KUMMER'S QUARTICS AND NUMERICALLY REFLECTIVE INVOLUTIONS OF ENRIQUES SURFACES

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Abstract. A (holomorphic) automorphism of an Enriques surface S is said to be *numerically reflective* if it acts on the cohomology group $H^2(S, \mathbb{Q})$ by reflection. We shall show that there are two lattice-types of numerically reflective involutions, and describe one type geometrically in terms of curves of genus 2 and Göpel subgroups of their Jacobians.

An automorphism of an Enriques surface S is *numerically trivial* if it acts on the cohomology group $H^2(S, \mathbb{Q}) \simeq \mathbb{Q}^{10}$ trivially. By [MN] and [M06], there are three types of numerically trivial involutions. An involution of S is *numerically reflective* if it acts on $H^2(S, \mathbb{Q})$ by reflection, that is, the eigenvalue -1 is of multiplicity one. In this article, we shall study numerically reflective involutions as the next case of the classification of involutions of Enriques surfaces.

We first explain an example, with which we started our investigation. Let C be a (smooth projective) curve of gnus 2 and $J = J(C)$ be its Jacobian variety. As is well known the quotient variety $J(C)/\{\pm 1_J\}$ is realized as a quartic surface with 16 nodes in $\mathbb{P}^3$, called *Kummer's quartic*. The minimal resolution of $J(C)/\{\pm 1_J\}$ is called *the Jacobian Kummer surface* of C and denoted by $Km\, C$.

Let $G \subset J(C)_{(2)}$ be a *Göpel subgroup* such that the four associated nodes $\bar{G} \subset J(C)/\{\pm 1_J\}$ are linearly independent in $\mathbb{P}^3$ (Proposition 5.2). Then the equation of Kummer's quartic $J(C)/\{\pm 1_J\} \subset \mathbb{P}^3$ referred to the four nodes has the form

$$(1) \qquad a(x^2t^2 + y^2z^2) + b(y^2t^2 + z^2x^2) + c(z^2t^2 + x^2y^2) + dxyzt$$

$$+f(yt + zx)(zt + xy) + g(zt + xy)(xt + yz) + h(xt + yz)(yt + zx) = 0$$

for constants $a, \dots, h \in \mathbb{C}$ by Hutchinson[H01]. The standard Cremona transformation

$$(x : y : z : t) \mapsto (x^{-1} : y^{-1} : z^{-1} : t^{-1})$$

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of $\mathbb{P}^3$ leaves the quartic invariant and induces a (holomophic) involution of $Km\,C$, which we denote by ε_G. As is observed in [Keu, §3], ε_G has no fixed points and the quotient $(Km\,C)/\varepsilon_G$ is an Enriques surface (Proposition 5.1 and Remark 5.3).

Projecting $J(C)/\{\pm 1_J\}$ from a node, we obtain a morphism of degree 2 from $Km\,C$ to $\mathbb{P}^2$. The covering involution of the projection $Km\,C \to \mathbb{P}^2$ from $(0:0:0:1)$, one of the four nodes, is given by

$$\beta : (x:y:z:t) \mapsto (x:y:z:\frac{q(yz,xz,xy)}{tq(x,y,z)}), \tag{2}$$

where we put

$$q(x,y,z) = ax^2 + by^2 + cz^2 + fyz + gxz + hxy.$$

The involution β commutes with ε_G and descends to an involution of the Enriques surface $(Km\,C)/\varepsilon_G$, which we denote by σ_G.

Our main purpose of this article is to characterize $((Km\,C)/\varepsilon_G, \sigma_G)$ as an Enriques surface with an involution. The following was found in our study of numerically trivial involutions:

Proposition 1. *σ_G is numerically reflective.*

Let S be an Enriques surface and $\tilde{S}$ the covering K3 surface of S. We denote the covering involution of $\tilde{S}/S$ by ε and the anti-invariant part of the cohomological action $\varepsilon \curvearrowright H^2(\tilde{S},\mathbb{Z})$ by $H^-(S,\mathbb{Z})$. Let σ be an involution of S. Then σ induces an involution σ_K^* of $H^-(S,\mathbb{Z})$ preserving a non-zero 2-form (Proposition 2.1).

If σ is numerically reflective, then the invariant part $H(S,\sigma;\mathbb{Z})$ of $\sigma_K^* \curvearrowright H^-(S,\mathbb{Z})$ is isomorphic to either $\langle -4\rangle \perp U(2) \perp U(2)$ or $\langle -4\rangle \perp U(2) \perp U$ as a lattice (Proposition 3.2). The above involution σ_G satisfies the former (Proposition 6.5). The following is our main theorem:

Theorem 2. *Let σ be a numerically reflective involution of an Enriques surface S such that $H(S,\sigma;\mathbb{Z})$ is isomorphic to $\langle -4\rangle \perp U(2) \perp U(2)$. Then the pair (S,σ) is isomorphic to $((Km\,C)/\varepsilon_G, \sigma_G)$ in Proposition 1.*

The case of $\langle -4\rangle \perp U(2) \perp U$ will be discussed elsewhere.

A Jacobiam Kummer surface $Km\,C$ is expressed as the intersection of three diagonal quadrics

$$\sum_{i=1}^{6} x_i^2 = \sum_{i=1}^{6} \lambda_i x_i^2 = \sum_{i=1}^{6} \lambda_i^2 x_i^2 = 0$$

in $\mathbb{P}^5$ for mutually distinct six constants $\lambda_1, \dots, \lambda_6$. Hence we have 10 fixed-point-free involutions, *e.g.*,

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (x_1 : x_2 : x_3 : -x_4 : -x_5 : -x_6),$$

corresponding to the 10 odd theta characteristics of C. A Jacobian Kummer surface $Km\, C$ has exactly 15 Göpel subgroups. A general $Km\, C$ is expressed as the quartic Hessian surfaces in 6 ways ([H99], [DK]) and has 6 fixed-point-free involutions of Hutchinson-Weber type.

Conjecture 3. *If the Picard group of $J(C)$ is infinitely cyclic, then a fixed-point-free involution ε of $Km\, C$ is conjugate to one of the above* $31(= 10 + 15 + 6)$ *involutions.*

In the situation of the conjecture, ε commutes with the covering involution β, and the quotient group

$$H_{\varepsilon=-1}(Km\, C, \mathbb{Z})/[H_{\varepsilon=-1,\beta=1}(Km\, C, \mathbb{Z}) + H_{\varepsilon=-1,\beta=-1}(Km\, C, \mathbb{Z})]$$

is of order 4. Our proof of Theorem 2 shows that the conjecture holds true when this (abelian) group is of type $(2, 2)$.

After a preparation on Kummer and Enriques surfaces in §§1 and 2, we compute the period of a numerically reflective involution in §§3 and 4. In §5, we construct a Hutchinson-Göpel involution ε_G of a Jacobian Kummer surface from its planar description. In §6, we compute the period of the Enriques surface $(Km\, C)/\varepsilon_G$ more explicitly, and prove Theorem 2 using an equivariant Torelli theorem for Enriques surfaces (Theorem 2.3).

Notations. For an abelian group A, $A_{(2)}$ denotes the 2-torsion subgroup. A free $\mathbb{Z}$-module with an integral symmetric bilinear form is simply called *a lattice*. U denotes the rank 2 lattice given by the symmetric matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. A_l, D_l and E_l are the *negative* definite root lattices of rank l of type A, D and E, respectively. $L(k)$ is the lattice obtained from a lattice L by replacing the bilinear form $(\ .\ )$ with $r(\ .\ )$, r being a rational number

1. Preliminary

We recall some basic facts on the cohomology of Kummer surfaces. Let T be a 2-dimensional complex torus. The minimal resolution of the quotient $T/\{\pm 1_T\}$ is called the Kummer surface of T and denoted by $Km(T)$. $Km(T)$ contains mutually disjoint $(-2)\mathbb{P}^1$'s N_a parametrized by the 2-torsion subgroup $T_{(2)} \simeq (\mathbb{Z}/2\mathbb{Z})^4$ of T. We denote by Γ_{KM} the primitive hull of the lattice generated by the 16 N_a's. Let Λ be the orthogonal complement of Γ_{KM} in $H^2(Km(T), \mathbb{Z})$. Then Λ is the image of $H^2(T,\mathbb{Z})$ by the quotient morphism from the blow-up of T at $T_{(2)}$ onto $Km(T)$. The following is well known ([BPV, Chap. VIII, §5]):

Lemma 1.1. *$\Lambda \subset H^2(Km(T))$ is isomorphic to $H^2(T,\mathbb{Z})$ as a Hodge structure and to $H^2(T,\mathbb{Z})(2) \simeq U(2) \perp U(2) \perp U(2)$ as a lattice.*

The discriminant group A_Λ of Λ is $(\frac{1}{2}\Lambda)/\Lambda \simeq H^2(T,\mathbb{Z}/2\mathbb{Z})$ and the discriminant form q_Λ is essentially the cup product, that is, $q_\Lambda(\bar{y}) = (y \cup y)/2 \bmod 2$ for $y \in H^2(T,\mathbb{Z})$.

Let $P \subset T_{(2)}$ be a subgroup of order 4, or equivalently, a 2-dimensional subspace of $T_{(2)}$ over the finite field $\mathbb{F}_2$. We put $N_{P'} = \sum_{a \in P'} N_a \in \Gamma_{Km}$ for a coset P' of $P \subset T_{(2)}$. We denote the Plücker coordinate of $P^\perp \subset T_{(2)}^\vee$ by $\pi_P \in \bigwedge^2 T_{(2)}^\vee \simeq H^2(T,\mathbb{Z}/2\mathbb{Z})$ and regard it as an element of $\Lambda/2\Lambda$. The following is known ([BPV, Chap. VIII, §5]):

Lemma 1.2. *$(N_{P'} \bmod 2) + \pi_P = 0$ holds in $H^2(Km(T),\mathbb{Z}/2\mathbb{Z})$ for every coset P' of $P \subset T_{(2)}$.*

Let (A,Θ) be a *principally polarized* abelian surface, that is, Θ is an ample divisor with $(\Theta^2) = 2$. The orthogonal complement of $[\Theta]$ in $H^2(A,\mathbb{Z})$ is equipped with a polarized Hodge structure. We denote it by $H(A,\Theta;\mathbb{Z})$. As a lattice it is isomorphic to $\langle -2\rangle \perp U \perp U$.

Proposition 1.3. *A polarized Hodge structure of weight 2 on the lattice $\langle -2\rangle \perp U \perp U$ is isomorphic to $H(A,\Theta;\mathbb{Z})$ for a principally polarized abelian surface (A,Θ). Moreover, such (A,Θ) is unique up to an isomorphism.*

Proof. A Hodge structure of weight 2 on the lattice $U \perp U \perp U$ is isomorphic to $H^2(T,\mathbb{Z})$ for a 2-dimensional complex torus T. Moreover, such T is unique up to an isomorphism and the dual ([Sh]). Our proposition is a direct consequence of these results. □

Let $A_{(2)} \times A_{(2)} \to \mathbb{Z}/2\mathbb{Z}$ be the Weil pairing with respect to 2Θ.

Definition 1.4. A subgroup G of the 2-torsion group $A_{(2)}$ is *Göpel* if it is of order 4 and totally isotropic with respect to the Weil pairing.

Let $P \subset A_{(2)}$ be a subgroup of order 4 and $\pi_P \in H^2(A,\mathbb{Z}/2\mathbb{Z})$ be the Plücker coordinate of $P^\perp$. π_P belongs to $H(A,\Theta;\mathbb{Z}/2\mathbb{Z})$ if and only if it is perpendicular to $\Theta \bmod 2$. Hence we have

Lemma 1.5. *The Plücker coordinate π_P belongs to $H(A,\Theta;\mathbb{Z}/2\mathbb{Z})$ if and only if P is Göpel.*

The Jacobian $J(C)$ of a curve C of genus 2 is a principally polarized abelian surface. An involution γ of C is called *bi-eliptic* if the quotient

C/γ is an elliptic curve E. E is embedded into $J(C)$ as the fixed locus of the action of γ on $J(C)$. The 2-torsion subgroup $E_{(2)}$ is a Göpel subgroup of $J(C)$, and denoted by G_γ.

Definition 1.6. A Göpel subgroup G, or more precisely, a pair (C, G) is *bi-elliptic* if C has a bi-elliptic involution γ with $G = G_\gamma$.

2. Involutions of Enriques surfaces

Let S be a (minimal) *Enriques surface*, that is, a compact complex surface with $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$ and $2K_S \sim 0$. Let $\tilde{S}$ and ε be as in Introduction. Consider the cohomological action of the involution ε on $H^2(\tilde{S}, \mathbb{Z}) \simeq \mathbb{Z}^{22}$. The invariant part coincides with the pull-back of $H^2(S, \mathbb{Z})$ by $\pi : \tilde{S} \to S$, and the anti-invariant part, denoted by $H^-(S, \mathbb{Z})$, is isomorphic to $E_8(2) \perp U(2) \perp U$ as a lattice ([BPV, Chap. VIII]).

Let σ be a (holomorphic) involution of S. σ is lifted to an automorphism $\tilde{\sigma}$ of the covering K3 surface $\tilde{S}$. Its square $\tilde{\sigma}^2$ is either the identiy or the covering involution ε. The latter is impossible since the order of $\tilde{\sigma}$ divides $\chi(\mathcal{O}_{\tilde{S}}) = 2$. Hence σ is lifted to two involutions $\tilde{\sigma}$ and $\tilde{\sigma}\varepsilon$ of $\tilde{S}$.

An involution of a K3 surface is called *symplectic* (resp. *anti-symplectic*) if it acts trivially (resp. as -1) on the space $H^0(\tilde{S}, \Omega^2)$ of holomorphic 2-forms. Distinguishing the two lifts by their actions on 2-forms, we have

Proposition 2.1. *There exists a unique symplectic involution σ_K of $\tilde{S}$ and a unique anti-symplectic one σ_R which lift σ.*

Let $H(S, \sigma; \mathbb{Z})$ (resp. $H_-(S, \sigma; \mathbb{Z})$) be the invariant (resp. the anti-invariant) part of the action of σ_K^* on $H^-(S, \mathbb{Z})$. $H(S, \sigma; \mathbb{Z})$ is endowed with a non-trivial polarized Hodge structure of weight 2, which we regard as the period of (S, σ). The lattice $H^-(S, \mathbb{Z})$ contains the orthogonal direct sum $H(S, \sigma; \mathbb{Z}) \perp H_-(S, \sigma; \mathbb{Z})$ as a sublattice of finite index. More precisely, the quotient group

$$D_\sigma := H^-(S, \mathbb{Z})/[H_-(S, \sigma; \mathbb{Z}) \oplus H(S, \sigma; \mathbb{Z})] \tag{3}$$

is 2-elementary. We call this quotient D_σ *the patching group* of σ.

The global Torelli theorems for K3 and Enriques surfaces are generalized to that for pairs of a K3 surface (resp. an Enriques surface) and an involution.

Theorem 2.2. *Let (X, τ) and (X', τ') be pairs of a K3 surface and its involution. If there exists an orientation preserving Hodge isometry*

$\alpha : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$ such that the diagram

$$\begin{array}{ccccc} & H^2(X',\mathbb{Z}) & \xrightarrow{\alpha} & H^2(X,\mathbb{Z}) & \\ \tau'^* & \downarrow & & \downarrow & \tau^* \\ & H^2(X',\mathbb{Z}) & \xrightarrow{\alpha} & H^2(X,\mathbb{Z}) & \end{array}$$

commutes, then there exists an isomorphism $\varphi : X \to X'$ such that $\varphi \circ \tau = \tau' \circ \varphi$.

Proof. If neither τ nor τ' has a base point, this is the global Torelli theorem for Enriques surfaces. The proof in [BPV, Chap. VIII, §21], especially its key Proposition (21.1), works in our general case too, as follows.

Let h' be a τ'-invariant ample divisor class of X' and put $h = \alpha(h')$. By our assumption, h is τ'-invariant and belongs to the positive cone of $H^{1,1}(X,\mathbb{Z})$. If h is ample, we are done by the global Torelli theorem for K3 surfaces. If not, there exists a $(-2)\mathbb{P}^1$ D with $(h.D) < 0$. By the Hodge index theorem, we have $(D.\tau(D)) = 1, 0$ or -2. Replace α with $r_{D+\tau(D)} \circ \alpha$ if $(D, \tau(D)) = 1$, with $r_D \circ r_{\tau(D)} \circ \alpha$ if $(D, \tau(D)) = 0$ and with $r_D \circ \alpha$ if $(D, \tau(D)) = -2$, where r_D is the reflection with respect to a (-2) divisor class D. Then we have $(\alpha(h').D) > 0$. Repeating this process, $\alpha(h')$ becomes ample after a finitely many steps. □

Theorem 2.3. *Let (S,σ) and (S',σ') be pairs of an Enriques surface and its involution. If there exists an orientation preserving Hodge isometry $\alpha : H^-(S',\mathbb{Z}) \to H^-(S,\mathbb{Z})$ such that the diagram*

$$\begin{array}{ccccc} & H^-(S',\mathbb{Z}) & \xrightarrow{\alpha} & H^-(S,\mathbb{Z}) & \\ \sigma'^* & \downarrow & & \downarrow & \sigma^* \\ & H^-(S',\mathbb{Z}) & \xrightarrow{\alpha} & H^-(S,\mathbb{Z}) & \end{array}$$

commutes, then there exists an isomorphism $\varphi : S \to S'$ such that $\varphi \circ \sigma = \sigma' \circ \varphi$.

Proof. Let $\tilde{S}$ and $\tilde{S}'$ be the covering K3 surfaces. Each has an action of $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. It suffices to show a G-equivariant Torelli theorem for K3 surfaces $\tilde{S}$ and $\tilde{S}'$. (The remaining part is the same as the usual global Torelli theorem for Enriques surfaces.) The proof goes as the preceding theorem if the G-orbit of D consists of one or two irreducible components. Assume that the the G-orbit of D has four irreducible components and let L be the sublattice spanned by them. If L is negative definite, then L is of type $4A_1$ or $2A_2$. Hence the same argument as the preceding theorem works. Otherwise $(h.D)$ is positive by the Hodge index theorem. □

3. Period of a numerically reflective involution

In this section and the next we assume that σ is numerically reflective and study the patching group D_σ in (3) in detail.

Let $H^2(S,\mathbb{Z})_f$ be the torsion free part of $H^2(S,\mathbb{Z})$. σ acts on $H^2(S,\mathbb{Z})_f$ by reflection with respect to a class $e = e_\sigma$. Since $H^2(S,\mathbb{Z})_f$ is an even unimodular lattice with respect to the intersection form, we have $(e^2) = -2$. Let N_R and N_K be the anti-invariant part of the action of σ_R and σ_K, respectively. Both N_R and N_K contains the pull-back $\tilde{e} \in H^2(\tilde{S},\mathbb{Z})$ of e. The orthogonal complement of $\tilde{e}$ in N_R is $H(S,\sigma;\mathbb{Z})$, and that in N_K is $H_-(S,\sigma;\mathbb{Z})$. Since N_K is isomorphic to $E_8(2)$ ([Mo, §5], [MN, Lemma 2.1]), we have

Lemma 3.1. *$H_-(S,\sigma;\mathbb{Z}) \simeq E_7(2)$.*

In particular, the discriminant group A_- of $H_-(S,\sigma;\mathbb{Z})$ is $u(2)^{\perp 3} \perp (4)$, whose underlying group is $(\mathbb{Z}/2\mathbb{Z})^{\oplus 6} \oplus (\mathbb{Z}/4\mathbb{Z})$.

There are two lattice-types of numerically reflective involutions:

Proposition 3.2. *The patching group D_σ is of order 2^a and $H(S,\sigma;\mathbb{Z})$ is isomorphic to $\langle -4\rangle \perp U(2) \perp U(a)$ for $a = 1$ or 2.*

Proof. The lattice $H_-(S,\sigma;\mathbb{Z})$ is not 2-elementary by the above lemma. Since $H^-(S,\mathbb{Z})$ is 2-elementary, D_σ is not trivial. Let $a \geq 1$ be the length of the patching group D_σ. Then we have

$$[\operatorname{disc} H^-(S,\mathbb{Z})] \cdot 2^{2a} = [\operatorname{disc} H_-(S,\sigma;\mathbb{Z})] \cdot [\operatorname{disc} H(S,\sigma;\mathbb{Z})].$$

The discriminant group of $H^-(S,\mathbb{Z})$ is an abelian groups of type (2^{10}). By the above lemma, the discriminant of $H(S,\sigma;\mathbb{Z})$ equals $-2^{(2+2a)}$. More precisely, the discriminant group A_+ of $H(S,\sigma;\mathbb{Z})$ is an abelian group of type $(2^{2a},4)$. Since $H(S,\sigma;\mathbb{Z})$ is of rank 5, we have $a \leq 2$.

If $a = 2$, then $H(S,\sigma;\mathbb{Z})(1/2)$ is an even (integral) lattice with discriminant -2. Hence $H(S,\sigma;\mathbb{Z}))(1/2)$ is isomorphic to $\langle -2\rangle \perp U \perp U$ by Nikulin's uniqueness theorem of indefinite lattices. If $a = 1$, then we have $H(S,\sigma;\mathbb{Z}) \simeq \langle -4\rangle \perp U(2) \perp U$ by the uniqueness theorem again. □

The lattice $H^-(S,\mathbb{Z})$ is a $\mathbb{Z}$-submodule of the direct sum $H_-(S,\sigma;\mathbb{Q}) \oplus H(S,\sigma;\mathbb{Q})$. Hence the patching group D_σ is a subgroup of the discriminant group $A_- \perp A_+$ of the lattice $H_-(S,\sigma;\mathbb{Z}) \perp H(S,\sigma;\mathbb{Z})$. The discriminant group A_+ is either $u(2)^{\perp 2} \perp (4)$ or $u(2) \perp (4)$.

Both A_- and A_+ contains exactly one copy of $\mathbb{Z}/4\mathbb{Z}$ as their direct summand. Let $\zeta_\pm \in A_\pm$ be the unique element which is twice an element $\eta_\pm$ of order 4. We call $(\zeta_-,\zeta_+) \in A_- \perp A_+$ *the canonical element.*

Lemma 3.3. *D_σ contains the canonical element (ζ_-, ζ_+).*

Proof. Both $H_-(S, \sigma; \mathbb{Q})$ and $H(S, \sigma; \mathbb{Q})$ are primitive in $H^-(S, \mathbb{Q})$. Hence D_σ does not contain $(0, \zeta_+)$ or $(\zeta_-, 0)$. Hence the intersection $D_\sigma \cap (2A_- \oplus 2A_+)$ is either 0 or generated by (ζ_-, ζ_+). We consider the intersection number of an element of D_σ and (η_-, η_+). Since the intersection number of (ζ_-, ζ_+) and (η_-, η_+) is zero (in $\mathbb{Q}/\mathbb{Z}$), the intersection number with (η_-, η_+) is a linear form on $\bar{D}_\sigma$, the image of D_σ in $A := (A_-)_{(2)}/\{0, \zeta_-\} \oplus (A_+)_{(2)}/\{0, \zeta_+\}$. Since the induced bilinear form on the group A is non-degenerate, there exists an element $(\beta_-, \beta_+) \in A_- \oplus A_+$ whose intersection number with D_σ is the same as (η_-, η_+). It follows that $(\eta_- + \beta_-, \eta_+ + \beta_+)$ is perpendicular to D_σ. Since $D_\sigma^\perp / D_\sigma$ is 2-elementary, $2 \times (\eta_- + \beta_-, \eta_+ + \beta_+) = (\zeta_-, \zeta_+)$ is contained in D_σ. □

The patching group D_σ is generated by the canonical element (ζ_-, ζ_+) when it is of order 2.

Lemma 3.4. *D_σ is generated by the canonical element and an element $(\pi_-, \pi_+) \in A_- \oplus A_+$ of order 2 such that $q_-(\pi_-) = q_+(\pi_+) = 0 \in \mathbb{Q}/2\mathbb{Z}$ if it is of order 4, where $q_\pm$ are the quadratic forms on $A_\pm$.*

Proof. $D_\sigma \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is generated by (ζ_-, ζ_+) and an element (π_-, π_+). Since D_σ is totally isotropic, we have $q_-(\pi_-) = q_+(\pi_+)$. This common value belongs to $\mathbb{Z}/2\mathbb{Z}$. If it is non-zero, replace $\pi_\pm$ with $\zeta_\pm + \pi_\pm$. Then we have $q_-(\pi_-) = q_+(\pi_+) = 0$. □

4. Numerically reflective involution with $\operatorname{ord} D_\sigma = 4$

Let σ be a numerically reflective involution of an Enriques surface S and assume that the patching group D_σ is of order 4. By Propositions 1.3 and 3.2, we have

Proposition 4.1. *There exists a principally polarized abelian surface (A, Θ) such that $H(S, \sigma; \mathbb{Z})$ is isomorphic to $H(A, \Theta; \mathbb{Z})(2)$ as a polarized Hodge structure.*

Let $\pi_+ \in (A_+)_{(2)}$ be as in Lemma 3.4. Since $q_+(\pi_+) = 0$, π_+ is the Plücker coordinate of a subgroup $G_\sigma \subset A_{(2)}$ of order 4. Since $(A_+)_{(2)}$ is the orthogonal complement of $[\Theta/2]$ in $H^2(A, (\frac{1}{2}\mathbb{Z})/\mathbb{Z})$, G_σ is Göpel (Definition 1.4 and Lemma 1.5).

Lemma 4.2. (A, Θ) *in Proposition* 4.1 *is not a product of two elliptic curves.*

Proof. Assume that (A, Θ) is the product $E_1 \times E_2$ (as a polarized abelian surface). Then $E_1 \times 0 - 0 \times E_2$, the difference of two fibers, is a (-2)-class in $H^2(A,\Theta;\mathbb{Z})$. Let D_+ be its image in $H(S,\sigma;\mathbb{Z})$. Then $(D_+^2)=-4$ and $D_+/4$ represents an element $\eta_+\in A_+$ of order 4. Hence $D_+/2$ represents the class ζ_+ in the discriminant group A_+. Let $\tilde{e}$ be the pull-back of $e=e_\sigma\in H^2(S,\mathbb{Z})_f$ as in §3. Then $\tilde{e}+D_+$ is divisible by 2 in $H^2(\tilde{S},\mathbb{Z})$ and $(\tilde{e}+D_+)/2$ is an algebraic (-2)-class in N_R. This is a contradiction since N_R is the anti-invariant part of the involution or σ_R. □

By the lemma (A,Θ) is the Jacobian of a curve C_σ of genus 2.

Lemma 4.3. *The pair (C_σ,G_σ) is not bi-elliptic (Definition 1.6).*

Proof. The proof is similar to the preceding lemma. Assume that (C_σ,G_σ) is bi-elliptic. Since $(\Theta.E)=2$, $\Theta-E$ is a (-2)-class in $H^2(A,\Theta;\mathbb{Z})$. Let D_+ be its image in $H(S,\sigma;\mathbb{Z})$. Then $(D_+^2)=-4$ and $D_+/2$ represents the class $\zeta_++\pi_+$ in A_+. $(D_-+D_+)/2$ belongs to $H^-(S,\mathbb{Z})$ if D_- belongs to $H_-(S,\sigma;\mathbb{Z})$ and $D_-/2$ represents $\zeta_-+\pi_-$. Since $q_-(\zeta_-+\pi_-)=1$ and since $H_-(S,\sigma;\mathbb{Z})$ is isomorphic to $E_7(2)$, there is such a D_- with $(D_-^2)=-4$. For this choice, $(D_-+D_+)/2$ is an algebraic (-2)-class. This is a contradiction since $H^-(S,\mathbb{Z})$ is the anti-invariant part of the involution ε. □

Summarizing this section, we have

Proposition 4.4. *There exists a unique non-bi-elliptic pair (C_σ,G_σ) of a curve C_σ and a Göpel subgroup G_σ of $J(C_\sigma)$ with the following properties:*

(1) *$H(S,\sigma;\mathbb{Z})\simeq H(J(C_\sigma),\Theta;\mathbb{Z})(2)$ as a polarized Hodge structure, and*

(2) *the patching subgroup D_σ is generated by the canonical element and an element (π_-,π_+) such that π_+ is the Plücker coordinate of G_σ.*

In the subsequent sections, we conversely construct a numerically reflective involutions σ_G of an Enriques surface from such a pair (C,G) as above (Proposition6.4).

5. Hutchinson-Göpel involution

Hutchinson[H01] discovered the equation (1) and the automorphism ε_G by means of theta functions. In this section we describe them in a more elementary manner.

Let C be a curve of genus 2 and $J(C)$ its Jacobian. By the natural morphism $Sym^2 C\to J(C)$ and Abel's theorem, the second symmetric product $Sym^2 C$ of C is the blow-up of $J(C)$ at the origin. Let $\overline{Sym^2 C}$ be the quotient of $Sym^2 C$ by the involution induced from the hyperelliptic involution.

Since C is a double cover of the projective line $\mathbb{P}^1$ with 6 branch points, $\overline{Sym^2}\, C$ is the double cover of $Sym^2\,\mathbb{P}^1 \simeq \mathbb{P}^2$ with branch 6 lines $l_1, \dots, l_6$. Moreover, these 6 lines are tangent lines of the conic Q corresponding to the diagonal $\mathbb{P}^1 \hookrightarrow Sym^2\,\mathbb{P}^1$. Note that the double cover has 15 nodes over 15 intersections $p_{i,j} = l_i \cap l_j$, $1 \le i < j \le 6$. These corresponds to the 15 non-zero 2-torsions of $J(C)$. The minimal resolution of this double cover $\overline{Sym^2}\, C$ is the Jacobian Kummer surface $Km\, C$.

Three nodes are called *Göpel* if they correspond to the three non-zero elements of a Göpel subgroup. More explicitly, a triple $(p_{ij}, p_{i'j'}, p_{i''j''})$ of nodes is Göpel if and only if all suffixes $i, j, \dots, j''$ are distinct. Hence the Göpel subgroups correspond to the decompositions of the 6 Weierstrass points of C into 3 pairs. Therefore, the number of Göpel subgroups is 15.

We now construct an involution of $Km\, C$ for each Göpel subgroup G. The construction differs a lot according as the Göpel triple is collinear or not. First we consider the non-collinear case, which we are most interested in.

A birational automorphism $\varphi : \mathbb{P}^2 \cdots \to \mathbb{P}^2$ is called *a Cremona involution with center p, q, r* if p, q, r are not collinear and φ is the quadratic Cremona transformation $(x : y : z) \mapsto (x^{-1} : y^{-1} : z^{-1})$, for a suitable system of coordinates $(x : y : z)$ such that p, q, r are the coordinate points.

Proposition 5.1. *Assume that a Göpel triple (p_{14}, p_{25}, p_{36}) of G is not collinear. Then there exists a unique quadratic Cremona transformation φ with center p_{14}, p_{25} and p_{36} which maps the line l_i onto l_{i+3} for $i = 1, 2, 3$.*

Proof. We choose a system of homogeneous coordinate $(x : y : z)$ of $\mathbb{P}^2$ such that p_{14}, p_{25} and p_{36} are the coordinate points. Then the six lines are

$$l_i : y = \alpha_i x \ (i = 1, 4), \quad l_j : z = \alpha_j y \ (j = 2, 5) \text{ and } l_k : x = \alpha_k z \ (k = 3, 6)$$

for $\alpha_1, \dots, \alpha_6 \in \mathbb{C}^*$. Let

$$\check{Q} : a'x^2 + b'y^2 + c'z^2 + f'yz + g'xz + h'xy = 0$$

be the dual of the conic Q to which the six lines tangent. Then we have

$$\alpha_1\alpha_4 = a'/b', \quad \alpha_2\alpha_5 = b'/c', \quad \alpha_3\alpha_6 = c'/a'$$

and hence $\prod_{i=1}^{6} \alpha_i = 1$. The Cemona involution $(x : y : z) \mapsto (A/x : B/y : 1/z)$ satisfies our requirement if and only if $A = \alpha_3\alpha_6$ and $B = \alpha_2^{-1}\alpha_5^{-1}$. □

The Cremona involution φ in the proposition is lifted to two involutions of $Km\,C$. One is symplectic and has eight fixed points over the four fixed points of φ. The other has no fixed points (Remark 5.3). We call the latter *the Hutchinson involution associated with a Göpel subgroup* G and denote by ε_G. Since the covering involution β commutes with ε_G, it induces an involution of the Enriques surface $(Km\,C)/\varepsilon_G$, which we denote by σ_G.

Now we assume that a Göpel triple, say (p_{14}, p_{25}, p_{36}), lies on a line l. Let p be the point whose polar with respect to the conic Q is l and $\tilde{\gamma}$ be the involution of $\mathbb{P}^2$ whose fixed locus is the union of l and p. Then $\tilde{\gamma}$ maps Q onto itself and interchanges p_i and p_{i+3} for $i = 1, 2$ and 3. $\tilde{\gamma}$ induces involutions of $Km\,C$ and C. The following is easily verified:

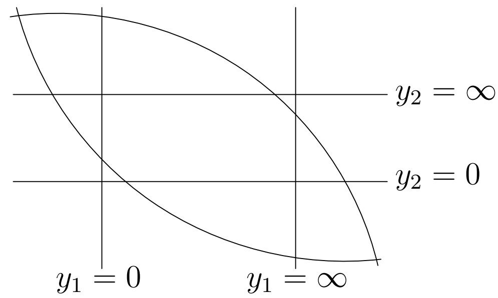
Proposition 5.2. *A Göpel triple of nodes are collinear if and only if* (C, G) *is bi-elliptic.*

Hence we have constructed an Enriques surface $(Km\,C)/\varepsilon_G$ with an involution σ_G for every non-bi-elliptic pair (C, G).

Remark 5.3. (Horikawa expression) By the proof of Proposition 5.1, $Km\,C$ is the minimal resolution of the double cover

$$\bar{S} : \tau^2 = (y - \alpha_1 x)(y - \alpha_4 x)(\alpha_2 y - 1)(\alpha_5 y - 1)(x - \alpha_3)(x - \alpha_6)$$

of $\mathbb{P}^1 \times \mathbb{P}^1$, where (x, y) is an inhomgeneous coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$.



The involution

$$\bar{\varepsilon}_G : (\tau, x, y) \mapsto \left(-\frac{AB\tau}{x^2 y^2}, \frac{A}{x}, \frac{B}{y} \right), \quad A = \alpha_3 \alpha_6,\ B = \alpha_2^{-1} \alpha_5^{-1}$$

of $\bar{S}$ has no fixed points. The K3 surface $\bar{S}$ has fourteen nodes and $(Km\,C)/\varepsilon_G$ is the minimal resolution of the Enriques surface $\bar{S}/\bar{\varepsilon}_G$ with seven nodes.

6. Period of $(Km\,C)/\varepsilon_G$

Returning to the case where (C, G) is not bi-elliptic, we compute the periods of the Enriques surface $(Km\,C)/\varepsilon_G$ and the involution σ_G.

The Jacobian Kummer surface $Km\, C$ is a double cover of the blow-up R of $\mathbb{P}^2$ at the 15 points p_{ij}, $1 \le i < j \le 6$. The pull-back of $H^2(R, \mathbb{Q})$ has $\{h, N_{ij},\ 1 \le i < j \le 6\}$ as a $\mathbb{Q}$-basis, where h is the pull-back of a line and N_{ij} is the $(-2)\mathbb{P}^1$ over p_{ij}.

We assume for simplicity that the Göpel triple is (p_{14}, p_{25}, p_{36}). Let $\bar{R}$ be the blow-up $\mathbb{P}^2$ at p_{14}, p_{25} and p_{36}. The Cremona involution φ in Proposition 5.1 acts on the Picard group of $\bar{R}$ by the reflection with respect to the (-2)-class $l - E_{14} - E_{25} - E_{36}$, where E_{14}, E_{25} and E_{36} are the exceptional curves. φ interchanges $p_{i,j}$ with $p_{i+3,j+3}$, and $p_{i,j+3}$ with $p_{j,i+3}$ for $1 \le i < j \le 3$. Hence we have

Proposition 6.1. *The action of ε_G on the pull-back of $H^2(R, \mathbb{Q})$ is the composite of the permutation*

$$N_{i,j} \leftrightarrow N_{i+3,j+3}, \quad N_{i,j+3} \leftrightarrow N_{j,i+3} \quad (1 \le i < j \le 3)$$

of type $(2)^6$ and the reflection with respect to the (-4)-class $h - N_{14} - N_{25} - N_{36}$.

By the proposition,

$$\{h - N_{14} - N_{25} - N_{36},\ N_{ij} - N_{i+3,j+3},\ N_{i,j+3} - N_{j,i+3}\}, \tag{4}$$

with $1 \le i < j \le 3$, is a $\mathbb{Q}$-basis of $H_-(Km\, C/\varepsilon_G, \sigma_G; \mathbb{Q})$. N_0, the $(-2)\mathbb{P}^1$ over the origin, maps onto the conic Q.

Proposition 6.2. *$h - N_0$ is invariant by ε_G and anti-invariant by β.*

Proof. There exists a cubic curve $D : r(x, y, z) = 0$ such that $D \cap C$ consists of the 6 tangent points $l_i \cap Q$, $1 \le i \le 6$. The union of 6 lines is defined by $r(x, y, z)^2 - q(x, y, z)s(x, y, z)$ for a suitable quartic form $s(x, y, z)$. Choose a cubic curve D such that it passes the Göpel triple. Then the quartic curve $s(x, y, z) = 0$ is singular at the Göpel triple. By the Cremona symmetry, $s(x, y, z)$ is a constant multiple of $q(yz, xz, xy)$. Hence the double cover $\overline{Sym^2}\, C$ is defined by

$$\tau^2 = r(x, y, z)^2 - cq(x, y, z)q(yz, xz, xy) \tag{5}$$

for a constant $c \in \mathbb{C}^*$. The rational function $\{r(x, y, z) + \tau\}/\{r(x, y, z) - \tau\}$ on $Km\, C$ gives a rational equivalence between two divisors $N_0 + \beta\varepsilon_G(N_0)$ and $\varepsilon_G(N_0) + \beta(N_0)$. Hence $\beta(N_0) - N_0$ is ε_G-invariant. Since $\beta(N_0) + N_0$ is linearly equivalent to $2h$, we have our proposition. $\square$

Remark 6.3. By (5) the linear system $|h + N_0|$ gives a birational morphism from the double cover $\overline{Sym^2}\, C$ to the quartic $cq(x, y, z)t^2 + 2r(x, y, z)t + q(yz, xz, xy) = 0$ in $\mathbb{P}^3$, which is essentialy the equation (1).

By Propositions 6.1 and 6.2,

(6) $\{h-N_0,\ h-N_{14},\ h-N_{25},\ h-N_{36},\ N_{ij}+N_{i+3,j+3},\ N_{i,j+3}+N_{j,i+3}\}$,

with $1 \le i < j \le 3$, is an orthogonal $\mathbb{Q}$-basis of $\pi^* H^2(Km\, C/\varepsilon_G, \mathbb{Q})$. In particular, σ_G acts on $\pi^* H^2(Km\, C/\varepsilon_G, \mathbb{Q})$ by the reflection with respect to $h - N_0$. Hence we have

Proposition 6.4. *The involution σ_G of the Enriques surface $(Km\, C)/\varepsilon_G$ is numerically reflective.*

Moreover, the inverse of the correspondence $(S, \sigma) \mapsto (C_\sigma, G_\sigma)$ of Proposition 4.4 is given by this construction $(C, G) \mapsto (Km\, C/\varepsilon_G, \sigma_G)$:

Proposition 6.5. (1) *The polarized Hodge structure $H(Km\, C/\varepsilon_G, \sigma_G; \mathbb{Z})$ is isomorphic to $H(J(C), \Theta; \mathbb{Z})(2)$.*

(2) *The patching group of σ_G is of order 4, and generated by the canonical element and (π_-, π_G), where π_G is the Plücker coordinate of G.*

Proof. By (4) and (6), $H(Km\, C/\varepsilon_G, \sigma_G; \mathbb{Z})$ is the orthogonal complement of the lattice generated by the 17 classes h, N_0 and N_{ij}, $1 \le i < j \le 6$, in $H^2(Km\, C, \mathbb{Z})$. Let $H \in H^2(Km\, C, \mathbb{Z})$ be the (4)-class in Λ corresponding to $\Theta \in H^2(J(C), \mathbb{Z})$ in the way of Lemma 1.1. It is easily checked that $H = h + N_0$. Hence we have (1).

The patching group is order 4 by (1) and Proposition 3.2 since $H(J(C), \Theta; \mathbb{Z})(2) \simeq \langle -4 \rangle \perp U(2) \perp U(2)$. By Proposition 6.1, both $N_{12} - N_{45}$ and $N_{15} - N_{24}$ belong to $H_-(Km\, C/\varepsilon_G, \sigma_G; \mathbb{Z})$. Since the 2-torsion points p_{12}, p_{45}, p_{15} and p_{24} form a coset of $G \subset J(C)_{(2)}$, $([(N_{12} - N_{45} + N_{15} - N_{24})/2], \pi_G)$ belongs to the patching group of σ_G by Lemma 1.2. □

Proof of Theorem 2. Let σ be a numerically reflective involution of an Enriques surface S and assume that the patching group D_σ is of order 4. Let (C_σ, G_σ) be as in Proposition 4.4 and σ' be the numerically reflective involution σ_G of the Enriques surface $S' := Km\, C/\varepsilon_G$ for $C = C_\sigma$ and $G = G_\sigma$. By Proposition 6.5, $H(S, \sigma; \mathbb{Z})$ is isomorphic to $H(S', \sigma'; \mathbb{Z})$ as a polarized Hodge structure. Moreover, the A_+-components of their patching groups are the same. Both are generated by ζ_+ and the Plücker coordinate π_G of G.

Now we look at the A_--components. Two lattices $H_-(S, \sigma; \mathbb{Z})$ and $H_-(S', \sigma'; \mathbb{Z})$ are $E_7(2)$ by Lemma 3.1. The A_--components of patching groups are generated by ζ_+ and π_- with $q_-(\pi_-) = 0$. The Weyl group W of E_7 acts on $A_- \simeq u(2)^{\perp 3} \perp (4)$ preserving ζ_-. There are 63 α's with $q_-(\alpha) = 0$ in $(A_-)_{(2)}$ and W acts transitively on them. Hence a Hodge isometry between $H(S, \sigma; \mathbb{Z})$ and $H(S', \sigma'; \mathbb{Z})$ extends to a

$\mathbb{Z}/2\mathbb{Z}$-equivariant Hodge isometry between $H^{-}(S, \mathbb{Z})$ and $H^{-}(S', \mathbb{Z})$. Now the theorem follows from Theorem 2.3. $\square$

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