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On competition numbers of complete multipartite graphs with partite sets of equal size

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Abstract

Let D be an acyclic digraph. The competition graph of D is a graph which has the same vertex set as D and has an edge between u and v if and only if there exists a vertex x in D such that (u, x) and (v, x) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of G is the smallest number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph G and it has been one of important research problems in the study of competition graphs to characterize a graph by its competition number.

In this paper, we compute the competition numbers of a complete multipartite graph in which each partite set has two vertices and a complete multipartite graph in which each partite set has three vertices.

Keywords: competition graphs, competition numbers, complete multipartite graphs

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1 Introduction

Suppose D is an acyclic digraph (for all undefined graph-theoretical terms, see [1] and [15]). The *competition graph* of D, denoted by $C(D)$, has the same set of vertices as D and an edge between vertices u and v if and only if there is a vertex x in D such that (u, x) and (v, x) are arcs of D. Roberts [14] observed that if G is any graph, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* $k(G)$ of a graph G to be the smallest number k such that G together with k isolated vertices added is the competition graph of an acyclic digraph.

The notion of competition graph was introduced by Cohen [3] as a means of determining the smallest dimension of ecological phase space. Since then, various variations have been defined and studied by many authors (see, for example, [2, 7, 8, 9, 11, 16]). Besides an application to ecology, the concept of competition graph can be applied to the study of communication over noisy channel (see Roberts [14] and Shannon [17]) and to problem of assigning channels to radio or television transmitters (see Cozzens and Roberts [4], Hale [6], or Opsut and Roberts [13]).

Roberts [14] observed that characterization of competition graph is equivalent to computation of competition number. It does not seem to be easy in general to compute $k(G)$ for all graphs G, as Opsut [12] showed that the computation of the competition number of a graph is an NP-hard problem (see [8, 9] for graphs whose competition numbers are known). It has been one of important research problems in the study of competition graphs to characterize a graph by its competition number.

In this paper, we shall compute competition numbers of a complete multipartite graph in which each partite set has two vertices and a complete multipartite graph in which each partite set has three vertices.

We denote by K_n^m the complete m-partite graph in which each partite set has n vertices.

For a digraph D, a sequence $v_1, v_2, \ldots, v_n$ of the vertices of D is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i > j$. It is well-known that a digraph D is acyclic if and only if there exists an acyclic ordering of D.

An *edge clique cover* (or an ECC for short) of a graph G is a family of cliques such that each edge of G is contained in some clique in the family. The smallest size of an ECC of G is called the *edge clique cover number* (or the ECC number for short), and is denoted by $\theta_e(G)$. Opsut [12] gave bounds of $k(G)$ for a graph G by showing that $\theta_e(G) - |V(G)| + 2 \leq k(G) \leq \theta_e(G)$. Dutton and Brigham [5] characterized the competition graphs of acyclic digraphs in terms of an ECC as follows:

Theorem 1 ([5])**.** *A graph G is the competition graph of an acyclic digraph if and only if there exists an ordering $v_1, \ldots, v_n$ of the vertices of G and an ECC $\{S_1, \ldots, S_n\}$ of G such that $v_i \in S_j$ implies $i < j$.*

For a vertex v in a graph G, let the open neighborhood of v be denoted by

$$N_G(v) = \{u \mid u \text{ is adjacent to } v\}$$

and the closed neighborhood of v be denoted by $N_G[v] = N_G(v) \cup \{v\}$. We denote the subgraph of G induced by $N_G(v)$ (resp. $N_G[v]$) by the same symbol $N_G(v)$ (resp. $N_G[v]$). For a digraph D, we define $N_D^-(v) = \{w \in V(D) \mid (w, v) \in A(D)\}$ and $N_D^+(v) = \{w \in V(D) \mid (v, w) \in A(D)\}$.

A *vertex clique cover* of a graph G is a family of cliques such that each vertex of G is contained in some clique in the family. The smallest size of a vetrex clicque cover of G is called the *vertex clique cover number*, and is denoted by $\theta_v(G)$. Opsut [12] showed the following:

Proposition 2 ([12])**.** *Let G be a graph. Then we have*

$$min\{\theta_v(N_G(v)) \mid v \in V(G)\} \leq k(G).$$

This proposition is true even if each open neighborhood is replaced with the closed neighborhood.

Proposition 3. *Let G be a graph. Then we have*

$$min\{\theta_v(N_G[v]) \mid v \in V(G)\} \leq k(G).$$

Proof. Let $t = \min\{\theta_v(N_G[v]) \mid v \in V(G)\}$ and $k = k(G)$. Let D be an acyclic digraph such that $C(D) = G \cup I_k = G \cup \{z_1, ..., z_k\}$. Let $z_1, ..., z_k, v_1, ..., v_n$ be an ordering of D so that (u, v) is an arc of D only if u is on the right hand side of v in the sequence. Then we have $|N_D^+(v_1)| \geq t$ since $\theta_v(N_G[v_1]) \geq t$. Since $N_D^+(v_1) \subseteq \{z_1, ..., z_k\}$, we have $t \leq k$ and thus the proposition holds. □

For some special graph families, we have explicit formulae for computing competition numbers. For example, if G is a choral graph with the minimum degree ≥ 1 then $k(G) = 1$, and if G is a triangle-free connected graph then

$$k(G) = |E(G)| - |V(G)| + 2$$

(see [14]). From this formula, it follows that for a complete bipartite graph K_{n_1,n_2}, we have $k(K_{n_1,n_2}) = n_1n_2 - (n_1 + n_2) + 2$. Kim and Sano [10] gave the exact competition number of a complete tripartite graph K_n^3:

Theorem 4 ([10]). *For $n \geq 2$,*

$$k(K_n^3) = n^2 - 3n + 4.$$

Then they proposed an open problem to compute competition numbers of various types of complete multipartite graphs. To answer their question, we study K_n^m. We give a lower bound for the competition number of K_n^m. Then we make use of it to compute the competition numbers of K_2^m and K_3^m.

2 A lower bound for the competition number of K_n^m

In this section, we give a lower bound for $k(K_n^m)$ for integers $m \geq 2$ and $n \geq 1$.

For each positive integer n, we denote the set $\{1, \dots, n\}$ by $[n]$.

Proposition 5. *For any vertex v of K_n^m with $m \geq 2$,*

$$\theta_v(N_{K_n^m}[v]) \geq n.$$

Proof. Let P_k denote the kth partite set of K_n^m and let $P_k = \{v_{k1}, \dots, v_{kn}\}$ for each $k \in [m]$. From the fact that v_{11} is adjacent to all the vertices in $V(K_n^m) \setminus P_1$ and that any two vertices in P_2 are not adjacent, we know that at least n cliques are needed to cover the n edges $v_{11}v_{21}, ..., v_{11}v_{2n}$ which are incident to v_{11}. Therefore we have $\theta_v(N_{K_n^m}[v_{11}]) \geq n$. By symmetry, we can conclude $\theta_v(N_{K_n^m}[v]) \geq n$ for any $v \in V(K_n^m)$. □

Given a digraph $D = (V, A)$, we use a symbol $u \to v$ for arc (u, v) in A. In addition, for $S \subset V$, we denote by $S \to w$ the arc set $\{(x, w) \mid x \in S\}$.

Theorem 6. *For an integer $m \geq 2$,*

$$k(K_n^m) \geq 2n - 2.$$

Proof. Let k be the competition number of K_n^m. Then there exists an acyclic digraph D such that $C(D) = K_n^m \cup I_k$ where $I_k = \{a_1, a_2, \dots, a_k\}$. Also, let $a_1, a_2, \dots, a_k, v_1, v_2, \dots, v_{mn}$ be an acyclic ordering of D. By Proposition 5, we have $\theta_v(N_{K_n^m}[v_i]) \geq n$ for $i = 1, \ldots, mn$. Thus v_i has at least n distinct prey, that is,

$$|N_D^+(v_i)| \geq n. \tag{1}$$

However, since v_1 and v_2 are the vertices of the lowest index and the second lowest index, respectively,

$$N_D^+(v_1) \subset I_k \quad \text{and} \quad N_D^+(v_2) \subset I_k \cup \{v_1\}.$$

Thus,

$$N_D^+(v_1) \cup N_D^+(v_2) \subset I_k \cup \{v_1\}. \tag{2}$$

Let P_1 and P_2 be the partite sets to which v_1 and v_2 belong, respectively. Then $P_1 = P_2$ if v_1 and v_2 are nonadjacent, and $P_1 \neq P_2$ if v_1 and v_2 are adjacent.

Firstly, assume that v_1 and v_2 are nonadjacent. Then v_1 and v_2 do not share a prey, that is,

$$N_D^+(v_1) \cap N_D^+(v_2) = \emptyset. \tag{3}$$

By (1), (2) and (3),

$$k \geq |N_D^+(v_1) \cup N_D^+(v_2)| - 1 = |N_D^+(v_1)| + |N_D^+(v_2)| - 1 \geq 2n - 1.$$

Now suppose that v_1 and v_2 are adjacent. Then $P_1 \neq P_2$. Let

$$A_1 = \{a \mid a \text{ is a common prey of } v_1 \text{ and } v \text{ for } v \in P_2 \setminus \{v_2\}\},$$

$$A_2 = \{a \mid \text{ is a common prey of } v_2 \text{ and } v \text{ for } v \in P_1 \setminus \{v_1\}\}.$$

Let b be a common prey of v_1 and v_2. Then $b \notin A_1 \cup A_2$. For, otherwise, v_1, v_2, v form a triangle for some $v \in P_1 \cup P_2$, which is a contradiction. Then $A_1 \cup \{b\} \subset N_D^+(v_1)$ and $A_2 \cup \{b\} \subset N_D^+(v_2)$. Since v_1 is adjacent to every vertex of P_2 and any pair of vertices in P_2 is nonadjacent, $|A_1| \geq n - 1$. For the same reason, $|A_2| \geq n - 1$. Suppose that there is a vertex a belonging to $A_1 \cap A_2$. Then $v \to a$, $v_1 \to a$, $w \to a$, $v_2 \to a$ for some $v \in P_2 \setminus \{v_2\}$ and $w \in P_1 \setminus \{v_1\}$. Then v_1, v_2, v, w form K_4. This implies that v_1 is adjacent to w and we reach a contradiction. Thus, $A_1 \cap A_2 = \emptyset$. Hence

$$k \geq |N_D^+(v_1) \cup N_D^+(v_2)| - 1 \geq (|A_1 \cup A_2| + 1) - 1 = |A_1| + |A_2| \geq 2n - 2.$$

□

3 The competition number of K_2^m

In this section, we compute the competition number of K_2^m, which is often called a ‘cocktail party graph’.

Theorem 7. *For $m \geq 2$,*

$$k(K_2^m) = 2.$$

Proof. By Theorem 6, it is true that $k(K_2^m) \geq 2$. Now we show that $k(K_2^m) \leq 2$. If $m = 2$, then K_2^2 is a 4-cycle and it is well-known that $k(K_2^2) = 2$. Now suppose that $m \geq 3$. Let $\{x_i, y_i\}$ be the ith partite set of K_2^m for each $i \in [m]$. We let

$$\begin{aligned} S_1 &= \{x_1, x_2, x_3, \dots, x_m\}; \\ S_2 &= \{x_1, y_2, y_3, \dots, y_m\}; \\ S_3 &= \{y_1, x_2, y_3, \dots, y_m\}; \\ S_4 &= \{y_1, y_2, x_3, \dots, y_m\}; \\ &\vdots \\ S_{m+1} &= \{y_1, y_2, y_3, \dots, x_m\}. \end{aligned}$$

We denote by $\mathcal{S}$ the family of those sets just defined. Since no two vertices in S_i belong to the same partite set, it is true that S_i forms a clique in K_2^m for each $i \in [m+1]$. Now we take an edge e of K_2^m. Then $e = x_i x_j$, $e = x_i y_j$, or $e = y_i y_j$ for some i, $j \in [m]$, $i \neq j$. If $e = x_i x_j$, then e is covered by the set S_1. If $e = x_i y_j$, then e is covered by S_{i+1}. If $e = y_i y_j$, then e is covered by S_k for some $k \in [m] \setminus \{1, i+1, j+1\}$. Thus $\mathcal{S}$ is an ECC of K_2^m.

We construct a digraph D as follows:

$$V(D) = V(K_2^m) \cup \{a, b\};$$

$$A(D) = \{(u, a) \mid u \in S_1\} \cup \{(u, b) \mid u \in S_2\} \cup \bigcup_{i=3}^{m+1} \{(u, x_{i-2}) \mid u \in S_i\}.$$

Since x_i is not contained in S_j for any $j \geq i+2$, it is true that D is acyclic. For each $x \in V(D)$, either $N_D^-(x) = \emptyset$ or $N_D^-(x) = S_i$ for some $i \in [m+1]$. Thus $C(D) = K_2^m \cup \{a, b\}$. $\square$

4 The competition number of K_3^m

In this section, we compute $k(K_3^m)$ for any $m \geq 2$. As K_3^2 is triangle-free, it is true that $k(K_3^2) = 9 - 6 + 2 = 5$. For $m \geq 3$, we present the following theorem.

Theorem 8. *For $m \geq 3$,*

$$k(K_3^m) = 4.$$

Proof. Given an integer $m \geq 3$, there exist a positive integer t and an integer r such that $m = 3t + r$ and $r \in \{0, 1, 2\}$. Now we take $3t$ partite sets of K_3^m and put them into t groups so that each group contains 3 partite sets. For each $i \in [t]$, we denote the jth partite set in the ith group by $P_{ij} = \{x_{ij}, y_{ij}, z_{ij}\}$ for each $j = 1, 2, 3$. In case of $r > 0$, we have r remaining partite sets and denote them by $P_{t+1,j} = \{x_{t+1,j}, y_{t+1,j}, z_{t+1,j}\}$ for $j \in [r]$.

Let $Q_i = P_{i1} \cup P_{i2} \cup P_{i3}$ for each $i \in [t]$ and $Q_{t+1} = P_{t+1,1} \cup \cdots \cup P_{t+1,r}$. Note that, for $i \in [t]$, the subgraph of K_3^m induced by Q_i is isomorphic to K_3^3, and the subgraph of K_3^m induced by Q_{t+1} is isomorphic to K_3^r.

For each $i \in [t]$, we let

$$\begin{aligned}
&S(x_{i1}) = \{x_{i1}, y_{i2}, y_{i3}\}, \quad S(y_{i1}) = \{y_{i1}, z_{i2}, z_{i3}\}, \quad S(z_{i1}) = \{z_{i1}, x_{i2}, x_{i3}\}, \\
&S(x_{i2}) = \{x_{i2}, y_{i1}, y_{i3}\}, \quad S(y_{i2}) = \{y_{i2}, z_{i1}, z_{i3}\}, \quad S(z_{i2}) = \{z_{i2}, x_{i1}, x_{i3}\}, \\
&S(x_{i3}) = \{x_{i3}, y_{i1}, y_{i2}\}, \quad S(y_{i3}) = \{y_{i3}, z_{i1}, z_{i2}\}, \quad S(z_{i3}) = \{z_{i3}, x_{i1}, x_{i2}\}.
\end{aligned}$$

We denote the collection of 9 sets defined above by $\mathcal{S}_i$. Note that any two vertices in each set in $\mathcal{S}_i$ belong to distinct partite sets. Thus each of the above sets forms a clique in K_3^m. It is also easy to check that $\mathcal{S}_i$ is an ECC of K_3^3 induced by Q_i. If $r > 0$, then we define 9 more sets: If $r = 1$, then

$$\begin{aligned}
S(x_{t+1,1}) &= \{x_{t+1,1}\}, \quad S(y_{t+1,1}) = \{y_{t+1,1}\}, \quad S(z_{t+1,1}) = \{z_{t+1,1}\}, \\
S(x_{t+1,2}) &= \{y_{t+1,1}\}, \quad S(y_{t+1,2}) = \{z_{t+1,1}\}, \quad S(z_{t+1,2}) = \{x_{t+1,1}\}, \\
S(x_{t+1,3}) &= S(x_{t+1,2}), \quad S(y_{t+1,3}) = S(y_{t+1,2}), \quad S(z_{t+1,3}) = S(z_{t+1,2}).
\end{aligned}$$

If $r = 2$, then

$$\begin{aligned}
&S(x_{t+1,1}) = \{x_{t+1,1}, y_{t+1,2}\}, \quad S(y_{t+1,1}) = \{y_{t+1,1}, z_{t+1,2}\}, \\
&S(z_{t+1,1}) = \{z_{t+1,1}, x_{t+1,2}\}, \quad S(x_{t+1,2}) = \{y_{t+1,1}, x_{t+1,2}\}, \\
&S(y_{t+1,2}) = \{z_{t+1,1}, y_{t+1,2}\}, \quad S(z_{t+1,2}) = \{x_{t+1,1}, z_{t+1,2}\}, \\
&S(x_{t+1,3}) = \{y_{t+1,1}, y_{t+1,2}\}, \quad S(y_{t+1,3}) = \{z_{t+1,1}, z_{t+1,2}\}, \\
&S(z_{t+1,3}) = \{x_{t+1,1}, x_{t+1,2}\}.
\end{aligned}$$

It is easy to check that the above sets form an ECC of K_3^r induced by Q_{t+1}. For convenience, if $r=0$, then we let

$$\begin{aligned}&S(x_{t+1,1})=S(y_{t+1,1})=S(z_{t+1,1})=S(x_{t+1,2})=S(y_{t+1,2})=S(z_{t+1,2})\\&=S(x_{t+1,3})=S(y_{t+1,3})=S(z_{t+1,3})=\emptyset.\end{aligned}$$

We also let $S(x_{t+2,j})=S(y_{t+2,j})=S(z_{t+2,j})=\emptyset$ for $j=1,\,2,\,3$.

Now we define a digraph D as follows:

$$\begin{aligned}V(D)&=V(K_3^m)\cup\{a_1,a_2,a_3,a_4\}\\A(D)&=\bigcup_{i=1}^{t+1}A_i\end{aligned}$$

where A_i is the union of arc sets

$$\begin{array}{rlrl}
\bigcup_{\ell=1}^{t+1}S(x_{\ell 1})\to a_1, & & S(x_{i2})\cup\bigcup_{\ell=i+1}^{t+1}S(y_{\ell 1})\to x_{i1},\\
S(x_{i3})\cup\bigcup_{\ell=i+1}^{t+1}S(z_{\ell 1})\to x_{i2}, & & S(y_{i1})\cup\bigcup_{\ell=i+1}^{t+1}S(x_{\ell 1})\to x_{i3},\\
S(y_{i2})\cup\bigcup_{\ell=i+1}^{t+1}S(y_{\ell 1})\to y_{i1}, & & S(y_{i3})\cup\bigcup_{\ell=i+1}^{t+1}S(z_{\ell 1})\to y_{i2},\\
S(z_{i1})\cup\bigcup_{\ell=i+1}^{t+1}S(x_{\ell 1})\to z_{i-1,1}, & & S(z_{i2})\cup\bigcup_{\ell=i+1}^{t+1}S(y_{\ell 1})\to z_{i-1,2},\\
S(z_{i3})\cup\bigcup_{\ell=i+1}^{t+1}S(z_{\ell 1})\to z_{i-1,3}. & & &
\end{array}$$

where $z_{01}=a_2$, $z_{02}=a_3$, and $z_{03}=a_4$.

We denote by D_i the subdigraph of D induced by $Q_i\cup\{a_1,z_{i-1,1},z_{i-1,2},z_{i-1,3}\}$ for each $i\in[t+1]$. Now we order the vertices of D_i as follows:

$$a_1,z_{i-1,1},z_{i-1,2},z_{i-1,3},x_{i1},x_{i2},x_{i3},y_{i1},y_{i2},y_{i3},z_{i1},z_{i2},z_{i3}.$$

Then we can easily check that $u\to v$ in D_i only if v is on the left hand side of u in the above sequence. Thus D_i is acyclic for each $i\in[t+1]$. Furthermore, an arc goes from a vertex in the jth partite set Q_j to the ith partite set Q_i in D only if $i\leq j$. Therefore D is acyclic.

Two nonadjacent vertices in G belong to Q_i for a unique $i \in [t+1]$. Moreover they cannot belong to the same clique in $\mathcal{S}_i$. However, no two vertices from distinct cliques in $\mathcal{S}_i$ prey on a common vertex in D and so they are nonadjacent in $C(D)$. Therefore $E(C(D)) \subset E(K_3^m)$.

We show that $E(C(D)) \supset E(K_3^m)$ in the following. We note that for each vertex v in D_i, $N^-_{D_i}(v)$ is either a clique in $\mathcal{S}_i$ or an empty set and that each clique in $\mathcal{S}_i$ is the neighborhood of a vertex in D_i for each $i \in [t]$. Thus, the competition graph of D_i is $K_3^3 \cup \{a_1, z_{i-1,1}, z_{i-1,2}, z_{i-1,3}\}$ if $i \in [t]$. Similarly, $C(D_{t+1}) = K_3^r \cup \{a_1, z_{t,1}, z_{t,2}, z_{t,3}\}$ if $r > 0$.

It remains to show that a vertex in Q_i and a vertex in Q_j are adjacent in $C(D)$ for $i, j \in [t]$ with $i < j$. We note that for any $i \in [t]$ and $k = 1, 2, 3$, it holds that Q_i is the disjoint union of $S(x_{ik}), S(y_{ik})$ and $S(z_{ik})$. Now we take a vertex $u \in Q_j$. Then $u \in S(x_{j1})$ or $u \in S(y_{j1})$ or $u \in S(z_{j1})$.

Firstly consider the case $u \in S(x_{j1})$. Then in D,

$$S(x_{i1}) \cup \{u\} \to a_1, \quad S(y_{i1}) \cup \{u\} \to x_{i3}, \quad \text{and} \quad S(z_{i1}) \cup \{u\} \to z_{i-1,1}.$$

Thus u is adjacent to every vertex in Q_i.

Now suppose that $u \in S(y_{j1})$. Then in D,

$$S(x_{i2}) \cup \{u\} \to x_{i1}, \quad S(y_{i2}) \cup \{u\} \to y_{i1}, \quad \text{and} \quad S(z_{i2}) \cup \{u\} \to z_{i-1,2}.$$

Thus u is adjacent to every vertex in Q_i.

Finally suppose that $u \in S(z_{j1})$. Then in D,

$$S(x_{i3}) \cup \{u\} \to x_{i2}, \quad S(y_{i3}) \cup \{u\} \to y_{i2}, \quad \text{and} \quad S(z_{i3}) \cup \{u\} \to z_{i-1,3}.$$

Thus u is adjacent to every vertex in Q_i. Therefore, $E(C(D)) \supset E(K_3^m)$. This competes the proof that $C(D) = K_3^m \cup \{a_1, a_2, a_3, a_4\}$. Hence $k(K_3^m) \leq 4$. From Theorem 6, it follows that $k(K_3^m) = 4$. □

5 Concluding remarks

In this paper, we give bounds for K_n^m and computed the competition numbers of K_3^m and K_2^m. Note that $k(K_3^m) = k(K_3^3) = 4$ for $m \geq 3$ and $k(K_2^m) = k(K_2^2) = 2$ for $m \geq 2$. We conjecture that $k(K_n^m) = k(K_n^n)$ for $m \geq n$.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North Holland, New York (1976).

[2] C. Cable, K. F. Jones, J. R. Lundgren, and S. Seager, “Niche graphs,” *Discrete Appl. Math.* **23** (1989) 231-241.

[3] J. E. Cohen, “Interval Graphs and Food Webs: A Finding and a Problem,” RAND Corporation Document 17696-PR, Santa Monica, CA (1968).

[4] M. B. Cozzens and F. S. Roberts, “T-Colorings of Graphs and the Channel Assignment Problem,” *Congressus Numerantium* **25** (1982) 191-208.

[5] R. D. Dutton and R. C. Brigham, “A characterization of competition graphs,” *Discrete Appl. Math.* **6** (1983) 315-317.

[6] W. K. Hale, “Frequency Assignment: Theory and Application,” *Proc. IEEE* **68** (1980) 1497-1514.

[7] P. C. Fishburn and W. V. Gehrlein, “Niche numbers,” *J. Graph Theory* **16** (1992) 131-139.

[8] S. -R. Kim, “The Competition Number and Its Variants,” in *Quo Vadis, Graph Theory?*, (J. Gimbel, J. W. Kennedy, and L. V. Quintas, eds.), *Annals of Discrete Mathematics* **55**, North Holland B. V., Amsterdam, the Netherlands, (1993) 313-326.

[9] S. -R. Kim and F. S. Roberts, “Competition numbers of graphs with a small number of triangles,” *Discrete Appl. Math.* **78** (1997) 153-162.

[10] S. -R. Kim and Y. Sano, “The competition numbers of complete tripartite graphs”, *Discrete Appl. Math.*, to appear (Available online from 18 June 2008).

[11] J. R. Lundgren, “Food Webs, Competition Graphs, Competition-Common Enemy Graphs, and Niche Graphs,” in *Applications of Combinatorics and Graph Theory to the Biological and Social Sciences*, (F. S. Roberts, ed.), IMH Volumes in Mathematics and Its Application, Vol. 17, Springer-Verlag, New York (1989) 221-243.

[12] R. J. Opsut, "On the Computation of the Competition Number of a Graph," *SIAM J. Alg. Discr. Meth.* **3** (1982) 420-428.

[13] R. J. Opsut and F. S. Roberts, "On the Fleet Maintenance, Mobile Radio frequency, Task Assignment and Traffic phasing Problem," *The Theory and Applications of Graphs*, (G. Chartrand, Y. Alavi, D. L. Goldsmith, L. Lesniak-Foster, and D. R. Lick, eds.), Wiley, New York (1981) 479-492.

[14] F. S. Roberts, "Food Webs, Competition Graphs, and the Boxicity of Ecological Phase Space," *Theory and Applications of Graphs*, (Y. Alavi and D. Lick, eds.), Springer Verlag, New York (1978) 477-490.

[15] F. S. Roberts, *Graph Theory and Its Applications to Problems of Society*, SIAM, Pennsylvania (1978).

[16] D. Scott, "The competition-common enemy graph of a digraph," *Discrete Appl. Math.* **17** (1987) 269-280.

[17] C. E. Shannon, "The Zero Error Capacity of a Noisy Channel," *IRE Trans. Inform. Theory* **IT-2** (1956) 8-19.