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A Structure Theory for the Parametric Submodular Intersection Problem

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Abstract

A linearly parameterized polymatroid intersection problem appears in the context of principal partitions. We consider a submodular intersection problem on a pair of strong-map sequences of submodular functions, which is an extension of the linearly parameterized polymatroid intersection problem to a nonlinearly parameterized one. We introduce the concept of a basis-frame on a finite nonempty set V that gives a mapping from the set of all base polyhedra in $\mathbb{R}^V$ into $\mathbb{R}^V$ such that each base polyhedron in $\mathbb{R}^V$ is mapped to one of its bases. We show the existence of a simple universal representation of all optimal solutions of the parameterized submodular intersection problem by means of basis-frames.

1 Introduction

The submodular intersection problem has been recognized as one of the most fundamental unifying problems in combinatorial optimization. It includes the (poly)matroid intersection problem [1] and the submodular function minimization problem [8, 11], and has a lot of applications [13, 17]. This paper deals with a nonlinearly parameterized submodular intersection problem and presents a structure theorem for it.

As an extension of monotone parametric maximum flow problems [3, 5, 10, 12], Iwata, Murota, and Shigeno [9] developed an algorithm for an intersection problem on a pair of nondecreasing and nonincreasing finite-length strong-map sequences of submodular functions, and they clarified an algorithmic advantage of the concept of a strong map. More generally, we consider a nonlinearly parameterized submodular intersection problem that is continuous in the parameter.

Many problems related to principal partitions (see [4, Section 7]), including a lexicographically optimal base problem [3], can be reduced to linearly parameterized submodular intersection problems. Iri and Nakamura [6, 7, 15, 16] developed the principal partition of a pair of polymatroids. In [15], it is shown that there exists a fixed pair of vectors, called a universal pair of bases, in terms of which optimal solutions to the linearly parameterized polymatroid intersection problem for all parameters are given in a simple form. The aim of

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this paper is to extend this result to a nonlinearly parameterized submodular intersection problem.

For a base polyhedron $\mathbf{B}(f) \subseteq \mathbb{R}^V$ associated with a submodular function f defined on the power set 2^V of a finite set V of cardinality n, each total order $\prec$ on V generates an extreme point $y^{\prec}$ of $\mathbf{B}(f)$ by the greedy algorithm [1, 19]. Thus, any point x in the base polyhedron can be generated by a set $\mathcal{O} = \{\prec_i : i \in I\}$ of total orders on V and a coefficient vector $\lambda = (\lambda(i) \in \mathbb{R} : i \in I)$ of a convex combination as $x = \sum_{i \in I} \lambda(i) y^{\prec_i}$, where I is a finite set of indices. We call such a triple $(I, \mathcal{O}, \lambda)$ a *basis-frame* on V. In this paper we will show that there exists a pair of basis-frames which provides us with a simple representation of optimal solutions to the parametric submodular intersection problem given in a nonlinear parametric form. Moreover, we point out a structural property of the parametric submodular function minimization [2, 14] by means of a basis-frame.

This paper is organized as follows. In Section 2 we provide basic results on submodular functions and show optimality conditions for submodular intersection problems. In Section 3 we describe a parametric submodular intersection problem and give some related problems. Section 4 shows our main result, i.e., a structure theorem for parametric submodular intersection problems.

2 Preliminaries

This section is devoted to basic properties about submodular functions. For more information on submodular functions, refer to, e. g., [4].

2.1 Basic Definitions

Let $V = \{1, \cdots, n\}$ for a positive integer n and $f : 2^V \to \mathbb{R}$ be a *submodular function*, i.e., we have

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$$

for each pair of subsets X, $Y \subseteq V$. Additionally, we assume that $f(\emptyset) = 0$. If $f(X) \leq f(Y)$ holds for all X and Y with $X \subseteq Y \subseteq V$, then f is called a *polymatroid rank function*.

For a vector $x \in \mathbb{R}^V$ we denote the component of x on $v \in V$ by $x(v)$. The *submodular polyhedron* $\mathbf{P}(f)$ and the *base polyhedron* $\mathbf{B}(f)$ associated with f are given by

$$\begin{aligned} \mathbf{P}(f) &= \{x \in \mathbb{R}^V : x(X) \leq f(X) \ (\forall X \subseteq V)\}, \\ \mathbf{B}(f) &= \{x \in \mathbb{R}^V : x(X) \leq f(X) \ (\forall X \subseteq V),\, x(V) = f(V)\}, \end{aligned}$$

where $x(X) = \sum_{v \in X} x(v)$ for any $X \subseteq V$. For any vector $y \in \mathbb{R}^V$ we write $\mathbf{P}(y) = \{x \in \mathbb{R}^V : x \leq y\}$. A point in $\mathbf{P}(f)$ is called a *subbase*, a point in $\mathbf{B}(f)$ a *base*, and an extreme point of $\mathbf{B}(f)$ an *extreme base*. For any base $x \in \mathbf{B}(f)$ and any $v \in V$ we have $f(V) - f(V \setminus \{v\}) \leq x(v) \leq f(\{v\})$. Therefore, $\mathbf{B}(f)$ is bounded. Consider any total order $\prec$ on V. The greedy algorithm [1, 19] finds an extreme base $y^{\prec} \in \mathbb{R}^V$ given by

$$y^{\prec}(v) = f(\{u \in V : u \prec v\} \cup \{v\}) - f(\{u \in V : u \prec v\}) \quad (v \in V). \tag{1}$$

Conversely, every extreme base can be obtained in this way. We say that vector $y^{\prec}$ defined by (1) is the *extreme base generated by* $\prec$. Let $\arg\min f \subseteq 2^V$ denote the collection of all the minimizers of f. It can easily be shown that $\arg\min f$ forms a *distributive lattice*, i.e., $\arg\min f$ is closed under the operations of set union and intersection. So there is the unique minimal (maximal) minimizer of f with respect to set inclusion.

We denote by $\mathcal{F}_n$ the collection of all submodular set functions $\widetilde{f} : 2^V \to \mathbb{R}$ such that $\widetilde{f}(\varnothing) = 0$. Recall that $V = \{1, \cdots, n\}$. Suppose $f_1, f_2 \in \mathcal{F}_n$. If $V \supseteq Y \supseteq X$ implies

$$f_1(Y) - f_1(X) \geq f_2(Y) - f_2(X) \tag{2}$$

for all X and Y, we denote this relation by $f_1 \to f_2$ or $f_2 \leftarrow f_1$ and we call the relation $f_1 \to f_2$ a *strong map*. It is easy to show the following property.

Lemma 1 (Topkis [20]) *Suppose $f_1 \to f_2$. Then the minimal (maximal) minimizer of f_1 is contained in the minimal (maximal) minimizer of f_2.*

Let $y_1^{\prec}$ and $y_2^{\prec}$ denote, respectively, the extreme bases of $\mathbf{B}(f_1)$ and $\mathbf{B}(f_2)$ generated by a total order $\prec$ on V. In view of (1) and (2), for any total order $\prec$ on V we have $y_1^{\prec} \geq y_2^{\prec}$ if $f_1 \to f_2$. Conversely, the relation $f_1 \to f_2$ can be characterized by the positional relationship between $y_1^{\prec}$ and $y_2^{\prec}$ for all $\prec$.

Lemma 2 (Iwata, Murota and Shigeno [9]) *The relation $f_1 \to f_2$ holds if and only if $y_1^{\prec} \geq y_2^{\prec}$ holds for all total orders $\prec$ on V.*

For any $X \subseteq V$ we define submodular functions $f^X : 2^X \to \mathbb{R}$ and $f_X : 2^{V \setminus X} \to \mathbb{R}$ by

$$\begin{aligned} f^X(Y) &= f(Y) \qquad (Y \subseteq X), \\ f_X(Y) &= f(Y \cup X) - f(X) \qquad (Y \subseteq V \setminus X). \end{aligned}$$

We call f^X and f_X the *restriction* of f to X and the *contraction* of f by X, respectively. For any $X, Y \subseteq V$ with $X \subset Y$ define a function $f_X^Y : 2^{Y \setminus X} \to \mathbb{R}$ by

$$f_X^Y = (f^Y)_X.$$

Clearly, $f_\varnothing^X = f^X$ and $f_X^V = f_X$.

For any $X \subseteq V$ and $x \in \mathbb{R}^V$ we denote by x^X the subvector $(x(v) : v \in X) \in \mathbb{R}^X$. Suppose that we are given disjoint finite sets U_1 and U_2. For vectors $x_1 \in \mathbb{R}^{U_1}$ and $x_2 \in \mathbb{R}^{U_2}$, the *direct sum* $x_1 \oplus x_2 \in \mathbb{R}^{U_1 \cup U_2}$ is defined by $(x_1 \oplus x_2)(v) = x_1(v)$ if $v \in U_1$ and $x_2(v)$ if $v \in U_2$. For sets $B_1 \subseteq \mathbb{R}^{U_1}$ and $B_2 \subseteq \mathbb{R}^{U_2}$, define their *direct sum* $B_1 \oplus B_2 = \{x_1 \oplus x_2 : x_1 \in B_1,\ x_2 \in B_2\} \subseteq \mathbb{R}^{U_1 \cup U_2}$. We shall use the following two lemmas about basic properties of submodular polyhedra and base polyhedra (see, e.g., [4]).

Lemma 3 *For $X \subseteq Y \subseteq Z \subseteq V$,*

$$\mathbf{B}(f_X^Y) \oplus \mathbf{B}(f_Y^Z) \subseteq \mathbf{B}(f_X^Z).$$

More specifically, $\mathbf{B}(f_X^Y) \oplus \mathbf{B}(f_Y^Z)$ is a face of $\mathbf{B}(f_X^Z)$, which can be represented as

$$\mathbf{B}(f_X^Y) \oplus \mathbf{B}(f_Y^Z) = \mathbf{B}(f_X^Z) \cap \{\widetilde{x} \in \mathbb{R}^{Z \setminus X} : \widetilde{x}(Y \setminus X) = f_X^Z(Y \setminus X)\}.$$

Lemma 4 *Suppose that $X \subseteq V$ and $x \in \mathbb{R}^V$ satisfy $x^X \in \mathbf{B}(f^X)$. Then, $x \in \mathbf{P}(f)$ if and only if $x^{V \setminus X} \in \mathbf{P}(f_X)$.*

2.2 The submodular intersection problem

Suppose ρ, $\sigma \in \mathcal{F}_n$. The *submodular intersection problem* associated with ρ and σ is to find a maximum common subbase, i.e., a common subbase $x \in \mathbf{P}(\rho) \cap \mathbf{P}(\sigma)$ that maximizes $x(V)$. If ρ and σ are polymatroid rank functions, the problem is called a *polymatroid intersection problem.* For any common subbase $x \in \mathbf{P}(\rho) \cap \mathbf{P}(\sigma)$ and any subset $X \subseteq V$, we have $x(V) = x(X) + x(V \setminus X) \leq \rho(X) + \sigma(V \setminus X)$. The inequality is tight for some x and X, i.e.,

Theorem 5 (Edmonds [1]) *For ρ, $\sigma \in \mathcal{F}_n$,*

$$\max\{x(V) : x \in \mathbf{P}(\rho) \cap \mathbf{P}(\sigma)\} = \min\{\rho(X) + \sigma(V \setminus X) : X \subseteq V\}. \tag{3}$$

Define $\sigma^{\#} : 2^V \to \mathbb{R}$ by $\sigma^{\#}(X) = \sigma(V) - \sigma(V \setminus X)$ $(X \subseteq V)$. Since $\rho - \sigma^{\#}$ is submodular, the minimizers of the right hand side of (3) form a distributive lattice. The following lemma gives a sufficient condition for optimality of the submodular intersection problem.

Lemma 6 *Suppose that a pair of $x \in \mathbb{R}^V$ and $X \subseteq V$ satisfies $x^X \in \mathbf{B}(\rho^X) \cap \mathbf{P}(\sigma_{V\setminus X})$ and $x^{V\setminus X} \in \mathbf{P}(\rho_X) \cap \mathbf{B}(\sigma^{V\setminus X})$. Then, x is optimal for $\max\{x(V) : x \in \mathbf{P}(\rho) \cap \mathbf{P}(\sigma)\}$ and $X \in \arg\min(\rho - \sigma^{\#})$.*

PROOF: By Lemma 4 we have $x \in \mathbf{P}(\rho) \cap \mathbf{P}(\sigma)$. Moreover, $x(V) = x(X) + x(V \setminus X) = \rho(X) + \sigma(V \setminus X)$. It follows from Theorem 5 that x maximizes $x(V)$ over $\mathbf{P}(\rho) \cap \mathbf{P}(\sigma)$ and X minimizes $\rho(X) - \sigma^{\#}(X)$. □

The next theorem describes the structure of the optimal solutions of the submodular intersection problem.

Theorem 7 (Iri and Nakamura [6, 7, 15, 16]) *Let $S_1 \subset \cdots \subset S_k$ be any chain of $\arg\min(\rho - \sigma^{\#})$ for an integer $k \geq 1$. For $x \in \mathbb{R}^V$, the following two statements* (7.a) *and* (7.b) *are equivalent:*

(7.a) *x is an optimal solution to $\max\{x(V) : x \in \mathbf{P}(\rho) \cap \mathbf{P}(\sigma)\}$.*

(7.b) *x satisfies:*

$$\begin{aligned} x^{S_1} &\in \mathbf{B}(\rho^{S_1}) \cap \mathbf{P}(\sigma_{V\setminus S_1}), \\ x^{S_{\ell+1}\setminus S_\ell} &\in \mathbf{B}(\rho_{S_\ell}^{S_{\ell+1}}) \cap \mathbf{B}(\sigma_{V\setminus S_{\ell+1}}^{V\setminus S_\ell}) \ \ \text{for each} \ \ \ell = 1, \cdots, k-1, \\ x^{V\setminus S_k} &\in \mathbf{P}(\rho_{S_k}) \cap \mathbf{B}(\sigma^{V\setminus S_k}). \end{aligned} \tag{4}$$

Here, let us see how we can show the implication (7.b) $\Rightarrow$ (7.a). Suppose (7.b) holds for $x \in \mathbb{R}^V$. Combining Lemmas 3 and 4, we have $x \in \mathbf{P}(\rho) \cap \mathbf{P}(\sigma)$. By Lemma 6 with $X = S_k$, it suffices to show $x^{S_k} \in \mathbf{B}(\rho^{S_k}) \cap \mathbf{P}(\sigma_{V\setminus S_k})$. It follows from Lemma 3 that $x^{S_k} \in \mathbf{B}(\rho^{S_k})$. Since $x^{V\setminus S_k} \in \mathbf{B}(\sigma^{V\setminus S_k})$ and $x \in \mathbf{P}(\sigma)$, with the aid of Lemma 4 we obtain $x^{S_k} \in \mathbf{P}(\rho_{V\setminus S_k})$.

In the following, we mainly deal with the case where $k = 2$ in Theorem 7.

3 The Parametric Submodular Intersection Problem

Suppose that we are given an interval K in $\mathbb{R}$ and two collections of submodular functions $\{\rho_\tau \in \mathcal{F}_n : \tau \in K\}$ and $\{\sigma_\tau \in \mathcal{F}_n : \tau \in K\}$ indexed by K such that

- for all $\tau \in K$ and $\tau' \in K$ with $\tau < \tau'$, we have $\rho_\tau \to \rho_{\tau'}$ and $\sigma_\tau \leftarrow \sigma_{\tau'}$;
- for each $X \subseteq V$, $\rho_\tau(X)$ and $\sigma_\tau(X)$ are continuous in τ.

Based on Theorem 5, for all $\tau \in K$ we consider the *parametric submodular intersection problem*

$$\max\{x(V) : x \in \mathbf{P}(\rho_\tau) \cap \mathbf{P}(\sigma_\tau)\} \tag{5}$$

and the associated submodular function minimization problem

$$\min\{\rho_\tau(X) + \sigma_\tau(V \setminus X) : X \subseteq V\}. \tag{6}$$

The parametric submodular intersection problem and related problems have appeared in the literature as follows.

Lexicographically optimal base problems. Consider a submodular function $\rho \in \mathcal{F}_n$ and a positive vector $w \in \mathbb{R}^V$. The *lexicographically optimal base* [3] for ρ and w is the base in $\mathbf{B}(\rho)$ that attains the lexicographic maximum among the sequences obtained from $(x(v)/w(v) : v \in V)$ arranged in the order of nondecreasing magnitude for all bases x. It is also the optimal solution to

$$\min\{\sum_{v \in V} x(v)^2/w(v) : x \in \mathbf{B}(\rho)\}.$$

This quadratic minimization problem was shown to be equivalent to the parametric optimization problem

$$\max\{x(V) : x \in \mathbf{P}(\rho) \cap \mathbf{P}(\tau w)\} \quad (\tau \in \mathbb{R}),$$

which is a special case of problem (5) in which $\rho_\tau = \rho$ and $\sigma = \tau w$ for all $\tau \in \mathbb{R}$.

Linearly parameterized polymatroid intersection problems. Suppose that functions ρ, $\sigma \in \mathcal{F}_n$ are polymatroid rank functions. Note that if $0 \leq \tau \leq \tau'$, then we have a strong map relation $\tau\sigma \leftarrow \tau'\sigma$. Iri and Nakamura [6, 7, 15, 16] studied the following polymatroid intersection problem with a parameter τ that appears linearly:

$$\max\{x(V) : x \in \mathbf{P}(\rho) \cap \mathbf{P}(\tau\sigma)\} \qquad (\tau \geq 0). \tag{7}$$

For vectors r, $s \in \mathbb{R}^V$, we denote by $r \wedge s$ the vector in $\mathbb{R}^V$ defined by

$$(r \wedge s)(v) = \min\{r(v),\ s(v)\} \qquad (v \in V).$$

It is known that the set of optimal solutions to the linearly parameterized polymatroid intersection problem has a simple representation as follows.

Theorem 8 (Nakamura [15]) *There exists a pair of bases $r \in \mathbf{B}(\rho)$ and $s \in \mathbf{B}(\sigma)$ such that for any $\tau \geq 0$, $r \wedge (\tau s)$ is an optimal solution to (7).*

The pair $(r, s) \in \mathbf{B}(\rho) \times \mathbf{B}(\sigma)$ in Theorem 8 is called a *universal pair of bases.* In the next section, we will give a further generalization of this result.

Discrete parametric submodular intersection problems. Suppose that we are given two sequences of submodular functions $\rho_1, \cdots, \rho_q \in \mathcal{F}_n$ and $\sigma_1, \cdots, \sigma_q \in \mathcal{F}_n$ such that $\rho_1 \to \cdots \to \rho_q$ and $\sigma_1 \leftarrow \cdots \leftarrow \sigma_q$. The associated base polyhedra when $n = 2$ and $q = 4$ are illustrated in Figure 1 by thick lines. Iwata, Murota, and Shigeno [9] treated the problem of finding a maximum common subbase $x \in \mathbf{P}(\rho_\ell) \cap \mathbf{P}(\sigma_\ell)$ for each $\ell = 1, \cdots, q$.

Set $K := [1, q] = \{\tau \in \mathbb{R} : 1 \leq \tau \leq q\}$. For any $\ell \in \{1, \cdots, q-1\}$ and any $\tau \in K$ with $\ell \leq \tau \leq \ell + 1$ define functions $\rho_\tau : 2^V \to \mathbb{R}$ and $\sigma_\tau : 2^V \to \mathbb{R}$ by

$$\rho_\tau = (\ell + 1 - \tau)\rho_\ell + (\tau - \ell)\rho_{\ell+1},$$
$$\sigma_\tau = (\ell + 1 - \tau)\sigma_\ell + (\tau - \ell)\sigma_{\ell+1}.$$

Base polyhedra $\mathbf{B}(\rho_\tau)$ and $\mathbf{B}(\sigma_\tau)$ for all $\tau \in [1, q]$ are illustrated in Figure 1. The given discrete problems for $\mathbf{B}(\rho_\ell) \cap \mathbf{B}(\sigma_\ell)$ for $\ell = 1, \cdots, q$ are thus embedded in the continuous problems of the form of (5).

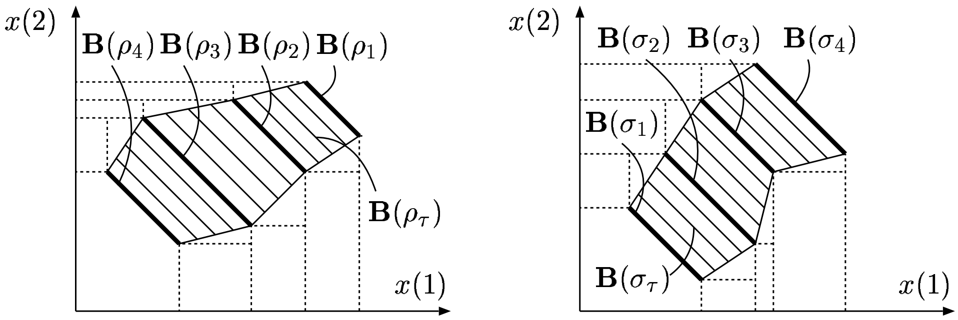

Figure 1: $\mathbf{B}(\rho_\tau)$ and $\mathbf{B}(\sigma_\tau)$ for all $\tau \in [1, q]$

Parametric submodular function minimization. For a sequence of submodular functions $\rho_1, \cdots, \rho_q \in \mathcal{F}_n$ such that $\rho_1 \to \cdots \to \rho_q$, the *parametric submodular function minimization* [2, 14] is the problem of finding a minimizer of ρ_ℓ for each $\ell = 1, \cdots, q$. For each $\ell = 1, \cdots, q$, using Theorem 5 with $\rho = \rho_\ell$ and $\sigma = \mathbf{0}$, we obtain

$$\min\{\rho_\ell(X) : X \subseteq V\} = \max\{x^-(V) : x \in \mathbf{B}(\rho_\ell)\}, \tag{8}$$

where $x^-(V) = \sum_{v \in V} \min\{0, x(v)\}$. Roughly speaking, if $x^-(V)$ is maximized over $\mathbf{B}(\rho_\ell)$, then we can find a minimizer of ρ_ℓ. Hence, the parametric submodular function minimization is a special case of the parametric submodular intersection problem.

4 Structure Theorem

In this section we present a structure theorem for the parametric submodular intersection problem (5).

4.1 Main Result

Consider a pair of triples $T_1 = (I_1, \mathcal{O}_1, \lambda_1)$ and $T_2 = (I_2, \mathcal{O}_2, \lambda_2)$ satisfying the following properties:

- I_1 and I_2 are finite index sets;
- $\mathcal{O}_1 = \{\prec_i : i \in I_1\}$ and $\mathcal{O}_2 = \{\prec_j : j \in I_2\}$ are sets of total orders on V;
- $\lambda_1 \in \mathbb{R}^{I_1}$ and $\lambda_2 \in \mathbb{R}^{I_2}$ satisfy $\lambda_1 \geq \mathbf{0}$, $\lambda_2 \geq \mathbf{0}$ and $\lambda_1(I_1) = \lambda_2(I_2) = 1$.

For all $\tau \in K$, the triples T_1 and T_2, respectively, generate the vectors

$$r_\tau = \sum_{i \in I_1} \lambda_1(i) y_\tau^{\prec_i}, \quad s_\tau = \sum_{j \in I_2} \lambda_2(j) z_\tau^{\prec_j}, \tag{9}$$

where $y_\tau^{\prec_i}$ is an extreme base of $\mathbf{B}(\rho_\tau)$ generated by $\prec_i \in \mathcal{O}_1$ and $z_\tau^{\prec_j}$ is an extreme base of $\mathbf{B}(\sigma_\tau)$ generated by $\prec_j \in \mathcal{O}_2$. Since r_τ is a convex combination of extreme bases of $\mathbf{B}(\rho_\tau)$, we have $r_\tau \in \mathbf{B}(\rho_\tau)$. Similarly, $s_\tau \in \mathbf{B}(\sigma_\tau)$ holds. Thus we can say that T_1 and T_2, respectively, define mappings $\mathbf{B}(\rho_\tau) \mapsto r_\tau \in \mathbf{B}(\rho_\tau)$ and $\mathbf{B}(\sigma_\tau) \mapsto s_\tau \in \mathbf{B}(\sigma_\tau)$ for each $\tau \in K$, through (9). We call such triples $T_1 = (I_1, \mathcal{O}_1, \lambda_1)$ and $T_2 = (I_2, \mathcal{O}_2, \lambda_2)$ *basis-frames* on V.

Lemma 9 *For vectors r_τ ($\tau \in K$) and s_τ ($\tau \in K$) defined in* (9), *the inequality $\tau < \tau'$ implies that $r_\tau \geq r_{\tau'}$ and $s_\tau \leq s_{\tau'}$.*

PROOF: The present lemma follows from (9) and Lemma 2. □

A typical example of a basis-frame appears in game theory. Let $\mathcal{O} = \{\prec_i : i \in I\}$ be the set of all total orders on V and hence the index set I has cardinality $n!$. Define $\lambda(i) = 1/n!$ for all $i \in I$. Then $T = (I, \mathcal{O}, \lambda)$ is a basis-frame on V, which gives a base for each base polyhedron that is known as the Shapley-value vector ([18, 19]) in cooperative n-person (convex) games. Note that base polyhedra are exactly cores in convex games (see [19]). It should be noted here that the basis-frames required in the sequel have index sets I of cardinality at most $n = |V|$.

We will show the existence of a pair of basis-frames T_1 and T_2 on V which generates optimal solutions to (5) for all $\tau \in K$. For some basis-frames T_1 and T_2, $r_\tau \wedge s_\tau$ is always a maximum common subbase of $\mathbf{P}(\rho_\tau)$ and $\mathbf{P}(\sigma_\tau)$ for all $\tau \in K$, which is our main result given as follows.

Theorem 10 *There exists a pair of basis-frames $T_1 = (I_1, \mathcal{O}_1, \lambda_1)$ and $T_2 = (I_2, \mathcal{O}_2, \lambda_2)$ on V such that $|I_1| \leq n$, $|I_2| \leq n$, and for all $\tau \in K$ $r_\tau \wedge s_\tau$ is a maximum common subbase of $\mathbf{P}(\rho_\tau)$ and $\mathbf{P}(\sigma_\tau)$, i.e., an optimal solution to* $\max\{x(V) : x \in \mathbf{P}(\rho_\tau) \cap \mathbf{P}(\sigma_\tau)\}$, *where r_τ and s_τ are defined by* (9).

This is a natural generalization of Theorem 8 for the linearly parameterized polymatroid intersection problem. When specified to problem (7), basis-frames T_1 and T_2 in Theorem 10, respectively, give r_τ and s_τ such that for all $\tau \geq 0$ we have $r_\tau = r$ and $s_\tau = \tau s$ for fixed bases $r \in \mathbf{B}(\rho)$ and $s \in \mathbf{B}(\sigma)$. The proof of Theorem 10 will be given in §4.2. Using this result, we immediately obtain a structure theorem for the parametric submodular function minimization problem as follows.

Corollary 11 *Suppose we are given submodular functions $\rho_1, \cdots, \rho_q \in \mathcal{F}_n$ satisfying the strong map relation $\rho_1 \to \cdots \to \rho_q$. Then, there exists a basis-frame $T = (I, \mathcal{O}, \lambda)$ on V such that $|I| \le n$ and $r_\ell := \sum_{i \in I} \lambda(i) y_\ell^{\prec_i}$ is an optimal solution to $\max_x\{x^-(V) : x \in \mathbf{B}(\rho_\ell)\}$ for each $\ell = 1, \cdots, q$, where $y_\ell^{\prec_i}$ is the extreme base of $\mathbf{B}(\rho_\ell)$ generated by $\prec_i \in \mathcal{O}$.*

PROOF: Since the parametric submodular function minimization can be converted to a parametric submodular intersection problem, the present corollary follows from Theorem 10. □

Before going into details of the proof of Theorem 10, let us consider the case where there is a real $\tau^* \in K$ such that $\mathbf{B}(\rho_{\tau^*}) \cap \mathbf{B}(\sigma_{\tau^*}) \ne \varnothing$. Then, we can see that a pair of basis-frames T_1 and T_2 of Theorem 10 can readily be constructed as follows. Choose a common base $w^* \in \mathbf{B}(\rho_{\tau^*}) \cap \mathbf{B}(\sigma_{\tau^*})$. Clearly, there is a pair of basis-frames $T_1 = (I_1, \mathcal{O}_1, \lambda_1)$ and $T_2 = (I_2, \mathcal{O}_2, \lambda_2)$ such that the vectors $r_\tau \in \mathbb{R}^V$ $(\tau \in K)$ and $s_\tau \in \mathbb{R}^V$ $(\tau \in K)$ defined in (9) satisfy $r_{\tau^*} = s_{\tau^*} = w^*$. By Carathéodory's Theorem, we can assume that $|I_1| \le n$ and $|I_2| \le n$. From Lemma 9, the inequality $\tau \le \tau^*$ implies $s_\tau \le w^* \le r_\tau$, and the inequality $\tau^* \le \tau$ implies $r_\tau \le w^* \le s_\tau$. Thus for all $\tau \in K$ we have

$$x_\tau := r_\tau \wedge s_\tau = \begin{cases} s_\tau \ \in \mathbf{B}(\sigma_\tau) & \text{if } \tau \le \tau^*, \\ r_\tau \ \in \mathbf{B}(\rho_\tau) & \text{if } \tau^* \le \tau. \end{cases}$$

Therefore, we obtain $x_\tau(V) = \min\{\rho_\tau(\varnothing) + \sigma_\tau(V),\ \rho_\tau(V) + \sigma_\tau(\varnothing)\}$, so that x_τ is an optimal solution to (5) for all $\tau \in K$. Hence, Theorem 10 is proved in this case. You might think that this setting is too simple, but we will see later that it is quite meaningful.

4.2 Representation of optimal solutions

Now we present a constructive proof of Theorem 10.

Minimizers of the parameterized minimization problem

We begin with an investigation into the parameterized minimization problem (6). W.l.o.g. we assume that $K \subseteq \mathbb{R}$ is of the form $(\kappa_1, \kappa_2] = \{\tau \in \mathbb{R} : \kappa_1 < \tau \le \kappa_2\}$. For all $\tau \in K$, define a parameterized submodular function $f_\tau : 2^V \to \mathbb{R}$ as $f_\tau = \rho_\tau - \sigma_\tau^\#$, that is,

$$f_\tau(X) = \rho_\tau(X) + \sigma_\tau(V \setminus X) - \sigma_\tau(V) \ (X \subseteq V).$$

By assumption, for each $X \subseteq V$ $f_\tau(X)$ is continuous in τ. For all $\tau \in K$, let $\mathcal{L}(\tau) = \arg\min f_\tau \subseteq 2^V$ and let $M_\tau \subseteq V$ be the unique minimal minimizer of f_τ.

Lemma 12 *For $\tau,\ \tau' \in K$ with $\tau' \le \tau$ we have $M_{\tau'} \subseteq M_\tau$.*

PROOF: It is easy to see that $f_{\tau'} \to f_\tau$. So, it follows from Lemma 1 that $M_{\tau'} \subseteq M_\tau$. □

Lemma 13 *For any $\tau \in K$ there exists some real number $\varepsilon > 0$ such that for all τ' with $\tau - \varepsilon < \tau' < \tau$ we have $M_{\tau'} = M_\tau$. That is to say, M_τ is left-continuous in τ.*

PROOF: Choose any $\tau \in K$. If $M_\tau = \varnothing$, the assertion follows from Lemma 12. So, we assume that $M_\tau \ne \varnothing$. Define

$$\delta = \min\{f_\tau(X) - f_\tau(Y) : X, Y \subseteq V,\ f_\tau(X) - f_\tau(Y) > 0\}.$$

Then, since $f_\tau(M_\tau) \neq f_\tau(\varnothing)$, we have $\delta > 0$. Because of the continuity of $f_\tau(X)$ in τ, we can choose a real number $\varepsilon > 0$ such that for all τ' with $\tau - \varepsilon < \tau' < \tau$ and any $X \subseteq V$, it holds that $|f_{\tau'}(X) - f_\tau(X)| < \frac{1}{2}\delta$. Take $\tau' \in K$ with $\tau - \varepsilon < \tau' < \tau$. Then we have

$$f_\tau(M_{\tau'}) - \tfrac{1}{2}\delta < f_{\tau'}(M_{\tau'}) \leq f_{\tau'}(M_\tau) < f_\tau(M_\tau) + \tfrac{1}{2}\delta,$$

which implies $M_{\tau'} \in \mathcal{L}(\tau)$. Therefore, we have $M_{\tau'} \supseteq M_\tau$. On the other hand, $M_{\tau'} \subseteq M_\tau$ follows from Lemma 12. Hence we obtain $M_{\tau'} = M_\tau$. □

By Lemma 12, there exist finitely many distinct M_τ ($\tau \in K$), which we suppose are given by

$$X_0 \subset X_1 \subset \cdots \subset X_d.$$

From Lemmas 12 and 13, $K \subseteq \mathbb{R}$ is divided into the intervals

$$K_0 = (\kappa_1,\, \tau_1],\ K_1 = (\tau_1,\, \tau_2],\ \cdots,\ K_{d-1} = (\tau_{d-1},\, \tau_d],\ K_d = (\tau_d,\, \kappa_2]$$

such that for any $\ell = 0, \cdots, d$ and any $\tau \in K_\ell$ we have $M_\tau = X_\ell$. We call τ_ℓ ($\ell = 1, \cdots, d$) *lower critical values*. To avoid messy discussions, we assume that $X_0 = \varnothing$ and $X_d = V$.

Lemma 14 *For any $\ell = 1, \cdots, d$, $X_{\ell-1} \subset X_\ell$ is a chain of $\mathcal{L}(\tau_\ell)$.*

PROOF: Take any $\ell \in \{1, \cdots, d\}$. Since $\tau_\ell \in K_{\ell-1}$, we have $X_{\ell-1} \in \mathcal{L}(\tau_\ell)$. For a sufficiently small $\varepsilon > 0$, $M_{\tau_\ell+\varepsilon} = X_\ell \in \mathcal{L}(\tau_\ell + \varepsilon)$. Due to the continuity of $f_\tau(X)$ in τ for all $X \subseteq V$, we obtain $X_\ell \in \mathcal{L}(\tau_\ell)$ by a similar argument as in the proof of Lemma 13. □

Construction and correctness

We first present a way of constructing r_τ, s_τ, T_1, and T_2 in Theorem 10, whose correctness will be verified later.

Take any $\ell \in \{1, \cdots, d\}$. For all $\tau \in K$, we define functions $\rho_\tau^{(\ell)} : 2^{X_\ell \setminus X_{\ell-1}} \to \mathbb{R}$ and $\sigma_\tau^{(\ell)} : 2^{X_\ell \setminus X_{\ell-1}} \to \mathbb{R}$ by

$$\rho_\tau^{(\ell)} = (\rho_\tau)_{X_{\ell-1}}^{X_\ell}, \quad \sigma_\tau^{(\ell)} = (\sigma_\tau)_{V \setminus X_\ell}^{V \setminus X_{\ell-1}}.$$

Recall that $X_{\ell-1} \subset X_\ell$ is a chain of $\mathcal{L}(\tau_\ell) = \arg\min(\rho_{\tau_\ell} - \sigma_{\tau_\ell}^{\#})$ (see Lemma 14). By Theorem 7 with $k = 2$, there exists a vector $w^{(\ell)} \in \mathbb{R}^{X_\ell \setminus X_{\ell-1}}$ such that

$$w^{(\ell)} \in \mathbf{B}(\rho_{\tau_\ell}^{(\ell)}) \cap \mathbf{B}(\sigma_{\tau_\ell}^{(\ell)}).$$

Then $w^{(\ell)}$ can be expressed by a convex combination of at most $|X_\ell \setminus X_{\ell-1}|$ extreme bases of $\mathbf{B}(\rho_{\tau_\ell}^{(\ell)})$ (resp. $\mathbf{B}(\sigma_{\tau_\ell}^{(\ell)})$). The following argument is similar to that of the latter part of §4.1. We can give a pair of basis-frames $T_1^{(\ell)} = (I_1^{(\ell)}, \mathcal{O}_1^{(\ell)}, \lambda_1^{(\ell)})$ and $T_2^{(\ell)} = (I_2^{(\ell)}, \mathcal{O}_2^{(\ell)}, \lambda_2^{(\ell)})$ on $X_\ell \setminus X_{\ell-1}$ which generates the vectors $r_\tau^{(\ell)} \in \mathbf{B}(\rho_\tau^{(\ell)})$ ($\tau \in K$) and $s_\tau^{(\ell)} \in \mathbf{B}(\sigma_\tau^{(\ell)})$ ($\tau \in K$) such that $r_{\tau_\ell}^{(\ell)} = s_{\tau_\ell}^{(\ell)} = w^{(\ell)}$, where $|I_1^{(\ell)}| \leq |X_\ell \setminus X_{\ell-1}|$ and $|I_2^{(\ell)}| \leq |X_\ell \setminus X_{\ell-1}|$.

By Lemma 9, the inequality $\tau \leq \tau_\ell$ implies $s_\tau^{(\ell)} \leq w^{(\ell)} \leq r_\tau^{(\ell)}$, and the inequality $\tau_\ell \leq \tau$ implies $r_\tau^{(\ell)} \leq w \leq s_\tau^{(\ell)}$. Thus we have

$$r_\tau^{(\ell)} \wedge s_\tau^{(\ell)} = \begin{cases} s_\tau^{(\ell)} \in \mathbf{B}(\sigma_\tau^{(\ell)}) & \text{if } \tau \leq \tau_\ell, \\ r_\tau^{(\ell)} \in \mathbf{B}(\rho_\tau^{(\ell)}) & \text{if } \tau_\ell \leq \tau. \end{cases} \tag{10}$$

For all $\tau \in K$, we set

$$r_\tau := r_\tau^{(1)} \oplus \cdots \oplus r_\tau^{(d)} \in \mathbb{R}^V, \quad (11)$$

$$s_\tau := s_\tau^{(1)} \oplus \cdots \oplus s_\tau^{(d)} \in \mathbb{R}^V, \quad (12)$$

$$x_\tau := r_\tau \wedge s_\tau \in \mathbb{R}^V. \quad (13)$$

Lemma 15 *For all $\tau \in K$, we have $r_\tau \in \mathbf{B}(\rho_\tau)$, $s_\tau \in \mathbf{B}(\sigma_\tau)$, and $x_\tau \in \mathbf{P}(\rho_\tau) \cap \mathbf{P}(\sigma_\tau)$.*

PROOF: By Lemma 3, for all $\tau \in K$ we have $\mathbf{B}(\rho_\tau^{(1)}) \oplus \cdots \oplus \mathbf{B}(\rho_\tau^{(d)}) \subseteq \mathbf{B}(\rho_\tau)$ and $\mathbf{B}(\sigma_\tau^{(1)}) \oplus \cdots \oplus \mathbf{B}(\sigma_\tau^{(d)}) \subseteq \mathbf{B}(\sigma_\tau)$. Thus, $r_\tau \in \mathbf{B}(\rho_\tau)$ and $s_\tau \in \mathbf{B}(\sigma_\tau)$. Therefore, $x_\tau = r_\tau \wedge s_\tau$ is a common subbase of $\mathbf{P}(\rho_\tau)$ and $\mathbf{P}(\sigma_\tau)$. □

The next lemma guarantees that among basis-frames T_1 and T_2 on V which generate the vectors r_τ and s_τ, respectively, we can choose the ones with index sets of small size.

Lemma 16 *There exists a basis-frame $T_1 = (I_1, \mathcal{O}_1, \lambda_1)$ such that T_1 generates r_τ in* (11) *for all $\tau \in K$ and $|I_1| \leq n$, and there exists a basis-frame $T_2 = (I_2, \mathcal{O}_2, \lambda_2)$ such that T_2 generates s_τ in* (12) *for all $\tau \in K$ and $|I_2| \leq n$.*

PROOF: It suffices to show the former statement. (The latter can be shown similarly.) We will prove the existence of the required basis-frame T_1 by combining $T_1^{(1)} = (I_1^{(1)}, \mathcal{O}_1^{(1)}, \lambda_1^{(1)})$, $\cdots$, $T_1^{(d)} = (I_1^{(d)}, \mathcal{O}_1^{(d)}, \lambda_1^{(d)})$. Note that

$$|I_1^{(1)}| + |I_1^{(2)}| + \cdots + |I_1^{(d)}| \leq |X_1| + |X_2 \setminus X_1| + \cdots + |X_d \setminus X_{d-1}| = n. \quad (14)$$

For each $v \in V$, let $h(v)$ be a number in $\{1, \cdots, d\}$ such that $v \in X_{h(v)} \setminus X_{h(v)-1}$. Given $\prec^{(1)} \in \mathcal{O}_1^{(1)}, \cdots, \prec^{(d)} \in \mathcal{O}_1^{(d)}$, the concatenation of $\prec^{(1)}, \cdots, \prec^{(d)}$ is a total order $\prec$ on V such that

$$u \prec v \quad \text{if } h(u) < h(v) \quad \text{or} \quad \text{if } h(u) = h(v) \text{ and } u \prec^{(h(u))} v.$$

The following procedure Combine constructs a desired basis-frame $T_1 = (I_1, \mathcal{O}_1, \lambda_1)$.

Procedure Combine($T_1^{(1)}, \cdots, T_1^{(d)}$)
  Initialize $I_1 := \varnothing$ and $\mathcal{O}_1 := \varnothing$;
  While $I_1^{(1)} \neq \varnothing$ **do begin**
    For $\ell := 1$ **to** d **do**: Choose an index $i_\ell \in I_1^{(\ell)}$;
    Let i be a new index and let $\prec_i$ be the concatenation of $\prec_{i_1}^{(1)}, \cdots, \prec_{i_d}^{(d)}$;
    Set $I_1 := I_1 \cup \{i\}$, $\mathcal{O}_1 := \mathcal{O}_1 \cup \{\prec_i\}$ and $\mu_i := \min\{\lambda_1^{(\ell)}(i_\ell) : 1 \leq \ell \leq d\}$;
    For $\ell := 1$ **to** d **do**:
      Set $\lambda_1^{(\ell)}(i_\ell) := \lambda_1^{(\ell)}(i_\ell) - \mu_i$;
      If $\lambda_1^{(\ell)}(i_\ell) = 0$ then $I_1^{(\ell)} := I_1^{(\ell)} \setminus i_\ell$;
  end;
  Let $\lambda_1 \in \mathbb{R}^{I_1}$ be a vector such that $\lambda_1(i) = \mu_i$ for each $i \in I_1$;
  Output $T_1 := (I_1, \mathcal{O}_1, \lambda_1)$;

In the execution of the procedure, observe that $I_1^{(1)}$ becomes empty if and only if $I_1^{(\ell)}$ becomes empty for all $\ell = 1, \cdots, d$. Let $\gamma = |I_1^{(1)}| + \cdots + |I_1^{(d)}|$. At the beginning of the execution of Combine, we have $\gamma \leq n$ because of (14). Since γ decreases by at least 1 in each iteration, finally we obtain $|I_1| \leq n$. In view of (1), the triple T_1 obtained by the procedure Combine is really a basis-frame that generates the vector r_τ in (11) for all $\tau \in K$. □

Finally, we show the correctness of r_τ, s_τ, T_1, and T_2 constructed as above.

PROOF OF THEOREM 10: To show the assertion, it is sufficient to prove that for all $\tau \in K$ the vector x_τ defined by (13) is optimal for $\max\{x(V) : x \in \mathbf{P}(\rho_\tau) \cap \mathbf{P}(\sigma_\tau)\}$. Therefore, by Lemma 6 with $X = M_\tau$, it suffices to show that for all $\tau \in K$

$$x_\tau^{M_\tau} \in \mathbf{B}(\rho_\tau^{M_\tau}) \cap \mathbf{P}((\sigma_\tau)_{V \setminus M_\tau}), \tag{15}$$

$$x_\tau^{V \setminus M_\tau} \in \mathbf{P}((\rho_\tau)_{M_\tau}) \cap \mathbf{B}(\sigma_\tau^{V \setminus M_\tau}). \tag{16}$$

Suppose $\tau \in K_0$. Then $M_\tau = \varnothing$ and hence trivially (15) holds. It follows from (10) that $x_\tau = s_\tau \in \mathbf{B}(\sigma_\tau)$. Moreover $x_\tau \in \mathbf{P}(\rho_\tau)$ holds, so that (16) also holds. If $\tau \in K_d$, we obtain (15) and (16) in a similar way.

We assume that $\tau \in K_\ell$ for some $\ell \in \{1, \cdots, d-1\}$. Since

$$\tau_1 \leq \cdots \leq \tau_\ell \leq \tau \leq \tau_{\ell+1} \leq \cdots \leq \tau_d,$$

it follows from (10) that

$$x_\tau^{M_\tau} = r_\tau^{(1)} \oplus \cdots \oplus r_\tau^{(\ell)} \in \mathbf{B}(\rho_\tau^{(1)}) \oplus \cdots \oplus \mathbf{B}(\rho_\tau^{(\ell)}),$$
$$x_\tau^{V \setminus M_\tau} = s_\tau^{(\ell+1)} \oplus \cdots \oplus s_\tau^{(d)} \in \mathbf{B}(\sigma_\tau^{(\ell+1)}) \oplus \cdots \oplus \mathbf{B}(\sigma_\tau^{(d)}).$$

By Lemma 3, we have $\mathbf{B}(\rho_\tau^{(1)}) \oplus \cdots \oplus \mathbf{B}(\rho_\tau^{(\ell)}) \subseteq \mathbf{B}(\rho_\tau^{M_\tau})$ and $\mathbf{B}(\sigma_\tau^{(\ell+1)}) \oplus \cdots \oplus \mathbf{B}(\sigma_\tau^{(d)}) \subseteq \mathbf{B}(\sigma_\tau^{V \setminus M_\tau})$. Hence, $x_\tau^{M_\tau} \in \mathbf{B}(\rho_\tau^{M_\tau})$ and $x_\tau^{V \setminus M_\tau} \in \mathbf{B}(\sigma_\tau^{V \setminus M_\tau})$. Recall that $x_\tau \in \mathbf{P}(\rho_\tau) \cap \mathbf{P}(\sigma_\tau)$. By Lemma 4 with $x = x_\tau$, $f = \rho_\tau$, and $X = M_\tau$, we have $x_\tau^{V \setminus M_\tau} \in \mathbf{P}((\rho_\tau)_{M_\tau})$. Similarly, by Lemma 4 with $x = x_\tau$, $f = \sigma_\tau$, and $X = V \setminus M_\tau$, we have $x_\tau^{M_\tau} \in \mathbf{P}((\sigma_\tau)_{V \setminus M_\tau})$. We thus obtain (15) and (16). □

5 Concluding Remarks

We have introduced the new concept of a basis-frame and have shown that the set of optimal solutions to a parametric submodular intersection problem for all parameters has a compact representation by means of basis-frames, which is a natural generalization of the results of [6, 7, 15, 16] for linearly parameterized polymatroid intersection problems. In addition, as a corollary we have observed a structure theorem for parametric submodular function minimization problems.

Apparently, the nonlinearity in the parameter makes the problem complicated. Hence, the present paper has reinforced the importance of strong map relations of submodular functions from a structural viewpoint, where an important rôle is played by the concept of a basis-frame.

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