TAME-BLIND EXTENSION OF MORPHISMS OF TRUNCATED BARSOTTI-TATE GROUP SCHEMES

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Abstract. The purpose of the present paper is to show that morphisms between the generic fibers of truncated Barsotti-Tate group schemes over mixed characteristic complete discrete valuation rings with perfect residue fields extend in a "tame-blind" fashion — i.e., under a condition which is unaffected by passing to a tame extension — to morphisms between the original truncated Barsotti-Tate group schemes. The "tame-blindness" of our extension result allows one to verify the analogue of a result of Tate for isogenies of Barsotti-Tate groups over the ring of integers of the p-adic completion of the maximal tamely ramified extension field.

Contents

0.	Introduction	1
1.	Discriminants and cotangent spaces	7
2.	Review of truncated p-adic Hodge theory	10
3.	Proof of the main theorem	15
References		26

0. Introduction

The purpose of the present paper is to show that morphisms between the generic fibers of truncated Barsotti-Tate group schemes over mixed characteristic complete discrete valuation rings with perfect residue fields extend in a "tame-blind" fashion — i.e., under a condition which is unaffected by passing to a tame extension — to morphisms between the original truncated Barsotti-Tate group schemes.

Throughout this paper, let R be a complete discrete valuation ring, k the residue field of R, K the field of fractions of R, $\overline{K}$ an algebraic closure of K, $\Gamma_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$, and v_p the valuation of $\overline{K}$ such that $v_p(p) = 1$. Assume, moreover, that K is of characteristic 0, and k is of characteristic $p > 0$ and *perfect*. Let e_K be the absolute ramification index of K.

By a result of Tate obtained in [11], for (p-)Barsotti-Tate groups (i.e., p-divisible groups) $\mathcal{G}$, $\mathcal{H}$ over R, every $\mathbb{Z}_p[\Gamma_K]$-equivariant morphism

$T_p(\mathcal{G}) \to T_p(\mathcal{H})$ of p-adic Tate modules arises from a morphism $\mathcal{G} \to \mathcal{H}$ of Barsotti-Tate groups over R (cf. [11], Theorem 4). Now one can consider the question of whether or not such a result can be generalized to finite level, i.e., whether or not for finite flat commutative group schemes G, H over R, any morphism $G \otimes_R K \to H \otimes_R K$ of the generic fibers extends to a morphism $G \to H$ of the original group schemes over R. For instance, a result of Raynaud obtained in [9] yields an affirmative answer to this question if $e_K < p-1$ (cf. [9], Corollaire 3.3.6, 1). On the other hand, one verifies immediately that this extension question cannot be resolved in the affirmative without some further assumption. Indeed, let K be the finite extension field of the field $\mathbb{Q}_p$ of p-adic rational numbers obtained by adjoining a primitive p-th root of unity to $\mathbb{Q}_p$, G the kernel μ_p of the endomorphism of the multiplicative group scheme $\mathbb{G}_m$ over R given by raising to the p-th power, and H the constant group scheme $\mathbb{Z}/(p)$ of order p over R. Then it is easily verified that although there exists an *isomorphism* $\mu_p \otimes_R K \xrightarrow{\sim} \mathbb{Z}/(p) \otimes_R K$ of group schemes over K, there is *no* nontrivial morphism of group schemes over R from μ_p to $\mathbb{Z}/(p)$ if $p \geq 3$.

In the present paper, we consider the following "Extension Problem":

> (**Extension Problem**) : *Find a sufficient condition for a morphism between the generic fibers of finite flat commutative group schemes over R to extend to a morphism between the original group schemes over R.*

In particular, in the present paper, we consider the following "Tame-blind Extension Problem":

> (**Tame-blind Extension Problem**) : *Find a sufficient condition which depends only on $v_p(e_K)$ for a morphism between the generic fibers of finite flat commutative group schemes over R to extend to a morphism between the original group schemes over R.*

Our main result yields a solution to this "Tame-blind Extension Problem" in the case where the morphisms in question are morphisms of truncated Barsotti-Tate group schemes (cf. Theorem 3.4):

Theorem 0.1. *Let G, H be truncated (p-)Barsotti-Tate group schemes over R, $f_K \colon G \otimes_R K \to H \otimes_R K$ a morphism of group schemes over K, n a natural number, and $\epsilon_K^{\mathrm{Fon}} \stackrel{\mathrm{def}}{=} 2 + v_p(e_K)$ (cf. the expression "$v_p(\mathcal{D}_{R/W(k)}) + 1/(p-1)$" in the statement of [2], Colloraire to Théorème 3). Assume that one of the following conditions is satisfied:*

(i) *The cokernel of the morphism $G(\overline{K}) \to H(\overline{K})$ determined by f_K is annihilated by p^n, and $4\epsilon_K^{\mathrm{Fon}} + n \leq \mathrm{lv}(H)$, where $\mathrm{lv}(H)$ is the level of H.*

(ii) *The kernel of the morphism* $G(\overline{K}) \to H(\overline{K})$ *determined by* f_K *is annihilated by* p^n, *and* $4\epsilon_K^{\mathrm{Fon}} + n \leq \mathrm{lv}(G)$, *where* $\mathrm{lv}(G)$ *is the level of* G.

Then the morphism f_K *extends uniquely to a morphism over* R.

The following result follows immediately from Theorem 0.1 (cf. Corollary 3.6, (iii)):

Corollary 0.2. *Let* G, H *be truncated Barsotti-Tate group schemes over* R, *and* $\mathrm{Isom}_R(G, H)$ (*respectively,* $\mathrm{Isom}_K(G \otimes_R K, H \otimes_R K)$) *the set of isomorphisms of* G (*respectively,* $G \otimes_R K$) *with* H (*respectively,* $H \otimes_R K$) *over* R (*respectively,* K). *Then if* $4\epsilon_K^{\mathrm{Fon}} \leq \mathrm{lv}(G)$, $\mathrm{lv}(H)$, *then the natural morphism*

$$\mathrm{Isom}_R(G, H) \longrightarrow \mathrm{Isom}_K(G \otimes_R K, H \otimes_R K)$$

is bijective.

Note that a number of results related to the above "Extension Problem" such as the result of Raynand referred to above have been obtained by various authors. Examples of such results are as follows:

Let G, H be finite flat commutative group schemes over R, and $f_K \colon G \otimes_R K \to H \otimes_R K$ a morphism of group schemes over K. Then the following hold:

(B) Let $\epsilon_K^{\mathrm{Bon}}$ be the smallest natural number which is $\geq \log_p(pe_K/(p-1))$. Then by a result of Bondarko obtained in [1], one can verify that if there exists a morphism $f'_K \colon G \otimes_R K \to H \otimes_R K$ of group schemes over K such that $f_K = p^{\epsilon_K^{\mathrm{Bon}}} \circ f'_K$, then the morphism f_K extends to a morphism $G \to H$ of the original group schemes over R (cf. [1], Theorem A). That is to say, any morphism between the generic fibers of finite flat commutative group schemes over R extends to a morphism between the original group schemes *after composition* with the endomorphism given by multiplication by $p^{\epsilon_K^{\mathrm{Bon}}}$.

(L) Let h be a natural number. Then by a result of Liu obtained in [5], one can verify that there exists a natural number $\epsilon_{K,h}^{\mathrm{Liu}}$ depending on e_K and h such that if G is a truncated Barsotti-Tate group scheme of height h, and $f_K = p^{\epsilon_{K,h}^{\mathrm{Liu}}} \circ f'_K$ for a morphism $f'_K \colon G \otimes_R K \to H \otimes_R K$ of group schemes over K, then the morphism f_K extends to a morphism $G \to H$ of the original group schemes over R (cf. [5], Theorem 1.0.5).

Theorem 0.1 is *weaker* than the above two results (B) and (L) in the sense that the class of group schemes considered in Theorem 0.1 are strictly *smaller* than the class of group schemes considered in the above two results. On the other hand, Theorem 0.1 is *stronger* than the above two results in the sense that *whereas the invariants $\epsilon_K^{\mathrm{Bon}}$ and $\epsilon_{K,h}^{\mathrm{Liu}}$ that appear in the above two results depend on e_K, our invariant $\epsilon_K^{\mathrm{Fon}}$ depends only on $v_p(e_K)$.*

It seems to the author that one of the reasons why the conditions for extending the morphisms in question in (B) and (L) depend on e_K (i.e., as $e_K/p^{v_p(e_K)}$ grows, the conditions become more stringent) is as follows:

In the arguments of [1], [5], *which appear to build on Tate's original argument, one must measure various integral structures by means of a "ruler graduated in units of size* $1/e_K$*". Thus, as the size* $1/e_K$ *of the units decreases (i.e., as* $e_K/p^{v_p(e_K)}$ *grows), it becomes more difficult to control the extent to which the integral structures in question converge.*

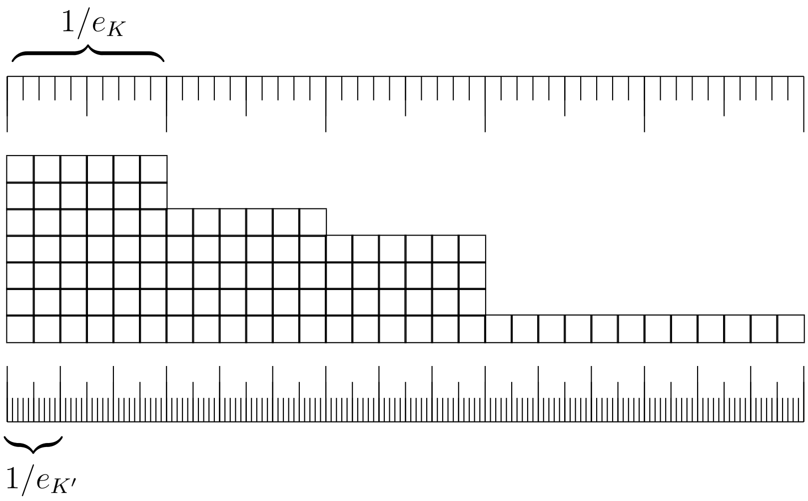

Figure 1: rulers graduated in units of sizes $1/e_K$, $1/e_{K'}$

From this point of view, the argument established in the present paper is an argument that does not rely on the use of a "ruler graduated in units of size $1/e_K$".

The "tame-blindness" of our extension result allows one to verify the analogue of the result of Tate referred to above for isogenies of Barsotti-Tate groups over the ring of integers of the p-adic completion of the maximal tamely ramified extension field (cf. Corollary 3.8). Note that this analogue does not follow from (B) and (L):

Corollary 0.3. *Let K^{tm} ($\subseteq \overline{K}$) be the maximal tamely ramified extension field of K, $(K^{\mathrm{tm}})^\wedge$ (respectively, $\widehat{\overline{K}}$) the p-adic completion of K^{tm} (respectively, $\overline{K}$), $(R^{\mathrm{tm}})^\wedge$ the ring of integers of $(K^{\mathrm{tm}})^\wedge$, and*

$\Gamma_{(K^{\mathrm{tm}})^\wedge} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widehat{\overline{K}}/(K^{\mathrm{tm}})^\wedge)$. *(Thus, by restricting elements of* $\Gamma_{(K^{\mathrm{tm}})^\wedge}$ *to the algebraic closure of* $(K^{\mathrm{tm}})^\wedge$ *in* $\widehat{\overline{K}}$*, one obtains a natural isomorphism of* $\Gamma_{(K^{\mathrm{tm}})^\wedge}$ *with the corresponding absolute Galois group of* $(K^{\mathrm{tm}})^\wedge$*.) Let* $\mathcal{G}$ *and* $\mathcal{H}$ *be Barsotti-Tate groups over* $(R^{\mathrm{tm}})^\wedge$*,* $T_p(\mathcal{G})$ *(respectively,* $T_p(\mathcal{H})$*) the p-adic Tate module of* $\mathcal{G}$ *(respectively,* $\mathcal{H}$*), and* $\mathrm{Isog}_{(R^{\mathrm{tm}})^\wedge}(\mathcal{G},\mathcal{H})$ *(respectively,* $\mathrm{Isog}_{\Gamma_{(K^{\mathrm{tm}})^\wedge}}(T_p(\mathcal{G}),T_p(\mathcal{H}))$*) the set of morphisms* ϕ *of Barsotti-Tate groups over* $(R^{\mathrm{tm}})^\wedge$ *(respectively,* $\mathbb{Z}_p[\Gamma_{(K^{\mathrm{tm}})^\wedge}]$*-equivariant morphisms* ϕ*) from* $\mathcal{G}$ *(respectively,* $T_p(\mathcal{G})$*) to* $\mathcal{H}$ *(respectively,* $T_p(\mathcal{H})$*) such that* ϕ *induces an isomorphism* $T_p(\mathcal{G})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \xrightarrow{\sim} T_p(\mathcal{H})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$*. Then the natural morphism*

$$\mathrm{Isog}_{(R^{\mathrm{tm}})^\wedge}(\mathcal{G},\mathcal{H}) \longrightarrow \mathrm{Isog}_{\Gamma_{(K^{\mathrm{tm}})^\wedge}}(T_p(\mathcal{G}),T_p(\mathcal{H}))$$

is bijective.

The present paper is organized as follows: In Section 1, we study the relationship between discriminants and cotangent spaces of finite flat group schemes. In Section 2, we review truncated p-adic Hodge theory for finite flat group schemes as established in [2] and prove lemmas needed later by means of this theory. In Section 3, we prove the main theorem and some corollaries which follow from the main theorem.

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Notations and Terminologies

Numbers. The notation $\mathbb{N}$ will be used to denote the set or (additive) monoid of nonnegative rational integers. The notation $\mathbb{Z}$ will be used to denote the set, group, or ring of rational integers. The notation $\mathbb{Q}$ will be used to denote the set, group, or field of rational numbers. The notation $\mathbb{Q}_{>0}$ will be used to denote the set or (additive) monoid of positive rational numbers. If l is a prime number, then the notation $\mathbb{Z}_l$ (respectively, $\mathbb{Q}_l$) will be used to denote the l-adic completion of $\mathbb{Z}$ (respectively, $\mathbb{Q}$).

Group schemes. In the present paper, by a *finite flat group scheme* over a scheme S we shall mean a *commutative* group scheme over S which is *finite* and *flat* over S, and by a *finite flat subgroup scheme* of a finite flat group scheme G over S we shall mean a *closed* subgroup scheme of G which is *finite* and *flat* over S.

Let G be a finite flat group scheme over a connected scheme S. Then we shall refer to the rank of the locally free $\mathcal{O}_S$-module $\phi_*\mathcal{O}_G$, where $\phi\colon G \to S$ is the structure morphism of G, as the *rank* of G over S. We shall denote by $\mathrm{rank}_S(G)$ the rank of G over S.

Let G be a finite flat group scheme over a scheme S, and n a natural number. Then we shall denote by $n_G\colon G \to G$ the endomorphism of G given by multiplication by n. Note that n_G is a *morphism of group schemes* over S.

Let $f\colon G \to H$ be a morphism of finite flat group schemes over a scheme S. Then we shall denote by $\mathrm{Ker}(f)$ the *group scheme* over S obtained as the fiber product of f and the identity section of H. We shall refer to $\mathrm{Ker}(f)$ as the *kernel* of the morphism f. Note that since H is separated over S, the kernel $\mathrm{Ker}(f)$ is a *closed* subgroup scheme of G.

Let G be a finite flat group scheme over a scheme S, and $H \subseteq G$ a finite flat subgroup scheme of G over S. Then a *quotient* of G by H, which is a *finite flat* group scheme over S, exists (cf. e.g., [8], Théorème 1, (iii)). We shall denote by G/H the quotient of G by H. Note that we have a natural morphism of group schemes $G \to G/H$ over S which is *finite* and *faithfully flat*; moreover, the kernel of this morphism $G \to G/H$ coincides with $H \subseteq G$.

Let G_1, G_2 be group schemes over a scheme S. Then we shall denote by

$$\mathrm{Hom}_{\mathrm{gp}/S}(G_1, G_2)$$

the set of morphisms of group schemes over S from G_1 to G_2.

Let G be a finite flat group scheme over a scheme S. Then we shall write G^D the Cartier dual of G, i.e., the finite flat group scheme over S which represents the functor over S

$$T \rightsquigarrow \mathrm{Hom}_{\mathrm{gp}/T}(G \times_S T, \mathbb{G}_{m,T})\,.$$

Note that for a morphism of finite flat group schemes $f\colon G \to H$ over S, it is easily verified that if f is *faithfully flat*, then the morphism of finite flat group schemes $f^D\colon H^D \to G^D$ over S induced by f is a *closed immersion*; moreover, if f is a *closed immersion*, then the morphism $f^D\colon H^D \to G^D$ is *faithfully flat*. Indeed, since G^D, H^D are finite and flat over S, it follows from [3], Corollaire 11.3.11, that by base-changing, we may assume that S is the spectrum of a field. On the other hand, since f is a closed immersion, it is verified that the morphism $\Gamma(G^D, \mathcal{O}_{G^D}) \to \Gamma(H^D, \mathcal{O}_{H^D})$ determined by f^D is *injective*. Thus, it follows from [12], Theorem in 14.1, that f^D is faithfully flat.

Let

$$0 \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \longrightarrow 0$$

be a sequence consisting of finite flat group schemes over a scheme S. Then we shall say that the above sequence is *exact* if f_1 is a closed immersion, and f_2 determines an isomorphism $G_2/G_1 \xrightarrow{\sim} G_3$. (In particular, f_2 is *faithfully flat*, and the kernel of f_2 coincides with $G_1 \subseteq G_2$.) Note that if the sequence of finite flat group schemes over S

$$0 \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \longrightarrow 0$$

is exact, then the sequence

$$0 \longrightarrow G_3^D \xrightarrow{f_2^D} G_2^D \xrightarrow{f_1^D} G_1^D \longrightarrow 0$$

is also *exact*.

Let G be a group scheme over a scheme S, and $\mathcal{M}$ an $\mathcal{O}_S$-module. Then we shall write $t^*_G(\mathcal{M}) \stackrel{\mathrm{def}}{=} e^*_G\Omega^1_{G/S} \otimes_{\mathcal{O}_S} \mathcal{M}$, where $e_G\colon S \to G$ is the identity section of G, and refer to $t^*_G(\mathcal{M})$ as the *$\mathcal{M}$-valued cotangent space* of G; moreover, we shall write $t_G(\mathcal{M}) \stackrel{\mathrm{def}}{=} \mathcal{H}om_{\mathcal{O}_S}(e^*_G\Omega^1_{G/S}, \mathcal{M})$ and refer to $t_G(\mathcal{M})$ as the *$\mathcal{M}$-valued tangent space* of G.

1. Discriminants and cotangent spaces

In this Section , we study the relationship between discriminants and cotangent spaces of finite flat group schemes.

Throughout this paper, let R be a complete discrete valuation ring, k the residue field of R, K the field of fractions of R, $\overline{K}$ an algebraic closure of K, and v_p the valuation of $\overline{K}$ such that $v_p(p) = 1$. Assume, moreover, that K is of characteristic 0, and k is of characteristic $p > 0$ and *perfect*.

Definition 1.1. Let G be a finite flat group scheme over R.

(i) We shall denote by $\mathrm{disc}_R(G) \subseteq R$ the ideal of R obtained as the discriminant of the finite flat R-algebra $\Gamma(G, \mathcal{O}_G)$ over R. Moreover, we shall write $D_R(G) \stackrel{\mathrm{def}}{=} v_p(\mathrm{disc}_R(G))$.

(ii) We shall write $d_G \stackrel{\mathrm{def}}{=} \dim_k(t^*_G(k))$ and refer to d_G as the *dimension* of G.

Lemma 1.2 (Finiteness of cotangent spaces). *Let G be a finite flat group scheme over R. Then the R-module $t^*_G(R)$ is of finite length and generated by exactly d_G elements.*

Proof. The assertion that $t^*_G(R)$ is of finite length follows from the *étaleness* of $G \otimes_R K$ over K; moreover, the assertion that $t^*_G(R)$ is generated by exactly d_G elements follows from the definition of dimension. □

Definition 1.3.

(i) Let M be an R-module of finite length. Then if $M \neq 0$, then there exists a *unique* element

$$(a_1, \cdots, a_{\dim_k(M\otimes_R k)}) \in \mathbb{Q}_{>0}^{\oplus \dim_k(M\otimes_R k)}$$

such that $a_i \leq a_j$ if $i \leq j$, and, moreover, there exists an isomorphism

$$M \simeq \bigoplus_{i=1}^{\dim_k(M \otimes_R k)} R/(p^{a_i})\,.$$

We shall write

$$L_R(M) \stackrel{\mathrm{def}}{=} \{\, a_1, \cdots, a_{\dim_k(M \otimes_R k)} \,\} \subseteq \mathbb{Q}_{>0}$$

for $M \neq 0$, $L_R(\{0\}) \stackrel{\mathrm{def}}{=} \{0\}$, and $|M|_R \stackrel{\mathrm{def}}{=} \sum_{a \in L_R(M)} a \in \mathbb{Q}_{>0}$. Moreover, for an integer n, we shall write $\underline{M}_R = n$ (respectively, $\underline{M}_R \leq n$; respectively, $\underline{M}_R < n$; respectively, $\underline{M}_R \geq n$; respectively, $\underline{M}_R > n$) if $a = n$ for any $a \in L_R(M)$ (respectively, $\max(L_R(M)) \leq n$; respectively, $\max(L_R(M)) < n$; respectively, $\min(L_R(M)) \geq n$; respectively, $\min(L_R(M)) > n$).

Note that it is immediate that if $M \neq 0$, then $v_p(\mathrm{length}_R(M)) = |M|_R$.

(ii) Let G be a finite flat group scheme over R. Then it follows from Lemma 1.2 that $t^*_G(R)$ is of finite length. We shall write $L(t^*_G) \stackrel{\mathrm{def}}{=} L_R(t^*_G(R))$ and $|t^*_G| \stackrel{\mathrm{def}}{=} |t^*_G(R)|_R$. Moreover, for an integer n, we shall write $\underline{t^*_G} = n$ (respectively, $\underline{t^*_G} \leq n$; respectively, $\underline{t^*_G} < n$; respectively, $\underline{t^*_G} \geq n$; respectively, $\underline{t^*_G} > n$) if $\underline{t^*_G(R)}_R = n$ (respectively, $\underline{t^*_G(R)}_R \leq n$; respectively, $\underline{t^*_G(R)}_R < n$; respectively, $\underline{t^*_G(R)}_R \geq n$; respectively, $\underline{t^*_G(R)}_R > n$).

Proposition 1.4 (Discriminants and cotangent spaces). *Let G be a finite flat group scheme over R. Then the following hold:*

$$D_R(G)\,(= (v_p(\mathrm{disc}_R(G))) = \mathrm{rank}_R(G) \cdot |t^*_G|\,.$$

Proof. By the transitivity of discriminant, we may assume without loss of generality that G is *connected*. Then it follows from [6], Lemma 6.1, that there exists an isomorphism of R-algebras

$$\Gamma(G, \mathcal{O}_G) \simeq R[t_1, \cdots, t_{d_G}]/(\Phi_1, \cdots, \Phi_{d_G})\,,$$

where the t_i's are indeterminates, and $(\Phi_1, \cdots, \Phi_{d_G})$ is a regular $R[t_1, \cdots, t_{d_G}]$-sequence. Thus, it follows from [7], Theorem 25.2, that there exists a natural exact sequence of $R[t_1, \cdots, t_{d_G}]$-modules

$$(\Phi_1, \cdots, \Phi_{d_G}) \xrightarrow{d} \bigoplus_{i=1}^{d_G} \Gamma(G, \mathcal{O}_G) \cdot dt_i \longrightarrow \Omega^1_{G/R} \longrightarrow 0\,.$$

Therefore, the assertion follows from [6], Corollary A. 13, together with the definition of $t^*_G(R)$. □

Lemma 1.5 (Isomorphisms of finite flat group schemes). *Let G, H be finite flat group schemes over R, and $f\colon G \to H$ a morphism of* *group schemes over R. Then f is an isomorphism if and only if the following two conditions are satisfied:*

(i) *The morphism $G \otimes_R K \to H \otimes_R K$ over K induced by f is an isomorphism.*
(ii) $|t^*_H| \leq |t^*_G|$.

Proof. By the definition of discriminant, we have that $D_R(G) \leq D_R(H)$. Thus, this follows immediately from Proposition 1.4. □

Lemma 1.6 (Exactness of sequences of cotangent spaces). *If a sequence of finite flat group schemes over R*

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

is exact, then the sequences of R-modules

$$0 \longrightarrow t^*_{G_3}(R) \longrightarrow t^*_{G_2}(R) \longrightarrow t^*_{G_1}(R) \longrightarrow 0\,;$$

$$0 \longrightarrow t_{G_1}(K/R) \longrightarrow t_{G_2}(K/R) \longrightarrow t_{G_3}(K/R) \longrightarrow 0$$

are also exact. In particular, for a morphism of finite flat group schemes $f\colon G \to H$ over R, if f is a closed immersion (respectively, faithfully flat morphism), then the morphism $t^*_H(R) \to t^*_G(R)$ induced by f is surjective (respectively, injective), and the morphism $t_G(K/R) \to t_H(K/R)$ induced by f is injective (respectively, surjective).

Proof. To prove Lemma 1.6, it is immediate that it is enough to show that the sequence

$$0 \longrightarrow t^*_{G_3}(R) \longrightarrow t^*_{G_2}(R) \longrightarrow t^*_{G_1}(R) \longrightarrow 0$$

is exact.

By the transitivity of discriminant, together with Proposition 1.4, we obtain that

$$D_R(G_2) = \mathrm{rank}_R(G_2)\cdot|t^*_{G_2}| = \mathrm{rank}_R(G_1)\cdot D_R(G_3)+\mathrm{rank}_R(G_3)\cdot D_R(G_1)$$

$$= \mathrm{rank}_R(G_2)\cdot(|t^*_{G_1}| + |t^*_{G_3}|)\,;$$

thus, we obtain that $|t^*_{G_2}| = |t^*_{G_1}| + |t^*_{G_3}|$. On the other hand, by definition, the exact sequence of group schemes appearing in the statement of Lemma 1.6 induces an exact sequence of R-modules

$$t^*_{G_3}(R) \longrightarrow t^*_{G_2}(R) \longrightarrow t^*_{G_1}(R) \longrightarrow 0\,.$$

Therefore, by the above equality $|t^*_{G_2}| = |t^*_{G_1}| + |t^*_{G_3}|$, the first arrow $t^*_{G_3}(R) \to t^*_{G_2}(R)$ is injective. This completes the proof of the assertion that the sequence in question is exact. □

2. Review of truncated p-adic Hodge theory

In this Section, we review truncated p-adic Hodge theory for finite flat group schemes as established in [2] and prove lemmas needed later by means of this theory.

We maintain the notation of the preceding Section. Moreover, let $\overline{R}$ be the ring of integers of $\overline{K}$, $\Gamma_K \overset{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$, and $\Omega \overset{\text{def}}{=} \Omega^1_{\overline{R}/R}$.

Definition 2.1.

(i) Let S be a connected scheme. Then we shall say that a finite flat group scheme G over S is a *p-group scheme* if its rank over S is a power of p.

(ii) Let n, h be natural numbers. Then we shall say that a finite flat group scheme G over R is *of p-rectangle-type of level n with height h* if $G(\overline{K})$ is isomorphic to $\bigoplus_h \mathbb{Z}/(p^n)$ as an abstract finite group (cf. Figure 2). Moreover, we shall denote by $\mathrm{lv}(G)$ the *level* of G, and by $\mathrm{ht}(G)$ the *height* of G.

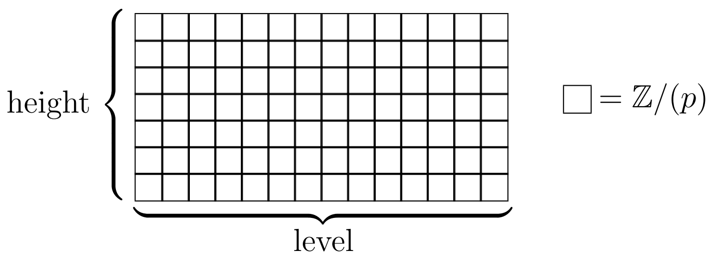

Figure 2: The group of $\overline{K}$-valued points of a group scheme of p-rectangle-type

Remark 2.2.

(i) Any *connected* finite flat group schemes over R are p-group schemes.

(ii) If G is of p-rectangle-type, then the Cartier dual G^D of G is also of p-rectangle-type. Moreover, $\mathrm{lv}(G) = \mathrm{lv}(G^D)$ and $\mathrm{ht}(G) = \mathrm{ht}(G^D)$.

The following lemma follows immediately from definition, together with Lemma 1.2:

Lemma 2.3 (Bound for the lengths of cotangent spaces). *Let G be a finite flat group scheme over R of p-rectangle-type. Then $\underline{t}^*_G \le \mathrm{lv}(G)$. In particular, $|t^*_G| \le d_G \cdot \mathrm{lv}(G)$.*

Definition 2.4. We shall write $\epsilon^{\mathrm{Fon}}_K \overset{\text{def}}{=} 2 + v_p(e_K)$, where e_K is the absolute ramification index of K.

Note that since as is well-known that

$$v_p(\mathfrak{D}_{R/W(k)}) \le 1 - (1/e_K) + v_p(e_K)\,,$$

where $W(k) \subseteq R$ is the ring of Witt vectors with coefficients in k, and $\mathfrak{D}_{R/W(k)} \subseteq R$ is the different of the extension $R/W(k)$ (cf. e.g., [10], Chapter III, Remarks following Proposition 13), we obtain that

$$v_p(\mathfrak{D}_{R/W(k)}) + 1/(p-1) \leq \epsilon_K^{\mathrm{Fon}} .$$

The following proposition follows from [2], Corollaire to Théorème 3, together with [2], Théorème 1′:

Proposition 2.5 (**Existence of functorial morphisms**). *Let G be a p-group scheme over R. Then there exists a functorial morphism of $\overline{R}[\Gamma_K]$-modules*

$$\phi_G \colon G(\overline{K}) \otimes_{\mathbb{Z}_p} \overline{R} \longrightarrow t^*_{G^D}(\overline{R}) \oplus t_G(\Omega) ,$$

where the kernel and cokernel are annihilated by $p^{\epsilon_K^{\mathrm{Fon}}}$; moreover, there exists a natural isomorphism of $\overline{R}[\Gamma_K]$-modules

$$(\overline{K}/\mathfrak{a})(1) \xrightarrow{\sim} \Omega ,$$

where

$$\mathfrak{a} \stackrel{\text{def}}{=} \{\, a \in \overline{K} \mid -v_p(\mathfrak{D}_{R/W(k)}) - 1/(p-1) \leq v_p(a) \,\} \subseteq \overline{K} .$$

Lemma 2.6 (**Bound for the orders of kernels and cokernels**). *Let G, H be p-group schemes over R, $f \colon G \to H$ a morphism of group schemes over R, and n a natural numbers. Then the following hold:*

(i) *If the kernel of the morphism $G(\overline{K}) \to H(\overline{K})$ induced by f is annihilated by p^n, then the cokernel* (*respectively, kernel*) *of the morphism*

$$t^*_H(R) \longrightarrow t^*_G(R) \ (\textit{respectively, } t^*_{G^D}(R) \longrightarrow t^*_{H^D}(R))$$

induced by f is annihilated by $p^{\epsilon_K^{\mathrm{Fon}}+n}$.

(ii) *If the cokernel of the morphism $G(\overline{K}) \to H(\overline{K})$ induced by f is annihilated by p^n, then the kernel* (*respectively, cokernel*) *of the morphism*

$$t^*_H(R) \longrightarrow t^*_G(R) \ (\textit{respectively, } t^*_{G^D}(R) \longrightarrow t^*_{H^D}(R))$$

induced by f is annihilated by $p^{\epsilon_K^{\mathrm{Fon}}+n}$.

Proof. First, we prove assertion (i). Now we have a commutative diagram:

$$\begin{array}{ccc} H^D(\overline{K}) \otimes_{\mathbb{Z}_p} \overline{R} & \xrightarrow{\text{via } f^D} & G^D(\overline{K}) \otimes_{\mathbb{Z}_p} \overline{R} \\ \phi_{H^D} \big\downarrow & & \big\downarrow \phi_{G^D} \\ t^*_H(\overline{R}) \oplus t_{H^D}(\Omega) & \xrightarrow[\text{via } f^D]{} & t^*_G(\overline{R}) \oplus t_{G^D}(\Omega) . \end{array}$$

Since the cokernel of the top horizontal arrow (respectively, right-hand vertical arrow) is annihilated by p^n (respectively, $p^{\epsilon_K^{\mathrm{Fon}}}$), the respective cokernels of the morphisms

$$t^*_H(\overline{R}) \longrightarrow t^*_G(\overline{R}) \ ; \ \ t_{H^D}(\Omega) \longrightarrow t_{G^D}(\Omega)$$

determined by f are annihilated by $p^{\epsilon_K^{\mathrm{Fon}}+n}$. Thus, the kernel of the morphism $t^*_{G^D}(R) \to t^*_{H^D}(R)$ determined by f is annihilated by $p^{\epsilon_K^{\mathrm{Fon}}+n}$. This completes the proof of assertion (i). Moreover, by taking "$(-)^D$", assertion (ii) follows from assertion (i). □

Lemma 2.7 (**Orders of generators of cotangent spaces**)**.** *Let G be a finite flat group scheme of p-rectangle-type of level $> 2\epsilon_K^{\mathrm{Fon}}$ over R, and $a \in L(t^*_G)$ (cf. Definition 1.3,* (ii)*). Then $0 \le a \le \epsilon_K^{\mathrm{Fon}}$ or $\mathrm{lv}(G) - \epsilon_K^{\mathrm{Fon}} \le a \le \mathrm{lv}(G)$ (cf. Figure 3).*

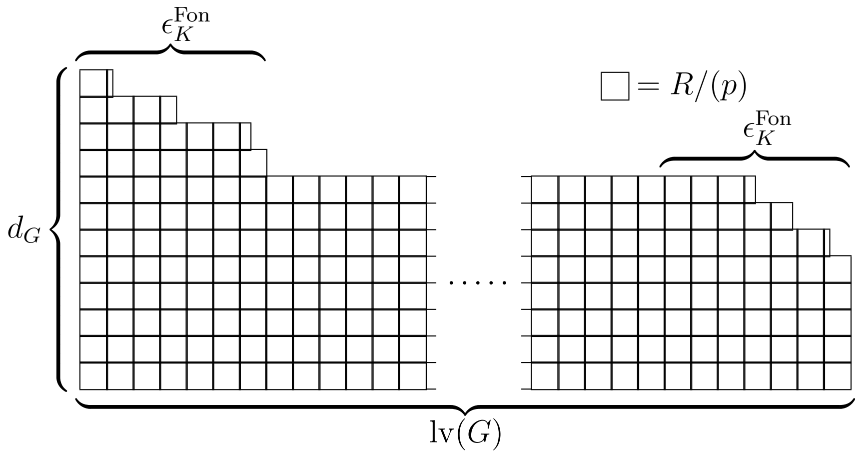

Figure 3: $t^*_G(R)$

Proof. If $\underline{t^*_G} \le \epsilon_K^{\mathrm{Fon}}$, then the assertion is immediate; thus, we may assume that $p^{\epsilon_K^{\mathrm{Fon}}} \cdot t^*_G(R) \neq 0$. By the definition of $L(t^*_G)$, to prove Lemma 2.7, it is immediate that it is enough to show that

$$\underline{p^{\epsilon_K^{\mathrm{Fon}}} \cdot t^*_G(R)}_R \ge \mathrm{lv}(G) - 2\epsilon_K^{\mathrm{Fon}}$$

(cf. Definition 1.3, (i)); thus, we prove this assertion.

Since the kernel and cokernel of the morphism

$$\phi_{G^D} \colon A_G \stackrel{\mathrm{def}}{=} G^D(\overline{R}) \otimes_{\mathbb{Z}_p} \overline{R} \longrightarrow T_G \stackrel{\mathrm{def}}{=} t^*_G(\overline{R}) \oplus t_{G^D}(\Omega)$$

are annihilated by $p^{\epsilon_K^{\mathrm{Fon}}}$, the composite $A_G \stackrel{\phi_{G^D}}{\to} T_G \twoheadrightarrow p^{\epsilon_K^{\mathrm{Fon}}} \cdot T_G$ is *surjective*, and the natural surjection $A_G \twoheadrightarrow p^{2\epsilon_K^{\mathrm{Fon}}} \cdot A_G$ ($\neq 0$, since $2\epsilon_K^{\mathrm{Fon}} < \mathrm{lv}(G)$) factors through this composite $A_G \stackrel{\phi_{G^D}}{\to} T_G \twoheadrightarrow p^{\epsilon_K^{\mathrm{Fon}}} \cdot T_G$, i.e., we obtain a sequence of *surjections*:

$$A_G \longrightarrow p^{\epsilon_K^{\mathrm{Fon}}} \cdot T_G \longrightarrow p^{2\epsilon_K^{\mathrm{Fon}}} \cdot A_G \, (\neq 0) \, .$$

Therefore, by the definition of the term "of p-rectangle-type", it is easily verified that $\underline{p^{\epsilon_K^{\mathrm{Fon}}} \cdot t^*_G(R)}_R \ge \mathrm{lv}(G) - 2\epsilon_K^{\mathrm{Fon}}$. This completes the proof of Lemma 2.7. □

Definition 2.8. Let G be a finite flat group scheme over R, M a module, and n a natural number.

(i) We shall write

$$d_G^\circ \stackrel{\text{def}}{=} \dim_k((p^{\epsilon_K^{\text{Fon}}} \cdot t_G^*(R)) \otimes_R k) \, (\leq d_G)$$

(cf. Figure 4).

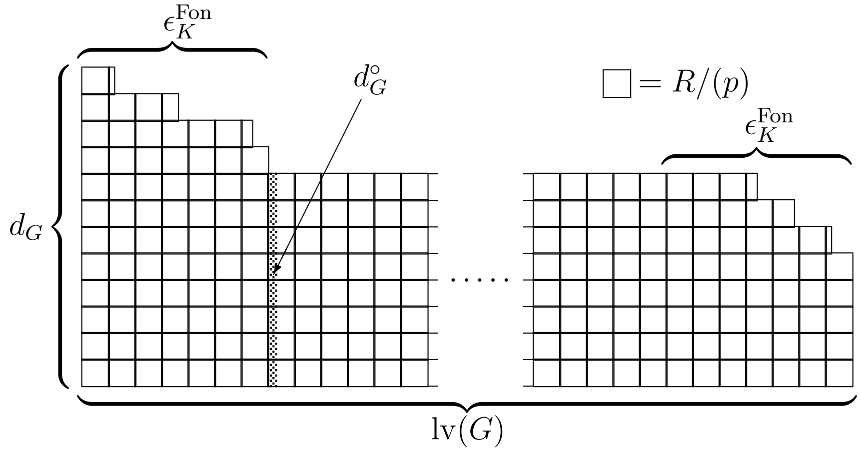

Figure 4: d_G°

Note that it follows from the proof of Lemma 2.7 that if G is of p-rectangle-type of level $> 2\epsilon_K^{\text{Fon}}$, then $d_G^\circ + d_{G^D}^\circ = \text{ht}(G)$.

(ii) Then we shall say that $x \in M$ is *n-primitive* if the following condition is satisfied:

$p^n x = 0$ *and* $x \notin p \cdot M$.

Note that for an R-module M of finite length, M has *no* n-primitive element if and only if $M = 0$ or $\underline{M}_R > n + 1$.

Remark 2.9. For a finite flat group scheme G of p-rectangle-type of level $> 2\epsilon_K^{\text{Fon}}$ over R which is *not* étale, the following conditions are equivalent:

(i) $\underline{t}_G^* > \epsilon_K^{\text{Fon}}$.
(ii) $\underline{t}_G^* \geq \text{lv}(G) - \epsilon_K^{\text{Fon}}$.
(iii) $d_G^\circ = d_G$.
(iv) $t_G^*(R)$ has no $(\epsilon_K^{\text{Fon}} - 1)$-primitive element.

Indeed, the assertion that (i) is equivalent to (ii) (respectively, (iii); respectively, (iv)) follows from Lemma 2.7 (respectively, Lemma 2.7; respectively, the definition of $(\epsilon_K^{\text{Fon}} - 1)$-primitive element).

Lemma 2.10 (**Facts concerning modified dimensions**)**.** *Let G, H be finite flat group schemes of p-rectangle-type over R, $f\colon G \to H$ a morphism of group schemes over R, and n a natural number. Then the following hold:*

(i) *If the kernel of the morphism $G(\overline{K}) \to H(\overline{K})$ is annihilated by p^n, and $3\epsilon_K^{\text{Fon}} + n \leq \text{lv}(G), \text{lv}(H)$, then $d_G^\circ \leq d_H^\circ$ and $d_{G^D}^\circ \leq d_{H^D}^\circ$.*

(ii) *If the cokernel of the morphism $G(\overline{K}) \to H(\overline{K})$ is annihilated by p^n, and $3\epsilon_K^{\text{Fon}} + n \leq \text{lv}(G), \text{lv}(H)$, then $d_H^\circ \leq d_G^\circ$ and $d_{H^D}^\circ \leq d_G^\circ$.*

Proof. First, we prove assertion (i). Let $L \stackrel{\text{def}}{=} \min\{\mathrm{lv}(G), \mathrm{lv}(H)\}$. Then it follows from Lemma 2.6 that the respective cokernels of the morphisms

$$(p^{\epsilon_K^{\mathrm{Fon}}} \cdot t_H^*(R)) \otimes_R R/(p^{L-2\epsilon_K^{\mathrm{Fon}}}) \longrightarrow (p^{\epsilon_K^{\mathrm{Fon}}} \cdot t_G^*(R)) \otimes_R R/(p^{L-2\epsilon_K^{\mathrm{Fon}}})\,;$$

$$(p^{\epsilon_K^{\mathrm{Fon}}} \cdot t_{H^D}(K/R)) \otimes_R R/(p^{L-2\epsilon_K^{\mathrm{Fon}}}) \longrightarrow (p^{\epsilon_K^{\mathrm{Fon}}} \cdot t_{G^D}(K/R)) \otimes_R R/(p^{L-2\epsilon_K^{\mathrm{Fon}}})$$

induced by f are annihilated by $p^{\epsilon_K^{\mathrm{Fon}}+n}$; moreover, it follows from Lemma 2.7 that $(p^{\epsilon_K^{\mathrm{Fon}}} \cdot t_G^*(R)) \otimes_R R/(p^{L-2\epsilon_K^{\mathrm{Fon}}})$ (respectively, $(p^{\epsilon_K^{\mathrm{Fon}}} \cdot t_H^*(R)) \otimes_R R/(p^{L-2\epsilon_K^{\mathrm{Fon}}})$; respectively, $(p^{\epsilon_K^{\mathrm{Fon}}} \cdot t_{G^D}(K/R)) \otimes_R R/(p^{L-2\epsilon_K^{\mathrm{Fon}}})$; respectively, $(p^{\epsilon_K^{\mathrm{Fon}}} \cdot t_{H^D}(K/R)) \otimes_R R/(p^{L-2\epsilon_K^{\mathrm{Fon}}})$) is a *free* $R/(p^{L-2\epsilon_K^{\mathrm{Fon}}})$-module of rank d_G° (respectively, d_H°; respectively, $d_{G^D}^\circ$; respectively, $d_{H^D}^\circ$). Therefore, since $\epsilon_K^{\mathrm{Fon}} + n \leq L - 2\epsilon_K^{\mathrm{Fon}}$, we obtain that $d_G^\circ \leq d_H^\circ$ and $d_{G^D}^\circ \leq d_{H^D}^\circ$. This completes the proof of assertion (i). Moreover, by taking "$(-)^D$", assertion (ii) follows from assertion (i). □

Lemma 2.11 (**Freeness of cotangent spaces of certain group schemes**). *Let G be a finite flat group scheme of p-rectangle-type over R. Then if $d_G + d_{G^D} = \mathrm{ht}(G)$, then $t_G^*(R)$ is free over $R/(p^{\mathrm{lv}(G)})$.*

Proof. To prove Lemma 2.11, it is immediate that it is enough to show that $|t_G^*| = \mathrm{lv}(G) \cdot d_G$, i.e., it is enough to show that $D_R(G) = p^{\mathrm{lv}(G)\cdot \mathrm{ht}(G)} \mathrm{lv}(G) \cdot d_G$ by Proposition 1.4. Now it follows from Proposition 1.4; Lemma 2.3 that $D_R(G) \leq p^{\mathrm{lv}(G)\cdot \mathrm{ht}(G)} \mathrm{lv}(G) \cdot d_G$. Moreover, again by Proposition 1.4; Lemma 2.3, we obtain that $D_R(G^D) \leq p^{\mathrm{lv}(G)\cdot \mathrm{ht}(G)} \mathrm{lv}(G) \cdot d_{G^D}$. On the other hand, it follows from [9], Proposition 9 in Appendice, that $D_R(G) + D_R(G^D) = \mathrm{rank}_R(G) \cdot v_p(\mathrm{rank}_R(G)) = p^{\mathrm{lv}(G)\cdot \mathrm{ht}(G)} \mathrm{lv}(G) \cdot \mathrm{ht}(G)$. Thus, since $d_G + d_{G^D} = \mathrm{ht}(G)$, we obtain that $D_R(G) = p^{\mathrm{lv}(G)\cdot \mathrm{ht}(G)} \mathrm{lv}(G) \cdot d_G$. □

Lemma 2.12 (**Isomorphisms of group schemes of *p*-rectangle-type**). *Let G, H be finite flat group schemes of p-rectangle-type of level $\geq 3\epsilon_K^{\mathrm{Fon}}$ over R, and $f: G \to H$ a morphism of group schemes over R. Assume that $t_G^*(R)$ is free over $R/(p^{\mathrm{lv}(G)})$, and $\underline{t}_H^* > \epsilon_K^{\mathrm{Fon}}$. Then f is an isomorphism if and only if the morphism $G \otimes_R K \to H \otimes_R K$ over K induced by f is an isomorphism.*

Proof. The "only if" part of the assertion is immediate; thus, we prove the "if" part of the assertion. Since the morphism $G \otimes_R K \to H \otimes_R K$ over K induced by f is an isomorphism, we obtain that $\mathrm{lv}(G) = \mathrm{lv}(H)$ and $d_G = d_H$ (cf. Remark 2.9; Lemma 2.10). Thus, it follows from Lemma 2.3 that $|t_H^*| \leq \mathrm{lv}(G) \cdot d_G$. On the other hand, since $|t_G^*| = \mathrm{lv}(G) \cdot d_G$, we obtain that $|t_H^*| \leq |t_G^*|$. Therefore, it follows from Lemma 1.5 that f is an isomorphism. □

Finally, we review the notion of *truncated Barsotti-Tate group schemes.*

Definition 2.13 (cf. e.g., [4], Définition 1.1)**.** Let S be a connected scheme. Then we shall say that a finite flat group scheme G over S is *truncated* (p-)*Barsotti-Tate* (*of level* ≥ 2) if there exist natural numbers n and h such that the following condition is satisfied:

> $n \geq 2$ and G is of rank p^{nh}. Moreover, for any natural number $m \leq n$, the morphism $G \to \mathrm{Im}(p_G^m)$, where $\mathrm{Im}(p_G^m)$ is the scheme-theoretic image of p_G^m, determined by p_G^m is *faithfully flat* (thus, $\mathrm{Ker}(p_G^m)$ is *flat* over S), and the finite flat group scheme $\mathrm{Ker}(p_G^m)$ over S is of rank p^{mh}.

For a truncated Barsotti-Tate group scheme G over S, and a natural number m, we shall write $G[p^m] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(p_G^m)$.

Remark 2.14.

(i) Any truncated Barsotti-Tate group schemes are of p-rectangle-type.

(ii) If G is truncated Barsotti-Tate, then the Cartier dual G^D of G is also truncated Barsotti-Tate.

Lemma 2.15 (**Freeness of cotangent spaces of truncated Barsotti-Tate group schemes**)**.** *Let G be a truncated Barsotti-Tate group schemes over R. Then $d_G + d_{G^D} = \mathrm{lv}(G)$. In particular, by Lemma 2.11, $t_G^*(R)$ is free over $R/(p^{\mathrm{lv}(G)})$.*

Proof. This follows from a similar argument to the argument used in the proof of [11], Proposition 3. □

Remark 2.16. The assertion that $t_G^*(R)$ is free over $R/(p^{\mathrm{lv}(G)})$ can be also proven by means of [2], Proposition 10, together with Proposition 2.17 below.

Proposition 2.17 (**Existence of certain Barsotti-Tate groups**)**.** *Let G be a truncated Barsotti-Tate group scheme over R. Then there exists a Barsotti-Tate group $\mathcal{G}$ over R such that G is isomorphic to $\mathrm{Ker}(p^{\mathrm{lv}(G)} \colon \mathcal{G} \to \mathcal{G})$.*

Proof. This follows from [4], Théorème 4.4, (e). □

3. Proof of the main theorem

In this Section, we prove the main theorem, i.e., Theorem 3.4 below. We maintain the notation of the preceding Section.

Lemma 3.1 (**Split injections of R-modules**)**.** *Let M, N be R-modules of finite length, $f \colon M \to N$ a morphism of R-modules, and m a natural number. Then the following hold:*

(i) *If M and N are free over $R/(p^m)$, and the morphism $M \otimes_R k \to N \otimes_R k$ induced by f is injective, then f is injective, and the image of f is a direct summand of N.*

(ii) *Assume that the following conditions are satisfied:*
   (1) *The morphism* $M \otimes_R R/(p^m) \to N \otimes_R R/(p^m)$ *is injective, and its image is a direct summand of* $N \otimes_R R/(p^m)$.
   (2) *The morphism* $p^m \cdot M \to p^m \cdot N$ *is injective, and its image is a direct summand of* $p^m \cdot N$.

Then f *is injective, and the image of* f *is a direct summand of* N.

Proof. Assertion (i) is immediate; thus, we prove assertion (ii). By assumptions (1), (2), it is immediate that f is injective. By means of this injectivity of f, we regard M as an R-submodule of N. First, observe that, by [7], Theorem 7.14, to prove assertion (ii), it is enough to show that for any $r \in (1/e_K) \cdot \mathbb{N}$, where e_K is the absolute ramification index of K, the natural inclusion $p^r \cdot N \hookrightarrow (p^r \cdot M) \cap N$ is an isomorphism.

If $r \leq m$, then it follows from assumption (1) that the natural inclusion $p^r \cdot N \hookrightarrow (p^r \cdot M) \cap N$ is an isomorphism. Assume that $m < r$. Since $(p^r \cdot M) \cap N \subseteq (p^m \cdot M) \cap N \subseteq p^m \cdot N$, we obtain that $(p^r \cdot M) \cap N \subseteq (p^{r-m} \cdot (p^m \cdot M)) \cap (p^m \cdot N)$. Now since $(p^{r-m} \cdot (p^m \cdot M)) \cap (p^m \cdot N) \subseteq p^{r-m} \cdot (p^m \cdot N) = p^r \cdot N$ by assumption (2), it follows that $(p^r \cdot M) \cap N \subseteq p^r \cdot N$. This completes the proof of assertion (ii). □

Lemma 3.2 (**Split injections of tangent spaces**). *Let* G, H *be finite flat group schemes over* R, *and* $f\colon G \to H$ *a morphism of group schemes. Assume that the following three conditions are satisfied:*

(i) *The morphism* $G \otimes_R K \to H \otimes_R K$ *determined by* f *is an isomorphism.*
(ii) G *is of* p*-rectangle-type, and* $2\epsilon_K^{\mathrm{Fon}} < \mathrm{lv}(G)$. (*Thus, by* (i), H *is also of* p*-rectangle-type, and* $2\epsilon_K^{\mathrm{Fon}} < \mathrm{lv}(H)$.)
(iii) *The morphism*

$$t_G(K/R) \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}+1}) \longrightarrow t_H(K/R) \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}+1})$$

(*cf. Figure* 5) *determined by* f *is injective, and its image is a direct summand of* $t_H(K/R) \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}+1})$.

Then the morphism

$$N_G \stackrel{\mathrm{def}}{=} t_G(K/R)\otimes_R R/(p^{\mathrm{lv}(G)-\epsilon_K^{\mathrm{Fon}}}) \longrightarrow N_H \stackrel{\mathrm{def}}{=} t_H(K/R)\otimes_R R/(p^{\mathrm{lv}(G)-\epsilon_K^{\mathrm{Fon}}})$$

(*cf. Figure* 6) *determined by* f *is injective, and its image is a direct summand of* N_H.

Proof. It follows from Lemmas 2.7, together with (ii), that $p^{\epsilon_K^{\mathrm{Fon}}} \cdot N_G$ and $p^{\epsilon_K^{\mathrm{Fon}}} \cdot N_H$ are *free* over $R/(p^{\mathrm{lv}(G)-2\epsilon_K^{\mathrm{Fon}}})$; moreover, by (iii), the morphism

$$(p^{\epsilon_K^{\mathrm{Fon}}} \cdot N_G) \otimes_R k \longrightarrow (p^{\epsilon_K^{\mathrm{Fon}}} \cdot N_H) \otimes_R k$$

determined by f is *injective*. Thus, it follows from Lemma 3.1, (i), that the morphism $p^{\epsilon_K^{\mathrm{Fon}}} \cdot N_G \to p^{\epsilon_K^{\mathrm{Fon}}} \cdot N_H$ is injective, and its image is a direct summand. On the other hand, again by (iii), the morphism $N_G \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}}) \to N_H \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}})$ is injective, and its image is a direct summand. Therefore, the assertion follows from Lemma 3.1, (ii).

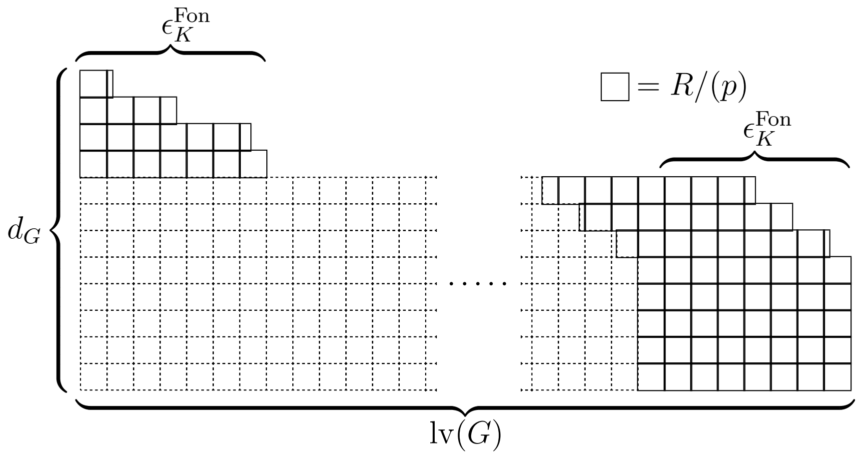

Figure 5 : $t_G(K/R) \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}+1})$

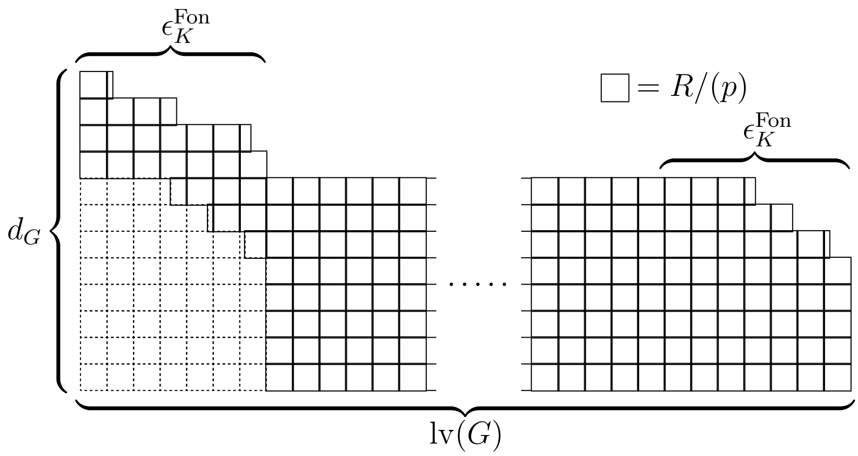

Figure 6 : N_G

$\square$

Lemma 3.3 (**Non-existence of** ($\boldsymbol{\epsilon_K^{\mathrm{Fon}}} - \mathbf{1}$)**-primitive elements of the cotangent spaces of certain group schemes**). *Let G, H be truncated Barsotti-Tate group schemes over R, X a finite flat group scheme over R, $G \times_R H \to X$ a morphism of group schemes which is faithfully flat, and n a natural number. Assume that the following four conditions are satisfed:*

(i) *The composite*

$$f_G \colon G \longrightarrow G \times_R H \longrightarrow X\,,$$

where the first arrow is the morphism induced by the identity section of H, determines an isomorphism $G(\overline{K}) \xrightarrow{\sim} X(\overline{K})$.

(ii) *The kernel of the morphism $H(\overline{K}) \to X(\overline{K})$ induced by the composite*

$$f_H \colon H \longrightarrow G \times_R H \longrightarrow X \,,$$

where the first arrow is the morphism induced by the identity section of G, is annihilated by p^n.

(iii) *The image of the morphism*

$$t_G(K/R) \otimes_R R/(p^{\mathrm{lv}(G)-\epsilon_K^{\mathrm{Fon}}}) \longrightarrow t_X(K/R) \otimes_R R/(p^{\mathrm{lv}(G)-\epsilon_K^{\mathrm{Fon}}})$$

(*cf. Figure 6*) *determined by f_G is a direct summand of $t_X(K/R) \otimes_R R/(p^{\mathrm{lv}(G)-\epsilon_K^{\mathrm{Fon}}})$.*

(iv) $4\epsilon_K^{\mathrm{Fon}} + n \leq \mathrm{lv}(H)$. (*Note that since* $\mathrm{lv}(H) - n \leq \mathrm{lv}(G)$ *by assumptions* (i), (ii), *it follows that* $4\epsilon_K^{\mathrm{Fon}} \leq \mathrm{lv}(G)$.)

Then $t_X^*(R)$ has no $(\epsilon_K^{\mathrm{Fon}} - 1)$-primitive element.

Proof. Assume that there exists an $(\epsilon_K^{\mathrm{Fon}} - 1)$-primitive element $\omega \in t_X^*(R)$. Then by the following steps, we obtain a contradiction.

(Step 1) *We shall write*

$$M_G \stackrel{\mathrm{def}}{=} \mathrm{Hom}_R(t_G(K/R) \otimes_R R/(p^{\mathrm{lv}(G)-\epsilon_K^{\mathrm{Fon}}}), K/R) \otimes_R \overline{R} \subseteq t_G^*(\overline{R}) \;;$$

$$M_X \stackrel{\mathrm{def}}{=} \mathrm{Hom}_R(t_X(K/R) \otimes_R R/(p^{\mathrm{lv}(G)-\epsilon_K^{\mathrm{Fon}}}), K/R) \otimes_R \overline{R} \subseteq t_X^*(\overline{R})$$

(*cf. Figure 7*).

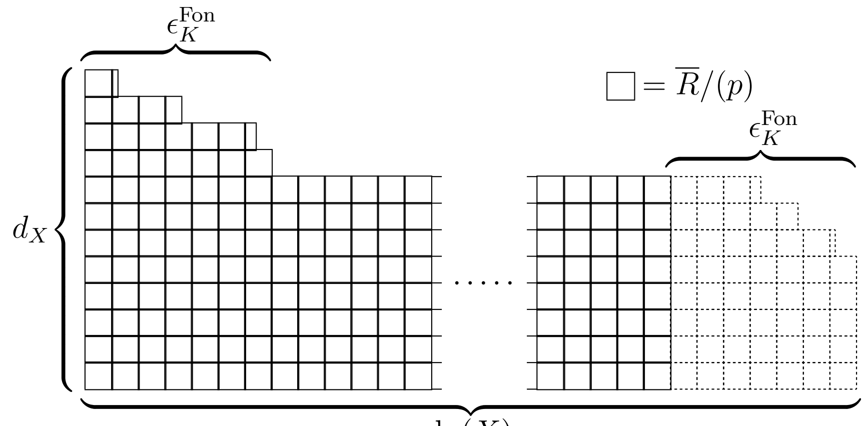

Figure 7 : M_X

Then the following hold:

(1-i) *The surjection*

$$M_X \longrightarrow \mathrm{Im}(f_G^* \colon M_X \to M_G)$$

determined by f_G splits.

(1-ii) $\mathrm{Ker}(p^{\mathrm{lv}(G)-\epsilon_K^{\mathrm{Fon}}} : t_X^*(\overline{R}) \to t_X^*(\overline{R})) = M_X$.

(1-iii) $p^{\epsilon_K^{\mathrm{Fon}}} \cdot t_X^*(\overline{R}) \subseteq M_X$.

Proof. Assertion (1-i) follows from assumption (iii). Assertion (1-ii) follows from the existence of the exact sequence

$$0 \longrightarrow M_X \longrightarrow t_X^*(\overline{R}) \longrightarrow \mathrm{Hom}_R(p^{\mathrm{lv}(G)-\epsilon_K^{\mathrm{Fon}}} \cdot t_X(K/R), K/R) \otimes_R \overline{R} \longrightarrow 0$$

(cf. the definition of M_X). Assertion (1-iii) follows assertion (1-ii), together with Lemma 2.3. ☐

(Step 2) *There exists an element* $\omega_H \in t_H^*(R)$ *such that* $p^{\mathrm{lv}(H)-\epsilon_K^{\mathrm{Fon}}+1}\omega_H = f_H^*(\omega)$.

Proof. Since $p^{\epsilon_K^{\mathrm{Fon}}-1}\omega = 0$, we obtain that $p^{\epsilon_K^{\mathrm{Fon}}-1} f_H^*(\omega) = 0$, i.e., $f_H^*(\omega) \in \mathrm{Ker}(p^{\epsilon_K^{\mathrm{Fon}}-1} \colon t_H^*(R) \to t_H^*(R)) = p^{\mathrm{lv}(H)-\epsilon_K^{\mathrm{Fon}}+1} \cdot t_H^*(R)$ (cf. Lemma 2.15). Thus, such an element exists. ☐

(Step 3) *There exists* $h \in H^D(\overline{K}) \otimes_{\mathbb{Z}_p} \overline{R}$ *such that the image of* h *via the morphism*

$$\phi_{H^D} \colon H^D(\overline{K}) \otimes_{\mathbb{Z}_p} \overline{R} \longrightarrow t_H^*(\overline{R}) \oplus t_{H^D}(\Omega)$$

(*cf. Proposition 2.5*) *is* $(p^{\epsilon_K^{\mathrm{Fon}}}\omega_H, 0)$; *moreover, there exists* $x \in X^D(\overline{K}) \otimes_{\mathbb{Z}_p} \overline{R}$ *such that the image of* x *via the morphism* $X^D(\overline{K}) \otimes_{\mathbb{Z}_p} \overline{R} \to H^D(\overline{K}) \otimes_{\mathbb{Z}_p} \overline{R}$ *induced by* f_H *is* $p^n h$.

Proof. This follows from the fact that the cokernel of ϕ_{H^D} (respectively, the morphism $X^D(\overline{K}) \to H^D(\overline{K})$ induced by f_H) is annihilated by $p^{\epsilon_K^{\mathrm{Fon}}}$ (respectively, p^n [cf. assumption (ii)]). ☐

(Step 4) *We shall write* $(\eta, \tau) \overset{\mathrm{def}}{=} \phi_{X^D}(x) \in t_X^*(\overline{R}) \oplus t_{X^D}(\Omega)$, *and* $\omega_1 \overset{\mathrm{def}}{=} \omega - p^{\mathrm{lv}(H)-2\epsilon_K^{\mathrm{Fon}}-n+1}\eta \in t_X^*(\overline{R})$. *Then the following hold:*

(4-i) $\omega_1 \notin p \cdot t_X^*(\overline{R})$; *in particular,* $\omega_1 \neq 0$.

(4-ii) $p^{\mathrm{lv}(G)-\mathrm{lv}(H)+2\epsilon_K^{\mathrm{Fon}}+n-1}\omega_1 = 0$. (*Note that since* $\mathrm{lv}(H) - n \leq \mathrm{lv}(G)$ *by assumption* (ii), $0 \leq 2\epsilon_K^{\mathrm{Fon}} - 1 \leq \mathrm{lv}(G) - \mathrm{lv}(H) + 2\epsilon_K^{\mathrm{Fon}} + n - 1$.)

(4-iii) *The image* $f_H^*(\omega_1) \in t_H^*(\overline{R})$ *of* ω_1 *in* $t_H^*(\overline{R})$ *vanishes.*

Proof. Assertion (4-i) follows from the assumption that ω is $(\epsilon_K^{\mathrm{Fon}} - 1)$-primitive, together with the assumption that $1 \leq \mathrm{lv}(H) - 2\epsilon_K^{\mathrm{Fon}} - n + 1$ (cf. assumption (iv)). Assertion (4-ii) follows from the assumption that ω is $(\epsilon_K^{\mathrm{Fon}} - 1)$-primitive, together with $p^{\mathrm{lv}(G)} \cdot t_X^*(\overline{R}) = 0$ (cf. assumption (i), also Lemma 2.3). Assertion (4-iii) follows from the fact that the images of ω and $p^{\mathrm{lv}(H)-2\epsilon_K^{\mathrm{Fon}}-n+1}\eta$ in $t_H^*(\overline{R})$ are $p^{\mathrm{lv}(H)-\epsilon_K^{\mathrm{Fon}}+1}\omega_H$ (cf. Steps 2 and 3). ☐

(Step 5) $f_G^*(\omega_1) \in p^{\mathrm{lv}(H)-2\epsilon_K^{\mathrm{Fon}}-n+1} \cdot t_G^*(\overline{R})$.

Proof. Since $p^{\mathrm{lv}(G)-\mathrm{lv}(H)+2\epsilon_K^{\mathrm{Fon}}+n-1} f_G^*(\omega_1) = 0$ (cf. Step 4, (4-ii)), $f_G^*(\omega_1) \in \mathrm{Ker}(p^{\mathrm{lv}(G)-\mathrm{lv}(H)+2\epsilon_K^{\mathrm{Fon}}+n-1} \colon t_G^*(\overline{R}) \to t_G^*(\overline{R})) = p^{\mathrm{lv}(H)-2\epsilon_K^{\mathrm{Fon}}-n+1} \cdot t_G^*(\overline{R})$ (cf. Lemma 2.15). □

(Step 6) $f_G^*(\omega_1) \notin p^{2\epsilon_K^{\mathrm{Fon}}+1} \cdot t_G^*(\overline{R})$.

Proof. Since $p^{\mathrm{lv}(G)-\mathrm{lv}(H)+2\epsilon_K^{\mathrm{Fon}}+n-1}\omega_1 = 0$ (cf. Step 4, (4-ii)), it follows from Step 1, (i-ii), together with assumption (iv), that $\omega_1 \in M_X \subseteq t_X^*(\overline{R})$ (cf. Step 1). Moreover, since the morphism $t_X^*(R) \to t_G^*(R) \oplus t_H^*(R)$ induced by the faithfully flat morphism $G \times_R H \to X$ is *injective* (cf. Lemma 1.6), it follows from Step 1, (1-i), together with Step 4, (4-iii), that we obtain an isomorphism

$$M_X \simeq \mathrm{Ker}(f_*^G \colon M_X \to M_G) \oplus \mathrm{Im}(f_*^G \colon M_X \to M_G)\,,$$

and

$$\omega_1 = (0, f_G^*(\omega_1)) \in \mathrm{Ker}(f_G^* \colon M_X \to M_G) \oplus \mathrm{Im}(f_G^* \colon M_X \to M_G) \simeq M_X\,.$$

Therefore, it follows from Step 4, (4-i), that $f_G^*(\omega_1) \notin p \cdot \mathrm{Im}(f_G^* : M_X \to M_G)$; moreover, since

$$p^{2\epsilon_K^{\mathrm{Fon}}+1} \cdot t_G^*(R) \subseteq p^{\epsilon_K^{\mathrm{Fon}}+1} \cdot \mathrm{Im}(f_G^* \colon t_X^*(R) \to t_G^*(R)) \subseteq p \cdot \mathrm{Im}(f_G^* \colon M_X \to M_G)$$

(cf. Lemma 2.6, (i), together with Step 1, (1-iii)), we obtain that $f_G^*(\omega_1) \notin p^{2\epsilon_K^{\mathrm{Fon}}+1} \cdot t_G^*(R)$. □

By Steps 5 and 6, together with the assumption that $2\epsilon_K^{\mathrm{Fon}} + 1 \leq \mathrm{lv}(H) - 2\epsilon_K^{\mathrm{Fon}} - n + 1$, we obtain a contradiction. This completes the proof of Lemma 3.3. □

Theorem 3.4 (**Extension of morphisms between generic fibers I**). *Let R be a complete discrete valuation ring whose residue field* (*respectively, field of fractions K*) *is of characteristic $p > 0$* (*respectively,* 0) *and perfect, $\overline{K}$ an algebraic closure of K, v_p the p-adic valuation of $\overline{K}$ such that $v_p(p) = 1$, e_K the absolute ramification index of K, $\epsilon_K^{\mathrm{Fon}} \stackrel{\mathrm{def}}{=} 2 + v_p(e_K)$, G and H truncated Barsotti-Tate group schemes over R, $f_K \colon G_K \stackrel{\mathrm{def}}{=} G \otimes_R K \to H_K \stackrel{\mathrm{def}}{=} H \otimes_R K$ a morphism of group schemes over K, and n a natural number. Assume that one of the following conditions is satisfied:*

- (i) *The cokernel of the morphism $G_K(\overline{K}) \to H_K(\overline{K})$ determined by f_K is annihilated by p^n, and $4\epsilon_K^{\mathrm{Fon}} + n \leq \mathrm{lv}(H)$.* (*Note that since* $\mathrm{lv}(H) \leq \mathrm{lv}(G) + n$, *it follows that* $4\epsilon_K^{\mathrm{Fon}} \leq \mathrm{lv}(G)$.)
- (ii) *The kernel of the morphism $G_K(\overline{K}) \to H_K(\overline{K})$ determined by f_K is annihilated by p^n, and $4\epsilon_K^{\mathrm{Fon}} + n \leq \mathrm{lv}(G)$.*

Then the morphism f_K extends uniquely to a morphism over R.

Proof. By taking "$(-)^D$" if necessary, we may assume that condition (i) is satisfied. Moreover, if H^D is étale over R, then the assertion is immediate; thus, we may assume that H^D is *not* étale over R.

Let $\mathcal{G}$ be a Barsotti-Tate group over R such that $G \simeq \mathrm{Ker}(p^{\mathrm{lv}(G)} : \mathcal{G} \to \mathcal{G})$ (cf. Proposition 2.17), and $\widetilde{G} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(p^{\mathrm{lv}(G)+\epsilon_K^{\mathrm{Fon}}+1} : \mathcal{G} \to \mathcal{G})$. Then the endomorphism $p_{\widetilde{G}}^{\epsilon_K^{\mathrm{Fon}}+1}$ of $\widetilde{G}$ factors through $G \subseteq \widetilde{G}$, and the resulting morphism fits into the following *exact* sequence

$$0 \longrightarrow \widetilde{G}[p^{\epsilon_K^{\mathrm{Fon}}+1}] \longrightarrow \widetilde{G} \stackrel{\mathrm{via}\ p_{\widetilde{G}}^{\epsilon_K^{\mathrm{Fon}}+1}}{\longrightarrow} G \longrightarrow 0$$

(cf. Definition 2.13). Now we shall denote by g_K the composite

$$\widetilde{G}_K \stackrel{\mathrm{def}}{=} \widetilde{G} \otimes_R K \stackrel{\mathrm{via}\ p_{\widetilde{G}}^{\epsilon_K^{\mathrm{Fon}}+1}}{\longrightarrow} G_K \stackrel{f_K}{\longrightarrow} H_K\,.$$

First, I *claim* that to prove Theorem 3.4, it is enough to show that g_K extends to a morphism over R. Indeed, since the morphism $\widetilde{G} \to G$ is faithfully flat, the morphism $\Gamma(G, \mathcal{O}_G) \to \Gamma(\widetilde{G}, \mathcal{O}_{\widetilde{G}})$ is injective, and its image is a direct summand. In particular, we obtain that

$$\Gamma(G, \mathcal{O}_G) = \Gamma(\widetilde{G}, \mathcal{O}_{\widetilde{G}}) \cap (\Gamma(G, \mathcal{O}_G) \otimes_R K)\,.$$

Now if g_K extends to a morphism $g : \widetilde{G} \to H$ over R, then by the construction of g, the morphism $\Gamma(H, \mathcal{O}_H) \to \Gamma(\widetilde{G}, \mathcal{O}_{\widetilde{G}})$ determined by g factors through $\Gamma(\widetilde{G}, \mathcal{O}_{\widetilde{G}}) \cap (\Gamma(G, \mathcal{O}_G) \otimes_R K)$; in particular, the morphism $\Gamma(H, \mathcal{O}_H) \to \Gamma(\widetilde{G}, \mathcal{O}_{\widetilde{G}})$ determined by g factors through $\Gamma(G, \mathcal{O}_G)$. This completes the proof of the first *claim*.

For a natural number $m \leq \mathrm{lv}(\widetilde{G})$ $(= \mathrm{lv}(G)+\epsilon_K^{\mathrm{Fon}}+1)$, we shall denote by X_m the scheme-theoretic image of the composite

$$\widetilde{G}_K[p^m] \stackrel{(\mathrm{id}, g_K)}{\longrightarrow} \widetilde{G}_K[p^m] \times_K H_K \stackrel{\subseteq}{\longrightarrow} \widetilde{G} \times_R H\,.$$

Then it is easily verified that $X_m \subseteq \widetilde{G} \times_R H$ is a *finite flat* subgroup scheme of $\widetilde{G} \times_R H$ over R, we have closed immersions

$$X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{\mathrm{lv}(\widetilde{G})-1} \subseteq X \stackrel{\mathrm{def}}{=} X_{\mathrm{lv}(\widetilde{G})} \subseteq \widetilde{G} \times_R H\,,$$

and the composite

$$X_m \longrightarrow \widetilde{G} \times_R H \stackrel{\mathrm{pr}_1}{\longrightarrow} \widetilde{G} \ \text{(respectively, } X_m \longrightarrow \widetilde{G} \times_R H \stackrel{\mathrm{pr}_2}{\longrightarrow} H)$$

factors through the subgroup scheme $\widetilde{G}[p^m] \subseteq \widetilde{G}$ (respectively, $H[p^m] \subseteq H$) of $\widetilde{G}$ (respectively, H). Now to prove Theorem 3.4, it is enough to show that the composite $X \hookrightarrow \widetilde{G} \times_R H \stackrel{\mathrm{pr}_1}{\to} \widetilde{G}$ is an *isomorphism*. (Indeed, then the composite

$$\widetilde{G} \stackrel{\sim}{\longleftarrow} X \stackrel{\subset}{\longrightarrow} \widetilde{G} \times_R H \stackrel{\mathrm{pr}_2}{\longrightarrow} H$$

is a morphism of the desired type.) Therefore, the rest of the proof of Theorem 3.4 is devoted to the proof of the assertion that the composite $X \hookrightarrow \widetilde{G} \times_R H \stackrel{\mathrm{pr}_1}{\to} \widetilde{G}$ is an isomorphism.

Now I *claim* that the morphism $X_{\epsilon_K^{\mathrm{Fon}}+1} \to \widetilde{G}[p^{\epsilon_K^{\mathrm{Fon}}+1}]$ determined by the composite $X_{\epsilon_K^{\mathrm{Fon}}+1} \hookrightarrow \widetilde{G} \times_R H \stackrel{\mathrm{pr}_1}{\to} \widetilde{G}$ is an *isomorphism*. Indeed, this *claim* is verified as follows: Let $Y_{\epsilon_K^{\mathrm{Fon}}+1} \subseteq \widetilde{G}[p^{\epsilon_K^{\mathrm{Fon}}+1}] \times_R H$ be the finite flat subgroup scheme of $\widetilde{G}[p^{\epsilon_K^{\mathrm{Fon}}+1}] \times_R H$ obtained as the scheme-theoretic image of the section of $\widetilde{G}[p^{\epsilon_K^{\mathrm{Fon}}+1}] \times_R H \stackrel{\mathrm{pr}_1}{\to} \widetilde{G}[p^{\epsilon_K^{\mathrm{Fon}}+1}]$ determined by the identity section of H. Then since $\widetilde{G}_K[p^{\epsilon_K^{\mathrm{Fon}}+1}] \subseteq \mathrm{Ker}(g_K)$, it is immediate that $X_{\epsilon_K^{\mathrm{Fon}}+1} \otimes_R K$ coincides with $Y_{\epsilon_K^{\mathrm{Fon}}+1} \otimes_R K$ in $(G \times_R H) \otimes_R K$. Therefore, we obtain that $X_{\epsilon_K^{\mathrm{Fon}}+1} = Y_{\epsilon_K^{\mathrm{Fon}}+1}$ in $G \times_R K$. In particular, the morphism in question is an isomorphism. This completes the proof of the second *claim*.

By taking "$(-)^D$", we obtain a commutative diagram

$$\begin{array}{ccc}
\widetilde{G}^D & = & \widetilde{G}^D \\
{\scriptstyle \mathrm{pr}_1^D}\downarrow & & \downarrow{\scriptstyle f_G} \\
\widetilde{G}^D \times_R H^D & \longrightarrow & X^D \\
{\scriptstyle \mathrm{pr}_2^D}\uparrow & & \uparrow{\scriptstyle f_H} \\
H^D & = & H^D ,
\end{array}$$

where the middle horizontal arrow is faithfully flat, the top right-hand vertical arrows f_G induces an isomorphism of group schemes $\widetilde{G}_K^D \stackrel{\sim}{\to} X_K^D$ over K, and the kernel of the morphism $H^D(\overline{K}) \to X^D(\overline{K})$ determined by the lower right-hand vertical arrows f_H is annihilated by p^n (cf. condition (i)); moreover, for a natural number $m \leq \mathrm{lv}(\widetilde{G})$, we obtain a commutative diagram

$$\begin{array}{ccc}
\widetilde{G}^D & \longrightarrow & \widetilde{G}[p^m]^D \\
{\scriptstyle f_G}\downarrow & & \downarrow \\
X^D & \longrightarrow & (X_m)^D ,
\end{array}$$

where the horizontal arrows are faithfully flat.

Now I *claim* that the morphism

$$t_{\widetilde{G}^D}(K/R) \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}+1}) \longrightarrow t_{X^D}(K/R) \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}+1})$$

is injective, and its image is a direct summand of $t_{X^D}(K/R) \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}+1})$; in particular, it follows from Lemma 3.2 that the morphism

$$t_{\widetilde{G}^D}(K/R) \otimes_R R/(p^{\mathrm{lv}(\widetilde{G})-\epsilon_K^{\mathrm{Fon}}}) \longrightarrow t_{X^D}(K/R) \otimes_R R/(p^{\mathrm{lv}(\widetilde{G})-\epsilon_K^{\mathrm{Fon}}})$$

is injective, and its image is a direct summand of $t_{X^D}(K/R) \otimes_R R/(p^{\mathrm{lv}(\widetilde{G})-\epsilon_K^{\mathrm{Fon}}})$. Indeed, this *claim* is verified as follows: It follows from Lemmas 1.6;

2.3 that we obtain a commutative diagram

$$\begin{array}{ccccc} t_{\widetilde{G}^D}(K/R) & \longrightarrow & t_{\widetilde{G}^D}(K/R) \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}+1}) & \longrightarrow & t_{\widetilde{G}[p^{\epsilon_K^{\mathrm{Fon}}+1}]^D}(K/R) \\ \downarrow & & \downarrow & & \downarrow \\ t_{X^D}(K/R) & \longrightarrow & t_{X^D}(K/R) \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}+1}) & \longrightarrow & t_{(X_{\epsilon_K^{\mathrm{Fon}}+1})^D}(K/R)\,, \end{array}$$

where the horizontal arrows are surjective. Now since $\widetilde{G}$ is truncated Barsotti-Tate, the right-hand top horizontal arrow $t_{\widetilde{G}^D}(K/R) \otimes_R R/(p^{\epsilon_K^{\mathrm{Fon}}+1}) \twoheadrightarrow t_{\widetilde{G}[p^{\epsilon_K^{\mathrm{Fon}}+1}]^D}(K/R)$ is an *isomorphism* (cf. Lemma 2.15); on the other hand, it follows from the second *claim* that the right-hand vertical arrow $t_{\widetilde{G}[p^{\epsilon_K^{\mathrm{Fon}}+1}]^D}(K/R) \to t_{(X_{\epsilon_K^{\mathrm{Fon}}+1})^D}(K/R)$ is also an *isomorphism*. This completes the proof of the third *claim*.

Next, I *claim* that $\underline{t}^*_{X^D} > \epsilon_K^{\mathrm{Fon}}$. Indeed, since H^D is not étale, it follows from condition (i), together with Lemma 2.10, (i), that X^D is *not* étale. Thus, the above *claim* follows from the third *claim*, Lemma 3.3, together with Remark 2.9. This completes the proof of the fourth *claim*.

Thus, it follows from the fourth *claim*, together with Lemma 2.12, that the morphism $\widetilde{G}^D \to X^D$, hence also the morphism $X \to \widetilde{G}$ in question is an isomorphism. This completes the proof of Theorem 3.4. $\square$

Remark 3.5. Let K^{tm} ($\subseteq \overline{K}$) be the maximal tamely ramified extension field of K, $(K^{\mathrm{tm}})^\wedge$ the p-adic completion of K^{tm}, and R^{tm} (respectively, $(R^{\mathrm{tm}})^\wedge$) the ring of integers of K^{tm} (respectively, $(K^{\mathrm{tm}})^\wedge$). Then it follows from the fact that the absolute Galois groups of K^{tm} and $(K^{\mathrm{tm}})^\wedge$ are naturally isomorphic, together with the faithfully flatness of the morphism $R^{\mathrm{tm}} \to (R^{\mathrm{tm}})^\wedge$, that any finite flat group scheme (respectively, truncated Barsotti-Tate group scheme; respectively, morphism of finite flat group schemes) over $(R^{\mathrm{tm}})^\wedge$ descends to a finite flat group scheme (respectively, truncated Barsotti-Tate group scheme; respectively, morphism of finite flat group schemes) over R^{tm}. Therefore, the following assertion follows from Theorem 3.4:

> *Let R be a complete discrete valuation ring whose residue field (respectively, field of fractions K) is of characteristic $p > 0$ (respectively, 0) and perfect, $\overline{K}$ an algebraic closure of K, v_p the p-adic valuation of $\overline{K}$ such that $v_p(p) = 1$, e_K the absolute ramification index of K, $\epsilon_K^{\mathrm{Fon}} \stackrel{\mathrm{def}}{=} 2 + v_p(e_K)$, K^{tm} the maximal tamely ramified extension field of K, $(K^{\mathrm{tm}})^\wedge$ (respectively, $\widehat{\overline{K}}$) the p-adic completion of K^{tm} (respectively, $\overline{K}$), $(R^{\mathrm{tm}})^\wedge$ the ring of integers of $(K^{\mathrm{tm}})^\wedge$, G and H truncated Barsotti-Tate group schemes over $(R^{\mathrm{tm}})^\wedge$, $f_K\colon G_K \stackrel{\mathrm{def}}{=} G \otimes_{(R^{\mathrm{tm}})^\wedge}$*

$(K^{\mathrm{tm}})^\wedge \to H_K \stackrel{\mathrm{def}}{=} H \otimes_{(R^{\mathrm{tm}})^\wedge} (K^{\mathrm{tm}})^\wedge$ a morphism of group schemes over $(K^{\mathrm{tm}})^\wedge$, and n a natural number. Assume that one of the following conditions is satisfied:

(i) *The cokernel of the morphism $G_K(\widehat{\overline{K}}) \to H_K(\widehat{\overline{K}})$ determined by f_K is annihilated by p^n, and $4\epsilon_K^{\mathrm{Fon}} + n \leq \mathrm{lv}(H)$.*

(ii) *The kernel of the morphism $G_K(\widehat{\overline{K}}) \to H_K(\widehat{\overline{K}})$ determined by f_K is annihilated by p^n, and $4\epsilon_K^{\mathrm{Fon}}+n \leq \mathrm{lv}(G)$.*

Then the morphism f_K extends uniquely to a morphism over $(R^{\mathrm{tm}})^\wedge$.

Corollary 3.6 (Extension of morphisms between generic fibers II). *Let R be a complete discrete valuation ring whose residue field (respectively, field of fractions K) is of characteristic $p > 0$ (respectively, 0) and perfect, $\overline{K}$ an algebraic closure of K, v_p the p-adic valuation of $\overline{K}$ such that $v_p(p) = 1$, e_K the absolute ramification index of K, $\epsilon_K^{\mathrm{Fon}} \stackrel{\mathrm{def}}{=} 2+v_p(e_K)$, and G and H truncated Barsotti-Tate group schemes over R. Assume that $4\epsilon_K^{\mathrm{Fon}} \leq \mathrm{lv}(G), \mathrm{lv}(H)$. Then the following hold:*

(i) *Let $\overline{K}\text{-}\mathrm{Inj}_R(G,H)$ (respectively, $\overline{K}\text{-}\mathrm{Inj}_K(G \otimes_R K, H \otimes_R K)$) the set of morphisms ϕ of group schemes over R (respectively, K) from G (respectively, $G \otimes_R K$) to H (respectively, $H \otimes_R K$) such that ϕ induces an injection $G(\overline{K}) \hookrightarrow H(\overline{K})$. Then the natural morphism*

$$\overline{K}\text{-}\mathrm{Inj}_R(G,H) \longrightarrow \overline{K}\text{-}\mathrm{Inj}_K(G \otimes_R K, H \otimes_R K)$$

is bijective.

(ii) *Let $\overline{K}\text{-}\mathrm{Surj}_R(G,H)$ (respectively, $\overline{K}\text{-}\mathrm{Surj}_K(G \otimes_R K, H \otimes_R K)$) the set of morphisms ϕ of group schemes over R (respectively, K) from G (respectively, $G \otimes_R K$) to H (respectively, $H \otimes_R K$) such that ϕ induces a surjection $G(\overline{K}) \twoheadrightarrow H(\overline{K})$. Then the natural morphism*

$$\overline{K}\text{-}\mathrm{Surj}_R(G,H) \longrightarrow \overline{K}\text{-}\mathrm{Surj}_K(G \otimes_R K, H \otimes_R K)$$

is bijective.

(iii) *Let $\mathrm{Isom}_R(G,H)$ (respectively, $\mathrm{Isom}_K(G \otimes_R K, H \otimes_R K)$) be the set of isomorphisms of G (respectively, $G \otimes_R K$) with H (respectively, $H \otimes_R K$) over R (respectively, K). Then the natural morphism*

$$\mathrm{Isom}_R(G,H) \longrightarrow \mathrm{Isom}_K(G \otimes_R K, H \otimes_R K)$$

is bijective.

Proof. This follows immediately from Theorem 3.4. □

Corollary 3.7 (Points of truncated Barsotti-Tate groups). *Let R be a complete discrete valuation ring whose residue field (respectively, field of fractions K) is of characteristic $p > 0$ (respectively, 0) and perfect, $\overline{K}$ an algebraic closure of K, v_p the p-adic valuation of $\overline{K}$ such that $v_p(p) = 1$, e_K the absolute ramification index of K, $\epsilon_K^{\mathrm{Fon}} \stackrel{\mathrm{def}}{=} 2 + v_p(e_K)$, K^{tm} ($\subseteq \overline{K}$) the maximal tamely ramified extension field of K, $\Gamma_{K^{\mathrm{tm}}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K^{\mathrm{tm}})$, and G a truncated Barsotti-Tate group scheme over R. Then the following hold:*

(i) *If G is of level $\geq 4\epsilon_K^{\mathrm{Fon}}$ and not étale over R, then $G(\overline{K}) \not\subseteq G(K^{\mathrm{tm}})$.*

(ii) *If G^D is connected, then the $\Gamma_{K^{\mathrm{tm}}}$-invariant part*

$$(G(\overline{K}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-1))^{\Gamma_{K^{\mathrm{tm}}}}$$

of the $\mathbb{Z}_p[\Gamma_{K^{\mathrm{tm}}}]$-module $G(\overline{K}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-1)$ is annihilated by $p^{4\epsilon_K^{\mathrm{Fon}}-1}$.

Proof. First, we prove assertion (i). By replacing G by its connected component, we may assume without loss of generality that G is *connected.* Then if $G(\overline{K}) = G(K^{\mathrm{tm}})$, it is easily verified that there exist a finite extension field K' of K which is *tamely ramified* over K, an *étale* truncated Barsotti-Tate group scheme H over the ring of integers R' of K', and an isomorphism of group schemes $G \otimes_R K' \xrightarrow{\sim} H \otimes_{R'} K'$ over K'. Thus, it follows from Theorem 3.4 that the isomorphism $G \otimes_R K' \xrightarrow{\sim} H \otimes_{R'} K'$ over K' extends to an isomorphism $G \otimes_R R' \xrightarrow{\sim} H$ over R'. On the other hand, since G is connected, any morphisms $G \otimes_R R' \to H$ over R' must be trivial. Thus, we obtain a contradiction. This completes the proof of assertion (i).

Next, we prove assertion (ii). If $(G(\overline{K}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-1))^{\Gamma_{K^{\mathrm{tm}}}}$ is *not* annihilated by $p^{4\epsilon_K^{\mathrm{Fon}}-1}$, then it is easily verified that there exist a finite extension field K' of K which is *tamely ramified* over K, an *étale* truncated Barsotti-Tate group scheme H of level $\geq 4\epsilon_K^{\mathrm{Fon}}$ with height 1 over the ring of integers R' of K', and a morphism of group schemes $G^D \otimes_R K' \to H \otimes_{R'} K'$ over K' such that the induced morphism $G^D(\overline{K}) \to H(\overline{K})$ is surjective. Thus, it follows from Theorem 3.4 that the morphism $G^D \otimes_R K' \to H \otimes_{R'} K'$ over K' extends to a morphism $G^D \otimes_R R' \to H$ over R'. On the other hand, since G^D is connected, any morphisms $G^D \otimes_R R' \to H$ over R' must be trivial. Thus, we obtain a contradiction. This completes the proof of assertion (ii). □

Corollary 3.8 (Extension of morphisms of Tate modules of Barsotti-Tate groups). *Let R be a complete discrete valuation ring whose residue field (respectively, field of fractions K) is of characteristic $p > 0$ (respectively, 0) and perfect, $\overline{K}$ an algebraic closure of K, K^{tm} ($\subseteq \overline{K}$) the maximal tamely ramified extension field of K, $(K^{\mathrm{tm}})^\wedge$ (respectively, $\widehat{\overline{K}}$) the p-adic completion of K^{tm} (respectively, $\overline{K}$), $(R^{\mathrm{tm}})^\wedge$ the ring of integers of $(K^{\mathrm{tm}})^\wedge$, and $\Gamma_{(K^{\mathrm{tm}})^\wedge} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widehat{\overline{K}}/(K^{\mathrm{tm}})^\wedge)$.*

(Thus, by restricting elements of $\Gamma_{(K^{\mathrm{tm}})^\wedge}$ *to the algebraic closure of* $(K^{\mathrm{tm}})^\wedge$ *in* $\widehat{\overline{K}}$*, one obtains a natural isomorphism of* $\Gamma_{(K^{\mathrm{tm}})^\wedge}$ *with the corresponding absolute Galois group of* $(K^{\mathrm{tm}})^\wedge$*.) Let* $\mathcal{G}$ *and* $\mathcal{H}$ *be Barsotti-Tate groups over* $(R^{\mathrm{tm}})^\wedge$*,* $T_p(\mathcal{G})$ *(respectively,* $T_p(\mathcal{H})$*) the p-adic Tate module of* $\mathcal{G}$ *(respectively,* $\mathcal{H}$*), and* $\mathrm{Isog}_{(R^{\mathrm{tm}})^\wedge}(\mathcal{G},\mathcal{H})$ *(respectively,* $\mathrm{Isog}_{\Gamma_{(K^{\mathrm{tm}})^\wedge}}(T_p(\mathcal{G}),T_p(\mathcal{H}))$*) the set of morphisms* ϕ *of Barsotti-Tate groups over* $(R^{\mathrm{tm}})^\wedge$ *(respectively,* $\mathbb{Z}_p[\Gamma_{(K^{\mathrm{tm}})^\wedge}]$*-equivariant morphisms* ϕ*) from* $\mathcal{G}$ *(respectively,* $T_p(\mathcal{G})$*) to* $\mathcal{H}$ *(respectively,* $T_p(\mathcal{H})$*) such that* ϕ *induces an isomorphism* $T_p(\mathcal{G})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \xrightarrow{\sim} T_p(\mathcal{H})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$*. Then the natural morphism*

$$\mathrm{Isog}_{(R^{\mathrm{tm}})^\wedge}(\mathcal{G},\mathcal{H}) \longrightarrow \mathrm{Isog}_{\Gamma_{(K^{\mathrm{tm}})^\wedge}}(T_p(\mathcal{G}),T_p(\mathcal{H}))$$

is bijective.

Proof. This follows from a similar argument to the argument used in the proof of [11], Theorem 4, together with Remark 3.5. □

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