Square-Free 2-Matchings in Bipartite Graphs and Jump Systems

Yusuke KOBAYASHI* Kenjiro TAKAZAWA†

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Abstract

For an undirected graph and a fixed integer k, a 2-matching is said to be C_k-free if it has no cycle of length k or less. In particular, a C_4-free 2-matching in a bipartite graph is called a square-free 2-matching. The problem of finding a maximum C_k-free 2-matching in a bipartite graph is NP-hard when $k \geq 6$, and polynomially solvable when $k = 4$. Also, the problem of finding a maximum-weight C_k-free 2-matching in a bipartite graph is NP-hard for any integer $k \geq 4$, and polynomially solvable when $k = 4$ and the weight function is vertex-induced on every cycle of length four.

In this paper, we prove that the degree sequences of the C_k-free 2-matchings in a bipartite graph form a jump system for $k = 4$, and do not always form a jump system for $k \geq 6$. This result is consistent with the polynomial solvability of the C_k-free 2-matching problem in bipartite graphs and partially proves the conjecture of Cunningham that the degree sequences of C_4-free 2-matchings form a jump system for any graph. We also show that the weighted square-free 2-matchings in a bipartite graph induce an M-concave (M-convex) function on the jump system if and only if the weight function is vertex-induced on every square. This result is also consistent with the polynomial solvability of the weighted square-free 2-matching problem.

1 Introduction

A jump system, introduced by Bouchet and Cunningham [3], is an extended concept of a matroid. A jump system is a set of integer lattice points with an exchange property (to be described in Section 2.2); see also [18, 23]. It is a generalization of a matroid [29, 33, 36], a delta-matroid [2, 4, 9], and a base polyhedron of an integral polymatroid (or a submodular system) [14]. The concept of M-concave (M-convex) functions on constant-parity jump systems [27] is a general framework of optimization problems on jump systems, and it is a generalization of valuated matroids [10, 12], valuated delta-matroids [11], and M-convex functions on base polyhedra [25] (see [26]).

Many efficiently solvable combinatorial optimization problems closely relate to these structures. For instance, the minsquare factor problem [1] is a special case of minimization of an M-convex function on a constant-parity jump system. The degree sequences of all matchings in an undirected graph form a delta-matroid, and the maximum-weight matchings induce a valuated delta-matroid (see [11, 27]). The even factor problem [7] (see also [6]) is NP-hard, and polynomially solvable if the given digraph has a certain property called odd-cycle-symmetric [7, 30]. This property is a necessary and sufficient condition for the degree sequences of the even factors to form a jump

*Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan. E-mail: `Yusuke_Kobayashi@mist.i.u-tokyo.ac.jp`

†Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan, and Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan. E-mail: `takazawa@misojiro.t.u-tokyo.ac.jp`

system [22]. A relation between weighted even factors and M-concave functions on constant-parity jump systems is also clarified in [22]. The objective of the present paper is to investigate the condition for the degree sequences of the C_k*-free* 2*-matchings* in a bipartite graph to have these matroidal structures.

Let $G = (V, E)$ be a simple undirected graph, that is, G has neither parallel edges nor self-loops. In what follows, we often omit to declare that the graph is undirected. The subset of edges incident to a vertex $v \in V$ is denoted by δv. A 2-matching is a subset of edges $M \subseteq E$ such that $|M \cap \delta v| \leq 2$ for every $v \in V$. For a 2-matching M, we say that M is C_k*-free* if M contains no cycle of length k or less. For a fixed integer k, the C_k-free 2-matching problem is to find a C_k-free 2-matching of maximum size in a given graph. Note that the case $k \leq 2$ is exactly the classical simple 2-matching problem, which can be solved efficiently.

Since the C_k-free 2-matching problem is a relaxation of the Hamiltonian cycle problem, it is easily seen that this problem is NP-hard when $|V|/2 \leq k \leq |V| - 1$. Moreover, Papadimitriou showed that the problem is NP-hard when $k \geq 5$ (see [8]). On the other hand, for the case $k = 3$, an augmenting path algorithm is given by Hartvigsen [16]. The C_4-free 2-matching problem is left open.

The relation between C_k-free 2-matchings and jump systems is studied by Cunningham [6]. For a graph $G = (V, E)$, the *degree sequence* $d_F \in \mathbf{Z}^V$ of an edge set $F \subseteq E$ is defined by

$$d_F(v) = |F \cap \delta v|, \quad v \in V.$$

Let $J_k(G) \subseteq \mathbf{Z}^V$ denote the set of all degree sequences of C_k-free 2-matchings in G, that is,

$$J_k(G) = \{d_M \mid M \text{ is a } C_k\text{-free 2-matching in } G\}.$$

When $k \leq 2$, $J_k(G)$ is the set of all degree sequences of 2-matchings, and hence it is a constant-parity jump system (see Section 2.2). Cunningham [6] showed the following theorem.

Theorem 1.1 ([6])**.** *For any graph G, $J_3(G)$ is a constant-parity jump system. For any integer $k \geq 5$, there exists a graph G such that $J_k(G)$ is not a jump system.*

Note that this result is consistent with the polynomial solvability of the C_k-free 2-matching problem. In [6], Cunningham conjectured that $J_4(G)$ is a jump system for any graph G and the C_4-free 2-matching problem is polynomially solvable.

The present paper focuses on bipartite graphs and discusses whether $J_k(G)$ is a jump system for a bipartite graph G and an integer k. Note that it suffices to consider the cases where k is even. The C_6-free 2-matching problem in bipartite graphs is known to be NP-hard [15]. On the other hand, for the C_4-free 2-matching problem in bipartite graphs, a min-max formula [19] and polynomial-time algorithms [17, 30] are proposed. We remark that a cycle of length four is called a square, and a C_4-free 2-matching in a bipartite graph is often referred to as a *square-free* 2*-matching*.

Our main contribution is the following theorem.

Theorem 1.2. *For any bipartite graph G, $J_4(G)$ is a constant-parity jump system. For any even integer $k \geq 6$, there exists a bipartite graph G such that $J_k(G)$ is not a jump system.*

Note that this theorem agrees with the polynomial solvability of the C_k-free 2-matching problem in bipartite graphs. Also, this theorem partially solves Cunningham's conjecture [6]. Table 1 summarizes the aforementioned results.

We also discuss the weighted version. Given a bipartite graph and a weight function on the edge set, consider the problem of finding a C_k-free 2-matching maximizing the total weight of its

Table 1: Relation between the C_k-free 2-matching problem and jump systems ($*$: our result).

	C_k-free 2-matching problem		Is $J_k(G)$ a jump system?	
	General graph	Bipartite graph	General graph	Bipartite graph
$k \geq 6$	NP-hard [8]	NP-hard [15]	No [6]	No*
$k = 5$	NP-hard [8]	—	No [6]	—
$k = 4$	Unknown	P [17, 30]	Unknown	Yes*
$k = 3$	P [16]	—	Yes [6]	—
$k \leq 2$	P	P	Yes	Yes

edges. When $k \geq 6$, this problem is NP-hard since the unweighted version is NP-hard. Moreover, Z. Király proved that the weighted square-free 2-matching problem is also NP-hard (see [13]). This problem is, however, tractable if the weight function is *vertex-induced* on every square.

Definition 1.3 (Vertex-induced weight)**.** Let (G, w) be a weighted graph with $G = (V, E)$ and $w : E \to \mathbf{R}$. For subgraph H of G, w is *vertex-induced on H* if there exists a function $\pi_H : V(H) \to \mathbf{R}$ such that $w(e) = \pi_H(u) + \pi_H(v)$ for every edge $e = (u, v) \in E(H)$. Here, $V(H)$ and $E(H)$ denote the vertex set and edge set of H, respectively, and (u, v) denotes an edge connecting $u, v \in V(H)$.

Makai [24] considered weight functions that are vertex-induced on every square in G. Note that such a weight function is not necessarily induced by a single potential function on V, and the potential functions may vary from one square to another. For this class of weight functions, Makai [24] showed a linear programming description of maximum-weight square-free 2-matchings and proved its dual integrality. By applying the ellipsoid method to this description, the weighted square-free 2-matching problem for this class of weight functions can be solved in polynomial time. Also, a combinatorial polynomial algorithm is given by Takazawa [32].

In this paper, we show a relation between the weighted square-free 2-matchings and M-concave functions on constant-parity jump systems. For a weighted bipartite graph (G, w), define a function f on $J_4(G)$ by

$$f(x) = \max \left\{ \sum_{e \in M} w(e) \;\middle|\; M \text{ is a square-free 2-matching, } d_M = x \right\}.$$

Theorem 1.4. *For a weighted bipartite graph (G, w), f is an M-concave function on the constant-parity jump system $J_4(G)$ if and only if w is vertex-induced on every square in G.*

This theorem suggests that assuming the weight function to be vertex-induced on every square is reasonable in considering the weighted square-free 2-matching problem in bipartite graphs.

As a generalization of the square-free 2-matching problem, Frank [13] introduced the $K_{t,t}$-free t-matching problem. A complete bipartite graph $K_{t,t}$ is a graph (V, E) such that V can be partitioned into two sets V_1 and V_2 with $|V_1| = |V_2| = t$ and $E = \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\}$. For a bipartite graph $G = (V, E)$, an edge set $M \subseteq E$ is a *$K_{t,t}$-free t-matching* if M is a t-matching that contains no $K_{t,t}$ as a subgraph. Note that the square-free 2-matching problem is a special case $t = 2$ of the $K_{t,t}$-free t-matching problem. In [13], a min-max formula for the $K_{t,t}$-free t-matching problem is given as an extension of that for the square-free 2-matchings [19]. Also, the results in [24, 30, 32] apply to $K_{t,t}$-free t-matchings. That is, the $K_{t,t}$-free t-matching problem is polynomially solvable, and so is the weighted problem if the weight function is vertex-induced on every $K_{t,t}$. We anticipate that Theorems 1.2 and 1.4 also extend to $K_{t,t}$-free t-matchings.

This paper is organized as follows. In Section 2, we give some definitions on C_k-free 2-matchings, jump systems and M-concave functions. In Sections 3 and 4, we prove Theorems 1.2 and 1.4, respectively. Finally, we discuss $K_{t,t}$-free t-matchings in Section 5.

2 Definitions

2.1 C_k-free 2-matchings

Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set E. An edge connecting $u, v \in V$ is denoted by (u, v). Recall that the set of edges incident to $v \in V$ is denoted by δv. A cycle C is a subgraph consisting of distinct vertices $v_1, \ldots, v_l$ and edges $(v_1, v_2), \ldots, (v_{l-1}, v_l), (v_l, v_1)$. A cycle C is often denoted by $C = (v_1, v_2, \ldots, v_l)$ and the length of C is defined by l, the number of its edges. For a subgraph H of G, the vertex set and edge set of H are denoted by $V(H)$ and $E(H)$, respectively. For a positive integer t, an edge set $M \subseteq E$ is said to be a *t-matching* if $|M \cap \delta v| \leq t$ for every $v \in V$. In particular, a 2-matching is a vertex-disjoint collection of paths and cycles. We denote a bipartite graph by $(V_1, V_2; E)$. That is, for any edge in E, one of its end vertices is in V_1 and the other in V_2.

Definition 2.1 (C_k-free 2-matching)**.** For a simple undirected graph $G = (V, E)$ and an integer k, an edge set $M \subseteq E$ is a *C_k-free 2-matching* if M is a 2-matching that contains no cycle of length k or less as a subgraph. In particular, if G is bipartite, a C_4-free 2-matching is called a *square-free 2-matching*.

2.2 Jump systems

Let V be a finite set. For $u \in V$, we denote by χ_u the *characteristic vector* of u, with $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for $v \in V \setminus \{u\}$. For $x, y \in \mathbf{Z}^V$, a vector $s \in \mathbf{Z}^V$ is called an *(x, y)-increment* if $x(u) < y(u)$ and $s = \chi_u$ for some $u \in V$, or $x(u) > y(u)$ and $s = -\chi_u$ for some $u \in V$.

Definition 2.2 (Jump system [3])**.** A nonempty set $J \subseteq \mathbf{Z}^V$ is said to be a *jump system* if it satisfies an exchange axiom, called the *2-step axiom*:

> For any $x, y \in J$ and for any (x, y)-increment s with $x + s \notin J$, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$.

A set $J \subseteq \mathbf{Z}^V$ is a *constant-parity system* if $\sum_{v \in V}(x(v) - y(v))$ is even for any $x, y \in J$. For constant-parity jump systems, J. F. Geelen observed a stronger exchange property:

(EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$ and $y - s - t \in J$.

This property characterizes a constant-parity jump system (see [27] for details).

Theorem 2.3. *A nonempty set J is a constant-parity jump system if and only if it satisfies* (EXC).

A constant-parity jump system is a generalization of the base family of a matroid, an even delta-matroid [34, 35], and a base polyhedron of an integral polymatroid. The degree sequences of all subgraphs in an undirected graph is a typical example of a constant-parity jump system. That is, for a graph $G = (V, E)$,

$$J_{\mathrm{SG}}(G) = \{d_F \mid F \subseteq E\}$$

is a constant-parity jump system [3, 23]. The set of all degree sequences of 2-matchings, or $J_2(G)$, is the intersection of $J_{\mathrm{SG}}(G)$ and a box $\{0,1,2\}^V$, and hence one can easily see that $J_2(G)$ is a jump system. However, Theorems 1.1 and 1.2 are not obvious since the additional condition "C_k-free" makes the situation more complicated when $k \geq 3$. Our contribution is to show how to deal with this condition when $k = 4$ and the graph is bipartite.

2.3 M-concave functions

An *M-concave* (*M-convex*) *function on a constant-parity jump system* is a quantitative extension of a jump system, which is a generalization of valuated matroids [10, 12], valuated delta-matroids [11], and M-concave (M-convex) functions on base polyhedra [25, 26].

Definition 2.4 (M-concave function on a constant-parity jump system [27])**.** For $J \subseteq \mathbf{Z}^V$, we call $f : J \to \mathbf{R}$ an *M-concave function on a constant-parity jump system* if it satisfies the following exchange axiom:

(M-EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$, $y - s - t \in J$, and $f(x) + f(y) \leq f(x + s + t) + f(y - s - t)$.

It directly follows from (M-EXC) that J satisfies (EXC), and hence J is a constant-parity jump system. We call a function $f : J \to \mathbf{R}$ an *M-convex function* if $-f$ is an M-concave function on a constant-parity jump system. M-concave functions on constant-parity jump systems appear in many combinatorial optimization problems such as the weighted matching problem, the minsquare factor problem [1], and the weighted even factor problem in odd-cycle-symmetric digraphs [6, 7, 22]. Some properties of M-concave functions are investigated in [20, 21], and efficient algorithms for maximizing an M-concave function on a constant-parity jump system are given in [28, 31].

3 C_k-free 2-matchings in bipartite graphs and jump systems

This section is devoted to the proof of Theorem 1.2. First, let us show the latter half of the theorem by presenting an example of a bipartite graph G such that $J_k(G)$ is not a jump system for $k \geq 6$.

Consider a bipartite graph $C_k = (V_k, E_k)$ consisting of a cycle $(v_1, v_2, v_3, \ldots, v_k)$. Define $x, y \in \mathbf{Z}^{V_k}$ by $x(v_1) = x(v_2) = 1$, $x(v) = 2$ for $v \in V_k \setminus \{v_1, v_2\}$, $y(v_4) = y(v_5) = 1$, and $y(v) = 2$ for $v \in V_k \setminus \{v_4, v_5\}$. Then, for an even integer $k \geq 6$, $x, y \in J_k(C_k)$ and $s = \chi_{v_2}$ is an (x, y)-increment, but there exists no $(x + s, y)$-increment t such that $x + s + t \in J_k(C_k)$ and $y - s - t \in J_k(C_k)$. Thus, for an even integer $k \geq 6$, $J_k(C_k)$ is not a jump system for this bipartite graph C_k.

We now focus on the first half of Theorem 1.2:

Proposition 3.1. *For any bipartite graph G, $J_4(G)$ is a constant-parity jump system.*

In the rest of this section, we prove Proposition 3.1 by presenting an algorithm for finding an $(x + s, y)$-increment t satisfying (EXC) for given $x, y \in J_4(G)$ and (x, y)-increment s. In what follows, we consider the case where $s = -\chi_u$ with $u \in V_1$. The other cases can be dealt with in a similar way.

3.1 Preliminaries

In this subsection, we prepare an operation and a notion that will be used in our algorithm.

3.1.1 Shrinking cycles

Definition 3.2. Let $C = (v_1, v_2, v_3, v_4)$ be a cycle of length four in $G = (V_1, V_2; E)$ with $v_1, v_3 \in V_1$ and $v_2, v_4 \in V_2$. *Shrinking* of C in G consists of the following operations:

- identify v_1 with v_3, and denote the corresponding vertex by u_1,
- identify v_2 with v_4, and denote the corresponding vertex by u_2, and
- identify all edges between u_1 and u_2.

In the obtained graph, the edge between u_1 and u_2 corresponding to $E(C)$ is called a *square-edge*.

Let $C_1, C_2, \ldots, C_p$ be edge disjoint cycles of length four, and let $G^\circ = (V_1^\circ, V_2^\circ; E^\circ)$ be the graph obtained from $G = (V_1, V_2; E)$ by shrinking $C_1, C_2, \ldots, C_p$. Note that G° might have some parallel edges, whereas G does not. For $F_1^\circ, F_2^\circ \subseteq E^\circ$, let $F_1^\circ \setminus F_2^\circ$ denote the usual difference set of F_1° and F_2°, and let $F_1^\circ - F_2^\circ$ denote the set of all edges $e \in F_1^\circ$ such that no parallel edge of e is in F_2°.

If an edge set $L^\circ \subseteq E^\circ$ is obtained from $L \subseteq E$ by shrinking $C_1, C_2, \ldots, C_p$ such that $|E(C_i) \cap L| = 3$ for $i = 1, 2, \ldots, p$, we say that L° is the *shrunk edge set* of L, and L is an *expanded edge set* of L°. Note that the shrunk edge set L° contains all square-edges in G°.

In a shrunk graph G°, a *square* is a cycle of length four whose corresponding edges in G form a cycle of length four. In particular, a square contains no square-edges. Obviously, when we shrink no edges, that is $G^\circ = G$, a square is exactly a cycle of length four. We say that an edge set in G° is *square-free* if it contains no square.

We now define a map $\phi : \mathbf{Z}^{V_1 \cup V_2} \to \mathbf{Z}^{V_1^\circ \cup V_2^\circ}$ by

$$(\phi(x))(u) = \sum \{x(v) \mid v \in V_1 \cup V_2,\ v \text{ corresponds to } u\} \\ - 2|\{\text{square-edges incident to } u\}| \qquad (1)$$

for $x \in \mathbf{Z}^{V_1 \cup V_2}$ and $u \in V_1^\circ \cup V_2^\circ$. One can see that for an edge set $L \subseteq E$ satisfying that $|E(C_i) \cap L| = 3$ for $i = 1, 2, \ldots, p$, $\phi(d_L)$ is the degree sequence of the shrunk edge set of L. Conversely, the following lemma holds.

Lemma 3.3. *Let $L^\circ \subseteq E^\circ$ be a 2-matching in G° that contains all square-edges and x be a vector in $\{0, 1, 2\}^{V_1 \cup V_2}$. If $\phi(x)$ is the degree sequence of L°, there exists an expanded edge set L of L° in G such that $d_L = x$. Furthermore, such L is unique.*

Proof. We show how to expand square-edges in L°. Let (u_1, u_2) be a square-edge in L°. Suppose that $v_1, v_3 \in V_1$ correspond to $u_1 \in V_1^\circ$, and $v_2, v_4 \in V_2$ correspond to $u_2 \in V_2^\circ$. Denote the cycle (v_1, v_2, v_3, v_4) in G by C.

For a given x and L°, $d_{L \cap E(C)}(v_1)$ should satisfy the following. Note that $x(v_1)$ is at least one, because $x(v_1) + x(v_3) \geq (\phi(x))(u_1) + 2 \geq 3$.

- If $x(v_1) = 1$, then $d_{L \cap E(C)}(v_1) = 1$.
- If $x(v_1) = 2$ and (u_1, u_2) is the unique incident edge of u_1 in L°, then $d_{L \cap E(C)}(v_1) = 2$.
- Suppose that $x(v_1) = 2$ and an edge $e \neq (u_1, u_2)$ in L° is incident to u_1. If the edge (or cycle of length four) corresponding to e in G contains v_1, then $d_{L \cap E(C)}(v_1) = 1$. Otherwise $d_{L \cap E(C)}(v_1) = 2$.

For each vertex $v \in V(C)$, $d_{L \cap E(C)}(v)$ is uniquely determined in the same way.

Since the degree sequence $d_{L \cap E(C)}$ defined as above satisfies that $\{d_{L \cap E(C)}(v_1), d_{L \cap E(C)}(v_3)\} = \{d_{L \cap E(C)}(v_2), d_{L \cap E(C)}(v_4)\} = \{1, 2\}$, there exists a unique set of three edges $L \cap E(C)$ satisfying this degree constraint. This shows the unique existence of a desired expanded edge set. $\square$

3.1.2 Semi-2-matching triple

In this subsection, we often denote a shrunk graph by $G = (V_1, V_2; E)$ to simplify the notation. Our algorithm to find an $(x+s, y)$-increment t satisfying (EXC), which is described in Section 3.2, keeps a triple (M, N, u) of $M, N \subseteq E$ and $u \in V_1 \cup V_2$ satisfying a certain condition. The purpose of this subsection is to define this condition and to show some properties of the triples. Note that the definitions in this subsection make sense only for the case where $s = -\chi_v$ with $v \in V_1$.

Definition 3.4. For two edge sets $M, N \subseteq E$ and a vertex $u \in V_1 \cup V_2$, we say that (M, N, u) is *a semi-2-matching triple* if M and N are square-free, both of them contain all square-edges in G, and one of the following holds:

- M and N are 2-matchings.
- $u \in V_1$, N is a 2-matching, $d_N(u) \leq 1$, $d_M(v) \leq 2$ for any $v \in (V_1 \cup V_2) \setminus \{u\}$, and $d_M(u) = 3$.
- $u \in V_2$, M is a 2-matching, $d_M(u) \leq 1$, $d_N(v) \leq 2$ for any $v \in (V_1 \cup V_2) \setminus \{u\}$, and $d_N(u) = 3$.

Definition 3.5. Let (M, N, u) be a semi-2-matching triple in G. For two vectors $x, y \in \{0, 1, 2\}^{V_1 \cup V_2}$, (x, y) is the *semi-degree* of (M, N, u) if one of the following holds:

- $u \in V_1$, $d_M - \chi_u = x$, and $d_N + \chi_u = y$.
- $u \in V_2$, $d_M + \chi_u = x$, and $d_N - \chi_u = y$.

We denote by $\mathcal{S}_G(x, y)$ the set of all semi-2-matching triples whose semi-degree is (x, y), and omit the subscript G when no confusion will arise.

Definition 3.6. For $(M_1, N_1, u_1), (M_2, N_2, u_2) \in \mathcal{S}(x, y)$, we say that (M_1, N_1, u_1) is *adjacent* to (M_2, N_2, u_2) if they satisfy one of the following conditions:

- $u_1 \in V_1$, $(u_1, u_2) \in M_1 - N_1$, $M_2 = M_1 \setminus \{(u_1, u_2)\}$, and $N_2 = N_1 \cup \{(u_1, u_2)\}$.
- $u_1 \in V_2$, $(u_1, u_2) \in N_1 - M_1$, $M_2 = M_1 \cup \{(u_1, u_2)\}$, and $N_2 = N_1 \setminus \{(u_1, u_2)\}$.

It is obvious that if (M_1, N_1, u_1) is adjacent to (M_2, N_2, u_2), then (M_2, N_2, u_2) is adjacent to (M_1, N_1, u_1).

We say that $(M, N, u) \in \mathcal{S}(x, y)$ is *active*, if $u \in V_1$ and $d_M(u) > d_N(u)$, or $u \in V_2$ and $d_M(u) < d_N(u)$. A semi-2-matching triple $(M, N, u) \in \mathcal{S}(x, y)$ is *stable* if M and N are 2-matchings and $|d_M(u) - d_N(u)| \leq 1$. This inequality means that χ_u or $-\chi_u$, say t, is an (x, y)-increment such that $d_M = x + t$ and $d_N = y - t$ (see Claim 3.18).

We now show some properties of the semi-2-matching triples, which will be used in our algorithm.

Lemma 3.7. *If $(M, N, u) \in \mathcal{S}(x, y)$, then neither M nor N has parallel edges.*

Proof. Suppose that M contains parallel edges e_1 and e_2, whose common end vertices are $u_1 \in V_1$ and $u_2 \in V_2$. Then, at least one of u_1 and u_2 is incident to a square-edge e, which satisfies $e \in M \cap N$ by Definition 3.4 and is distinct from e_1 and e_2. By the degree constraint in Definition 3.4, the only possibility is that $u = u_1$ is incident to e and u_2 is incident to no square-edge. Suppose that e corresponds to a square (v_1, v_2, v_3, v_4), u_1 corresponds to v_1 and v_3, and e_1 and e_2 correspond to (v_1, u_2) and (v_3, u_2) in the original graph. Then an expanded edge set of M contains a square (u_2, v_1, v_2, v_3) or (u_2, v_1, v_4, v_3), which contradicts that M is square-free. Similarly, N has no parallel edges. □

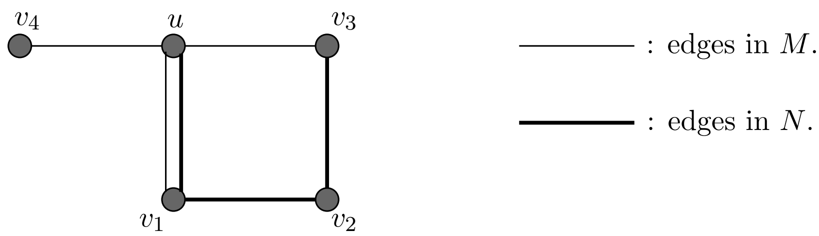

Figure 1: An illustration of Lemma 3.9.

Lemma 3.8. *Suppose that (M, N, u_1) is a semi-2-matching triple in $\mathcal{S}(x, y)$, $e = (u_1, u_2)$ is an edge in $M - N$, and $u_1 \in V_1$. If $N \cup \{e\}$ is square-free, then $(M \setminus \{e\}, N \cup \{e\}, u_2)$ is in $\mathcal{S}(x, y)$ and adjacent to (M, N, u_1).*

Proof. Since $N \cup \{e\}$ is square-free, $(M \setminus \{e\}, N \cup \{e\}, u_2)$ is a semi-2-matching triple. The semi-degree of $(M \setminus \{e\}, N \cup \{e\}, u_2)$ is

$$(d_{M\setminus\{e\}} + \chi_{u_2}, d_{N\cup\{e\}} - \chi_{u_2}) = (d_M - \chi_{u_1}, d_N + \chi_{u_1}) = (x, y),$$

which means $(M\setminus\{e\}, N\cup\{e\}, u_2) \in \mathcal{S}(x, y)$. It is obvious that $(M\setminus\{e\}, N\cup\{e\}, u_2)$ and (M, N, u_1) are adjacent by the definition. □

Next we show the following.

Lemma 3.9. *Suppose that (M, N, u) is a semi-2-matching triple in $\mathcal{S}(x, y)$, $u \in V_1$, and $d_M(u) - d_N(u) \geq 2$. Then, one of the following conditions holds:*

- *(M, N, u) is adjacent to at least two semi-2-matching triples in $\mathcal{S}(x, y)$.*
- *(M, N, u) is adjacent to exactly one semi-2-matching triple in $\mathcal{S}(x, y)$ and there exists a square $C = (u, v_1, v_2, v_3)$ in G such that $\{(u, v_1), (u, v_3)\} \subseteq M$ and $\{(u, v_1), (v_1, v_2), (v_2, v_3)\} \subseteq N$ (see Figure 1).*

Proof. First, a 2-matching N consists of some disjoint paths and cycles, and $|N \cap \delta u| \leq 1$ by the assumption of $d_M(u) - d_N(u) \geq 2$. Hence, for at most one edge $e \in \delta u$, $N \cup \{e\}$ contains a square.

Suppose that (M, N, u) is adjacent to at most one semi-2-matching triple in $\mathcal{S}(x, y)$. By Lemma 3.8, there exists at most one edge $e \in (M \cap \delta u) - N$ such that $N \cup \{e\}$ has no square. Since $|(M \cap \delta u) - N| \geq d_M(u) - d_N(u) \geq 2$, we have that $|(M \cap \delta u) - N| = 2$ and $N \cup \{e\}$ has a square for some edge $e \in (M \cap \delta u) - N$.

Therefore, there exist a square $C = (u, v_1, v_2, v_3)$ and an edge (u, v_4) in G such that $\{(u, v_3), (u, v_4)\} \subseteq M$ and $\{(u, v_1), (v_1, v_2), (v_2, v_3)\} \subseteq N$. To the end, since $d_M(u) - d_N(u) \geq 2$ and $|(M \cap \delta u) - N| = 2$, (u, v_1) is contained in M. □

Lemma 3.10. *Suppose that (M, N, u) is a semi-2-matching triple in $\mathcal{S}(x, y)$, $u \in V_1$, and $d_M(u) - d_N(u) \geq 1$. Then, one of the following conditions holds:*

- *(M, N, u) is adjacent to at least one semi-2-matching triple in $\mathcal{S}(x, y)$.*
- *(M, N, u) is adjacent to no semi-2-matching triple in $\mathcal{S}(x, y)$ and there exists a square $C = (u, v_1, v_2, v_3)$ in G such that $\{(u, v_1), (u, v_3)\} \subseteq M$ and $\{(u, v_1), (v_1, v_2), (v_2, v_3)\} \subseteq N$.*

Proof. Almost the same as the proof of Lemma 3.9. $\square$

The next lemma is a generalization of Lemma 3.3. As the proof is almost the same as Lemma 3.3, we omit it.

Lemma 3.11. *Let $G^\circ = (V_1^\circ, V_2^\circ; E^\circ)$ be a shrunk graph of the original graph $G = (V_1, V_2; E)$. Suppose that $x^\circ, y^\circ \in \{0,1,2\}^{V_1^\circ \cup V_2^\circ}$ and $(M^\circ, N^\circ, u^\circ) \in \mathcal{S}_{G^\circ}(x^\circ, y^\circ)$. For any vectors $x, y \in \{0,1,2\}^{V_1 \cup V_2}$ with $\phi(x) = x^\circ$ and $\phi(y) = y^\circ$, there exists a semi-2-matching triple $(M, N, u) \in \mathcal{S}_G(x, y)$ such that M and N are expanded edge sets of M° and N°, respectively, and u corresponds to u°. Furthermore, if u° corresponds to a unique vertex u in G, such a semi-2-matching triple (M, N, u) is unique.*

3.2 Proof for Proposition 3.1

In this section, we give a constructive proof for Proposition 3.1. More precisely, we give an algorithm for finding edge sets M', N' and an $(x+s, y)$-increment t such that M' and N' are square-free 2-matchings, $d_{M'} = x + s + t$, and $d_{N'} = y - s - t$.

3.2.1 Updating a semi-2-matching triple

In this subsection, we consider a procedure of updating a given semi-2-matching triple in a shrunk graph $G = (V_1, V_2; E)$, which is a subroutine of our main algorithm. Roughly speaking, when a semi-2-matching triple (M, N, u) is given as the input, this procedure increases $|M \cap N|$, maintaining its semi-degree. In the procedure, the shrunk graph G and edge sets M and N satisfy the following assumption.

Assumption 3.12. Both edge sets M and N contain all square-edges in G, and G has no square C such that $E(C) \subseteq M \cup N$ and $|E(C) \cap M| = |E(C) \cap N| = 3$.

The procedure is described as follows.

Procedure A
Input: A shrunk bipartite graph $G = (V_1, V_2; E)$, vectors $x, y \in \{0,1,2\}^{V_1 \cup V_2}$, and an active semi-2-matching triple $(M, N, u) \in \mathcal{S}(x, y)$ satisfying Assumption 3.12.
Output: A stable semi-2-matching triple $(M', N', u') \in \mathcal{S}(x, y)$ with $|M' \cap N'| \geq |M \cap N|$, or a non-stable semi-2-matching triple $(M', N', u') \in \mathcal{S}(x, y)$ with $|M' \cap N'| > |M \cap N|$.

Step 0. Set $\tau := 0$, $M^{(0)} := M$, $N^{(0)} := N$, and $u^{(0)} := u$. Then, go to Step 1.

Step 1. If $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ has an adjacent semi-2-matching triple $(M', N', u') \in \mathcal{S}(x, y)$ which is different from $(M^{(\tau-1)}, N^{(\tau-1)}, u^{(\tau-1)})$ (we ignore this condition if $\tau = 0$), then set $(M^{(\tau+1)}, N^{(\tau+1)}, u^{(\tau+1)}) := (M', N', u')$ and $\tau := \tau + 1$, and go to Step 2. Otherwise, go to Step 4.

Step 2. If $u^{(\tau)} = u^{(\tau')}$ for some $\tau' < \tau$, then output a semi-2-matching triple $(M^{(\tau')}, N^{(\tau)}, u^{(\tau)}) \in \mathcal{S}(x, y)$, which satisfies that $|M^{(\tau')} \cap N^{(\tau)}| > |M \cap N|$ (see Claim 3.14), and stop the procedure. Otherwise, go to Step 3.

Step 3. If $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ is a stable semi-2-matching triple, then output $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)}) \in \mathcal{S}(x, y)$ and stop the procedure. Otherwise, go to Step 1.

Step 4. If $u^{(\tau)} \in V_1$, then execute Step 4-1. Otherwise, execute Step 4-2.
Step 4-1. In this case, if $\tau \geq 1$, then $d_{M^{(\tau)}}(u^{(\tau)}) - d_{N^{(\tau)}}(u^{(\tau)}) \geq 2$, because $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ is not stable by Step 3. On the other hand, $d_{M^{(\tau)}}(u^{(\tau)}) - d_{N^{(\tau)}}(u^{(\tau)}) \geq 1$ if $\tau = 0$ by the activeness

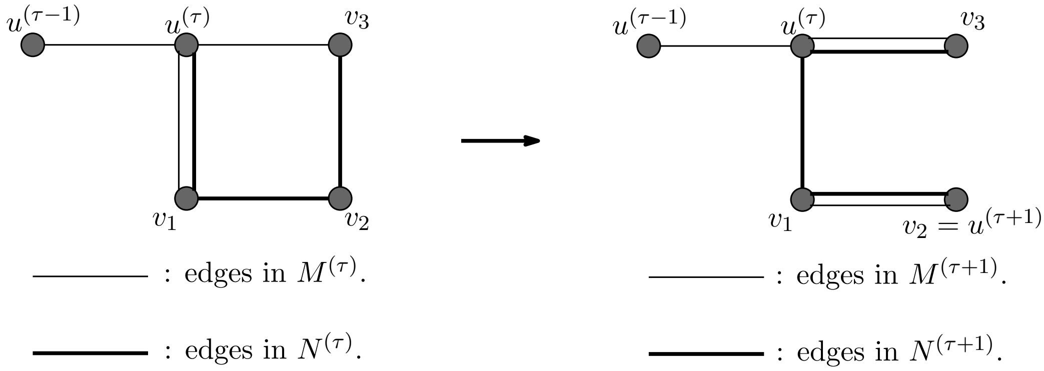

Figure 2: Definitions of $M^{(\tau+1)}$, $N^{(\tau+1)}$, and $u^{(\tau+1)}$.

of the input. By Lemmas 3.9 and 3.10, there exists a square $C = (u^{(\tau)}, v_1, v_2, v_3)$ in G such that $\{(u^{(\tau)}, v_1), (u^{(\tau)}, v_3)\} \subseteq M^{(\tau)}$ and $\{(u^{(\tau)}, v_1), (v_1, v_2), (v_2, v_3)\} \subseteq N^{(\tau)}$. Hence, by Assumption 3.12, $E(C) \cap M^{(\tau)} = \{(u^{(\tau)}, v_1), (u^{(\tau)}, v_3)\}$ and $E(C) \cap N^{(\tau)} = \{(u^{(\tau)}, v_1), (v_1, v_2), (v_2, v_3)\}$.

As shown in Figure 2, define $M^{(\tau+1)}$, $N^{(\tau+1)}$, and $u^{(\tau+1)}$ by

$$\begin{aligned} M^{(\tau+1)} &:= (M^{(\tau)} \setminus \{(u^{(\tau)}, v_1)\}) \cup \{(v_1, v_2)\}, \\ N^{(\tau+1)} &:= (N^{(\tau)} \setminus \{(v_2, v_3)\}) \cup \{(u^{(\tau)}, v_3)\}, \\ u^{(\tau+1)} &:= v_2. \end{aligned}$$

If $M^{(\tau+1)}$ is square-free, then output a semi-2-matching triple $(M^{(\tau+1)}, N^{(\tau+1)}, u^{(\tau+1)}) \in \mathcal{S}(x, y)$, which satisfies that $|M^{(\tau+1)} \cap N^{(\tau+1)}| = |M \cap N| + 1$ (see Claim 3.15), and stop the procedure.

Otherwise, there exists a square $C' = (v_2, v_1, v_4, v_5)$ in $M^{(\tau+1)}$, where $\{u^{(\tau)}, v_3\} \cap \{v_4, v_5\} = \emptyset$ (see Figure 3). Then define

$$\begin{aligned} M^{(\tau+2)} &:= M^{(\tau+1)} \setminus \{(v_2, v_5)\}, \\ N^{(\tau+2)} &:= N^{(\tau+1)} \cup \{(v_2, v_5)\}, \\ u^{(\tau+2)} &:= v_5. \end{aligned}$$

Output a semi-2-matching triple $(M^{(\tau+2)}, N^{(\tau+2)}, u^{(\tau+2)}) \in \mathcal{S}(x, y)$, which satisfies that $|M^{(\tau+2)} \cap N^{(\tau+2)}| = |M \cap N| + 1$ (see Claim 3.16), and stop the procedure.

Step 4-2. Execute a similar procedure to Step 4-1 by switching $M^{(\tau)}$ and $N^{(\tau)}$.

If $u^{(\tau_1)} = u^{(\tau_2)}$ for distinct τ_1 and τ_2, then Procedure A stops in Step 2, which assures that each step is executed at most $|V_1| + |V_2|$ times. We now show the correctness of the procedure. First, one can easily see the following claim.

Claim 3.13. *In Step 1,* $|M^{(\tau)} \cap N^{(\tau)}| = |M^{(\tau+1)} \cap N^{(\tau+1)}|$.

By this claim, if Procedure A outputs a stable semi-2-matching triple $(M', N', u') \in \mathcal{S}(x, y)$ in Step 3, then $|M' \cap N'| = |M \cap N|$, which shows that (M', N', u') is a desired output. The correctness of termination in Step 2 and Step 4 is guaranteed by the following claims.

Claim 3.14. *In Step 2,* $(M^{(\tau')}, N^{(\tau)}, u^{(\tau)})$ *is in* $\mathcal{S}(x, y)$ *and* $|M^{(\tau')} \cap N^{(\tau)}| > |M \cap N|$.

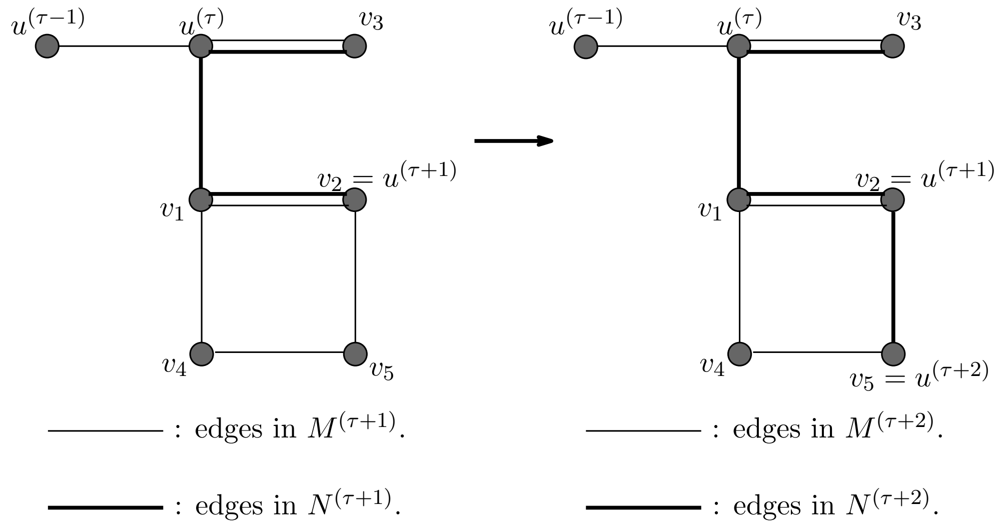

Figure 3: Definitions of $M^{(\tau+2)}$, $N^{(\tau+2)}$, and $u^{(\tau+2)}$.

Proof. Since both $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ and $(M^{(\tau')}, N^{(\tau')}, u^{(\tau')})$ are in $\mathcal{S}(x, y)$, $(M^{(\tau')}, N^{(\tau)}, u^{(\tau)})$ is also in $\mathcal{S}(x, y)$. On the other hand, $M^{(\tau')} \setminus M^{(\tau)} \subseteq N^{(\tau)}$ and $(M^{(\tau)} \setminus M^{(\tau')}) \cap N^{(\tau)} = \emptyset$ by the definition of the procedure, and hence

$$\begin{aligned} |M^{(\tau')} \cap N^{(\tau)}| &= |M^{(\tau)} \cap N^{(\tau)}| - |(M^{(\tau)} \setminus M^{(\tau')}) \cap N^{(\tau)}| + |(M^{(\tau')} \setminus M^{(\tau)}) \cap N^{(\tau)}| \\ &= |M \cap N| + |M^{(\tau')} \setminus M^{(\tau)}| \\ &> |M \cap N|, \end{aligned}$$

which completes the proof. □

Claim 3.15. *In Step 4-1, $N^{(\tau+1)}$ is square-free and $|M^{(\tau+1)} \cap N^{(\tau+1)}| = |M \cap N| + 1$.*

Proof. First, $(u^{(\tau)}, v_1)$ is the unique edge in $N^{(\tau)} \cap \delta u^{(\tau)}$ and $N^{(\tau)} \cap \delta v_1 = \{(u^{(\tau)}, v_1), (v_1, v_2)\}$. Thus, $N^{(\tau+1)}$ does not have a square containing $(u^{(\tau)}, v_3)$, and hence $N^{(\tau+1)}$ is square-free because $N^{(\tau)}$ does not have a square. Furthermore, by Claim 3.13, $|M^{(\tau+1)} \cap N^{(\tau+1)}| = |M^{(\tau)} \cap N^{(\tau)}| + 1 = |M \cap N| + 1$. □

Claim 3.16. *In Step 4-1, $(M^{(\tau+2)}, N^{(\tau+2)}, u^{(\tau+2)}) \in \mathcal{S}(x, y)$ and $|M^{(\tau+2)} \cap N^{(\tau+2)}| = |M \cap N| + 1$.*

Proof. Since $C = (v_2, v_1, v_4, v_5)$ is the unique square in $M^{(\tau+1)}$, $M^{(\tau+2)}$ is square-free. On the other hand, since $N^{(\tau+2)} \cap \delta v_2 = \{(v_1, v_2), (v_2, v_5)\}$, $N^{(\tau+2)} \cap \delta v_1 = \{(u^{(\tau)}, v_1), (v_1, v_2)\}$, and $(u^{(\tau)}, v_5) \notin N^{(\tau+2)}$, $N^{(\tau+2)}$ does not have a square containing (v_2, v_5). Thus, by Claim 3.15, $N^{(\tau+2)}$ is square-free, and hence $(M^{(\tau+2)}, N^{(\tau+2)}, u^{(\tau+2)}) \in \mathcal{S}(x, y)$. Furthermore, by Claim 3.15, $|M^{(\tau+2)} \cap N^{(\tau+2)}| = |M^{(\tau+1)} \cap N^{(\tau+1)}| = |M \cap N| + 1$. □

We can show the correctness of Step 4-2 in the same way. The above claims show the correctness of Procedure A.

3.2.2 Main algorithm

In this subsection, we give an algorithm for finding an $(x+s, y)$-increment t using Procedure A. In order to avoid confusion, let $G = (V_1, V_2; E)$ denote the original graph and let $G^\circ = (V_1^\circ, V_2^\circ; E^\circ)$ denote a shrunk graph. The algorithm is described as follows.

Algorithm FIND-INCREMENT
Input: Square-free 2-matchings M and N in a bipartite graph $G = (V_1, V_2; E)$ with $d_M = x$ and $d_N = y$, and an (x, y)-increment $s = -\chi_u$ with $u \in V_1$.
Output: An $(x + s, y - s)$-increment t and square-free 2-matchings M' and N' in G such that $d_{M'} = x + s + t$ and $d_{N'} = y - s - t$.

Step 1. Let $C_1, C_2, \ldots, C_p$ be edge disjoint cycles of length four in G such that $E(C_i) \subseteq M \cup N$ and $|E(C_i) \cap M| = |E(C_i) \cap N| = 3$ for $i = 1, 2, \ldots, p$. We take such $C_1, C_2, \ldots, C_p$ maximally, and shrink them. Let $G^\circ = (V_1^\circ, V_2^\circ; E^\circ)$ be the obtained graph satisfying Assumption 3.12, and let $M^\circ, N^\circ, x^\circ, y^\circ, u^\circ$ and s° be counterparts in G° to M, N, x, y, u and s, respectively.

Step 2. Execute Procedure A for $(M^\circ, N^\circ, u^\circ) \in \mathcal{S}_{G^\circ}(x^\circ + s^\circ, y^\circ - s^\circ)$. Then we obtain either (i) a non-stable semi-2-matching triple $(M^*, N^*, u^*) \in \mathcal{S}_{G^\circ}(x^\circ + s^\circ, y^\circ - s^\circ)$ with $|M^* \cap N^*| > |M^\circ \cap N^\circ|$, or (ii) a stable semi-2-matching triple $(M^*, N^*, u^*) \in \mathcal{S}_{G^\circ}(x^\circ + s^\circ, y^\circ - s^\circ)$ with $|M^* \cap N^*| \geq |M^\circ \cap N^\circ|$. Go to Step 3 in the case (i), and go to Step 4 in the case (ii).

Step 3. Update M°, N°, and u° as $M^\circ := M^*$, $N^\circ := N^*$, and $u^\circ := u^*$. While there exists a square C such that $E(C) \subseteq M^\circ \cup N^\circ$ and $|E(C) \cap M^\circ| = |E(C) \cap N^\circ| = 3$, shrink C. Then, go to Step 2.

Step 4. Let t be the $(x + s, y - s)$-increment corresponding to an $(x^\circ + s^\circ, y^\circ - s^\circ)$-increment $t^\circ = d_{M^*} - x^\circ - s^\circ$. Output 2-matchings M' and N' in G which are expanded edge sets of M^* and N^*, respectively, such that $d_{M'} = x + s + t$ and $d_{N'} = y - s - t$, and stop the algorithm.

In the algorithm, $|M^\circ \setminus N^\circ| + |N^\circ \setminus M^\circ|$ decreases monotonically, which shows that FIND-INCREMENT terminates in finite steps. The correctness of FIND-INCREMENT is guaranteed by the following claims.

Claim 3.17. *In the case (i) of Step 2, the non-stable semi-2-matching triple (M^*, N^*, u^*) is active.*

Proof. Suppose that $u^* \in V_1$. Since $x^\circ + s^\circ, y^\circ - s^\circ \in \{0, 1, 2\}^{V_1^\circ \cup V_2^\circ}$, $d_{M^*}(u^*) = (x^\circ + s^\circ)(u^*) + 1$, and $d_{N^*}(u^*) = (y^\circ - s^\circ)(u^*) - 1$, we have that $d_{M^*}(u^*) \geq 1$ and $d_{N^*}(u^*) \leq 1$. Therefore, if (M^*, N^*, u^*) is not active, then $d_{M^*}(u^*) = d_{N^*}(u^*) = 1$, which contradicts that (M^*, N^*, u^*) is not stable. For the case when $u^* \in V_2^\circ$, we can show the claim in the same way. □

Claim 3.18. *In Step 4, $t^\circ = d_{M^*} - x^\circ - s^\circ$ is an $(x^\circ + s^\circ, y^\circ - s^\circ)$-increment.*

Proof. Suppose that $u^* \in V_1^\circ$. Then, $d_{M^*} - x^\circ - s^\circ = -d_{N^*} + y^\circ - s^\circ = \chi_{u^*}$ by the definition of $\mathcal{S}_{G^\circ}(x^\circ + s^\circ, y^\circ - s^\circ)$. Since $d_{M^*}(u^*) - d_{N^*}(u^*) \leq 1$ by the definition of a stable semi-2-matching triple, $(y^\circ - s^\circ)(u^*) - (x^\circ + s^\circ)(u^*) = (d_{N^*}(u^*) + 1) - (d_{M^*}(u^*) - 1) \geq 1$, which means that $t^\circ = \chi_{u^*}$ is an $(x^\circ + s^\circ, y^\circ - s^\circ)$-increment. We can deal with the case when $u^* \in V_2^\circ$ in the same way. □

Claim 3.19. *In Step 4, there exist an $(x + s, y - s)$-increment t corresponding to t° and 2-matchings M' and N' such that $d_{M'} = x + s + t$ and $d_{N'} = y - s - t$. Furthermore, such t, M', and N' are unique.*

Proof. Without loss of generality, we assume that $u^* \in V_1^\circ$. Then, $t^\circ = \chi_{u^*}$ is an $(x^\circ + s^\circ, y^\circ - s^\circ)$-increment by Claim 3.18.

First we show the existence of an unique $(x+s, y-s)$-increment t corresponding to t°. We observe that at most one square-edge is incident to u^*, because if two square-edges are incident to u^* then they are never updated in the algorithm.

If there exists no square-edge incident to u^* and u^* corresponds to $u' \in V_1$, then obviously, $t = \chi_{u'}$ is a desired $(x+s, y-s)$-increment.

Suppose that one square-edge is incident to u^*, and $u^* \in V_1^\circ$ corresponds to $v_1, v_2 \in V_1$. Since $(x^\circ + s^\circ)(u^*) = 1$ and $(y^\circ - s^\circ)(u^*) = 2$, we have that $\{(x+s)(v_1), (x+s)(v_2)\} = \{1, 2\}$ and $(y-s)(v_1) = (y-s)(v_2) = 2$. Hence, exactly one of χ_{v_1} and χ_{v_2} is a desired $(x+s, y-s)$-increment.

The $(x+s, y-s)$-increment t defined as above satisfies that $\phi(x+s+t) = x^\circ + s^\circ + t^\circ$ and $\phi(y-s-t) = y^\circ - s^\circ - t^\circ$, where ϕ is defined by (1). Furthermore, $x^\circ + s^\circ + t^\circ$ and $y^\circ - s^\circ - t^\circ$ are the degree sequences of M^* and N^*, respectively. Hence, by Lemma 3.3, there exist 2-matchings M' and N' in G such that they are expanded edge sets of M^* and N^*, respectively, and $d_{M'} = x+s+t$ and $d_{N'} = y - s - t$. The uniqueness of M' and N' is also guaranteed by Lemma 3.3. ☐

We have already presented an algorithm to find an $(x+s, y-s)$-increment. To obtain an $(x+s, y)$-increment t, we need the following assumption.

Assumption 3.20. For $x, y \in J_4(G)$, let M and N be square-free 2-matchings with $d_M = x$ and $d_N = y$ maximizing $|M \cap N|$.

We now prove that under Assumption 3.20 the output t of FIND-INCREMENT is an $(x+s, y)$-increment, that is, $d_{M'} \neq x$, $d_{N'} \neq y$.

Proposition 3.21. *Let M and N be inputs of FIND-INCREMENT satisfying Assumption 3.20. Then, the output (M', N') of FIND-INCREMENT satisfies that $d_{M'} \neq d_M$ and $d_{N'} \neq d_N$.*

Proof. If Procedure A outputs $(M^*, N^*, u^*) \in \mathcal{S}_{G^\circ}(x^\circ + s^\circ, y^\circ - s^\circ)$ with $|M^* \cap N^*| > |M^\circ \cap N^\circ|$ at least once in Step 2 of FIND-INCREMENT, then the output (M', N') satisfies that $|M' \cap N'| > |M \cap N|$, which implies $d_{M'} \neq d_M$ and $d_{N'} \neq d_N$ by Assumption 3.20.

Otherwise, when we execute Procedure A for the first time, it outputs a stable semi-2-matching triple $(M^*, N^*, u^*) \in \mathcal{S}_{G^\circ}(x^\circ + s^\circ, y^\circ - s^\circ)$ with $|M^* \cap N^*| = |M^\circ \cap N^\circ|$, where $(M^\circ, N^\circ, u^\circ)$ is the input of Procedure A. Since $|M^* \cap N^*| = |M^\circ \cap N^\circ|$, (M^*, N^*, u^*) is not output in Step 2 of Procedure A, and hence $u^* \neq u^\circ$. This shows that $t^\circ = \pm\chi_{u^*}$ is different from $-s^\circ = \chi_{u^\circ}$, which means $t \neq -s$. ☐

By Proposition 3.21, we complete the proof of Proposition 3.1.

Finally, we give a short remark. The above arguments show that FIND-INCREMENT finds an $(x+s, y)$-increment in polynomial time if we are given square-free 2-matchings M and N with $d_M = x$ and $d_N = y$ maximizing $|M \cap N|$. When we are given square-free 2-matchings M and N with $d_M = x$ and $d_N = y$ which do not necessarily maximize $|M \cap N|$, by executing FIND-INCREMENT, we obtain either an $(x+s, y)$-increment or square-free 2-matchings M' and N' such that $d_{M'} = x$, $d_{N'} = y$, and $|M' \cap N'| > |M \cap N|$. Thus, by repeating FIND-INCREMENT, we can also find an $(x+s, y)$-increment in polynomial time in this case.

4 Weighted square-free 2-matchings and M-concave functions

In this section, we prove Theorem 1.4, which is a generalization of Proposition 3.1. We prove the sufficiency (Proposition 4.1) and the necessity (Proposition 4.4) separately in the rest of this section.

4.1 Sufficiency

This subsection is devoted to proving the sufficiency in Theorem 1.4.

Proposition 4.1. *For a weighted bipartite graph (G, w), if w is vertex-induced on every square in G, then f is an M-concave function on the constant-parity jump system $J_4(G)$.*

Let $G = (V_1, V_2; E)$ be a bipartite graph with a weight function w and $G^\circ = (V_1^\circ, V_2^\circ; E^\circ)$ be its shrunk bipartite graph. For $F \subseteq E$, we define $w(F) = \sum_{e \in F} w(e)$. In a similar way as Proposition 3.1, we give an algorithm for finding an $(x + s, y)$-increment t satisfying (M-EXC) for given x, y, and s. In our algorithm, we keep a semi-2-matching triple in $\mathcal{S}_{G^\circ}(x^\circ + s^\circ, y^\circ - s^\circ)$, where x°, y°, and s° are the counterparts in G° of x, y, and s. We define the weight of shrunk edge sets with the aid of Lemma 3.11. Let $(M^\circ, N^\circ, u^\circ) \in \mathcal{S}_{G^\circ}(x^\circ + s^\circ, y^\circ - s^\circ)$ be a semi-2-matching triple such that u° corresponds to a unique vertex u in G, and let $(M, N, u) \in \mathcal{S}_G(x + s, y - s)$ be the corresponding semi-2-matching triple as in Lemma 3.11. For this case, we define the weights of edge sets M° and N° by $w'(M^\circ) = w(M)$ and $w'(N^\circ) = w(N)$.

With this terminology, to deal with the weighted case, we modify Step 2 of Procedure A as follows:

Step 2′. If $u^{(\tau)} = u^{(\tau')}$ for some $\tau' < \tau$, then output the following semi-2-matching triple and stop the procedure: output $(M^{(\tau')}, N^{(\tau)}, u^{(\tau)}) \in \mathcal{S}(x, y)$ if $w'(M^{(\tau')}) \geq w'(M^{(\tau)})$, and output $(M^{(\tau)}, N^{(\tau')}, u^{(\tau)}) \in \mathcal{S}(x, y)$ if $w'(M^{(\tau')}) < w'(M^{(\tau)})$. Otherwise, go to Step 3.

Note that when Step 2′ of Procedure A is executed, the weight of each edge set can be defined by the following claim.

Claim 4.2. *When Step 2′ of Procedure A is executed, $u^{(\tau)} = u^{(\tau')}$ is incident to no square-edges.*

Proof. Without loss of generality, we assume $u^{(\tau')} = u^{(\tau)} \in V_1^\circ$. Then, $(u^{(\tau')}, u^{(\tau'+1)}) \in N^{(\tau'+1)} \setminus M^{(\tau'+1)}$ and $(u^{(\tau)}, u^{(\tau-1)}) \in N^{(\tau-1)} \setminus M^{(\tau-1)}$. Since $u^{(\tau'+1)} \neq u^{(\tau-1)}$, two edges $(u^{(\tau')}, u^{(\tau'+1)})$ and $(u^{(\tau)}, u^{(\tau-1)})$ are contained in $N^{(\tau-1)}$. If $u^{(\tau)} = u^{(\tau')}$ is incident to a square-edge, then $d_{N^{(\tau-1)}}(u^{(\tau)}) \geq 3$, which contradicts the definition of $\mathcal{S}(x, y)$. □

By modifying Step 2 of Procedure A as above, Algorithm FIND-INCREMENT is also modified for weighted graphs. Note that the modified algorithm runs correctly and outputs an $(x + s, y - s)$-increment t and square-free 2-matchings M' and N' such that $d_{M'} = x + s + t$ and $d_{N'} = y - s - t$. We now discuss the weight of the edges of the output of FIND-INCREMENT.

Lemma 4.3. *Let (G, w) be a weighted bipartite graph such that w is vertex-induced on every square in G, M and N be square-free 2-matchings in G, and $s = -\chi_u$ be a (d_M, d_N)-increment with $u \in V_1$. If we execute the modified algorithm of FIND-INCREMENT to obtain new square-free 2-matchings M', N' and a $(d_M + s, d_N - s)$-increment t such that $d_{M'} = d_M + s + t$ and $d_{N'} = d_N - s - t$, then it holds that $w(M') + w(N') \geq w(M) + w(N)$.*

Proof. When $w'(M^{(\tau)})$ and $w'(N^{(\tau)})$ can be defined, let $M_{(\tau)}$ and $N_{(\tau)}$ be expanded edge sets of $M^{(\tau)}$ and $N^{(\tau)}$ which are used to define $w'(M^{(\tau)})$ and $w'(N^{(\tau)})$, that is $w'(M^{(\tau)}) = w(M_{(\tau)})$ and $w'(N^{(\tau)}) = w(N_{(\tau)})$. In Step 1 of Procedure A, $M^{(\tau)} \cup N^{(\tau)}$ does not change even if we consider the multiplicity of the edges. Thus, for each shrunk square C, $d_{M_{(\tau)} \cap E(C)} + d_{N_{(\tau)} \cap E(C)}$ is invariable. Since w is vertex-induced on every square in G, Step 1 of Procedure A does not change $w'(M^{(\tau)}) + w'(N^{(\tau)})$.

By modification of Step 2 of Procedure A as above, the output in Step 2′ does not decrease $w'(M^{(\tau)}) + w'(N^{(\tau)})$, in fact, $w'(M^{(\tau')}) + w'(N^{(\tau)}) \geq w'(M^{(\tau)}) + w'(N^{(\tau)})$ if $w'(M^{(\tau')}) \geq w'(M^{(\tau)})$, and $w'(M^{(\tau)}) + w'(N^{(\tau')}) > w'(M^{(\tau)}) + w'(N^{(\tau)})$ if $w'(M^{(\tau')}) < w'(M^{(\tau)})$, because $w'(M^{(\tau)}) + w'(N^{(\tau)}) = w'(M^{(\tau')}) + w'(N^{(\tau')})$ by the above argument for Step 1.

When Procedure A outputs a semi-2-matching triple $(M^{(\tau)}, N^{(\tau)}, u^{(\tau)})$ in Step 3, by the argument in Claim 3.19, the expanded edge sets M' and N' of $M^{(\tau)}$ and $N^{(\tau)}$ are determined uniquely. Then, by the same argument as for Step 1, the total weight of the expanded edge sets is invariable.

Moreover, the total weight of the expanded edge sets of $M^{(\tau)}$ and $N^{(\tau)}$ does not change in Step 4 of Procedure A, because w is vertex-induced on every square in G.

Therefore, the total weight does not decrease in Procedure A, which means that $w(M') + w(N') \geq w(M) + w(N)$. □

We are now ready to show Proposition 4.1.

Proof for Proposition 4.1. For $x, y \in J_4(G)$ and an (x, y)-increment s, let M and N be square-free 2-matchings such that $d_M = x$, $d_N = y$, $w(M) = f(x)$, and $w(N) = f(y)$. As with Assumption 3.20, we assume that M and N maximize $|M \cap N|$ among such 2-matchings.

By executing the modified algorithm of FIND-INCREMENT, we find new square-free 2-matchings M' and N' and an $(x + s, y)$-increment t that satisfy $d_{M'} = x+s+t$ and $d_{N'} = y-s-t$. By Lemma 4.3, we have

$$\begin{aligned} f(x) + f(y) &= w(M) + w(N) \\ &\leq w(M') + w(N') \\ &\leq f(x + s + t) + f(y - s - t). \end{aligned}$$

Hence f is an M-concave function on $J_4(G)$. □

4.2 Necessity

This subsection is devoted to proving the necessity in Theorem 1.4.

Proposition 4.4. *For a weighted bipartite graph (G, w), if f is an M-concave function on the constant-parity jump system $J_4(G)$, then w is vertex-induced on every square in G.*

Proof. Let $C = (v_1, v_2, v_3, v_4)$ be a square in G. Let $x = \chi_{v_1} + 2\chi_{v_2} + 2\chi_{v_3} + \chi_{v_4} \in J_4(G)$ and $y = 2\chi_{v_1} + \chi_{v_2} + \chi_{v_3} + 2\chi_{v_4} \in J_4(G)$. Then, $M = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$ and $N = \{(v_1, v_2), (v_3, v_4), (v_4, v_1)\}$ are the unique edge sets such that $d_M = x$ and $d_N = y$, and hence $f(x) = w(M)$ and $f(y) = w(N)$.

For an (x, y)-increment $s = \chi_{v_1}$, one can see that $t = -\chi_{v_3}$ is the only $(x + s, y)$-increment such that $x+s+t \in J_4(G)$ and $y-s-t \in J_4(G)$. Then, $x+s+t = 2\chi_{v_1} + 2\chi_{v_2} + \chi_{v_3} + \chi_{v_4}$ and $y-s-t = \chi_{v_1} + \chi_{v_2} + 2\chi_{v_3} + 2\chi_{v_4}$. Since $M' = \{(v_1, v_2), (v_2, v_3), (v_4, v_1)\}$ and $N' = \{(v_2, v_3), (v_3, v_4), (v_4, v_1)\}$ are the unique edge sets such that $d_{M'} = x + s + t$ and $d_{N'} = y - s - t$, it holds that $f(x + s + t) =$

$w(M')$ and $f(y-s-t) = w(N')$. If f is an M-concave function on $J_4(G)$, by (M-EXC), we have $w(M) + w(N) \leq w(M') + w(N')$, which means that

$$w(v_1, v_2) + w(v_3, v_4) \leq w(v_2, v_3) + w(v_4, v_1). \tag{2}$$

Similarly, it holds that

$$w(v_1, v_2) + w(v_3, v_4) \geq w(v_2, v_3) + w(v_4, v_1). \tag{3}$$

By (2) and (3),

$$w(v_1, v_2) + w(v_3, v_4) = w(v_2, v_3) + w(v_4, v_1),$$

which shows that w is vertex-induced on C. □

5 Concluding remarks

Finally, in this section, we discuss extensions of Theorems 1.2 and 1.4 to the $K_{t,t}$-free t-matchings.

A min-max formula for the square-free 2-matching problem [19] extends to the $K_{t,t}$-free t-matching problem [13], and Pap's maximum square-free 2-matching algorithm [30] is applicable to the $K_{t,t}$-free t-matching problem. Thus, we expect that $K_{t,t}$-free t-matchings in bipartite graphs also have some sort of matroidal structure. Namely, as an extension of the first half of Theorem 1.2, we conjecture the following.

Conjecture 5.1. *For any bipartite graph G,*

$$J_{t,t}(G) = \{d_M \mid M \text{ is a } K_{t,t}\text{-free } t\text{-matching in } G\}$$

is a constant-parity jump system.

We can also consider the extension of Theorem 1.4. For a weighted bipartite graph (G, w), define a function $f_{t,t}$ on $J_{t,t}(G)$ by

$$f_{t,t}(x) = \max \left\{ \sum_{e \in M} w(e) \;\middle|\; M \text{ is a } K_{t,t}\text{-free } t\text{-matching, } d_M = x \right\}.$$

For the weighted $K_{t,t}$-free t-matching problem in a weighted bipartite graph (G, w) where w is vertex-induced on every $K_{t,t}$ in G, we have a linear programming description with dual integrality [24] and a polynomial-time algorithm [32]. Thus, the following conjecture naturally arises.

Conjecture 5.2. *For a weighted bipartite graph (G, w), $f_{t,t}$ is an M-concave function on the constant-parity jump system $J_{t,t}(G)$ if and only if w is vertex-induced on every $K_{t,t}$ in G.*

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