1}^{(0)}}{d\tilde{x}}\cdots\frac{dx_{\mu_l+1}^{(0)}}{d\tilde{x}}\left(-\frac{dx_0^{(0)}}{d\tilde{x}}\right)^{-(l+1)}$$

$$+\frac{3}{4}\sum_{k_1+k_2+l+\mu=n-2}\sum_{\mu_1+\cdots+\mu_l=\mu}(l+1)\frac{d^2x_{k_1}^{(0)}}{d\tilde{x}^2}\frac{d^2x_{k_2}^{(0)}}{d\tilde{x}^2}$$

$$\times\frac{dx_{\mu_1+1}^{(0)}}{d\tilde{x}}\cdots\frac{dx_{\mu_l+1}^{(0)}}{d\tilde{x}}\left(-\frac{dx_0^{(0)}}{d\tilde{x}}\right)^{-(l+2)}.$$

First we note that the assumption (0.3) together with the requirements (1.4), (1.5) and (1.9) forces

(1.13) $$E_0^{(0)}=0.$$

Hence (1.9) entails

(1.14) $$x_0^{(0)}(\tilde{x})=2\left(\int_0^{\tilde{x}}\sqrt{-Q(\tilde{x})}d\tilde{x}\right)^{1/2},$$

and conditions (0.3) and (1.2) guarantee that $x_0^{(0)}(\tilde{x})$ is holomorphic on U and that it satisfies (1.4) and (1.5).

Next we evaluate the right-hand side of (1.10) at $\tilde{x}=0$ to find

(1.15) $$E_1^{(0)}=0$$

should hold if $x_1^{(0)}(\tilde{x})$ is holomorphic near $\tilde{x}=0$. On the other hand, if (1.15) holds, then by dividing (1.10) by $x_0^{(0)}(dx_0^{(0)}/d\tilde{x})^2$ we obtain

(1.16) $$x_0^{(0)}\frac{dx_1^{(0)}}{dx_0^{(0)}}+x_1^{(0)}=0.$$

In view of (1.4) and (1.5) we may use

(1.17) $$x\underset{\text{def}}{=}x_0^{(0)}(\tilde{x})$$

as a new coordinate near $\tilde{x} = 0$, and the inverse function of $x_0^{(0)}(\tilde{x})$ is denoted by $g(x)$, i.e.,

$$g(x_0^{(0)}(\tilde{x})) = \tilde{x}. \tag{1.18}$$

Regarding (1.16) as an equation on x-space, we find that it is an equation with regular singularity at $x = 0$ with characteristic index -1. Hence a holomorphic solution $x_1^{(0)}$ of (1.16) should vanish identically near $x = 0$. To fix the notation, let us choose a small disk ω_0 in x-space that is bi-holomorphically mapped by $g(x)$ to a neighborhood ω of the origin of $\tilde{x}$-space which is contained in U.

Now, as the structure of the principal part of (1.11.n) is the same as that of (1.10), the argument for $x_n^{(0)}$ is basically the same as above; the only difference is that, instead of (1.15), we obtain

$$E_n^{(0)} = -R_n^{(0)} \left(\frac{dx_0^{(0)}}{d\tilde{x}} \right)^{-2} \Bigg|_{\tilde{x}=0}, \tag{1.19}$$

and that, instead of (1.16), we find

$$x \frac{dx_n^{(0)}}{dx} + x_n^{(0)} = 2 \frac{E_n^{(0)} + \tilde{R}_n^{(0)}}{x_0^{(0)}(\tilde{x})} \tag{1.20}$$

where $\tilde{R}_n^{(0)} = (dx_0^{(0)}/d\tilde{x})^{-2} R_n^{(0)}$. The above choice of $E_n^{(0)}$ guarantees that $(E_n^{(0)} + \tilde{R}_n^{(0)})/x_0^{(0)}(\tilde{x})$ is holomorphic on ω, and hence a holomorphic solution $x_n^{(0)}(x)$ of (1.20) exists on ω_0. Furthermore, (1.12) entails that $R_{2p+1}^{(0)}$ is a sum of terms each of which contains $E_{2q+1}^{(0)}$, $x_{2q+1}^{(0)}$ or its derivative as its factor with $q < p$. Since we have confirmed $(E_1^{(0)}, x_1^{(0)}) = 0$, we find by the induction on p that $R_{2p+1}^{(0)}$ vanishes identically. Hence (1.19) implies that $E_{2p+1}^{(0)}$ also vanishes. Thus we have constructed $x^{(0)}(\tilde{x}, \eta)$ and $E^{(0)}(\eta)$ which satisfy (1.3) $\sim$ (1.7). □

Remark 1.1. If the potential Q contains lower order terms in η, i.e., if Q has the form

$$Q = \sum_{k \geq 0} \eta^{-k} Q_k(\tilde{x}, t), \tag{1.21}$$

the reasoning proceeds equally as well on the condition that $Q_0(\tilde{x}, t)$ satisfies (1.1) and (1.2) and that $Q_k(\tilde{x}, t)$'s have the common domain of definition which contains the origin; it suffices to add $-Q_{n+2}(\tilde{x}, 0)$ to $R_n^{(0)}$ in (1.12).

As is well-known ([AKT1], [KT1], [KT2]), Theorem 1.1 entails the following structure theorem for a WKB solution of an MTP equation restricted to $t = 0$.

Theorem 1.2. *In the situation considered in Theorem 1.1, the infinite series* $x^{(0)}(\tilde{x}, \eta)$ *and* $E^{(0)}(\eta)$ *satisfy*

$$\begin{aligned} \tilde{S}(\tilde{x}, \eta) = &\left(\frac{dx^{(0)}}{d\tilde{x}}\right) S(x^{(0)}(\tilde{x}, \eta), E^{(0)}(\eta), \eta) \\ &- \frac{1}{2}\left(\frac{d^2 x^{(0)}(\tilde{x}, \eta)}{d\tilde{x}^2}\right) \Big/ \left(\frac{dx^{(0)}(\tilde{x}, \eta)}{d\tilde{x}}\right), \end{aligned} \tag{1.22}$$

where $\tilde{S}$ *and* S *are formal series in* η^{-1} *beginning with respectively* $\tilde{S}_{-1}(\tilde{x})\eta$ *and* $S_{-1}(x)\eta$ *which solve*

$$\tilde{S}^2 + \frac{d\tilde{S}}{d\tilde{x}} = \eta^2 Q(\tilde{x}, 0) \tag{1.23}$$

and

$$S^2 + \frac{dS}{dx} = \eta^2 (E^{(0)}(\eta) - \frac{1}{4}x^2), \tag{1.24}$$

and for which

$$\arg \tilde{S}_{-1}(\tilde{x}) = \arg \left(\frac{dx_0^{(0)}}{d\tilde{x}} S_{-1}(x_0^{(0)}(\tilde{x}))\right) \tag{1.25}$$

holds (and hence $\tilde{S}_{-1}(\tilde{x})$ *and* $\dfrac{dx_0^{(0)}}{d\tilde{x}} S_{-1}(x_0^{(0)}(\tilde{x}))$ *coincide).*

Proof. First we note that the relation (1.3) together with the definition of S entails the following relation (1.26). Here, and in what follows, we often omit $E^{(0)}(\eta)$ in the symbol $S(x, E^{(0)}(\eta), \eta)$.

(1.26)

$$\begin{aligned}
&\left(\frac{dx^{(0)}}{d\tilde{x}} S(x^{(0)}(\tilde{x},\eta),\eta) - \frac{1}{2}\left(\frac{d^2x^{(0)}}{d\tilde{x}^2}\right)\Big/\left(\frac{dx^{(0)}}{d\tilde{x}}\right)\right)^2 \\
&\quad + \frac{d}{d\tilde{x}}\left(\frac{dx^{(0)}}{d\tilde{x}} S(x^{(0)}(\tilde{x},\eta),\eta) - \frac{1}{2}\left(\frac{d^2x^{(0)}}{d\tilde{x}^2}\right)\Big/\left(\frac{dx^{(0)}}{d\tilde{x}}\right)\right) \\
&= \left(\frac{dx^{(0)}}{d\tilde{x}}\right)^2 S(x^{(0)},\eta)^2 - \frac{d^2x^{(0)}}{d\tilde{x}^2} S(x^{(0)},\eta) + \frac{1}{4}\left(\frac{d^2x^{(0)}}{d\tilde{x}^2}\right)^2\Big/\left(\frac{dx^{(0)}}{d\tilde{x}}\right)^2 \\
&\quad + \frac{d^2x^{(0)}}{d\tilde{x}^2} S(x^{(0)},\eta) + \left(\frac{dx^{(0)}}{d\tilde{x}}\right)^2 \frac{\partial S}{\partial x}(x^{(0)},\eta) \\
&\quad + \frac{1}{2}\left(\frac{d^2x^{(0)}}{d\tilde{x}^2}\right)^2\Big/\left(\frac{dx^{(0)}}{d\tilde{x}}\right)^2 - \frac{1}{2}\left(\frac{d^3x^{(0)}}{d\tilde{x}^3}\right)\Big/\left(\frac{dx^{(0)}}{d\tilde{x}}\right) \\
&= \left(\frac{dx^{(0)}}{d\tilde{x}}\right)^2 \eta^2 (E^{(0)}(\eta) - \frac{1}{4} x^{(0)}(\tilde{x},\eta)^2) - \frac{1}{2}\{x^{(0)};\tilde{x}\} \\
&= \eta^2 Q(\tilde{x},0).
\end{aligned}$$

Comparing (1.26) with (1.23), we find that

$$\frac{dx^{(0)}}{d\tilde{x}} S(x^{(0)}(\tilde{x},\eta),\eta) - \frac{1}{2}\left(\frac{d^2x^{(0)}}{d\tilde{x}^2}\right)\Big/\left(\frac{dx^{(0)}}{d\tilde{x}}\right) \tag{1.27}$$

and $\tilde{S}(\tilde{x},\eta)$ satisfy the same equation. Then, in view of the assumption (1.25), we conclude

$$\tilde{S}(\tilde{x},\eta) = \frac{dx^{(0)}}{d\tilde{x}} S(x^{(0)}(\tilde{x},\eta),\eta) - \frac{1}{2}\left(\frac{d^2x^{(0)}}{d\tilde{x}^2}\right)\Big/\left(\frac{dx^{(0)}}{d\tilde{x}}\right). \tag{1.28}$$

□

To proceed to discuss the structure of wave functions, let us now recall the following definition of S_{odd}, the odd part of a solution S of the Riccati equation.

Definition 1.1 ([AKT3, Definition 2.1])**.** Consider the following Riccati equation with η-dependent potential (like $E^{(0)}(\eta) - x^2/4$):

$$S(x,\eta)^2 + \frac{dS}{dx}(x,\eta) = \eta^2 \left(\sum_{k\geq 0} Q_k(x)\eta^{-k} \right). \tag{1.29}$$

Let $S^{\pm}$ respectively denote the solution of (1.29) that begins with $\pm\eta\sqrt{Q_0(x)}$. Then the odd part S_{odd} of S is, by definition, given by

$$S_{\mathrm{odd}} = \frac{1}{2}(S^+ - S^-). \tag{1.30}$$

Using this definition of the odd part of S, we obtain the following result from (1.22).

Corollary 1.3. *The odd part $\tilde{S}_{\mathrm{odd}}$ is reduced to the odd part of S of the Weber equation with $E_0^{(0)} = 0$, that is,*

$$\tilde{S}_{\mathrm{odd}}(\tilde{x},\eta) = \left(\frac{dx^{(0)}}{d\tilde{x}} \right) S_{\mathrm{odd}}(x^{(0)}(\tilde{x},\eta),\eta) \tag{1.31}$$

holds with the appropriate choice of the branch of $S_{-1}(x)$.

Using these transformation results for a WKB solution of the Riccati equation associated with the Schrödinger equation, we can relate a WKB solution of the Weber equation itself with that of the MTP equation at $t = 0$. To discuss this point in detail, we first note the following relation:

$$(S^+)^2 - (S^-)^2 + \frac{d}{dx}(S^+ - S^-) = 0. \tag{1.32}$$

Hence we obtain

$$2(S^+ + S^-)S_{\text{odd}} + 2\frac{d}{dx}S_{\text{odd}} = 0, \tag{1.33}$$

i.e.,

$$S^+ + S^- = -\frac{\frac{d}{dx}S_{\text{odd}}}{S_{\text{odd}}} = -\frac{d}{dx}\log S_{\text{odd}}. \tag{1.34}$$

This means that, for a generic point a,

$$\psi_\pm = \frac{1}{\sqrt{S_{\text{odd}}}}\exp(\pm\int_a^x S_{\text{odd}}dx) \tag{1.35}$$

satisfy the equation

$$\frac{d^2\psi_\pm}{dx^2} = \eta^2\left(\sum_{k\geq 0} Q_k(x)\eta^{-k}\right)\psi_\pm, \tag{1.36}$$

though the definition of the odd part S_{odd} is not a naive one based on the oddness of the degree in η.

Now, using this normalization of a WKB solution, we find the following.

Theorem 1.4. *Let us consider the situation assumed in Theorem 1.1, and let ψ be a WKB solution of the ∞-Weber equation*

$$\left(\frac{d^2}{dx^2} - \eta^2(E^{(0)}(\eta) - \frac{1}{4}x^2)\right)\psi = 0 \tag{1.37}$$

defined with the infinite series $E^{(0)}(\eta)$ constructed there; in particular, we have

$$E_0^{(0)} = 0. \tag{1.38}$$

Then with the infinite series $x^{(0)}(\tilde{x},\eta)$ constructed there we find

$$\varphi(\tilde{x},\eta) = \left(\frac{dx^{(0)}(\tilde{x},\eta)}{d\tilde{x}}\right)^{-1/2}\psi(x^{(0)}(\tilde{x},\eta),\eta) \tag{1.39}$$

satisfies the following MTP equation at $t = 0$:

$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x}, 0)\right)\varphi(\tilde{x}, \eta) = 0. \tag{1.40}$$

Proof. It follows from (1.3) and (1.37) that

(1.41)

$$\begin{aligned}
\frac{d^2\varphi}{d\tilde{x}^2} &= \left(\frac{3}{4}\left(\frac{dx^{(0)}}{d\tilde{x}}\right)^{-5/2}\left(\frac{d^2x^{(0)}}{d\tilde{x}^2}\right)^2 - \frac{1}{2}\left(\frac{dx^{(0)}}{d\tilde{x}}\right)^{-3/2}\frac{d^3x^{(0)}}{d\tilde{x}^3}\right)\psi \\
&\quad + \left(\frac{dx^{(0)}}{d\tilde{x}}\right)^{3/2}\frac{d^2\psi}{dx^2}\bigg|_{x=x^{(0)}(\tilde{x},\eta)} \\
&= \left(\frac{dx^{(0)}}{d\tilde{x}}\right)^2 \eta^2\left(E^{(0)} - \frac{x^{(0)2}}{4}\right)\left(\frac{dx^{(0)}}{d\tilde{x}}\right)^{-1/2}\psi \\
&\quad + \left(\frac{3}{4}\left(\frac{dx^{(0)}}{d\tilde{x}}\right)^{-2}\frac{d^2x^{(0)}}{d\tilde{x}^2} - \frac{1}{2}\left(\frac{dx^{(0)}}{d\tilde{x}}\right)^{-1}\frac{d^3x^{(0)}}{d\tilde{x}^3}\right)\left(\frac{dx^{(0)}}{d\tilde{x}}\right)^{-1/2}\psi \\
&= \eta^2 Q(\tilde{x}, 0)\varphi.
\end{aligned}$$

Thus we find (1.40). □

Concerning the structure of the function φ, by considering the logarithmic derivative of both sides of (1.39) we obtain the following relation (1.42) by (1.22):

$$\begin{aligned}
\frac{d\log\varphi}{d\tilde{x}} &= -\frac{1}{2}\frac{d}{d\tilde{x}}\log\left(\frac{\partial x^{(0)}}{\partial\tilde{x}}\right) + \frac{dx^{(0)}}{d\tilde{x}}S(x^{(0)}(\tilde{x}, \eta), \eta) \\
&= \tilde{S}(\tilde{x}, \eta).
\end{aligned} \tag{1.42}$$

This means that the wave function φ is also represented in the form of (1.35). Thus the infinite series $x^{(0)}(\tilde{x}, \eta)$ defines a transformation of WKB solutions via (1.39) in the case of a double turning point problem, just like in the case of a simple turning point ([S], [AKT1]).

Furthermore, the growth order condition (A.3) on $\{E_k^{(0)}\}_{k\geq 0}$ implies that $E^{(0)}(\eta)$ is a symbol of a microdifferential operator; this means that the Borel transform of the ∞-Weber equation

$$\left(\frac{\partial^2}{\partial x^2} - \left(E^{(0)}(\partial/\partial y) - \frac{x^2}{4}\right)\frac{\partial^2}{\partial y^2}\right)\psi_B = 0 \tag{1.43}$$

is a well-defined microdifferential equation defined on

$$\{(x, y; \xi, \eta) \in T^*\mathbb{C}^2;\ \eta \neq 0\}. \tag{1.44}$$

In what follows we let M denote the microdifferential operator in (1.44), that is,

$$M = \frac{\partial^2}{\partial x^2} - \left(E^{(0)}(\partial/\partial y) - \frac{x^2}{4}\right)\frac{\partial^2}{\partial y^2}. \tag{1.45}$$

On the other hand, the growth order condition (A.4) on $\{x_k^{(0)}(\tilde{x})\}_{k\geq 0}$ guarantees that the relation (1.39) turns out to be a microdifferential relation through the Borel transformation. The proof of this fact is basically the same as that given in [AKT1, Section 2], where a simple turning point problem is discussed. In this paper, by following the presentation of [AY], we formulate our result as the microlocal equivalence between the Borel transformed MTP equation at $t = 0$ and the Borel transformed ∞-Weber equation with $E_0^{(0)} = 0$. (Theorem 1.5 below.) To state Theorem 1.5, we first introduce

$$r_k(x) = x_k^{(0)}(g(x)) \quad (k \geq 0). \tag{1.46}$$

In particular, we have

$$r_0 = x. \tag{1.47}$$

We note that the Borel transformed MTP operator at $t = 0$, i.e.,

$$\frac{\partial^2}{\partial \tilde{x}^2} - Q(\tilde{x}, 0)\frac{\partial^2}{\partial y^2} \tag{1.48}$$

can be rewritten in (x, y)-variable as follows:

$$(g')^{-2}\left(\frac{\partial^2}{\partial x^2}+\frac{x^2}{4}\frac{\partial^2}{\partial y^2}-\frac{g''}{g'}\frac{\partial}{\partial x}\right). \tag{1.49}$$

Here, and in what follows, we let g' and g'' denote respectively dg/dx and d^2g/dx^2, and we let L denote the operator given by (1.49). As the Taylor expansion of $\psi(x^{(0)}(\tilde{x},\eta),\eta)$ is

$$\sum_{n\geq 0}\frac{(r_1(x)\eta^{-1}+r_2(x)\eta^{-2}+\cdots)^n}{n!}\frac{\partial^n}{\partial x^n}\psi(x,\eta), \tag{1.50}$$

its Borel transform is given by

$$:\exp(r(x,\eta)\xi):\psi_B(x,y), \tag{1.51}$$

where the ideograph : : designates the normal ordered product ([A]),

$$r(x,\eta)=\sum_{k\geq 1}r_k(x)\eta^{-k} \tag{1.52}$$

and ψ_B denotes the Borel transform of ψ. Hence the Borel transform of the right-hand side of (1.39) is expressed as

$$:g'(x)^{1/2}\left(1+\frac{dr}{dx}\right)^{-1/2}\exp(r(x,\eta)\xi):\psi_B. \tag{1.53}$$

Let us now denote the microdifferential operator in (1.53) by $\mathcal{X}$, that is,

$$\mathcal{X}=:g'(x)^{1/2}(1+\frac{dr}{dx})^{-1/2}\exp(r(x,\eta)\xi):. \tag{1.54}$$

Since Theorem 1.4 asserts that the Borel transformed MTP operator L at $t=0$ annihilates $\mathcal{X}\psi_B$ for a solution ψ_B of the Borel transformed ∞-Weber equation with $E_0^{(0)}=0$, we may naturally expect the following Theorem 1.5 to hold. We now prove that our expectation is correct.

Theorem 1.5. *There exists a microdifferential operator* $\mathcal{Y}$ *on*

(1.55) $$\Omega_0 = \{(x, y; \xi, \eta) \in T^*\mathbb{C}^2; x \in \omega_0, \eta \neq 0\}$$

which satisfies

(1.56) $$L\mathcal{X} = \mathcal{Y}M,$$

and both $\mathcal{X}$ *and* $\mathcal{Y}$ *are invertible.*

Proof. Let us try to find $\mathcal{Y}$ in the form

(1.57) $$: C_1(x, \eta) \exp(r(x, \eta)\xi) :$$

Note that $\mathcal{X}$ has a similar form with C_1 replaced by

(1.58) $$C = g'(x)^{1/2} \left(1 + \frac{dr}{dx}\right)^{-1/2}.$$

By a straightforward symbol calculus we find that (1.56) is satisfied if the following three conditions are satisfied:

(1.59) $$C\left(1 + \frac{dr}{dx}\right)^2 = C_1,$$

(1.60) $$2\left(1 + \frac{dr}{dx}\right)\frac{dC}{dx} + \left(\frac{d^2r}{dx^2} - \frac{g''}{g'}\left(1 + \frac{dr}{dx}\right)\right) - C = 0,$$

(1.61) $$\frac{d^2C}{dx^2}\,\frac{g''}{g'}\,\frac{dC}{dx} = -C_1\left(\sum_{k\geq 1} E_k^{(0)}\eta^{-k} - \frac{1}{4}(x + r)^2\right)\eta^2.$$

As C satisfies (1.60), we should have

(1.62) $$C_1 = g'(x)^{1/2}\left(1 + \frac{dr}{dx}\right)^{3/2}.$$

Using these concrete expressions together with (1.9), we can rewrite (1.61) as an equation in $\tilde{x}$-coordinate:

(1.63)
$$Q(\tilde{x}, 0) = \left(\frac{dx^{(0)}(\tilde{x}, \eta)}{d\tilde{x}}\right)^2 \left(E^{(0)}(\eta) - \frac{x^{(0)}(\tilde{x}, \eta)^2}{4}\right) - \frac{\eta^{-2}}{2}\{x; \tilde{x}\}.$$

Here we have used the relation

$$C = \left(\frac{dx}{d\tilde{x}}\right)^{-1/2} \tag{1.64}$$

to find

$$\frac{d^2C}{dx_2} - \frac{g''}{g'}\frac{dC}{dx} = \frac{d^2C}{d\tilde{x}^2} = -\frac{1}{2}C\{x;\tilde{x}\}. \tag{1.65}$$

The relation (1.61) is nothing but (1.3). Thus (1.61) has a solution C_1 of the form (1.62), which proves the existence of required operator $\mathcal{Y}$. Since the principal symbol of $\mathcal{X}$ and that of $\mathcal{Y}$ are both

$$g'(x)^{1/2}\exp(r_1(x)\xi\eta^{-1}), \tag{1.66}$$

they are different from 0 on Ω_0. Hence they are invertible as microdifferential operators. □

Remark 1.2. The microlocal result formulated as in Theorem 1.5 is a special case of Theorem 2.6 in Section 2; the point is that the transformation of an MTP operator to the ∞-Weber equation is constructed as a perturbation of the transformation that brings the MTP operator at $t = 0$ to a particular (i.e., $E_0^{(0)} = 0$) ∞-Weber equation and that the perturbation series in t are convergent ones (Proposition B.1 and Proposition B.2 in Appendix B).

2 Reduction of an MTP equation to the canonical form for $t \neq 0$

The purpose of this section is to find the canonical form of an MTP equation for $t \neq 0$ by making use of the result in the preceding section. Before entering the detailed analysis of an MTP equation, we first make its definition precise concerning the merging speed of two simple turning points in (0.4) so that we may avoid unnecessary complications.

Definition 2.1. A Schrödinger operator P of the form

$$\frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x}, t) \quad (\eta : \text{a large parameter}) \tag{2.1}$$

is called a merging-paired-simple-turning-points operator, or, for short, a merging-turning-points (MTP) operator, if its potential $Q(\tilde{x}, t)$ satisfies the following conditions (2.2) $\sim$ (2.5).

(2.2) $Q(\tilde{x}, t)$ is holomorphic near the origin $(\tilde{x}, t) = (0, 0)$,

(2.3) $Q(\tilde{x}, 0) = c\tilde{x}^2 + O(\tilde{x}^3)$ (c : a non-zero constant),

(2.4) for each t $(\neq 0)$, the equation $Q(\tilde{x}, t) = 0$ in $\tilde{x}$ has two distinct simple roots $s_\pm(t)$ which merge together at $t = 0$, whereas other roots of the equation stay uniformly away from 0 for sufficiently small t,

(2.5) there exists a positive constant σ_0 for which

$$\left| \frac{s_\pm(t)}{\sqrt{t}} \right| > \sigma_0$$

holds on a neighborhood of the origin $t = 0$.

Remark 2.1. Condition (2.4) means that the points $x = s_\pm(t)$ are simple turning points of the operator in question, and Condition (2.5) guarantees that the situation considered is a generic one under the assumption (2.3), as the following Proposition 2.1 shows.

Proposition 2.1. *Let P be an MTP operator. Then its potential $Q(\tilde{x}, t)$ has the following form on a sufficiently small neighborhood of the origin $(\tilde{x}, t) = (0, 0)$:*

$$Q(\tilde{x}, t) = Q^{(0)}(\tilde{x}) + tQ^{(1)}(\tilde{x}) + t^2 Q^{(2)}(\tilde{x}) + \cdots \tag{2.6}$$

with

$$Q^{(1)}(0) \neq 0. \tag{2.7}$$

Proof. Using (2.3), we apply the Weierstrass preparation theorem to $Q(\tilde{x}, t)$ to find holomorphic functions $h(\tilde{x}, t)$ and $g_j(t)$ $(j = 1, 2)$ for which the following hold:

$$Q(\tilde{x}, t) = h(\tilde{x}, t)(\tilde{x}^2 + g_1(t)\tilde{x} + g_2(t)), \tag{2.8}$$

$$h(0, 0) \neq 0, \tag{2.9}$$

$$g_1(0) = g_2(0) = 0. \tag{2.10}$$

Then we find

$$s_\pm(t) = \frac{-g_1(t) \pm \sqrt{g_1(t)^2 - 4g_2(t)}}{2} \tag{2.11}$$

near $t = 0$, and hence (2.10) and (2.5) imply

$$g_2(t) = g_2^{(1)} t + \sum_{j \geq 2} g_2^{(j)} t^j \tag{2.12}$$

with

$$g_2^{(1)} \neq 0. \tag{2.13}$$

Expanding $h(\tilde{x}, t)$ and $g_1(t)$ as

$$h(\tilde{x}, t) = h^{(0)}(\tilde{x}) + th^{(1)}(\tilde{x}) + t^2 h^{(2)}(\tilde{x}) + \cdots \tag{2.14}$$

and

$$g_1(t) = g_1^{(1)} t + g_1^{(2)} t^2 + \cdots, \tag{2.15}$$

respectively, we find

$$\begin{aligned} Q(\tilde{x}, t) = {} & h^{(0)}(\tilde{x})\tilde{x}^2 \\ & + t(h^{(1)}(\tilde{x})\tilde{x}^2 + h^{(0)}(\tilde{x})(g_1^{(1)}\tilde{x} + g_2^{(1)})) + O(t^2). \end{aligned} \tag{2.16}$$

Thus Q has the expansion of the form (2.6) and the coefficient $Q^{(1)}(\tilde{x})$ of t^1 in the expansion has the form

$$h^{(0)}(\tilde{x})(g_2^{(1)} + g_1^{(1)}\tilde{x}) + h^{(1)}(\tilde{x})\tilde{x}^2. \tag{2.17}$$

Then (2.9) and (2.13) guarantee that $Q^{(1)}(0)$ is different from 0. □

Remark 2.2. As the holomorphic function $g_2(t)$ vanishes at $t = 0$, the relation (2.13) entails that

$$g_1(t)^2 \neq 4g_2(t) \tag{2.18}$$

holds near $t = 0$. Thus (2.13) guarantees that $s_+(t)$ and $s_-(t)$ are distinct simple turning points near $t = 0$.

We now state the core result in this section.

Theorem 2.2. *Let $Q(\tilde{x}, t)$ be the potential of an MTP operator. Then we can find an open neighborhood ω_0 of the origin $\tilde{x} = 0$, holomorphic functions $x_k^{(j)}(\tilde{x})$ $(j, k \geq 0)$ on ω_0 and constants $E_k^{(j)}$ $(j, k \geq 0)$ such that the formal series*

$$x(\tilde{x}, t, \eta) = \sum_{j,k\geq 0} x_k^{(j)}(\tilde{x}) t^j \eta^{-k} \tag{2.19}$$

and

$$E(t, \eta) = \sum_{j,k\geq 0} E_k^{(j)} t^j \eta^{-k} \tag{2.20}$$

satisfy the following relations (2.21) $\sim$ (2.26)*:*

(2.21)

$$Q(\tilde{x}, t) = \left(\frac{\partial x(\tilde{x}, t, \eta)}{\partial \tilde{x}}\right)^2 (E(t, \eta) - \frac{x(\tilde{x}, t, \eta)^2}{4}) - \frac{\eta^{-2}}{2}\{x(\tilde{x}, t, \eta); \tilde{x}\},$$

(2.22) $$x_0^{(0)}(0) = 0,$$

(2.23) $$\frac{dx_0^{(0)}}{d\tilde{x}}(0) \neq 0,$$

(2.24) $$E_0^{(0)} = 0,$$

(2.25) $$E_{2p+1}^{(j)} = 0 \quad (j, p = 0, 1, 2, \cdots),$$

(2.26) $$x_{2p+1}^{(j)}(\tilde{x}), = 0 \quad (j, p = 0, 1, 2, \cdots).$$

Proof. Using the expansion (2.6), we construct the required $(x_k^{(j)}, E_k^{(j)})$ by regarding the relation (2.21) as a perturbation of the relation (1.3); we start with our reasoning by regarding $(x^{(0)}(\tilde{x}, \eta), E^{(0)}(\eta))$ constructed in Theorem 1.1 as the initial term of the series $(x(\tilde{x}, t, \eta), E(t, \eta)) = (\sum_{j \geq 0} x^{(j)}(\tilde{x}, \eta)t^j, \sum_{j \geq 0} E^{(j)}(\eta)t^j)$. Then the comparison of the coefficients of like powers of t in (2.21) yields the following relations:

(2.27.j)

$$Q^{(j)}(\tilde{x}) = \left(2\frac{\partial x^{(0)}}{\partial \tilde{x}}\frac{\partial x^{(j)}}{\partial \tilde{x}}\right)(E^{(0)} - \frac{1}{4}x^{(0)^2}) + \left(\frac{\partial x^{(0)}}{\partial \tilde{x}}\right)^2 (E^{(j)} - \frac{1}{2}x^{(0)}x^{(j)}) - \frac{\eta^{-2}}{2}\{x^{(j)}; \tilde{x}\} + R^{(j)} \quad (j \geq 1),$$

where

(2.28)

$$R^{(j)} = \sum_{\substack{j_1+j_2+j_3=j \\ j_1, j_2, j_3 < j}} \frac{\partial x^{(j_1)}}{\partial \tilde{x}}\frac{\partial x^{(j_2)}}{\partial \tilde{x}} E^{(j_3)} - \frac{1}{4}\sum_{\substack{j_1+j_2+j_3+j_4=j \\ j_1, j_2, j_3, j_4 < j}} \frac{\partial x^{(j_1)}}{\partial \tilde{x}}\frac{\partial x^{(j_2)}}{\partial \tilde{x}} x^{(j_3)}x^{(j_4)}$$

$$+\frac{\eta^{-2}}{2}\sum_{p+l+\mu=j}\sum_{\mu_1+\cdots+\mu_l=\mu}\frac{\partial^3 x^{(p)}}{\partial\tilde{x}^3}\frac{\partial x^{(\mu_1+1)}}{\partial\tilde{x}}\cdots\frac{\partial x^{(\mu_l+1)}}{\partial\tilde{x}}\left(-\frac{\partial x^{(0)}}{\partial\tilde{x}}\right)^{-(l+1)}$$

$$+\frac{3\eta^{-2}}{4}\sum_{p_1+p_2+l+\mu=j}\sum_{\mu_1+\cdots+\mu_l=\mu}(l+1)\frac{\partial^2 x^{(p_1)}}{\partial\tilde{x}^2}\frac{\partial^2 x^{(p_2)}}{\partial\tilde{x}^2}$$

$$\times\frac{\partial x^{(\mu_1+1)}}{\partial\tilde{x}}\cdots\frac{\partial x^{(\mu_l+1)}}{\partial\tilde{x}}\left(-\frac{\partial x^{(0)}}{\partial\tilde{x}}\right)^{-(l+2)}.$$

Since $R^{(j)}$ depends only on $\{E^{(j_1)}, x^{(j_2)}$ or its derivatives$\}_{j_1,j_2<j}$, we may try to find a solution $(E^{(j)}, x^{(j)})$ of $(2.27.j)$ recursively, i.e., using $\{E^{(j_1)}, x^{(j_2)}\}_{j_1,j_2<j}$ as given data. As each equation $(2.27.j)$ consists of infinitely many terms, finding a solution $(E^{(j)}, x^{(j)})$ of $(2.27.j)$ amounts to finding out infinitely many quantities $\{E^{(j)}_k, x^{(j)}_k\}_{k\geq 0}$. In order to construct a holomorphic function $x^{(j)}_k(\tilde{x})$ on ω_0 we have to choose a constant $E^{(j)}_k$ appropriately, just in the same way as was done in the proof of Theorem 1.1. To illustrate the point, we write down the degree 0 in η part of (2.27.1); it reads as follows:

$$Q^{(1)}(\tilde{x}) = 2\frac{dx^{(0)}_0}{d\tilde{x}}\frac{dx^{(1)}_0}{d\tilde{x}}E^{(0)}_0 - \frac{1}{2}x^{(0)2}_0\frac{dx^{(0)}_0}{d\tilde{x}}\frac{dx^{(1)}_0}{d\tilde{x}} + \left(\frac{dx^{(0)}_0}{d\tilde{x}}\right)^2 E^{(1)}_0 - \frac{1}{2}\left(\frac{dx^{(0)}_0}{d\tilde{x}}\right)^2 x^{(0)}_0 x^{(1)}_0. \tag{2.29}$$

Since $E^{(0)}_0$ vanishes by (1.6) and since $x^{(0)}_0$ vanishes linearly at the origin by (1.4) and (1.5), generally speaking, we find (2.29) to be with an irregular singularity at $\tilde{x}=0$. But, if we choose $E^{(1)}_0$ so that it satisfies

$$Q^{(1)}(0) = \left(\frac{dx^{(0)}_0}{d\tilde{x}}(0)\right)^2 E^{(1)}_0, \tag{2.30}$$

we can divide both sides of (2.29) by $x_0^{(0)}(dx_0^{(0)}/d\tilde{x})^2$ to find

$$(2.31)\qquad x_0^{(0)}\frac{dx_0^{(1)}}{dx_0^{(0)}}+x_0^{(1)}=\frac{2}{x_0^{(0)}}\left(E_0^{(1)}-\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^{-2}Q^{(1)}(\tilde{x})\right),$$

which is with regular singularity at $x_0^{(0)}=0$. We also note that (2.7) implies

$$(2.32)\qquad E_0^{(1)}\neq 0.$$

The equation for $(E_k^{(1)},x_k^{(1)})_{k>0}$ is exactly of the same form as (2.29), i.e.,

$$(2.33)\quad x_0^{(0)2}\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_k^{(1)}}{d\tilde{x}}+x_0^{(0)}\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2x_k^{(1)}=2\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2E_k^{(1)}+R_k^{(1)},$$

where $R_k^{(1)}$ depends only on $\{E_{k_1}^{(0)},E_{k_2}^{(1)}\}_{k_1,k_2<k}$ and $\{x_{k_1}^{(0)},x_{k_2}^{(1)}$ and their derivatives$\}_{k_1,k_2<k}$. Thus an appropriate choice of the constant $E_k^{(1)}$ enables us to divide both sides of (2.33) by $x_0^{(0)}(dx_0^{(0)}/d\tilde{x})^2$ to find an equation with regular singularity at $x_0^{(0)}=0$ with the characteristic index -1. We can then find a holomorphic solution $x_k^{(1)}$ on ω_0. It is now clear that we can proceed further in a similar way to find $\{x_k^{(j)}(\tilde{x}),E_k^{(j)}\}_{j,k\geq 0}$ so that $x(\tilde{x},t,\eta)$ and $E(t,\eta)$ may satisfy (2.21). The relations (2.22), (2.23) and (2.24) are then immediate consequences of Theorem 1.1. Since $R_{2p+1}^{(j)}$, the coefficient of $\eta^{-(2p+1)}$ in $R^{(j)}$, is a sum of terms each of which contains $E_{2q+1}^{(i)}$, $x_{2q+1}^{(i)}$ or its derivative as its factor with either

(i) $i<j$ and $q\leq p$

or

(ii) $i=j$ and $q<p$.

Hence by the induction on p (and also on j as a subsidiary step), we find they satisfy (2.25) and (2.26). This completes the proof of Theorem 2.2. □

Remark 2.3. In the above proof we arranged our argument so that we may construct $\{x_k^{(j)}, E_k^{(j)}\}$ by assuming that $\{x_{k_1}^{(j_1)}, x_{k_2}^{(j_2)}\}_{j_1,j_2<j;k_1,k_2\geq 0}$ and $\{x_{k_1}^{(j)}, E_{k_2}^{(j)}\}_{0\leq k_1,k_2<k}$ have been constructed. But we may arrange our argument equally well by constructing $\{x_k^{(j)}, E_k^{(j)}\}$ by assuming that $\{x_{k_1}^{(j_1)}, E_{k_2}^{(j_2)}\}_{k_1,k_2<k;j_1,j_2\geq 0}$ and $\{x_k^{(j_1)}, E_k^{(j_2)}\}_{0\leq j_1,j_2<j}$ have been constructed. Actually our argument in Appendix B is arranged in the second way.

The infinite series (2.19) and (2.20) are convergent with respect to t as Proposition B.1 and Proposition B.2 in Appendix B show. Hence Theorem 2.2 (together with the results in Appendix B) entails the following structure theorem (Theorem 2.4 below) for a WKB solution of an MTP equation for $t \neq 0$. Note that, for $t \neq 0$, the assumption (2.4) enables us to describe explicitly the structure of a wave function for an MTP operator, besides a solution of the attached Riccati equation, in terms of that for the ∞-Weber equation; the key point of the discussion is the following lemma.

Lemma 2.3 (Cf. [AKT2, Proposition 1.6]). *For $\tilde{S}_{\text{odd}}$ given in Definition 1.1, we find that $\tilde{S}_{\text{odd}}$ consists of terms with a half odd integer power of $Q^{(0)}$ multiplied by a holomorphic function. In particular, if a point $\tilde{x} = a$ is a simple zero of $Q^{(0)}(\tilde{x}) = 0$, the singularity of $\tilde{S}_{\text{odd}}$ is of square-root type.*

Proof. Using the induction on l, we can readily confirm that the coefficient of η^{-l} in $\tilde{S}^{+}$ (resp., $\tilde{S}^{-}$) is of the form $a_l^{+}(x)(Q^{(0)})^{-(3l+2)/2}$ (resp., $a_l^{-}(x)(Q^{(0)})^{-(3l+2)/2}$) with a holomorphic function $a_l^{+}(x)$ (resp.,

$a_l^-(x))$. Thus the assertion immediately follows from the definition of $\tilde{S}_{\rm odd}$. □

An important implication of this lemma is that the integral

$$\int_a^{\tilde{x}} \tilde{S}_{\rm odd} d\tilde{x} \tag{2.34}$$

is a well-defined series for a simple turning point a if we interpret the integral as

$$\frac{1}{2}\int_{\check{\tilde{x}}}^{\tilde{x}} \tilde{S}_{\rm odd} d\tilde{x}, \tag{2.35}$$

where $\check{\tilde{x}}$ denotes the point corresponding to $\tilde{x}$ on the "second" (near a) sheet of the Riemann surface of $\sqrt{Q^{(0)}(\tilde{x})}$. Thus a normalization of a WKB solution of an MTP equation ($t \neq 0$) can be given as

$$\frac{1}{\sqrt{\tilde{S}_{\rm odd}}} \exp(\pm \int_{s_+(t)}^{\tilde{x}} \tilde{S}_{\rm odd} d\tilde{x}), \tag{2.36}$$

or

$$\frac{1}{\sqrt{\tilde{S}_{\rm odd}}} \exp(\pm \int_{s_-(t)}^{\tilde{x}} \tilde{S}_{\rm odd} d\tilde{x}). \tag{2.37}$$

This normalization is most appropriate for our subsequent discussions.

Theorem 2.4. *In the situation considered in Theorem 2.2 the constructed sequences $x(\tilde{x}, t, \eta)$ and $E(t, \eta)$ enjoy the following properties:*
(i) *For a WKB solution $\tilde{S}$ of the Riccati equation*

$$\tilde{S}^2 + \frac{\partial \tilde{S}}{\partial \tilde{x}} = \eta^2 Q(\tilde{x}, t) \tag{2.38}$$

and a WKB solution S of the Riccati equation

$$S^2 + \frac{\partial S}{\partial x} = \eta^2 (E(t, \eta) - \frac{1}{4} x^2), \tag{2.39}$$

we find that

$$\tilde{S}_{\text{odd}}(\tilde{x}, t, \eta) = \frac{\partial x(\tilde{x}, t, \eta)}{\partial \tilde{x}} S_{\text{odd}}(x(\tilde{x}, t, \eta), \eta; E(t, \eta)) \tag{2.40}$$

holds if the branches of $\tilde{S}_{-1}$ *and* S_{-1} *are chosen so that*

$$\arg \tilde{S}_{-1}(\tilde{x}, t) = \arg \left(\frac{\partial x_0(\tilde{x}, t)}{\partial \tilde{x}} S_{-1}(x_0(\tilde{x}, t); E_0(t)) \right) \tag{2.41}$$

may hold.

(ii) *For a WKB solution* $\tilde{\psi}_+(\tilde{x}, t, \eta)$ *of the MTP equation*

$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x}, t) \right) \tilde{\psi}_+ = 0 \quad (t \neq 0) \tag{2.42}$$

that is normalized as in (2.36), *we can find a WKB solution* $\psi_+(x, \eta; E(t, \eta))$ *of the* ∞*-Weber equation*

$$\left(\frac{d^2}{dx^2} - \eta^2 (E(t, \eta) - \frac{1}{4} x^2) \right) \psi_+(x, \eta; E(t, \eta)) = 0 \tag{2.43}$$

for which the following relation holds:

$$\tilde{\psi}_+(\tilde{x}, t, \eta) = \left(\frac{\partial x(\tilde{x}, t, \eta)}{\partial \tilde{x}} \right)^{-1/2} \psi_+(x(\tilde{x}, t, \eta), \eta; E(t, \eta)). \tag{2.44}$$

Proof. The first assertion (i) is proved in exactly the same manner as in the proof of Theorem 1.2 and Corollary 1.3.

To prove (ii) let us introduce the following symbols:

$$x_0(\tilde{x}, t) = \sum_{j \geq 0} x_0^{(j)}(\tilde{x}) t^j, \tag{2.45}$$

$$E_0(t) = \sum_{j \geq 0} E_0^{(j)} t^j, \tag{2.46}$$

and

$$w(\tilde{x},t,\eta) = \sum_{j\geq 0, k\geq 1} x_k^{(j)}(\tilde{x}) t^j \eta^{-k}. \tag{2.47}$$

Note that we may assume

$$x_0(s_+(t),t) = 2\sqrt{E_0(t)} \tag{2.48}$$

holds; in fact, since $x_0(\tilde{x},t)$ and $E_0(t)$ are holomorphic by Proposition B.1, the comparison of the coefficients of η^0 in (2.21) shows

$$Q(\widetilde{x},t) = \left(\frac{\partial x_0}{\partial \tilde{x}}\right)^2 (E_0(t) - \frac{1}{4} x_0(\tilde{x},t)^2). \tag{2.49}$$

Now using these symbols, we find

(2.50)

$$\begin{aligned}
&S_{\mathrm{odd}}(x(\tilde{x},t,\eta),\eta;E(t,\eta))\frac{\partial x}{\partial \tilde{x}} \\
&= \left(\sum_{n\geq 0} \frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0(\tilde{x},t),\eta;E(t,\eta))\frac{w(\tilde{x},t,\eta)^n}{n!}\right)\left(\frac{\partial x_0}{\partial \tilde{x}} + \frac{\partial w}{\partial \tilde{x}}\right) \\
&= \sum_{n\geq 0} \frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0,\eta;E(t,\eta))\frac{w^n}{n!}\frac{\partial x_0}{\partial \tilde{x}} \\
&\quad + \sum_{n\geq 0} \frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0,\eta;E(t,\eta))\frac{\partial}{\partial \tilde{x}}\left(\frac{w^{n+1}}{(n+1)!}\right).
\end{aligned}$$

We then obtain the following relation from (2.50):

(2.51)

$$\begin{aligned}
&\frac{1}{2}\int_{\check{\tilde{x}}}^{\tilde{x}} S_{\mathrm{odd}}(x(\tilde{x},t,\eta),\eta;E(t,\eta))\frac{\partial x}{\partial \tilde{x}} d\tilde{x} \\
&= \frac{1}{2}\int_{\check{\tilde{x}}}^{\tilde{x}} \Big(\sum_{n\geq 0} \frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0,\eta;E(t,\eta))\frac{w^n}{n!}\frac{\partial x_0}{\partial \tilde{x}} d\tilde{x}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n\geq 0}\frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0,\eta;E(t,\eta))\frac{w^{n+1}}{(n+1)!}\\
&-\frac{1}{2}\int_{\check{\tilde{x}}}^{\tilde{x}}\Big(\sum_{n\geq 0}\frac{\partial^{n+1} S_{\mathrm{odd}}}{\partial x^{n+1}}(x_0,\eta;E(t,\eta))\frac{\partial x_0}{\partial\tilde{x}}\frac{w^{n+1}}{(n+1)!}d\tilde{x}\\
&=\frac{1}{2}\int_{\check{\tilde{x}}}^{\tilde{x}} S_{\mathrm{odd}}(x_0,\eta;E(t,\eta))\frac{\partial x_0}{\partial\tilde{x}}d\tilde{x}+\sum_{n\geq 0}\frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0,\eta;E(t,\eta))\frac{w^{n+1}}{(n+1)!}\\
&=\frac{1}{2}\int_{\check{x}_0}^{x_0} S_{\mathrm{odd}}(x,\eta;E(t,\eta))dx+\sum_{n\geq 0}\frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0,\eta;E(t,\eta))\frac{w^{n+1}}{(n+1)!}\\
&=\frac{1}{2}\int_{\check{x}}^{x} S_{\mathrm{odd}}(x,\eta;E(t,\eta))dx\Big|_{x=x(\tilde{x},t,\eta)}.
\end{aligned}
$$

Furthermore, the relation (2.48) entails that this can be written as

$$
(2.52)\qquad \int_{2\sqrt{E_0(t)}}^{x} S_{\mathrm{odd}}(x,\eta;E(t,\eta))dx\Big|_{x=x(\tilde{x},t,\eta)}.
$$

Thus, by choosing

$$
(2.53)\qquad \frac{1}{\sqrt{S_{\mathrm{odd}}(x,\eta;E(t,\eta))}}\exp\Big(\int_{2\sqrt{E_0(t)}}^{x} S_{\mathrm{odd}}(x,\eta;E(t,\eta))dx\Big)
$$

as $\psi_+(x,\eta;E(t,\eta))$, we obtain (2.44) from (2.40), (2.51) and (2.52). □

Corollary 2.5 ([KT1, Proposition A.6]). *For a WKB solution $\widetilde{S}$ of* (2.38) *we find*

$$
(2.54)\qquad \oint_{\widetilde{\gamma}(t)} \widetilde{S}_{\mathrm{odd}}(\widetilde{x},t,\eta)d\widetilde{x}=2\pi i E(t,\eta),
$$

where $\widetilde{\gamma}(t)$ designates the closed curve in the cut plane shown in Figure 2.1.

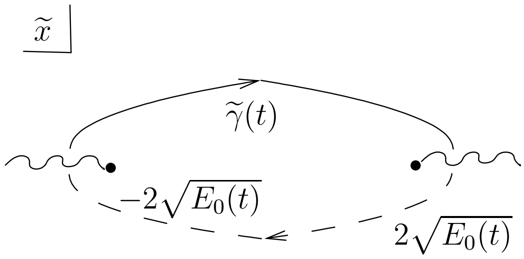

Figure 2.1.

Proof. Using the convergence proof of $\sum_{j\geq 0} x_k^{(j)}(\widetilde{x})t^j$ (Propositions B.1 and B.2), we find from (2.40) that

$$\oint_{\widetilde{\gamma}(t)} \widetilde{S}_{\mathrm{odd}} d\widetilde{x} = \oint_{x_0(\widetilde{\gamma}(t),t)} S_{\mathrm{odd}} dx. \tag{2.55}$$

Then a straightforward computation shows that the right-hand side of (2.55) coincides with $2\pi i E(t,\eta)$. □

The similarity between Theorem 1.4 and Theorem 2.4 (ii) indicates that the Borel transformation of the relation (2.44) may provide us with a microdifferential relation, and it is really the case. To show this fact we introduce a holomorphic function $g(x,t)$, instead of $g(x)$ given by (1.18), which satisfies

$$x = x_0(g(x,t),t) \tag{2.56}$$

on a neighborhood of the origin $(x,t) = (0,0)$. The unique existence of such a function g is guaranteed by (2.23). In particular, $g(x,0) = g(x)$ holds. Then, by defining $r_k = r_k(x,t)$ $(k \geq 0)$ by

$$r_k = \sum_{j\geq 0} x_k^{(j)}(g(x,t))t^j \tag{2.57}$$

this time, we find that the proof of Theorem 1.5 applies to the current situation, almost word for word.

First, the Borel transformed MTP operator for $t \neq 0$ is seen to assume the form

$$(2.58)\qquad L \underset{\text{def}}{=} (g')^{-2}\left(\frac{\partial^2}{\partial x^2} - \left(E_0(t) - \frac{x^2}{4}\right)\frac{\partial^2}{\partial y^2} - \frac{g''}{g'}\frac{\partial}{\partial x}\right)$$

in (x, y, t)-coordinate; here g' and g'' respectively stand for $\partial g/\partial x$ and $\partial^2 g/\partial x^2$. Next we define a microdifferential operator $\mathcal{X}$ by

$$(2.59)\qquad :g'(x,t)^{1/2}\left(1 + \frac{\partial r}{\partial x}\right)^{-1/2} \exp(r(x,t,\eta)\xi):,$$

where

$$(2.60)\qquad r = r(x,t,\eta) = \sum_{k\geq 1} r_k(x,t)\eta^{-k}.$$

Then (2.44) implies

$$(2.61)\qquad \widetilde{\psi}_{+,B}(x,t,y) = \mathcal{X}\psi_{+,B}(x,y).$$

By letting M denote the Borel transformed ∞-Weber operator, i.e.,

$$(2.62)\qquad \frac{\partial^2}{\partial x^2} - \left(E\left(t, \frac{\partial}{\partial y}\right) - \frac{x^2}{4}\right)\frac{\partial^2}{\partial y^2},$$

and defining another microdifferential operator $\mathcal{Y}$ by

$$(2.63)\qquad :g'(x,t)^{1/2}\left(1 + \frac{\partial r}{\partial x}\right)^{3/2} \exp(r(x,t,\eta)\xi):,$$

we obtain the following Theorem 2.6 that generalizes Theorem 1.5; Theorem 1.5 is a special case of Theorem 2.6 in the sense that it is nothing but Theorem 2.6 where t is set to be 0.

Theorem 2.6. *We find*

$$(2.64)\qquad L\mathcal{X} = \mathcal{Y}M$$

holds for invertible microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$.

Theorem 2.6 shows that the operators L and M are microlocally intertwined. This fact indicates that the singularity structure of $\psi_{+,B}$ should be inherited to $\widetilde{\psi}_{+,B}$. A more precise statement (Theorem 5.1) will be given in Section 5 after some detailed analysis of singularity structure of $\psi_{+,B}$ to be done in Section 4. Here we only note that we can find an integral operator to represent the action of the microdifferential operator $\mathcal{X}$ upon the multi-valued analytic function $\psi_{+,B}(x,t,y)$, as is discussed in Appendix C. Here we summarize the core of Appendix C as the following

Theorem 2.7. *The action of the microdifferential operator $\mathcal{X}$ upon the multi-valued analytic function $\psi_{+,B}(x,y)$ is represented as an integro-differential operator of the following form.*

$$\mathcal{X}\psi_{+,B} = \int_{y_0}^{y} K(x,t,y-y',d/dx)\psi_{+,B}(x,t,y')dy', \tag{2.65}$$

where $K(x,t,y,d/dx)$ is a differential operator of infinite order that is defined on $\{(x,t,y) \in \mathbb{C}^3; (x,t) \in \omega$ for an open neighborhood ω of the origin and $|y| < C$ for some positive constant $C\}$, and y_0 is a constant that fixes the action of $(\partial/\partial y)^{-1}$ as an integral operator. (See Figure 2.2.)

The proof of Theorem 2.7 is based on Theorem B.4 and Proposition C.1. Here we emphasize that a differential operator of infinite order is of local character ([SKK]). Thus the location of singularities of $\mathcal{X}\psi_{+,B}$ can be immediately read off from the location of singularities of $\psi_{+,B}(x,y')$ in y'-plane for each fixed (x,t).

Remark 2.4. It follows from the reasoning in Appendix C that ω may be assumed to have the form $\omega_0 \times D$, where

(2.66) ω_0 is a simply connected open set in $\mathbb{C}_x$ that contains $s_+(t)$ and $s_-(t)$,

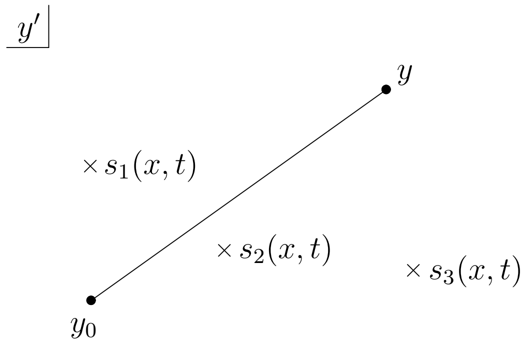

Figure 2.2 : $y' = s_j(x,t)(j = 1, 2, \cdots)$ are the singular points of $\psi_{+,B}(x, t, y')$; the local character of K implies the singularities of $K\psi_{+,B}$ are confined to these points.

and

(2.67) $\quad D = \{t \in \mathbb{C}; |t| < \delta$ for some positive constant $\delta\}$.

Then, as long as t is in D, the integral operator in the right-hand side of (2.65), which is obtained through the Borel transformation of the right-hand side of (2.44) written down in (x, y, t)-coordinate, acts on any multi-valued analytic function φ defined on a neighborhood of $\omega_0 \times \{t\} \times \{y \in \mathbb{C}; |y - \delta_0| < C\}$; the domain of definition of the acted function φ contains a product set $\omega_0 \times D \times \{y \in \mathbb{C}; |y - \delta_0| < C\}$. This is what we mean by saying that the transformation given by (2.44) is "uniform" with respect to t; the uniformity is primarily concerned with the uniformity in the Borel-plane, i.e., y-plane. This uniformity, which is not immediately visible from (2.44), guarantees that each individual fixed singular point of $\psi_{+,B}$ is contained in the domain of definition of the integral operator (2.65) for sufficiently small t. Note that, as we will see in Section 4, a fixed singular point of $\psi_{+,B}$ is of the form

(2.68) $$y = -y_{\pm}(x,t) + 2m\pi E_0(t) \quad (m = 0, \pm 1, \pm 2, \cdots),$$

where

$$(2.69) \qquad y_{\pm}(x,t) = \pm \int_{2\sqrt{E_0(t)}}^{x} \sqrt{E_0(t) - \frac{x^2}{4}} dx.$$

Note also that $E_0(t)$ tends to 0 as t tends to 0 by (2.24).

3 Analytic properties of WKB solutions of the Weber equation

To analyze WKB solutions of the ∞-Weber equation in Section 4, we first recall several basic facts about WKB solutions of the Weber equation. In this section the Weber equation means, by definition, the following Schrödinger equation:

$$(3.1) \qquad \left(\frac{d^2}{dx^2} + \eta^2\left(\frac{x^2}{4} - E\right)\right)\psi = 0.$$

In choosing the above potential $-(x^2/4-E)$ we have followed [KT1]. Via the scaling

$$(3.2) \qquad x = \sqrt{2}z,$$

Equation (3.1) is reduced to

$$(3.3) \qquad \left(\frac{d^2}{dz^2} + \eta^2(z^2 - 2E)\right)\psi = 0,$$

the equation used in [SS] with the difference of the sign in front of $2E$. Note that we use the inverted-parabola potential to find the model equation for the situation where two simple turning points are connected by a Stokes curve ([AKT1, Section 3], [KT1], [SS]). We also note that this choice forces us to employ the coordinate transformation

$$(3.4) \qquad w = \exp\left(-\frac{\pi}{4}i\right)\sqrt{\eta}x$$

to relate WKB solutions of (3.1) with Whittaker's principal parabolic cylinder function $D_{i\eta E-1/2}(w)$. (The rotation by $-\pi/4$ has the effect of bringing the inverted-parabola potential to the ordinary parabola potential.) As we emphasized in Introduction, the core object of this section is Sato's conjecture ([KT1, p.95]); originally it related a WKB solution of (3.1) with the parabolic cylinder function, and Shen and Silverstone elucidated its WKB-theoretic meaning by observing that the parabolic cylinder function is a finite constant (versus infinite series; see (3.6) below) multiple of a Borel resummed WKB solution of the Weber equation that is normalized at infinity in the sense of [DDP1] and [DP], that is,

(3.5)

$$\psi_{\pm}^{(\infty)}(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm\left\{\eta\int_{2\sqrt{E}}^{x} S_{-1}dx + \int_{\infty}^{x}(S_{\mathrm{odd}}-\eta S_{-1})dx\right\}\right).$$

Here we note that the meaning of the symbols η and S is different from that used in [SS]. There are two important points to be noted in the relation presented in [SS, (44), (45)]. First it manifests the well-definedness of the Borel sum of the WKB solution normalized at infinity when $\arg\eta = 0$; secondly it enables us to analyze Sato's conjecture completely in the framework of exact WKB analysis in the following manner: the numerical factor relating the parabolic cylinder function and the particular WKB solution in question is a "huge" but explicit one, i.e.,

$$\exp\left(\frac{i\pi}{8}\right)\left(\frac{-E}{e\hbar}\right)^{\frac{i(-E)}{2\hbar}} \tag{3.6}$$

(cf. [SS, (43)]; $1/\hbar$ is our large parameter η), and setting aside this factor we find that Sato's conjecture is reduced to finding out the explicit form of the logarithm ϕ of the ratio of a WKB solution

$$\psi_+(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\eta \int_{2\sqrt{E}}^{x} S_{\mathrm{odd}} dx\right) \tag{3.7}$$

and the normalized at infinity WKB solution (3.5), that is,

$$\psi_+(x,\eta) = (\exp \phi(E,\eta))\psi_+^{(\infty)}(x,\eta). \tag{3.8}$$

Note that $\phi(E,\eta)$ is independent of x; actually it is known in exact WKB analysis by the name of Voros' coefficient after [V]. Its explicit form is

$$\int_{2\sqrt{E}}^{\infty} (S_{\mathrm{odd}} - \eta S_{-1}) dx, \tag{3.9}$$

and the problem is to show that it is equal to

$$\frac{1}{2} \sum_{n \geq 1} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n} (-i\eta E)^{1-2n}, \tag{3.10}$$

where B_{2n} designates the $2n$-th Bernoulli number, i.e.,

$$\frac{w}{e^w - 1} = 1 - \frac{w}{2} + \sum_{n \geq 1} \frac{B_{2n}}{(2n)!} w^{2n}. \tag{3.11}$$

In what follows we use "Sato's conjecture" in its WKB theoretic form, that is, we begin our discussion with the expression (3.10) of $\phi(E,\eta)$. At the same time we note that the proof of "Sato's conjecture" given by Shen and Silverstone makes full use of analytic properties of the parabolic cylinder function.

It is known ([DDP1], [DP, Theorem 1.2.2 (c)]) that $\psi_{+,B}^{(\infty)}(x,y)$, the Borel transform of $\psi_+^{(\infty)}(x,\eta)$, is free from singularities on the real 1-dimensional half line $\{y \in \mathbb{C};\ y = -y_+(x) + \rho,\ \rho > 0\}$, where $y_+(x)$ is, by definition,

$$\int_{2\sqrt{E}}^{x} S_{-1}(x) dx. \tag{3.12}$$

Hence (3.8) implies that the study of singularity structure of $\psi_{+,B}(x,y)$ is reduced to that of the Borel transform of $\exp\phi(E,\eta)$. To study its singularity structure we first give a concrete description of the Borel transform $\phi_B(E,y)$ of ϕ. It then follows from (3.10) and the definition of the Borel transformation that

$$\begin{aligned}
\phi_B(E,y) &= \frac{1}{2}\sum_{n\geq 1}\frac{2^{1-2n}-1}{2n(2n-1)}B_{2n}(-iE)^{1-2n}\frac{y^{2n-2}}{(2n-2)!} \\
&= \frac{-iE}{y^2}\sum_{n\geq 1}\frac{B_{2n}(-2iE)^{-2n}y^{2n}}{(2n)!} \\
&\quad - \frac{-iE}{2y^2}\sum_{n\geq 1}\frac{B_{2n}}{(2n)!}(-iE)^{-2n}y^{2n} \\
&= \frac{-iE}{y^2}\left(\frac{iy/(2E)}{\exp(iy/(2E))-1}-1+\frac{iy}{4E}\right) \\
&\quad + \frac{iE}{2y^2}\left(\frac{iy/E}{\exp(iy/E)-1}-1+\frac{iy}{2E}\right).
\end{aligned} \tag{3.13}$$

Setting

$$\sigma = iy/(2E) \quad \text{and} \quad X = \exp\sigma, \tag{3.14}$$

we find

$$\begin{aligned}
\phi_B(E,y) &= \frac{1}{2y}\left(\frac{1}{X-1}-\frac{1}{X^2-1}-\frac{1}{2\sigma}\right) \\
&= \frac{1}{4y}\left(-\frac{1}{\sigma}+\frac{1}{X-1}+\frac{1}{X+1}\right).
\end{aligned} \tag{3.15}$$

In a neighborhood of $y=0$, the Taylor expansion shows

$$\phi_B(E,y) = \frac{-i}{48E} + O(y), \tag{3.16}$$

whereas, near $y = 4n\pi E$ $(n \neq 0)$,

$$\phi_B(E,y) = \frac{1}{8n\pi i}\,\frac{1}{y - 4n\pi E} + O(1) \tag{3.17}$$

and, near $y = 2(2n+1)\pi E$,

$$\phi_B(E,y) = \frac{-1}{4(2n+1)\pi i}\,\frac{1}{y - 2(2n+1)\pi E} + O(1). \tag{3.18}$$

Thus $\phi_B(E,y)$ is a single-valued analytic function with simple poles at $y = 2m\pi E$ $(m \neq 0)$ with its residue $(-1)^m/(4m)$ there.

We next consider the alien derivative $\Delta_{y=2m\pi E}\ \phi$ of $\phi(E,\eta)$. The alien derivative is, by definition, given by

$$\begin{aligned} \Delta\phi &= \mathcal{B}^{-1}\log(\mathcal{L}_-^{-1}\mathcal{L}_+)\mathcal{B}\phi \\ &= \mathcal{B}^{-1}\log(1 + (\mathcal{L}_-^{-1}\mathcal{L}_+ - 1))\mathcal{B}\phi \\ &= \mathcal{B}^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(\mathcal{L}_-^{-1}\mathcal{L}_+ - 1)^n\mathcal{B}\phi, \end{aligned} \tag{3.19}$$

where $\mathcal{B}$ denotes the Borel transformation and $\mathcal{L}_+$ (resp., $\mathcal{L}_-$) denotes the Laplace transformation along a path which avoids the singular points from the above (resp., from the below). It is known (cf., e.g., [DP]) that (3.19) can be expressed also as

$$\Delta\phi = \sum_{m=1}^{\infty}\Delta_{y=2m\pi E}\ \phi \tag{3.20}$$

with
(3.21)

$$\Delta_{y=2m\pi E}\ \phi = \mathcal{B}^{-1}\left[(\gamma_+^{(m)} - \gamma_-^{(m)})\sum_{\varepsilon_j=\pm}\frac{p_+!p_-!}{m!}\gamma_{\varepsilon_{m-1}}^{(m-1)}\cdots\gamma_{\varepsilon_1}^{(1)}\right]\mathcal{B}\phi,$$

where $\gamma_+^{(j)}$ (resp., $\gamma_-^{(j)}$) designates analytic continuation along a path avoiding the j-th singular point $y = 2j\pi E$ from the above (resp., from the below) and p_+ (resp., p_-) denotes the number of indices j for which $1 \leq j \leq m-1$ and $\varepsilon_j = +$ (resp., $\varepsilon_j = -$) hold. In the case of $\phi(E, \eta)$ in question, as its Borel transform is a single-valued analytic function with simple poles at $y = 2m\pi E$ ($m \neq 0$), its alien derivative $\Delta_{y=2m\pi E}\, \phi$ is the residue of $\phi_B(E, y)$ at $y = 2m\pi E$, that is,

$$\Delta_{y=2m\pi E}\, \phi = \frac{(-1)^m}{4m}. \tag{3.22}$$

(Cf. [P1], [CNP], [Sa]). Then, by the alien calculus, we find

$$\Delta_{y=2m\pi E}(\exp\phi) = \frac{(-1)^m}{4m} \exp\phi. \tag{3.23}$$

Since

$$\Delta(\exp(-y_+(x)\eta)\psi_+^{(\infty)}(x, \eta)) = 0 \tag{3.24}$$

holds when x is in the interior of each region bounded by Stokes curves associated with the Weber equation (cf. Figure 3.1), say in region I, we

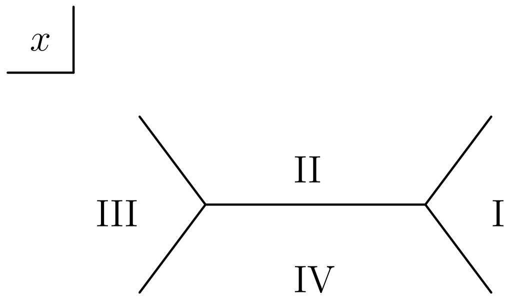

Figure 3.1.

find that

(3.25)

$$\Delta_{y=-y_+(x)+2m\pi E}\ (\exp(-y_+(x)\eta)\psi_+(x, \eta))$$

$$= \Delta_{y=-y_+(x)+2m\pi E} \left(\exp(-y_+(x)\eta)\exp(\phi(E,\eta))\psi_+^{(\infty)}(x,\eta)\right)$$
$$= \frac{(-1)^m}{4m} \left(\exp(-y_+(x)\eta)\exp(\phi(E,\eta))\ \psi_+^{(\infty)}(x,\eta)\right)$$
$$= \frac{(-1)^m}{4m} \left(\exp(-y_+(x)\eta)\psi_+(x,\eta)\right)$$

holds for x in I.

Thus we find the following Theorem 3.1 on the singularity structure of $\psi_{+,B}(x,y)$.

Theorem 3.1. *Let $\psi_+(x,\eta)$ denote the WKB solution of the Weber equation that is normalized as in* (3.7). *Then its Borel transform $\psi_{+,B}(x,y)$ is singular at*

(3.26) $$y = -y_+(x) + 2m\pi E \quad (m = 0, \pm 1, \pm 2, \cdots),$$

where

(3.27) $$y_+(x) = \int_{2\sqrt{E}}^{x} \sqrt{E - \frac{x^2}{4}}dx,$$

and its alien derivative there, i.e, $\Delta_{y=-y_+(x)+2m\pi E}\ \psi_+$ satisfies the following relation (3.28) *for x in region I :*

(3.28) $$(\Delta_{y=-y_+(x)+2m\pi E}\ \psi_+)_B(x,y) = \frac{(-1)^m}{4m}\psi_{+,B}(x, y + 2m\pi E).$$

4 WKB solutions of the ∞-Weber equation

As Theorems 2.2 and 2.4 show, the WKB theoretic canonical form of an MTP equation is the ∞-Weber equation

(4.1) $$\left(\frac{d^2}{dx^2} - \eta^2(E(t,\eta) - \frac{1}{4}x^2)\right)\tilde{\psi}(x,\eta;E(t,\eta)) = 0.$$

In analyzing WKB solutions of (4.1), we wish to relate them with WKB solutions of the Weber equation

$$\left(\frac{d^2}{dx^2} - \eta^2(E - \frac{1}{4}x^2)\right)\psi(x,\eta;E) = 0. \tag{4.2}$$

For this purpose we again use the core idea of Sections 1 and 2, that is, we relate the Borel transform $\tilde{\psi}_B$ of $\tilde{\psi}$ and the Borel transform ψ_B of ψ by a microdifferential operator and then deduce analytic properties of $\tilde{\psi}_B$ from that of ψ_B. To be more concrete, we interpret a WKB solution $\tilde{\psi}(x,\eta;E(t,\eta))$ of (4.1) as follows:

$$\begin{aligned}\tilde{\psi}(x,\eta) =& \tilde{\psi}(x,\eta;E(t,\eta)) \\ =& \sum_{n\geq 0} \frac{(E_1\eta^{-1} + E_2\eta^{-2} + \cdots)^n}{n!}\frac{\partial^n}{\partial E_0^n}\psi(x,\eta;E_0),\end{aligned} \tag{4.3}$$

where $\psi(x,\eta;E_0)$ is a WKB solution of (4.2) with $E = E_0(t)$. As (2.32) guarantees that

$$\left.\frac{\partial E_0}{\partial t}\right|_{t=0} \neq 0 \tag{4.4}$$

holds for an MTP operator, we may use E_0 as an independent variable; E_j's may be regarded as functions of E_0. In view of the growth order condition (B.107) that E_j's satisfy we find

$$\mathcal{E}\left(E_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial E_0}\right) = \sum_{n\geq 0} \frac{(E_1(\partial/\partial y)^{-1} + E_2(\partial/\partial y)^{-2} + \cdots)^n}{n!}\frac{\partial^n}{\partial E_0^n} \tag{4.5}$$

is a well-defined microdifferential operator on

$$\{(y,E_0;\eta,\theta) \in T^*\mathbb{C}^2; |E_0| < \delta_0, \eta \neq 0\} \tag{4.6}$$

for some $\delta_0 > 0$. In what follows we identify η and θ respectively with the symbol $\sigma(\partial/\partial y)$ and the symbol $\sigma(\partial/\partial E_0)$; using these symbols we may write

$$\mathcal{E} =: \sum_{n\geq 0} \frac{(E_1\eta^{-1} + E_2\eta^{-2} + \cdots)^n\theta^n}{n!} : . \tag{4.7}$$

Now, through the Borel transformation the relation (4.3) reads as follows:

$$\tilde{\psi}_B(x,y) = \mathcal{E}(E_0, \partial/\partial y, \partial/\partial E_0)\psi_B(x,y;E_0). \tag{4.8}$$

We also note that a similar relation (4.11) holds for the Borel transform $\tilde{S}_B$ (resp., S_B) of a WKB solution $\tilde{S}$ (resp., S) of the Riccati equation (4.9) (resp., (4.10)) associated with (4.1) (resp., (4.2)), that is,

$$\tilde{S}^2 + \frac{\partial \tilde{S}}{\partial x} = \eta^2(E(t,\eta) - \frac{1}{4}x^2), \tag{4.9}$$

$$S^2 + \frac{dS}{dx} = \eta^2(E_0 - \frac{1}{4}x^2), \tag{4.10}$$

$$\tilde{S}_B(x,y) = \mathcal{E}(E_0, \partial/\partial y, \partial/\partial E_0)S_B(x,y;E_0). \tag{4.11}$$

It is also clear that a similar relation holds for $\tilde{S}_{\mathrm{odd}}$, the odd part of $\tilde{S}$;

$$\tilde{S}_{\mathrm{odd},B}(x,y) = \mathcal{E}(E_0, \partial/\partial y, \partial/\partial E_0)S_{\mathrm{odd},B}(x,y;E_0). \tag{4.12}$$

Furthermore, in parallel with the above treatment of WKB solutions of the ∞-Weber equation, we can give an analytic meaning to the exponential of the Voros coefficient for the ∞-Weber equation via its Borel transform, i.e., $V_B \underset{\mathrm{def}}{=} (\exp\phi(E(t,\eta),\eta))_B$ in the following manner:

$$V_B(y) = \mathcal{E}(E_0, \partial/\partial y, \partial/\partial E_0)(\exp\phi(E_0,\eta))_B. \tag{4.13}$$

Remark 4.1. As in Section 2, the right-hand sides of (4.8), (4.11), (4.12) and (4.13) should be understood as multi-valued analytic functions acted upon by the integral operator determined by the microdifferential operator $\mathcal{E}$. While the estimation (B.107) guarantees the existence of a common domain of definition as t tends to 0, the quantity $E_0(t)$ tends to 0 as t tends to 0. On the other hand, (3.26) implies that a fixed singular point of $\psi_{+,B}(x,y)$ (with respect to $y = -y_+(x)$) is located at $y = -y_+(x) + 2m\pi E_0$. Thus each individual fixed singular point of $\psi_{+,B}(x,y)$ is contained, for sufficiently small t, in the domain of definition of the integral operator in question. Hence, in this section, we do not worry about the existence of a sufficiently large domain of definition of the integral operator in question; if necessary, we assume that t (or, equivalently E_0) is sufficiently close to 0.

Concerning the analytic structure of $\tilde{\psi}_{+,B}$ and V_B we find the following.

Theorem 4.1. *Let* $\tilde{\psi}_+(x,\eta)$ *and* $\phi(E(t,\eta),\eta)$ *respectively denote*

(4.14) $$\frac{1}{\sqrt{\tilde{S}_{\mathrm{odd}}}} \exp \int_{2\sqrt{E_0}}^{x} \tilde{S}_{\mathrm{odd}}\, dx$$

and

(4.15) $$\int_{2\sqrt{E_0}}^{\infty} (\tilde{S}_{\mathrm{odd}} - \eta \tilde{S}_{-1})\, dx,$$

where $\tilde{S}_{\mathrm{odd}}$ *designates the odd part* (*in the sense of Definition 1.1*) *of a WKB solution* $\tilde{S}$ *of the Riccati equation attached to* (4.1), *i.e.,*

(4.16) $$\tilde{S}^2 + \frac{d\tilde{S}}{dx} = E(t,\eta) - \frac{1}{4}x^2.$$

Then the Borel transform $\tilde{\psi}_{+,B}(x,y)$ *and* $V_B = (\exp\phi)_B$ *satisfy the following relations:*

(4.17)

$$\begin{aligned}(\Delta_{y=-y_+(x)+2m\pi E_0}\tilde{\psi}_+)_B(x,y) \\ = \frac{(-1)^m}{4m} : \exp(2m\pi(E_1 + E_2\eta^{-1} + \cdots)) : \tilde{\psi}_{+,B}(x, y + 2m\pi E_0),\end{aligned}$$

(4.18)

$$(\Delta_{y=2m\pi E_0}V)_B(y)$$
$$= \frac{(-1)^m}{4m} : \exp(2m\pi(E_1 + E_2\eta^{-1} + \cdots)) : V_B(y + 2m\pi E_0),$$

where $m = 1, 2, 3, \cdots,$ *and* $y_+(x)$ *denotes*

$$\int_{2\sqrt{E_0}}^{x} \tilde{S}_{-1}(x)dx. \tag{4.19}$$

Proof. By (4.8) and the definition of the alien derivative, we find

(4.20)

$$(\Delta_{y=-y_+(x)+2m\pi E_0}\widetilde{\psi}_+)_B(x,y)$$
$$=(\Delta_{y=-y_+(x)+2m\pi E_0}\mathcal{B}^{-1}(\mathcal{E}(E_0,\partial/\partial y,\partial/\partial E_0)\psi_{+,B}(x,y;E_0)))_B(x,y)$$
$$=\mathcal{E}(E_0,\partial/\partial y,\partial/\partial E_0)((\Delta_{y=-y_+(x)+2m\pi E_0}\psi_+)_B(x,y;E_0))(x,y).$$

It then follows from Theorem 3.1 that this can be rewritten further as follows:

$$\mathcal{E}(E_0,\partial/\partial y,\partial/\partial E_0)\frac{(-1)^m}{4m}\psi_{+,B}(x,y+2m\pi E_0;E_0). \tag{4.21}$$

To relate this function with $\widetilde{\psi}_{+,B}(x, y + 2m\pi E_0)$, we introduce the following coordinate transformation from (y, E_0) to $(\widetilde{y}, \widetilde{E}_0)$:

$$\begin{cases} \widetilde{y} = y + 2m\pi E_0 \\ \widetilde{E}_0 = E_0. \end{cases} \tag{4.22}$$

Correspondingly we then have

$$\begin{cases} \eta = \widetilde{\eta} \\ \theta = 2m\pi\widetilde{\eta} + \widetilde{\theta}. \end{cases} \tag{4.23}$$

Using $(\widetilde{y}, \widetilde{E}_0)$-variable, we find

(4.24)

$$
\begin{aligned}
&\mathcal{E}(E_0, \partial/\partial y, \partial/\partial E_0)\psi_{+,B}(x, y + 2m\pi E_0; E_0) \\
&=: \sum_{n\geq 0} \frac{(E_1\widetilde{\eta}^{-1} + E_2\widetilde{\eta}^{-2} + \cdots)^n(\widetilde{\theta} + 2m\pi\widetilde{\eta})^n}{n!} : \psi_{+,B}(x, \widetilde{y}; \widetilde{E}_0) \\
&=: \sum_{n\geq 0} \frac{1}{n!}(E_1\widetilde{\eta}^{-1} + E_2\widetilde{\eta}^{-2} + \cdots)^n \sum_{\substack{k+l=n \\ k,l\geq 0}} \frac{n!}{k!l!}\widetilde{\theta}^k(2m\pi\widetilde{\eta})^l : \psi_{+,B}(x, \widetilde{y}; \widetilde{E}_0) \\
&=: \sum_{l\geq 0} \frac{1}{l!}(2m\pi(E_1 + E_2\widetilde{\eta}^{-1} + \cdots))^l : \\
&\qquad\qquad : \sum_{k\geq 0} \frac{1}{k!}(E_1\widetilde{\eta}^{-1} + E_2\widetilde{\eta}^{-2} + \cdots)^k\widetilde{\theta}^k : \psi_{+,B}(x, \widetilde{y}; \widetilde{E}_0) \\
&=: \exp(2m\pi(E_1 + E_2\widetilde{\eta}^{-1} + \cdots)) : \mathcal{E}(E_0, \partial/\partial\widetilde{y}, \partial/\partial\widetilde{E}_0)\psi_{+,B}(x, \widetilde{y}; \widetilde{E}_0) \\
&=: \exp(2m\pi(E_1 + E_2\eta^{-1} + \cdots)) : \widetilde{\psi}_{+,B}(x, y + 2m\pi E_0).
\end{aligned}
$$

Combining (4.20), (4.21) and (4.24), we obtain (4.17). The proof of (4.18) can be given in exactly the same manner. □

Remark 4.2. From the viewpoint of applications, it should be most appropriate to understand (4.18) to be the content of the mathematical assertion called "Sato's conjecture".

Remark 4.3. Although we have presented the result in full generality for the future reference, all E_k (k : odd) vanish in our actual problem discussed in this article. (See (2.25).) However, if the potential Q has the form (1.21), then E_k (k : odd) appears in general.

5 Analytic properties of Borel transformed WKB solutions of an MTP equation

In the preceding section we have seen that the Borel transform ψ_B of a WKB solution ψ of the ∞-Weber equation

$$(5.1)\qquad \left(\frac{d^2}{dx^2} - \eta^2(E(t,\eta) - \frac{1}{4}x^2)\right)\psi(x,\eta) = 0$$

can be represented in the form

$$(5.2)\qquad \mathcal{E}(E_0, \partial/\partial y, \partial/\partial E_0)\varphi_B(x,y;E_0),$$

with a microdifferential operator $\mathcal{E}$ and the Borel transform φ_B of a WKB solution φ of the Weber equation

$$(5.3)\qquad \left(\frac{d^2}{dx^2} - \eta^2(E_0 - \frac{1}{4}x^2)\right)\varphi(x,\eta;E_0) = 0.$$

(For the convenience of the presentation in this section, here we have changed the symbol $(\tilde{\psi}, \psi)$ to (ψ, φ).) On the other hand, Theorem 2.4 (ii) shows that the study of each WKB solution $\tilde{\psi}_+(\tilde{x}, t, \eta)$ of an MTP equation for $t \neq 0$ can be reduced to that of a WKB solution ψ_+ of the ∞-Weber equation in that they are related as in (5.4) below with the infinite series $x(\tilde{x}, t, \eta)$ and $E(t, \eta)$ constructed in Theorem 2.2:

$$(5.4)\qquad \tilde{\psi}_+(\tilde{x}, t, \eta) = \left(\frac{\partial x(\tilde{x}, t, \eta)}{\partial x}\right)^{-1/2} \psi_+(x(\tilde{x}, t, \eta), \eta; E(t, \eta)).$$

Furthermore, the growth order condition (B.108) that $\{x_k(\tilde{x}, t, \eta)\}_{k\geq 0}$ satisfies has enabled us to rewrite (5.4) as a microdifferential relation (2.61), that is,

$$(5.5)\qquad \tilde{\psi}_{+,B}(x, t, y) = \mathcal{X}\psi_{+,B}(x, y)$$

for the microdifferential operator $\mathcal{X}$ given by

$$(5.6) \qquad : g'(x,t)^{1/2}\left(1+\frac{\partial r}{\partial x}\right)^{-1/2} \exp(r(x,t,\eta)\xi) :,$$

with the notations in Section 2. (See (2.59).) In view of the concrete expression (2.65) together with Theorem 4.1, we find that the singularities of $\tilde{\psi}_{+,B}$ are confined to

$$(5.7) \qquad y = -y_+(x,t) + 2m\pi E_0(t) \quad (m = 0, \pm 1, \pm 2, \cdots)$$

in a sufficiently small neighborhood of the origin $(x,y,t) = (0,0,0)$, where

$$(5.8) \qquad y_+(x,t) = \int_{2\sqrt{E_0(t)}}^{x} \sqrt{E_0(t) - \frac{x^2}{4}}dx.$$

Then, in view of (2.48) and (2.49), we find that the corresponding singular point in $(\tilde{x},t,y)$-coordinate is

$$(5.9) \qquad y = -y_+(\tilde{x},t) + 2m\pi E_0(t),$$

where

$$(5.10) \qquad y_+(\tilde{x},t) = \int_{s_+(t)}^{\tilde{x}} \sqrt{Q(\tilde{x},t)}d\tilde{x}.$$

Since the alien derivative (or the discontinuity) of $\psi_{+,B}$ at the point is given by (4.17) (with $E_{2p+1} = 0$), the application of $\mathcal{X}$ entails the following

Theorem 5.1. *For an integer m and the Borel transform $\tilde{\psi}_{+,B}$ of the WKB solution $\tilde{\psi}_+$ of an MTP equation ($t \neq 0$) that is normalized as in* (2.36)*, the following relation* (5.11) *holds for sufficiently small t ($\neq 0$).*

$$(5.11) \qquad (\Delta_{y=-y_+(\tilde{x},t)+2m\pi E_0(t)}\tilde{\psi}_+)_B(\tilde{x},t,y)$$

$$= \frac{(-1)^m}{4m} : \exp(2m\pi(E_2(t)\eta^{-1} + E_4(t)\eta^{-3} + \cdots)) : \tilde{\psi}_{+,B}(\tilde{x}, t, y + 2m\pi E_0(t)),$$

where

$$y_+(\tilde{x}, t) = \int_{s_+(t)}^{\tilde{x}} \sqrt{Q(\tilde{x}, t)}\, d\tilde{x} \tag{5.12}$$

and

$$E_j = \frac{1}{2\pi i} \oint_{\tilde{\gamma}} \tilde{S}_j(\tilde{x}, t)\, d\tilde{x} \tag{5.13}$$

with $\tilde{\gamma}$ being the closed path given in Figure 2.1 and with $\tilde{S}_j$ denoting the coefficient of η^{-j} in $\tilde{S}_{\mathrm{odd}}$, the odd part of a WKB solution $\tilde{S}$ of the Riccati equation

$$\tilde{S}^2 + \frac{\partial \tilde{S}}{\partial \tilde{x}} = \eta^2 Q(\tilde{x}, t). \tag{5.14}$$

Appendix A. Estimation of the transformation to the canonical form near a double turning point

In Appendix A we show that the transformation

$$x^{(0)}(\tilde{x}, \eta) = \sum_{k=0}^{\infty} x_k^{(0)}(\tilde{x})\eta^{-k} \tag{A.1}$$

constructed in Theorem 1.1 is Borel transformable in the sense of [KT2]. That is, we prove the following

Theorem A.1. *Let*

$$x^{(0)}(\tilde{x}, \eta) = \sum_{k=0}^{\infty} x_k^{(0)}(\tilde{x})\eta^{-k} \quad \textit{and} \quad E^{(0)}(\eta) = \sum_{k=0}^{\infty} E_k^{(0)}\eta^{-k} \tag{A.2}$$

be the transformation and the coefficient of the canonical form of an MTP operator at $t = 0$ *constructed in Theorem 1.1, respectively. Then there exist a positive number* ρ_0*, an open neighborhood* ω *of* $\tilde{x} = 0$ *and a positive constant* C_0 *for which* $\omega \supset \{\tilde{x};\, |\tilde{x}| \le \rho_0\}$, $x_k^{(0)}(\tilde{x})$ $(k = 0, 1, 2, \dots)$ *are holomorphic and* $dx_0^{(0)}/d\tilde{x} \neq 0$ *in* ω *and the following inequalities hold for* $k = 1, 2, 3, \cdots$:

$$|E_k^{(0)}| \le k!\, C_0^k, \tag{A.3}$$

$$\sup_{|\tilde{x}| \le \rho_0} |x_k^{(0)}(\tilde{x})| \le k!\, C_0^k. \tag{A.4}$$

$$\sup_{|\tilde{x}| \le \rho_0} \left| \frac{dx_k^{(0)}(\tilde{x})}{d\tilde{x}} \right| \le k!\, C_0^k. \tag{A.5}$$

To prove this theorem, we show the following proposition by induction:

Proposition A.2. *Let*

$$x^{(0)}(\tilde{x}, \eta) = \sum_{k=0}^{\infty} x_k^{(0)}(\tilde{x}) \eta^{-k} \quad \textit{and} \quad E^{(0)}(\eta) = \sum_{k=0}^{\infty} E_k^{(0)} \eta^{-k} \tag{A.6}$$

be the transformation and the coefficient of the canonical form of an MTP operator at $t = 0$ *constructed in Theorem 1.1, respectively. Let* ρ *be a positive constant for which* $x_k^{(0)}(\tilde{x})$ $(k = 0, 1, 2, \dots)$ *are holomorphic and* $dx_0^{(0)}/d\tilde{x} \neq 0$ *holds in an open set containing the disc* $|\tilde{x}| \le \rho$*. Then there exists a positive constant* A *so that for each small positive number* ε*, the following inequalities hold for* $k = 1, 2, 3, \cdots$:

$$|E_k^{(0)}| \le k!\, \varepsilon^{-k} A^k, \tag{A.7}$$

$$\sup_{|\tilde{x}|\le\rho-\varepsilon} |x_k^{(0)}(\tilde{x})| \le k!\,\varepsilon^{-k}A^k, \tag{A.8}$$

$$\sup_{|\tilde{x}|\le\rho-\varepsilon} \left|\frac{dx_k^{(0)}(\tilde{x})}{d\tilde{x}}\right| \le k!\,\varepsilon^{-k}A^k. \tag{A.9}$$

Proof. First we recall the construction of $x_k^{(0)}(\tilde{x})$ and $E_k^{(0)}$ in Section 1. For the sake of simplicity of notation, we abbreviate the superscript. That is, $x_k^{(0)}(\tilde{x})$ and $E_k^{(0)}$ are denoted by $x_k(\tilde{x})$ and E_k, respectively. The leading term E_0 of E is taken to be 0 and that of x is defined by the relation

$$Q(\tilde{x}) = Q(0,\tilde{x}) = -\frac{1}{4}(x_0')^2 x_0^2, \tag{A.10}$$

which entails

$$x_0(\tilde{x}) = 2\left(\int_0^{\tilde{x}} \sqrt{-Q(\tilde{x})}\,d\tilde{x}\right)^{1/2}. \tag{A.11}$$

Here, and in what follows, x_0' designates the differentiation of x_0 with respect to $\tilde{x}$. As is discussed in the proof of Theorem 1.1, $E_{2p+1} = 0$ and $x_{2p+1} = 0$ for $p = 0, 1, 2, \ldots$. Hence (A.7), (A.8) and (A.9) trivially hold for odd k and the statements seem to be redundant. We prove (A.7), (A.8) and (A.9)) by induction on k, however, because our argument works in the case $Q(\tilde{x})$ contains lower order terms with respect to η, where some of E_k or x_k are not equal to zero for odd k. The higher order terms x_n and E_n ($n \ge 1$) are determined so that the following relation is satisfied:

$$x_0^2 x_0' \frac{dx_n}{d\tilde{x}} + x_0(x_0')^2 x_n = 2(x_0')^2 E_n + 2R_n, \tag{A.12}$$

where

$$R_n = R_{n,1} + R_{n,2} + R_{n,3} + R_{n,4} \tag{A.13}$$

with

$$R_{n,1} = \sum_{\substack{k_1+k_2+l=n \\ k_1,k_2,l<n}} x'_{k_1} x'_{k_2} E_l, \tag{A.14}$$

$$R_{n,2} = -\frac{1}{4} \sum_{\substack{k_1+k_2+l_1+l_2=n \\ k_1,k_2,l_1,l_2<n}} x'_{k_1} x'_{k_2} x_{l_1} x_{l_2}, \tag{A.15}$$

$$R_{n,3} = \frac{1}{2} \sum_{k+l+\mu=n-2} \sum_{\mu_1+\mu_2+\cdots+\mu_l=\mu} \frac{x'''_k x'_{\mu_1+1} x'_{\mu_2+1} \cdots x'_{\mu_l+1}}{(-x'_0)^{l+1}}, \tag{A.16}$$

(A.17)

$$R_{n,4} = \frac{3}{4} \sum_{k_1+k_2+l+\mu=n-2} \sum_{\mu_1+\mu_2+\cdots+\mu_l=\mu} \frac{(l+1) x''_{k_1} x''_{k_2} x'_{\mu_1+1} \cdots x'_{\mu_l+1}}{(-x'_0)^{l+2}}.$$

We take $z = x_0(\tilde{x})$ as a new independent variable. Then (A.12) is rewritten as follows:

$$z^2 \frac{dx_n}{dz} + z x_n = 2E_n + 2\frac{R_n}{(x'_0)^2}. \tag{A.18}$$

To obtain the estimation of x_n and E_n from that of R_n, we use the following lemma:

Lemma A.3. *Let $v(z)$ be a given holomorphic function on $\Delta = \{z;\, |z| < r_0\}$, and consider the following differential equation for $u(z)$:*

$$\left(z^2 \frac{d}{dz} + z \right) u(z) = 2E + 2v(z), \tag{A.19}$$

where E is a constant to be determined. Then there uniquely exist a constant E and a holomorphic function $u(z)$ on Δ that satisfy

(A.19), *and the following inequalities hold for any positive constant* r *which is smaller than* r_0*:*

$$|E| \leq \sup_{|z|\leq r} |v(z)|, \tag{A.20}$$

$$\sup_{|z|\leq r} |u(z)| \leq \frac{4}{r} \sup_{|z|\leq r} |v(z)|, \tag{A.21}$$

$$\sup_{|z|\leq r} \left|\frac{du(z)}{dz}\right| \leq \frac{8}{r^2} \sup_{|z|\leq r} |v(z)|. \tag{A.22}$$

Proof. The unique solution $u(z)$ and E are given as follows:

$$u(z) = \frac{2}{z} \int_0^z \frac{v(z) - v(0)}{z} dz, \tag{A.23}$$

$$E = -v(0). \tag{A.24}$$

Hence (A.20) immediately follows from (A.24).

Let $w(z)$ denote

$$w(z) = \frac{v(z) - v(0)}{z} = \frac{v(z) + E}{z}. \tag{A.25}$$

Then (A.23) entails

$$u(z) = 2 \int_0^1 w(zs) ds. \tag{A.26}$$

Hence, by using the maximum principle for $w(z)$, we obtain

$$\begin{aligned} \sup_{|z|\leq r} |u(z)| &\leq 2 \sup_{|z|\leq r} |w(z)| \\ &= \frac{2}{r} \sup_{|z|\leq r} |v(z) + E| \end{aligned} \tag{A.27}$$

$$\leq \frac{2}{r}\left(\sup_{|z|\leq r}|v(z)|+|E|\right)$$
$$\leq \frac{4}{r}\sup_{|z|\leq r}|v(z)|.$$

Finally, using the differential equation (A.19) together with (A.25) and (A.27), we find

$$\begin{aligned}\text{(A.28)}\qquad \sup_{|z|\leq r}\left|z\frac{du(z)}{dz}\right| &\leq \sup_{|z|\leq r}|u(z)|+2\sup_{|z|\leq r}|w(z)| \\ &\leq 4\sup_{|z|\leq r}|w(z)| \\ &\leq \frac{8}{r}\sup_{|z|\leq r}|v(z)|.\end{aligned}$$

Hence it follows from the Schwarz lemma that

$$\text{(A.29)}\qquad \sup_{|z|\leq r}\left|\frac{du(z)}{dz}\right| \leq \frac{8}{r^2}\sup_{|z|\leq r}|v(z)|.$$

This completes the proof of Lemma A.3. □

We assume that (A.7), (A.8) and (A.9) hold for $k < n$. To obtain the estimation of R_n from the hypothesis of induction, we need the following lemma:

Lemma A.4. *The following inequality holds for all positive integers j and k satisfying $k \leq j$:*

$$\text{(A.30)}\qquad \sum_{\substack{j_1+j_2+\cdots+j_k=j\\ j_1,\ldots,j_k\geq 1}} j_1!j_2!\cdots j_k! \leq 4^{k-1}(j-k+1)!.$$

Proof. If j is smaller than or equal to 3, (A.30) trivially holds. Suppose that j is greater than 3. The case where $k=1$ is trivial. If $k=2$, we find

$$\text{(A.31)} \sum_{\substack{j_1+j_2=j\\ j_1,j_2\geq 1}} j_1!j_2! = (j-1)!\left(1+\frac{2!}{j-1}+\frac{3!}{(j-1)(j-2)}+\cdots+\frac{2!}{j-1}+1\right) \leq 2(j-1)!\left(1+\frac{j-3}{j-1}\right) \leq 4(j-1)!.$$

Hence we have (A.30) for $k=2$. If k is greater than 2, (A.30) can be reduced to the case $k-1$:

(A.32)

$$\begin{aligned}\sum_{\substack{j_1+j_2+\cdots+j_k=j\\ j_1,\ldots,j_k\geq 1}} j_1!j_2!\cdots j_k! &= \sum_{\substack{j'+j_k=j\\ j'\geq k-1, j_k\geq 1}} \sum_{\substack{j_1+\cdots j_{k-1}=j'\\ j_1,\ldots,j_{k-1}\geq 1}} j_1!j_2!\cdots j_{k-1}!j_k! \\ &\leq \sum_{\substack{j'+j_k=j\\ j'\geq k-1, j_k\geq 1}} 4^{k-2}(j'-k+2)!j_k! \\ &= 4^{k-2} \sum_{\substack{j_0+j_k=j-k+2\\ j_0\geq 1, j_k\geq 1}} j_0!j_k! \\ &\leq 4^{k-1}(j-k+1)!.\end{aligned}$$

This completes the proof of Lemma A.4. □

Let C denote the constant satisfying

$$\text{(A.33)} \qquad \sup_{|\tilde{x}|\leq\rho}\left|\frac{d^j x_0(\tilde{x})}{d\tilde{x}^j}\right| \leq C \quad (j=0,1,2,3),$$

$$\text{(A.34)} \qquad \sup_{|\tilde{x}|\leq\rho}\left|\frac{1}{x_0'(\tilde{x})}\right| \leq C$$

and

$$\sup_{|\tilde{x}|\leq\rho}\left|\frac{1}{(x_0'(\tilde{x}))^2}\right|\leq C. \tag{A.35}$$

By the definition of $R_{n,1}$, we have

$$R_{n,1}=\sum_{\substack{k_1+k_2+l=n\\k_1,k_2,l\geq1}}x_{k_1}'x_{k_2}'E_l+2x_0'\sum_{\substack{k+l=n\\k,l\geq1}}x_k'E_l \tag{A.36}$$

and hence

$$\begin{aligned}\sup_{|\tilde{x}|\leq\rho-\varepsilon}|R_{n,1}(\tilde{x})| &\leq \sum_{\substack{k_1+k_2+l=n\\k_1,k_2,l\geq1}}k_1!\,k_2!\,l!\,\varepsilon^{-k_1-k_2-l}A^{k_1+k_2+l}\\ &\qquad +2C\sum_{\substack{k+l=n\\k,l\geq1}}k!\,l!\,\varepsilon^{-k-l}A^{k+l}\\ &\leq 16\varepsilon^{-n}A^n(n-2)!+8C\varepsilon^{-n}A^n(n-1)!\\ &= n!\,\varepsilon^{-n}A^n\cdot\frac{1}{n}\left(\frac{16}{n-1}+8C\right).\end{aligned} \tag{A.37}$$

We set

$$B_{n,1}=\frac{1}{n}\left(\frac{16}{n-1}+8C\right). \tag{A.38}$$

Similarly, we divide the sum in the definition of $R_{n,2}$ as follows:

$$\begin{aligned}R_{n,2}=&-\frac{1}{4}\sum_{\substack{k_1+k_2+l_1+l_2=n\\k_1,k_2,l_1,l_2\geq1}}x_{k_1}'x_{k_2}'x_{l_1}x_{l_2}-\frac{x_0'}{2}\sum_{\substack{k+l_1+l_2=n\\k,l_1,l_2\geq1}}x_k'x_{l_1}x_{l_2}\\ &-\frac{x_0}{2}\sum_{\substack{k_1+k_2+l=n\\k_1,k_2,l\geq1}}x_{k_1}'x_{k_2}'x_l-x_0'x_0\sum_{\substack{k+l=n\\k,l\geq1}}x_k'x_l\\ &-\frac{(x_0')^2}{4}\sum_{\substack{l_1+l_2=n\\l_1,l_2\geq1}}x_{l_1}x_{l_2}-\frac{(x_0)^2}{4}\sum_{\substack{k_1+k_2=n\\k_1,k_2\geq1}}x_{k_1}'x_{k_2}'.\end{aligned} \tag{A.39}$$

Estimating each term by using the hypothesis of induction, we have

(A.40)
$$\sup_{|\tilde{x}|\le\rho-\varepsilon} |R_{n,2}(\tilde{x})|$$
$$\le \Bigg(\frac{1}{4} \sum_{\substack{k_1+k_2+l_1+l_2=n \\ k_1,k_2,l_1,l_2\ge 1}} k_1!\,k_2!\,l_1!\,l_2! + \frac{C}{2} \sum_{\substack{k+l_1+l_2=n \\ k,l_1,l_2\ge 1}} k!\,l_1!\,l_2!$$
$$+ \frac{C}{2} \sum_{\substack{k_1+k_2+l=n \\ k_1,k_2,l\ge 1}} k_1!\,k_2!\,l! + C^2 \sum_{\substack{k+l=n \\ k,l\ge 1}} k!\,l!$$
$$+\frac{C^2}{4} \sum_{\substack{l_1+l_2=n \\ l_1,l_2\ge 1}} l_1!\,l_2! + \frac{C^2}{4} \sum_{\substack{k_1+k_2=n \\ k_1,k_2\ge 1}} k_1!\,k_2! \Bigg) \varepsilon^{-n} A^n.$$

Applying Lemma A.4, we obtain

(A.41)
$$\sup_{|\tilde{x}|\le\rho-\varepsilon} |R_{n,2}(\tilde{x})|$$
$$\le \big(16(n-3)! + 8C(n-2)! + 8C(n-2)!$$
$$+4C^2(n-1)! + C^2(n-1)! + C^2(n-1)!\big)\,\varepsilon^{-n} A^n$$
$$\le n!\,\varepsilon^{-n} A^n \cdot \frac{1}{n} \left(\frac{16}{(n-1)(n-2)} + \frac{16C}{n-1} + 6C^2 \right).$$

Now we set

(A.42)
$$B_{n,2} = \frac{1}{n} \left(\frac{16}{(n-1)(n-2)} + \frac{16C}{n-1} + 6C^2 \right).$$

To have the estimation of $R_{n,3}$ and $R_{n,4}$, we need that of $x_k''(\tilde{x})$ and $x_k'''(\tilde{x})$ for $k < n$. Replacing ε by $k\varepsilon/(k+1)$ in (A.9), we obtain the following estimate for $|\tilde{x}| \le \rho - k\varepsilon/(k+1)$:

(A.43)
$$|x_k'(\tilde{x})| \ \le\ k! \left(\frac{k\varepsilon}{k+1} \right)^{-k} A^k$$

$$\leq \ ek!\,\varepsilon^{-k}A^k.$$

If $|\tilde{x}| \leq \rho - \varepsilon$, we can write

(A.44)
$$x_k''(\tilde{x}) = \frac{1}{2\pi i}\int_{|\zeta-\tilde{x}|=\frac{1}{k+1}\varepsilon} \frac{x_k'(\zeta)}{(\zeta-\tilde{x})^2}d\zeta.$$

Combining this with (A.43), we have

(A.45)
$$\sup_{|\tilde{x}|\leq\rho-\varepsilon} |x_k''(\tilde{x})| \leq e(k+1)!\,\varepsilon^{-k-1}A^k.$$

Similar argument shows

(A.46)
$$\sup_{|\tilde{x}|\leq\rho-\varepsilon} |x_k'''(\tilde{x})| \leq e^2(k+2)!\,\varepsilon^{-k-2}A^k.$$

We divide the sum appearing in the definition of $R_{n,3}$ as follows:

(A.47)
$$\begin{aligned} R_{n,3} =& \frac{1}{2}\sum_{\substack{k+l+\mu=n-2\\ k\geq 1}}\ \sum_{\mu_1+\mu_2+\cdots+\mu_l=\mu} \frac{x_k'''x_{\mu_1+1}'x_{\mu_2+1}'\cdots x_{\mu_l+1}'}{(-x_0')^{l+1}} \\ &+ \frac{x_0'''}{2}\sum_{\substack{l+\mu=n-2\\ l\geq 1}}\ \sum_{\mu_1+\mu_2+\cdots+\mu_l=\mu} \frac{x_{\mu_1+1}'x_{\mu_2+1}'\cdots x_{\mu_l+1}'}{(-x_0')^{l+1}}. \end{aligned}$$

The hypothesis of induction and (A.46) yield

(A.48)
$$\begin{aligned} &\sup_{|\tilde{x}|\leq\rho-\varepsilon} |R_{n,3}(\tilde{x})| \\ &\quad\leq \frac{e^2}{2}\sum_{\substack{k+l+\mu=n-2\\ k\geq 1}}\ \sum_{\mu_1+\mu_2+\cdots+\mu_l=\mu} (k+2)!\,\varepsilon^{-k-2}A^kC^{l+1} \\ &\qquad\times(\mu_1+1)!\cdots(\mu_l+1)!\,\varepsilon^{-\mu-l}A^{\mu+l} \\ &\quad+\frac{C}{2}\sum_{\substack{l+\mu=n-2\\ l\geq 1}}\ \sum_{\mu_1+\mu_2+\cdots+\mu_l=\mu} C^{l+1}(\mu_1+1)!\cdots(\mu_l+1)!\varepsilon^{-\mu-l}A^{\mu+l}. \end{aligned}$$

By Lemma A.4, the right-hand side is dominated by

$$\text{(A.49)}\qquad \frac{e^2C}{2}\, n!\, \varepsilon^{-n} A^{n-2} \sum_{l=0}^{n-3} \frac{(n-l)!}{n!}(4C)^l + \frac{C^3}{2}\,(n-2)!\,\varepsilon^{-n+2} A^{n-2} \sum_{l=0}^{n-3} \frac{(n-2-l)!}{(n-2)!}(4C)^l.$$

Since

$$\text{(A.50)}\qquad \sum_{l=0}^{m} \frac{(n-l)!}{n!} s^l \le e^s \quad \text{and} \quad \sum_{l=0}^{m} \frac{(n-l-1)!\,(l+1)}{n!} s^l \le e^s$$

hold for any positive s and $m \le n-1$, (A.49) is not greater than

$$\text{(A.51)}\qquad \left(\frac{e^2C}{2}\, n!\, \varepsilon^{-n} A^{n-2} + \frac{C^3}{2}\,(n-2)!\,\varepsilon^{-n+2} A^{n-2} \right) e^{4C}.$$

Hence we have

$$\text{(A.52)}\qquad \sup_{|\tilde{x}| \le \rho - \varepsilon} |R_{n,3}(\tilde{x})| \le n!\, \varepsilon^{-n} A^n\, B_{n,3},$$

where we set

$$\text{(A.53)}\qquad B_{n,3} = \frac{C}{2A^2} \left(e^2 + \frac{C^2 \varepsilon^2}{n(n-1)} \right) e^{4C}.$$

Finally, we rewrite the definition of $R_{n,4}$ as follows:

(A.54)

$$R_{n,4} = \frac{3}{4} \sum_{\substack{k_1+k_2+l+\mu=n-2 \\ k_1,k_2,l\ge 1}} \frac{(l+1) x''_{k_1} x''_{k_2}}{(-x'_0)^{l+2}} \sum_{\mu_1+\cdots+\mu_l=\mu} x'_{\mu_1+1} \cdots x'_{\mu_l+1} + \frac{3}{4} \sum_{\substack{k_1+k_2=n-2 \\ k_1,k_2\ge 1}} \frac{x''_{k_1} x''_{k_2}}{(x'_0)^2}$$

$$+\frac{3}{2}x_0'' \sum_{\substack{k+l+\mu=n-2\\ k,l\geq 1}} \frac{(l+1)x_k''}{(-x_0')^{l+2}} \sum_{\mu_1+\cdots+\mu_l=\mu} x'_{\mu_1+1}\cdots x'_{\mu_l+1}$$

$$+\frac{3x_0''x_{n-2}''}{2(x_0')^2}+\frac{3(x_0'')^2}{4}\sum_{\substack{l+\mu=n-2\\ l\geq 1}} \frac{(l+1)}{(-x_0')^{l+2}} \sum_{\mu_1+\cdots+\mu_l=\mu} x'_{\mu_1+1}\cdots x'_{\mu_l+1}.$$

It follows from the hypothesis of induction and (A.45) that the following inequality holds:

(A.55)

$$\sup_{|\tilde{x}|\leq\rho-\varepsilon} |R_{n,4}(\tilde{x})| \leq \frac{3e^2C^2}{4}\varepsilon^{-n}A^{n-2}\sum_{l=1}^{n-4}(l+1)C^l$$

$$\times \sum_{k_1+k_2+\mu_1+\cdots+\mu_l=n-l-2} (k_1+1)!(k_2+1)!(\mu_1+1)!\cdots(\mu_l+1)!$$

$$+\frac{3e^2C^2}{4}\varepsilon^{-n}A^{n-2}\sum_{\substack{k_1+k_2=n-2\\ k_1,k_2\geq 1}}(k_1+1)!(k_2+1)!$$

$$+\frac{3eC^3}{2}\varepsilon^{-n+1}A^{n-2}\sum_{l=1}^{n-3}(l+1)C^l$$

$$\times \sum_{k+\mu_1+\cdots+\mu_l=n-l-2}(k+1)!(\mu_1+1)!\cdots(\mu_l+1)!$$

$$+\frac{3eC^3}{2}(n-1)!\,\varepsilon^{-n+1}A^{n-2}$$

$$+\frac{3C^4}{4}\varepsilon^{-n+2}A^{n-2}\sum_{l=1}^{n-2}(l+1)C^l \sum_{\mu_1+\cdots+\mu_l=n-l-2}(\mu_1+1)!\cdots(\mu_l+1)!.$$

By Lemma A.4, the right-hand side is dominated by

(A.56) $$3e^2C^2\varepsilon^{-n}A^{n-2}\sum_{l=1}^{n-4}(l+1)(4C)^l(n-l-1)!$$

$$
\begin{aligned}
&+ 3e^2C^2(n-1)!\,\varepsilon^{-n}A^{n-2} \\
&+ \frac{3eC^3}{2}\varepsilon^{-n+1}A^{n-2}\sum_{l=1}^{n-3}(l+1)(4C)^l(n-l-1)! \\
&+ \frac{3eC^3}{2}(n-1)!\,\varepsilon^{-n+1}A^{n-2} \\
&+ \frac{3C^4}{16}\varepsilon^{-n+2}A^{n-2}\sum_{l=1}^{n-2}(l+1)(4C)^l(n-l-1)!.
\end{aligned}
$$

Using (A.50), we see that (A.56) is smaller than

$$
\begin{aligned}
\text{(A.57)}\qquad & 3e^2C^2n!\,\varepsilon^{-n}A^{n-2}e^{4C} + 3e^2C^2(n-1)!\,\varepsilon^{-n}A^{n-2} \\
&+ \frac{3eC^3}{2}n!\,\varepsilon^{-n+1}A^{n-2}e^{4C} + \frac{3eC^3}{2}(n-1)!\,\varepsilon^{-n+1}A^{n-2} \\
&+ \frac{3C^4}{16}n!\,\varepsilon^{-n+2}A^{n-2}e^{4C}.
\end{aligned}
$$

Thus we have the following estimation for $R_{n,4}$:

$$
\text{(A.58)}\qquad \sup_{|\tilde{x}|\le\rho-\varepsilon}|R_{n,4}(\tilde{x})| \le n!\,\varepsilon^{-n}A^n\,B_{n,4},
$$

where we set

$$
\text{(A.59)}\qquad B_{n,4} = \frac{3C^2}{A^2}\left(\left(e^2 + \frac{eC\varepsilon}{2} + \frac{C^2\varepsilon^2}{16}\right)e^{4C} + \frac{e}{2n}(2e + C\varepsilon)\right).
$$

Therefore we have the following estimation for R_n:

$$
\text{(A.60)}\qquad \sup_{|\tilde{x}|\le\rho-\varepsilon}|R_n(\tilde{x})| \le n!\,\varepsilon^{-n}A^n\,B_n,
$$

with

$$
\text{(A.61)}\qquad B_n = B_{n,1} + B_{n,2} + B_{n,3} + B_{n,4}.
$$

We apply Lemma A.3 for

$$
\text{(A.62)}\qquad v = \frac{R_n}{(x_0')^2}.
$$

Then we have the estimation of E_n:

$$|E_n| \le n!\,\varepsilon^{-n} A^n\, C^2 B_n. \tag{A.63}$$

We go back to the original coordinate $\tilde{x}$. Taking supremum in $\tilde{x}$ instead of $z = x_0(\tilde{x})$, we have the estimation for x_n and x_n':

$$\sup_{|\tilde{x}|\le\rho-\varepsilon} |x_n(\tilde{x})| \le n!\,\varepsilon^{-n} A^n\, \frac{4C^2 B_n}{\rho-\varepsilon}, \tag{A.64}$$

$$\sup_{|\tilde{x}|\le\rho-\varepsilon} |x_n'(\tilde{x})| \le \frac{2}{\rho-\varepsilon} n!\,\varepsilon^{-n} A^n\, \frac{4C^3 B_n}{\rho-\varepsilon}. \tag{A.65}$$

Since we may assume $0 < \varepsilon < \rho/3$, we obtain

$$\sup_{|\tilde{x}|\le\rho-\varepsilon} |x_n(\tilde{x})| \le n!\,\varepsilon^{-n} A^n\, \frac{6C^2 B_n}{\rho}, \tag{A.66}$$

$$\sup_{|\tilde{x}|\le\rho-\varepsilon} |x_n'(\tilde{x})| \le n!\,\varepsilon^{-n} A^n\, \frac{18\, C^3 B_n}{\rho^2}. \tag{A.67}$$

We note that B_n are bounded when $\varepsilon > 0$ tends to zero and

$$\lim_{n\to\infty} B_{n,1} = 0, \quad \lim_{n\to\infty} B_{n,2} = 0. \tag{A.68}$$

Hence there exists $n_0 \in \mathbb{N}$ which depends only on x_0 and ρ so that

$$B_{n,1} + B_{n,2} \le \min\left\{\frac{1}{2C^2}, \frac{\rho}{12C^2}, \frac{\rho^2}{36C^3}\right\} \tag{A.69}$$

holds for $n \ge n_0$. On the other hand, we can choose $A > 0$ from the beginning for which

$$B_{n,3} + B_{n,4} \le \min\left\{\frac{1}{2C^2}, \frac{\rho}{12C^2}, \frac{\rho^2}{36C^3}\right\} \tag{A.70}$$

holds. Then we have

$$\max \left\{ C^2 B_n, \frac{6C^2 B_n}{\rho}, \frac{18C^3 B_n}{\rho^2} \right\} \leq 1 \tag{A.71}$$

and induction proceeds when $n \geq n_0$.

The above argument shows that there exist positive numbers K_n ($n = 1, 2, \ldots, n_0 - 1$) independent of ε so that

$$|E_n| \leq \varepsilon^{-n} K_n, \tag{A.72}$$

$$\sup_{|\tilde{x}| \leq \rho - \varepsilon} |x_n(\tilde{x})| \leq \varepsilon^{-n} K_n, \tag{A.73}$$

and

$$\sup_{|\tilde{x}| \leq \rho - \varepsilon} |x_n'(\tilde{x})| \leq \varepsilon^{-n} K_n \tag{A.74}$$

hold for $n = 1, 2, \ldots, n_0 - 1$. Now we set

$$A' = \max_{1 \leq n \leq n_0 - 1} \left\{ \left(\frac{K_n}{n!} \right)^{\frac{1}{n}} \right\} \tag{A.75}$$

and take A' as A if A' is greater than A. This completes the proof of Proposition A.2. □

Appendix B. Estimation of the transformation that brings an MTP equation to an ∞-Weber equation

In Appendix B we discuss estimation of growth order of the transformation

$$x(\tilde{x}, t, \eta) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} x_k^{(j)}(\tilde{x}) t^j \right) \eta^{-k} \tag{B.1}$$

constructed in Section 2, that is, the transformation that brings an MTP equation

$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x},t)\right)\tilde{\psi} = 0, \quad Q(\tilde{x},t) = \sum_{j=0}^{\infty} Q^{(j)}(\tilde{x})t^j \tag{B.2}$$

satisfying (2.2) $\sim$ (2.5) to an ∞-Weber equation

$$\left(\frac{d^2}{dx^2} - \eta^2\left(E(t,\eta) - \frac{1}{4}x^2\right)\right)\psi = 0 \tag{B.3}$$

with

$$E(t,\eta) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} E_k^{(j)} t^j\right)\eta^{-k}. \tag{B.4}$$

Estimation of growth order of $E(t,\eta)$ will be simultaneously given also.

We first verify holomorphy (in t) of the top order part with respect to η. That is, we prove the following

Proposition B.1. *Let*

$$x_0(\tilde{x},t) = \sum_{j=0}^{\infty} x_0^{(j)}(\tilde{x})t^j \quad \textit{and} \quad E_0(t) = \sum_{j=0}^{\infty} E_0^{(j)} t^j \tag{B.5}$$

be the top order part (*with respect to* η^{-1}) *of the transformation and the coefficient of the* ∞*-Weber equation constructed in Section 2, respectively. Then there exists a small positive constant* δ *so that both* $x_0(\tilde{x},t)$ *and* $E_0(t)$ *are convergent for* $|\tilde{x}| \le \delta$ *and* $|t| \le \delta$.

Proof. We will prove the convergence of $x_0(\tilde{x},t)$ and $E_0(t)$ by using the method of majorant series.

Before discussing the convergence, we now recall the construction of $x_0^{(j)}(\tilde{x})$ and $E_0^{(j)}$ explained in Section 2. (For the sake of simplicity

$x_0^{(j)}(\tilde{x})$ and $E_0^{(j)}$ are denoted by $x^{(j)}(\tilde{x})$ and $E^{(j)}$, respectively, and the differentiation with respect to $\tilde{x}$ are often abbreviated by $'$ in this proof hereafter.) Comparing the coefficients of like powers of η^{-1} in (2.21), i.e.,

(B.6)

$$Q(\tilde{x},t) = \left(\frac{\partial x(\tilde{x},t,\eta)}{\partial \tilde{x}}\right)^2 (E(t,\eta) - \frac{x(\tilde{x},t,\eta)^2}{4}) - \frac{\eta^{-2}}{2}\{x(\tilde{x},t,\eta);\tilde{x}\},$$

we find that $x_0(\tilde{x},t)$ and $E_0(t)$ are determined in such a way that

$$Q(\tilde{x},t) = \left(\frac{\partial x_0}{\partial \tilde{x}}\right)^2 (E_0(t) - \frac{1}{4}(x_0)^2) \tag{B.7}$$

is satisfied. We now set $E^{(0)} = 0$ and define $x^{(0)}(\tilde{x})$ by the relation

$$Q^{(0)}(\tilde{x}) = -\frac{1}{4}(x^{(0)\prime})^2(x^{(0)})^2, \tag{B.8}$$

that is,

$$x^{(0)}(\tilde{x}) = 2\left(\int_0^{\tilde{x}} \sqrt{-Q^{(0)}(\tilde{x})}d\tilde{x}\right)^{1/2}. \tag{B.9}$$

Hence, thanks to the assumption (2.3), $x^{(0)}(\tilde{x})$ is holomorphic in $\{\tilde{x}; |\tilde{x}| \le \delta\}$ for some small positive constant δ and $x^{(0)\prime}(0) \neq 0$ holds. Next the higher order terms $x^{(j)}(\tilde{x})$ and $E^{(j)}$ $(j \ge 1)$ are determined so that the following relation is satisfied:

(B.10)

$$Q^{(j)}(\tilde{x}) = -\frac{1}{2}(x^{(0)})^2(x^{(0)})'(x^{(j)})' + (x^{(0)\prime})^2(E^{(j)} - \frac{1}{2}x^{(0)}x^{(j)}) + R^{(j)},$$

where $R^{(1)} = 0$ and

(B.11)

$$R^{(j)} = \sum_{\substack{k_1+k_2+l=j \\ k_1,k_2,l\le j-1}} (x^{(k_1)})'(x^{(k_2)})'E^{(l)} - \frac{1}{4}\sum_{\substack{k_1+k_2+l_1+l_2=j \\ k_1,k_2,l_1,l_2\le j-1}} (x^{(k_1)})'(x^{(k_2)})'x^{(l_1)}x^{(l_2)}$$

for $j \geq 2$. If we take $z = x^{(0)}(\tilde{x})$ as a new independent variable in $\{\tilde{x}; |\tilde{x}| \leq \delta\}$ for a sufficiently small constant δ, then (B.10) can be equivalently written as follows:

$$z^2 \frac{dx^{(j)}}{dz} + zx^{(j)} = 2E^{(j)} + 2(x^{(0)\prime})^{-2}\widetilde{R^{(j)}}, \tag{B.12}$$

where $\widetilde{R^{(j)}} = R^{(j)} - Q^{(j)}$. The differential equation (B.12) uniquely determines $x^{(j)}$ and $E^{(j)}$, as was already discussed in Section 2. Furthermore, the estimation of them can be obtained through Lemma A.3. In fact, putting

$$C = \sup_{|\tilde{x}| \leq \delta} \left| \frac{1}{(x^{(0)\prime}(\tilde{x}))^2} \right|, \tag{B.13}$$

we obtain the following estimates for $E^{(j)}$ and $x^{(j)}$ through Lemma A.3.

$$|E^{(j)}| \leq C \sup_{|z| \leq r} |\widetilde{R^{(j)}}|, \tag{B.14}$$

$$\sup_{|z| \leq r} |x^{(j)}| \leq \frac{4C}{r} \sup_{|z| \leq r} |\widetilde{R^{(j)}}|, \tag{B.15}$$

$$\sup_{|z| \leq r} \left| \frac{dx^{(j)}}{dz} \right| \leq \frac{8C}{r^2} \sup_{|z| \leq r} |\widetilde{R^{(j)}}|. \tag{B.16}$$

Based on these estimates (B.14) $\sim$ (B.16), the construction of a majorant series $A(t) = \sum_j A^{(j)} t^j$ of $x_0(t, \tilde{x}) = \sum_j x^{(j)}(\tilde{x}) t^j$ and $E_0(t) = \sum_j E^{(j)} t^j$ is done in the following manner.

We first take sufficiently large $A^{(0)}$ and $A^{(1)}$ so that they satisfy

$$|E^{(0)}| \leq \frac{r}{4} A^{(0)}, \quad \sup_{|z| \leq r} |x^{(0)}| \leq A^{(0)}, \quad \sup_{|z| \leq r} \left| \frac{dx^{(0)}}{dz} \right| \leq \frac{2}{r} A^{(0)}, \tag{B.17}$$

$$|E^{(1)}| \leq \frac{r}{4} A^{(1)}, \quad \sup_{|z| \leq r} |x^{(1)}| \leq A^{(1)}, \quad \sup_{|z| \leq r} \left| \frac{dx^{(1)}}{dz} \right| \leq \frac{2}{r} A^{(1)}. \tag{B.18}$$

Next we define $A^{(j)}$ for $j \geq 2$ by the following recursive relation:

(B.19)

$$A^{(j)} = C\left(\frac{2C}{r}\right)^2 \sum_{\substack{k_1+k_2+l=j \\ k_1,k_2,l\leq j-1}} A^{(k_1)}A^{(k_2)}A^{(l)} + \frac{1}{2}\left(\frac{2C}{r}\right)^3 \sum_{\substack{k_1+k_2+l_1+l_2=j \\ k_1,k_2,l_1,l_2\leq j-1}} A^{(k_1)}A^{(k_2)}A^{(l_1)}A^{(l_2)} + \frac{4CC_1}{r}r_1^{-j},$$

where C_1 and r_1 are positive constants satisfying

$$\sup_{|z|\leq r} |Q^{(j)}| \leq C_1 r_1^{-j}. \tag{B.20}$$

Then we can verify that $A(t) = \sum_j A^{(j)}t^j$ is a majorant series of $x_0(t,\tilde{x}) = \sum_j x^{(j)}(\tilde{x})t^j$ and $E_0(t) = \sum_j E^{(j)}t^j$ in the sense that the following inequalities hold for any $j \geq 0$:

$$|E^{(j)}| \leq \frac{r}{4}A^{(j)}, \quad \sup_{|z|\leq r} |x^{(j)}| \leq A^{(j)}, \quad \sup_{|z|\leq r} \left|\frac{dx^{(j)}}{dz}\right| \leq \frac{2}{r}A^{(j)}. \tag{B.21}$$

To prove this, let us assume that (B.21) holds up to $j-1$. Replacing the definition (B.13) of the constant C by

$$C = \max\left\{\sup_{|\tilde{x}|\leq\delta} \left|\frac{1}{(x^{(0)\prime}(\tilde{x}))^2}\right|, \sup_{|\tilde{x}|\leq\delta} \left|x^{(0)\prime}(\tilde{x})\right|\right\}, \tag{B.22}$$

we find that

$$|(x^{(k)})'| = \left|(x^{(0)})'\frac{dx^{(k)}}{dz}\right| \leq C\left|\frac{dx^{(k)}}{dz}\right| \leq \frac{2C}{r}A^{(k)} \tag{B.23}$$

holds for $k \leq j-1$. In view of (B.11), induction hypotheses together with (B.23) entail

$$\sup_{|z|\leq r} |R^{(j)}| \leq \sum_{\substack{k_1+k_2+l=j \\ k_1,k_2,l\leq j-1}} |(x^{(k_1)})'||(x^{(k_2)})'||E^{(l)}| \tag{B.24}$$

$$+\frac{1}{4}\sum_{\substack{k_1+k_2+l_1+l_2=j\\ k_1,k_2,l_1,l_2\le j-1}} |(x^{(k_1)})'||(x^{(k_2)})'||x^{(l_1)}||x^{(l_2)}|$$

$$\le \frac{C^2}{r}\sum_{\substack{k_1+k_2+l=j\\ k_1,k_2,l\le j-1}} A^{(k_1)}A^{(k_2)}A^{(l)}$$

$$+\frac{1}{4}\left(\frac{2C}{r}\right)^2\sum_{\substack{k_1+k_2+l_1+l_2=j\\ k_1,k_2,l_1,l_2\le j-1}} A^{(k_1)}A^{(k_2)}A^{(l_1)}A^{(l_2)}.$$

Hence we have

(B.25)

$$\sup_{|z|\le r}|\widetilde{R^{(j)}}| \le \frac{C^2}{r}\sum_{\substack{k_1+k_2+l=j\\ k_1,k_2,l\le j-1}} A^{(k_1)}A^{(k_2)}A^{(l)}$$

$$+\frac{1}{4}\left(\frac{2C}{r}\right)^2\sum_{\substack{k_1+k_2+l_1+l_2=j\\ k_1,k_2,l_1,l_2\le j-1}} A^{(k_1)}A^{(k_2)}A^{(l_1)}A^{(l_2)} + C_1 r_1^{-j}$$

$$= \frac{r}{4C}A^{(j)}.$$

Combining (B.25) with (B.14) $\sim$ (B.16), we thus obtain (B.21) for j. That is, (B.21) holds for any $j \ge 0$ by the induction.

To verify the holomorphy of $x_0(\tilde{x}, t)$ and $E_0(t)$, it now suffices to prove that $A(t) = \sum_j A^{(j)} t^j$ is convergent for sufficiently small t. The recursive relation (B.19) implies that $A(t)$ satisfies the following algebraic equation:

(B.26)

$$A = C\left(\frac{2C}{r}\right)^2 (A^3 - 3(A^{(0)})^2 A) + \frac{1}{2}\left(\frac{2C}{r}\right)^3 (A^4 - 4(A^{(0)})^3 A)$$

$$+\frac{4CC_1}{r}\left[\left(1-\frac{t}{r_1}\right)^{-1}-\left(1+\frac{t}{r_1}\right)\right]$$
$$+A^{(0)}+\frac{1}{2}\left(\frac{2C}{r}\right)^3(2r(A^{(0)})^3+3(A^{(0)})^4)+A^{(1)}t.$$

(Here the last few terms of (B.26) are added so that the degree 0 and 1 parts (with respect to t) of (B.26) are automatically satisfied.) Note that $A(t)=\sum_j A^{(j)}t^j$ is a unique formal power series solution of (B.26) with the degree 0 part $A^{(0)}$. On the other hand, if we introduce $\Phi=\Phi(t,A)$ by

(B.27)
$$\begin{aligned}\Phi(t,A)=&\left[1+\frac{1}{2}\left(\frac{2C}{r}\right)^3(3r(A^{(0)})^2+4(A^{(0)})^3)\right]A\\&-C\left(\frac{2C}{r}\right)^2A^3-\frac{1}{2}\left(\frac{2C}{r}\right)^3A^4\\&-\frac{4CC_1}{r}\left[\left(1-\frac{t}{r_1}\right)^{-1}-\left(1+\frac{t}{r_1}\right)\right]\\&-A^{(0)}-\frac{1}{2}\left(\frac{2C}{r}\right)^3(2r(A^{(0)})^3+3(A^{(0)})^4)-A^{(1)}t,\end{aligned}$$

(B.26) can be equivalently written as $\Phi(t,A)=0$. Since

$$\text{(B.28)}\qquad \Phi(0,A^{(0)})=0 \quad\text{and}\quad \frac{\partial\Phi}{\partial A}(0,A^{(0)})=1\neq 0$$

are readily verified, it follows from the implicit function theorem that $\Phi(t,A)=0$ has a local holomorphic solution satisfying $A(0)=A^{(0)}$. Hence $A(t)=\sum_j A^{(j)}t^j$ is convergent for sufficiently small t. This completes the proof of Proposition B.1. □

Remark B.1. It is interesting to compare the above proof of Proposition B.1 with the core part of the proof of Lemma 3.4 of [AKT1], namely the existence of $(q_0(\tilde{q}), E_0)$, which corresponds to $(x_0(\tilde{x}, t_0), E_0(t_0))$ (where t_0 is a fixed non-zero constant) in our current context.

First of all, in [AKT1] we made essential use of the assumption that two simple turning points p_0 and p_1 are connected by a Stokes curve γ, whereas we have not made this assumption in this paper. Actually the argument in [AKT1] was a subtle and geometric one; we first constructed a bi-holomorphic function $z(\tilde{q})$ on a neighborhood of the interior of γ, and then confirmed that

$$q_0(\tilde{q}) = -2\sqrt{E_0}\cos(z(\tilde{q})/2) \tag{B.29}$$

is well-defined and holomorphic near p_0 and p_1. Compared with this delicate argument, the reasoning in this paper seems to be more straightforward. The reason why we do not need such a geometric condition in this paper is that an MTP operator has a double turning point at $t = 0$. This fact does not imply that $s_+(t)$ and $s_-(t)$ in (2.4) are connected by a Stokes curve, but it still excludes the "bad" case mentioned before Theorem 3.3 in [AKT1], namely the situation where some Stokes curve runs across the interval between $s_+(t)$ and $s_-(t)$; if such a Stokes curve persists as t tends to 0, $s_+(t)$ and $s_-(t)$ cannot smoothly coalesce into a double turning point.

Another interesting difference between the reasoning in [AKT1] and that in this paper is that we do not require

$$x_0(s_\pm(t), t) = \pm 2\sqrt{E_0(t)} \tag{B.30}$$

in our reasoning but that it is a consequence of Proposition B.1, i.e., the convergence of the series $\sum_j x_0^{(j)}(\tilde{x})t^j$.

Summing up, we observe that all the subtleties in the reasoning of [AKT1] are automatically built in our double series (i.e., a series in t and η^{-1}) construction scheme, particularly in the convergence proof.

Thus we have verified the holomorphy of the top order part $x_0(\tilde{x}, t)$ and $E_0(t)$ of the transformation (B.1) and the coefficient (B.4) of the ∞-Weber equation constructed in Section 2. We next consider their higher order parts

$$\text{(B.31)}\quad x_n(\tilde{x}, t) = \sum_{j=0}^{\infty} x_n^{(j)}(\tilde{x}) t^j \quad \text{and} \quad E_n(t) = \sum_{j=0}^{\infty} E_n^{(j)} t^j \quad (n \geq 1).$$

Comparison of the coefficients of like powers of η^{-1} in (B.6) yields that $x_n = x_n(\tilde{x}, t)$ and $E_n = E_n(t)$ $(n \geq 1)$ are determined by the following relation:

$$\text{(B.32)}\quad 2\frac{\partial x_0}{\partial \tilde{x}}\frac{\partial x_n}{\partial \tilde{x}}(E_0 - \frac{1}{4}(x_0)^2) + \left(\frac{\partial x_0}{\partial \tilde{x}}\right)^2 (E_n - \frac{1}{2}x_0 x_n) + R_n = 0,$$

where

$$\text{(B.33)}\qquad R_n = R_{n,1} + R_{n,2} + R_{n,3} + R_{n,4},$$

with

(B.34)

$$R_{n,1} = \sum_{\substack{k_1+k_2+l=n \\ k_1,k_2,l \leq n-1}} \frac{\partial x_{k_1}}{\partial \tilde{x}}\frac{\partial x_{k_2}}{\partial \tilde{x}} E_l,$$

(B.35)

$$R_{n,2} = -\frac{1}{4} \sum_{\substack{k_1+k_2+l_1+l_2=n \\ k_1,k_2,l_1,l_2 \leq n-1}} \frac{\partial x_{k_1}}{\partial \tilde{x}}\frac{\partial x_{k_2}}{\partial \tilde{x}} x_{l_1} x_{l_2},$$

(B.36)

$$R_{n,3} = \frac{1}{2} \sum_{k+l+\mu=n-2} \sum_{\mu_1+\cdots+\mu_l=\mu} \frac{\partial^3 x_k}{\partial \tilde{x}^3}\frac{\partial x_{\mu_1+1}}{\partial \tilde{x}} \cdots \frac{\partial x_{\mu_l+1}}{\partial \tilde{x}} \left(-\frac{\partial x_0}{\partial \tilde{x}}\right)^{-(l+1)},$$

(B.37)
$$R_{n,4} = \frac{3}{4} \sum_{k_1+k_2+l+\mu=n-2} \sum_{\mu_1+\cdots+\mu_l=\mu} (l+1) \frac{\partial^2 x_{k_1}}{\partial \tilde{x}^2} \frac{\partial^2 x_{k_2}}{\partial \tilde{x}^2} \times \\ \times \frac{\partial x_{\mu_1+1}}{\partial \tilde{x}} \cdots \frac{\partial x_{\mu_l+1}}{\partial \tilde{x}} \left(-\frac{\partial x_0}{\partial \tilde{x}} \right)^{-(l+2)}.$$

Since $x_0(\tilde{x}, t)$ is holomorphic at $(\tilde{x}, t) = (0, 0)$ and $(\partial x_0/\partial \tilde{x})(0, 0) \neq 0$ holds, $x_0(\tilde{x}, t)$ can be taken as a new independent variable near $\tilde{x} = 0$. Furthermore, as $E_0(t)$ is also holomorphic and $(dE_0(t)/dt)(0) \neq 0$ holds thanks to (2.32), we can take $E_0(t)$ as a new parameter instead of t. Thus, in what follows we use $(z, s) = (x_0(\tilde{x}, t), E_0(t))$ as a new coordinate in a neighborhood of $(\tilde{x}, t) = (0, 0)$. Relation (B.32) then becomes of the following form:

(B.38)
$$(4s - z^2) \frac{\partial x_n}{\partial z} - z x_n = -2E_n - 2 \left(\frac{\partial x_0}{\partial \tilde{x}} \right)^{-2} R_n.$$

As a higher order version of Proposition B.1 we can prove the following

Proposition B.2. *Let* $n \geq 1$ *and*

(B.39)
$$x_n(z, s) = \sum_{j=0}^{\infty} x_n^{(j)}(z) s^j \quad \text{and} \quad E_n(s) = \sum_{j=0}^{\infty} E_n^{(j)} s^j$$

be the n*-th order part* (*with respect to* η^{-1}) *of the transformation and the coefficient of the* ∞*-Weber equation determined by* (B.38). *Then both* $x_n(z, s)$ *and* $E_n(s)$ *are holomorphic at* $(z, s) = (0, 0)$. *Consequently they are holomorphic also in the original variables* $(\tilde{x}, t)$.

Proof. We now prove that x_n and E_n are holomorphic under the assumption that R_n in the right-hand side of (B.38) is holomorphic.

Then, as R_n depends only on x_j and E_j for $0 \le j \le n-1$, Proposition B.2 immediately follows by an inductive argument. Similarly to the proof of Proposition B.1, we will prove the holomorphy of x_n and E_n by using the method of majorant series.

Let

$$\left(\frac{\partial x_0}{\partial \tilde{x}}\right)^{-2} R_n = \sum_{j=0}^{\infty} \tilde{R}_n^{(j)}(z) s^j \tag{B.40}$$

be a power series expansion of $(\partial x_0/\partial \tilde{x})^{-2} R_n$ around $s = 0$. If we also expand x_n and E_n as in (B.39), the comparison of the coefficients of like powers of s in (B.38) yields the following relations:

$$\left(z^2 \frac{d}{dz} + z\right) x_n^{(0)} = 2E_n^{(0)} + 2\tilde{R}_n^{(0)}, \tag{B.41}$$

$$\left(z^2 \frac{d}{dz} + z\right) x_n^{(j)} = 2E_n^{(j)} + 4\frac{dx_n^{(j-1)}}{dz} + 2\tilde{R}_n^{(j)} \quad (j \ge 1). \tag{B.42}$$

In view of Lemma A.3 we find that these relations uniquely determine $x_n^{(j)}$ and $E_n^{(j)}$ in a recursive manner. Furthermore, Lemma A.3 also entails the following inequalities for every $j \ge 0$ and any sufficiently small positive number r:

$$|E_n^{(j)}| \le \sup_{|z| \le r} |F^{(j)}|, \tag{B.43}$$

$$\sup_{|z| \le r} |x_n^{(j)}| \le \frac{4}{r} \sup_{|z| \le r} |F^{(j)}|, \tag{B.44}$$

$$\sup_{|z| \le r} \left|\frac{dx_n^{(j)}}{dz}\right| \le \frac{8}{r^2} \sup_{|z| \le r} |F^{(j)}|, \tag{B.45}$$

where $F^{(0)} = \tilde{R}_n^{(0)}$ and $F^{(j)} = 2(dx_n^{(j-1)}/dz) + \tilde{R}_n^{(j)}$ for $j \ge 1$.

Let us define $B^{(j)}$ $(j \geq 0)$ by

$$B^{(0)} = \frac{4}{r} \sup_{|z| \leq r} |\tilde{R}_n^{(0)}(z)|, \tag{B.46}$$

$$B^{(j)} = \frac{4}{r} \left(\frac{4}{r} B^{(j-1)} + \sup_{|z| \leq r} |\tilde{R}_n^{(j)}(z)| \right) \quad (j \geq 1). \tag{B.47}$$

We now verify that $B(s) = \sum_j B^{(j)} s^j$ is a majorant series of $x_n(z,s) = \sum_j x_n^{(j)}(z) s^j$ and $E_n(s) = \sum_j E_n^{(j)} s^j$ in the sense that the following inequalities hold for any $j \geq 0$:

$$|E_n^{(j)}| \leq \frac{r}{4} B^{(j)}, \quad \sup_{|z| \leq r} |x_n^{(j)}| \leq B^{(j)}, \quad \sup_{|z| \leq r} \left| \frac{dx_n^{(j)}}{dz} \right| \leq \frac{2}{r} B^{(j)}. \tag{B.48}$$

To prove (B.48), we first note that (B.48) for $j = 0$ immediately follows from (B.43) $\sim$ (B.45) for $j = 0$ and the definition (B.46) of $B^{(0)}$. Next let us assume that (B.48) holds up to $j - 1$. Then we have

$$\begin{aligned} |F^{(j)}| &\leq 2 \left| \frac{dx_n^{(j-1)}}{dz} \right| + \left| \tilde{R}_n^{(j)} \right| \\ &\leq \frac{4}{r} B^{(j-1)} + \sup_{|z| \leq r} |\tilde{R}_n^{(j)}(z)| \\ &= \frac{r}{4} B^{(j)} \end{aligned} \tag{B.49}$$

for $|z| \leq r$. Hence, combining (B.49) with (B.43) $\sim$ (B.45), we obtain (B.48) for j. Thus (B.48) holds for any $j \geq 0$ by the induction.

The convergence of the majorant series $B(s) = \sum_j B^{(j)} s^j$ can be confirmed in the following way: Let $|\tilde{R}_n|(s)$ denote a power series of s defined by

$$|\tilde{R}_n|(s) = \sum_{j=0}^{\infty} \left(\sup_{|z| \leq r} |\tilde{R}_n^{(j)}(z)| \right) s^j. \tag{B.50}$$

Then the definition (B.46) $\sim$ (B.47) of $B^{(j)}$ implies that $B(s)$ and $|\tilde{R}_n|(s)$ should satisfy the following algebraic equation:

$$B(s) = \frac{16}{r^2} s B(s) + \frac{4}{r} |\tilde{R}_n|(s), \tag{B.51}$$

that is,

$$B(s) = \frac{4}{r} \left(1 - \frac{16}{r^2} s\right)^{-1} |\tilde{R}_n|(s). \tag{B.52}$$

If $\tilde{R}_n(z, s)$ is holomorphic in $\{(z, s)\,;\, |z| \le r, |s| \le \delta\}$, then

$$\sup_{|z|\le r} |\tilde{R}_n^{(j)}(z)| \le \sup_{|z|\le r, |s|\le\delta} |\tilde{R}_n(z, s)|\delta^{-j} \tag{B.53}$$

holds and hence $|\tilde{R}_n|(s)$ is convergent for $|s| < \delta$. Thus, as we may assume $0 < \delta < r^2/16$, $B(s)$ is also convergent for $|s| < \delta$ thanks to (B.52). This verifies the holomorphy of $x_n(z, s)$ and $E_n(s)$ at $(z, s) = (0, 0)$, completing the proof of Proposition B.2. $\square$

Remark B.2. (i) Since $R_1 = 0$, we have $x_1(z, s) = 0$ and $E_1(s) = 0$.
(ii) Assume that $x_0(z, s)$ and $E_0(s)$ are holomorphic and $\partial x_0/\partial\tilde{x} \neq 0$ holds in $\{(z, s)\,;\, |z| \le r, |s| \le \delta\}$ for some positive constants r and δ satisfying $\delta < r^2/16$. The above proof of Proposition B.2 then verifies that $x_n(z, s)$ and $E_n(s)$ are holomorphic in $\{(z, s)\,;\, |z| \le r, |s| < \delta_n\}$ for an arbitrarily chosen decreasing sequence $\delta = \delta_0 > \delta_1 > \delta_2 > \cdots$. In particular, if we choose δ_n so that $\delta > \delta_1 > \delta_2 > \cdots > \delta_n > \cdots \ge \delta/2$ holds, we find that x_n and E_n are holomorphic for every n in a domain $\{(z, s)\,;\, |z| \le r, |s| \le \delta/2\}$ independent of n.

Finally, combining the holomorphy of x_n and E_n verified above with the reasoning employed in Appendix A, we discuss estimation of growth order of x_n and E_n as n tends to ∞.

For that purpose, we first seek for integral representations of x_n and E_n when $s \neq 0$. Equation (B.38) that determines x_n and E_n can be written as

$$\frac{\partial}{\partial z}\left(\sqrt{4s - z^2}\,x_n\right) = -\frac{2}{\sqrt{4s - z^2}}\left(E_n + \tilde{R}_n\right), \tag{B.54}$$

where

$$\tilde{R}_n = \left(\frac{\partial x_0}{\partial \tilde{x}}\right)^{-2} R_n. \tag{B.55}$$

We then find that, under the assumption $s \neq 0$, a solution x_n of (B.54) which is holomorphic at $z = -2\sqrt{s}$ is given by

$$x_n = -\frac{2}{\sqrt{4s - z^2}} \int_{-2\sqrt{s}}^{z} \frac{E_n + \tilde{R}_n}{\sqrt{4s - w^2}}\,dw. \tag{B.56}$$

If this solution (B.56) is holomorphic also at $z = 2\sqrt{s}$, E_n should satisfy

$$\int_{-2\sqrt{s}}^{2\sqrt{s}} \frac{E_n + \tilde{R}_n}{\sqrt{4s - w^2}}\,dw = 0, \tag{B.57}$$

that is,

$$E_n = -\frac{1}{\pi} \int_{-2\sqrt{s}}^{2\sqrt{s}} \frac{\tilde{R}_n}{\sqrt{4s - w^2}}\,dw. \tag{B.58}$$

Since x_n and E_n that are constructed in Proposition B.2 enjoy the holomorphy both at $z = \pm 2\sqrt{s}$, we can conclude that they have the integral representations (B.56) and (B.58) for any sufficiently small $s \neq 0$.

Using these integral representations (B.56) and (B.58), we prove the following

Lemma B.3. *Assume that* $\tilde{R}_n(z,s)$ *is holomorphic on* $\{(z,s); |z| < 4r_0, |s| < r_0^2\}$ *for a positive constant* r_0*. Let* (x_n, E_n) *be a solution of* (B.54) *which is given by the integral representations* (B.56) *and* (B.58)*. Then the following estimates hold for any* s *satisfying* $|s| = r^2$ *with* $0 < r < r_0$*:*

$$|E_n(s)| \le \sup_{|z|\le 4r} |\tilde{R}_n(z,s)|, \tag{B.59}$$

$$\sup_{|z|\le 4r} |x_n(z,s)| \le \frac{4}{r} \sup_{|z|\le 4r} |\tilde{R}_n(z,s)|, \tag{B.60}$$

$$\sup_{|z|\le 4r} \left|\frac{\partial x_n(z,s)}{\partial z}\right| \le \frac{5}{r^2} \sup_{|z|\le 4r} |\tilde{R}_n(z,s)|. \tag{B.61}$$

Proof. (i) By making a change of variable $w = 2\sqrt{s}p$ in (B.58), we obtain

$$E_n(s) = -\frac{1}{\pi}\int_{-1}^{1} \frac{\tilde{R}_n(2\sqrt{s}p, s)}{\sqrt{1-p^2}}dp. \tag{B.62}$$

Hence we have

(B.63)

$$|E_n(s)| \le \frac{1}{\pi} \sup_{|z|\le 2r} |\tilde{R}_n(z,s)| \int_{-1}^{1} \frac{1}{\sqrt{1-p^2}}dp \le \sup_{|z|\le 4r} |\tilde{R}_n(z,s)|.$$

(ii) Let $\tilde{z}$ denote $z/(2\sqrt{s})$. Note that the region $\{|z| \le 4r\}$ corresponds to $\{|\tilde{z}| \le 2\}$ in $\tilde{z}$-variable. It then follows from (B.56) that

$$x_n = -\frac{1}{\sqrt{s}\sqrt{1-\tilde{z}^2}} \int_{-1}^{\tilde{z}} \frac{E_n + \tilde{R}_n(2\sqrt{s}p, s)}{\sqrt{1-p^2}}dp. \tag{B.64}$$

We first consider the estimation of x_n in the region $\Omega_- = \{\tilde{z}\,; |\tilde{z}| \le 2, \operatorname{Re}\tilde{z} \le 0\}$. When $\tilde{z}$ belongs to Ω_-, the integration variable p may be assumed to satisfy $|1-p| \geq 1$. Hence, by making a change of variable $1+p=(1+\tilde{z})t$, we have

$$\left|\int_{-1}^{\tilde{z}} \frac{1}{\sqrt{1-p^2}} dp\right| \leq |1+\tilde{z}|^{1/2} \int_0^1 \frac{1}{\sqrt{t}} dt = 2|1+\tilde{z}|^{1/2}. \tag{B.65}$$

Hence, as $|1-\tilde{z}| \geq 1$ holds when $\tilde{z} \in \Omega_-$, we obtain

$$\begin{aligned} \sup_{\tilde{z}\in\Omega_-} |x_n| &\leq \frac{2}{r} \sup_{|z|\leq 4r} |E_n + \tilde{R}_n(z,s)| \\ &\leq \frac{4}{r} \sup_{|z|\leq 4r} |\tilde{R}_n(z,s)|. \end{aligned} \tag{B.66}$$

Since x_n can be expressed also as

$$x_n = -\frac{1}{\sqrt{s}\sqrt{1-\tilde{z}^2}} \int_1^{\tilde{z}} \frac{E_n + \tilde{R}_n(2\sqrt{s}p, s)}{\sqrt{1-p^2}} dp \tag{B.67}$$

in view of (B.57), we can obtain the same estimate for $\tilde{z} \in \Omega_+ = \{\tilde{z}\,;\, |\tilde{z}| \leq 2,\ \mathrm{Re}\,\tilde{z} \geq 0\}$ as well. Thus (B.60) holds.
(iii) It follows from (B.38) that

$$2\sqrt{s}(1-\tilde{z}^2)\frac{\partial x_n}{\partial \tilde{z}} - 2\sqrt{s}\tilde{z}x_n = -2(E_n + \tilde{R}_n), \tag{B.68}$$

that is,

$$(1-\tilde{z}^2)\frac{\partial x_n}{\partial \tilde{z}} = g_n \quad \text{with} \quad g_n = \tilde{z}x_n - \frac{E_n + \tilde{R}_n}{\sqrt{s}}. \tag{B.69}$$

Let $\Omega_{-,1}$ (resp. $\Omega_{-,2}$) be $\{\tilde{z}\,;\, |\tilde{z}+1| \leq 1\} \cap \Omega_-$ (resp. $\{\tilde{z}\,;\, |\tilde{z}+1| \geq 1\} \cap \Omega_-$). In $\Omega_{-,1}$, as $|1-\tilde{z}| \geq 1$ holds in Ω_-, the maximum principle and (B.69) entail that

$$\sup_{\tilde{z}\in\Omega_{-,1}} \left|\frac{\partial x_n}{\partial \tilde{z}}\right| \leq \sup_{\tilde{z}\in\Omega_{-,1}} \frac{|g_n|}{|1-\tilde{z}|} \leq \sup_{\tilde{z}\in\Omega_{-,1}} |g_n|. \tag{B.70}$$

This inequality also holds in $\Omega_{-,2}$ since $|1+\tilde{z}| \geq 1$ holds there. Thus we have

$$\sup_{\tilde{z}\in\Omega_-} \left|\frac{\partial x_n}{\partial \tilde{z}}\right| \leq \sup_{\tilde{z}\in\Omega_-} |g_n|. \tag{B.71}$$

Note that in Ω_- we have

$$\begin{aligned} |g_n| &\leq 2\sup_{\tilde{z}\in\Omega_-} |x_n| + \frac{1}{\rho}(|E_n| + \sup_{\tilde{z}\in\Omega_-} |\tilde{R}_n|) \\ &\leq \left(\frac{8}{\rho}+\frac{1}{\rho}+\frac{1}{\rho}\right) \sup_{\tilde{z}\in\Omega_-} |\tilde{R}_n| = \frac{10}{\rho}\sup_{\tilde{z}\in\Omega_-} |\tilde{R}_n|. \end{aligned} \tag{B.72}$$

Combining (B.71) with (B.72), we obtain

$$\sup_{\tilde{z}\in\Omega_-} \left|\frac{\partial x_n}{\partial \tilde{z}}\right| \leq \frac{10}{\rho}\sup_{\tilde{z}\in\Omega_-} |\tilde{R}_n|. \tag{B.73}$$

By a similar argument we can confirm the same inequality also in Ω_+. Hence we conclude

$$\sup_{|z|\leq 4r} \left|\frac{\partial x_n}{\partial z}\right| = \frac{1}{2\rho}\sup_{|\tilde{z}|\leq 2} \left|\frac{\partial x_n}{\partial \tilde{z}}\right| \leq \frac{5}{\rho^2}\sup_{|z|\leq 4r} |\tilde{R}_n|. \tag{B.74}$$

This completes the proof of Lemma B.3. □

We now take a small positive constant $\rho > 0$ so that $x_n(z,s)$ and $E_n(s)$ $(n = 0,1,2,\ldots)$ are holomorphic and $\partial x_0/\partial \tilde{x} \neq 0$ holds in a polydisk $\{(z,s)\,;\,|z| \leq 4\rho, |s| \leq \rho^2\}$. Combining Lemma B.3 with the reasoning employed in Appendix A, we can obtain the following estimation of growth order of x_n and E_n as n tends to ∞: There exist positive constants A so that for each small positive number ε

$$\sup_{|s|\leq(\rho-\varepsilon)^2} |E_n(s)| \leq n!\,\varepsilon^{-n}A^n, \tag{B.75}$$

$$\sup_{|z|\leq 4(\rho-\varepsilon),|s|\leq(\rho-\varepsilon)^2} |x_n(z,s)| \leq n!\,\varepsilon^{-n}A^n, \tag{B.76}$$

$$\text{(B.77)} \qquad \sup_{|z|\le 4(\rho-\varepsilon),|s|\le(\rho-\varepsilon)^2} \left|\frac{\partial x_n(z,s)}{\partial z}\right| \le n!\,\varepsilon^{-n}A^n$$

hold for $n = 1, 2, 3, \ldots$ and any $\varepsilon > 0$.

To verify (B.75) $\sim$ (B.77), let us assume that they hold up to $n-1$. By the assumption we can find a constant C satisfying

$$\text{(B.78)} \qquad \sup_{|s|\le\rho^2} |E_0(s)| \le C,$$

$$\text{(B.79)} \qquad \sup_{|z|\le 4\rho,|s|\le\rho^2} \left|\frac{\partial^l x_0}{\partial \tilde{x}^l}\right| \le C \quad (l = 0, 1, 2, 3),$$

$$\text{(B.80)} \qquad \sup_{|z|\le 4\rho,|s|\le\rho^2} \left|\left(\frac{\partial x_0}{\partial \tilde{x}}\right)^l\right| \le C \quad (l = 2, 3 \text{ and } l = -1, -2),$$

$$\text{(B.81)} \qquad \sup_{|z|\le 4\rho,|s|\le\rho^2} \left|\frac{\partial x_0}{\partial \tilde{x}}\frac{\partial^2 x_0}{\partial \tilde{x}^2}\right| \le C.$$

On the other hand, by the same reasoning as in Appendix A (cf. (A.45) and (A.46) in Appendix A) we obtain

$$\text{(B.82)} \qquad \sup_{|z|\le 4(\rho-\varepsilon),|s|\le(\rho-\varepsilon)^2} \left|\frac{\partial^2 x_m(z,s)}{\partial z^2}\right| \le e(m+1)!\,\varepsilon^{-(m+1)}A^m$$

and

$$\text{(B.83)} \qquad \sup_{|z|\le 4(\rho-\varepsilon),|s|\le(\rho-\varepsilon)^2} \left|\frac{\partial^3 x_m(z,s)}{\partial z^3}\right| \le e^2(m+2)!\,\varepsilon^{-(m+2)}A^m$$

for $m \le n-1$ from (B.77). Using these estimates, we find that the relations

$$\text{(B.84)} \qquad \frac{\partial x_m}{\partial \tilde{x}} = \frac{\partial x_0}{\partial \tilde{x}}\frac{\partial x_m}{\partial z},$$

$$\text{(B.85)} \qquad \frac{\partial^2 x_m}{\partial \tilde{x}^2} = \frac{\partial^2 x_0}{\partial \tilde{x}^2}\frac{\partial x_m}{\partial z} + \left(\frac{\partial x_0}{\partial \tilde{x}}\right)^2 \frac{\partial^2 x_m}{\partial z^2},$$

(B.86) $$\frac{\partial^3 x_m}{\partial \tilde{x}^3} = \frac{\partial^3 x_0}{\partial \tilde{x}^3}\frac{\partial x_m}{\partial z} + 3\frac{\partial^2 x_0}{\partial \tilde{x}^2}\frac{\partial x_0}{\partial \tilde{x}}\frac{\partial^2 x_m}{\partial z^2} + \left(\frac{\partial x_0}{\partial \tilde{x}}\right)^3 \frac{\partial^3 x_m}{\partial z^3}$$

yield

(B.87) $$\sup_{|z|\leq 4(\rho-\varepsilon),|s|\leq(\rho-\varepsilon)^2} \left|\frac{\partial x_m}{\partial \tilde{x}}\right| \leq Cm!\,\varepsilon^{-m}A^m,$$

(B.88) $$\sup_{|z|\leq 4(\rho-\varepsilon),|s|\leq(\rho-\varepsilon)^2} \left|\frac{\partial^2 x_m}{\partial \tilde{x}^2}\right| \leq C(1+e)(m+1)!\,\varepsilon^{-(m+1)}A^m,$$

(B.89) $$\sup_{|z|\leq 4(\rho-\varepsilon),|s|\leq(\rho-\varepsilon)^2} \left|\frac{\partial^3 x_m}{\partial \tilde{x}^3}\right| \leq C(1+e)^2(m+2)!\,\varepsilon^{-(m+2)}A^m$$

for $1 \leq m \leq n-1$ and $\varepsilon \leq 1$. Then, through an argument similar to that of Appendix A, we obtain the following estimates for $R_{n,k}$ ($k = 1,2,3,4$) defined by (B.34) $\sim$ (B.37):

(B.90)
$$\sup_{|z|\leq 4(\rho-\varepsilon),|s|\leq(\rho-\varepsilon)^2} |R_{n,k}(z,s)| \leq n!\,\varepsilon^{-n}A^n B_{n,k} \qquad (k = 1,2,3,4),$$

where

(B.91) $$B_{n,1} = \frac{C^2}{n}\left(\frac{16}{n-1} + 8\right),$$

(B.92) $$B_{n,2} = \frac{C^2}{n}\left\{\frac{16}{(n-1)(n-2)} + \frac{8(1+C)}{n-1} + \left(1 + 4C + C^2\right)\right\},$$

(B.93) $$B_{n,3} = \frac{C^2}{2A^2}\left((1+e)^2 + \frac{C^2\varepsilon^2}{n(n-1)}\right)e^{4C^2},$$

(B.94) $$B_{n,4} = \frac{3C^4}{A^2}\left\{\left((1+e)^2 + \frac{(1+e)\varepsilon}{2} + \frac{\varepsilon^2}{16}\right)e^{4C^2}\right.$$

$$+\frac{(1+e)}{2n}\,(2(1+e)+\varepsilon)\Bigg\}.$$

It then follows from (B.33), (B.55), (B.80) and (B.90) that

$$\text{(B.95)} \qquad \sup_{|z|\le 4(\rho-\varepsilon),|s|\le(\rho-\varepsilon)^2} \left|\tilde{R}_n(z,s)\right| \le n!\,\varepsilon^{-n}A^n\left(C\sum_{k=1}^{4} B_{n,k}\right).$$

Thanks to the maximum principle (with respect to the s-variable for each fixed z),

$$\text{(B.96)} \qquad \sup_s |E_n(s)|,\ \sup_{z,s}|x_n(z,s)| \text{ and } \sup_{z,s}\left|\frac{\partial x_n(z,s)}{\partial z}\right|$$

for $|z|\le 4(\rho-\varepsilon)$ and $|s|\le(\rho-\varepsilon)^2$ should be attained on $\{s\,;\,|s|=(\rho-\varepsilon)^2\}$. Hence we can apply Lemma B.3 to obtain

$$\text{(B.97)} \qquad \sup_{|s|\le(\rho-\varepsilon)^2} |E_n(s)| \le n!\,\varepsilon^{-n}A^n\left(C\sum_{k=1}^{4} B_{n,k}\right),$$

$$\text{(B.98)} \qquad \sup_{|z|\le 4(\rho-\varepsilon),|s|\le(\rho-\varepsilon)^2} |x_n(z,s)| \le n!\,\varepsilon^{-n}A^n\left(\frac{4C}{\rho-\varepsilon}\sum_{k=1}^{4} B_{n,k}\right),$$

(B.99)

$$\sup_{|z|\le 4(\rho-\varepsilon),|s|\le(\rho-\varepsilon)^2}\left|\frac{\partial x_n(z,s)}{\partial z}\right| \le n!\,\varepsilon^{-n}A^n\left(\frac{5C}{(\rho-\varepsilon)^2}\sum_{k=1}^{4} B_{n,k}\right)$$

from (B.95). Since we may assume $0<\varepsilon<\rho/3$ and $8\rho/15\le 1$, we thus obtain

$$\text{(B.100)} \qquad \sup_{|s|\le(\rho-\varepsilon)^2} |E_n(s)| \le n!\,\varepsilon^{-n}A^n\tilde{C},$$

$$\text{(B.101)} \qquad \sup_{|z|\le 4(\rho-\varepsilon),|s|\le(\rho-\varepsilon)^2} |x_n(z,s)| \le n!\,\varepsilon^{-n}A^n\tilde{C},$$

$$\text{(B.102)} \qquad \sup_{|z|\le 4(\rho-\varepsilon),|s|\le(\rho-\varepsilon)^2} \left|\frac{\partial x_n(z,s)}{\partial z}\right| \le n!\,\varepsilon^{-n}A^n\tilde{C}$$

with

$$\text{(B.103)} \qquad \tilde{C} = \frac{45C}{4\rho^2}\sum_{k=1}^{4} B_{n,k}.$$

Hence, if we take n and A so large that $\tilde{C} \le 1$ is satisfied, (B.75) $\sim$ (B.77) hold also for n. That is, the induction proceeds (for sufficiently large n) and we have completed the proof of (B.75) $\sim$ (B.77).

In conclusion, we have proved

Theorem B.4. *Let*

$$\text{(B.104)} \qquad x(\tilde{x},t,\eta) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} x_k^{(j)}(\tilde{x})t^j\right)\eta^{-k}$$

be the transformation that brings an MTP equation (B.2) *to an* ∞*-Weber equation* (B.3) *with*

$$\text{(B.105)} \qquad E(t,\eta) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} E_k^{(j)}t^j\right)\eta^{-k}.$$

We also let

$$\text{(B.106)} \qquad x_k(\tilde{x},t) = \sum_{j=0}^{\infty} x_k^{(j)}(\tilde{x})t^j \quad \text{and} \quad E_k(t) = \sum_{j=0}^{\infty} E_k^{(j)}t^j$$

denote the k-th order part (with respect to η^{-1}) of $x(\tilde{x},t,\eta)$ and $E(t,\eta)$, respectively. Then there exist positive constants δ and C_0 for which the following hold:
(i) *for every k both $x_k(\tilde{x},t)$ and $E_k(t)$ are holomorphic in*

$\{(\tilde{x},t); |\tilde{x}| \le \delta, |t| \le \delta\}$ *and* $\{\, t\,; \, |t| \le \delta\}$, *respectively.*
(ii) *the following inequalities hold for* $k = 1, 2, 3, \ldots$*:*

$$\sup_{|t|\le\delta} |E_k(t)| \le k!\, C_0^k, \tag{B.107}$$

$$\sup_{|\tilde{x}|,|t|\le\delta} |x_k(\tilde{x},t)| \le k!\, C_0^k, \tag{B.108}$$

$$\sup_{|\tilde{x}|,|t|\le\delta} \left| \frac{\partial x_k(\tilde{x},t)}{\partial \tilde{x}} \right| \le k!\, C_0^k. \tag{B.109}$$

Appendix C. Representation of the action of $\mathcal{X}$ as an integro-differential operator

In Appendix C, using the estimates of the transformation obtained in Appendix B (Theorem B.4), we investigate the microdifferential operator $\mathcal{X}$ defined by (2.59) in detail and represent its action upon a multi-valued analytic function in the form of (2.65). Throughout this appendix, to simplify the notation, we do not write the dependence of the transformation on the parameter t explicitly. As a matter of fact, the parameter t does not play an important role in the reasoning below and it applies equally to the situation where no parameter is introduced such as the transformation near a double turning point discussed in Section 1.

The microdifferential operator $\mathcal{X}$ is defined by (2.59), that is,

$$\mathcal{X} =: g'(x)^{1/2} \left(1 + \frac{\partial r}{\partial x}\right)^{-1/2} \exp(r(x,\eta)\xi) :, \tag{C.1}$$

where

$$r = r(x,\eta) = \sum_{j\ge1} \eta^{-j} r_j(x) \tag{C.2}$$

and ξ stands for the symbol of $\partial/\partial x$. We also define $r_j^\dagger(x)$ by

$$g'(x)^{-1}\left(1+\frac{\partial r}{\partial x}\right)=\sum_{j=0}^{\infty}\eta^{-j}r_j^\dagger(x). \tag{C.3}$$

Then, by expanding in the powers of η, we may write

$$g'(x)^{1/2}\left(1+\frac{\partial r}{\partial x}\right)^{-1/2}=\sum_{j=0}^{\infty}\eta^{-j}h_j(x) \tag{C.4}$$

and

$$\exp\left(r(x,\eta)\xi\right)=1+\sum_{1\le l\le j}\eta^{-j}\xi^l\, f_{l,j}(x). \tag{C.5}$$

Here we set

$$\begin{cases} h_0=(r_0^\dagger)^{1/2}, \\ h_j=(r_0^\dagger)^{1/2}\displaystyle\sum_{l=1}^{j}\frac{(-1)^l\Gamma(l+\frac{1}{2})}{l!\,\Gamma(\frac{1}{2})}\sum_{\substack{j_1+\cdots+j_l=j\\ j_k\ge 1}}\frac{r_{j_1}^\dagger\cdots r_{j_l}^\dagger}{(r_0^\dagger)^l} \quad (j\ge 1), \end{cases} \tag{C.6}$$

and

$$f_{l,j}=\frac{1}{l!}\sum_{\substack{j_1+\cdots+j_l=j\\ j_k\ge 1}} r_{j_1}\cdots r_{j_l}. \tag{C.7}$$

Thus the total symbol of $\mathcal{X}$ has the form

$$\sum_{j=0}^{\infty}\eta^{-j}\left(h_j+\sum_{j'=1}^{j}\sum_{l=1}^{j'}\xi^l\, h_{j-j'}f_{l,j'}\right). \tag{C.8}$$

We first consider the kernel function of $\mathcal{X}$, which is, by definition, given by

(C.9) $K(x,y,x',y')$

$$= \frac{1}{4\pi^2}\left(\frac{-h_0}{(x'-x)(y'-y)} + \sum_{j=1}^{\infty}\frac{h_j(y-y')^{j-1}}{x'-x}\log(y-y') + \sum_{j=1}^{\infty}\sum_{j'=1}^{j}\sum_{l=1}^{j'}\frac{l!\,h_{j-j'}f_{l,j'}(y-y')^{j-1}}{(j-1)!\,(x'-x)^{l+1}}\log(y-y')\right).$$

It follows from Theorem B.4 and its proof that there exist a neighborhood ω_1 of $(x,t)=(0,0)$ and a constant $C_0>0$ so that we have

$$\sup_{\omega_1}|r_j| \le j!\,C_0^j \quad (j=1,2,\dots), \tag{C.10}$$

$$\sup_{\omega_1}|r_j^\dagger| \le j!\,C_0^j \quad (j=1,2,\dots) \tag{C.11}$$

and

$$\max\left\{\sup_{\omega_1}\left|r_0^\dagger\right|,\ \sup_{\omega_1}\left|(r_0^\dagger)^{-1}\right|\right\} \le C_0. \tag{C.12}$$

Using Lemma A.4, we find

$$\sup_{\omega_1}|h_j| \le C_0^{3/2}\,e^{4C_0}\,j!\,C_0^j \quad (j\ge 1) \tag{C.13}$$

and

$$\sup_{\omega_1}|f_{l,j}| \le \frac{(j-l+1)!}{l!}\,4^{l-1}C_0^j \quad (1\le l\le j). \tag{C.14}$$

Hence the coefficient of $\log(y-y')$ in the right-hand side of (C.9) converges in the set

$$D = \left\{(x,y,x',y')\in\mathbb{C}^4;\ x\in\omega_1, x\ne x',\ |y-y'|<\frac{1}{C_0}\right\}. \tag{C.15}$$

Thus $K(x,y,x',y')$ has the following form:

(C.16)

$$K(x,y,x',y') = \frac{1}{4\pi^2}\left(\frac{-h_0}{(x'-x)(y'-y)} + L(x,y,x',y')\log(y-y')\right),$$

where L is a holomorphic function defined in D.

Next we discuss the representation of the action of $\mathcal{X}$ as an integro-differential operator. For that purpose we prepare the following

Proposition C.1. *For a domain U in $\mathbb{C}_x$ let Ω denote*

$$\Omega = \{(x, y; \xi, \eta) \in T^*(U \times \mathbb{C}_y) \,;\, \eta \neq 0\}, \tag{C.17}$$

and let $P = P(x, \partial/\partial x, \partial/\partial y)$ be a microdifferential operator of order 0 on Ω with the total symbol

$$\sigma(P) = \sum_{j=0}^{\infty} \eta^{-j} P_j(x, \eta^{-1}\xi). \tag{C.18}$$

Here we assume that each $P_j(x, \zeta)$ is an entire function of ζ and that the following growth order condition should hold: There exists a constant $C_0 > 0$ so that for any compact subset K of $U \times \mathbb{C}_\zeta$ we can find another constant M_K satisfying

$$\sup_{(x,\zeta)\in K} |P_j(x, \zeta)| \leq M_K j!\, C_0^j \tag{C.19}$$

for $j = 0, 1, 2, \ldots$. Then the action of P upon a (multi-valued) analytic function $\phi(x, y)$ is represented in the following form:

$$P\phi(x, y) = \int_{y_0}^{y} K(x, y - y', d/dx)\phi(x, y')dy', \tag{C.20}$$

where $K(x, y, d/dx)$ is a differential operator of infinite order that is defined on $\{(x, y); x \in U$ and $|y| < 1/C_0\}$ and y_0 is an arbitrarily chosen point that fixes the action of $(\partial/\partial y)^{-1}$ as an integral operator.

Proof. Let

$$P_j(x, \zeta) = \sum_{k=0}^{\infty} a_{jk}(x)\zeta^k \tag{C.21}$$

be a power series expansion of $P_j(x,\zeta)$ around $\zeta = 0$. Since $a_{jk}(x)$ is given by

(C.22)
$$a_{jk}(x) = \frac{1}{2\pi i}\oint_{|\zeta|=R} \frac{P_j(x,\zeta)}{\zeta^{k+1}} d\zeta,$$

it follows from (C.19) that for $K = K' \times \{\zeta; |\zeta| \le R\}$ with a compact subset K' of U and a positive constant R we have

(C.23)
$$\sup_{x\in K'} |a_{jk}(x)| \le M_{K',R}\, j!\, C_0^j R^{-k}$$

with some constant $M_K = M_{K',R}$.

Now, in terms of $\{a_{jk}(x)\}_{j,k}$, the action of P can be expressed as

(C.24)
$$\begin{aligned} P\phi(x,y) &=: \sum_{j,k=0}^{\infty} \eta^{-j} a_{jk}(x) \eta^{-k} \xi^k : \phi(x,y) \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} : \eta^{-j-k} a_{jk}(x) : \left(\frac{\partial}{\partial x}\right)^k \phi(x,y) \\ &= \int_{y_0}^{y} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{jk}(x) \frac{(y-y')^{j+k-1}}{(j+k-1)!} \left(\frac{\partial}{\partial x}\right)^k \phi(x,y') dy'. \end{aligned}$$

We set

(C.25)
$$K\left(x,y,\frac{\partial}{\partial x}\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{jk}(x)\frac{y^{j+k-1}}{(j+k-1)!}\right)\left(\frac{\partial}{\partial x}\right)^k$$

and let $c_k = c_k(x,y)$ $(k = 0,1,2,\ldots)$ denote its coefficient of $(\partial/\partial x)^k$, i.e.,

(C.26)
$$c_k(x,y) = \sum_{j=0}^{\infty} a_{jk}(x)\frac{y^{j+k-1}}{(j+k-1)!}.$$

To prove Proposition C.1, it then suffices to confirm the following estimate for c_k: For any compact subset K' of U, any constant r satisfying $r < 1/C_0$ and any positive constant ε there exists a constant M so that

$$\sup_{x\in K', |y|\le r} |c_k(x,y)| \le M\frac{\varepsilon^k}{(k-1)!} \tag{C.27}$$

holds for $k = 1, 2, \ldots$.

In view of (C.23) we find

$$\begin{aligned} |c_k(x,y)| &\le \sum_{j=0}^{\infty} |a_{jk}(x)| \frac{|y|^{j+k-1}}{(j+k-1)!} \\ &\le M_{K',R} R^{-k} \sum_{j=0}^{\infty} \frac{j!\, C_0^j |y|^{j+k-1}}{(j+k-1)!} \\ &\le \frac{M_{K',R} R^{-k}}{(k-1)!} \sum_{j=0}^{\infty} C_0^j |y|^{j+k-1} \end{aligned} \tag{C.28}$$

for $x \in K'$. Hence, if y satisfies $|y| \le r < 1/C_0$, it follows from (C.28) that

$$\sup_{x\in K', |y|\le r} |c_k(x,y)| \le \frac{M_{K',R} R^{-k} C_0^{1-k}}{(k-1)!\,(1-rC_0)}. \tag{C.29}$$

Thus, as R can be taken arbitrarily large, we obtain (C.27) by setting $\varepsilon = (RC_0)^{-1}$ and $M = M_{K',R}C_0(1-rC_0)^{-1}$. This completes the proof of Proposition C.1. □

Remark C.1. The condition (C.19) is slightly stronger than the growth order condition used in the definition of "a microdifferential operator of WKB type" introduced in [AKKT]. For the reference of the reader we recall the definition of a microdifferential operator of WKB type, which is an operator P of order 0 on Ω whose total symbol is of the form (C.18) with $P_j(x,\zeta)$ satisfying the following condition: For any compact subset K of $U \times \mathbb{C}_\zeta$ we can find a constant $M_K > 0$ so that

$$\sup_{(x,\zeta)\in K} |P_j(x,\zeta)| \le j!\, M_K^{j+1} \tag{C.30}$$

holds for $j = 0, 1, 2, \ldots$.

To apply Proposition C.1 to the microdifferential operator $\mathcal{X}$ in question, we rewrite the total symbol (C.8) of $\mathcal{X}$ in the following manner:

$$\begin{aligned}
&\left(\sum_{j=0}^{\infty} \eta^{-j} h_j\right)\left(1 + \sum_{1\le l\le j} \eta^{-j}\xi^l f_{l,j}\right) \\
&= \left(\sum_{j=0}^{\infty} \eta^{-j} h_j\right)\left(1 + \sum_{k=0}^{\infty} \eta^{-k} \sum_{l=1}^{\infty} f_{l,l+k}(\eta^{-1}\xi)^l\right) \\
&= \sum_{j=0}^{\infty} \eta^{-j} h_j + \sum_{j,k=0}^{\infty} \eta^{-(j+k)} h_j \sum_{l=1}^{\infty} f_{l,l+k}(\eta^{-1}\xi)^l \\
&= \sum_{m=0}^{\infty} \eta^{-m}\left[h_m + \sum_{l=1}^{\infty}\left(\sum_{j+k=m} h_j f_{l,l+k}\right)(\eta^{-1}\xi)^l\right].
\end{aligned} \tag{C.31}$$

Thus, letting $P_m(x,\zeta)$ denote

$$P_m(x,\zeta) = h_m + \sum_{l=1}^{\infty}\left(\sum_{j+k=m} h_j f_{l,l+k}\right)\zeta^l, \tag{C.32}$$

we find that the total symbol of $\mathcal{X}$ has the form (C.18). Using (C.13), (C.14) and Lemma A.4, we then obtain the following estimate for P_m:

$$|P_m| \le |h_m| + \sum_{l=1}^{\infty}\left(\sum_{j+k=m} |h_j f_{l,l+k}|\right)|\zeta|^l \tag{C.33}$$

$$
\begin{aligned}
&\le C_0^{3/2} e^{4C_0} m!\, C_0^m \\
&\quad + \sum_{l=1}^{\infty} \left(\sum_{j+k=m} C_0^{3/2} e^{4C_0} \frac{j!(k+1)!}{l!} 4^{l-1} C_0^{j+k+l} \right) |\zeta|^l \\
&\le C_0^{3/2} e^{4C_0} (m+1)!\, C_0^m \left(1 + \frac{5}{4} \sum_{l=1}^{\infty} \frac{1}{l!} (4C_0|\zeta|)^l \right) \\
&= C_0^{3/2} e^{4C_0} \left(1 + \frac{5}{4} (e^{4C_0|\zeta|} - 1) \right) (m+1)!\, C_0^m .
\end{aligned}
$$

This implies that $P_m(x, \zeta)$ defined by (C.32) is an entire function of ζ and satisfies the growth order condition (C.19). Hence it follows from Proposition C.1 that the action of $\mathcal{X}$ can be represented in the form (C.20), which completes the proof of Theorem 2.7.

Remark C.2. In estimating the transformation to the Airy equation near a simple turning point in [AKT1], we used the following inequality:

(C.34) $$\sum_{\substack{j_1+j_2+\cdots+j_k=j \\ j_1,\ldots,j_k \ge 1}} j_1! j_2! \cdots j_k! \le j!$$

(cf. [AKT1, Sublemma A.2.2]). Lemma A.4 is a refined version of (C.34) and this refinement is essential in the above proof of Theorem 2.7. In fact, if we were to use (C.34) instead of Lemma A.4, (C.13) and (C.14) would then be

(C.13)′ $$\sup_{\omega_1} |h_j| \le \frac{C_0^{3/2}}{C_0 - 1} j!\, C_0^{2j} \quad (j \ge 1)$$

and

(C.14)′ $$\sup_{\omega_1} |f_{l,j}| \le \frac{j!}{l!} C_0^j \quad (1 \le l \le j),$$

respectively, and hence the estimate (C.33) for $|P_m|$ would become

$$(\text{C.33})' \quad |P_m| \le \frac{C_0^{3/2}}{C_0 - 1} m!\, C_0^{2m} \left[1 + \sum_{l=1}^{\infty} (C_0|\zeta|)^l \left(1 + 2\frac{(m+l)!}{m!l!} \right) \right].$$

Consequently, in order that the right-hand side of (C.33)$'$ should converge, it would be necessary for $|\zeta|$ to be sufficiently small. Thus, without Lemma A.4, we cannot apply Proposition C.1 and the proof of Theorem 2.7 fails.

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