ON ENDOMORPHISMS OF FANO MANIFOLDS OF PICARD NUMBER ONE

JUN-MUK HWANG* AND NOBORU NAKAYAMA

Abstract. Let X be a Fano manifold of Picard number one different from the projective space. It has been conjectured that a surjective endomorphism $X \to X$ must be bijective. In this article, we will prove a weaker version of the conjecture: a surjective endomorphism $X \to X$ which is étale outside a completely invariant divisor is bijective. As applications of this result, the conjecture is confirmed in the case where the variety of minimal rational tangents of X is linear and in the case where X is quasi-homogeneous.

1. Introduction

We will work over the complex number field $\mathbb{C}$. It seems that the following has been a folklore since 1980's.

Conjecture 1.1. Let X be a Fano manifold of Picard number one different from the projective space. Then a surjective endomorphism $X \to X$ must be bijective.

Up to our knowledge, no general strategy to this conjecture has been suggested. Even testing the conjecture for a specifically given Fano manifold of Picard number one is not easy. For that reason, it is worth studying the conjecture with some additional assumptions on X. For example, Conjecture 1.1 was proved for homogeneous spaces in [PS], for hypersurfaces of the projective space in [Be] and for Fano manifolds containing a rational curve with trivial normal bundle in [HM03]; the last work solves Conjecture 1.1 in case $\dim X = 3$.

On the other hand, since Conjecture 1.1 predicts that all surjective endomorphisms are just automorphisms, it is somewhat artificial and aesthetically repulsive to work on the conjecture with additional assumptions on the endomorphism. Notwithstanding this, we will study Conjecture 1.1 for a special class of endomorphisms in this paper. We say that an endomorphism $f\colon X \to X$ is *étale outside a completely invariant divisor* if there exists a reduced divisor $D \subset X$ such that $f^{-1}(D) := f^*(D)_{\mathrm{red}} = D$ and $f|_{X\setminus D}\colon X \setminus D \to X \setminus D$ is étale. We will prove the following.

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Theorem 1.2. *Let X be a Fano manifold of Picard number one different from the projective space. If an endomorphism of X is étale outside a completely invariant divisor, it is bijective.*

In fact, our result is slightly stronger. See Theorem 2.1 for the precise statement.

What is our excuse for making this special assumption on the endomorphism? We believe that Theorem 1.2 will be useful in attacking Conjecture 1.1. It seems that, for many examples of X, the geometry of rational curves on X forces an arbitrary endomorphism $X \to X$ to be étale outside a completely invariant divisor. To illustrate this idea, we will use Theorem 1.2 to prove the following.

Theorem 1.3. *Let X be a Fano manifold of Picard number one different from the projective space. Suppose that the variety of minimal rational tangents of X is linear. Then a surjective endomorphism $X \to X$ is bijective.*

See Section 6 for the meaning of the assumption on the variety of minimal rational tangents. In practice, the only known examples of Fano manifolds of Picard number one whose variety of minimal rational tangents is linear are those having rational curves with trivial normal bundles. For the latter class of Fano manifolds, Theorem 1.3 were already proved in [HM03]. In this sense, Theorem 1.3 is a generalization of a result of [HM03]. However, the proof given here is different from that of [HM03] and conceptually simpler. In particular, the calculation involving discriminantal orders, which was the hardest part in [HM03], is not needed here. Moreover, as far as endomorphisms are concerned, Theorem 1.3 has a theoretical advantage which makes it more useful than [HM03]. As an example, we will use Theorem 1.2 and Theorem 1.3 to prove the following, for which the result of [HM03] is not sufficient.

Theorem 1.4. *Let X be a Fano manifold of Picard number one different from the projective space. Assume that X is quasi-homogeneous, i.e., the connected component $\mathrm{Aut}_o(X)$ of the group of biregular automorphisms of X has an open orbit in X. Then a surjective endomorphism $X \to X$ is bijective.*

This verifies Conjecture 1.1 for quasi-homogeneous cases. Note that quasi-homogeneous Fano manifolds of Picard number one cover a large class of examples, much larger than the homogeneous cases of [PS]. Even when $\mathrm{Aut}_o(X)$ is reductive, this class of Fano manifolds have not yet been classified.

2. Proof of Theorem 1.2

In this section, we will prove the following stronger version of Theorem 1.2.

Theorem 2.1. *Let X be an n-dimensional Fano manifold of Picard number one and $D \subset X$ a reduced divisor. Assume that there exist a non-isomorphic surjective endomorphism $f\colon X \to X$ such that $f^{-1}(D) = D$ and $f|_{X\setminus D}\colon X \setminus D \to X \setminus D$ is étale. Then X is isomorphic to the projective space $\mathbb{P}_n$ and D is a simple normal crossing divisor consisting of $n+1$ hyperplanes.*

Given a reduced divisor D in a projective manifold X, we define the sheaf $\hat{\Omega}^1_X(\log D)$ as follows. Let $U \subset X$ be an open subset with $\operatorname{codim}(X \setminus U) \geq 2$ and $D \cap U$ being a smooth divisor. Denote by $\Omega^1_U(\log(D \cap U))$ the locally free sheaf of germs of logarithmic 1-forms on U with poles only along $U \cap D$. Using the open immersion $j\colon U \subset X$, we define

$$\hat{\Omega}^1_X(\log D) := j_*\Omega^1_U(\log(D \cap U)).$$

This is a reflexive coherent sheaf on X.

Proposition 2.2. *In the setting of* Theorem 2.1*, let A be the ample generator of $\operatorname{Pic}(X) \cong \mathbb{Z}$ and let $q > 1$ be the integer with $f^*A = qA$ in $\operatorname{Pic}(X)$; in particular, $\deg f = q^n$. Then the following hold.*

(i) *$f^*(c_1(A)^i) = q^i c_1(A)^i \in \mathrm{H}^{2i}(X, \mathbb{Z})$ and $f_*(c_1(A)^j) = q^{n-j}c_1(A)^j \in \mathrm{H}^{2j}(X, \mathbb{Z})$ for any $0 \leq i, j \leq n$.*

(ii) *There is a natural isomorphism $f^*\hat{\Omega}^1_X(\log D) \cong \hat{\Omega}^1_X(\log D)$. In particular, $K_X + D = f^*(K_X + D)$.*

(iii) *For any $i > 0$, $c_i(\hat{\Omega}^1_X(\log D))\, c_1(A)^{n-i} = 0$.*

Proof. (i) is direct from the projection formula. (ii) follows from the fact that f is flat, $f^{-1}(D) = D$ and f is étale outside D. For (iii), denoting $\hat{\Omega}^1_X(\log D)$ by $\mathcal{F}$, (ii) gives $f^*(c_i(\mathcal{F})) = c_i(\mathcal{F})$ for any i. Then

$$q^{n-i}c_i(\mathcal{F})c_1(A)^{n-i} = f^*c_i(\mathcal{F})(f^*c_1(A))^{n-i} = (\deg f)c_i(\mathcal{F})c_1(A)^{n-i} = q^n c_i(\mathcal{F})c_1(A)^{n-i}$$

implies $c_i(\mathcal{F})c_1(A)^{n-i} = 0$. □

Proposition 2.3. *In the setting of* Proposition 2.2*, the sheaf $\hat{\Omega}^1_X(\log D)$ is semi-stable with respect to A. In fact, $\hat{\Omega}^1_X(\log D) \cong \mathcal{O}_X^{\oplus n}$.*

Proof. Suppose it is not semi-stable with respect to A. Then, there is a non-zero coherent sheaf $\mathcal{F} \subset \hat{\Omega}^1(\log D)$ such that

$$\mu_A(\mathcal{F}) := \frac{c_1(\mathcal{F})c_1(A)^{n-1}}{\operatorname{rank}\mathcal{F}} > 0.$$

Then $\mu_A(f^*\mathcal{F}) = q\mu_A(\mathcal{F})$ by $f_*(c_1(A)^{n-1}) = qc_1(A) \in \mathrm{H}^2(X, \mathbb{Z})$ and by the projection formula. For the iterated power $f^k = f\circ\cdots\circ f$ ($k \geq 1$) of f, we have $\mu_A((f^k)^*\mathcal{F}) = q^k\mu_A(\mathcal{F})$. Note that the set

$$\{\mu_A(\mathcal{F}) \mid 0 \neq \mathcal{F} \subset \hat{\Omega}^1(\log D)\}$$

is bounded from above. Since $(f^k)^*\mathcal{F} \subset \hat{\Omega}^1_X(\log D)$ by Proposition 2.2 (ii), we get a contradiction.

Now if a reflexive sheaf $\mathcal{G}$ on a projective manifold X is semi-stable with respect to an ample line bundle A, satisfying $c_1(\mathcal{G}) = 0$ and $c_2(\mathcal{G})c_1(A)^{n-2} = 0$, then $\mathcal{G}$ is locally free and there exists a sequence

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_l = \mathcal{G}$$

of subbundles such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is a projectively flat vector bundle with $c_1(\mathcal{E}_i/\mathcal{E}_{i-1}) = 0$ for any $1 \leq i \leq l$ (cf. [Na, IV.4.1]). We can apply it to $\mathcal{G} = \hat{\Omega}^1_X(\log D)$ by Proposition 2.2 (iii). Then, $\hat{\Omega}^1_X(\log D)$ is free, since X is simply connected and

$$\dim \operatorname{Ext}^1_X(\mathcal{O}_X, \mathcal{O}_X) = \dim \mathrm{H}^1(X, \mathcal{O}_X) = 0. \qquad \square$$

The proof of the following result is taken from [NZ, Lemma 5.3 and Proposition 5.4].

Proposition 2.4. *In the setting of* Theorem 2.1*, there is an open subset $U \subset X$ such that $D \cap U$ is a normal crossing divisor and* $\operatorname{codim}(X \setminus U) \geq 3$.

Proof. Let $\nu\colon \widetilde{D} \to D \subset X$ be the normalization of D and $\mathbf{c}$ be the conductor of D, regarded as a Weil divisor on $\widetilde{D}$. The adjunction formula gives

$$K_{\widetilde{D}} + \mathbf{c} = \nu^*(K_X + D).$$

There is an endomorphism $h\colon \widetilde{D} \to \widetilde{D}$ such that $\nu \circ h = f \circ \nu$ and its ramification divisor R_h is $h^*(\mathbf{c}) - \mathbf{c}$. In fact, we have $K_{\widetilde{D}} + \mathbf{c} = h^*(K_{\widetilde{D}} + \mathbf{c})$ from $K_X + D = f^*(K_X + D)$ in Proposition 2.2 (ii). Moreover, $\deg h = q^{n-1}$ by $h^*\nu^*(A) = \nu^* f^*(A) = q\nu^*(A)$, where q is as in Proposition 2.2.

We will show that $\mathbf{c}$ is reduced. Let Γ be an irreducible component of $\mathbf{c}$ and Θ be an irreducible component of $h^{-1}(\Gamma)$. We set $a := \operatorname{mult}_\Theta h^*(\Gamma)$. Then

$$a - 1 = \operatorname{mult}_\Theta(R_h) = \operatorname{mult}_\Theta(h^*\mathbf{c}) - \operatorname{mult}_\Theta(\mathbf{c}) = a\ \operatorname{mult}_\Gamma(\mathbf{c}) - \operatorname{mult}_\Theta(\mathbf{c}).$$

Consequently,

$$\operatorname{mult}_\Theta(\mathbf{c}) - 1 = a\ (\operatorname{mult}_\Gamma(\mathbf{c}) - 1). \tag{1}$$

Thus, Θ is contained in $\mathbf{c}$. By considering the number of irreducible components of $\mathbf{c}$, we infer that $\Theta \mapsto h(\Theta)$ induces a permutation of the set of irreducible components of $\mathbf{c}$. In particular, $h^*(\Gamma) = a\Theta$. Replacing h by some iteration h^m, we may assume that $h^*(\Gamma) = a\Gamma$. Since $h^*\nu^*(A) \sim q\nu^*(A)$ and $\deg h = q^{n-1}$, we have $a = q$ by

$$q^{n-1}\Gamma\nu^*(A)^{n-2} = h^*(\Gamma)h^*\nu^*(A)^{n-2} = aq^{n-2}\Gamma\nu^*(A)^{n-2} > 0.$$

Thus $h^*(\Gamma) = q\Gamma$ and for each positive integer k, $(h^k)^*(\Gamma) = q^k\Gamma$ for the iterated endomorphism h^k. Then

$$\operatorname{mult}_\Gamma(\mathbf{c}) - 1 = q^k(\operatorname{mult}_\Gamma(\mathbf{c}) - 1)$$

by (1). Thus, $\mathrm{mult}_\Gamma(\mathbf{c}) = 1$, proving that $\mathbf{c}$ is reduced.

If a plane curve has a reduced conductor over a singular point, then the singularity is nodal. Hence, D has only normal crossing singularities in codimension one. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let U be the open subset in Proposition 2.4. Since $\mathrm{codim}(X \setminus U) \geq 3$, we have an isomorphism

$$\mathbb{C} \cong \mathrm{H}^1(X, \Omega^1_X) \cong \mathrm{H}^1(U, \Omega^1_U).$$

On the other hand, we have an exact sequence

$$0 \longrightarrow \Omega^1_U \longrightarrow \Omega^1_U(\log(D \cap U)) \longrightarrow \nu_* \mathcal{O}_{\widetilde{D}}|_U \longrightarrow 0$$

for the normalization $\nu\colon \widetilde{D} \to D$. Here, $\widetilde{D}$ is the disjoint union of the normalizations $\widetilde{D}_i$ of irreducible components D_i of D. Thus, $\dim \mathrm{H}^0(U, \nu_* \mathcal{O}_{\widetilde{D}})$ is the number l of irreducible components of D. The connecting homomorphism

$$\mathrm{H}^0(U, \nu_* \mathcal{O}_{\widetilde{D}}) \longrightarrow \mathrm{H}^1(U, \Omega^1_U) \cong \mathrm{H}^1(X, \Omega^1_X)$$

essentially sends a generator 1 of $\mathcal{O}_{\widetilde{D}_i}$ for each component D_i to the first Chern class $c_1(D_i)$. From

$$\dim \mathrm{H}^0(U, \Omega^1_U) = \dim \mathrm{H}^0(X, \Omega^1_X) = 0, \quad \dim \mathrm{H}^1(U, \Omega^1_U) = \dim \mathrm{H}^1(X, \Omega^1_X) = 1,$$

we get

$$l - 1 = \dim \mathrm{H}^0(U, \Omega^1_U(\log(D \cap U))) = \dim \mathrm{H}^0(X, \hat{\Omega}^1_X(\log D)) = n$$

where the last equality is from Proposition 2.3. Since $K_X + D = 0$ by Proposition 2.3, $-K_X = mA$ for some positive integer $m \geq n + 1$. Thus, $X \cong \mathbb{P}_n$ by Kobayashi–Ochiai's criterion [KO]. Moreover, $m = l = n + 1$ implies that each irreducible component D_i is a hyperplane. The normal crossing property of $D = \sum D_i$ is verified in [NZ, Proposition 5.6]. □

3. Free immersed submanifolds with trivial normal bundle

Let X be a non-singular projective variety.

Definition 3.1. A finite morphism $\nu\colon V \to X$ is called an *immersion from a projective manifold* if V is a non-singular projective variety with $\dim V < \dim X$ and ν is unramified and generically injective. If the normal bundle $N_\nu = N_{V/X}$ is a trivial bundle of rank $\dim X - \dim V = \mathrm{codim}\, \nu(V) > 0$ in addition, then ν is called an *immersion with trivial normal bundle*. In this case, the image $\nu(V)$ is called an *immersed submanifold with trivial normal bundle*.

For an immersion $\nu\colon V \to X$ from a projective manifold with trivial normal bundle, we have an exact sequence

$$0 \to T_V \to \nu^* T_X \to N_{V/X} \simeq \mathcal{O}_V^{\oplus c} \to 0 \tag{2}$$

for $c = \operatorname{codim} \nu(V)$. In particular, $K_V = \nu^* K_X$. Therefore, if X is a Fano manifold, then so is V. A projective space $\mathbb{P}_n$ does not have an immersed submanifold with trivial normal bundle since the tangent bundle is ample.

Definition 3.2. Let $\nu\colon \mathcal{V} \to X \times \mathcal{W}$ be a finite morphism for algebraic schemes $\mathcal{V}$ and $\mathcal{W}$. Let $\varphi\colon \mathcal{V} \to X$ and $\pi\colon \mathcal{V} \to \mathcal{W}$ be the morphisms induced by projections. If π is a smooth morphism with connected fibers and $\nu|_{V_w} = \varphi|_{V_w}\colon V_w := \pi^{-1}(w) \to X$ is an immersion, then ν is called a *family of immersions from projective manifolds* parametrized by $\mathcal{W}$

Let $\nu\colon \mathcal{V} \to X \times \mathcal{W}$ be a family of immersions from projective manifolds. Then, we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \pi^*\Omega^1_{\mathcal{W}} & = & \pi^*\Omega^1_{\mathcal{W}} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^\vee_{\mathcal{V}/X\times\mathcal{W}} & \longrightarrow & \nu^*\Omega^1_{X\times\mathcal{W}} & \longrightarrow & \Omega^1_{\mathcal{V}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^\vee_{\mathcal{V}/X\times\mathcal{W}} & \longrightarrow & \nu^*\Omega^1_{X\times\mathcal{W}/\mathcal{W}} & \longrightarrow & \Omega^1_{\mathcal{V}/\mathcal{W}} & \longrightarrow & 0
\end{array} \tag{3}$$

of exact sequences for the dual $N^\vee_{\mathcal{V}/X\times\mathcal{W}}$ of the normal bundle $N_\nu = N_{\mathcal{V}/X\times\mathcal{W}}$. For any point $w \in \mathcal{W}$ and $V_w = \pi^{-1}(w)$, we have $N_{V_w/X} \simeq N_{\mathcal{V}/X\times\mathcal{W}}|_{V_w}$. We can consider ν as a deformation of the holomorphic map $\nu|_{V_w}$ varying the source and fixing the target. In particular, we have the characteristic map

$$T_w(\mathcal{W}) \to \mathrm{H}^0(V_w, N_{V_w/X})$$

for the deformation $\mathcal{V} \to X \times \mathcal{W}$ of the non-degenerate holomorphic map $\nu|_{V_w}$ in the sense of Horikawa [Ho], where $T_w(\mathcal{W})$ denotes the tangent space of $\mathcal{W}$ at w. On the other hand, we can consider the push-forward $\nu_*(\mathcal{V}_w)$ as an algebraic cycle of X associated with the subvariety $\nu(\mathcal{V}_w)$ for any w. Moreover, if $\mathcal{W}$ is normal, then the push-forward $\nu_*(\mathcal{V})$ is regarded as a family of algebraic cycles of X parametrized by $\mathcal{W}$. Thus, in this case, we have an associated morphism from $\mathcal{W}$ to the Chow variety $\mathrm{Chow}(X)$ of X.

Lemma 3.3. *Let $\nu\colon \mathcal{V} \to X \times \mathcal{W}$ be a finite morphism for algebraic schemes $\mathcal{V}$, $\mathcal{W}$, and let $\varphi\colon \mathcal{V} \to X$ and $\pi\colon \mathcal{V} \to \mathcal{W}$ be the morphisms induced from projections. Assume that π is a smooth morphism with connected fibers. For a given point $w \in \mathcal{W}$ and $V = V_w = \pi^{-1}(w)$, the following three conditions are mutually equivalent*:

(i) $\nu|_V\colon V \to X$ *is an immersion with trivial normal bundle and* $\varphi\colon \mathcal{V} \to X$ *is étale at a point of* V.

(ii) $\mathcal{W}$ *is non-singular at* w, ν *is a family of immersions over an open neighborhood of* w *in* $\mathcal{W}$, $\nu|_V$ *has trivial normal bundle, and the characteristic map* $T_w(\mathcal{W}) \to \mathrm{H}^0(V, N_{V/X})$ *is an isomorphism.*

(iii) ν *is a family of immersions over an open neighborhood of* w *in* $\mathcal{W}$ *and* φ *is étale along* V.

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{G}$ be the cokernel of $\mathcal{O}_{X\times\mathcal{W}} \to \nu_*\mathcal{O}_{\mathcal{V}}$. Then $\dim(\mathrm{Supp}(\mathcal{G}) \cap (X \times \{w\})) < \dim V$. Since $\nu|_V$ is unramified, $\mathrm{Supp}(\Omega^1_{\mathcal{V}/(X\times\mathcal{W})}) \cap V = \emptyset$. Thus, there is an open neighborhood $\mathcal{W}'$ of w in $\mathcal{W}$ such that

$$\dim \mathrm{Supp}(\mathcal{G}) \cap (X \times \{w'\}) < \dim V \quad \text{and} \quad \mathrm{Supp}(\Omega^1_{\mathcal{V}/(X\times\mathcal{W})}) \cap \pi^{-1}(w') = \emptyset$$

for any $w' \in \mathcal{W}'$. Thus, ν is a family of immersions over $\mathcal{W}'$. We may replace $\mathcal{W}$ with $\mathcal{W}'$. The composition of the pullback homomorphism $\varphi^*\Omega^1_X \to \nu^*\Omega^1_{X\times\mathcal{W}}$ and an homomorphism $\nu^*\Omega^1_{X\times\mathcal{W}} \to \Omega^1_{\mathcal{V}}$ appearing in the diagram (3) is an isomorphism at a point of V, by assumption. Thus, the composition

$$\Phi\colon N^\vee_{\mathcal{V}/X\times\mathcal{W}} \to \pi^*\Omega^1_{\mathcal{W}}$$

of a homomorphism $N^\vee_{\mathcal{V}/X\times\mathcal{W}} \to \nu^*\Omega^1_{X\times\mathcal{W}}$ in the diagram (3) and the natural projection $\nu^*\Omega^1_{X\times\mathcal{W}} \to \pi^*\Omega^1_{\mathcal{W}}$ is an isomorphism at the point. Therefore, w is a non-singular point of W, since $\pi^*\Omega^1_{\mathcal{W}}$ is a free sheaf at a point over w. Thus, we may assume that $\mathcal{W}$ is non-singular. Then we have an exact sequence

$$0 \to T_{\mathcal{V}} \to \nu^*T_{X\times\mathcal{W}} = \varphi^*T_X \oplus \pi^*T_{\mathcal{W}} \to N_{\mathcal{V}/X\times\mathcal{W}} \to 0 \tag{4}$$

as the dual of the middle exact sequence of the diagram (3). The dual

$$\Phi^\vee\colon \pi^*T_{\mathcal{W}} \to N_{\mathcal{V}/X\times\mathcal{W}}$$

of Φ is related to the characteristic map, i.e., the induced morphism

$$\mathrm{H}^0(\Phi^\vee|_V)\colon T_w(\mathcal{W}) = \mathrm{H}^0(V, \pi^*T_{\mathcal{W}}|_V) \to \mathrm{H}^0(V, N_{V/X}) = \mathrm{H}^0(V, N_{\mathcal{V}/X\times\mathcal{W}}|_V)$$

is just the characteristic map at w for the deformation $\nu\colon \mathcal{V} \to X \times \mathcal{W}$. Since $\Phi^\vee$ is an isomorphism at a point of V, the characteristic map is injective. Since $N_{V/X} \simeq \mathcal{O}_V^{\oplus c}$ for $c = \dim \mathcal{W}$, the characteristic map is an isomorphism.

(ii) $\Rightarrow$ (iii): By assumption, the restriction

$$\Phi^\vee|_V\colon \pi^*T_{\mathcal{W}}|_V = T_w(\mathcal{W}) \otimes \mathcal{O}_V \to N_{\mathcal{V}/X\times\mathcal{W}}|_V = N_{V/X}$$

of $\Phi^\vee$ is an isomorphism. By replacing $\mathcal{W}$ with an open neighborhood of w, we may assume that $\Phi^\vee$ is surjective. Then $\Phi^\vee$ is an isomorphism since the rank of $N_{\mathcal{V}/X\times\mathcal{W}}$ equals $\dim \mathcal{W}$.

Hence, $T_{\mathcal{V}} \to \varphi^* T_X$ is an isomorphism by (4). Since $\mathcal{V}$ is non-singular along V and X are non-singular, φ is an étale morphism along V.

(iii) $\Rightarrow$ (i): We may assume that ν is a family of immersions and that φ is étale. Then $T_{\mathcal{V}} \simeq \varphi^* T_X$. This is equivalent to: $\pi^* T_{\mathcal{W}} \simeq N_{\mathcal{V}/X\times\mathcal{W}}$ by (4). Therefore, $N_{V/X} \simeq T_w(W) \otimes \mathcal{O}_V$. □

A family $\mathcal{V} \to X \times \mathcal{W}$ of immersions from projective manifolds is called *étale* if the first projection $\mathcal{V} \to X$ is étale. Let $\nu = (\varphi, \pi)\colon \mathcal{V} \to X \times \mathcal{W}$ be an étale family of immersions from projective manifolds. Then, $\nu|_{V_w} = \varphi|_{V_w}\colon V_w = \pi^{-1}(w) \to X$ is an immersion with trivial normal bundle for any $w \in \mathcal{W}$ by Lemma 3.3. Moreover, $\mathcal{V} \to X \times \mathcal{W}$ is a versal family of the deformation of $\nu|_{V_w}$ for any $w \in \mathcal{W}$ by Lemma 3.3 and [Ho].

Definition 3.4. A positive-dimensional closed subvariety M of $\mathrm{Chow}(X)$ is called a *component of the locus of free immersed submanifolds with trivial normal bundle* (FIT, for short) if M is the closure of the image of $\mathcal{W} \to \mathrm{Chow}(X)$ induced from an étale family $\mathcal{V} \to X \times \mathcal{W}$ of immersions from projective manifolds. The set of FITs of X is denoted by $\mathrm{FIT}(X)$.

An example of an FIT is provided by the fibers of a surjective morphism $X \to Y$ with $\dim X > \dim Y$. More interesting examples arise from minimal rational curves on Fano manifolds, as we will see in Proposition 6.1. For example, any Fano threefold X of Picard number one, excepting $\mathbb{P}_3$ and the quadric hypersurface $X \subset \mathbb{P}_4$, admits an FIT, as noted in [HM03, p.628].

Theorem 3.5. *Let X be a Fano manifold. Then $\mathrm{FIT}(X)$ is a finite set.*

Proof. We recall the fact that the Fano manifolds of fixed dimension are bounded (cf. [Kr, V.2]). Thus, there exist finitely many smooth families $\Pi_i\colon \mathcal{X}_i \to \mathcal{S}_i$ of Fano manifolds such that, for any $M \in \mathrm{FIT}(X)$, there exist an étale family $\nu = (\varphi, \pi)\colon \mathcal{V} \to X \times \mathcal{W}$ of immersions from projective manifolds defining M as the closure of the image of $\mathcal{W} \to \mathrm{Chow}(X)$, a morphism $\sigma\colon \mathcal{W} \to \mathcal{S}_i$ for some i, and an isomorphism $\mathcal{V} \simeq \mathcal{X}_i \times_{\mathcal{S}_i} \mathcal{W}$ over $\mathcal{W}$. We fix M, $\nu\colon \mathcal{V} \to X \times \mathcal{W}$, and $\sigma\colon \mathcal{W} \to \mathcal{S}_i$. We write $\Pi\colon \mathcal{X} \to \mathcal{S}$ for $\Pi_i\colon \mathcal{X}_i \to \mathcal{S}_i$ for the i. The morphism ν defines a morphism $[\nu]\colon \mathcal{W} \to \mathcal{H}$ into the relative Home scheme $\mathcal{H} := \mathrm{Hom}_{\mathcal{S}}(\mathcal{X}, X \times \mathcal{S})$ over $\mathcal{S}$. Here, $\sigma = q \circ [\nu]$ for the structure morphism $q\colon \mathcal{H} \to \mathcal{S}$. Note that $\mathcal{H}$ is regarded as an open subscheme of the relative Hilbert scheme of $\mathcal{X} \times_{\mathcal{S}} (X \times \mathcal{S}) \simeq \mathcal{X} \times X$ over $\mathcal{S}$. Let

$$\Lambda\colon \mathcal{X} \times_{\mathcal{S}} \mathcal{H} \to (X \times \mathcal{S}) \times_{\mathcal{S}} \mathcal{H} = X \times \mathcal{H}$$

be the universal family for the Hom scheme $\mathcal{H}$ and let

$$\Psi = \Pi \times_{\mathcal{S}} \mathrm{id}_{\mathcal{H}}\colon \mathcal{X} \times_{\mathcal{S}} \mathcal{H} \to \mathcal{H}$$

be the second projection. For a point $t \in \mathcal{H}$, let

$$\Lambda_t \colon \mathcal{X}_t := \Psi^{-1}(t) = (\mathcal{X} \times_{\mathcal{S}} \mathcal{H}) \times_{\mathcal{H}} \{t\} \simeq \Pi^{-1}(q(t)) \to (X \times \mathcal{H}) \times_{\mathcal{H}} \{t\} = X$$

be the base change of Λ by $\{t\} \to \mathcal{H}$.

Claim 3.6. Let $\mathcal{T}$ be the set of points $t \in \mathcal{H}$ such that Λ_t is an immersion with trivial normal bundle. Then $\mathcal{T}$ is an open subset and $\Lambda \colon \Psi^{-1}(\mathcal{T}) \to X \times \mathcal{T}$ is a family of immersions.

Proof. Let t be a point of $\mathcal{T}$. For the proof, we may replace $\mathcal{H}$ with an open neighborhood of t in $\mathcal{H}$. Then Λ is a finite morphism over an open neighborhood of t in $\mathcal{H}$. Moreover, by an argument in the proof of (i) $\Rightarrow$ (ii) of Lemma 3.3, Λ is an immersion over an open neighborhood of t in $\mathcal{H}$. Thus, we may assume that Λ is a family of immersions. It is enough to show that $\mathcal{T}$ is an open subset. For the normal bundle

$$\mathcal{N} := N_{\Lambda} = N_{\mathcal{X} \times_{\mathcal{S}} \mathcal{H}/X \times \mathcal{H}}$$

and for the fiber $\mathcal{X}_t = \Psi^{-1}(t)$, we have an isomorphism $\mathcal{N}|_{\mathcal{X}_t} \simeq N_{\mathcal{X}_t/X}$. Since this is trivial of rank $c := \dim X - \dim \mathcal{X}_t$ and $\mathcal{X}_t$ is Fano, $\mathrm{H}^p(\mathcal{X}_t, \mathcal{N}|_{\mathcal{X}_t}) = 0$ for any $p > 0$. Applying the upper semi-continuity theorem and the base change theorem to Ψ and $\mathcal{N}$, we see that $\Psi_* \mathcal{N}$ is a locally free sheaf of rank c and

$$\Psi_* \mathcal{N} \otimes \mathbb{C}(t) \simeq \mathrm{H}^0(\mathcal{X}_t, \mathcal{N}|_{\mathcal{X}_t}).$$

Thus, $\Psi^* \Psi_* \mathcal{N} \to \mathcal{N}$ is isomorphism along $\mathcal{X}_t$. Therefore, $\Lambda_{t'}$ is an immersion with trivial normal bundle for any point t' of an open neighborhood of t in $\mathcal{W}$. Thus, $\mathcal{T}$ is open. □

Claim 3.7. $\mathcal{T}$ has only finitely many irreducible components.

Proof. We consider the relative ample divisors $-K_{\mathcal{X}}$ on $\mathcal{X}$ and $p_1^*(-K_X)$ on $X \times \mathcal{S}$ respectively with respect to $\mathcal{S}$, where $p_1 \colon X \times \mathcal{S} \to X$ denotes the first projection. These two divisors define a relative ample divisor on $\mathcal{X} \times X$ over $\mathcal{S}$. Note that $-K_{\mathcal{X}_t} = \Lambda_t^*(-K_X)$ for $t \in \mathcal{T}$. By the boundedness of $(-K_V)^{\dim V}$ for the Fano manifolds V, for any $s \in \mathcal{S}$, the open subset $\mathcal{T} \cap q^{-1}(s) \subset \mathrm{Hilb}(\Pi^{-1}(s) \times X)$ is contained in a union of finitely many projective subvarieties. Thus, the closure $\overline{\mathcal{T}} \subset \mathrm{Hilb}_{\mathcal{S}}(\mathcal{X} \times X)$ is proper over $\mathcal{S}$. Hence, the Claim follows. □

Proof of Theorem 3.5 *continued.* By Claim 3.7, there exist finitely many families of immersions $\nu_j = (\varphi_j, \pi_j) \colon \mathcal{U}_j \to X \times \mathcal{T}_j$ $(j = 1, 2, \ldots, m)$ from projective manifolds satisfying the following conditions:

- $\mathcal{T}_j$ is an irreducible algebraic variety for any j.
- $\pi_j \colon \mathcal{U}_j \to \mathcal{T}_j$ is a smooth family of Fano manifolds for any j.
- The restriction $\nu_j|_{\mathcal{U}_t} \colon \mathcal{U}_t \to X$ for $\mathcal{U}_t := \pi_j^{-1}(t)$ is an immersion with trivial normal bundle for any j and any $t \in \mathcal{T}_j$.

- For any $M \in \mathrm{FIT}(X)$, there exist an étale family $\nu\colon \mathcal{V} \to X \times \mathcal{W}$ of immersions from projective manifolds and a morphism $\mathcal{W} \to \mathcal{T}_j$ for some j such that M is the closure of the image of $\mathcal{W} \to \mathrm{Chow}(X)$ and that $\nu\colon \mathcal{V} \to X \times \mathcal{W}$ is just the pullback of ν_j by $\mathcal{W} \to \mathcal{T}_j$.

For a given $M \in \mathrm{FIT}(X)$, let $\nu = (\varphi, \pi)\colon \mathcal{V} \to X \times \mathcal{W}$ and $\mathcal{W} \to \mathcal{T}_j$ be as above. We shall show that M is just the closure of the image of the morphism $\mathcal{T}_j \to \mathrm{Chow}(X)$ associated with the family ν_j of immersions. If this is proved, then the finiteness of $\mathrm{FIT}(X)$ follows. Let $t \in \mathcal{T}_j$ be the image of $w \in \mathcal{W}$ by $\mathcal{W} \to \mathcal{T}_j$ and let V be the fiber $\pi^{-1}(w) = \pi_j^{-1}(t)$. Now, $\nu\colon \mathcal{V} \to X \times \mathcal{W}$ is a versal family of the deformation of the immersion $\varphi|_V\colon V \to X$. Thus, $\mathcal{U}_j \to \mathcal{T}_j$ is isomorphic to the pullback of the versal family on an analytic open neighborhood $\mathcal{T}_j^\sharp$ of t in $\mathcal{T}_j$. Hence, the image of $\mathcal{T}_j^\sharp \to \mathrm{Chow}(X)$ is contained in the image of $\mathcal{W} \to \mathrm{Chow}(X)$. Therefore, the closure of the image of $\mathcal{T}_j \to \mathrm{Chow}(X)$ is M. Thus, we are done. □

4. Divisors univalent with respect to an FIT

We will use the notation of the previous section. Let M be an FIT of X and let $\mathcal{V} \to X \times \mathcal{W}$ be an étale family of immersions from projective manifolds defining M. Let $Z \to M$ be the normalization. Then $\mathcal{W} \to M \subset \mathrm{Chow}(X)$ factors through Z. There is a family $\mathcal{Y}$ of algebraic cycles of X parametrized by Z. For a point $w \in \mathcal{W}$ and its image z in Z, $\mathcal{Y} \cap (X \times \{z\})$ coincides with the cycle $\nu_*(\mathcal{V}_w)$. Let Y be the normalization of the irreducible component of $\mathrm{Supp}(\mathcal{Y})$ which dominates Z. Then we have a finite birational morphism $\mathcal{V} \to Y \times_Z \mathcal{W}$. In particular, $\mathcal{V} \to X \times \mathcal{W}$ is determined by the morphism $\mathcal{W} \to Z$. Let $Y \to X \times Z$ be the induced generically injective morphism. Let $\mu\colon Y \to X$ and $\rho\colon Y \to Z$ be the morphisms induced from the projections. We define $Z_o \subset Z$ to be the maximum open subset such that $\rho\colon Y_o \to Z_o$ is smooth and $\mu\colon Y_o \to X$ is étale for $Y_o := \rho^{-1}(Z_o)$. Then Z_o is a non-singular dense Zariski open subset of Z and $Y_o \to X \times Z_o$ is an étale family of immersions.

Definition 4.1. The morphism $(\mu, \rho)\colon Y \to X \times Z$ above is called the *normalized realization* of M. The étale family $Y_o \to X \times Z_o$ of immersions is called the *smooth realization* of M. A *smooth member* of M means the immersion $\mu\colon \rho^{-1}(z) \to X$ with trivial normal bundle for a point $z \in Z_o$.

Lemma 4.2. *Let X be a Fano manifold of Picard number one. Let $(\mu, \rho)\colon Y \to X \times Z$ be the normalized realization of an FIT of X. Then* $\deg \mu > 1$.

Proof. Assume that μ is birational. Then $\mu\colon Y_o \to X$ is an open immersion. For a non-zero effective Cartier divisor Θ of Z, we have

$$\mu^*\mu_*\rho^*(\Theta) = \rho^*(\Theta) + E$$

for an effective divisor E supported on $Y \setminus Y_o$. The non-zero effective divisor $\mu_*(\rho^*(\Theta))$ is not ample since it does not intersect $\mu(\rho^{-1}(z))$ for a point $z \in Z_o \setminus \Theta$. This is a contradiction. Thus, $\deg \mu > 1$. □

Definition 4.3. Let M be an FIT of X and let $(\mu, \rho)\colon Y \to X \times Z$ be the normalized realization of M.

(i) For a prime divisor H of X, let $\mu^*(H) = H^h + H^v$ be the decomposition into the horizontal part H^h and the vertical part H^v with respect to $\rho\colon Y \to Z$. Then, $\rho(\operatorname{Supp} H^v) \neq Z$ and any irreducible component of H^h dominates Z.

(ii) A prime divisor H is said to be *univalent* with respect to M if H^h is irreducible and μ induces a birational morphism $H^h \to H$.

Lemma 4.4. *Let H be a univalent prime divisor with respect to an FIT M. Then, for a general point $x \in H$, there exists a unique smooth member $\nu\colon V \to X$ of M such that $x \in \nu(V)$ and $\nu(V) \not\subset H$. Moreover, the image $\nu(V)$ is non-singular at x.*

Proof. Let $(\mu, \rho)\colon Y \to X \times Z$ be the normalized realization of M. Let D be an irreducible component of H^v such that $D \cap Y_o \neq \emptyset$. Then $D \cap Y_o = \rho^{-1}(\rho(D) \cap Z_o)$ since ρ is smooth over Z_o. Therefore,

$$\mu^*(H)|_{Y_o} = H^h|_{Y_o} + \rho^*(\Theta)$$

for an effective divisor Θ on Z_o. Let $x \in H \cap \mu(Y_o)$ be a general point. Then $\mu^{-1}(x) \cap H^h = \{y\}$ for a unique point $y \in Y_o$. If $y' \in \mu^{-1}(x) \setminus \{y\}$, then $y' \in \rho^{-1}(\Theta)$, $\rho^{-1}(\rho(y')) \subset H^v$ and $\mu(\rho^{-1}(\rho(y'))) \subset H$. Thus, $\rho^{-1}(\rho(y)) \to X$ is the unique smooth member $\nu\colon V \to X$ of M with $x \in \nu(V)$ and $\nu(V) \not\subset H$. Since $\rho^{-1}(y) \cap \mu^{-1}(x) = \{y\}$, x is a non-singular point of $\nu(V) = \mu(\rho^{-1}(\rho(y)))$. □

Let $f\colon \tilde{X} \to X$ be a generically finite surjective morphism from another projective manifold $\tilde{X}$. We consider pulling back of FITs of X to $\tilde{X}$. Let M be an FIT of X and $(\mu, \rho)\colon Y \to X \times Z$ the normalized realization of M. Let $\tilde{Y}$ be the normalization of an irreducible components of the fiber product $\tilde{X} \times_X Y$ which dominates Y and let $\tilde{\mu}\colon \tilde{Y} \to \tilde{X}$ and $f_Y\colon \tilde{Y} \to Y$ be the induced morphisms. As the Stein factorization of the composition $\rho \circ f_Y\colon \tilde{Y} \to Y \to Z$, we have a proper surjective morphism $\tilde{\rho}\colon \tilde{Y} \to \tilde{Z}$ to a normal projective variety $\tilde{Z}$ with connected fibers and a finite morphism $f_Z\colon \tilde{Z} \to Z$ such that $f_Z \circ \tilde{\rho} = \rho \circ f_Y$.

Lemma 4.5. *In this situation, let $Z_\flat \subset Z_o$ be the maximum open subset over which $\rho \circ f_Y\colon \tilde{Y} \to Z$ is smooth. Set $Y_\flat = \rho^{-1}(Z_\flat)$, $\tilde{Z}_\flat = f_Z^{-1}(Z_\flat)$, and $\tilde{Y}_\flat = f_Y^{-1}(Y_\flat)$. Then $(\tilde{\mu}, \tilde{\rho})\colon \tilde{Y}_\flat \to \tilde{X} \times \tilde{Z}_\flat$ is an étale family of immersions from projective manifolds and the associated morphism $\tilde{Z}_\flat \to \operatorname{Chow}(\tilde{X})$ is generically injective. Let $\tilde{M}$ be the FIT of $\tilde{X}$ defined by $\tilde{Y}_\flat \to \tilde{X} \times \tilde{Z}_\flat$. If f is a finite morphism, then $(\tilde{\mu}, \tilde{\rho})\colon \tilde{Y} \to \tilde{X} \times \tilde{Z}$ is the normalized realization of $\tilde{M}$.*

Proof. By a property of the fiber product, we see that $(\tilde{\mu}, \rho \circ f_Y)\colon \tilde{Y} \to \tilde{X} \times Z$ is generically injective. Thus so is $(\tilde{\mu}, \tilde{\rho})\colon \tilde{Y} \to \tilde{X} \times \tilde{Z}$. Considering the Stein factorization of a smooth morphism, we see that $\tilde{\rho}$ is smooth over $\tilde{Z}_\flat$ and f_Z is étale over $Z_\flat$. Since $\tilde{X} \times_X Y_o \to \tilde{X}$ is étale, $\tilde{Y}_\flat \to \tilde{X} \times_X Y_\flat$ is an isomorphism onto a connected component, and $\tilde{\mu}\colon \tilde{Y}_\flat \to \tilde{X}$ is étale. Therefore, $(\tilde{\mu}, \tilde{\rho})\colon \tilde{Y}_\flat \to \tilde{X} \times \tilde{Z}_\flat$ is an étale family of immersions. We shall show that the associated morphism $\tilde{Z}_\flat \to \mathrm{Chow}(\tilde{X})$ is generically injective. Let $Z'_\flat \subset Z_\flat$ be an open dense subset such that $Z'_\flat \to \mathrm{Chow}(X)$ is injective. Assume that $\tilde{\mu}(\tilde{\rho}^{-1}(z_1)) = \tilde{\mu}(\tilde{\rho}^{-1}(z_2))$ for two points z_1, z_2 of $\tilde{Z}_\flat$ such that $f_Z(z_1)$, $f_Z(z_2) \in Z'_\flat$. There exist points $y_1 \in \tilde{\rho}^{-1}(z_1)$ and $y_2 \in \tilde{\rho}^{-1}(z_2)$ such that $\tilde{\mu}(y_1) = \tilde{\mu}(y_2)$. Thus, y_1 and y_2 define the same point in $\tilde{X} \times_X Y$. Since $\tilde{Y}_\flat$ is a connected component of the fiber product $\tilde{X} \times_X Y_\flat$, we have $y_1 = y_2$ and $z_1 = z_2$. Hence, $\tilde{Z}_\flat \to \mathrm{Chow}(\tilde{X})$ is generically injective.

The closure $\tilde{M}$ of the image of $\tilde{Z}_\flat \to \mathrm{Chow}(\tilde{X})$ is an FIT of $\tilde{X}$. We have a morphism $j\colon \tilde{Z}_\flat \to \overline{Z}$ for the normalization $\overline{Z}$ of $\tilde{M}$. We shall prove that j extends to an isomorphism $\tilde{Z} \to \overline{Z}$ provided that f is finite. It is enough for the proof of the rest. Since f is finite, considering the push-forward f_* of cycles, we have a finite morphism $f_*\colon \overline{Z} \to Z$ such that $f_* \circ j$ coincides with the composition of $f_Z\colon \tilde{Z}_\flat \to Z_\flat$ and $Z_\flat \subset Z$. Thus, j extends to a finite morphism $\tilde{Z} \to \overline{Z}$ since $\tilde{Z} \to Z$ is also a finite morphism. Hence, $\tilde{Z} \simeq \overline{Z}$ since it is a birational morphism of normal projective varieties. Thus, we are done. □

Let $f\colon \tilde{X} \to X$ be a generically finite surjective morphism between non-singular projective varieties. Let $\tilde{X} \to \overline{X} \to X$ be the Stein factorization. The branch locus B of the finite morphism $\overline{X} \to X$ is purely of codimension one. We call B the *branch divisor* of f.

Lemma 4.6. *Let $f\colon \tilde{X} \to X$, M, $(\mu, \rho)\colon Y \to X \times Z$ be the same as before. For the branch divisor B of f, assume that the horizontal part B^h of B with respect to $\rho\colon Y \to Z$ is not zero. Then there is an irreducible component of $\tilde{X} \times_X Y$ such that, for the normalization $\tilde{Y}$ of the component and for the induced morphisms $f_Y\colon \tilde{Y} \to Y$, $f_Z\colon \tilde{Z} \to Z$ as above, the inequality $\deg(\tilde{Y}/Y) > \deg(\tilde{Z}/Z)$ holds.*

Proof. Assume that $\deg(\tilde{Y}/Y) = \deg(\tilde{Z}/Z)$ for any $\tilde{Y}$. Then

$$\tilde{Y}_\flat \simeq Y_\flat \times_{Z_\flat} \tilde{Z}_\flat,$$

since everything is smooth over $Z_\flat$. Thus, $f_Y\colon \tilde{Y}_\flat \to Y_\flat$ is étale over $\rho^{-1}(Z'_\flat)$ for a non-empty Zariski open subset $Z'_\flat \subset Z_\flat$. Since it holds for any $\tilde{Y}$, the divisor $\mu^{-1}(B) \cap Y_\flat$ does not dominate $Z_\flat$. This is a contradiction to the assumption: $B^h \neq 0$. □

Proposition 4.7. *Let X be a Fano manifold of Picard number one. For a given $M \in \mathrm{FIT}(X)$, there exist at most finitely many univalent prime divisors of X with respect to M.*

Proof. Let $(\mu, \rho)\colon Y \to X \times Z$ be the normalized realization of M. We apply Lemmata 4.5 and 4.6 to the generically finite morphism $\mu = f\colon Y = \tilde{X} \to X$. Let $\tilde{Y} \to \tilde{X} \times \tilde{Z}$ be the

same as in Lemma 4.6. Also let the symbol $\flat$ be as in Lemma 4.5. Let U be the open subset $X \setminus \mu(Y \setminus Y_\flat)$. Then $\mu^{-1}(U) \subset Y_\flat$. It is enough to show that $U \cap H = \emptyset$ for any univalent divisor H of X with respect to M. Assume the contrary. Since $\mu^*(H) = H^h + H^v$ and $\deg \mu > 1$, there is an irreducible component $D \subset H^v$ such that $\mu(D) = H$ and $D \cap Y_\flat \neq \emptyset$. Then $D \cap Y_\flat = \rho^{-1}(\Theta \cap Z_\flat)$ for the prime divisor $\Theta = \rho(D)$ since ρ is smooth over $Z_\flat$. There is an irreducible component $\Gamma_\flat$ of $f_Y^*(D \cap Y_\flat)$ such that the closure of $\tilde{\mu}(\Gamma_\flat)$ is H^h for the induced morphism $\tilde{\mu}\colon \tilde{Y} \to \tilde{X}$. Then $\Gamma_\flat \to D \cap Y_\flat$ is a non-birational generically finite morphism by Lemma 4.6. On the other hand, $\Gamma_\flat \to D \cap Y_\flat$ is birational, since

$$\Gamma_\flat \subset H^h \times_X (D \cap Y_\flat) \subset \tilde{X} \times_X Y_\flat$$

and $H^h \to H$ is birational. This is a contradiction. Hence, any univalent divisor with respect to M is an irreducible component of $X \setminus U$. Thus, we are done. □

5. Webs and tangentially special divisors

Let X be a non-singular projective variety.

Definition 5.1. Let M be an FIT and let $\nu = (\mu, \rho)\colon Y_o \to X \times Z_o$ be the smooth realization of M. We call M a *web* if the following two conditions are satisfied for an open dense subset $U_M \subset X$:

(i) For any $y \in \mu^{-1}(U_M) \cap Y_o$, $\mu(y)$ is a non-singular point of $\mu(\rho^{-1}(\rho(y)))$.

(ii) For any point $(y_1, y_2) \in (Y_o \times_X Y_o) \setminus (Y \times_Z Y)$ with $x = \mu(y_1) = \mu(y_2) \in U_M$,

$$\mu_* T_{y_1}(V_1) \cap \mu_* T_{y_2}(V_2) = 0 \quad \text{in} \quad T_x(X), \tag{5}$$

where $V_i = \rho^{-1}(\rho(y_i))$ for $i = 1, 2$.

Remark 5.2. The condition (i) of Definition 5.1 is equivalent to that the projection $\nu(Y_o) \cap (U_M \times Z_o) \to Z_o$ is smooth for the image $\nu(Y_o)$. The condition (ii) of Definition 5.1 is equivalent to that the composition

$$T_{Y_o \times Y_o / Z_o \times Z_o}|_{Y_o \times_X Y_o} \to T_{Y_o \times Y_o}|_{Y_o \times_X Y_o} \to N_{Y_o \times_X Y_o / Y_o \times Y_o} \tag{6}$$

of natural homomorphisms is injective and has maximal rank at any point $(y_1, y_2) \in (Y_o \times_X Y_o) \setminus (Y \times_Z Y)$ with $x = \mu(y_1) = \mu(y_2) \in U_M$. Indeed, the fiber of the homomorphism (6) at $(y_1, y_2) \in (Y_o \times_X Y_o) \setminus (Y \times_Z Y)$ is expressed as

$$T_{y_1}(V_1) \oplus T_{y_2}(V_2) \ni (v_1, v_2) \mapsto \mu_*(v_1) - \mu_*(v_2) \in T_x(X),$$

where $x = \mu(y_1) = \mu(y_2) \in X$ and $V_i = \rho^{-1}(\rho(y_i))$ for $i = 1, 2$. This is because $\mu_*\colon T_{y_i}(Y) \to T_x(X)$ is an isomorphism for $i = 1, 2$ and the tangent space $T_{(x,x)}(\Delta_X)$ of the diagonal locus $\Delta_X \subset X \times X$ at (x, x) is just the diagonal locus of $T_x(X) \oplus T_x(X) = T_{(x,x)}(X \times X)$.

Definition 5.3. Let M be a web and let $(\mu, \rho)\colon Y \to X \times Z$ be the normalized realization of M. A prime divisor H of X is called *tangentially special* if $H^h \neq \emptyset$ (cf. Definition 4.3) and there is a coherent subsheaf $\mathcal{L}$ of $T_X|_H$ of rank one satisfying the following conditions:

(i) $\mathcal{L} \cap T_H = 0$ for the subsheaf $T_H \subset T_X|_H$.

(ii) $\mu_H^*\mathcal{L} \cap (T_{Y_o/Z_o}|_{H_o^h})$ is of rank one on $H_o^h := H^h \cap Y_o$ for the restriction $\mu_H\colon H_o^h \to H$ of μ and for the subsheaf

$$T_{Y_o/Z_o}|_{H_o^h} \subset T_{Y_o}|_{H_o^h} \simeq \mu^*(T_X)|_{H_o^h} \simeq \mu_H^*(T_X|_H).$$

Remark. In the situation of Definition 5.3, the injection $\mathcal{L} \oplus T_H \subset T_X|_H$ is an isomorphism over an open dense subset of H. Thus, $\mathcal{L} \otimes \mathbb{C}(x) = \mathcal{L}_x \otimes_{\mathcal{O}_{H,x}} \mathbb{C}(x)$ is a one-dimensional subspace of $T_x(X)$ for general $x \in X$.

Lemma 5.4. *For a web M and for the open subset U_M in* Definition 5.1*, if H is a tangentially special prime divisor with respect to M with $U_M \cap H \neq \emptyset$, then H is univalent with respect to M.*

Proof. Let H be a tangentially special prime divisor with respect to M. Let $\mathcal{L}$ be the subsheaf of $T_X|_H$ as in Definition 5.3. For a smooth point $x \in H \cap U_M$, assume that the image of $\mathcal{L} \otimes \mathbb{C}(x)$ is a one-dimensional subspace of $T_x(X)$ and that $x = \mu(y_1) = \mu(y_2)$ for two points y_1, $y_2 \in H_o^h$. For $i = 1, 2$, let v_i be a non-zero element of

$$\left(\mu_H^*\mathcal{L} \cap (T_{Y_o/Z_o}|_{H_o^h})\right) \otimes \mathbb{C}(y_i) \subset T_{Y_o/Z_o} \otimes \mathbb{C}(y_i) = T_{y_i}(V_i),$$

where $V_i := \rho^{-1}(\rho(y_i))$. Then the images $\mu_*(v_1)$ and $\mu_*(v_2)$ in $T_x(X)$ are non-zero elements contained in $\mathcal{L} \otimes \mathbb{C}(x)$. Therefore, the equality (5) is not satisfied. Hence, $\rho(y_1) = \rho(y_2)$. Thus, $\mu(V_1) = \mu(V_2)$ is singular at x. This is a contradiction to the condition (i) of Definition 5.1. Therefore, $H^h \to H$ is birational. □

Corollary 5.5. *If X is a Fano manifold of Picard number one, then, for a given web, there exist at most finitely many tangentially special prime divisors.*

Proof. A tangentially special divisor is an irreducible component of $X \setminus U_M$ or univalent with respect to M by Lemma 5.4. Thus, the assertion follows from Proposition 4.7. □

Lemma 5.6. *Let M be a web of X. Let $f\colon \tilde{X} \to X$ be a finite surjective morphism from another non-singular projective variety $\tilde{X}$. Let $\tilde{M}$ be an FIT of $\tilde{X}$ arising from M and f as in* Lemma 4.5*. Then $\tilde{M}$ is a web.*

Proof. Let $U_{\tilde{M}}$ be the open subset $f^{-1}(U_M \setminus B)$ for the branch divisor B of f. Then the condition (i) of Definition 5.1 is satisfied for $\tilde{M}$ and $U_{\tilde{M}}$, since $f^{-1}(X \setminus B) \to X \setminus B$ is étale. Let $(\tilde{\mu}, \tilde{\rho}) : \tilde{Y}_o \to \tilde{X} \times \tilde{Z}_o$ be the smooth realization of $\tilde{M}$. For $(\tilde{y}_1, \tilde{y}_2) \in (\tilde{Y}_o \times_{\tilde{X}} \tilde{Y}_o) \setminus (\tilde{Y} \times_{\tilde{Z}} \tilde{Y})$ with $\tilde{x} = \tilde{\mu}(\tilde{y}_1) = \tilde{\mu}(\tilde{y}_2) \in U_{\tilde{M}}$, if we set $y_1 = f_Y(\tilde{y}_1)$ and $y_2 = f_Y(\tilde{y}_2)$, then $(y_1, y_2) \in$

$(Y_o \times_X Y_o) \setminus (Y \times_Z Y)$ with $x = f(\tilde{x}) = \mu(y_1) = \mu(y_2) \in U_M$. Since f is unramified at $\tilde{x}$, (5) implies that

$$\tilde{\mu}_* T_{\tilde{y}_1}(\tilde{V}_1) \cap \tilde{\mu}_* T_{\tilde{y}_2}(\tilde{V}_2) = 0 \quad \text{in} \quad T_{\tilde{x}}(\tilde{X})$$

where $\tilde{V}_i = \tilde{\rho}^{-1}(\tilde{\rho}(\tilde{y}_i))$ for $i = 1, 2$. Thus, the condition (ii) of Definition 5.1 is also satisfied for $\tilde{M}$. □

Lemma 5.7. *In the situation of* Lemma 5.6*, let $\tilde{H}$ be a prime divisor of $\tilde{X}$ such that $H^h \neq 0$ for $H := f(\tilde{H})$. Either if f is ramified along $\tilde{H}$ or if $f(\tilde{H})$ is tangentially special with respect to M, then $\tilde{H}$ is tangentially special with respect to any web $\tilde{M}$ obtained as in* Lemma 5.6.

Proof. The horizontal part $\tilde{H}^h$ with respect to $\tilde{\rho}$ is not zero for the normalized realization $(\tilde{\mu}, \tilde{\rho})\colon \tilde{Y} \to \tilde{X} \times \tilde{Z}$ of $\tilde{M}$.

Assume that f is ramified along $\tilde{H}$. The kernel $\widetilde{\mathcal{L}}$ of

$$T_{\tilde{X}}|_{\tilde{H}} \to f^* T_X|_{\tilde{H}}$$

is of rank one and $\widetilde{\mathcal{L}} \cap T_{\tilde{H}} = 0$. Let $\tilde{H}^h_\flat$ be the open subset $\tilde{H}^h \cap \tilde{Y}_\flat$. Then

$$\mathrm{Ker}(T_{\tilde{Y}/\tilde{Z}}|_{\tilde{H}^h} \to f_Y^* T_{Y/Z}|_{\tilde{H}^h})|_{\tilde{H}^h_\flat} = \mathrm{Ker}(T_{\tilde{Y}}|_{\tilde{H}^h} \to f_Y^* T_Y|_{\tilde{H}^h})|_{\tilde{H}^h_\flat} \simeq \tilde{\mu}^*_{\tilde{H}} \widetilde{\mathcal{L}}$$

for the morphism $\tilde{\mu}_{\tilde{H}}\colon \tilde{H}^h_\flat \to \tilde{H}$ induced from $\tilde{\mu}$, since $f_Z\colon \tilde{Z}_\flat \to Z_\flat$ and $\tilde{\mu}\colon \tilde{Y}_\flat \to \tilde{Y}$ are étale. Thus, $\tilde{H}$ is tangentially special with respect to $\tilde{M}$.

Next assume that H is tangentially special with respect to M and that f is not ramified along $\tilde{H}$. Let $\tilde{U} \subset \tilde{X}$ be an open subset such that $f\colon \tilde{U} \to X$ is étale and $\tilde{U} \cap \tilde{H} \neq \emptyset$. Let $\mathcal{L}$ be the subsheaf of $T_X|_H$ in Definition 5.3. For the induced morphism $f_H\colon \tilde{H} \to H$, we set

$$\widetilde{\mathcal{L}} := f_H^* \mathcal{L} \cap T_{\tilde{X}}|_{\tilde{H}} \subset f^* T_X|_{\tilde{H}} = f_H^*(T_X|_H).$$

Then $\widetilde{\mathcal{L}} \cap T_{\tilde{H}} = 0$ and $\widetilde{\mathcal{L}}$ is of rank one, since $T_{\tilde{X}} \to f^* T_X$ is isomorphism over $\tilde{U}$. Since $\tilde{Y}_\flat \to Y_\flat$ is étale along $\tilde{\mu}^{-1}(\tilde{U})$, the condition (ii) of Definition 5.3 also holds for $\widetilde{\mathcal{L}}$ and $\tilde{Y} \to \tilde{X} \times \tilde{Z}$. Thus, we are done. □

Theorem 5.8. *Let X be a Fano manifold of Picard number one admitting a web. Then any surjective endomorphism $X \to X$ is bijective.*

Proof. For a web M of X, let E_M be the union of tangentially special prime divisors with respect to M. Then E_M is a divisor by Corollary 5.5. The set of webs of X is a finite set by Proposition 4.7. Thus, the union E of E_M for all the webs M of X is also a divisor. Suppose that there is a surjective endomorphism $f\colon X \to X$ of degree > 1. Then any irreducible component of the ramification divisor of f is contained in E and $f^{-1}(E) \subset E$ by Lemma 5.7. Thus, $f^{-1}(E) = E$ and $f\colon X \setminus E \to X \setminus E$ is étale. Then X is a projective space by Theorem 1.2. This contradicts that X has a web. □

6. Proofs of Theorems 1.3 and 1.4

Let X be a Fano manifold of Picard number one. An irreducible component $\mathcal{K}$ of the space RatCurves(X) of rational curves (in the sense of [Kr]) on X is called a *minimal component* if for a general point $x \in X$, the subscheme $\mathcal{K}_x$ of $\mathcal{K}$ consisting of members passing through x is non-empty and complete. In this case, the subvariety $\mathcal{C}_x$ of the projectivized tangent space $\mathbb{P}T_x(X)$ consisting of the tangent directions at x of members of $\mathcal{K}_x$ is called the *variety of minimal rational tangents* at x (see [HM04] for more details). We say that the variety of minimal rational tangents of X is *linear* if $\mathcal{C}_x$ is a union of linear subspaces of $\mathbb{P}T_x(X)$ for a general $x \in X$. This includes the case when $\mathcal{C}_x$ is a finite set. Then we have the following results from [Hw, Propositions 2.1 and 2.2].

Proposition 6.1. *Let X be a Fano manifold of Picard number one different from the projective space. Suppose that the variety of minimimal tangents of X is linear of dimension $p \geq 0$. Then X has a web M such that the projection $\rho\colon Y_o \to Z_o$ is a $\mathbb{P}_{p+1}$-bundle for the smooth realization $(\mu, \rho)\colon Y_o \to X \times Z_o$ of M.*

Therefore, we have completed the proof of Theorem 1.3 by Theorem 5.8 and Proposition 6.1.

We recall the following result from [HM04, p. 62, Corollary 2].

Proposition 6.2. *Let $f\colon X' \to X$ be a surjective generically finite morphism from a projective manifold X' to a Fano manifold X of Picard number one. Assume that the variety of minimal rational tangents of X is not linear. Then any holomorphic vector field on X' descends to a holomorphic vector field on X such that f is equivariant with respect to the 1-parameter groups of automorphisms of X' and X generated by the holomorphic vector fields.*

Combining Theorem 1.3 and Proposition 6.2, we have the following.

Proposition 6.3. *Let X be a Fano manifold of Picard number one different from the projective space. Let $f\colon X \to X$ be a surjective endomorphism. Then f is equivariant with respect to* $\mathrm{Aut}_o(X)$*, in the sense that it induces a homomorphism $\Phi\colon \mathrm{Aut}_o(X) \to \mathrm{Aut}_o(X)$ such that $f(\sigma x) = \Phi(\sigma) f(x)$ for $\sigma \in \mathrm{Aut}_o(X)$ and $x \in X$.*

Proof. If the variety of minimal rational tangents of X is linear, f is biregular by Theorem 1.3. Otherwise, we apply Proposition 6.2 to get the equivariance. □

A projective manifold X is *quasi-homogeneous* if $\mathrm{Aut}_o(X)$ has an open orbit $X_o \subset X$. The complement of X_o is called the *boundary* of X. Now Proposition 6.3 implies the following, which proves Theorem 1.4 by Theorem 1.2.

Corollary 6.4. *Let X be a quasi-homogeneous Fano manifold of Picard number one with the boundary $E \subset X$. Then any surjective endomorphism $f\colon X \to X$ satisfies $f^{-1}(E) = E$ and $f|_{X\setminus E}$ is étale.*

Let us finish with a final remark. As noted in [HM04, p. 62], the variety of minimal rational tangents is not linear for homogeneous spaces of Picard number one, excepting the projective space. However, it is linear for some quasi-homogeneous Fano manifolds of Picard number one (e.g. [HM03, Corollary 2]). So it is essential to use Theorem 1.3 for the proof of Theorem 1.4.

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(Jun-Muk Hwang) Korea Institute for Advanced Study
Seoul 130-722 Korea
E-mail address: jmhwang@kias.re.kr

(Noboru Nakayama) Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606-8502 Japan
E-mail address: nakayama@kurims.kyoto-u.ac.jp