Graph equation for line graphs and m-step graphs

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Abstract

Given a graph G, the *m-step graph* of G, denoted by $S_m(G)$, has the same vertex set as G and an edge between two distinct vertices u and v if there is a walk of length m from u to v. The *line graph* of G, denoted by $L(G)$, is a graph such that the vertex set of $L(G)$ is the edge set of G and two vertices u and v of $L(G)$ are adjacent if the edges corresponding to u and v share a common end vertex in G. In this paper, we characterize connected graphs G satisfying graph equation $S_m(G) = L(G)$.

Key words and phrases: graph equations, m-step graphs, line graphs

1 Introduction

Throughout this paper, we only consider simple graphs. Given a graph G, the following two notions of well-known graphs can be defined: The *m-step graph* of G, denoted by $S_m(G)$, has the same vertex set as G and an edge between two vertices u and v if there is a walk of length m from u to v. The *line graph* of G, denoted by $L(G)$, is the intersection graph of the edge set of G. That is, the line graph of G is a graph such that the vertex set of $L(G)$ is the edge set of G and two vertices u and v of $L(G)$ are adjacent if the edges corresponding to u and v share a common end vertex in G. For all undefined graph-theoretical terms, see [2].

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Graph equations are equations in which the unknowns are graphs. Since many problems and results in graph theory can be formulated in terms of graph equations, graph equations have been studied by many authors (see [3] for surveys of the literature of graph equations). Especially, graph equations seeking a solution graph G whose line graph $L(G)$ is isomorphic to another graph structure built from G have been widely studied. For example, Akiyama *et al.* [1] studied graph equations for line graphs and nth power graphs and Simić [6] studied graph equations for line graphs and nth distance graphs. The authors may refer to [4] and [7] for the work on $S_m(G)$.

In this paper, we study graph equation for line graphs and m-step graphs. We characterize the graphs whose m-step graphs and whose line graphs are isomorphic. If $S_m(G)$ is isomorphic to $L(G)$ for a graph G, we say that G *satisfies* $S_m(G) = L(G)$.

A graph with exactly one cycle is said to be *unicyclic*.

Proposition 1.1. *If a connected graph $G = (V, E)$ satisfies $S_m(G) = L(G)$, then G is unicyclic. Especially, if m is even, the length of the cycle contained in G is odd.*

Proof. Since $S_m(G) = L(G)$, $V(S_m(G)) = V(L(G))$. By definition, $|V(S_m(G))| = |V|$ and $|V(L(G))| = |E|$ and so $|V| = |E|$. Since G is connected, it is true that G contains exactly one cycle C.

Suppose that m is even. If the length n of C is even, then G is a bipartite graph. Then any two vertices the distance between which is odd are disconnected in $S_m(G)$ and so $S_m(G)$ is disconnected. However, $L(G)$ is still connected and we reach a contradiction. Thus the length of C must be odd. □

We call a cycle of length ≥ 4 without a chord a *hole*. Now we present a simple but useful property of $L(G)$ for a unicyclic graph G:

Proposition 1.2. *Suppose that a connected graph G is a unicyclic graph having cycle $C = v_0e_1v_1 \cdots v_{n-1}e_nv_0$ for $n \geq 5$. Then $L(G)$ has a unique hole $e_1e_2 \cdots e_ne_1$.*

Proof. Since C is a hole, it is true that $e_1e_2 \cdots e_ne_1$ is a hole in $L(G)$. If $f_1f_2 \cdots f_m$ is a hole in $L(G)$, then $x_0x_1 \cdots x_{m-1}x_0$ is a hole in G where x_{i-1} and x_i are the end vertices of f_i for $i = 1,\ldots, m$. (Identify x_m with x_0). Since C is the only cycle of G, we have $m = n$ and $x_i = v_i$ for $i = 0, \ldots, n-1$. Thus $L(G)$ has a unique hole, which has length n. □

Note that $E(S_m(G)) \subset E(S_{m+2}(G))$ for any positive integer m. To see why, take an edge $e = xy$ in $S_m(G)$. Then there exists an (x, y)-walk W of length m in G. Let z be the vertex immediately preceding y in W. Then Wzy is an (x, y)-walk of length $m + 2$. Thus e is an edge of $S_{m+2}(G)$.

We denote by $d_G(u,v)$ the distance between u and v in a connected graph G and by $diam(G)$ the diameter of G.

Lemma 1.3. *Suppose that G is a connected unicyclic graph with $diam(G) \geq 4$. Then $diam(S_m(G)) \leq diam(G) - 2$ for odd $m \geq 3$.*

Proof. Take two vertices u, v in G. Since $E(G) \subset E(S_m(G))$ as noted above, we have $d_{S_m(G)}(u,v) \leq d_G(u,v)$. If $d_G(u,v) \leq 2$, then $d_{S_m(G)}(u,v) \leq d_G(u,v) \leq 2 \leq diam(G) - 2$. Now suppose that $d_G(u,v) \geq 3$. Let $uv_1v_2 \cdots v_{l-1}v$ (Identify v_l with v.) be a shortest (u,v)-path in G. Then $l \geq 3$, and u and v_3 are adjacent in $S_m(G)$ since m is odd. Therefore, $d_{S_m(G)}(u,v) \leq d_G(u,v) - 2$. Thus, $d_{S_m(G)}(u,v) \leq diam(G) - 2$ for any u, v in G and so $diam(S_m(G)) \leq diam(G) - 2$. □

Given a cycle C of a connected graph G, we call a path of G *C-avoiding path* if its internal vertices are not on C. A *spiked cycle* is a connected graph whose non-pendant vertices form a cycle.

For a vertex v of a graph G, we denote by $N_G(v)$ (resp. $N_G[v]$) the set of vertices adjacent to v in G (resp. the set of v and vertices adjacent to v in G) or the subgraph of G induced by the those adjacent vertices.

Theorem 1.4. *Let m be an odd integer greater than or equal to 3. Then for a connected graph G, G satisfies $S_m(G) = L(G)$ if and only if G is either a 3-cycle or a 4-cycle.*

Proof. The 'if' part is obviously true. Now we show the 'only if' part. Since a path of G of length l as an induced subgraph corresponds to a path of $L(G)$ of $l-1$ as an induced subgraph, $diam(G) \leq diam(L(G)) + 1$. By Proposition 1.1, G is unicyclic. If $diam(G) \geq 4$, then,by Lemma 1.3, $diam(L(G)) \geq diam(G) - 1 > diam(S_m(G))$, which contradicts the hypothesis that $S_m(G) = L(G)$. Thus $diam(G) \leq 3$. This implies that G is a spiked cycle with a cycle C of length from 3 up to 7, or a unicyclic graph with a 3-cycle C such that only one vertex x on C has degree ≥ 3 and a longest C-avoiding path starting at x is unique and has length 2.

If C is a 5-cycle, a 6-cycle, or a 7-cycle, then C is a hole in $L(G)$ while C has a chord in $S_m(G)$ since any two vertices at distance 3 in G are adjacent in $S_m(G)$.

If C is a 4-cycle, then the degree of each vertex on C is 2. For otherwise, there is a vertex v of degree at least 3 on C, and the neighbors of v on C and a neighbor of v not on C form an independent set of size 3 in $S_3(G)$. Thus the edge clique cover number of $N_{S_3(G)}[v]$ is at least 3. It is impossible for $L(G) = S_m(G)$ since it is known that the edge clique cover number of the closed neighborhood of a vertex in a line graph is at most 2. Hence G itself is 4-cycle if C is a 4-cycle.

Now consider the case where C is a 3-cycle. Since $diam(G) \leq 3$, $S_m(G)$ is a complete graph for odd $m \geq 3$. If there is a vertex x not on C that is joined to a vertex y on C by an edge e, then e is not adjacent to the edge joining the two vertices on C other than y in $L(G)$. Thus it is impossible that $S_m(G) = L(G)$. Therefore G has to be a 3-cycle. $\square$

Now it remains to characterize a connected graph G satisfying $S_m(G) = L(G)$ for an even integer m. We begin by presenting the following theorem:

Theorem 1.5. *Let G be a connected graph. If $S_m(G) = L(G)$ for an even integer $m \geq 4$, then the girth of G is* 3.

Proof. By Proposition 1.1, G contains a unique odd cycle $C = v_0v_1 \cdots v_{l-1}v_0$. Suppose that $l \geq 5$. We will reach a contradiction. Since G is C_4-free,

$$|E(L(G))| = \sum_{v \in V(G)} \binom{\deg_G(v)}{2}$$

where $\deg_G(v)$ denotes the degree of v in G.

Since $E(S_2(G)) \subset E(S_m(G))$ for even m, $S_m(G)$ has at least

$$\sum_{v \in V(G)} \binom{\deg_G(v)}{2} \qquad (\star)$$

edges. However, since C is not a 6-cycle, $S_m(G)$ contains edge v_0v_4 that is not counted in $(\star)$. This implies that $|E(S_m(G))| > |E(L(G))|$, which is contradiction. Thus, $l \leq 3$. $\square$

We have characterized a connected graph G satisfying $S_m(G) = L(G)$ for an odd integer m. Now it remains to characterize a connected graph G satisfying $S_m(G) = L(G)$ for an even integer m. We consider a connected graph G with girth greater than 3 in Section 2 and a connected graph G with girth 3 in Section 3.

2 Graph with girth$>$ 3 satisfying $S_m(G) = L(G)$ for even m

Theorem 1.5 tells us that if a connected graph G with girth greater than 3 satisfies $S_m(G) = L(G)$ for even m, then $m = 2$. Thus, it is sufficient to consider the case where $m = 2$.

Proposition 2.1. *Suppose that a connected graph G is a unicyclic graph having cycle $C = v_0e_1v_1 \cdots v_{n-1}e_nv_0$ for $n \geq 5$. Then $S_2(G)$ and $L(G)$ have unique holes $v_0v_2 \cdots v_{n-1}v_1v_3 \cdots v_{n-2}v_0$ and $e_1e_2 \cdots e_ne_1$, respectively.*

Proof. Let $C = v_0e_1v_1e_2 \cdots v_{n-1}e_nv_0$ $(n \geq 5)$. Then delete an edge e_n on C from G. Then the resulting graph $G - e_n$ is a tree. Phelps [5] showed that the 2-step graph of a tree with at least 2 vertices has exactly two components and is chordal. It is easy to see that $v_0v_2 \cdots v_{n-1}$ and $v_1v_3 \cdots v_{n-2}$ are paths of length at least one as induced subgraphs of $S_2(G - e_n)$ which belong to different components of $S_2(G - e_n)$. Note that

$$E(S_2(G)) = E(S_2(G - e_n)) \cup \{xv_{n-1} \mid x \in N_G(v_0)\} \cup \{yv_0 \mid y \in N_G(v_{n-1})\}.$$

Thus $S_2(G)$ has hole $C^* = v_0v_2 \cdots v_{n-1}v_1v_3 \cdots v_{n-2}v_0$. Any vertex in $N_G(v_0)$ (resp. $N_G(v_{n-1})$) other than v_1 (resp. v_{n-2}) forms a triangle with v_1 and v_{n-1} (resp. v_0 and v_{n-2}) in $S_2(G)$. Hence C^* is the only hole in $S_2(G)$.

By Proposition 1.2, $L(G)$ contains a unique hole $e_1e_2 \cdots e_ne_1$. $\square$

Proposition 2.2. *If a connected graph G has girth greater than 3 and satisfies $S_2(G) = L(G)$, then G is a spiked odd cycle.*

Proof. By contradiction. Suppose that there is a connected graph G such that $S_2(G) = L(G)$ and G is not a spiked odd cycle. Since $S_2(G) = L(G)$, there exists a bijection ϕ from $V(S_2(G))$ to $V(L(G))$ such that $uv \in E(S_2(G))$ if and only if $\phi(u)\phi(v) \in E(L(G))$ for vertices u, v in G. By Proposition 1.1, G has a unique odd cycle $C = v_0e_1v_1 \ldots e_{n-1}v_{n-1}e_nv_0$ $(n \geq 5)$. Let $P = x_1x_2 \cdots x_k$ be a longest C-avoiding path which shares an initial vertex x_1 with C. Then $k \geq 3$ by the assumption that G is not a spiked odd cycle. Without loss of generality, we may assume that $v_0 = x_1$. By Proposition 2.1, $S_2(G)$ and $L(G)$ have unique holes $C_1 = v_0v_2 \cdots v_{n-1}v_1v_3 \cdots v_{n-2}v_0$ and $C_2 = e_1e_2 \cdots e_ne_1$, respectively. Then $S_2(G) = L(G)$ implies that $\phi(V(C_1)) = V(C_2)$. It is easy to check that

$$\begin{aligned} r &= \max\{d_{S_2(G)}(v, w) \mid v \in V(C_1), w \in V(G) \setminus V(C_1)\} \\ &= \begin{cases} d_{S_2(G)}(v_1, x_k) & \text{if } k \text{ is odd;} \\ d_{S_2(G)}(v_0, x_k) & \text{if } k \text{ is even;} \end{cases} \\ &= \frac{n-1}{2} + \left\lfloor \frac{k}{2} \right\rfloor \end{aligned}$$

and

$$\begin{aligned} s &= \max\{d_{L(G)}(e, f) \mid e \in V(C_2), f \in E(G) \setminus V(C_2)\} \\ &= d_{L(G)}(e_{(n+1)/2}, x_{k-1}x_k) \\ &= \frac{n-1}{2} + (k-1). \end{aligned}$$

Since $\phi(V(C_1)) = V(C_2)$, we have $r = s$. Thus $\lfloor k/2 \rfloor = k-1$. However, $k-1 > \lfloor k/2 \rfloor$ for $k \geq 3$ and we reach a contradiction. $\square$

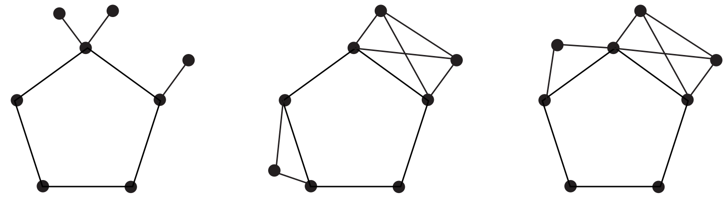

Figure 1: A spiked odd cycle G not satisfying $S_2(G) = L(G)$.

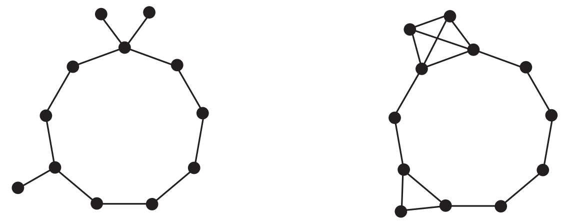

Figure 2: A spiked odd cycle G satisfying $S_2(G) = L(G)$.

It is natural to ask whether or not the converse of Proposition 2.2 is true. It is not true as we can see from Figure 1. However, the spiked odd cycles of certain types such as the one in Figure 2 satisfy $S_2(G) = L(G)$:

Theorem 2.3. *Let G be a spiked odd cycle with cycle $C = v_0v_1 \cdots v_{n-1}v_0$ $(n \geq 5)$ and a_i be the number of pendant vertices adjacent to vertex v_i for each $i = 0, \ldots, n-1$. Suppose that there exists $k \in \{0, \ldots, n-1\}$ such that either $a_0 = a_k$, $a_2 = a_{k+1}$, $\ldots$, $a_{n-1} = a_{k+(n-1)/2}$, $a_1 = a_{k+(n+1)/2}$, $\ldots$, $a_{n-2} = a_{k+(n-1)}$ or $a_0 = a_k$, $a_2 = a_{k-1}$, $a_4 = a_{k-2}$, $\ldots$, $a_{n-1} = a_{k-(n-1)/2}$, $a_1 = a_{k-(n+1)/2}$, $\ldots$, $a_{n-2} = a_{k-(n-1)}$ where the subscripts are reduced to modulo n. Then G satisfies $S_2(G) = L(G)$.*

Proof. Let $C = v_0e_1v_1e_2 \cdots v_{n-1}e_nv_0$. We denote by $w_{i1}, \ldots, w_{ia_i}$ the pendant vertices adjacent to v_i in G and by $f_{i1}, \ldots, f_{ia_i}$ the edges $v_iw_{i1}, \ldots, v_iw_{ia_i}$, respectively. Suppose that $a_0 = a_k$, $a_2 = a_{k+1}$, $\ldots$, $a_{n-1} = a_{k+(n-1)/2}$, $a_1 = a_{k+(n+1)/2}$, $\ldots$, $a_{n-2} = a_{k+(n-1)}$. We define a map ϕ from $V(S_2(G))$ to $V(L(G))$ as follows:

$$\phi(v_i) = \begin{cases} e_{k+(i+1)/2} & \text{if } i \text{ is odd;} \\ e_{k+(n+i+1)/2} & \text{if } i \text{ is even;} \end{cases}$$

and if w_{ij} exists, then

$$\phi(w_{ij}) = \begin{cases} f_{k+i/2,j} & \text{if } i \text{ is even;} \\ f_{k+(n+i)/2,j} & \text{if } i \text{ is odd;} \end{cases}$$

where all the subscripts are reduced to modulo n. It is easy to check that ϕ is a bijection.

To show that ϕ is an isomorphism, take an edge xy in $S_2(G)$. If both are vertices on the hole in $S_2(G)$, then $x = v_i$ and $y = v_{i+2}$. If i is odd, then $\phi(v_i) = e_{k+(i+1)/2}$ and $\phi(v_{i+2}) = e_{k+(i+3)/2}$, and they are adjacent in $L(G)$. If i is even, then $\phi(v_i) = e_{k+(n+i+1)/2}$ and $\phi(v_{i+2}) = e_{k+(n+i+3)/2}$, and they are adjacent in $L(G)$. If one is on the hole while the other one is not, then we may assume that $x = v_{i-1}$ and $y = w_{ij}$ for some $j \in \{1, \dots, a_i\}$. Then $\phi(v_{i-1}) = e_{k+i/2}$ and $\phi(w_{ij}) = f_{k+i/2,j}$ if i is even, and $\phi(v_{i-1}) = e_{k+(n+i)/2}$ and $\phi(w_{ij}) = f_{k+(n+i)/2,j}$ if i is odd. In both cases, $\phi(x)$ and $\phi(y)$ are adjacent in $L(G)$. Finally suppose that none of x and y is on the hole in $S_2(G)$. Then $x = w_{ij}$ and $y = w_{ij'}$. Then $\phi(w_{ij}) = f_{k+i/2,j}$ and $\phi(w_{ij'}) = f_{k+i/2,j'}$ if i is even, and $\phi(w_{ij}) = f_{k+(n+i)/2,j}$ and $\phi(w_{ij'}) = f_{k+(n+i)/2,j'}$ if i is odd. In both cases, $\phi(x)$ and $\phi(y)$ are adjacent in $L(G)$.

Now take an edge ef in $L(G)$. Suppose that e and f both are on the hole. Then $e = e_i$ and $f = e_{i+1}$. Let $A = \{k + j \pmod{n} \mid j = 1, \dots, k + (n-1)/2\}$. If $\{i, i+1\} \subset A$, then $\phi(v_{2(i-k)-1}) = e_i$ and $\phi(v_{2(i-k)+1}) = e_{i+1}$, and it is true that $v_{2(i-k)-1}$ and $v_{2(i-k)+1}$ are adjacent in $S_2(G)$. If exactly one of i, $i+1$ belongs to A, then either $i = k$ or $i = k + (n-1)/2 \pmod{n}$. Now it is true that $\phi(v_{n-1}) = e_k$, $\phi(v_1) = e_{k+1}$, $\phi(v_{n-2}) = e_{k+(n-1)/2}$, and $\phi(v_0) = e_{k+(n+1)/2}$ and that v_{n-1} and v_1 are adjacent and so are v_{n-2} and v_0 in $S_2(G)$. If neither i nor $i+1$ belongs to A, then $\phi(v_{2(i-k)-n-1}) = e_i$ and $\phi(v_{2(i-k+1)-n-1}) = e_{i+1}$. It is true that $v_{2(i-k)-n-1}$ and $v_{2(i-k+1)-n-1}$ are adjacent in $S_2(G)$. Suppose that e is on the hole while f is not. Then $e = e_i$ and $f = f_{i-1,j}$ for some $j \in \{1, 2, \dots, a_{i-1}\}$ or f_{ij} for some $j \in \{1, 2, \dots, a_i\}$. If $\{i-1, i\} \subset A$, then $\phi(v_{2(i-k)-1}) = e_i$, $\phi(w_{2(i-k-1),j}) = f_{i-1,j}$, $\phi(w_{2(i-k),j}) = f_{ij}$, and it is true that $v_{2(i-k)-1}$ is adjacent to both $w_{2(i-k-1),j}$ and $w_{2(i-k),j}$ in $S_2(G)$. If exactly one of $i-1$, i belongs to A, then either $i = k+1$ or $i = k+(n+3)/2 \pmod{n}$. It is true that $\phi(v_1) = e_{k+1}$, $\phi(w_{0j}) = f_{k,j}$, $\phi(w_{2j}) = f_{k+1,j}$, and it is true that v_1 is adjacent to both w_{0j} and w_{2j}. In addition, it is true that $\phi(v_2) = e_{k+(n+3)/2}$, $\phi(w_{1j}) = f_{k+(n+1)/2,j}$, $\phi(w_{3j}) = f_{k+(n+3)/2}$, and that v_2 is adjacent to both w_{1j} and w_{3j}. If neither $i-1$ nor i belongs to A, then $\phi(v_{2(i-k)-n-1}) = e_i$ and $\phi(w_{2(i-k-1)-n-1,j}) = f_{i-1,j}$, and $\phi(w_{2(i-k)-n,j}) = f_{ij}$, and it is true that $v_{2(i-k)-n-1}$ is adjacent to both $w_{2(i-k-1)-n,j}$ and $w_{2(i-k)-n,j}$. Suppose that neither e nor f is on the hole in $L(G)$. Then $e = f_{ij}$ and $f = f_{ij'}$. Then if $i \in A$, then $\phi(w_{2(i-k),j}) = f_{ij}$ and $\phi(w_{2(i-k),j'}) = f_{ij'}$, and it is true that $w_{2(i-k),j}$ and $w_{2(i-k),j'}$ are adjacent in $S_2(G)$. If $i \notin A$, then $\phi(w_{2(i-k)-n,j}) = f_{ij}$ and $\phi(w_{2(i-k)-n,j'}) = f_{ij'}$, and it is true that $w_{2(i-k)-n,j}$ and $w_{2(i-k)-n,j'}$ are adjacent in $S_2(G)$. Hence we have shown that ϕ is an isomorphism.

In the case where $a_0 = a_k$, $a_2 = a_{k-1}$, $a_4 = a_{k-2}$, $\dots$, $a_{n-1} = a_{k-(n-1)/2}$, $a_1 = a_{k-(n+1)/2}$, $\dots$, $a_{n-2} = a_{k-(n-1)}$, we define a map ψ from $S_2(G)$ to $L(G)$

as follows:

$$\psi(v_i) = \begin{cases} e_{k-(i-1)/2} & \text{if } i \text{ is odd;} \\ e_{k-(n+i-1)/2} & \text{if } i \text{ is even;} \end{cases}$$

and if w_{ij} exists, then

$$\psi(w_{ij}) = \begin{cases} f_{k-i/2,j} & \text{if } i \text{ is even;} \\ f_{k-(n+i)/2,j} & \text{if } i \text{ is odd.} \end{cases}$$

Then by applying a similar argument as above, we can show that ψ is an isomorphism. Hence in any case, G satisfies $S_2(G) = L(G)$. □

In fact, it is also a necessary condition for a connected graph G with girth greater than 3 satisfying $S_2(G) = L(G)$:

Lemma 2.4. *Let G be a spiked odd cycle with cycle $C = v_0v_1 \cdots v_{n-1}v_0$ ($n \geq 5$) and a_i be the number of pendant vertices adjacent to vertex v_i for each $i = 0, \ldots, n-1$. Then if G satisfies $S_2(G) = L(G)$, then there exists $k \in \{0, \ldots, n-1\}$ such that either $a_0 = a_k$, $a_2 = a_{k+1}$, $\ldots$, $a_{n-1} = a_{k+(n-1)/2}$, $a_1 = a_{k+(n+1)/2}$, $\ldots$, $a_{n-2} = a_{k+(n-1)}$ or $a_0 = a_k$, $a_2 = a_{k-1}$, $a_4 = a_{k-2}$, $\ldots$, $a_{n-1} = a_{k-(n-1)/2}$, $a_1 = a_{k-(n+1)/2}$, $\ldots$, $a_{n-2} = a_{k-(n-1)}$ where the subscripts are reduced to modulo n.*

Proof. Since $S_2(G) = L(G)$, there exists an isomorphism ϕ from $S_2(G)$ to $L(G)$. We denote $v_{i-1}v_i$ by e_i. By Proposition 2.1, $S_2(G)$ and $L(G)$ have unique holes $v_1v_3 \cdots v_{n-2}v_0v_2 \cdots v_{n-1}v_1$ and $e_{k+1}e_{k+2} \cdots e_ke_{k+1}$, respectively. Thus for some k, either

$$\phi(v_i) = \begin{cases} e_{k+(i+1)/2} & \text{if } i \text{ is odd;} \\ e_{k+(n+i+1)/2} & \text{if } i \text{ is even;} \end{cases} \tag{$*$}$$

or

$$\phi(v_i) = \begin{cases} e_{k-(i-1)/2} & \text{if } i \text{ is odd;} \\ e_{k-(n+i-1)/2} & \text{if } i \text{ is even;} \end{cases} \tag{$**$}$$

where all the subscripts are reduced modulo n.

Now the maximal cliques containing v_i in $S_2(G)$ are $N_G(v_{i-1})$ and $N_G(v_{i+1})$ sizes of which are $a_{i-1}+2$ and $a_{i+1}+2$, respectively. Similarly, we can check that the sizes of maximal clique containing e_j in $L(G)$ are $a_{j-1}+2$ and a_j+2. Since the sizes of maximal cliques containing each of two corresponding vertices under isomorphism must be the same, the following are true: If ϕ satisfies ($*$), then

$$a_{i+1} = \begin{cases} a_{k+(i+1)/2} & \text{if } i \text{ is odd;} \\ a_{k+(n+i+1)/2} & \text{if } i \text{ is even;} \end{cases}$$

or

$$a_i = \begin{cases} a_{k+i/2} & \text{if } i \text{ is even;} \\ a_{k+(n+i)/2} & \text{if } i \text{ is odd.} \end{cases}$$

Similarly, if ϕ satisfies $(**)$, then

$$a_i = \begin{cases} a_{k-i/2} & \text{if } i \text{ is even;} \\ a_{k-(n+i)/2} & \text{if } i \text{ is odd.} \end{cases}$$

□

Now we are ready to characterize a connected graph G with girth greater than 3 satisfying graph equality $S_2(G) = L(G)$:

Theorem 2.5. *Let G be a connected graph G with girth greater than 3. Then G satisfies $S_2(G) = L(G)$ if and only if G is a spiked odd cycle such that for some $k \in \{0, \dots, n-1\}$, either $a_0 = a_k$, $a_2 = a_{k+1}$, $\dots$, $a_{n-1} = a_{k+(n-1)/2}$, $a_1 = a_{k+(n+1)/2}$, $\dots$, $a_{n-2} = a_{k+(n-1)}$ or $a_0 = a_k$, $a_2 = a_{k-1}$, $a_4 = a_{k-2}$, $\dots$, $a_{n-1} = a_{k-(n-1)/2}$, $a_1 = a_{k-(n+1)/2}$, $\dots$, $a_{n-2} = a_{k-(n-1)}$ where the subscripts are reduced to modulo n, $C = v_0v_1 \cdots v_{n-1}v_0$ $(n \geq 5)$ is the cycle of G, and a_i is the number of pendant vertices adjacent to vertex v_i for each $i = 0, \dots, n-1$.*

Proof. The 'if' part is just Theorem 2.3. To show the 'only if' part, assume that $S_2(G) = L(G)$. Then by Proposition 2.2, G is a spiked odd cycle and Lemma 2.4 completes the proof. □

3 Graph with girth 3 satisfying $S_m(G) = L(G)$

In Sections 1 and 2, we characterized a connected graph G satisfying graph equation $S_m(G) = L(G)$ for an odd integer $m \geq 3$ and for an even integer m with girth greater than 3. Now to complete the characterization of a connected graph satisfying $S_m(G) = L(G)$ for $m \geq 2$, it remains to characterize a connected graph G with girth 3 satisfying graph equation $S_m(G) = L(G)$ for an even integer m. By Proposition 1.1, a connected graph G with girth 3 satisfying $S_m(G) = L(G)$ is still unicyclic.

For a unicyclic graph with girth 3, the following holds:

Lemma 3.1. *Given a connected unicyclic graph G with girth 3, a vertex e of $L(G)$ is a cut vertex in $L(G)$ if and only if e is neither incident with a pendant vertex nor on the cycle in G.*

Proof. To show the 'only if' part, suppose that e is an edge incident with a pendant vertex v. Then the neighbors of e form a clique in $L(G)$ and so $L(G)-e$ is connected. Suppose that e is on the cycle in G. Let e_1 and e_2 be the edges on C which are adjacent to e. Then every neighbor of e in $L(G)$ is adjacent to either e_1 or e_2. Since vertices e_1 and e_2 are connected in $L(G) - e$, it is true that $L(G) - e$ is connected.

To show the converse, take an edge $f = xy$ that is neither incident with a pendant vertex nor on the cycle. Since f is not incident with a pendant vertex, there exist vertices w and z distinct from x and y such that x is adjacent to w and y is adjacent z. Since f is not on the cycle, it is true that $wxyz$ is the only path between w and z. This implies that the vertex sequence wx, f, yz in $L(G)$ is a unique path between wx and yz. Thus deleting f from $L(G)$ disconnects vertex xw from vertex yz and so f is a cut vertex of $L(G)$. $\square$

For a vertex $v \notin V(C)$ of a graph G, we let $d_G(v, C) = \min\{d_G(v, w) \mid w \in V(C)\}$.

Lemma 3.2. *If a connected unicyclic graph G with a 3-cycle $C = xyzx$ has a C-avoiding path length at least 2 starting at x, then $diam(L(G)) \geq k + l$ where k is the length of a longest C-avoiding path and l is the length of a C-avoiding path starting at y or z.*

Proof. It is true that

$$diam(L(G)) \geq d_{L(G)}(e, f) = d_{L(G)}(e, C) + d_{L(G)}(f, C) = k + l$$

for edges e and f incident to pendant vertices on a longest C-avoiding path starting x and a C-avoiding path starting y or z, respectively. Thus we have $diam(L(G)) \geq k + l$. $\square$

Lemma 3.3. *Suppose that G is a connected unicyclic graph with girth 3 and has a C-avoiding path of length at least 2. Then if m is even, then*

$$diam(S_m(G)) = \left\lceil \frac{k}{m} \right\rceil + 1 + \left\lfloor \frac{k}{m} \right\rfloor$$

where k is the length of a longest C-avoiding path.

Proof. Let $C = xyzx$ and P be a longest C-avoiding path starting at x. Let $P = xx_1 \cdots x_k$. Then $k \geq 2$ by the hypothesis. It is not difficult to check that x_k and x_{k-1} are farthest vertices in $S_m(G)$ with

$$d_{S_m(G)}(x_k, x_{k-1}) = d_{S_m(G)}(x_k, C) + 1 + d_{S_m(G)}(x_{k-1}, C) = \left\lceil \frac{k}{m} \right\rceil + 1 + \left\lfloor \frac{k}{m} \right\rfloor.$$

$\square$

Given a graph G, we denote the edge clique number of a graph G by $\theta_E(G)$.

Lemma 3.4. *Suppose that a connected unicyclic graph G with cycle $C = xyzx$ satisfies $S_m(G) = L(G)$ for an even integer m. If G has a C-avoiding path P with an end vertex x on C, then $m = 2$ and the degree of each of y, z is 2.*

Proof. It follows from Lemmas 3.2 and 3.3 that $m \leq 3$ and $l \leq 2$ where l is the length of a C-avoiding path starting at y or z. Since m is even, we have $m = 2$. If $l \geq 1$, then one of y or z is adjacent to a vertex w not on C. Without loss of generality, we may assume that w is adjacent to y. In addition, we denote by v the vertex on P at distance 2. Then y, w, v are neighbors of x in $S_2(G)$. However, no two of these vertices belong to the same edge clique in $S_2(G)$ and so $\theta_E(N_{S_2(G)}(x)) \geq 3$, which is impossible for a line graph. This completes the proof. □

From Lemma 3.1 and Lemma 3.4, the following proposition follows:

Proposition 3.5. *If a connected unicyclic graph G with a 3-cycle C satisfies $S_m(G) = L(G)$ for an even integer m, then any C-avoiding path with an end vertex on C has length at most 2.*

Proof. By contradiction. Suppose that there exists a C-avoiding path of length ≥ 3 with an end vertex x on C. Let $\mathcal{P}$ be the set of paths of length ≥ 3 from x to a pendant vertex. Let y, z be the other vertices on C. By Lemma 3.4, $m = 2$, and y and z have degree 2. Thus, by Lemma 3.1, the set of cut vertices of $L(G)$ equals

$$\bigcup_{P \in \mathcal{P}} E(P) \setminus \{e_P\}$$

where e_P is an incident with the pendant vertex on P. Thus the subgraph of $L(G)$ induced by its cut vertices is connected.

Let $P = xx_1x_2 \cdots x_k$ for some $k \geq 3$. Then every walk between x_2 and x_3 in $S_2(G)$ contains x_1 and so x_1 is a cut vertex of $S_2(G)$. In addition, every walk between x_1 and x_2 in $S_2(G)$ contains x and so x is a cut vertex of $S_2(G)$. On the other hand, since any vertex adjacent to y or z is a pendant vertex, the neighbors of y form a clique, and so do the neighbors of z. This implies that neither y nor z is a cut vertex. However, every walk between x and x_1 in $S_2(G)$ contains either y or z. Thus the subgraph of $S_2(G)$ induced by its cut vertices contains two disconnected vertices x and x_1. Hence the subgraph of $S_2(G)$ induced by its cut vertices is disconnected. However, we have shown that the subgraph of $L(G)$ induced by its cut vertices is connected, which contradict the hypothesis that $S_2(G) = L(G)$. □

Lemma 3.6. *Suppose that a connected unicyclic graph G with cycle $C = xyzx$ satisfies $S_m(G) = L(G)$ for an even integer m. If P is a longest C-avoiding path with an end vertex x on C, then any C-avoiding paths of length 2 with an end vertex x shares an edge incident to x with P and there is no pendant vertex adjacent to x.*

Proof. By contradiction. Suppose that there exist a C-avoiding path Q of length 2 that is edge-disjoint from P. By Proposition 3.5, the length of P is 2. Let $P = xwv$.

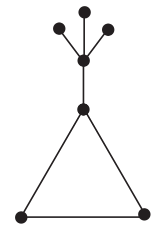

Figure 3: A graph obtained from identifying a vertex of a 3-cycle with a pendant vertex of $K_{1,4}$.

Let u be the other end vertex of Q than x. Then u, v, y are neighbors of x in $S_2(G)$ which form an independent set. Hence $\theta_E(N_{S_2(G)}(x)) \geq 3$, which is impossible for a line graph.

Suppose that there exists a pendent vertex u adjacent to x. By Lemma 3.1, xw is the only cut vertex of $L(G)$. Since x is on every walk connecting w and v in $S_2(G)$, x is a cut vertex of $S_2(G)$. Since $S_2(G) = L(G)$, x corresponds to xw under any isomorphism from $S_2(G)$ to $L(G)$. However, the degree of x in $S_2(G)$ is $k+2$ while the degree of xw in $L(G)$ is at least $k+3$ where k is the number pendant vertices adjacent to w. Hence we reach a contradiction and so there is no pendant vertex adjacent to x. □

Now we are ready to characterize a connected graph G with girth 3 satisfying $S_m(G) = L(G)$.

Theorem 3.7. *Let G be a connected graph with girth 3. Then G satisfies $S_m(G) = L(G)$ for an even integer m if and only if*

(1) G is a 3-cycle, or

(2) $m = 2$ and either G is a spiked odd cycle with a 3-cycle or G is a graph obtained from identifying a vertex of a 3-cycle with a pendant vertex of $K_{1,n}$. (see Figure 3 for an illustration).

Proof. The 'only if' part immediately follows from Proposition 3.5 and Lemmas 3.4 and 3.6. If (1) is true, then it is obvious that $S_m(G) = L(G)$. Now suppose that (2) holds. Suppose that G is a spiked odd cycle with a 3-cycle. Let $v_0v_1v_2v_0$ be the 3-cycle of G and P_i be the set of pendant vertices adjacent to v_i for $i = 0, 1, 2$. Define a map φ from $V(G)$ to $E(G)$ as follows:

$$\varphi(v_i) = v_{i+1}v_{i+2} \text{ (the subscripts are reduced modulo 3);}$$

For each $x \in P_i$,

$$\varphi(x) = v_i x.$$

Then it can easily be checked that φ is a bijection such that $uv \in E(S_2(G))$ if and only if $\varphi(u)\varphi(v) \in E(L(G))$.

Let G be a graph G obtained from identifying a vertex of a 3-cycle with a pendant vertex of $K_{1,n}$. Then it is easy to check that $S_2(G)$ and $L(G)$ both are the graphs obtained from identifying one end of e of $K_4 - e$ and a vertex of K_n. Thus we have shown that the 'if' part is true. $\square$

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