Growth series for Artin monoids of finite type

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1 Introduction

Let G_M^+ be the Artin monoid ([B-S] 1.2) generated by the letters a_i, $i \in I$ with respect to a Coxeter matrix $M = (m_{ij})_{i,j \in I}$. That is, G_M^+ is a monoid generated by the letters a_i, $i \in I$ which are subordinate to the relation generated by

$$(1.1) \qquad a_i a_j a_i \cdots \ = \ a_j a_i a_j \cdots \qquad i, j \in I,$$

where both hand sides are words of alternating sequences of letters a_i and a_j of the same length $m_{ij} = m_{ji}$. More precisely, G_M^+ is the quotient of the free monoid generated by the letters a_i ($i \in I$) by the equivalence relation: two words U and V in the letters are equivalent, if there exists a sequence $U_0 := U, U_1, \cdots, U_m := V$ such that the word U_k ($k = 1, \cdots, m$) is obtained by replacing a phrase in U_{k-1} of the form on LHS of (1.1) by RHS of (1.1) for some $i, j \in I$. We write by $U \underset{\bullet}{=} V$ if U and V are equivalent. The equivalence class (i.e. an element of G_M^+) of a word W is denoted by the same notation W. By the definition, equivalent words have the same length. Hence, we define the degree homomorphism:

$$(1.2) \qquad \deg \ : \ G_M^+ \longrightarrow \mathbb{Z}_{\geq 0}$$

by assigning the length to each equivalence class of words.

In [S] §12, we considered the growth series for the Artin monoid, defined by

$$(1.3) \qquad P_{G_M^+, I}(t) \ := \sum_{n \in \mathbb{Z}_{\geq 0}} \#\{W \in G_M^+ \mid \deg(W) \leq n\} \ t^n,$$

and asked to determine the space $\Omega(P_{G_M^+, I})$ of opposite series [ibid] for a *finite type* M.[1] In the present paper, we, conjecturally, answer to the question. Namely, in §2, we show that the growth series has an expression

$$(1.4) \qquad P_{G_M^+, I}(t) \ = \ 1/(1-t) N_M(t),$$

where $N_M(t)$ is a polynomial determined explicitly from the Coxeter-Dynkin graph of M. In §3, we show that zeroes of $N_M(t) = 0$ lie in the disc of radius $1+\varepsilon$ for a small ε. We further give a conjectural description of the distribution of the zero-loci in §4, which concludes that $\Omega(P_{G_M^+, I})$ consists of a single element, answering to the question posed above.

[1] In the present paper, we call a Coxeter matrix M is of finite type if it is indecomposable and the associated Coxeter group $\overline{G}_M$ ([B] Ch.IV §1) is finite.

2 Growth series $P_{G_M^+,I}(t)$

For a Coxeter matrix M, consider the *spherical growth series of the monoid* G_M^+:

$$\dot{P}_{G_M^+,I}(t) \ := \sum_{n\in\mathbb{Z}_{\geq 0}} \#(\deg^{-1}(n))\ t^n, \tag{2.1}$$

so that $P_{G_M^+,I}(t) = \dot{P}_{G_M^+,I}(t)/(1-t)$. The goal of the present § is the following.

Theorem. *Let G_M^+ be the Artin monoid with respect to a Coxeter matrix M of finite type. Then the spherical growth series of the monoid is given by the Taylor expansion of the rational function of the form*

$$\dot{P}_{G_M^+,I}(t) \ = \ 1/N_M(t), \tag{2.2}$$

where the denominator $N_M(t)$ is a monic polynomial in t given by

$$N_M(t) \ := \sum_{J\subset I} (-1)^{\#(J)}\ t^{\deg(\Delta_J)}. \tag{2.3}$$

Here, the summation index J runs over all subsets of I, and Δ_J is the fundamental element in G_M^+ associated to the set J ([B-S] §5 Definition. See also Lemma-Definition *and* Remark 2.1 *of the present note).*

Proof. The proof is achieved by a recursion formula on the coefficients of the growth series. For the proof of the recursion formula, we use the method used to solve the word problem for the Artin monoid ([B-S] 6.1), which we recall below.

A word U is said to be divisible (from the left) by a word V, and denoted by $V|U$, if there exists a word W such that $U \underset{\bullet}{=} VW$. There exists an algorithm, which terminates in finite steps, to decide whether $V|U$ or not for given words U and V (see [B-S]§3). Since $V \underset{\bullet}{=} V'$, $U \underset{\bullet}{=} U'$ and $V|U$ implies $V'|U'$, we use the notation "|" of divisibility also between elements of the monoid G_M^+.

We have the following basic concepts ([B-S] §5 Definition and §6 6.1)

Lemma-Definition 1. For any subset $J \subset I$, there exists a unique element $\Delta_J \in G_M^+$, called the *fundamental element*, such that i) $a_i|\Delta_J$ for all $i \in J$, and ii) if W is a word such that $a_i|W$ for all $i \in J$, then $\Delta_J|W$.

2. For a word W, we associate a subset of I:

$$I(W) \ := \ \{i \in I \ \mid \ a_i|W\}. \tag{2.4}$$

One has $\Delta_{I(W)}|W$ and if $\Delta_J|W$ then $J \subset I(W)$. If $W \underset{\bullet}{=} W'$ then $I(W) = I(W')$. Therefore, we use the same notation $I(W)$ for an element W in G_M^+.

We return to the proof of Theorem. For $n \in \mathbb{Z}_{\geq 0}$ and a subset $J \subset I$, put

$$G_n^+ \ := \ \{W \in G_M^+ \mid \deg(W) = n\} \tag{2.5}$$

$$(2.6) \qquad G^+_{n,J} \ := \ \{W \in G^+_n \mid I(W) = J\}.$$

By the definition, we have the disjoint decomposition:

$$(2.7) \qquad G^+_n \ = \ \amalg_{J\subset I}\, G^+_{n,J},$$

where J runs over all subsets of I. Note that $G^+_{n,\emptyset}=\emptyset$ if $n>0$ but $G^+_{0,\emptyset}=\{\emptyset\}\neq\emptyset$. For any subset J of I, the union $\amalg_{J\subset K\subset I} G^+_{n,K}$, where the index K runs over all subsets of I containing J, is equal to the subset of G^+_n consisting of elements divisible by Δ_J. That is, it is the image of $G^+_{n-\deg(\Delta_J)}$ under the multiplication by Δ_J from the left. On the other hand, since G^+_M is injectively embedded in the Artin group G_M ([B-S] Proposition 5.5), the multiplication map of Δ_J is injective. Hence we obtain a bijection: $G^+_{n-\deg(\Delta_J)} \ \simeq \ \amalg_{J\subset K\subset I} G^+_{n,K}$. This implies a numerical relation:

$$(2.8) \qquad \#(G^+_{n-\deg(\Delta_J)}) \ = \ \textstyle\sum_{J\subset K\subset I} \#(G^+_{n,K}).$$

Then, for $n > 0$, using this formula, we get the recursion relation:

$$(2.9) \qquad \sum_{J\subset I} (-1)^{\#(J)} \#(G^+_{n-\deg(\Delta_J)}) \ = \ 0.$$

Together with $\#(G^+_0)=1$ for $n=0$, this is equivalent to the formula:

$$(2.10) \qquad \dot{P}_{G^+_M,I}(t) N_M(t) \ = \ 1.$$

This completes the proof of Theorem. □

By the definition (2.3), one has $N_M(1) = \sum_{J\subset I}(-1)^{\#J} = 0$. That is, $N_M(t)$ has the factor $1-t$. Then, we conjecture the following.

Conjecture 1. The polynomial $\tilde{N}_M(t) := N_M(t)/(1-t)$ is irreducible over $\mathbb{Z}$ for any indecomposable Coxeter matrix M of finite type.

The conjecture is explicitly confirmed (using computer) for the types A_l, B_l, C_l ($l \le 7$), D_l ($l \le 5$), E_6, E_7, E_8, F_4 G_2 and $I_2(p)$ ($p \le 6$). The conjecture shall play an important role when we study the global space $V(G^+_M, I)$ of limit elements [S] 11.4.10, on which the Galois group of the splitting field of $\tilde{N}_M(t)$ acts.

Remark 2.1. Actually, $\deg(\Delta_J)$ is equal to the number of reflections in the Coxeter group $\overline{G}_{M|J}$ associated to $M|_J := (m_{ij})_{i,j\in J}$ ([B-S] 5.7).

Remark 2.2. The fact that the growth series for the Artin monoids are rational functions with the numerator equal to 1 was first observed by Fuchiwaki for some examples of low rank [F].

3 Zeroes of the polynomial $N_M(t)$

The following lemma gives a bound of zero-loci of the polynomial $N_M(t)$.

Lemma 3.1. *For a Coxeter matrix M, define a numerical invariant:*

$$a_M := \frac{\deg(\Delta_I) - \max\{\deg(\Delta_J) \mid J \subset I,\ J \neq I\}}{\#I} \quad . \tag{3.1}$$

Then, one has

1. *$a_M \geq 1$ for any finite type Coxeter matrix M,*
2. *all the roots of $N_M(t) = 0$ are contained in the open disc of radius $2^{1/a_M}$ centered at the origin.*

Proof. 1. This is shown by using the classification of finite Coxeter groups:
(3.2)

$$\begin{array}{llll}
A_{l\geq 1}: & \deg(\Delta_{A_l}) = (l+1)l/2 & \max\{\deg(\Delta_J)\} = l(l-1)/2 & a_{A_l} = 1, \\
B_{l\geq 2}: & \deg(\Delta_{B_l}) = l^2 & \max\{\deg(\Delta_J)\} = (l-1)^2 & a_{B_l} = (2l-1)/l, \\
D_{l\geq 4}: & \deg(\Delta_{D_l}) = l(l-1) & \max\{\deg(\Delta_J)\} = (l-1)(l-2) & a_{D_l} = 2(l-1)/l, \\
E_6 \ : & \deg(\Delta_{E_6}) = 36 & \max\{\deg(\Delta_J)\} = 20 & a_{E_6} = 8/3, \\
E_7 \ : & \deg(\Delta_{E_7}) = 63 & \max\{\deg(\Delta_J)\} = 36 & a_{E_7} = 27/7, \\
E_8 \ : & \deg(\Delta_{E_8}) = 120 & \max\{\deg(\Delta_J)\} = 63 & a_{E_8} = 57/8, \\
F_4 \ : & \deg(\Delta_{F_4}) = 24 & \max\{\deg(\Delta_J)\} = 9 & a_{F_4} = 15/4, \\
G_2 \ : & \deg(\Delta_{G_2}) = 6 & \max\{\deg(\Delta_J)\} = 1 & a_{G_2} = 5/2, \\
H_3 \ : & \deg(\Delta_{H_3}) = 15 & \max\{\deg(\Delta_J)\} = 5 & a_{H_3} = 10/3, \\
H_4 \ : & \deg(\Delta_{H_3}) = 60 & \max\{\deg(\Delta_J)\} = 15 & a_{H_4} = 45/4, \\
I_2(p_{\geq 3}): & \deg(\Delta_{I_2(p)}) = p & \max\{\deg(\Delta_J)\} = 1 & a_{I_2(p)} = (p-1)/2,
\end{array}$$

2. We compare the roots of $N_M(t) = 0$ with that of $t^{\deg(\Delta_I)} = 0$ by Rouché's theorem as follows. Let $t \in \mathbb{C}$ be a point with $|t| = 2^{1/a_M}$. Then

$$\begin{array}{rl}
& |N_M(t) - (-1)^{\#(I)} t^{\deg(\Delta_I)}| = |\sum_{J\subset I, J\neq I} (-1)^{\#(J)} t^{\deg(\Delta_J)}| \\
\leq & \sum_{J\subset I, J\neq I} |t^{\deg(\Delta_J)}| \ \leq \ (2^{\#(I)} - 1)|t|^{\max\{\Delta_J | J\subset I,\ J\neq I\}} \ < \ |t|^{\deg(\Delta_I)}.
\end{array}$$

Due to Rouché's theorem, the number of roots of $N_M = 0$ in the disc of radius $2^{1/a_M}$ is equal to that of $t^{\deg(\Delta_I)} = 0$. That is, all the roots of $N_M = 0$ are in the disc $\{|t| < 2^{1/a_M}\}$. □

4 The space of opposite series $\Omega(P_{G^+_M, I})$

We give conjectures on the distribution of the zeroes of $N_M(t)$ to determine $\Omega(P_{G^+_W, I})$. We formulate them more than necessary for the purpose, because of possible applications (see §2 Conjecture 1 and its following explanations).

Conjecture 2. There are $l - 1$ mutually distinct real roots of $N_M(t) = 0$ on the interval $(0, 1)$, where $l := \#I$ is the rank of G_M.

Conjecture 3. Let r_M (resp. R_M) be the smallest (resp. largest) roots of $N_M(t) = 0$ on the interval $(0, 1)$. Then, all the remaining roots lie in the annulus $R_w < |t| < R_W^{-1}$.

The conjectures are directly confirmed (using Sturm criterion) for the types A_l, B_l, C_l ($l \le 7$), D_l ($l \le 5$), E_6, E_7, E_8, F_4 G_2 and $I_2(p)$ ($p \ge 3$).

Remark 4.1. An evidence of Conjecture **2** is the following.

Put $N_{M,j}(t) = \sum_{J \subset I, \# J = j} t^{\deg(\Delta_J)}$ for $0 \le j \le l := \# I$, and consider the sequence of polynomials $f_i(t) := \sum_{j=0}^{i} (-1)^j N_{M,j}(t)$ $(i = 0, 1, \cdots, l)$. Explicitly,

$$f_0 = 1,\ f_1 = 1 - l \cdot t,\ f_2 = 1 - l \cdot t + \tfrac{(l-1)(l-2)}{2} t^2 + \sum_{\substack{\text{edges of}\\ \text{type } p}} t^p\ ,\ \cdots\ ,\ f_l = N_M(t).$$

Obviously, $f_0(0) = f_1(0) = \cdots = f_l(0) = 1$. Their values at $t = 1$ are given by

$$f_i(1) = \sum_{j=0}^{i} (-1)^j N_{M,j}(1) = \sum_{j=0}^{i} (-1)^j C_j^l = (-1)^i C_i^{l-1}.$$

So, they form a *sign alternating sequence*, except the vanishing $f_l(1) = 0$ at the end. Therefore, we may expect to show inductively that

Assertion. *The equation $f_i = 0$ has i distinct real roots on the interval* $(0, 1]$.

Remark 4.2. Examples show that the angles of the roots in the annulus in Conjecture **3** are somehow equally distributed. However, we do not know how we can precisely formulate this phenomenon.

Assuming Conjecture 2 and 3, we are able to determine the space $\Omega(P_{G_M^+, I})$ of *opposite series* for $P_{G_M^+, I}$ ([S] (11.2.3)), where we recall that a series in $\mathbb{R}[[s]]$ is called an opposite series for $P_{G_M^+, I}$ if it is an accumulation point (with respect to the classical topology) of the sequence of polynomials

$$X_n(P_{G_M^+, I}) := \sum_{k=0}^{n} \frac{\# \deg^{-1}(n-k)}{\# \deg^{-1}(n)} s^k\ , \qquad n = 0, 1, 2, \cdots . \tag{4.1}$$

Recall that $\Delta^{top}_{P_{G_M^+, I}}(t)$ denotes the reduced polynomial vanishing at the loci of the poles of $P_{G_M^+, I}$ of the smallest radius and highest order (among them). Actually, Conjectures 2 and 3 imply $\Delta^{top}_{P_{G_M^+, I}}(t) = t - r_M$. The Duality Theorem ([S] §11 Theorem) says, in general, that if $\deg(\Delta^{top}_{P_{G_M^+, I}}) = h$, then putting $\Delta^{op}_{P_{G_M^+, I}}(s) := s^h \Delta^{top}_{P_{G_M^+, I}}(s^{-1})$, the opposite series have the form $b(s)/\Delta^{op}_{P_{G_M^+, I}}(s)$ for suitable polynomial $b(s)$ of degree $< h$ ([S] (11.2) Assertion). This, in particular in our case, implies the the sequence X_n converges to a unique element in $\mathbb{R}[[s]]$ of the form

$$a(s) \ = \ \frac{1}{1 - r_M s}, \tag{4.2}$$

where $r_M \in \mathbb{R}_{>0}$ is given by

$$r_M = \lim_{n \to \infty} \tfrac{\# \deg^{-1}(n-1)}{\# \deg^{-1}(n)}. \tag{4.3}$$

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