Twisted cohomology and homology groups associated to the Riemann-Wirtinger integral

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Abstract

The Riemann-Wirtinger integral is a function defined by a definite integral on a complex torus whose integrand is a power product of the exponential function and theta functions. It was found as a special solution of the system of differential equations which governs the monodromy-preserving deformation of Fuchsian differential equations on the complex torus [11], [12]. The main purpose of this paper is to give an interpretation to this integral as the pairing between twisted cohomology and homology groups with coefficients in rank-one local systems on the complex torus minus finitely many points. The local systems contain a parameter corresponding to the Jacobian of the torus. We also give an explicit description of the cohomology groups valid for any value of the parameter. This result shall be applied to studies on generalization of the Wirtinger integral, which is another integral representation of Gauss' hypergeometric function in terms of the power product of theta functions [14].

1. Introduction.

The twisted de Rham theory developed by Aomoto [1], [3] has brought a unified treatment, and a systematic way of generalization, of various hypergeometric integrals which were invented and investigated by many authors. According to Aomoto, any such integral is formed as a paring of a homology class and a cohomology class on a complex projective space $\mathbf{P}^n$ minus an effective divisor D (in most cases D is a union of hyperplanes) with coefficients in a local system (or its dual) of rank one which is defined by a multi-valued function on $\mathbf{P}^n$ ramified just along D. Moreover, knowing the structures of the corresponding homology and cohomology groups (e.g., their vanishing or bases) enables us not only to produce systematically a system of differential equations satisfied by such integrals, and a system of fundamental solutions of such a system of differential equations, but also to determine the connection formulae and the monodromy representation of such integrals [2], [4], [5].

In the recent paper [14], one of the authors gave a new derivation of the connection formulae and the monodromy representation for Gauss' hypergeometric function $F(\alpha,\beta,\gamma,z)$ by investigating the behaviour of the following integral representation on the complex torus, which we call the Wirtinger integral (see also [16]),

$$\lambda(\tau)^{\frac{\gamma-1}{2}}(1-\lambda(\tau))^{\frac{\alpha+\beta-\gamma}{2}}F(\alpha,\beta,\gamma,\lambda(\tau)) \\ = \frac{2\pi\Gamma(\gamma)\theta_3^2}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^{\frac{1}{2}}\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+1}\theta_3(u)^{-2\beta+1}du$$

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by the action of modular transformations, where $\lambda(\tau)$ denotes the lambda function: $\lambda(\tau) = \theta_1^4/\theta_3^4$ (for the definition of theta functions, see *Notation for theta functions* below). On the other hand, the other author found, in [11] and [12], the following integral, which we call the Riemann-Wirtinger integral (cf. [13]),

$$(1) \qquad f_j = \int_\gamma e^{2\pi\sqrt{-1}c_0 u}\theta(u-t_1)^{c_1}\cdots\theta(u-t_n)^{c_n}\mathfrak{s}(u-t_j;\lambda)du, \quad j=1,\cdots,n,$$

where we assume that $c_1 + \cdots + c_n = 0$ and $\lambda + c_0\tau + c_1t_1 + \cdots + c_nt_n + c_\infty = 0$ for a complex number c_∞ (λ in (1) is not the lambda function), as a special solution of the system of partial differential equations which is the integrability condition of the monodromy-preserving deformation of Fuchsian differential equations with n singularities $t_1, \ldots, t_n$ on the complex torus. It may be considered as an analogue on the torus to the hypergeometric solutions of the sixth Painlevé equation and the Garnier systems. The Wirtinger integral appears when we restrict the configuration $t_1, \ldots, t_n$ on the torus to the two-torsion points: $t_1 = 0$, $t_2 = -1/2$, $t_3 = -\tau/2$, $t_4 = -(\tau+1)/2$, and the parameter λ to zero in the Riemann-Wirtinger integral (1). (Precisely, the integral (1) diverges at $\lambda = 0$. But we shall prove that we can avoid the divergence successfully.) Then, it is an interesting problem to generalize the Wirtinger integral to the case where the configuration is restricted to the N-torsion points on the torus (where N is an integer greater than two) because it is plausible that the generalized Wirtinger integral solves a Fuchsian differential equation defined on the modular curve of level N with singularities at the cusps (see also [17]).

The purpose of this paper is to give a foundation for the connection problem and the monodromy representation of the Riemann-Wirtinger integral (especially keeping the generalization of the Wirtinger integral in mind) by studying the twisted cohomology and homology groups on the complex torus minus n distinct points $t_1, ..., t_n$ with coefficients in a local system (or its dual) of rank one defined by the multi-valuedness of the function $T(u) = e^{2\pi\sqrt{-1}c_0 u}\theta(u-t_1)^{c_1}\cdots\theta(u-t_n)^{c_n}$ and a one-dimensional representation of the fundamental group of the complex torus. K. Ito [9] studied the twisted homology groups with coefficients in the local system defined by a multi-valued function on the complex torus minus four points. The present work may be also regarded as a generalization of [9]. In Section 2, we study the twisted cohomology groups. Differently from the usual theory of hypergeometric integrals on $\mathbf{P}^n$ defined by rank-one local systems, since there exist non-trivial holomorphic line bundles with Chern class zero on the complex torus (they are parameterized by the Jacobian of the complex torus), we need to consider the tensor product of the local system defined by a multi-valued function and a holomorphic line bundle with Chern class zero on the complex torus. Then the variable λ in (1) is a parameter runs over the Jacobian of the torus, and the (generalized) Wirtinger integrals appear in the case where $\lambda = 0$, namely the trivial bundle. The structure of the cohomology groups depend on the value of λ. When $\lambda \neq 0$, we can take logarithmic differential forms as a basis of the first twisted cohomology group. When $\lambda = 0$, there is no basis consisting of only logarithmic forms (see also [15]). But, whether λ is equal to 0 or not, we can describe the cohomology group by taking generators and a relation among them appropriately (Theorem 2.6 and Corollary 2.7). In Section 3, we study the twisted homology groups with coefficients in the local system dual to the one treated in Section 2. We determine generators of twisted cycles and relations among them explicitly by giving a cell decomposition (Theorem 3.1). In Section 4, regarding configurations of n points $t_1, \ldots, t_n$ on tori and their periods τ as independent variables, we describe a natural connection on the twisted cohomology groups, i.e. the Gauss-Manin connection

[10] (Proposition 4.1).

Notation for theta functions. In this paper, we follow Chandrasekharan's notation for theta functions [6]:

$$\begin{aligned}
\theta(u) = \theta(u,\tau) &= -\sqrt{-1}\sum_{n=-\infty}^{+\infty}(-1)^n e^{\pi\sqrt{-1}(n+1/2)^2\tau}e^{2\pi\sqrt{-1}(n+1/2)u},\\
\theta_1(u) = \theta_1(u,\tau) &= \sum_{n=-\infty}^{+\infty} e^{\pi\sqrt{-1}(n+1/2)^2\tau}e^{2\pi\sqrt{-1}(n+1/2)u},\\
\theta_2(u) = \theta_2(u,\tau) &= \sum_{n=-\infty}^{+\infty}(-1)^n e^{\pi\sqrt{-1}n^2\tau}e^{2\pi\sqrt{-1}nu},\\
\theta_3(u) = \theta_3(u,\tau) &= \sum_{n=-\infty}^{+\infty} e^{\pi\sqrt{-1}n^2\tau}e^{2\pi\sqrt{-1}nu}.
\end{aligned}$$

Then $\theta(u)$ is an odd function and $\theta_1(u)$, $\theta_2(u)$, $\theta_3(u)$ are even functions. These functions are related to each other as follows:

$$\begin{aligned}
\theta_1(u) &= \theta(u+\frac{1}{2}),\\
\theta_2(u) &= -\sqrt{-1}e^{\pi\sqrt{-1}(\tau/4+u)}\theta(u+\frac{\tau}{2}),\\
\theta_3(u) &= e^{\pi\sqrt{-1}(\tau/4+u)}\theta(u+\frac{\tau+1}{2}).
\end{aligned}$$

We also use the following symbols: $\theta' = \theta'(0,\tau)$, $\theta_i = \theta_i(0,\tau)$, $i=1,2,3$. We introduce the following function:

$$\mathfrak{s}(u;\lambda) = \frac{\theta(u-\lambda)\theta'}{\theta(u)\theta(-\lambda)}.$$

Then $\mathfrak{s}(u;\lambda)$ has the quasi-periodicity

$$\mathfrak{s}(u+1;\lambda) = \mathfrak{s}(u;\lambda), \quad \mathfrak{s}(u+\tau;\lambda) = e^{2\pi\sqrt{-1}\lambda}\mathfrak{s}(u;\lambda).$$

2. Twisted cohomology groups.

Let $\mathbf{H}$ be the upper half plane. For $\tau \in \mathbf{H}$, we set $\Gamma = \Gamma_\tau = \mathbf{Z} + \mathbf{Z}\tau$, a subgroup of the additive group $\mathbf{C}$ of complex numbers, where $\mathbf{Z}$ is the additive group of integers. The fundamental group of the torus $E = E_\tau = \mathbf{C}/\Gamma$ is isomorphic to the group Γ. Let λ be a complex number. We define a one-dimensional representation of the fundamental group, which we denote by $e_\lambda : \Gamma \ni \gamma \to e_\lambda(\gamma) \in \mathbf{C}^*$ ($\mathbf{C}^*$ denotes the multiplicative group of non-zero complex numbers), by the assignment of generators of Γ into $\mathbf{C}^*$: $e_\lambda(1) = 1$, $e_\lambda(\tau) = e^{2\pi\sqrt{-1}\lambda}$. Let R_λ be the local system of rank one on E determined by this representation e_λ. Let $\mathcal{O}_E$ be the sheaf of holomorphic functions on E. We set $\mathcal{O}_{E,\lambda} = \mathcal{O}_E \otimes_{\mathbf{C}} R_\lambda$, the tensor product of $\mathcal{O}_E$ and R_λ. Let L_λ be the line bundle on E whose local sections are generated by the sheaf $\mathcal{O}_{E,\lambda}$. The sheaf $\mathcal{O}_{E,\lambda}$ is also denoted by $\mathcal{O}_E(L_\lambda)$ in a literature. Then we have

Lemma 2.1. Assume that $\lambda \notin \Gamma$. Then $H^0(E,\mathcal{O}_{E,\lambda}) = H^1(E,\mathcal{O}_{E,\lambda}) = 0$.
Proof. Let f be in $H^0(E,\mathcal{O}_{E,\lambda})$. By definition, $f(u)$ is holomorphic on E, satisfying the

relations $f(u+1) = f(u)$ and $f(u+\tau) = e^{2\pi\sqrt{-1}\lambda} f(u)$. By the definition of $\mathfrak{s}(u;-\lambda)$, the product $f(u)\mathfrak{s}(u;-\lambda)$ must be an elliptic function on E with at most a single simple pole at $u=0$, from which we have $f=0$. Therefore $H^0(E, \mathcal{O}_{E,\lambda}) = 0$. Now the Riemann-Roch theorem holds: $\dim H^0(E, \mathcal{O}_{E,\lambda}) - \dim H^1(E, \mathcal{O}_{E,\lambda}) - c(L_\lambda) = 1-g$, where $c(L_\lambda)$ is the first Chern class of the line bundle L_λ. Since we can take $\mathfrak{s}(u;\lambda)$ as a global meromorphic section of L_λ, we conclude $c(L_\lambda) = 0$. Since $g=1$ and $H^0(E, \mathcal{O}_{E,\lambda}) = 0$, we have $\dim H^1(E, \mathcal{O}_{E,\lambda}) = 0$ by the Riemann-Roch theorem, which completes the proof of Lemma 2.1. Q.E.D.

Let $t_1, \ldots, t_n$ be n distinct points of E, where $n \geq 2$. We set $D = \{t_1, \ldots, t_n\}$ and $M = M(t_1, \cdots, t_n, \tau) = E \setminus D$. Let $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ be an open covering of E. Without loss of generality we may assume that $\mathcal{U}$ is a Leray open covering such that every open set $U_i \in \mathcal{U}$ contains at most one point t_k of D. Let us consider a collection $\mu = \{h_i\}_{i \in \Lambda}$, where h_i is a holomorphic section of L_λ on U_i if U_i contains no point of D, or it is a holomorphic section of L_λ on $U_i - \{t_k\}$ and may have an isolated singularity (i.e., a pole or an essential singularity) at t_k if U_i contains a point $t_k \in D$. Such a collection μ is called a *Mittag-Leffler distribution* if, $\rho_{U,V}$ denoting the restriction map of $H^0(U, \mathcal{O}_{E,\lambda})$ to $H^0(V, \mathcal{O}_{E,\lambda})$ for $U \supset V$, the differences $\rho_{U_i, U_i \cap U_j}(h_i) - \rho_{U_j, U_i \cap U_j}(h_j)$ are holomorphic on $U_i \cap U_j$. By a *solution* of μ we mean a global analytic section h of L_λ holomorphic on M such that the difference $\rho_{E,U_i}(h) - h_i$ is a holomorphic section on U_i for every $i \in \Lambda$. Here we note that our definition of Mittag-Leffler distribution is a little wider than the one given in [8]. Then we have

Lemma 2.2. If $\lambda \notin \Gamma$, every Mittag-Leffler distribution μ has a solution.

Proof. We set $h_i|_{U_i \cap U_j} = \rho_{U_i, U_i \cap U_j}(h_i)$. The collection $\{h_i|_{U_i \cap U_j} - h_j|_{U_i \cap U_j}\}_{i \neq j}$ forms a 1-cocycle: $\{h_i|_{U_i \cap U_j} - h_j|_{U_i \cap U_j}\}_{i \neq j} \in Z^1(\mathcal{U}, \mathcal{O}_{E,\lambda})$. Since $H^1(E, \mathcal{O}_{E,\lambda}) = 0$ by Lemma 2.1 and $\mathcal{U}$ is a Leray covering, there exists a local holomorphic section g_i on every U_i such that $h_i|_{U_i \cap U_j} - h_j|_{U_i \cap U_j} = g_i|_{U_i \cap U_j} - g_j|_{U_i \cap U_j}$. Then we have $h_i|_{U_i \cap U_j} - g_i|_{U_i \cap U_j} = h_j|_{U_i \cap U_j} - g_j|_{U_i \cap U_j}$. Gluing the local sections $h_i - g_i$ over E, we have a global section on E, which is the desired solution of μ. Q.E.D.

When $\lambda \in \Gamma$, we may assume $\lambda = 0$ without loss of generality.

Lemma 2.3. Assume that $\lambda = 0$. Let a_k $(1 \leq k \leq n)$ be the residue of the 1-form $h_i du$ at the singularity $u = t_k$ if $t_k \in U_i$. Then the Mittag-Leffler distribution μ has a solution if and only if $\sum_{k=1}^n a_k = 0$.

For the proof, see [8], Chap.2, §18.

Let $\mathcal{O}_M$ be the sheaf of holomorphic functions on M and let Ω^1_M be the sheaf of holomorphic 1-forms on M. We define the sheaves $\mathcal{O}_\lambda$ and Ω^1_λ on M by $\mathcal{O}_\lambda = \mathcal{O}_M \otimes_{\mathbf{C}} R_\lambda$ and $\Omega^1_\lambda = \Omega^1_M \otimes_{\mathbf{C}} R_\lambda$, where we denote the restriction of R_λ to M by the same symbol R_λ by abuse of notation. Since R_λ is locally constant and without torsion, we have the exact sequence of sheaves on M:

$$0 \to R_\lambda \to \mathcal{O}_\lambda \xrightarrow{d} \Omega^1_\lambda \to 0,$$

where d denotes the sheaf mapping induced by the differential $d : \mathcal{O}_M \to \Omega^1_M$. Let c_0 be an arbitrary complex number, and $c_1, \ldots, c_n$ $(n \geq 2)$ be non-integral complex numbers satisfying $c_1 + \cdots + c_n = 0$. We define a multi-valued function $T(u)$ on M by $T(u) = e^{2\pi\sqrt{-1}c_0 u}\theta(u-t_1)^{c_1} \cdots \theta(u-t_n)^{c_n}$. We set $\omega = d(\log T(u))$. We define a connection ∇ by $\nabla\varphi = d\varphi + \omega \wedge \varphi$. Then we have $\nabla\nabla = 0$ and $\nabla(1) = \omega$. The connection ∇ defines a sheaf morphism $\mathcal{O}_M \to \Omega^1_M$, and therefore $\mathcal{O}_\lambda \to \Omega^1_\lambda$. Let $\mathcal{L}$ and $\check{\mathcal{L}}$ be the local systems on M defined by the multi-valuedness of $T(u)^{-1}$ and $T(u)$, respectively: $\mathcal{L} = \mathbf{C}T(u)^{-1}$ and $\check{\mathcal{L}} = \mathbf{C}T(u)$. They are dual to each other. In case where we emphasize the dependence on the parameters c_i's contained in $\mathcal{L}$ and $\check{\mathcal{L}}$, we also write $\mathcal{L} = \mathcal{L}(c_0, \cdots, c_n)$ and $\check{\mathcal{L}} = \check{\mathcal{L}}(c_0, \cdots, c_n)$. Since the local system $\mathcal{L}$ is locally constant and without torsion, we have the following exact sequence on M:

$$0 \to \mathcal{L} \otimes_{\mathbf{C}} R_\lambda \to \mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_\lambda \xrightarrow{\mathrm{id}\otimes d} \mathcal{L} \otimes_{\mathbf{C}} \Omega^1_\lambda \to 0. \tag{2}$$

Let φ be a local section of $\mathcal{O}_\lambda$. Then the assignment $\varphi \mapsto T(u)\varphi$ defines a sheaf isomorphism $\mathcal{O}_\lambda \to \mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_\lambda$. Since $d(T(u)\varphi) = T(u)\nabla\varphi$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_\lambda & \xrightarrow{\nabla} & \Omega^1_\lambda \\ \downarrow & & \downarrow \\ \mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_\lambda & \xrightarrow[\mathrm{id}\otimes d]{} & \mathcal{L} \otimes_{\mathbf{C}} \Omega^1_\lambda, \end{array} \tag{3}$$

where the vertical arrows represent isomorphisms. Combining this commutative diagram and the exact sequence (2), we have the exact sequence

$$0 \to \mathcal{L} \otimes_{\mathbf{C}} R_\lambda \to \mathcal{O}_\lambda \xrightarrow{\nabla} \Omega^1_\lambda \to 0, \tag{4}$$

from which we have the following long exact sequence of the cohomology groups:

$$\begin{aligned} 0 \to & H^0(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \to H^0(M, \mathcal{O}_\lambda) \xrightarrow{\nabla} H^0(M, \Omega^1_\lambda) \to H^1(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \to H^1(M, \mathcal{O}_\lambda) \\ \xrightarrow{\nabla} & H^1(M, \Omega^1_\lambda) \to H^2(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \to H^2(M, \mathcal{O}_\lambda) \xrightarrow{\nabla} H^2(M, \Omega^1_\lambda) \to \cdots. \end{aligned}$$

Proposition 2.4. We have $H^i(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda) = 0$ for $i \neq 1$, and

$$H^1(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \cong H^0(M, \Omega^1_\lambda)/\nabla(H^0(M, \mathcal{O}_\lambda)).$$

Proof. The local system $\mathcal{L} \otimes_{\mathbf{C}} R_\lambda$ has no global section on M but zero section. So we have $H^0(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda) = 0$. Note that $\mathcal{O}_\lambda$ and Ω^1_λ are coherent $\mathcal{O}_M$-modules. Since the open Riemann surface M is Stein, we have $H^i(M, \mathcal{O}_\lambda) = H^i(M, \Omega^1_\lambda) = 0$ $(i > 0)$. Combining these results with (3), we have the short exact sequence

$$0 \to H^0(M, \mathcal{O}_\lambda) \xrightarrow{\nabla} H^0(M, \Omega^1_\lambda) \to H^1(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \to 0$$

and $H^i(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda) = 0$ $(i \geq 2)$, from which the proposition follows. Q.E.D.

Let $\tilde{\mathcal{K}}^0$ be the sheaf of meromorphic functions on E, and $\tilde{\mathcal{K}}^1$ be the sheaf of meromorphic 1-forms on E. Let $\mathcal{K}^0 = H^0(E, \tilde{\mathcal{K}}^0)$ be the field of global meromorphic functions on E, and $\mathcal{K}^1 = H^0(E, \tilde{\mathcal{K}}^1)$ be the $\mathcal{K}^0$-vector space of global meromorphic 1-forms on E. We define sheaves $\tilde{\mathcal{K}}^p_\lambda$ $(p = 0, 1)$ on E by $\tilde{\mathcal{K}}^p_\lambda = \tilde{\mathcal{K}}^p \otimes_{\mathbf{C}} R_\lambda$. Let $\mathcal{S}^p$ $(p = 0, 1)$ be the sheaf on M whose sections on an open set $U \subset M$ form the $\mathbf{C}$-vector space $\mathcal{S}^p(U) = \{s \in \mathcal{K}^p \mid s \text{ is holomorphic on } U\}$. By definition, $\mathcal{S}^0$ is a subsheaf of $\mathcal{O}_M$, and $\mathcal{S}^1$ a subsheaf of Ω^1_M. We define sheaves $\mathcal{S}^0_\lambda$ and $\mathcal{S}^1_\lambda$ on M by $\mathcal{S}^0_\lambda = \mathcal{S}^0 \otimes_{\mathbf{C}} R_\lambda$ and $\mathcal{S}^1_\lambda = \mathcal{S}^1 \otimes_{\mathbf{C}} R_\lambda$. Let us consider the subcomplex of sheaves

$$0 \to \mathcal{S}^0_\lambda \xrightarrow{\nabla} \mathcal{S}^1_\lambda \to 0 \tag{5}$$

of the complex

$$0 \to \mathcal{O}_\lambda \xrightarrow{\nabla} \Omega^1_\lambda \to 0. \tag{6}$$

The inclusion ι of (5) to (6) induces the natural homomorphism ι_* of the de Rham cohomologies:

$$\iota_* : H^0(M, \mathcal{S}^1_\lambda)/\nabla(H^0(M, \mathcal{S}^0_\lambda)) \to H^0(M, \Omega^1_\lambda)/\nabla(H^0(M, \mathcal{O}_\lambda)). \tag{7}$$

Let $\mathcal{O}_E(*D)$ be the sheaf of functions meromorphic on E and holomorphic on M, and $\Omega^1_E(*D)$ be the sheaf of 1-forms meromorphic on E and holomorphic on M. We set $\mathcal{O}_\lambda(*D) = \mathcal{O}_E(*D) \otimes_{\mathbf{C}} R_\lambda$ and $\Omega^1_\lambda(*D) = \Omega^1_E(*D) \otimes_{\mathbf{C}} R_\lambda$. Since $H^0(M, \mathcal{S}^1_\lambda) = H^0(E, \Omega^1_\lambda(*D))$ and $H^0(M, \mathcal{S}^0_\lambda) = H^0(E, \mathcal{O}_\lambda(*D))$, (7) is written by

$$\iota_* : H^0(E, \Omega^1_\lambda(*D))/\nabla(H^0(E, \mathcal{O}_\lambda(*D))) \to H^0(M, \Omega^1_\lambda)/\nabla(H^0(M, \mathcal{O}_\lambda)). \tag{8}$$

In fact we have

Proposition 2.5. For any $\lambda \in \mathbf{C}$, ι_* is an isomorphism.

Proof. Since we have

$$\mathcal{L}(c_0, c_1, \ldots, c_n) \otimes_{\mathbf{C}} R_{\lambda + m\tau + n} \cong \mathcal{L}(c_0 + m, c_1, \ldots, c_n) \otimes_{\mathbf{C}} R_\lambda,$$

for $m, n \in \mathbf{Z}$, we may assume that $\lambda \in P = \{a\tau + b | 0 \le a, b < 1\}$ without loss of generality. For the proof we need the following claim:

$(*)$ For $\varphi \in H^0(M, \Omega^1_\lambda)$, there exist $\psi \in H^0(E, \Omega^1_\lambda(*D))$ and $\tilde{Q}_* \in H^0(M, \mathcal{O}_\lambda)$ such that $\varphi = \psi + \nabla \tilde{Q}_*$.

In fact, let φ be in $H^0(M, \Omega^1_\lambda)$. Since the canonical line bundle of E is trivial, we may set $\varphi = f(u)du$, where $f(u)$ is a section belonging to $H^0(M, \mathcal{O}_\lambda)$ and may have isolated essential singularities at $u = t_k$ $(k = 1, \ldots, n)$. Let $P_k(u)$ be the principal part of the Laurent expansion of $f(u)$ at $u = t_k$. Let us find a function $Q_k(u)$ single-valued around $u = t_k$ satisfying the equation $P_k(u)du = \nabla Q_k$, that is, $P_k(u) = \dfrac{dQ_k}{du} + Q_k \dfrac{d}{du}(\log T(u))$. Here we may assume that $P_k(u) = \sum_{n \le -1} a^{(k)}_n (u - t_k)^n$. By quadrature we have a general solution of this equation: $Q_k = T(u)^{-1}[\int T(u)P_k(u)du + C]$ for some constant C. Since Q_k is single-valued at t_k, the condition $C = 0$ is necessary. Let us investigate the behaviour of the solution $Q_k = Q_k(u)$ with $C = 0$ around $u = t_k$. Since c_k is not an integer, the multi-valuedness of $T(u)$ around $u = t_k$ comes from the factor $(u - t_k)^{c_k}$. Then we can write $T(u) = (u - t_k)^{c_k} \times$ (single-valued holomorphic function) around $u = t_k$. Moreover, since we can write $T(u)P_k(u) = \sum_{n=-\infty}^{+\infty} e^{(k)}_n (u - t_k)^{c_k + n}$ around $u = t_k$, we have $\displaystyle\int T(u)P_k(u)du = \sum_{n=-\infty}^{+\infty} \frac{e^{(k)}_n}{c_k + n + 1}(u - t_k)^{c_k + n + 1}$, which is of the form $(u - t_k)^{c_k} \times$ (single-valued analytic function which may have an isolated singularity at $u = t_k$) around $u = t_k$. Consequently, the function $Q_k(u) = T(u)^{-1} \int T(u)P_k(u)du$ is a single-valued analytic function around $u = t_k$ which may have an isolated singularity at $u = t_k$, and therefore can be expanded in Laurent series at $u = t_k$. We set $Q_k(u) = \sum_{n=-\infty}^{+\infty} b^{(k)}_n (u - t_k)^n$, the Laurent expansion at $u = t_k$. Moreover we set $Q_{k-}(u) = \sum_{n \le 0} b^{(k)}_n (u - t_k)^n$ and $Q_{k+}(u) = \sum_{n \ge 1} b^{(k)}_n (u - t_k)^n$. Substituting $Q_k = Q_{k-} + Q_{k+}$ into the original equation above, we have $P_k = Q'_{k-} + Q'_{k+} + Q_{k-} \cdot (\log T(u))' + Q_{k+} \cdot (\log T(u))'$. Since $(\log T(u))'$

has a pole of order one at $u = t_k$ and Q_{k+} has a zero of order one at $u = t_k$, the product $Q_{k+} \cdot (\log T(u))'$ is holomorphic at $u = t_k$, and so is Q'_{k+}. Consequently, we see that in the right-hand side of the preceding relation the sum $Q'_{k-} + Q_{k-} \cdot (\log T(u))'$ contributes to the principal part P_k. Therefore, setting $\nabla Q_{k-} = g_k(u)du$, we see that the principal part of the Laurent expansion of $g_k(u)$ at $u = t_k$ is equal to P_k. Let $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ be a Leray open covering of E such that every open set U_i contains at most one point t_k of D. To conclude the proof of the claim $(*)$, we consider the two cases: (i) $\lambda \neq 0$ and (ii) $\lambda = 0$.

(i) Assume $\lambda \neq 0$. Let $\mu = \{Q_{*i}\}_{i\in\Lambda}$ be a Mittag-Leffler distribution subordinate to the open covering $\mathcal{U}$ consisting of local sections Q_{*i} of L_λ on U_i such that for every open set U_i containing $t_k \in D$ the section Q_{*i} coincides with the branch containing $Q_{k-} - b_0^{(k)}$ and for every open set U_j not containing any point of D the section Q_{*j} is holomorphic. Then, by Lemma 2.2, there exists a global analytic section Q_* of L_λ on E, holomorphic on M such that the difference $Q_{k-} - b_0^{(k)} - Q_*$ is holomorphic around $u = t_k$. We note that for every k the 1-form $P_k(u)du - \nabla b_0^{(k)} - \nabla Q_*$ is holomorphic at $u = t_k$, and that the 1-form $\nabla b_0^{(k)}$ locally defined around $u = t_k$ has a pole of order one at $u = t_k$ with residue $c_k b_0^{(k)}$. We set $\psi = \sum_{k=1}^n b_0^{(k)} c_k \mathfrak{s}(u - t_k, \lambda)du \in H^0(E, \Omega^1_\lambda(*D))$. Then the 1-form $f(u)du - \psi - \nabla Q_*$ is a global holomorphic section on E, and therefore by Lemma 2.1 we have $f(u)du = \psi + \nabla Q_*$, which proves the claim $(*)$.

(ii) Assume $\lambda = 0$. Then the claim $(*)$ follows if we apply Lemma 2.3 and the reasoning of the proof of Lemma 2.2 in [15]. We omit the detail.

We are now in a position to prove the proposition with the aid of the claim $(*)$. Let us take $[\varphi] \in H^0(M, \Omega^1_\lambda)/\nabla(H^0(M, \mathcal{O}_\lambda))$ arbitrarily, where $\varphi \in H^0(M, \Omega^1_\lambda)$. If we form $[\psi] \in H^0(E, \Omega^1_\lambda(*D))/\nabla(H^0(E, \mathcal{O}_\lambda(*D)))$ from the element $\psi \in H^0(E, \Omega^1_\lambda(*D))$ whose existence is guaranteed by the claim $(*)$, then we have $\iota_*[\psi] = [\varphi]$, which proves the surjectivity of ι_*. The proof of the injectivity of ι_* is as follows. For $[\psi] \in H^0(E, \Omega^1_\lambda(*D))/\nabla(H^0(E, \mathcal{O}_\lambda(*D)))$, assume that $\iota_*[\psi] = 0$. We set $\psi = f(u)du$, where $f(u)$ is a section of $H^0(E, \mathcal{O}_\lambda(*D))$ may have poles at points of D if $f(u)$ is not holomorphic there. The equation $\iota_*[\psi] = 0$ is translated into the assertion that there exists a section $g \in H^0(M, \mathcal{O}_\lambda)$ such that $f(u)du = \nabla g$. The equation is rewritten as $f(u) = \dfrac{dg}{du} + g(u)\dfrac{d}{du}(\log T(u))$, from which we have the solution $g(u) = T(u)^{-1} \int T(u) f(u) du$. By the same argument as when we constructed Q_k from P_k and investigated the behaviour of Q_k at $u = t_k$, we see that $g(u)$ is in $H^0(E, \mathcal{O}_\lambda(*D))$, and $[\psi] = 0$ as the equality in $H^0(E, \Omega^1_\lambda(*D))/\nabla(H^0(E, \mathcal{O}_\lambda(*D)))$, which proves the injectivity of ι_*. Q.E.D.

Remark. It is well-known that this lemma is also proved algebro-geometrically by the comparison theorem by Deligne, [7], II, Section 6.

Let D and D' be the divisors on E defined by $D = [t_1] + \cdots + [t_n]$ and $D' = 2[t_1] + [t_2] + \cdots + [t_n]$. Then we have the following natural homomorphisms:

$$I : H^0(E, \Omega^1_\lambda(D)) \to H^0(E, \Omega^1_\lambda(*D))/\nabla H^0(E, \mathcal{O}_\lambda(*D)),$$

$$I' : H^0(E, \Omega^1_\lambda(D')) \to H^0(E, \Omega^1_\lambda(*D))/\nabla H^0(E, \mathcal{O}_\lambda(*D)),$$

where, for a divisor F on E, we denote $H^0(E, \Omega^p_\lambda(F)) = \{f \in H^0(E, \tilde{\mathcal{K}^p_\lambda}) \mid (f) + F \geq 0\}$.

Theorem 2.6. In this setting, the following assertions hold:

(i) For $\lambda \in P \setminus \{0\}$, I is an isomorphism.

(ii) For $\lambda \in P$, I' is surjective and $\dim \ker I' = 1$.

Proof. First, we prove (i). We introduce a filtration to the vector spaces $H^0(E, \Omega^p_\lambda(*D))$ $(p = 0, 1)$ by the following:

$$F_k = F_k H^0(E, \Omega^p_\lambda(*D)) = H^0(E, \Omega^p_\lambda(kD)), \quad k = 0, 1, 2, \dots.$$

It is obvious that

$$F_k H^0(E, \Omega^p_\lambda(*D)) \subset F_{k+1} H^0(E, \Omega^p_\lambda(*D)),$$
$$\bigcup_{k=0}^{\infty} F_k H^0(E, \Omega^p_\lambda(*D)) = H^0(E, \Omega^p_\lambda(*D)),$$

and

$$F_0 H^0(E, \Omega^p_\lambda(*D)) = H^0(E, \Omega^p_\lambda) = 0.$$

For $k \geq 1$, it is easy to see that the set of n functions $\{\frac{\partial^{k-1}}{\partial u^{k-1}} \mathfrak{s}(u - t_i; \lambda)\}_{i=1}^n$ forms a basis of $\mathrm{Gr}^F_k H^0(E, \mathcal{O}_\lambda(*D)) = F_k / F_{k-1}$. From this fact and the assumptions $c_i \notin \mathbf{Z}$ $(i = 1, \dots, n)$, we can check that the induced homomorphism

$$\mathrm{Gr}^F_k \nabla : \mathrm{Gr}^F_k H^0(E, \mathcal{O}_\lambda(*D)) \to \mathrm{Gr}^F_{k+1} H^0(E, \Omega^1_\lambda(*D))$$

is isomorphic. Hence we have

$$H^0(E, \Omega^1_\lambda(*D)) / \nabla H^0(E, \mathcal{O}_\lambda(*D)) \cong F_1 H^0(E, \Omega^1_\lambda(*D)) = H^0(E, \Omega^1_\lambda(D)),$$

which concludes the assertion (i).

Next, we prove (ii). We introduce another filtration to $H^0(E, \Omega^p_\lambda(*D))$:

$$F'_k H^0(E, \Omega^p_\lambda(*D)) = H^0(E, \Omega^p_\lambda(D' + (k-1)D)), \ k = 0, 1, 2, \dots.$$

For $\lambda \in P$, we define $n + 1$ functions $\varphi_j(u; \lambda)$ $(j = 0, \dots, n)$ by

$$\varphi_0(u; \lambda) = -\lambda \mathfrak{s}(u - t_1; \lambda),$$
$$\varphi_1(u; \lambda) = \frac{\partial \mathfrak{s}}{\partial u}(u - t_1; \lambda),$$
$$\varphi_j(u; \lambda) = \mathfrak{s}(u - t_j; \lambda) - \mathfrak{s}(u - t_1; \lambda), \ j = 2, \dots, n.$$

Then we see that $F'_0 H^0(E, \mathcal{O}_\lambda(*D)) = \mathbf{C}\varphi_0(u; \lambda)$, and that $\{\frac{\partial^{k-1}}{\partial u^{k-1}} \varphi_i(u; \lambda)\}_{i=1}^n$ forms a basis of $\mathrm{Gr}^{F'}_k H^0(E, \mathcal{O}_\lambda(*D))$. The induced homomorphism

$$\mathrm{Gr}^{F'}_k \nabla : \mathrm{Gr}^{F'}_k H^0(E, \mathcal{O}_\lambda(*D)) \to \mathrm{Gr}^{F'}_{k+1} H^0(E, \Omega^1_\lambda(*D))$$

is isomorphic for $k \geq 1$. Hence we have

$$H^0(E, \Omega^1_\lambda(*D)) / \nabla H^0(E, \mathcal{O}_\lambda(*D)) \cong F'_1 H^0(E, \Omega^1_\lambda(*D)) / \nabla F'_0 H^0(E, \mathcal{O}_\lambda(*D)).$$

Noting that $\nabla \varphi_0(u; \lambda) \neq 0$, we can conclude the assertion (ii). Q.E.D.

Corollary 2.7. For any $\lambda \in P$, we have $\dim H^1(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda) = n$. The set of $n + 1$ classes $\{[\varphi_i(u; \lambda) du]\}_{i=0}^n$ generates the $\mathbf{C}$-vector space $H^1(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda)$, and these classes are subject to the unique relation $[\nabla \varphi_0(u; \lambda)] = 0$. In particular, for $\lambda \in P \setminus \{0\}$, the set of n classes $\{[\mathfrak{s}(u - t_i; \lambda) du]\}_{i=1}^n$ forms a basis of $H^1(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda)$.

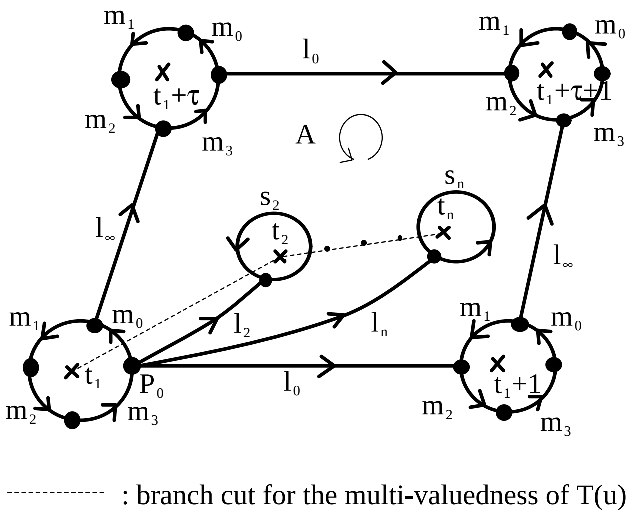

Figure 1: Cell decomposition of M.

3. Twisted homology groups.

Let $\check{\mathcal{L}}$ and $\check{R}_\lambda$ be the local systems dual to $\mathcal{L}$ and R_λ respectively. The universal coefficient theorem implies that the natural pairings $H_i(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda) \times H^i(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \to \mathbf{C}$ $(i = 0, 1, 2)$ are non-degenerate. From the results of the last section, we have

$$H_0(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda) = H_2(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda) = 0,$$

$$\dim H_1(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda) = n.$$

In this section, we shall find generators of $H_1(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda)$ and relations among them. We give a cell decomposition of M in Figure 1: A is a 2-cell, $l_0, l_2. \dots, l_n, l_\infty$, $s_2, \dots, s_n$, m_0, m_1, m_2, m_3 are 1-cells, and the dots $\bullet$'s stand for 0-cells. By looking at the boundary map for 1-cells, we see that the following $n+1$ elements generate the group $Z_1(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda)$ of 1-cycles:

$$\gamma_\infty = \mathrm{reg}(t_1, t_1 + \tau) = l_\infty + \frac{1 - e^{-2\pi\sqrt{-1}c_\infty}}{e^{2\pi\sqrt{-1}c_1} - 1}(m_0 + m_3) + \frac{1 - e^{2\pi\sqrt{-1}(c_1 - c_\infty)}}{e^{2\pi\sqrt{-1}c_1} - 1}(m_1 + m_2),$$

$$\gamma_0 = \mathrm{reg}(t_1, t_1{+}1) = l_0 {+} \frac{1 - e^{-2\pi\sqrt{-1}c_0}}{e^{2\pi\sqrt{-1}c_1} - 1}(m_0 {+} e^{2\pi\sqrt{-1}c_1} m_1) {+} \frac{e^{2\pi\sqrt{-1}c_1} - e^{2\pi\sqrt{-1}c_0}}{e^{2\pi\sqrt{-1}c_1} - 1}(m_2 {+} m_3),$$

$$\gamma_j = \mathrm{reg}(t_1, t_j) = l_j - \frac{s_j}{e^{2\pi\sqrt{-1}c_j} - 1} + \frac{m_0 + e^{2\pi\sqrt{-1}c_1}(m_1 + m_2 + m_3)}{e^{2\pi\sqrt{-1}c_1} - 1}, \ j = 2, \dots, n,$$

where we put $c_\infty = -\lambda - c_0\tau - c_1 t_1 - \cdots - c_n t_n$. On the other hand, the group $B_1(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda)$ of 1-boundaries is generated by the single element

$$\partial A = (e^{2\pi\sqrt{-1}c_0} - 1)\gamma_\infty + (1 - e^{-2\pi\sqrt{-1}c_\infty})\gamma_0 - \sum_{j=2}^{n} e^{-2\pi\sqrt{-1}(c_1 + \cdots + c_j)}(1 - e^{2\pi\sqrt{-1}c_j})\gamma_j.$$

Theorem 3.1. The twisted homology group $H_1(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda)$ is generated by $n+1$ twisted cycles $\gamma_\infty, \gamma_0, \gamma_2, \dots, \gamma_n$ subject to the unique relation

$$(e^{2\pi\sqrt{-1}c_0} - 1)\gamma_\infty + (1 - e^{-2\pi\sqrt{-1}c_\infty})\gamma_0 - \sum_{j=2}^{n} e^{-2\pi\sqrt{-1}(c_1+\cdots+c_j)}(1 - e^{2\pi\sqrt{-1}c_j})\gamma_j = 0.$$

4. Connection on $H^1(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda)$.

We put $\mathcal{T} = \mathbf{C}^n \times \mathbf{H} \setminus D_{\mathcal{T}}$, where $D_{\mathcal{T}}$ is the divisor on $\mathbf{C}^n \times \mathbf{H}$ defined by

$$D_{\mathcal{T}} = \{(t_1, \dots, t_n, \tau) \in \mathbf{C}^n \times \mathbf{H} | \exists\, i \neq j \text{ s.t. } t_i + \Gamma = t_j + \Gamma\}.$$

Let $f : \mathcal{M} \to \mathcal{T}$ be the family of curves over $\mathcal{T}$ whose fiber over $(t_1, \dots, t_n, \tau) \in \mathcal{T}$ is $M(t_1, \dots, t_n, \tau)$. The local system $\mathcal{L} \otimes_{\mathbf{C}} R_\lambda$ on M extends naturally to the one on the total space $\mathcal{M}$ under the constraint $\lambda + c_0\tau + c_1 t_1 + \cdots + c_n t_n + c_\infty = 0$ for a constant $c_\infty \in \mathbf{C}$, which we denote by the same symbol $\mathcal{L} \otimes_{\mathbf{C}} R_\lambda$ for notational simplicity. The coherent $\mathcal{O}_{\mathcal{T}}$-module $R^1 f_*(\mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \otimes_{\mathbf{C}} \mathcal{O}_{\mathcal{T}}$ is locally free and we have the isomorphism

$$R^1 f_*(\mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \otimes_{\mathbf{C}} \mathcal{O}_{\mathcal{T}} \cong f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D)/\nabla_{\mathcal{M}/\mathcal{T}} f_*\mathcal{O}_{\mathcal{M},\lambda}(*D), \tag{9}$$

where $\nabla_{\mathcal{M}/\mathcal{T}} = d_{\mathcal{M}/\mathcal{T}} + \omega$. We wish to describe the connection on the right hand side of (9) whose horizontal sections form the image of $R^1 f_*(\mathcal{L} \otimes_{\mathbf{C}} R_\lambda)$, namely the Gauss-Manin connection on the de Rham cohomology [10].

Proposition 4.1. For a local section $\varphi(u;\lambda)du$ of the sheaf $f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D)$, we define differential operators ∇_{t_i} $(i = 1, \dots, n)$ and ∇_τ by

$$\nabla_{t_i}\varphi(u;\lambda)du = \frac{\partial\varphi}{\partial t_i}(u;\lambda)du + \omega_i(u)\varphi(u;\lambda)du - c_i\frac{\partial\varphi}{\partial\lambda}(u;\lambda)du,$$

$$\nabla_\tau\varphi(u;\lambda)du = \frac{\partial\varphi}{\partial\tau}(u;\lambda)du + \omega_0(u)\varphi(u;\lambda)du - c_0\frac{\partial\varphi}{\partial\lambda}(u;\lambda)du + \frac{1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\lambda}\nabla_{\mathcal{M}/\mathcal{T}}\varphi(u;\lambda),$$

where we put $\omega_i(u) = (\partial/\partial t_i)\log T(u)$ and $\omega_0(u) = (\partial/\partial\tau)\log T(u)$. Then ∇_{t_i} and ∇_τ define $\mathbf{C}$-homomorphisms on $f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D)$ and $\nabla_{\mathcal{M}/\mathcal{T}} f_*\mathcal{O}_{\mathcal{M},\lambda}(*D)$:

$$\nabla_{t_i}, \nabla_\tau : f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D) \to f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D),$$
$$\nabla_{t_i}, \nabla_\tau : \nabla_{\mathcal{M}/\mathcal{T}} f_*\mathcal{O}_{\mathcal{M},\lambda}(*D) \to \nabla_{\mathcal{M}/\mathcal{T}} f_*\mathcal{O}_{\mathcal{M},\lambda}(*D).$$

Consequently, ∇_{t_i} and ∇_τ induce differential operators on $f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D)/\nabla_{\mathcal{M}/\mathcal{T}} f_*\mathcal{O}_{\mathcal{M},\lambda}(*D)$.

Proof. Note that $\varphi(u;\lambda)$ has the following quasi-periodicity:

$$\varphi(u+1;\lambda) = \varphi(u;\lambda),\ \ \varphi(u+\tau;\lambda) = e^{2\pi\sqrt{-1}\lambda}\varphi(u;\lambda).$$

Therefore we have the following formulae:

$$
\begin{gathered}
\frac{\partial \varphi}{\partial t_i}(u+1;\lambda) = \frac{\partial \varphi}{\partial t_i}(u;\lambda), \\
\omega_i(u+1)\varphi(u+1;\lambda) = \omega_i(u)\varphi(u;\lambda), \\
\frac{\partial \varphi}{\partial \lambda}(u+1;\lambda) = \frac{\partial \varphi}{\partial \lambda}(u;\lambda), \\
\frac{\partial \varphi}{\partial t_i}(u+\tau;\lambda) = e^{2\pi\sqrt{-1}\lambda}\frac{\partial \varphi}{\partial t_i}(u;\lambda), \\
\omega_i(u+\tau)\varphi(u+\tau;\lambda) = e^{2\pi\sqrt{-1}\lambda}(\omega_i(u) + 2\pi\sqrt{-1}c_i)\varphi(u;\lambda), \\
\frac{\partial \varphi}{\partial \lambda}(u+\tau;\lambda) = e^{2\pi\sqrt{-1}\lambda}(\frac{\partial \varphi}{\partial \lambda}(u;\lambda) + 2\pi\sqrt{-1}\varphi(u;\lambda)),
\end{gathered}
$$

from which we see that $\nabla_{t_i}\varphi(u;\lambda)du$ is also a section of $f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D)$. Besides, we have

$$
\begin{aligned}
\nabla_{t_i}\nabla_{\mathcal{M}/\mathcal{T}} &= (\frac{\partial}{\partial t_i} + \omega_i(u) - c_i\frac{\partial}{\partial \lambda})(d_{\mathcal{M}/\mathcal{T}} + \omega) \\
&= \nabla_{\mathcal{M}/\mathcal{T}}\nabla_{t_i} - d_{\mathcal{M}/\mathcal{T}}\omega_i(u) + \frac{\partial \omega}{\partial t_i} \\
&= \nabla_{\mathcal{M}/\mathcal{T}}\nabla_{t_i},
\end{aligned}
$$

which implies that $\nabla_{t_i}(\nabla_{\mathcal{M}/\mathcal{T}}f_*\mathcal{O}_{\mathcal{M},\lambda}(*D)) \subset \nabla_{\mathcal{M}/\mathcal{T}}f_*\mathcal{O}_{\mathcal{M},\lambda}(*D)$. We can also prove the assertions for ∇_τ in the similar way. Q.E.D.

For $\lambda \in P\setminus\{0\}$, we take the basis $\{\mathfrak{s}(u-t_i;\lambda)du\}_{i=1}^n$ of $H^1(M,\mathcal{L}\otimes_{\mathbf{C}} R_\lambda)$ and fix a basis $\{\gamma_{(1)},\ldots,\gamma_{(n)}\}$ of $H_1(M,\check{\mathcal{L}}\otimes_{\mathbf{C}}\check{R}_\lambda)$. We consider the pairing

$$
f_i^{(j)}(t_1,\ldots,t_n,\tau) = <\gamma_{(j)}, \mathfrak{s}(u-t_i;\lambda)du> .
$$

By the well-known procedure, this pairing is written by the integral:

$$
f_i^{(j)}(t_1,\ldots,t_n,\tau) = \int_{\gamma_{(j)}} T(u)\mathfrak{s}(u-t_i;\lambda)du.
$$

Then, by the action of the Gauss-Manin connection, it follows that $\mathbf{f}^{(j)} = \begin{pmatrix} f_1^{(j)} \\ \vdots \\ f_n^{(j)} \end{pmatrix}$ $(j=1,\ldots,n)$ form a fundamental system of solutions to a system of linear partial differential equations with respect to the independent variables $t_1,\ldots,t_n,\tau$. Such a differential system is given by the following (see [12] for details):

$$
\begin{cases}
\dfrac{\partial f_i}{\partial t_j} = -c_j\mathfrak{s}(t_j-t_i;\lambda)f_j + c_j\rho(t_j-t_i)f_i, \quad j\neq i, \\
\dfrac{\partial f_i}{\partial t_i} = \displaystyle\sum_{k\neq i} c_k\mathfrak{s}(t_k-t_i;\lambda)f_k + (2\pi\sqrt{-1}c_0 - \sum_{k\neq i} c_k\rho(t_k-t_i))f_i, \\
2\pi\sqrt{-1}\dfrac{\partial f_i}{\partial \tau} = \displaystyle\sum_{k=1}^n \frac{c_k}{2}(\rho(t_i-t_k)^2 - \wp(t_i-t_k))f_i + \sum_{k=1}^n c_k\frac{\partial \mathfrak{s}}{\partial \lambda}(t_k-t_i;\lambda)f_k,
\end{cases}
$$

where we assume $\lambda + c_0\tau + c_1t_1 + \cdots + c_nt_n + c_\infty = 0$ and we put $\rho(u) = \theta'(u)/\theta(u)$. This is the system of differential equations satisfied by the Riemann-Wirtinger integral (1).

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