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Bounded fractionality of the multiflow feasibility problem for demand graph $K_3 + K_3$ and related maximization problems

By

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Abstract

We consider the multiflow feasibility problem whose demand graph is the vertex-disjoint union of two triangles. We show that this problem has a 1/12-integral solution or no solution under the Euler condition. This solves a conjecture raised by Karzanov, and completes the classification of the demand graphs having bounded fractionality. We reduce this problem to the multiflow maximization problem whose terminal weight is the graph metric of the complete bipartite graph $K_{n,m}$, and show that it always has a 1/12-integral optimal multiflow for every inner Eulerian graph.

1 Introduction

Let $G = (VG, EG)$ be an undirected graph with nonnegative edge capacity $c : EG \to \mathbf{R}_+$, and let $S \subseteq VG$ be a set of *terminals*. Let $H = (S, R)$ be another (simple) graph on S, which is called a *demand graph*. A (simple) path P in G is called an *S-path* if its ends belong to distinct nodes of S. A *multiflow* $f = (\mathcal{P}, \lambda)$ is a pair of a set $\mathcal{P}$ of S-paths and its nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying the capacity constraint

$$\sum \{\lambda(P) \mid P \in \mathcal{P} : P \text{ contains } e\} \leq c(e) \quad (e \in EG).$$

For a demand function $q : R \to \mathbf{R}_+$, the *multiflow feasibility problem* with respect to $(G, c; H, q)$ is:

(1.1) Find a multiflow f satisfying the demand requirement

$$\sum \{\lambda(P) \mid P \in \mathcal{P} : P \text{ is an } (s,t)\text{-path}\} = q(st) \quad (st \in R),$$

or establish that there is no such a multiflow.

The classical max-flow min-cut theorem, due to Ford-Fulkerson [2], says that if H is one edge K_2 (or a star), c and q are integral and if a feasible solution exists, then an integral feasible solution also exists. Hu [6] extended this result to two-commodity flows, saying that if $H = K_2 + K_2$ (a matching of size 2), c and q are integral and if a feasible solution exists, then a half-integral feasible solution also exists. On the other hand, the 3-commodity flow problem, that corresponds to $H = K_2 + K_2 + K_2$ (a matching of size 3), does not have such a property. Lomonosov [14] gave an infinite series of the feasible integer-capacitated 3-commodity flow problems with integer demands in which there is no fixed integer k such that all these problems have a $1/k$-integral feasible solution; see [17, Chapter 70, p.1232].

Motivated by these examples, following [9], we define the *fractionality* of a (simple) demand graph H by the least positive integer k with the property that for every integer-capacitated graph having H as a demand graph with every integral demand a $1/k$-integral feasible solution exists whenever a feasible solution exists. If such a integer k never exists, we define the fractionality to be the infinity. Karzanov raised the problem:

(1.2) *Characterize the demand graph H having bounded fractionality.*

Lomonosov's 3-commodity example above implies that if H has a matching of size 3, then the fractionality of H is infinity. Therefore, for the problem (1.2), we may restrict to consider the demand graphs without a matching of size 3. Such a graph (except a star) falls into one of the following three classes:

(i) K_4, C_5, or the union of two stars.

(ii) K_5 or the union of a star and a triangle K_3.

(iii) $K_3 + K_3$, i.e., the vertex-disjoint sum of two triangles.

For the class (i), the works by Rothschild and Winston [15], Seymour [18] and Lomonosov [14] imply the following. Here, we say "$(G, c; H, q)$ *satisfies the Euler condition*" if the graph $G + H = (VG, EG \cup R)$ with capacity $c + q$ is Eulerian.

Theorem 1.1 ([15, 18, 14]). *Suppose that H is K_4, C_5, or the union of two stars, and $(G, c; H, q)$ satisfies the Euler condition. If a feasible multiflow exists, then an integral feasible multiflow exists.*

In particular, the graphs of the class (i) have fractionality 2. Karzanov [8] showed that the same result holds for the graphs of the class (ii).

Theorem 1.2 ([8]). *Suppose that H is K_5 or the union of a star and a triangle, and $(G, c; H, q)$ satisfies the Euler condition. If a feasible multiflow exists, then an integral feasible multiflow exists.*

For the remaining last class (iii): $H = K_3 + K_3$, it is known that the fractionality is greater than or equal to 4; see [17, p. 1275]. Karzanov [10] conjectured that $K_3 + K_3$ has bounded fractionality, and also conjectured, more strongly, that under the Euler condition the existence of a feasible multiflow implies the existence of a half-integral feasible multiflow, and in particular the fractionality of $H = K_3 + K_3$

equals the lower bound 4. These two conjectures are also raised as Problem 52 and Problem 51 in Schrijver's book [17]; also see p. 1274. The main result of this paper solves the weaker conjecture (Problem 52) affirmatively as follows.

Theorem 1.3. *Suppose that H is $K_3 + K_3$, and $(G, c; H, q)$ satisfies the Euler condition. If a feasible multiflow exists, then a $1/12$-integral feasible multiflow exists.*

This result completes the classification of the demand graphs having bounded fractionality. In particular, the fractionality of $H = K_3 + K_3$ is one of $4, 8, 12, 24$. We however do not know whether the constant $1/12$ is tight.

$K_{n,m}$-metric-weighted maximum multiflow problem. In fact, the multiflow feasibility problem for $H = K_3 + K_3$ reduces to a certain multiflow maximization problem. Let G be an undirected graph with nonnegative edge capacity c and terminals $S \subseteq VG$. Let $K_{n,m}$ be the complete bipartite graph having S as vertices. Consider the following multiflow maximization problem (*$K_{n,m}$-metric-weighted maximum multiflow problem*):

$$\text{(1.3)} \qquad \begin{array}{ll} \text{Maximize} & \displaystyle\sum_{P \in \mathcal{P}} \mathrm{dist}_{K_{n,m}}(s_P, t_P)\lambda(P) \\ \text{subject to} & f = (\mathcal{P}, \lambda)\text{: multiflow for } (G, c; S), \end{array}$$

where s_P and t_P are the ends of P, and $\mathrm{dist}_{K_{n,m}}$ denotes the graph metric induced by $K_{n,m}$. Suppose that the bipartition of $K_{n,m}$ is $\{A, B\}$. If a path $P \in \mathcal{P}$ is an A-path or a B-path, then P contributes $2\lambda(P)$ for the objective value of (1.3). If P connects A and B, then P contributes $\lambda(P)$. (G, c) is said to be *inner Eulerian* (with respect to S) if c is integral and every node except the terminals S has even degree. For the case of $\min(n, m) = 2$, Karzanov and Mannoussakis [13] showed that (1.3) has an integral optimal multiflow for every inner Eulerian graph having S as terminals. For the case of $\min(n, m) \geq 3$, however, such an integrality result does not hold. For example, S is a six-set having $K_{3,3}$, G is a star having S as the leafs, and all six edges EG have unit capacity. Then (1.3) has no integral optimal multiflow (although a half-integral optimal multiflow exists). We will derive the main theorem (Theorem 1.3) from:

Theorem 1.4. *$K_{n,m}$-metric-weighted maximum multiflow problem* (1.3) *has a* $1/12$*-integral optimal multiflow for every inner Eulerian graph.*

For μ-weighted maximum multiflow problems for a general terminal weight $\mu : S \times S \to \mathbf{R}_+$, we can define the *fractionality* of μ in a similar way. Hence $\mathrm{dist}_{K_{n,m}}$ has bounded fractionality. This result is a step toward the classification of the terminal weights having bounded fractionality. We will further investigate this subject in the next paper [5].

This paper is organized as follows. In Section 2, we describe a combinatorial duality theorem (Theorem 2.1) for (1.3) due to Karzanov [12], and its two optimality criterions: the first one (Lemma 2.2) is well-known and the second one (Proposition 2.3) is new. We explain a reduction of the feasibility problem for $H = K_3 + K_3$ to the maximization problem for $K_{3,3}$ in Section 2.5. The proof of the combinatorial duality theorem together with the second optimality criterion is given in Section 2.7. Our proof of Theorem 1.4 is based on a fractional variation of the splitting-off method together with optimality criterions. Section 3 is devoted to the proof. A basic idea and an overview of the proof are described at Section 3.1. Section 4 gives some concluding remarks.

Notation. $\mathbf{R}$ and $\mathbf{R}_+$ denote the sets of reals and nonnegative reals, respectively. Similarly, $\mathbf{Z}$ and $\mathbf{Z}_+$ denote the sets of integers and nonnegative integers, respectively. The set of functions from a set V to $\mathbf{R}$ (resp. $\mathbf{R}_+$) is denoted by $\mathbf{R}^V$ (resp. $\mathbf{R}_+^V$). For a subset S of V, the characteristic vector $\chi_S \in \mathbf{R}^V$ is defined by: $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ otherwise. As usual, $\chi_{\{s\}}$ is denoted by χ_s for a singleton set $\{s\}$.

In this paper, by a graph we mean an undirected graph with possible parallel edges and loops. For a graph G, the set of vertices is denoted by VG, and the set of edges is denoted by EG. For an edge e and vertices x, y, the notation $e = xy$ means that e connects x and y. If e is a unique edge connecting vertices x and y, then e is also denoted by xy. We will treat two types of graphs: one is a supply graph G in which multiflows flow, and the other one is a simple graph Γ that represents dual variables (*potentials*) as its vertices. To distinguish the roles of G and Γ, a vertex of a supply graph G is particularly called a *node*. A node that is not a terminal is called an *inner node*.

A path P in G is an alternating sequence $(x_1, e_1, x_2, e_2, x_3, \dots, x_m)$ of nodes and edges with $e_i = x_i x_{i+1}$. Without noted, a path P means a simple path, i.e., there are no repeated nodes and edges in P. For two nodes x_i and x_j $(i < j)$ in P, the subpath of P between x_i and x_j is denoted by $P(x_i, x_j)$. For two paths P $=(x_1, e_1, \dots, e_{m-1}, x_m)$ and $Q = (x_m, e_m, \dots, e_{n-1}, x_n)$ having exactly one common end x_m, the concatenation $(x_1, e_1 \dots, x_m, e_m, \dots, x_n)$ of P and Q is denoted by $P \cdot Q$. When $P \cdot Q$ is nonsimple, we redefine $P \cdot Q$ by its simplification. For subsets $A_1, A_2, \dots A_m$ of nodes, a path P passing $A_1, A_2, \dots, A_m$ in order is called an $(A_1, A_2, \dots, A_m)$-path. As usual, if A_j is a singleton set $\{a_j\}$, we simply say that P is an $(A_1, A_2, \dots, a_j, \dots, A_m)$-path. In this paper, the terminal set S is partitioned into two sets A and B. For $a \in A$ and $b \in B$, $A \setminus a$ and $B \setminus b$ are simply denoted by $\bar{a}$ and $\bar{b}$, respectively. Clearly an A-path P is an $(a, \bar{a})$-path for some $a \in A$. For a path P and a function d on edges set EG, $d(P)$ denotes the sum of $d(e)$ over edges e in P.

For a multiflow $f = (\mathcal{P}, \lambda)$, $\mathcal{P}$ is allowed to be a multiset. Without noted, λ is supposed to be positive, i.e., $\lambda(P) > 0$ for every $P \in \mathcal{P}$. When $\lambda(P) = 0$ occurs in some multiflow manipulation, we always delete P from $\mathcal{P}$. For an edge e, the subset of paths in $\mathcal{P}$ passing e is denoted by $\mathcal{P}(e)$, and the total sum of its flow-values is denoted by f^e, i.e., $f^e = \sum_{P \in \mathcal{P}(e)} \lambda(P)$. Similarly, for two edges e, e', the subset of paths in $\mathcal{P}$ passing both e and e' is denoted by $\mathcal{P}(e, e')$, and the total sum of its flow-values is denoted by $f^{e,e'}$.

By a metric d on a set S we mean a function defined on $S \times S$ satisfying $d(s,t) = d(t,s) \geq d(t,t) = 0$ and the triangle inequalities $d(s,t) + d(t,u) \geq d(s,u)$ for $s, t, u \in S$. We often regard a metric d on VG of a graph G as $d : EG \to \mathbf{R}_+$ by $d(e) = d(x,y)$ for $e = xy$. For a graph Γ, the shortest path metric on $V\Gamma$ by Γ (with unit length) is denoted by dist_Γ.

2 $K_{n,m}$-metric-weighted maximum multiflow problem

Let G be a graph with terminals $S \subseteq VG$. Suppose that the terminal set S is partitioned into two sets A and B with $\min\{\#A, \#B\} \geq 3$. Let $\mu_{A,B}$ be the metric on S defined by

$$\mu_{A,B}(s,t) = \begin{cases} 4 & \text{if } s \neq t,\ s,t \in A \text{ or } s,t \in B, \\ 2 & \text{if } (s,t) \in A \times B \text{ or } (t,s) \in A \times B, \\ 0 & \text{if } s = t. \end{cases} \tag{2.1}$$

Namely $\mu_{A,B}$ is twice the graph metric of the complete bipartite graph with bipartition $\{A, B\}$. For a technical reason, instead of (1.3) we consider the following scaled version:

$$\begin{array}{ll} \text{Maximize} & \displaystyle\sum_{P \in \mathcal{P}} \mu_{A,B}(s_P, t_P)\lambda(P) \\ \text{subject to} & f = (\mathcal{P}, \lambda) \text{ is a multiflow for } (G, c; S), \end{array} \tag{2.2}$$

where s_P and t_P are ends of P. The optimal value of (2.2) is denoted by $\nu(G, c)$.

2.1 A combinatorial duality theorem

First we describe a combinatorial duality theorem for (2.2), which was (implicitly) described by Karzanov [12]. Let Γ be a simple graph whose vertices $V\Gamma$ are

$$p^O,\ p^a,\ p^b,\ p^{ab} \quad ((a,b) \in A \times B),$$

and edges $E\Gamma$ are

$$p^O p^{ab},\ p^a p^{ab},\ p^b p^{ab} \quad ((a,b) \in A \times B).$$

Namely, Γ is the graph obtained by subdividing the complete bipartite graph with bipartition $\{\{p^a\}_{a \in A}, \{p^b\}_{b \in B}\}$ and joining a new point p^O and each subdivided point p^{ab}. See Figure 1. Note that Γ has $\mu_{A,B}$ as a submetric, i.e.,

$$\mu_{A,B}(s,t) = \mathrm{dist}_\Gamma(p^s, p^t) \quad (s, t \in S). \tag{2.3}$$

Consider the following discrete location problem (the *minimum 0-extension problem*) on Γ:

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{e = xy \in EG} c(e)\mathrm{dist}_\Gamma(\rho(x), \rho(y)) \\ \text{subject to} & \rho : VG \to V\Gamma, \\ & \rho(s) = p^s \ (s \in S = A \cup B). \end{array} \tag{2.4}$$

Theorem 2.1 ([12])**.** *The maximum value of* (2.2) *is equal to the minimum value of* (2.4).

Note that the weak duality is easily seen from (2.3). We call a feasible solution ρ of (2.4) a *potential*.

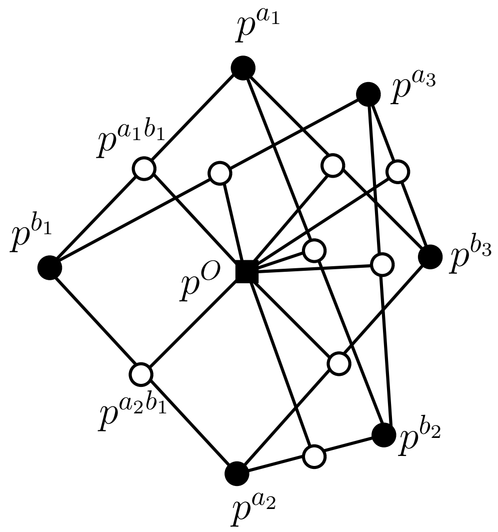


Figure 1: Graph Γ for $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$

2.2 Optimality criterion I

Second we describe the optimality criterion of primal-dual type, which involves both multiflow and potential. For a potential ρ, a metric d^ρ on VG is defined by

$$d^\rho(x, y) = \mathrm{dist}_\Gamma(\rho(x), \rho(y)) \quad (x, y \in VG).$$

For a multiflow $f = (\mathcal{P}, \lambda)$ and a potential ρ, the objective values of (2.2) and (2.4) are denoted by $\mu_{A,B} \circ f$ and $\langle c, d^\rho \rangle = \langle c, d^\rho \rangle_G$, respectively. The weak duality states

$$\mu_{A,B} \circ f \leq \langle c, d^\rho \rangle.$$

The duality gap $\langle c, d^\rho \rangle - \mu_{A,B} \circ f$ is given by

$$\sum_{e \in EG} d^\rho(e)(c(e) - f^e) + \sum_{P \in \mathcal{P}} \lambda(P)(d^\rho(P) - \mu_{A,B}(s_P, t_P)). \tag{2.5}$$

Therefore the optimality criterion is given as:

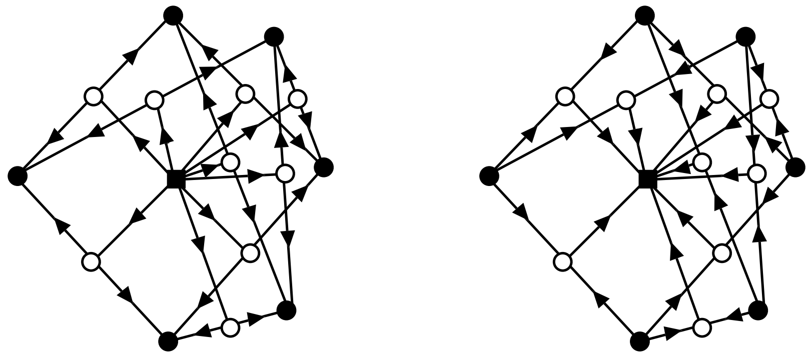
Lemma 2.2. *A multiflow $f = (\mathcal{P}, \lambda)$ and a potential ρ are optimal to* (2.2) *and* (2.4)*, respectively, if and only if*

$$\begin{aligned} \forall e = xy \in EG : d^\rho(x, y) > 0 \;&\Rightarrow\; f^e = c(e), \\ \forall P \in \mathcal{P} : \lambda(P) > 0 \;&\Rightarrow\; d^\rho(P) = \mu_{A,B}(s_P, t_P). \end{aligned} \tag{2.6}$$

Let $f = (\mathcal{P}, \lambda)$ and ρ be an optimal multiflow and an optimal potential, respectively. Let x be an inner node and P an (s, x, t)-path in $\mathcal{P}$ passing x. By (2.6), the ends s and t of P must satisfy

$$d^\rho(s, x) + d^\rho(x, t) = d^\rho(s, t) = \mathrm{dist}_\Gamma(p^s, p^t).$$

From this formula, we can determine the ends s, t of P. For example, we have:



(a) forward (b) backward

Figure 2: (a) forward orientation and (b) backward orientation

(2.7) If $\rho(x) = p^O$, then P is an A-path or a B-path.

If $\rho(x) = p^a$, then P is an (a, B)-path, an $(a, \bar{a})$-path, or a B-path.

If $\rho(x) = p^{ab}$, then P is an (a, b)-path, an $(a, \bar{a})$-path, or a $(b, \bar{b})$-path.

Let $e = xy$ be an edge with $\rho(x) \neq \rho(y)$ and P an (s, x, y, t)-path in $\mathcal{P}(e)$. Similarly, the ends s and t of P must satisfy

$$d^\rho(s, x) + d^\rho(x, y) + d^\rho(y, t) = \mathrm{dist}_\Gamma(p^s, p^t).$$

Therefore we have the following.

(2.8) If $\{\rho(x), \rho(y)\} = \{p^a, p^O\}$, then P is an $(a, \bar{a})$-path.

If $\{\rho(x), \rho(y)\} = \{p^{ab}, p^O\}$, then P is an $(a, \bar{a})$-path or a $(b, \bar{b})$-path.

If $\{\rho(x), \rho(y)\} = \{p^{ab}, p^{a'b}\}$, then P is an (a, a')-path.

If $\{\rho(x), \rho(y)\} = \{p^{ab}, p^{a'b'}\}$, then P is an (a, a')-path or a (b, b')-path.

If $\{\rho(x), \rho(y)\} = \{p^a, p^{a'b}\}$, then P is an (a, a')-path.

If $\{\rho(x), \rho(y)\} = \{p^a, p^{a'}\}$, then P is an (a, a')-path.

2.3 Optimality criterion II

Third we describe the optimality criterion of dual type, involving potentials only. We endow Γ with two orientations. The *forward orientation* of Γ is an orientation such that p^s are sinks and p^O is the unique source. The *backward orientation* of Γ is the reverse of the forward orientation. See Figure 2. For a potential ρ, a potential ρ' is called a *forward neighbor* to ρ if for $x \in VG$ with $\rho(x) \neq \rho'(x)$, $\overrightarrow{\rho(x)\rho'(x)}$ is an edge of the forward orientation, or $(\rho(x), \rho'(x)) = (p^O, p^s)$ for some $s \in S$. Similarly, a potential ρ' is called a *backward neighbor* to ρ if for

$x \in VG$ with $\rho(x) \neq \rho'(x)$, $\overrightarrow{\rho(x)\rho'(x)}$ is an edge of the backward orientation, or $(\rho(x), \rho'(x)) = (p^s, p^O)$ for some $s \in S$. A potential ρ' is called a *neighbor* to ρ if ρ' is a forward neighbor or a backward neighbor to ρ. We give an optimality criterion to (2.4) using the notion of neighbors as follows.

Proposition 2.3. *A potential ρ is optimal to* (2.4) *if and only if*

$$\langle c, d^\rho \rangle \leq \langle c, d^{\rho'} \rangle$$

holds for every neighbor ρ' to ρ.

Namely, if ρ is not optimal, there is another potential ρ' *close to* ρ such that $\langle c, d^\rho \rangle > \langle c, d^{\rho'} \rangle$.

2.4 Euler condition

Recall that a graph (G, c) is called *inner Eulerian* if c is integral and the degree of each inner node is even.

Lemma 2.4. *For an inner Eulerian graph (G, c) and two potentials ρ, ρ', we have*

$$\langle c, d^{\rho'} \rangle - \langle c, d^\rho \rangle \in 2\mathbf{Z}.$$

Proof. Since (G, c) is inner Eulerian, $c \in \mathbf{Z}_+^{EG}$ can be decomposed into the sum of the characteristic vectors of cycles C_i and S-paths P_j. Then we have

$$\langle c, d^{\rho'} \rangle - \langle c, d^\rho \rangle = \sum_i \{d^{\rho'}(C_i) - d^\rho(C_i)\} + \sum_j \{d^{\rho'}(P_j) - d^\rho(P_j)\} = 0 \mod 2,$$

where $d^{\rho'}(C_i) = d^\rho(C_i) \mod 2$ and $d^{\rho'}(P_j) = d^\rho(P_j) \mod 2$ follow from the bipartiteness of Γ. □

2.5 Reducing the feasibility problem for $K_3 + K_3$ to the maximization problem for $K_{3,3}$

Here we describe a reduction of the multiflow feasibility problem for $H = K_3 + K_3$ to $K_{3,3}$-metric-weighted maximum multiflow problem (2.2).

Let G be a graph with capacity c. Let $H = (S, R)$ be the demand graph defined by $S = \{s_1, s_2, s_3, t_1, t_2, t_3\}$ and $R = \{s_i s_j\}_{1 \leq i < j \leq 3} \cup \{t_i t_j\}_{1 \leq i < j \leq 3}$, and $q : R \to \mathbf{R}_+$ a demand function. Construct a new graph (G', c') from (G, c) by adding new terminals $S' = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ with $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$ and connecting a_i and s_i by an edge with capacity $q(s_i s_j) + q(s_i s_k)$ and connecting b_i and t_i by an edge with capacity $q(t_i t_j) + q(t_i t_k)$ $(i = 1, 2, 3; \{i, j, k\} = \{1, 2, 3\})$. Then (G', c') is inner Eulerian with respect to S' if $(G, c; H, q)$ satisfies the Euler condition. Consider (2.2) for $(G', c'; S', \mu_{A,B})$. Let ρ^* be a potential defined by

$$\text{(2.9)} \qquad \rho^*(x) = \begin{cases} p^x & \text{if } x \in S', \\ p^O & \text{otherwise,} \end{cases} \qquad (x \in VG).$$

Then we have the following.

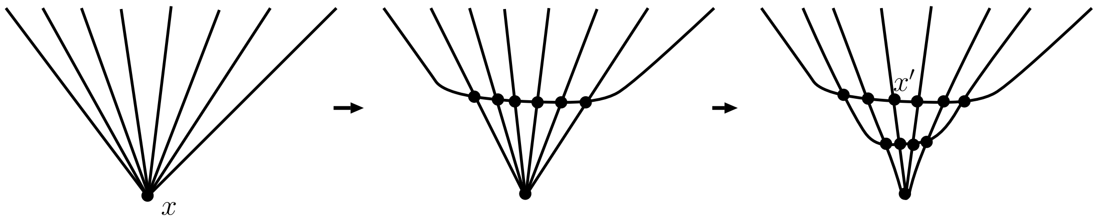


Figure 3: Reduction of an inner node

Lemma 2.5. *The multiflow feasibility problem for* $(G, c; H, q)$ *is feasible if and only if potential* ρ^* *defined by* (2.9) *is optimal to* (2.4) *for* $(G', c'; S', \mu_{A,B})$. *Moreover, if feasible, then the restriction of any optimal multiflow* f^* *to* $(G, c; S)$ *is a feasible multiflow.*

Proof. Suppose that $(G, c; H, q)$ is feasible. From a feasible multiflow $f = (\mathcal{P}, \lambda)$, we can construct an optimal multiflow as follows. For each path $P \in \mathcal{P}$, if the ends of P are s_i and s_j (resp. t_i and t_j), then extend P by adding edges $a_i s_i$ and $s_j a_j$ (resp. $b_i t_i$ and $t_j b_j$). Let f' be the resulting multiflow to $(G', c'; S')$. By construction, $a_i s_i$ and $b_i t_i$ for $i = 1, 2, 3$ are saturated by f'. Clearly f' and ρ^* satisfies the optimality criterion I (Lemma 2.2).

Conversely, suppose that ρ^* is optimal. Take an optimal multiflow f^*. Then edges $s_i a_i$ and $t_i b_i$ for $i = 1, 2, 3$ are saturated by f^*. By (2.8), each path passing $s_i a_i$ is necessarily an $(a_i, \{a_j, a_k\})$-path, and each path passing $t_i b_i$ is necessarily a $(b_i, \{b_j, b_k\})$-path. Then f^* consists of (a_i, a_j)-paths of the total flow-value $q(s_i s_j)$ for $1 \leq i < j \leq 3$ and (b_i, b_j)-paths of the total flow-value $q(t_i t_j)$ for $1 \leq i < j \leq 3$. Restricting f^* to $(G, c; S)$, we obtain a feasible multiflow. □

Therefore Theorem 1.4 implies Theorem 1.3.

2.6 Making each inner node have degree four

Here we describe a standard method reducing (2.2) to the problem on a graph with small-degree; see [3, p. 50] for example. Suppose that (G, c) is inner Eulerian. By multiplying edges, we can make each edge have unit capacity. Take an inner node $x \in VG$ of degree greater than 4. Transform (G, c) into (G', c') by changing the incidence at x as in Figure 3.

Then we can easily see that any $1/k$-integral multiflow in (G', c') can be transformed into a $1/k$-integral multiflow in (G, c) having the same objective value, and any $1/k$-integral multiflow in (G, c) can also be transformed into a $1/k$-integral multiflow in (G', c') having the same objective value. Furthermore,

(2.10) any optimal potential ρ for (G, c) is extended to an optimal potential ρ for (G', c') by setting $\rho(x') := \rho(x)$ for each new node x' in (G', c'),

which is an easy consequence of the optimality criterion I (Lemma 2.2).

2.7 proof

The combinatorial duality theorem (Theorem 2.1) was essentially given in [12, p. 241] as a corollary of Karzanov's modular closure construction together with the orbit splitting method. Here, we describe a more direct geometric proof by calculating the explicit coordinate of the *tight span* of $\mu_{A,B}$. This approach is suitable to prove the optimality criterion II (Proposition 2.3).

2.7.1 T-dual to μ-problem

We start with a general framework. Let μ be a metric defined on the terminals S. The *μ-weighted maximum multiflow problem* (for short, *μ-problem*) is:

$$\text{(2.11)} \qquad \text{Maximize} \sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P) \text{ over all multiflows } f = (\mathcal{P}, \lambda) \text{ in } (G, c),$$

where s_P and t_P are the ends of P. As is well-known in the multiflow theory [14], the LP-dual to μ-problem (2.11) is given as follows:

$$\text{(2.12)} \qquad \begin{array}{rl} \text{Minimize} & \displaystyle\sum_{e \in EG} c(e)d(e) \\ \text{subject to} & d\text{: metric on } VG, \\ & d(s,t) = \mu(s,t) \quad (s,t \in S). \end{array}$$

Since c is nonnegative, we can always take an optimal metric from the *minimal set of the feasible region* of (2.12). Such a minimal metric is called a *tight extension* of μ. Now let us introduce the formal definitions below. A metric d on V is called an *extension* of a metric μ on S if $S \subseteq V$ and $d(s,t) = \mu(s,t)$ for $s,t \in S$. An extension d on V of μ is said to be *tight* if there is no extension d' on V of μ with $d' \leq d$ and $d' \neq d$. Therefore (2.12) might be regarded as the problem of finding a tight extension d on VG of μ with $\langle c, d \rangle$ minimum.

Isbell [7] and Dress [1] independently showed that for any metric μ there is an essentially unique *universal* tight extension of μ such that every tight extension of μ is a subspace of it. We shall describe it. For a metric μ on S, we define two polyhedral sets P_μ and T_μ in $\mathbf{R}^S_+$ by

$$\text{(2.13)} \qquad \begin{array}{rcl} P_\mu & = & \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\}, \\ T_\mu & = & \text{the set of minimal elements of } P_\mu. \end{array}$$

T_μ is called the *tight span* of μ. For $s \in S$, let $\mu_s \in \mathbf{R}^S$ be a point in T_μ defined by

$$\mu_s(t) = \mu(s,t) \quad (t \in S).$$

Namely, μ_s is the s-th row vector of distance matrix μ. One can easily see that

$$\|\mu_s - \mu_t\|_\infty = \mu(s,t) \quad (s,t \in S).$$

Therefore μ is isometrically embedded into (T_μ, l_∞), and thus (T_μ, l_∞) is regarded as an extension of μ. Moreover, every tight extension of μ is embedded into (T_μ, l_∞) as follows.

Theorem 2.6 ([7, 1])**.** *(T_μ, l_∞) is a tight extension of μ. Moreover, for any tight extension d on V of μ, there is a unique map $\rho : V \to T_\mu$ such that*

(i) *$\rho(s) = \mu_s$ for $s \in S$, and*

(ii) *$\|\rho(x) - \rho(y)\|_\infty = \mu(x, y)$ for $x, y \in V$.*

In particular, the map ρ is given explicitly as $(\rho(x))(s) = d(x, s)$ for $x \in V, s \in S$.

Consider the following continuous location problem on (T_μ, l_∞).

$$\text{(2.14)} \qquad \begin{array}{rl} \text{Minimize} & \displaystyle\sum_{e=xy\in EG} c(e)\|\rho(x) - \rho(y)\|_\infty \\ \text{subject to} & \rho : VG \to T_\mu \quad (x \in VG), \\ & \rho(s) = \mu_s \quad (s \in S). \end{array}$$

We call it *T-dual*; see also [4] for a general version. By the previous theorem, we have a sharper duality theorem for μ-problem:

Corollary 2.7. *The maximum value of μ-problem* (2.11) *is equal to the minimum value of T-dual* (2.14)*.*

A map $\rho : VG \to T_\mu$ satisfying the constraint of T-dual is called a *potential.* The relationship between LP-dual (2.12) and T-dual (2.14) is summarized as follows:

(2.15) (i) For a metric d *minimal* in the feasible region of LP-dual (2.12), a map $\rho^d : VG \to \mathbf{R}^S$ defined by

$$\rho^d(x)(s) = d(s, x) \quad (s \in S, x \in VG)$$

is a potential to (2.14).

(ii) For a potential ρ to (2.14), a metric d^ρ defined by

$$d^\rho(x, y) = \|\rho(x) - \rho(y)\|_\infty \quad (x, y \in VG)$$

is minimal in the feasible region to LP-dual (2.12).

In particular, we can always take an optimal solution d^ρ of the LP-dual for some potential $\rho : VG \to T_\mu$ of T-dual.

2.7.2 The tight span for $\mu_{A,B}$

Let $\mu_{A,B}$ be the metric defined by (2.1). Let us calculate $T_{\mu_{A,B}}$ explicitly. Let q^O, q^a $(a \in A)$, q^b $(b \in B)$ be the points in $T_{\mu_{A,B}}$ defined by

$$\begin{aligned} q^O &= 2\chi_S, \\ q^a &= q^O + 2(-\chi_a + \chi_{\bar{a}}) = (\mu_{A,B})_a, \\ q^b &= q^O + 2(-\chi_b + \chi_{\bar{b}}) = (\mu_{A,B})_b. \end{aligned}$$

Recall $\bar{a} = A \setminus a$ and $\bar{b} = B \setminus b$. Then we have:

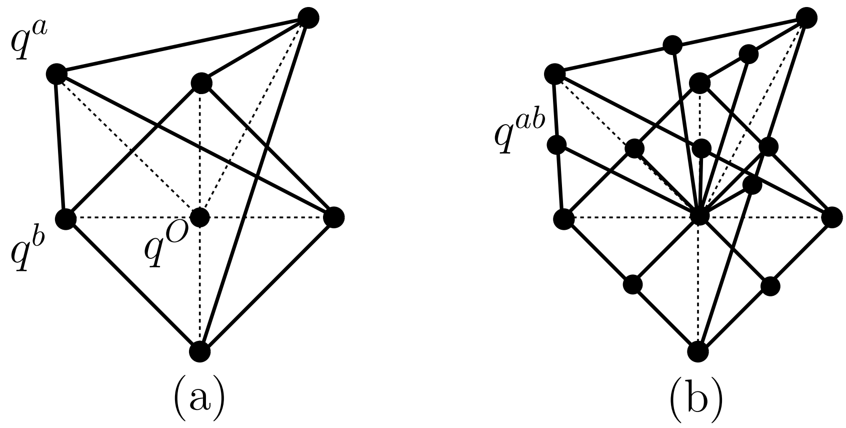


Figure 4: (a) $T_{\mu_{A,B}}$ and (b) $L^1 \cap T_{\mu_{A,B}}$ with graph Γ^1

Lemma 2.8. $T_{\mu_{A,B}}$ *coincides with*

$$\bigcup_{(a,b)\in A\times B} \text{convex hull of } \{q^O, q^a, q^b\}. \tag{2.16}$$

Proof. For a general metric μ, a point $p \in P_\mu$ is minimal in P_μ (i.e., $p \in T_\mu$) if and only if $p - \epsilon\chi_s \notin P_\mu$ for any $\epsilon > 0$ and $s \in S$. From this, to see (2.16) $\subseteq T_{\mu_{A,B}}$ is straightforward. We show the converse. Take $p \in T_{\mu_{A,B}}$. By (2.13), there is $a' \in A$ with $p(a') \geq 2$. By minimality, there is $a \in A\backslash a'$ with $p(a')+p(a) = \mu_{A,B}(a', a) = 4$. In particular, $p(a) \leq 2$. Again we have $p(a'') \geq p(a)$ for $a'' \in A \setminus \{a, a'\}$ by $p(a) + p(a'') \geq 4$. By minimality, $p(a'') = p(a')$ holds. Similarly, there is $b \in B$ such that $p(b) \leq 2$ and $p(b') = p(b'') \geq 2$ for $b', b'' \in B \setminus b$. Then we have

$$q^O = p - (2 - p(a))(-\chi_a + \chi_{\bar{a}}) - (2 - p(b))(-\chi_b + \chi_{\bar{b}}).$$

By calculation, we have

$$p = \left(\frac{p(a) + p(b)}{2} - 1\right) q^O + \frac{2 - p(a)}{2} q^a + \frac{2 - p(b)}{2} q^b.$$

Since $p(a) + p(b) \geq \mu_{A,B}(a, b) = 2$, all coefficients are nonnegative. □

Therefore $T_{\mu_{A,B}}$ is isomorphic to the join of one point q^O and the complete bipartite graph with bipartition $\{\{q^a\}_{a\in A}, \{q^b\}_{b\in B}\}$; see Figure 4 (a).

2.7.3 Drawing graphs Γ^k on $T_{\mu_{A,B}}$

Here we draw the graph Γ on $T_{\mu_{A,B}}$. For a positive integer k, let L^k be the lattice (a discrete subgroup) in $\mathbf{R}^S$ defined by

$$L^k = \{p \in \mathbf{R}^S \mid p(s) + p(t) \in 2\mathbf{Z}/k \ (s, t \in S)\}.$$

Let Γ^k be a graph whose vertices are $L^k \cap T_{\mu_{A,B}}$ and edges are given as $pq \in E\Gamma^k \Leftrightarrow \|p - q\|_\infty = 1/k$. Then $L^1 \cap T_{\mu_{A,B}}$ consists of q^O, q^s $(s \in S)$, and

$$q^{ab} := q^O + (-\chi_a + \chi_{\bar{a}}) + (-\chi_b + \chi_{\bar{b}}) \quad ((a, b) \in A \times B).$$

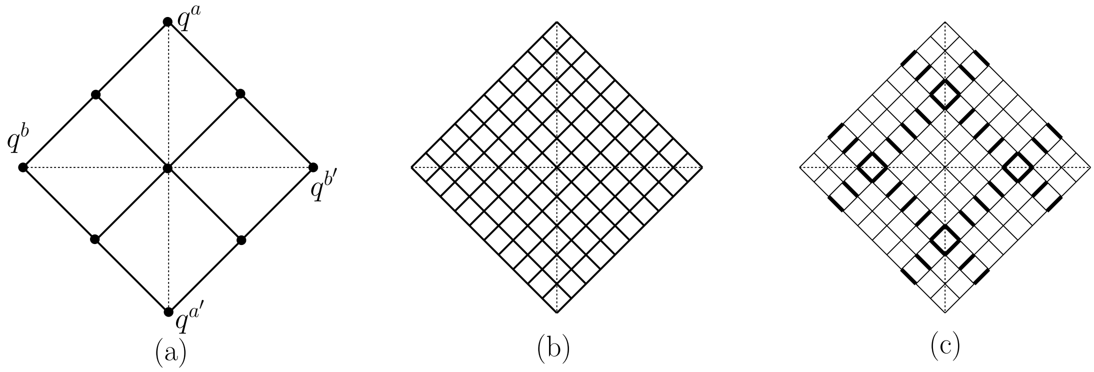


Figure 5: (a) an apartment, (b) $L^5 \cap T_{\mu_{A,B}}$ with Γ^5 on an apartment, and (c) orbit O_3

In particular, Γ^1 is isomorphic to Γ by the natural correspondence $q \mapsto p$, and thus we can identify a potential ρ of (2.4) with a potential ρ of T-dual (2.14) satisfying $\rho(VG) \subseteq T_\mu \cap L^1$; see Figure 4 (b).

For $(a, b), (a', b') \in A \times B$ with $a \neq a'$ and $b \neq b'$, consider the following subset of $T_{\mu_{A,B}}$:

$$\bigcup_{(u,v) \in \{a,a'\} \times \{b,b'\}} \text{convex hull of } \{q^O, q^u, q^v\}.$$

We call it the (a, b, a', b')-*apartment*; the name stems from the building theory. We easily see the following properties of apartments:

(2.17) (i) For every pair $p, q \in T_{\mu_{A,B}}$, there is an apartment containing both p and q.

(ii) Each apartment is a geodesic subspace of $(T_{\mu_{A,B}}, l_\infty)$, i.e., each pair of points p, q in the apartment has a path of length $\|p - q\|_\infty$ within it.

(iii) The projection of the (a, b, a', b')-apartment to $(\mathbf{R}^{\{a,b\}}, l_\infty)$ is an injective isometry, and its image is a square with edge length 2.

See Figure 5 (a). Recall the well-known fact that the l_∞-plane is isomorphic to the l_1-plane by 45-degree rotation. Viewing from the rotated plane, the subgraph of Γ^k induced by the apartment is exactly the grid graph of size $(2k, 2k)$; see Figure 5 (b). By these observations, we see that the graph Γ^k realizes the l_∞-distances among $L^k \cap T_{\mu_{A,B}}$ as follows.

$$\|p - q\|_\infty = \frac{1}{k} \text{dist}_{\Gamma^k}(p, q) \quad (p, q \in V\Gamma^k = L^k \cap T_{\mu_{A,B}}). \tag{2.18}$$

Indeed, take an apartment containing p, q, and project it as in Figure 5. Then we can take a zig-zag shortest path in the grid graph induced by the apartment.

2.7.4 Constructing a convex combination

Here we show the following statement.

(2.19) For any potential $\rho : VG \to L^k \cap T_{\mu_{A,B}} (= V\Gamma^k)$, the corresponding metric d^ρ can be represented as a convex combination of $\{d^{\rho_i}\}$ for some potentials $\rho_i : VG \to L^1 \cap T_{\mu_{A,B}} (= V\Gamma^1)$,

where the metric d^ρ for a potential ρ is defined by (2.15). This immediately yields Theorem 2.1. Indeed, for any rational potential ρ, there is k such that $\rho(VG) \subseteq L^k \cap T_{\mu_{A,B}}$ since $\mathbf{Z}^S/l \subseteq L^{2l}$. To prove (2.19), we use the notion of orbits and related concepts introduced by [11]. Two edges $e, e' \in E\Gamma^k$ are called *mates* if there is a 4-cycle containing e and e' as a nonadjacent pair. Two edges $e, e' \in E\Gamma^k$ are called *projective* if there is a sequence of edges $e = e_1, e_2, \ldots, e_m = e'$ such that e_i and e_{i+1} are mates. The projectiveness defines an equivalence relation on the set of edges $E\Gamma^k$. An equivalence class is called *an orbit*. Γ^k has k orbits $\{O_1, O_2, \ldots, O_k\}$. Here we order $O_1, O_2, \ldots, O_k$ so that

(2.20) O_i is the orbit containing the edge connecting

$$\frac{k-i+1}{k} q^O + \frac{i-1}{k} q^{ab} \text{ and } \frac{k-i}{k} q^O + \frac{i}{k} q^{ab}.$$

For an orbit O_i, the *orbit graph* Γ^k_i is the graph obtained by contracting all edges not in O_i and deleting multiple edges appeared. Then the orbit graph Γ^k_i is isomorphic to $\Gamma^1 = \Gamma$. This construction naturally gives a map $\phi_i : L^k \cap T_{\mu_{A,B}} \to L^1 \cap T_{\mu_{A,B}}$ by defining $\phi_i(p)$ to be the contracted point and identifying Γ^k_i with Γ^1 so that ϕ_i is identity on $L^1 \cap T_{\mu_{A,B}}$. In particular, if $\rho : VG \to L^k \cap T_{\mu_{A,B}}$ is a potential, then the composition $\phi_i \circ \rho$ is also a potential by $\phi_i(q^s) = q^s$. By considering each shortest path in some apartment, we easily see that the following relation holds:

$$\mathrm{dist}_{\Gamma^k}(p, q) = \sum_{i=1}^{k} \mathrm{dist}_{\Gamma^1}(\phi_i(p), \phi_i(q)) \quad (p, q \in V\Gamma^k).$$

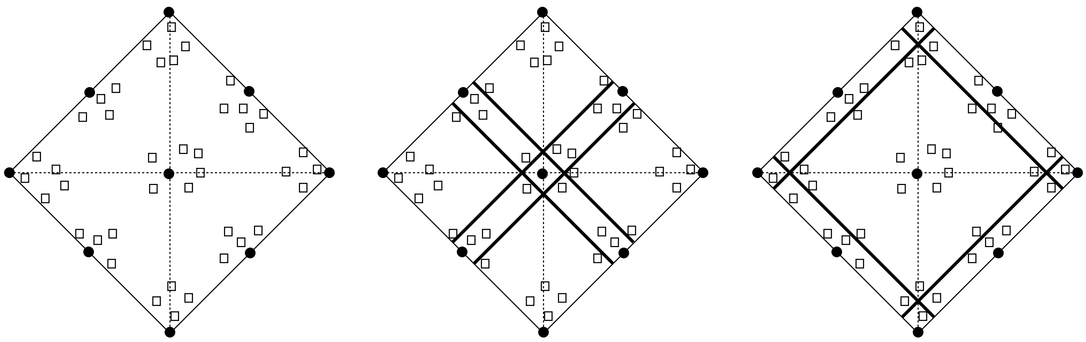
This can also be derived from a general property of orbits in the modular graph [12]. By (2.18), for any $x, y \in VG$, we have

$$\text{(2.21)} \quad \begin{aligned} d^\rho(x,y) &= \|\rho(x) - \rho(y)\|_\infty = \frac{1}{k}\mathrm{dist}_{\Gamma^k}(\rho(x), \rho(y)) \\ &= \frac{1}{k}\sum_{i=1}^{k} \mathrm{dist}_{\Gamma^1}(\phi_i \circ \rho(x), \phi_i \circ \rho(y)) = \frac{1}{k}\sum_{i=1}^{k} d^{\phi_i \circ \rho}(x,y). \end{aligned}$$

Then we obtain a desired convex combination.

2.7.5 Proof of Proposition 2.3

Take a potential $\rho : VG \to V\Gamma$ in (2.4). We can identify $V\Gamma$ with $L^1 \cap T_{\mu_{A,B}}$ by the argument above, and thus we regard ρ as $VG \to L^1 \cap T_{\mu_{A,B}}$. Suppose that ρ is not optimal. Then d^ρ is not optimal to (2.12). By convexity, there is a rational metric d' *sufficiently close to* d^ρ such that



(a) (b) (c)

Figure 6: Perturbation to d^ρ

(i) d' is minimal in the feasible region of (2.12),

(ii) $\langle c, d' \rangle < \langle c, d^\rho \rangle$,

(iii) $|d^\rho(x,y) - d'(x,y)| < 1/2$ for any $x, y \in VG$.

The property (iii) means that d' is sufficiently close to d^ρ. By (i) and the correspondence (2.15), there uniquely exists $\rho' : VG \to T_{\mu_{A,B}}$ such that $d' = d^{\rho'}$. Since $d^{\rho'}$ is rational, there is a positive even integer k such that $\rho'(VG) \subseteq L^k \cap T_{\mu_{A,B}}$. By (2.21), we can decompose $d^{\rho'}$ as $d^{\rho'} = 1/k \sum_{i=1}^k d^{\phi_i \circ \rho'}$. By (ii), at least one of $d^{\phi_i \circ \rho'}$ satisfies $\langle c, d^{\phi_i \circ \rho'} \rangle < \langle c, d^\rho \rangle$. By (iii) and (2.15), we have $\|\rho(x) - \rho'(x)\|_\infty < 1/2$. Therefore, by (2.20) and by the construction of ϕ_i, if $1 \leq i \leq k/2$, then $\phi_i \circ \rho'$ is a forward neighbor to ρ, and if $k/2 < i \leq k$, then $\phi_i \circ \rho'$ is a backward neighbor to ρ. Thus we are done. Figure 6 illustrates this situation restricted to some apartment. In this figure, a small square box represents $\rho'(x)$, which is sufficiently close to $\rho(x)$ that belongs to $L^1 \cap T_{\mu_{A,B}}$ represented by black dot points. Consider orbit O_i, which is represented by bold lines in (b) for $1 \leq i \leq k/2$ and in (c) for $k/2 < i \leq k$. Then $(\phi_i \circ \rho')(x) \neq \rho(x)$ if and only if a shortest path between $\rho'(x)$ and $\rho(x)$ crosses O_i. In (b), the change from $\rho(x)$ to $(\phi_i \circ \rho')(x)$ produced by such a crossing is one of $p^O \to p^{ab}$, $p^O \to p^a$, $p^O \to p^b$, $p^{ab} \to p^a$, and $p^{ab} \to p^b$ for some a, b. This implies that $\phi_i \circ \rho'$ for $1 \leq i \leq k/2$ is a forward neighbor. In (c), the change occurs in the reverse way, which implies that $\phi_i \circ \rho'$ for $k/2 < i \leq k$ is a backward neighbor.

3 Fractional splitting-off

Let (G, c) be an integer-capacitated graph (allowing multiple edges and loops). We begin with some notation. For two consecutive edges e and e' incident to y, a triple (e, y, e') is called a *fork*. If both e and e' have no multiple edge and $e = xy$ and $e' = yz$, then (e, y, e') is also simply denoted by xyz. For a fork $\tau = (e, y, e')$, the *splitting-off* operation at τ is to decrease the capacity of edges e, e' by one, and add a new edge e^* connecting the end of e and e' different from y with unit capacity; see Figure 7. If this operation keeps the optimal value $\nu(G, c)$ invariant,

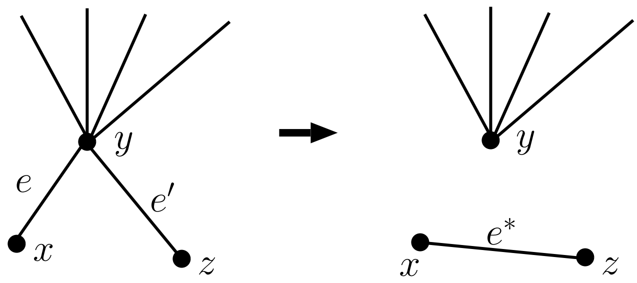


Figure 7: Splitting-off operation

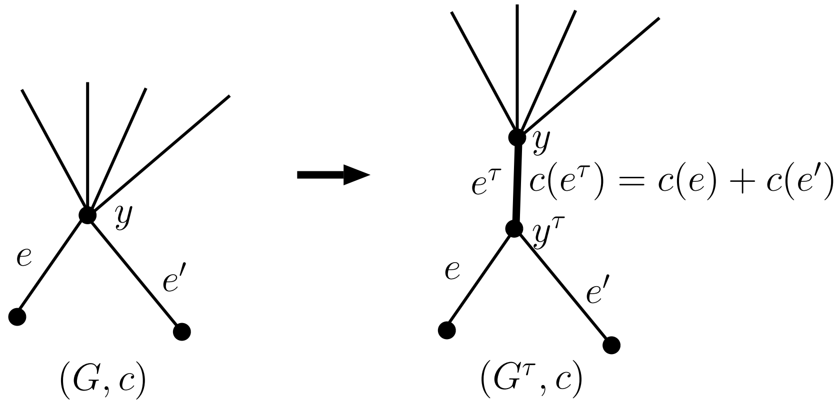


Figure 8: Construction of (G^τ, c)

then we say that τ is *splittable*. In addition, if the new graph has a $1/k$-integral optimal multiflow, then so does the original graph. Consequently, if we can succeed the splitting-off operations until there is no inner node, then the resulting graph clearly has an integral optimal multiflow (by metricity of $\mu_{A,B}$), and so does the original graph.

As seen in the introduction, our problem (2.2) may have no integral optimal solution. So we consider a fractional variant of the splitting-off operation. Our approach is slightly different from Karzanov's one in [8, 13]. For a fork $\tau = (e, y, e')$ with $e = xy$ and $e' = yz$, consider the graph (G^τ, c) obtained from (G, c) by adding a new node y^τ and a new edge $e^\tau = yy^\tau$ and reconnecting e and e' to y^τ. The capacity of e^τ is defined by $c(e) + c(e')$; see Figure 8. Then multiflows in (G, c) and in (G^τ, c) are in one-to-one correspondence as follows.

(3.1) (i) For a multiflow $f = (\mathcal{P}, \lambda)$ in (G^τ, c), contract e^τ (to y) for each path in $\mathcal{P}(e^\tau)$. Then the resulting f is a multiflow to (G, c).

(ii) For a multiflow $f = (\mathcal{P}, \lambda)$ in (G, c), replace subpath (x, e, y) of each path in $\mathcal{P}(e) \setminus \mathcal{P}(e')$ by $(x, e, y^\tau, e^\tau, y)$, replace subpath (z, e', y) of each path in $\mathcal{P}(e') \setminus \mathcal{P}(e)$ by $(z, e', y^\tau, e^\tau, y)$, and replace subpath (x, e, y, e', z) of each path in $\mathcal{P}(e, e')$ by (x, e, y^τ, e', z). Then the resulting f is a multiflow in (G^τ, c).

We shall often identify a multiflow for (G, c) with a multiflow for (G^τ, c) by this correspondence. For also a potential, by optimality criterion I, we have:

(3.2) Let ρ be an optimal potential for (G, c). Extend ρ to $VG^\tau \to V\mathit{\Gamma}$ by setting $\rho(y^\tau) := \rho(y)$. Then the resulting ρ is also optimal to (G^τ, c).

Therefore we shall often identify a potential for G with a potential for G^τ by (3.2). For a fork τ and a nonnegative real $\alpha \leq c(e^\tau)$, we call the operation replacing (G, c) by $(G^\tau, c - \alpha\chi_{e^\tau})$ the *fractional splitting-off* operation at τ. Clearly, if $\nu(G, c) = \nu(G^\tau, c - \alpha\chi_{e^\tau})$ and $(G^\tau, c - \alpha\chi_\tau)$ has a $1/k$-integer optimal multiflow, then so does (G, c). Motivated by this fact, we define the *splitting capacity* $\alpha(\tau) = \alpha^{G,c}(\tau)$ at τ by

$$\alpha(\tau) = \max\{0 \leq \alpha \leq c(e^\tau) \mid \nu(G^\tau, c - \alpha\chi_{e^\tau}) = \nu(G, c)\}.$$

Clearly, for a fork $\tau = (e, y, e')$ with $c(e) = c(e') = 1$, τ is splittable if and only if $\alpha(\tau) = 2$; see Lemma 3.3 for general case. A key to our proof of main result is the following formula of $\alpha(\tau)$ in terms of neighbors.

Proposition 3.1. *Let τ be a fork and ρ an optimal potential. Then we have the following.*

$$\text{(3.3)} \qquad \alpha(\tau) = \min\left\{ \frac{\langle c, d^{\rho'}\rangle - \langle c, d^\rho\rangle}{d^{\rho'}(e^\tau)} \;\middle|\; \rho'\text{: neighbor to } \rho \text{ with } d^{\rho'}(e^\tau) > 0 \right\},$$

where ρ is extended to an optimal potential for (G^τ, c) by (3.2). *In particular, if (G, c) is inner Eulerian, then we have*

$$\alpha(\tau) \in \left\{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2, \ldots\right\} = \frac{1}{2}\mathbf{Z}_+ \cup \frac{2}{3}\mathbf{Z}_+.$$

Proof. For $0 \leq \alpha \leq c(e^\tau)$, we have the following equivalence:

(i) $\alpha \leq \alpha(\tau)$, i.e., $\nu(G, c) = \nu(G^\tau, c - \alpha\chi_{e^\tau})$.

(ii) ρ is optimal to $(G^\tau, c - \alpha\chi_{e^\tau})$ (by regarding ρ as $VG^\tau \to V\mathit{\Gamma}$).

(iii) For each neighbor ρ' to ρ with $d^{\rho'}(e^\tau) > 0$, we have

$$\langle c - \alpha\chi_{e^\tau}, d^{\rho'}\rangle \geq \langle c - \alpha\chi_{e^\tau}, d^\rho\rangle = \langle c, d^\rho\rangle.$$

(iv) For each neighbor ρ' to ρ with $d^{\rho'}(e^\tau) > 0$, we have

$$\alpha \leq \frac{\langle c, d^{\rho'}\rangle - \langle c, d^\rho\rangle}{d^{\rho'}(e^\tau)}.$$

The equivalence between (i) and (ii) follows from $\nu(G, c) = \langle c, d^\rho\rangle = \langle c - \alpha\chi_{e^\tau}, d^\rho\rangle$ by $d^\rho(e^\tau) = 0$. The equivalence between (ii) and (iii) follows from Proposition 2.3 together with the fact that for each neighbor ρ' to ρ with $d^{\rho'}(e^\tau) = 0$, the inequality trivially holds by $\langle c - \alpha\chi_{e^\tau}, d^{\rho'}\rangle = \langle c, d^{\rho'}\rangle \geq \langle c, d^\rho\rangle = \langle c - \alpha\chi_{e^\tau}, d^\rho\rangle$. Hence we obtain the desired formula. The latter part immediately follows from $\mathrm{dist}_\mathit{\Gamma}(p, q) \in \{0, 1, 2, 3, 4\}$ and Lemma 2.4. □

A neighbor ρ' that attains (3.3) is called *critical*. Note that both ρ and ρ' are optimal to $(G^\tau, c - \alpha(\tau)\chi_{e^\tau})$.

3.1 A basic idea and an overview of the proof

Here we give a basic idea and an overview to the proof of the main theorem (Theorem 1.4). The previous proposition (Proposition 3.1) implies that

> *the splitting properties of a node y depend crucially on the position $\rho(y)$ in Γ for an optimal potential ρ.*

For example, we easily verify (from definition of neighbors) that for a fork τ at y,

(3.4) (i) if $\rho(y) = p^s$ or p^{ab} and (G, c) is inner Eulerian, then $\alpha(\tau) \in \mathbf{Z}_+$, and

(ii) if $\rho(y) = p^O$, then any critical neighbor ρ' to ρ at τ is forward.

Motivated by these facts, for an optimal potential ρ, we partition VG into the following three sets:

(3.5)
$$\begin{aligned} S_\rho &= \{x \in VG \mid \rho(x) = p^s \text{ for some } s \in S\}, \\ M_\rho &= \{x \in VG \mid \rho(x) = p^{ab} \text{ for some } (a, b) \in A \times B\}, \\ C_\rho &= \{x \in VG \mid \rho(x) = p^O\}. \end{aligned}$$

Nodes in S_ρ have a particular nice property, which we will show in Section 3.3, that

(3.6) if $y \in S_\rho$, then y has a splittable fork.

An immediate corollary is:

(3.7) if $M_\rho \cup C_\rho = \emptyset$ for some optimal potential ρ, then there exists an integral optimal multiflow.

So we have to consider the case where $M_\rho \cup C_\rho$ is nonempty. To describe the basic idea of our proof, we consider an illustrative situation below. Take $y \in C_\rho$ and a fork τ at y. Take a critical neighbor ρ' to ρ with respect to τ. Then ρ' is necessarily forward by (ii). Suppose $\alpha(\tau) = 3/2$ (say). Then $d^{\rho'}(e^\tau) = 4$ and thus $(\rho'(y), \rho'(y^\tau)) = (p^a, p^{a'})$ (or $(p^b, p^{b'})$). Update $(G, c) \leftarrow (G^\tau, c - \alpha(\tau)\chi_{e^\tau})$ and $\rho \leftarrow \rho'$. Then *the cardinality of C_ρ strictly decreases.* Therefore, if $M_\rho \cup C_\rho = \emptyset$ (luckily), then $(G, 4c)$ has an integral optimal multiflow, and thus the original graph has a 1/4-integral optimal multiflow. Suppose that there still exists a node $x \in C_\rho$. Again, take a fork τ' at x, and consider $\alpha(\tau')$. Although (G, c) is *not inner Eulerian*, the following still holds:

(3.8)
$$\alpha(\tau') \in \frac{1}{2}\mathbf{Z}_+ \cup \frac{2}{3}\mathbf{Z}_+.$$

Indeed, take a critical neighbor ρ' to ρ with respect to τ', and compare ρ' with ρ. Since ρ' is forward, $(\rho(y^\tau), \rho(y)) = (\rho'(y^\tau), \rho'(y)) = (p^a, p^{a'})$ holds. Therefore $c(e^\tau)d^\rho(e^\tau)$ and $c(e^\tau)d^{\rho'}(e^\tau)$ cancel out in $\langle c, d^\rho \rangle - \langle c, d^{\rho'} \rangle$. Since the deletion of e^τ makes $(G^{\tau'}, c)$ inner Eulerian, *the difference $\langle c, d^\rho \rangle - \langle c, d^{\rho'} \rangle$ is an even integer,* and thus (3.8) holds. This observation suggests a possibility to repeat such a procedure until $M_\rho \cup C_\rho = \emptyset$ with keeping (G, kc) inner Eulerian for a fixed integer k.

Our proof will be carried out in this way. We always keep a graph (G, c) together with its optimal potential ρ; we denote it by $(G, c; \rho)$. We will pick a node $x \in M_\rho \cup C_\rho$, and a fork τ at x. If τ is splittable, then apply the splitting-off operation at τ, update (G, c), and keep ρ, that is also optimal to the new graph (Lemma 3.4). Suppose that τ is unsplittable. Then take a critical neighbor ρ', and update the graph *together with the optimal potential* as

$$(G, c; \rho) \leftarrow (G^\tau, c - \alpha(\tau)\chi_{e^\tau}; \rho').$$

We call this operation the *SPUP (Splitting-off with Potential-UPdate)* at τ with respect to a critical neighbor ρ'. In particular, if ρ' is forward, the corresponding SPUP is said to be *forward*. In the sequential forward SPUP operations, C_ρ is nonincreasing, and M_ρ is nonincreasing if $C_\rho = \emptyset$. We will try to repeat the forward SPUP operations until $M_\rho \cup C_\rho = \emptyset$ with keeping (G, kc) Eulerian for a fixed integer k.

The remaining of this section is organized as follows. Section 3.2 describes several basic properties of the fractional splitting-off. Section 3.3 proves (3.6). Section 3.4 introduces two notions "Eulerianness" and "admissibility" for $(G, c; \rho)$ to keep (3.8) in the forward SPUP. Section 3.5 investigates the splitting properties at nodes in M_ρ, and shows that if C_ρ is empty, then there exists a half-integral optimal multiflow. Section 3.6 shows the existence of a half-integral optimal multiflow under Eulerianness and the *ring condition*, which is reached after the forward SPUP operations are applied to all nodes of degree four in C_ρ starting from the graph each of whose inner nodes has degree four. The final Section 3.7 completes the proof by showing that the forward SPUP operations at C_ρ succeed with keeping $(G, 6c; \rho)$ Eulerian until $(G, c; \rho)$ reaches the ring condition.

Remark 3.2. The emptiness of C_ρ or M_ρ is the property of the face F of the polyhedron of LP-dual (2.12) that contains d^ρ as its relative interior. Therefore a geometric interpretation to (3.6) is: the set of characteristic vectors of S-paths forms a *Hilbert basis* of the normal cone at such a face F with respect to the lattice of inner Eulerian capacities; see [16, Section 22.3].

3.2 Basic properties

In this section, we list several basic properties of the fractional splitting-off. We first verify:

Lemma 3.3. *τ is splittable if and only if $\alpha(\tau) \geq 2$.*

Proof. The only-if part is easy. We show the if part. Let $\tau = (e, y, e')$ with $e = xy$ and $e = yz$. Take an optimal multiflow $f = (\mathcal{P}, \lambda)$ for $(G^\tau, c - 2\chi_{e^\tau})$, and regard it as an optimal multiflow for (G, c) (by shrinking e^τ to y). Then $f^{e^\tau} = f^{e,e^\tau} + f^{e',e^\tau} \leq c(e) + c(e') - 2$. If both $c(e) - f^{e,e^\tau}$ and $c(e') - f^{e',e^\tau}$ are greater than or equal to 1, then τ is clearly splittable. Suppose that $c(e) - f^{e,e^\tau} > 1 > c(e') - f^{e',e^\tau}$. Then the unsaturation $c(e) - f^e$ is at least $\gamma = c(e) - c(e') + f^{e',e^\tau} - f^{e,e^\tau}$. Let (G', c') be the graph resulted by the splitting-off operation at τ with new edge $e^* = xz$. Take a set $\mathcal{Q} \subseteq \mathcal{P}(e', e^\tau)$ together with its flow-value function $\kappa : \mathcal{Q} \to \mathbf{R}_+$ such that $\kappa(P) \leq \lambda(P)$ for $P \in \mathcal{Q}$ and the total sum of flow-values of $\mathcal{Q}$ is $\gamma/2$. Then f is decomposed into two multiflows $(\mathcal{P}, \lambda - \kappa)$ and $(\mathcal{Q}, \kappa)$. Replace subpath (z, e', y) of each path in $\mathcal{Q}$ by (z, e^*, x, e, y), and replace subpath (x, e, y, e', z) of each path in $\mathcal{P}(e, e')$ by (x, e^*, z). Then the resulting f is a multiflow to (G', c') having the same objective value, which implies that τ is splittable. □

Lemma 3.4. *Let ρ be an optimal potential for (G, c) and τ a splittable fork. Then ρ is also optimal to the graph (G', c') obtained by the splitting-off at τ.*

Proof. Let $\tau = (e, y, e')$ with $e = xy$ and $e' = yz$, and let $e^* = xz$. By the triangle inequality, we have $\nu(G', c') \leq \langle c - \chi_e - \chi_{e'} + \chi_{e^*}, d^\rho \rangle = \langle c, d^\rho \rangle - d^\rho(x, y) - d^\rho(y, z) + d^\rho(x, z) \leq \langle c, d^\rho \rangle = \nu(G, c) = \nu(G', c')$. □

For a multiflow $f = (\mathcal{P}, \lambda)$ and a fork $\tau = (e, y, e')$, we note the following obvious relations:

$$\mathcal{P}(e^\tau) = \mathcal{P}(e) \cup \mathcal{P}(e') \setminus \mathcal{P}(e, e') \text{ and } f^{e^\tau} = f^e + f^{e'} - 2f^{e,e'}.$$

Lemma 3.5. *Let $\tau = (e, y, e')$ be a fork, and let f be an optimal multiflow. Then we have*

$$\alpha(\tau) \geq c(e^\tau) - f^{e^\tau} \geq 2f^{e,e'}.$$

Proof. The first inequality is obvious. The second follows from $c(e^\tau) - f^{e^\tau} = (c(e) - f^e) + (c(e') - f^{e'}) + 2f^{e,e'} \geq 2f^{e,e'}$. □

Lemma 3.6. *Let τ and τ' be two forks at distinct nodes, and let $(G', c') = (G^\tau, c - \alpha^{G,c}(\tau)\chi_{e^\tau})$. Then we have*

$$\alpha^{G',c'}(\tau') \leq \alpha^{G,c}(\tau').$$

Proof. Take an optimal flow f in $((G')^{\tau'}, c' - \alpha^{G',c'}(\tau')\chi_{e^{\tau'}})$. By shrinking e^τ and $e^{\tau'}$, we obtain an optimal flow f in (G, c). Then $f^{e^{\tau'}} \leq c(e^{\tau'}) - \alpha^{G',c'}(\tau')$ (in fact the equality holds). Lemma 3.5 implies the desired inequality. □

Lemma 3.7. *Let ρ be an optimal potential. Let e be an edge with $d^\rho(e) = 0$. If e is saturated by every optimal multiflow, then there is a neighbor ρ' to ρ such that $d^{\rho'}(e) > 0$ and ρ' is optimal.*

Proof. Consider $\max\{0 \leq \alpha \leq c(e) \mid \nu(G, c) = \nu(G, c - \alpha\chi_e)\}$. Then this must be zero by the hypothesis. The same argument as in the proof of Proposition 3.1 implies the existence of such a neighbor. □

3.2.1 A key lemma

Let ρ be an optimal potential. For a fork τ, let ρ' be a critical neighbor to ρ with respect to τ. Take an optimal multiflow $f = (\mathcal{P}, \lambda)$ for $(G^\tau, c - \alpha(\tau)\chi_{e^\tau})$. By regarding f as a multiflow for (G, c) and by the optimality criterion I, we have:

(3.9) (i) f is also optimal to (G, c),

(ii) $f^{e^\tau} = c(e^\tau) - \alpha(\tau)$, i.e., e^τ is saturated in $(G^\tau, c - \alpha(\tau)\chi_{e^\tau})$, and

(iii) every path P in $\mathcal{P}(e^\tau)$ satisfies

$$d^{\rho'}(s_P, y^\tau) + d^{\rho'}(e^\tau) + d^{\rho'}(y, t_P) = \mu_{A,B}(s_P, t_P),$$

where P is supposed to be an (s_P, y^τ, y, t_P)-path.

Conversely,

(3.10) if an optimal multiflow f for (G, c) satisfies $\alpha(\tau) = 2f^{e,e'}$, then f can be regarded as an optimal multiflow for $(G^\tau, c - \alpha(\tau)\chi_{e^\tau})$, and thus f satisfies (ii) and (iii) in (3.9) and both e and e' are saturated.

As will be seen in Section 3.3, (3.10) is a favorable situation for us. Indeed we can completely determine the ends of paths in $\mathcal{P}(e^\tau)$ according to (2.8). In many cases, however, we need to estimate the ingredients of $\mathcal{P}(e^\tau)$ for an (arbitrary) optimal flow $f = (\mathcal{P}, \lambda)$ satisfying $\alpha(\tau) > 2f^{e,e'}$.

For a critical neighbor ρ' to ρ with respect to τ, we define $\mathcal{P}(e^\tau : \rho') \subseteq \mathcal{P}(e^\tau)$ together with its flow-value $f^{e^\tau:\rho'}$ by

$$\begin{aligned}\mathcal{P}(e^\tau : \rho') &= \{P \in \mathcal{P}(e^\tau) \mid d^{\rho'}(s_P, y^\tau) + d^{\rho'}(e^\tau) + d^{\rho'}(y, t_P) = \mu_{A,B}(s_P, t_P)\}, \\ f^{e^\tau:\rho'} &= \sum\{\lambda(P) \mid P \in \mathcal{P}(e^\tau : \rho')\},\end{aligned}$$

where P is supposed to be an (s_P, y^τ, y, t_P)-path. The following lemma plays a crucial role in the sequel.

Lemma 3.8. *Let f and ρ be an optimal multiflow and an optimal potential, respectively. Let $\tau = (e, y, e')$ be a fork and ρ' a critical neighbor to ρ with respect to τ. Then we have*

$$d^{\rho'}(e^\tau)f^{e^\tau:\rho'} + (d^{\rho'}(e^\tau) - 2)(f^{e^\tau} - f^{e^\tau:\rho'}) \geq d^{\rho'}(e^\tau)(c(e^\tau) - \alpha(\tau)). \tag{3.11}$$

In addition, if $d^{\rho'}(e^\tau) \geq 2$, then we have

$$f^{e^\tau:\rho'} \geq c(e^\tau) - 2f^{e,e'} - \frac{d^{\rho'}(e^\tau)}{2}(\alpha(\tau) - 2f^{e,e'}). \tag{3.12}$$

Proof. We use the formula (2.5) of the duality gap. By definition of α, we have

$$\nu(G, c) = \nu(G^\tau, c - \alpha(\tau)\chi_{e^\tau}) = \langle d^{\rho'}, c - \alpha(\tau)\chi_{e^\tau}\rangle.$$

Let f' be the multiflow for $(G^\tau, c - \alpha(\tau)\chi_{e^\tau})$ obtained by deleting all paths in $\mathcal{P}(e^\tau)$ from $f = (\mathcal{P}, \lambda)$. Then the duality gap between f' and ρ' in $(G^\tau, c - \alpha(\tau)\chi_{e^\tau})$ is

$$\langle d^{\rho'}, c - \alpha(\tau)\chi_{e^\tau}\rangle - \mu_{A,B} \circ f' = \sum_{P \in \mathcal{P}(e^\tau)} \mu_{A,B}(s_P, t_P)\lambda(P). \tag{3.13}$$

We next estimate the first term $\delta_1 := \sum_{e \in EG} d^{\rho'}(e)(c(e) - (f')^e)$ of (2.5), which means the unsaturation of edges. Since there is no path passing e^τ in $(G^\tau, c - \alpha(\tau)\chi_{e^\tau})$, this contributes $d^{\rho'}(e^\tau)(c(e^\tau) - \alpha(\tau))$ for δ_1. The deletion of an (s_P, y^τ, y, t_P)-path $P \in \mathcal{P}(e^\tau)$ contributes at least $\lambda(P)(d^{\rho'}(s_P, y^\tau) + d^{\rho'}(y, t_P))$ for the unsaturation of edges except e^τ. Therefore we have

$$\delta_1 \geq d^{\rho'}(e^\tau)(c(e^\tau) - \alpha(\tau)) + \sum_{P \in \mathcal{P}(e^\tau)} \lambda(P)\{d^{\rho'}(s_P, y^\tau) + d^{\rho'}(y, t_P)\}.$$

Since the duality gap (3.13) is greater than or equal to δ_1, we have

$$\sum_{P \in \mathcal{P}(e^\tau)} \lambda(P)\{\mu_{A,B}(s_P, t_P) - d^{\rho'}(s_P, y^\tau) - d^{\rho'}(y, t_P)\} \geq d^{\rho'}(e^\tau)(c(e^\tau) - \alpha(\tau)). \tag{3.14}$$

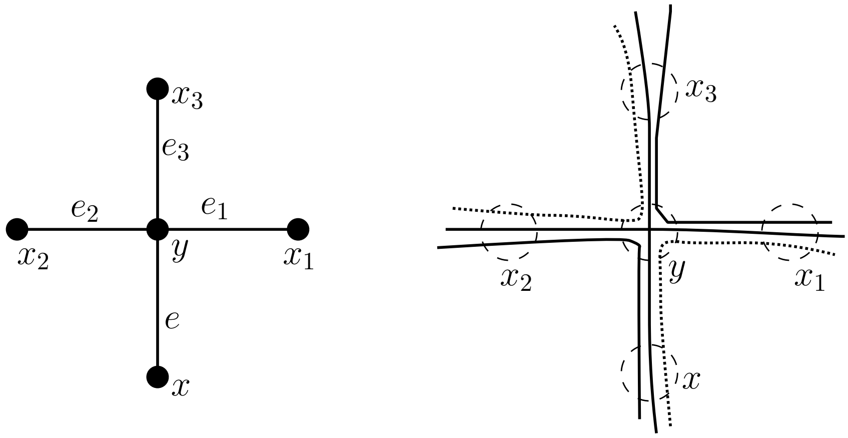


Figure 9: Flow configuration at an inner node of degree four

Since $\mathrm{dist}_\Gamma(p^s, p^t) = \mu_{A,B}(s,t)$ and Γ is bipartite, we have

$$d^{\rho'}(s_P, y^\tau) + d^{\rho'}(e^\tau) + d^{\rho'}(y, t_P) - \mu_{A,B}(s_P, t_P) \in 2\mathbf{Z}_+.$$

Therefore

$$\sum_{P \in \mathcal{P}(e^\tau:\rho')} \lambda(P) d^{\rho'}(e^\tau) + \sum_{P \in \mathcal{P}(e^\tau) \setminus \mathcal{P}(e^\tau:\rho')} \lambda(P)(d^{\rho'}(e^\tau) - 2) \geq \text{LHS of (3.14)}.$$

Thus we obtain the first inequality (3.11). The second (3.12) follows from substituting $f^{e^\tau} = f^e + f^{e'} - 2f^{e,e'} \leq c(e) + c(e') - 2f^{e,e'} = c(e^\tau) - 2f^{e,e'}$ to (3.11). □

3.2.2 Splitting at an inner node of degree four

As seen in Section 2.6, we may consider the problem (2.2) for an inner Eulerian graph with unit capacity all of whose inner node have degree four. Here we consider splitting properties at an inner node of degree four. Let (G, c) be an integer-capacitated graph. Let $y \in VG$ be an inner node of degree four. We easily see that

(3.15) if y has multiple edges, then y has a splittable fork.

Suppose that y has no splittable fork. Then the four nodes incident to y are all distinct. We assume that y is incident to four nodes x, x_1, x_2, x_3 by edges $e = xy$, $e_1 = x_1y$, $e_2 = x_2y$, $e_3 = x_3y$. The right of Figure 9 represents a flow configuration at y, where each line represents a path or a subset of $\mathcal{P}$ for some multiflow $f = (\mathcal{P}, \lambda)$. We shall often use such a figure. We note the following symmetry.

(3.16) Since (G^τ, c) is identified with $(G^{\check\tau}, c)$ for $\tau = (e, y, e_1)$ and $\check\tau = (e_2, y, e_3)$, we have $\alpha(\tau) = \alpha(\check\tau)$, and an optimal potential ρ for (G^τ, c) is regarded as an optimal potential for $(G^{\check\tau}, c)$ by replacing $(y, y^\tau; \rho(y), \rho(y^\tau))$ by $(y^{\check\tau}, y; \rho(y^{\check\tau}), \rho(y))$.

Therefore it suffices to consider three forks $\tau_i = (e, y, e_i)$ $(i = 1, 2, 3)$. By (3.1), we have

$$\mathcal{P}(e^{\tau_i}) = \mathcal{P}(e, e_j) \cup \mathcal{P}(e, e_k) \cup \mathcal{P}(e_i, e_j) \cup \mathcal{P}(e_i, e_k) \tag{3.17}$$

for $\{i, j, k\} = \{1, 2, 3\}$.

(3.18) For a optimal multiflow $f = (\mathcal{P}, \lambda)$ for $(G^{\tau_1}, c - \alpha(\tau_1)\chi_{e^{\tau_1}})$, if $\mathcal{P}(e^{\tau_1})$ consists of either $(a, \bar{a})$-paths or $(b, \bar{b})$-paths, then we have

$$\max\{\alpha(\tau_2), \alpha(\tau_3)\} \geq 2 - \alpha(\tau_1).$$

By the condition, the restriction of f to $\mathcal{P}(e^{\tau_1})$ is regarded as a *single commodity flow*. Therefore we can rearrange f with keeping optimality so that

$$\max\{f^{e,e_2}, f^{e,e_3}, f^{e_1,e_2}, f^{e_1,e_3}\} \geq 1 - \alpha(\tau_1)/2.$$

Indeed, by $f^{e^{\tau_1}} = f^{e^{\tau_1},e} + f^{e^{\tau_1},e_1} = f^{e^{\tau_1},e_2} + f^{e^{\tau_1},e_3} = 2 - \alpha(\tau_1)$, we may assume $f^{e^{\tau_1},e_2} \geq f^{e^{\tau_1},e} \geq 1 - \alpha(\tau_1)/2 \geq f^{e^{\tau_1},e_1} \geq f^{e^{\tau_1},e_3}$ by relabeling and symmetry (3.16). Take two paths $P_1 \in \mathcal{P}(e, e_3)$ and $P_2 \in \mathcal{P}(e_1, e_2)$. We may assume that P_1 is an (a, x, y, x_3, a')-path and P_2 is an (a, x_1, y, x_2, a'')-path.

(3.19) (i) Decrease $\lambda(P_1)$ and $\lambda(P_2)$ by $\epsilon := \min(\lambda(P_1), \lambda(P_2))$.

(ii) Append two paths $P_1' = P_1(a, y) \cdot P_2(y, a'')$ and $P_2' = P_2(a, y) \cdot P_1(y, a')$ with flow-values $\lambda(P_1') = \lambda(P_2') = \epsilon$.

Clearly, the resulting f is a multiflow having the same objective value. We can repeat it until $\mathcal{P}(e, e_3) = \emptyset$ by $f^{e_1,e_2} - f^{e,e_3} = f^{e^{\tau_1},e_2} - f^{e^{\tau_1},e} \geq 0$. By $f^{e,e^{\tau_1}} = f^{e,e_2} + f^{e,e_3}$, we have $f^{e,e_2} = f^{e,e^{\tau_1}} \geq 1 - \alpha(\tau_1)/2$. By Lemma 3.5, we have $\alpha(\tau_2) \geq 2 - \alpha(\tau_1)$.

3.3 Splitting at S_ρ

Recall the partition $\{S_\rho, M_\rho, C_\rho\}$ of VG defined by (3.5).

Proposition 3.9. *Let (G, c) be an inner Eulerian graph and ρ an optimal potential. For a node $y \in S_\rho$ of degree four, there is a splittable fork at y.*

Corollary 3.10. *Let (G, c) be an inner Eulerian graph, and let ρ be an optimal potential. If $M_\rho \cup C_\rho = \emptyset$, then there exists an integral optimal multiflow.*

Proof. Make each node in S_ρ have degree four by the method in Section 2.6 together with (2.10), and apply the previous proposition. □

It suffices to show the statement for the case $\rho(y) = p^a$ for some $a \in A$. By (3.15), we may assume that y is incident to four distinct nodes x, x_1, x_2, x_3; we use the notation of Section 3.2.2. Figure 10 (a) illustrates the graph structure of Γ around p^a; this is the complete bipartite graph $K_{2,m}$. Then we can combine our idea of neighbors (Proposition 3.1) and Karzanov's splitting-off proof used in [8, 13].

We can take a fork τ at y with $\alpha(\tau) > 0$. Indeed, if $\alpha(\tau) = 0$, then $f^{e^\tau} = 2$ and thus y has another fork $\tau' = (e, y, e')$ with $f^{e,e'} > 0$ for an optimal multiflow f,

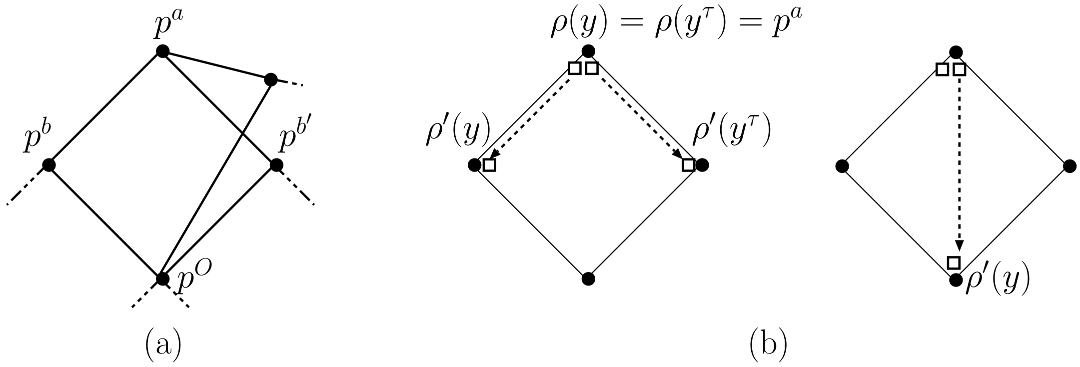


Figure 10: (a) the graph structure around p^a and (b) behavior of neighbors

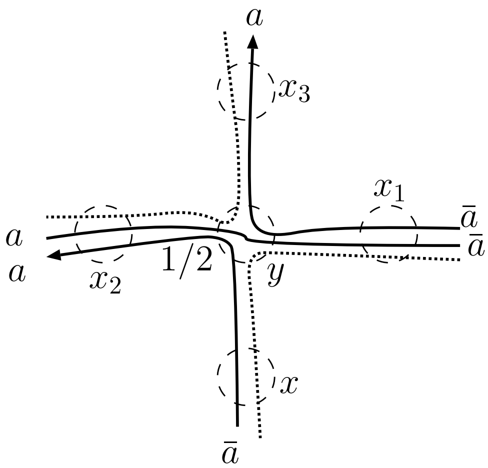


Figure 11: Flow configuration at y

and $\alpha(\tau') > 0$ (by Lemma 3.5). By (3.4), we have $\alpha(\tau) \in \{1, 2\}$. We may assume that $\tau_1 = (e, y, e_1)$ is unsplittable and thus $\alpha(\tau_1) = 1$. Take a critical neighbor ρ' to ρ with respect to τ_1. Then ρ' is necessarily backward and satisfies

(i) $\{\rho'(y^{\tau_1}), \rho'(y)\} = \{p^O, p^a\}$ or

(ii) $\{\rho'(y^{\tau_1}), \rho'(y)\} = \{p^{ab}, p^{ab'}\}$ for distinct $b, b' \in B$.

See Figure 10 (b). Note that if $\{\rho'(y^{\tau_1}), \rho'(y)\} = \{p^a, p^{ab}\}$, then $d^{\rho'}(e^\tau) = 1$, $\alpha(\tau) = 2$, and τ is splittable. Take an optimal multiflow $f = (\mathcal{P}, \lambda)$ for $(G^{\tau_1}, c - \chi_{e^{\tau_1}})$ such that $\sum_{e \in EG^{\tau_1}} f^e$ is minimum, and regard it as an optimal multiflow for (G, c). By (2.8), the restriction of f to $\mathcal{P}(e^{\tau_1})$ is a single commodity flow; $\mathcal{P}(e^{\tau_1})$ consists of $(a, \bar{a})$-paths for the case (i) and consists of (b, b')-paths for the case (ii). By $f^{e^{\tau_1}} = 1$, as in Section 3.2.2, we can rearrange f (with keeping the optimality and the minimality) so that

$$f^{e,e_2} \geq 1/2 \geq f^{e_1,e_2} + f^{e_1,e_3} \text{ and } f^{e,e_3} = 0$$

by relabeling x, x_1, x_2, x_3 if necessary; see Figure 11 for case (i). We may assume that f^{e,e_2} is equal to 1/2. Otherwise $\alpha(\tau_2) > 1$ (by Lemma 3.5) and $\tau_2 = (e, y, e_2)$

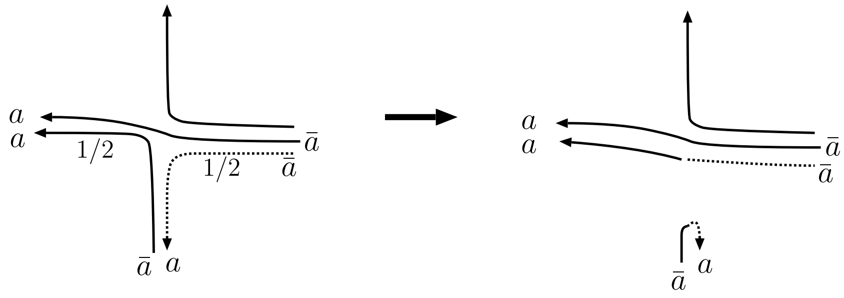


Figure 12: Flow modification

is splittable. Suppose the case (i) with $(\rho'(y^{\tau_1}), \rho'(y)) = (p^O, p^a)$. Then $\mathcal{P}(e^{\tau_1})$ consists of $(\bar{a}, y^\tau, y, a)$-paths of flow-value 1. First we show:

(3.20) $\mathcal{P}(e, e_1)$ cannot contain A-paths.

Indeed, suppose to the contrary that $\mathcal{P}(e, e_1)$ contain an A-paths P_1. By (2.7), P_1 is necessarily an $(\bar{a}, a)$-path, say, P_1 is an (a, x, y, x_1, a')-path for $a' \neq a$ (by changing roles of x and x_1 if necessarily). $\mathcal{P}(e, e_2)$ is nonempty. Take a path P_2 from $\mathcal{P}(e, e_2) \subseteq \mathcal{P}(e^{\tau_1})$; recall (3.17). Then P_2 is an (a'', x, y, x_2, a)-path for $a'' \neq a$.

(3.21) (i) Decrease $\lambda(P_1)$ and $\lambda(P_2)$ by $\epsilon := \min(\lambda(P_1), \lambda(P_2))$.

(ii) Append two paths $P_1' = P_1(a, x) \cdot P_2(x, a'')$ and $P_2' = P_2(a, y) \cdot P_1(y, a')$ with flow-values $\lambda(P_1') = \lambda(P_2') = \epsilon$.

Then the resulting f' is also optimal; see Figure 12. However $(f)'^e < f^e$ contradicts to the minimality assumption.

Since $\alpha(\tau_2) = 2f^{e,e_2} = 1$, by (3.10), f is also optimal to $(G^{\tau_2}, c - \chi_{e^{\tau_2}})$. Then $f^{e^{\tau_2}} = 1$, and e, e_2, and e_1 are all saturated. In particular $f^{e,e_1} = 1/2$. Take a critical neighbor ρ'' to ρ with respect to τ_2. Suppose $f^{e_1,e_2} > 0$. Then $\mathcal{P}(e^{\tau''})$ contains A-paths, and therefore ρ'' is of the case (i). Then $\mathcal{P}(e, e_1)$ consists of A-paths, which contradicts to (3.20).

Therefore we have $f^{e,e_2} = f^{e_1,e_3} = f^{e,e_1} = f^{e_2,e_3} = 1/2$, and ρ'' is of the case (ii) Then both $\mathcal{P}(e^{\tau_1}) = \mathcal{P}(e, e_2) \cup \mathcal{P}(e_1, e_3)$ and $\mathcal{P}(e^{\tau_2}) = \mathcal{P}(e_2, e_3) \cup \mathcal{P}(e, e_1)$ are single commodity flows. We can rearrange f so that $f^{e,e_3} = f^{e_1,e_2} = 1$ as in Figure 13. Then $\tau_3 = (e, y, e_3)$ is splittable.

For the other cases, e.g., $(\rho'(y^{\tau_1}), \rho'(y)) = (p^a, p^O)$ and $\{\rho'(y^{\tau_1}), \rho'(y)\} = \{p^{ab}, p^{ab'}\}$, the completely same argument (by changing roles of $\bar{a}, a, b, b'$) implies that τ_3 is splittable.

3.4 Keeping $\alpha(\tau)$ half- or 2/3-integral

Let (G, c) be a graph and ρ an optimal potential. $(G, c; \rho)$ is called *Eulerian* if c is integral and each node in $M_\rho \cup C_\rho$ has even degree.

Lemma 3.11. *Suppose that $(G, c; \rho)$ is Eulerian. For a fork τ, if a critical neighbor*

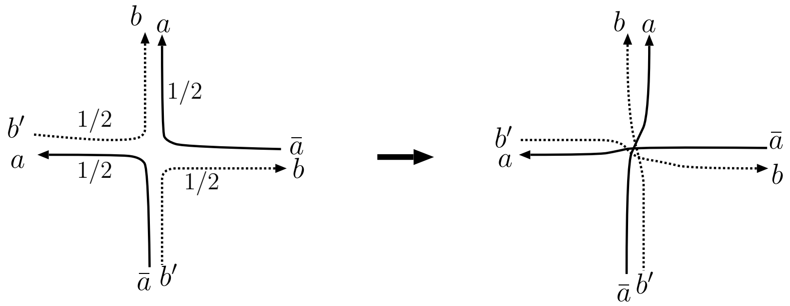


Figure 13: Flow modification

ρ' to ρ with respect to τ is forward, then we have $\langle c, d^{\rho'}\rangle - \langle c, d^{\rho}\rangle \in 2\mathbf{Z}_+$, and thus $\alpha(\tau)$ is half- or 2/3-integral.

We prove it under a more general condition. We give three definitions below. An edge $e = xy \in EG$ is called *mixed* if it satisfies

(1) $(\rho(x), \rho(y)) = (p^{ab}, p^{a'b'})$ for some $a \neq a'$ and $b \neq b'$, and

(2) for every optimal flow $f = (\mathcal{P}, \lambda)$, $\mathcal{P}(e)$ contains both A-paths and B-paths.

An inner node x is called *tri-fixed* if it satisfies

(0) degree of x is odd,

(1) $\rho(x) = p^O$, and

(2) there exist distinct $a, a', a'' \in A$ or distinct $b, b', b'' \in B$ such that every optimal flow has (a, a')-, (a', a'')-, and (a'', a)-paths passing x, or (b, b')-, (b', b'')-, and (b'', b)-paths passing x.

For a graph (G, c) together with an optimal potential ρ, $(G, c; \rho)$ is called *admissible* if it satisfies:

(1) each edge incident to $M_\rho \cup C_\rho$ has integer capacity, and

(2) for some set $\tilde{E}$ of mixed edges, each node in $M_\rho \cup C_\rho$ except tri-fixed nodes has even degree in the graph obtained by deleting $\tilde{E}$ from G.

Clearly, if $(G, c; \rho)$ is Eulerian, then it is also admissible since there is no tri-fixed node and $\tilde{E}$ can be taken to be empty.

Lemma 3.12. *Suppose that $(G, c; \rho)$ is admissible. Let y be an inner node that is not tri-fixed. For a fork τ at y, if a critical neighbor ρ' to ρ with respect to τ is forward, then we have $\langle c, d^{\rho'}\rangle - \langle c, d^{\rho}\rangle \in 2\mathbf{Z}_+$, and thus $\alpha(\tau)$ is half- or 2/3-integral.*

Proof. First, we claim that for a mixed edge e we have

$$d^{\rho'}(e) = d^{\rho}(e) = 2. \tag{3.22}$$

Indeed, the possible situations are $d^{\rho'}(e) = 2$, 3, and 4 by the definition of forward neighbors. The case $d^{\rho'}(e) \in \{3, 4\}$ implies that for every optimal flow f to $(G^\tau, c-$

$\alpha(\tau)\chi_{e^\tau}$), f is a single commodity flow on edge e^τ by (2.8). Since f can also be regarded as an optimal multiflow to (G,c), this is a contradiction to the definition of mixed edges.

Second, we claim that

(3.23) for each tri-fixed node x, its potentials $\rho(x)$ and $\rho'(x)$ belong to the same color class of bipartite graph Γ.

Indeed, if f contains (a,a')-, (a',a'')-, and (a'',a)-paths passing x, then $\rho'(x)$ must be p^O or p^b for some $b \in B$ by (2.7).

Third, for each $s \in S$, contract all nodes x with $\rho(x) = p^s$ into s, and delete $\tilde{E}$ and all edges connecting nodes $x, y \in S_\rho$ with $\rho(x) \neq \rho(y)$. The resulting graph is denoted by (G', c'). By (3.22) and the fact that ρ' is forward, we have

$$\langle c, d^{\rho'} \rangle_G - \langle c, d^{\rho} \rangle_G = \langle c', d^{\rho'} \rangle_{G'} - \langle c', d^{\rho} \rangle_{G'},$$

where we restrict ρ' and ρ to VG' (well-defined). The set of tri-fixed nodes is denoted by T. By construction, (G', c') is inner Eulerian with respect to $S \cup T$. Therefore c' is decomposed into the integral sum of the characteristic vectors of cycles C_i and $(S \cup T)$-paths P_j.

$$\langle c', d^{\rho'} \rangle_{G'} - \langle c', d^{\rho} \rangle_{G'} = \sum_i (d^{\rho'}(C_i) - d^{\rho}(C_i)) + \sum_j (d^{\rho'}(P_j) - d^{\rho}(P_j)) = 0 \mod 2,$$

where the last equality follows from the bipartiteness of Γ and the fact (3.23). □

3.5 Splitting at M_ρ

Here we study the splitting properties of nodes in M_ρ.

Proposition 3.13. *Let (G,c) be a graph and ρ an optimal potential. Suppose that $(G,c;\rho)$ is Eulerian. For a node $y \in M_\rho$ of degree four with $\rho(y) = p^{ab}$, at least one of the following holds:*

(0) *there is a splittable fork at y.*

(1) *there is an optimal forward neighbor ρ' to ρ with $\rho'(y) \neq \rho(y)$.*

(2) *there is a fork τ at y such that a critical neighbor ρ' to ρ with respect to τ is forward and satisfies $\alpha(\tau) = 1$ and $\{\rho'(y^\tau), \rho'(y)\} = \{p^a, p^b\}$, and thus the corresponding SPUP is forward and keeps $(G,c;\rho)$ Eulerian.*

Corollary 3.14. *Let (G,c) be a graph and ρ its optimal potential. Suppose that $(G,c;\rho)$ is Eulerian and $C_\rho = \emptyset$. Then there exists a half-integer optimal multiflow.*

Proof. Make each node in M_ρ have degree four. According to the previous proposition. for each node in M_ρ, apply the splitting-off or the forward SPUP, or replace ρ by its optimal forward neighbor, which keeps $(G,c;\rho)$ Eulerian. Since C_ρ is empty, the set M_ρ strictly decreases. Repeat this process until $M_\rho \cup C_\rho$ is empty. Now $(G,2c)$ is inner Eulerian with $M_\rho \cup C_\rho = \emptyset$. By Corollary 3.10, (G,c) has a half-integral optimal multiflow, and so does the original graph. □

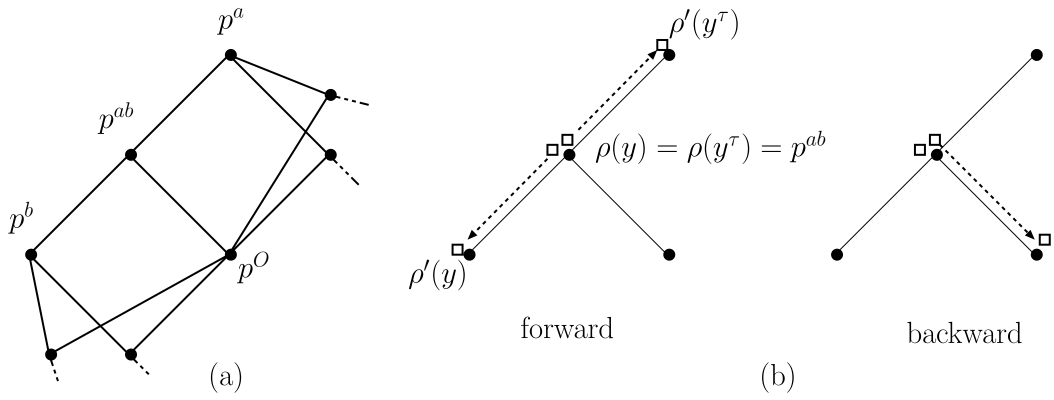


Figure 14: (a) the graph structure around p^{ab} and (b) behavior of neighbors

We may assume that y has four distinct nodes x, x_1, x_2, x_3; we use the notation of Section 3.2.2. We first show the following statement.

(3.24) Suppose that y has no splittable fork. Then (2) in Proposition 3.13 occurs, or there is a fork τ at y with $\alpha(\tau) = 0$.

Suppose to the contrary that each fork τ at y satisfies $0 < \alpha(\tau) < 2$ and any critical neighbor ρ' to ρ with respect to τ is backward. Therefore we have

$$\{\rho'(y^\tau), \rho'(y)\} = \{p^O, p^{ab}\}.$$

In particular, $d^{\rho'}(e^\tau) = 1$ and $\alpha(\tau) = 1$ since c is integral (and not necessarily inner Eulerian) and thus $\langle c, d^{\rho'}\rangle - \langle c, d^\rho\rangle \in \mathbf{Z}_+$. Let $\tau_i = (e, y, e_i)$ for $i = 1, 2, 3$. Let ρ_i be a critical backward neighbor to ρ with respect to τ_i for $i = 1, 2, 3$. By relabeling and symmetry (3.16), we may assume that

(i) $(\rho_i(y^{\tau_i}), \rho_i(y)) = (p^O, p^{ab})$ for $i = 1, 2, 3$, or

(ii) $(\rho_i(y^{\tau_i}), \rho_i(y)) = (p^{ab}, p^O)$ for $i = 1, 2, 3$.

We show a contradiction for the first case (i); a contradiction for the second (ii) can be obtained by interchanging roles of (a, b) and $(\bar{a}, \bar{b})$. Take an optimal multiflow f for $(G^{\tau_1}, c - \chi_{e^{\tau_1}})$ and regard it as an optimal multiflow for (G, c). Then, by (2.8), $\mathcal{P}(e^{\tau_1})$ consists of $(\bar{a}, y^{\tau_1}, y, a)$-paths and $(\bar{b}, y^{\tau_1}, y, b)$-paths. Decompose f^{e,e_1} into $(f^{e,e_1})_i$ $(i = 1, 2)$ defined by

$$(f^{e,e_1})_1 = \sum_{P \in \mathcal{P}(e,e_1)} \{\lambda(P) \mid P \text{ is an } (\bar{a}, x, y, x_1, a)\text{-path or a } (\bar{b}, x, y, x_1, b)\text{-path}\},$$

$$(f^{e,e_1})_2 = \sum_{P \in \mathcal{P}(e,e_1)} \{\lambda(P) \mid P \text{ is an } (a, x, y, x_1, \bar{a})\text{-path or a } (b, x, y, x_1, \bar{b})\text{-path}\}.$$

Note that $\mathcal{P}(e, e_1)$ has no (a, b)-paths by $\rho_1(y^{\tau_1}) = p^O$ and (2.7). Similarly, decom-

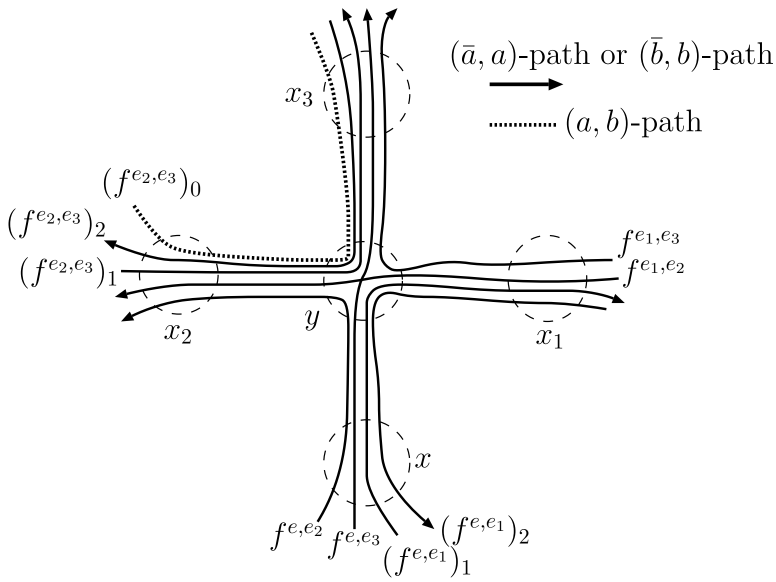


Figure 15: Flow configuration at y

pose f^{e_2,e_3} into:

$$(f^{e_2,e_3})_0 = \sum_{P \in \mathcal{P}(e_2,e_3)} \{\lambda(P) \mid P \text{ is an } (a,b)\text{-path}\},$$

$$(f^{e_2,e_3})_1 = \sum_{P \in \mathcal{P}(e_2,e_3)} \{\lambda(P) \mid P \text{ is an } (\bar{a},x_2,y,x_3,a)\text{-path or a } (\bar{b},x_2,y,x_3,b)\text{-path}\},$$

$$(f^{e_2,e_3})_2 = \sum_{P \in \mathcal{P}(e_2,e_3)} \{\lambda(P) \mid P \text{ is an } (a,x_2,y,x_3,\bar{a})\text{-path or a } (b,x_2,y,x_3,\bar{b})\text{-path}\}.$$

See Figure 15. We use the inequality (3.11) in Lemma 3.8 for (τ_2, ρ_2) and (τ_3, ρ_3). Since $\mathcal{P}(e^{\tau_i} : \rho_i)$ consists of $(\bar{a}, y^{\tau_i}, y, a)$- and $(\bar{b}, y^{\tau_i}, y, b)$-paths for $i = 2, 3$ by (i) and (2.8), we have $f^{e^{\tau_2}:\rho_2} = f^{e,e_3} + (f^{e,e_1})_1 + (f^{e_2,e_3})_1$ and $f^{e^{\tau_3}:\rho_3} = f^{e,e_2} + (f^{e,e_1})_1 + (f^{e_2,e_3})_2$ (by (3.17)). By $d^{\rho_i}(e^{\tau_i}) = 1$, $\alpha(\tau_i) = 1$, and $c(e^{\tau_i}) = 2$, the inequality (3.11) yields

$$\begin{aligned} f^{e,e_3} + (f^{e,e_1})_1 + (f^{e_2,e_3})_1 - f^{e_1,e_2} - (f^{e,e_1})_2 - (f^{e_2,e_3})_0 - (f^{e_2,e_3})_2 &\geq 1, \\ f^{e,e_2} + (f^{e,e_1})_1 + (f^{e_2,e_3})_2 - f^{e_1,e_3} - (f^{e,e_1})_2 - (f^{e_2,e_3})_0 - (f^{e_2,e_3})_1 &\geq 1. \end{aligned}$$

Summing up the two inequalities yields

$$f^{e,e_3} + f^{e,e_2} + 2(f^{e,e_1})_1 \geq 2 + f^{e_1,e_2} + f^{e_1,e_3} + 2(f^{e,e_1})_2 + 2(f^{e_2,e_3})_0.$$

Substitute $f^{e,e_2} + f^{e,e_3} + f^{e_1,e_2} + f^{e_1,e_3} = f^{e^{\tau_1}} = 1$. Then we have

$$f^{e,e_3} + f^{e,e_2} + (f^{e,e_1})_1 \geq 3/2 + (f^{e,e_1})_2 + (f^{e_2,e_3})_0.$$

However, this contradicts to $f^{e,e_3} + f^{e,e_2} + (f^{e,e_1})_1 + (f^{e,e_1})_2 = f^e \leq 1$.

Suppose that (0) and (2) in Proposition 3.13 do not occur. Then, by (3.24), there is a fork τ with $\alpha(\tau) = 0$. We may assume $\tau = \tau_1 = (e, y, e_1)$. In this case,

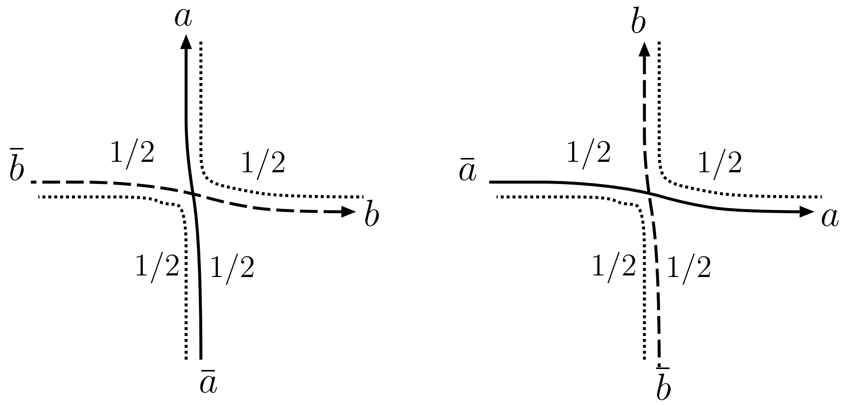


Figure 16: Flow configuration at y when $\alpha(\tau) = 0$ occurs

the flow configuration at y can be completely determined as follows.

(3.25) For any optimal multiflow f, we have

$$f^{e,e_1} = f^{e_2,e_3} = 0 \text{ and } f^{e,e_2} = f^{e,e_3} = f^{e_1,e_2} = f^{e_1,e_3} = 1/2.$$

Indeed, $\alpha(\tau_1) = 0$ implies $f^{e^{\tau_1}} = 2$, and thus $f^{e,e_1} = f^{e_2,e_3} = 0$ and $f^{e,e_2} + f^{e,e_3} = f^{e_1,e_2} + f^{e_1,e_3} = 1$. $f^{e,e_2} > 1/2 > f^{e,e_3}$ implies that (e, y, e_2) is splittable. Then we obtain (3.25). Let $\tau_2 = (e, y, e_2)$. Take a critical neighbor ρ_2 w.r.t. τ_2. By the assumption, ρ_2 is backward. We may assume that $(\rho_2(y^\tau), \rho_2(y)) = (p^O, p^{ab})$ by symmetry (3.16). By $\alpha(\tau_2) = 2f^{e,e_2} = 1$, f can be regarded as an optimal multiflow for $(G^{\tau_2}, c - \chi_{e^{\tau_2}})$. Therefore, by (3.10) with a help of (2.8), $\mathcal{P}(e^{\tau_2})$ consists of $(\bar{a}, y^{\tau_2}, y, a)$-paths and $(\bar{b}, y^{\tau_2}, y, b)$-paths. If both $\mathcal{P}(e, e_3)$ and $\mathcal{P}(e_1, e_2)$ contain $(\bar{a}, a)$-paths (or $(\bar{b}, b)$-paths), then we can rearrange f (as in (3.21)) so that $f^{e,e_1} > 0$ and this is a contradiction to $\alpha(\tau_1) = 0$. Therefore, we have

(3.26) (i) $\mathcal{P}(e, e_3)$ consists of $(\bar{a}, x, y, x_3, a)$-paths and $\mathcal{P}(e_1, e_2)$ consists of $(b, x_1, y, x_2, \bar{b})$-paths, or

(ii) $\mathcal{P}(e, e_3)$ consists of $(\bar{b}, x, y, x_3, a)$-paths and $\mathcal{P}(e_1, e_2)$ consists of $(a, x_1, y, x_2, \bar{a})$-paths.

See Figure 16. Now the edge e^{τ_1} in (G^{τ_1}, c) is saturated by every optimal multiflow by $\alpha(\tau_1) = 0$. By Lemma 3.7, we can take an optimal neighbor ρ' to ρ with $\rho(y^{\tau_1}) \neq \rho(y)$. Suppose ρ' is backward. Then $\mathcal{P}(e^{\tau_1})$ consists of $(\bar{a}, y^{\tau_1}, y, a)$-paths and $(\bar{b}, y^{\tau_1}, y, b)$-paths if $(\rho'(y^{\tau_1}), \rho'(y)) = (p^O, p^{ab})$, and $\mathcal{P}(e^{\tau_1})$ consists of $(a, y^{\tau_1}, y, \bar{a})$-paths and $(b, y^{\tau_1}, y, \bar{b})$-paths if $(\rho'(y^{\tau_1}), \rho'(y)) = (p^{ab}, p^O)$. Both cases contradict to both (i) and (ii) in (3.26). Therefore ρ' is necessarily forward. We may assume that $\rho'(y) = p^a$ or p^b, say, $\rho'(y) = p^a$. In (G^{τ_1}, c), node y^{τ_1} has three incident edges e, e_1, e^{τ_1} with $c(e^{\tau_1}) = 2$. Then, fork $(e, y^{\tau_1}, e^{\tau_1})$ is (trivially) splittable by (3.15). After the splitting-off at $(e, y^{\tau_1}, e^{\tau_1})$ (and $(e_1, y^{\tau_1}, e^{\tau_1})$), the resulting graph coincides with (G, c) and ρ' is an optimal forward neighbor to ρ with $\rho'(y) \neq \rho(y)$.

3.6 Half-integrality under the ring condition

For a graph (G, c) and its optimal potential ρ, consider the following condition.

(3.27) The connected components of the subgraph of G induced by C_ρ consist of paths and cycles.

We call it the *ring condition*.

Proposition 3.15. *Let (G, c) be a graph and ρ its optimal potential. Suppose that $(G, c; \rho)$ is Eulerian and satisfies the ring condition. Then there exists a half-integral optimal multiflow.*

Let us explain a motivation behind it. We start with inner Eulerian graph (G, c) each of whose inner nodes has degree four. Let ρ be an optimal potential. Apply (necessarily forward) SPUP at each node of *degree four* in C_ρ until no such a node exists. Then each node in C_ρ is one of y and y^τ produced by the SPUP. Therefore each node in C_ρ has three incident nodes with at least one of them not belonging to C_ρ. Then each node in the subgraph induced by C_ρ has at most two incident nodes. Therefore $(G, c; \rho)$ satisfies the ring condition, and $(G, kc; \rho)$ is Eulerian for some integer k. In Section 3.7, we will show that k can be taken as 6.

Now we begin the proof. We use the induction on the sum of capacity of edges incident to C_ρ. If $C_\rho = \emptyset$, then we are done by Corollary 3.14. Suppose $C_\rho \neq \emptyset$. By using the method in Section 2.6 together with (2.10), make each node in M_ρ have degree four. According to Proposition 3.13, at each node in M_ρ, apply the splitting-off or the forward SPUP, or replace ρ to an optimal forward neighbor until $M_\rho = \emptyset$. Note that the forward SPUP never increases C_ρ. In this process, if the cardinality of C_ρ strictly decreases, then $(G, c; \rho)$ still satisfies the condition of Proposition 3.15 and the induction follows. We may assume that C_ρ keeps invariant. Consider $(G, 2c)$, which is inner Eulerian, and apply splitting-off to each node in S_ρ in $(G, 2c; \rho)$; it is always applicable by Proposition 3.9. Then c is half-integer. Make (G, c) simple. Then $(G, c; \rho)$ again satisfies the condition of Proposition 3.15, and

(3.28) each inner node belongs to C_ρ, i.e., $VG = S \cup C_\rho$.

Here (G, c) is simple, and thus we will use a simplified notation xyz for a fork (xy, y, yz) in the following. We may assume that

(3.29) any fork $\tau = syu$ for $s \in S$, $y \in C_\rho$, $u \in VG$ is unsplittable, and thus

$$\alpha(\tau) \in \left\{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}\right\}$$

by the Eulerianness of (G, c) (Lemma 3.11) and Lemma 3.3.

Otherwise, the graph resulted by the splitting-off at τ clearly satisfies the ring condition and the induction follows.

(3.30) For a fork $\tau = aya'$ with $y \in C_\rho$ and $a, a' \in A$, if $0 < \alpha(\tau) < 2$, then any critical neighbor ρ' w.r.t. τ satisfies $(\rho'(y^\tau), \rho'(y)) = (p^O, p^{a''})$ or $(p^b, p^{a''})$ for $a'' \in A \setminus \{a, a'\}$ and $b \in B$, and thus $\alpha(\tau) = 1$.

Proof. Take an optimal multiflow f for $(G^\tau, c - \alpha(\tau)\chi_{e^\tau})$. Then edges ay^τ, $a'y^\tau$, and e^τ are all saturated, and thus f^{ay^τ, e^τ}, $f^{a'y^\tau, e^\tau}$, and $f^{ay^\tau, y^\tau a'}$ are all nonzero. Then $\mathcal{P}(ay^\tau, e^\tau)$, $\mathcal{P}(a'y^\tau, e^\tau)$, and $\mathcal{P}(ay^\tau, y^\tau a')$ consists of $(a, A \setminus \{a, a'\})$-paths, $(a', A \setminus \{a, a'\})$-paths, and (a, a')-paths, respectively. Indeed, if $\mathcal{P}(ay^\tau, e^\tau)$ has an (a, a')-path, then we can rearrange f (as in Figure 12) to make e^τ unsaturated, which contradicts to $d^{\rho'}(e^\tau) > 0$ and the optimality criterion I. By (2.8), $(\rho'(y^\tau), \rho'(y))$ is $(p^O, p^{a''})$ or $(p^b, p^{a''})$. □

If $(\rho'(y^\tau), \rho'(y)) = (p^b, p^{a''})$, then $(G^\tau, c - \chi_{e^\tau}; \rho')$ satisfies the assumption of Proposition 3.15; the corresponding SPUP succeeds, $\# C_{\rho'} < \# C_\rho$, and thus the induction follows. Suppose that $(\rho'(y^\tau), \rho'(y)) = (p^O, p^{a''})$. Replace $\rho'(y^\tau)$ by p^b for arbitrary $b \in B$. Then the resulting ρ' is also optimal, and it reduces to the case above.

Therefore, we may assume that

(3.31) any fork $\tau = syt$ for $y \in C_\rho$, $s, t \in S$ satisfies $\alpha(\tau) = 0$.

By $d^\rho(a, y) + d^\rho(y, b) > d^\rho(a, b)$ for $(a, b) \in A \times B$, there is no optimal multiflow passing a, y, b in order. From this, we have $\alpha(ayb) = 0$.

(3.32) An inner node $y \in C_\rho$ is incident to exactly two nodes in C_ρ, and is incident to at least two terminals.

Proof. Suppose that y is incident to exactly one inner node x. For an optimal multiflow f, we have $c(sy) = f^{sy} = f^{sy, yx}$ by (3.31). Therefore syx is splittable, which contradicts to (3.29). Suppose that y is incident to exactly one terminal s. y is incident to exactly two nodes $x, z \in C_\rho$. By Eulerianness, we may assume that $c(zy) > c(xy)$. Then syz is (trivially) splittable. A contradiction to (3.29)

□

Therefore, we may assume that

(3.33) for a node $y \in C_\rho$ incident to $x, z \in C_\rho$ and $s \in S$ and every optimal multiflow f, we have

$$c(sy) = f^{sy} = f^{sy, yz} + f^{sy, yx}, \text{ and } f^{sy, yz} > 0, f^{sy, yx} > 0.$$

In particular, $c(sy) = 1$.

Indeed, $c(sy) \geq 2$ implies that $\max\{f^{sy, yz}, f^{sy, yx}\} \geq 1$ and thus one of syz an syx is splittable (a contradiction to (3.29)).

(3.34) Each fork $\tau = syz$ for $s \in S$, $y, z \in C_\rho$ satisfies $\alpha(\tau) = 1$.

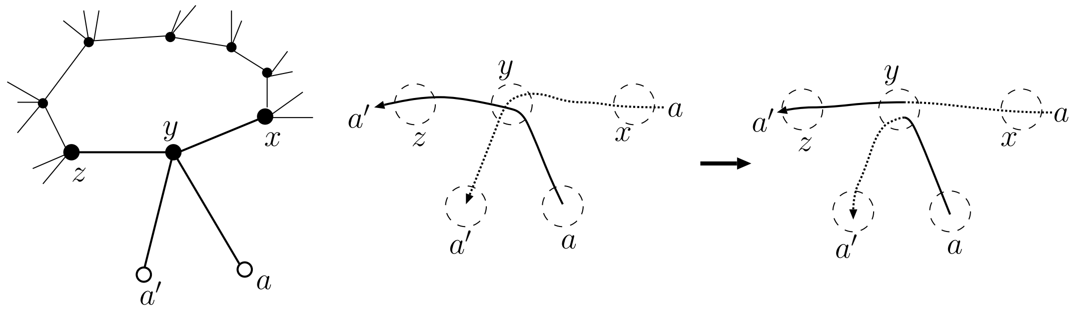


Figure 17: Nodes x, y, z, a, a' and flow modification

Proof. We may assume that $s = a$ for $a \in A$. Take a critical neighbor ρ'. Suppose to the contrary that $d^{\rho'}(e^\tau) = 3$ or 4. If $d^{\rho'}(e^\tau) = 3$, then $(\rho'(y^\tau), \rho'(y)) = (p^a, p^{a'b})$ or $(p^{ab}, p^{a'})$ for some $a' \in \bar{a}$ and $b \in B$. If $d^{\rho'}(e^\tau) = 4$, then $(\rho'(y^\tau), \rho'(y)) = (p^a, p^{a'})$ for $a' \in \bar{a}$. Indeed, take an optimal multiflow f to $(G^\tau, c - \alpha(\tau)\chi_\tau)$. Then $f^{ay^\tau, e^\tau} > 0$, which implies $d^{\rho'}(a, y) = d^{\rho'}(a, y^\tau) + d^{\rho'}(y^\tau, y)$, and thus $\rho'(y)$ and $\rho'(y^\tau)$ are determined as above. In both cases, $\mathcal{P}(e^\tau)$ consists of (a, a')-paths. By (3.32) and (3.33), y has another terminal s' with $f^{s'y, e^\tau} = f^{s'y, yz} > 0$, which implies $s' = a'$. Then we can rearrange f so that $f^{ay, ya'} > 0$ as in Figure 17. This is a contradiction to (3.31). $\square$

In particular, $f^{sy,yz} = f^{sy,yx} = 1/2$ necessarily holds for every optimal multiflow f. By $\alpha(syx) = 2f^{sy,yx} = 1$ from (3.34) and by (3.10), yx is saturated by every optimal multiflow. By Lemma 3.7, we can take an optimal neighbor ρ' to ρ with $d^{\rho'}(yx) > 0$, which is necessarily forward. Then $\#C_{\rho'} < \#C_\rho$ holds, and the induction follows.

3.7 Splitting at C_ρ

Here we complete the proof of Theorem 1.4. Let (G, c) be an inner Eulerian graph. By using the method in Section 2.6, we may assume that each edge has unit capacity and each inner node has degree four. Apply the splitting-off operations to all inner nodes if applicable. We may assume that there is no splittable fork. Take an optimal potential ρ. If C_ρ is empty, then there exists a half-integer optimal multiflow by Corollary 3.14. Therefore we may assume that C_ρ is nonemtpy.

We will repeat the forward SPUP operations to nodes of degree four in C_ρ keeping the following condition:

(3.35) $(G, 3c; \rho)$ is admissible and $(G, 6c; \rho)$ is Eulerian.

In this process, an inner node x is said to be *untouched* if x has not been split yet, or equivalently, x has degree four. In the initial step, the subset $\tilde{E}$ of mixed edges is set to be empty. If there is no untouched node in C_ρ, then $(G, c; \rho)$ necessarily satisfies the ring condition (3.27), as described in the previous subsection.

Now we begin the proof. Take an (untouched) inner node $y \in C_\rho$. We may assume that $0 < \alpha(\tau) < 2$ for every fork τ at y. Indeed, if $\alpha(\tau) = 0$, then we can take an optimal neighbor ρ' to ρ with $\#C_{\rho'} < \#C_\rho$ by the same argument as

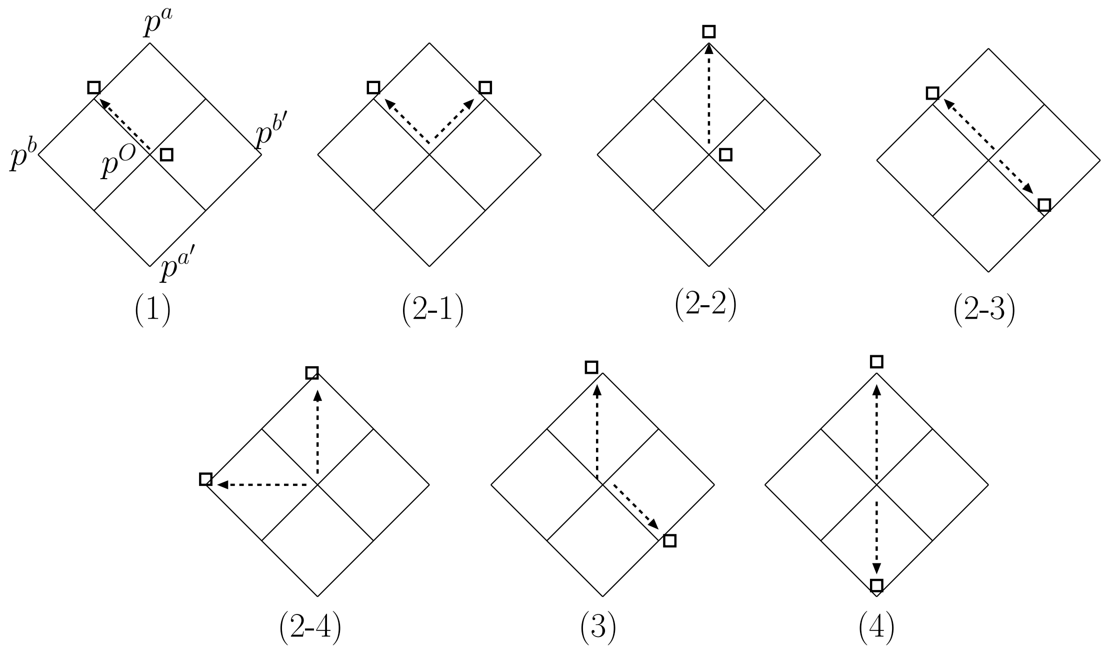


Figure 18: Behavior of neighbor at p^O

that in the proof of Proposition 3.13 (1). By Proposition 3.1 and Lemma 3.3, the possible values of $\alpha(\tau)$ are

$$\frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}.$$

Let ρ' be a critical neighbor to ρ with respect to τ.

(3.36) If $d^{\rho'}(e^\tau) = 4$, then the SPUP at τ keeps (G, c) admissible (with $\tilde{E} = \emptyset$) and $(G, 2c)$ Eulerian.

Indeed, this immediately follows from $\{\rho'(y), \rho'(y^\tau)\} = \{p^a, p^{a'}\}$ or $\{p^b, p^{b'}\}$ and $\alpha(\tau) \in \{1/2, 1, 3/2\}$.

Apply the SPUP to all such forks τ with $d^{\rho'}(e^\tau) = 4$ if exists. At this moment, if C_ρ is empty, then there exists a 1/4-integral optimal multiflow.

We may assume that C_ρ is nonempty. Then, by Lemma 3.6,

(3.37) any fork τ at any untouched node in C_ρ satisfies $\alpha(\tau) < 3/2$.

Now suppose $(G, 3c; \rho)$ is admissible (with $\tilde{E}$) and $(G, 6c; \rho)$ is Eulerian. In the SPUP, if some edge $e = xy$ in $\tilde{E}$ moves, i.e., $(\rho'(x), \rho'(y)) \neq (\rho(x), \rho(y))$, then update $\tilde{E}$ by deleting all such edges; both $\rho'(x)$ and $\rho'(y)$ fall into S_ρ.

Let τ be a fork at an untouched node $y \in C_\rho$. Let ρ' be a critical neighbor to ρ with respect to τ. Then, by Lemma 3.12, we have

$$\alpha(\tau) \in \frac{1}{d^{\rho'}(e^\tau)} \frac{2}{3} \mathbf{Z}_+.$$

Therefore the possible situations of ρ' are classified into the following; also see Figure 18.

(1) $\{\rho'(y^\tau), \rho'(y)\} = \{p^O, p^{ab}\}$, $d^{\rho'}(e^\tau) = 1$, and $\alpha(\tau) \in \left\{\frac{2}{3}, \frac{4}{3}\right\}$.

(2-1) $\{\rho'(y^\tau), \rho'(y)\} = \{p^{ab}, p^{ab'}\}$ or $\{p^{ab}, p^{a'b}\}$, $d^{\rho'}(e^\tau) = 2$, and $\alpha(\tau) \in \left\{\frac{2}{3}, 1, \frac{4}{3}\right\}$.

(2-2) $\{\rho'(y^\tau), \rho'(y)\} = \{p^O, p^s\}$, $d^{\rho'}(e^\tau) = 2$, and $\alpha(\tau) \in \left\{\frac{2}{3}, 1, \frac{4}{3}\right\}$.

(2-3) $\{\rho'(y^\tau), \rho'(y)\} = \{p^{ab}, p^{a'b'}\}$, $d^{\rho'}(e^\tau) = 2$, and $\alpha(\tau) \in \left\{\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}\right\}$.

(2-4) $\{\rho'(y^\tau), \rho'(y)\} = \{p^a, p^b\}$, $d^{\rho'}(e^\tau) = 2$, and $\alpha(\tau) \in \left\{\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}\right\}$.

(3) $\{\rho'(y^\tau), \rho'(y)\} = \{p^a, p^{a'b}\}$ or $\{p^b, p^{ab'}\}$, $d^{\rho'}(e^\tau) = 3$, and $\alpha(\tau) \in \left\{\frac{2}{3}, \frac{8}{9}, \frac{10}{9}, \frac{4}{3}\right\}$.

(4) $\{\rho'(y^\tau), \rho'(y)\} = \{p^a, p^{a'}\}$ or $\{p^b, p^{b'}\}$, $d^{\rho'}(e^\tau) = 4$, and $\alpha(\tau) \in \left\{\frac{2}{3}, \frac{5}{6}, 1, \frac{7}{6}, \frac{4}{3}\right\}$.

In the statements, a and a' (resp. b and b') are distinct. Here $\alpha(\tau) < 3/2 = 1.5$ necessarily holds by (3.37). Therefore $\alpha(\tau) \geq 5/3 = 1.666\cdots$ never occurs in the cases (2-1), (2-2), (2-3), and (2-4). Similarly, $\alpha(\tau) \geq 14/9 = 1.555\cdots$ never occurs in the case (3). In the cases (2-1), (2-2), (3), and (4), $\alpha(\tau) \leq 1/2$ never occurs. Otherwise, by (2.8) and (3.18), y has another fork τ' with $\alpha(\tau') \geq 3/2$.

There are several cases such that the SPUP at τ succeeds, i.e., it keeps (3.35). (1), (4), and (2-4) are such cases. Also, in all cases, if $\alpha(\tau) \in \{2/3, 4/3\}$, then the SPUP is clearly successful. If the case (2-3) with $\alpha(\tau) = 1$ occurs and e^τ is mixed in $(G^\tau, c - \alpha(\tau)\chi_{e^\tau}; \rho')$, then the SPUP succeeds with adding e^τ to $\tilde{E}$. In the case (2-3) with $\alpha(\tau) = 1/3$, if e^τ is not mixed in $(G^\tau, c - \alpha(\tau)\chi_{e^\tau}; \rho')$, then y has another fork τ' such that $\alpha(\tau') \geq 5/3 > 3/2$ by (3.18), and this is a contradiction to (3.37). Therefore if the case (2-3) with $\alpha(\tau) = 1/3$ occurs, then e^τ is necessarily mixed in $(G^\tau, c - \alpha(\tau)\chi_{e^\tau}; \rho')$ and thus the SPUP succeeds with adding e^τ to $\tilde{E}$.

Apply SPUP at all such forks. Now, if there is no untouched node in C_ρ, then $(G, 6c; \rho)$ satisfies the condition of Proposition 3.15, and thus we are done. Suppose that C_ρ still has an untouched node y. For each fork τ at y, the remaining possible situations are the following.

(2-1) $\{\rho'(y^\tau), \rho'(y)\} = \{p^{ab}, p^{ab'}\}$ or $\{p^{ab}, p^{a'b}\}$, $d^{\rho'}(e^\tau) = 2$, and $\alpha(\tau) = 1$.

(2-2) $\{\rho'(y^\tau), \rho'(y)\} = \{p^O, p^s\}$, $d^{\rho'}(e^\tau) = 2$, and $\alpha(\tau) = 1$.

(2-3) $\{\rho'(y^\tau), \rho'(y)\} = \{p^{ab}, p^{a'b'}\}$, $d^{\rho'}(e^\tau) = 2$, $\alpha(\tau) = 1$,
and e^τ is not mixed in $(G^\tau, c - \chi_{e^\tau}; \rho')$.

(3) $\{\rho'(y^\tau), \rho'(y)\} = \{p^a, p^{a'b}\}$ or $\{p^b, p^{ab'}\}$, $d^{\rho'}(e^\tau) = 3$, and $\alpha(\tau) \in \left\{\frac{8}{9}, \frac{10}{9}\right\}$.

We show that (2-1), (2-3), and (3) never occur.

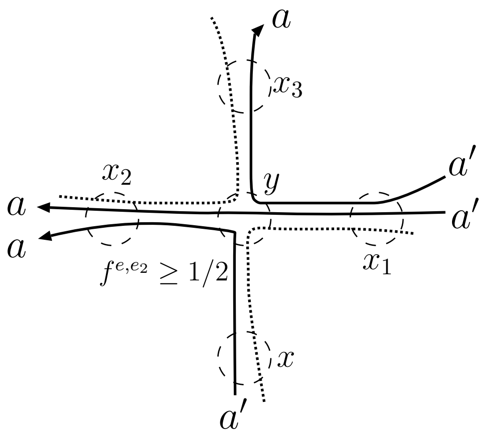


Figure 19: Flow configurations at y

The case (2-3). First we show that (2-3) never occurs. An untouched inner node $y \in C_\rho$ has four distinct node x, x_1, x_2, x_3 with edges $e = xy$, $e_1 = x_1y$, $e_2 = x_2y$, $e_3 = x_3y$. Suppose to the contrary that (2-3) occurs for $\tau_1 = (e, y, e_1)$. Take a critical neighbor ρ_1 to ρ w.r.t. τ_1. We may assume that $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^{a'b'}, p^{ab})$. We can take an optimal multiflow f for $(G^{\tau_1}, c - \chi_{e^{\tau_1}})$ such that

(3.38) (i) $\mathcal{P}(e^{\tau_1})$ consists of (a', y^{τ_1}, y, a)-paths (by changing roles of A and B if necessarily), and

(ii) among such optimal multiflows, $\sum_{e \in EG^{\tau_1}} f^e$ is minimum.

We may assume that $f^{e_2,e^{\tau_1}} \geq f^{e,e^{\tau_1}} \geq 1/2 \geq f^{e_1,e^{\tau_1}} \geq f^{e_3,e^{\tau_1}}$ by relabeling. By rearranging $\mathcal{P}(e^{\tau_1})$ as (3.19) in Section 3.2.2, we have $f^{e,e_2} = f^{e,e^{\tau_1}} \geq 1/2$, and $f^{e,e_3} = 0$ (, which keeps the minimality); see Figure 19. By the minimality assumption in (3.38) and by the same argument as that used for (3.20) in Section 3.3,

(3.39) both $\mathcal{P}(e, e_1)$ and $\mathcal{P}(e_2, e_3)$ cannot contain A-paths.

Consider fork $\tau_2 = (e, y, e_2)$. Then $\alpha(\tau_2) \geq 1$ by $f^{e,e_2} \geq 1/2$ (Lemma 3.5). Therefore,

$$\alpha(\tau_2) \in \{1, 10/9\}.$$

Suppose $\alpha(\tau_2) = 10/9$ (case (3)). Take a critical neighbor ρ_2 w.r.t. τ_2. Put $\epsilon = f^{e,e_2} - 1/2 \geq 0$. Applying the inequality (3.12) in Lemma 3.8 for (τ_2, ρ_2), we have

$$f^{e^{\tau_2}:\rho_2} \geq 2 - 2(1/2 + \epsilon) - (3/2)(10/9 - 2(1/2 + \epsilon)) = 5/6 + \epsilon.$$

If $\{\rho_2(y^{\tau_2}), \rho_2(y)\} = \{p^{a''}, p^{a'''b''}\}$, then $\mathcal{P}(e^{\tau_2} : \rho_2)$ consists of A-paths of flow-value at least $5/6 + \epsilon$, and thus $\mathcal{P}(e, e_1)$ also contains A-paths (of flow-value at least $5/6 + \epsilon - (1 - f^{e,e_2}) = 5/6 - 1/2 + 2\epsilon > 0$), which contradicts to (3.39). Therefore we may assume that $\{\rho_2(y^{\tau_2}), \rho_2(y)\} = \{p^{b''}, p^{a''b'''}\}$. Then $\mathcal{P}(e^{\tau_2} : \rho_2)$ consists of B-paths of flow-value at least $5/6 + \epsilon$. Since $\mathcal{P}(e_1, e_2)$ consists of A-paths (if nonempty), we have $\mathcal{P}(e^{\tau_2} : \rho_2) \subseteq \mathcal{P}(e, e_1) \cup \mathcal{P}(e_2, e_3)$; recall (3.17). Therefore both $\mathcal{P}(e, e_1)$ and $\mathcal{P}(e_2, e_3)$ have B-paths of flow-value at least $5/6 + \epsilon - (1 - f^{e,e_2}) = 5/6-$

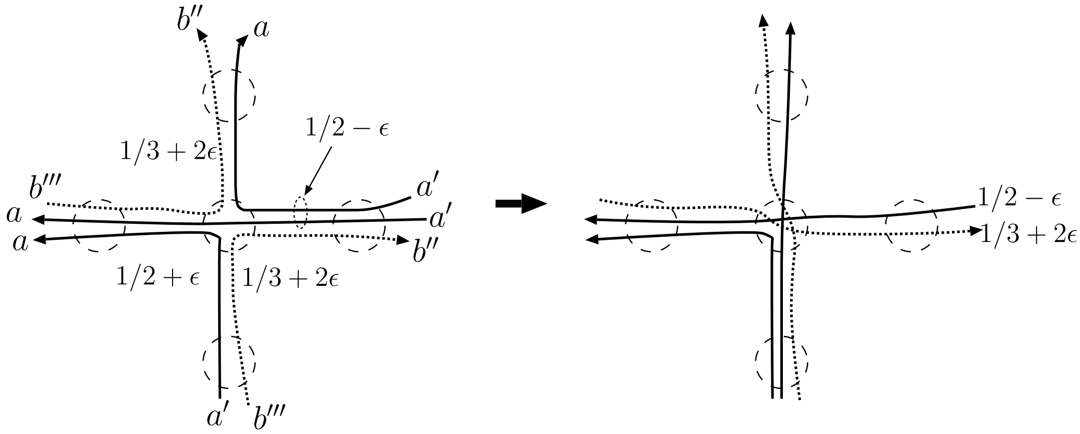


Figure 20: Flow modification

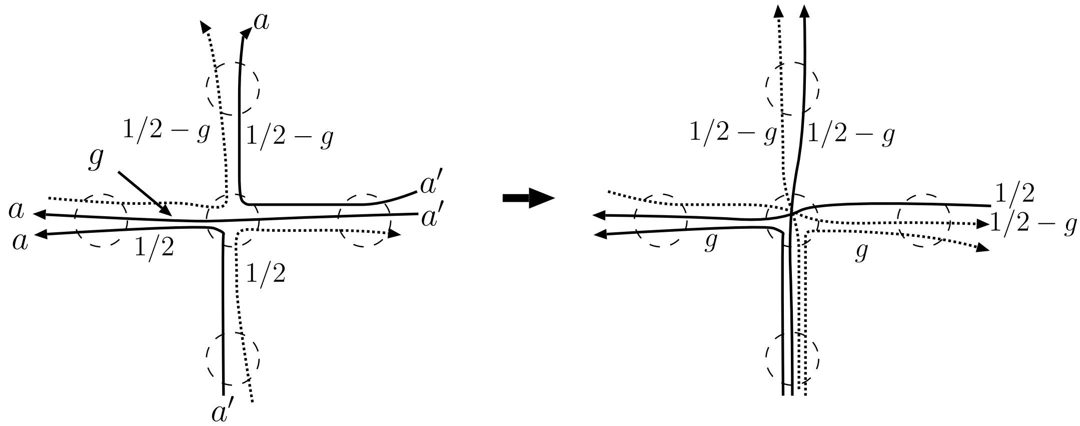


Figure 21: Flow modification

$1/2+2\epsilon = 1/3+2\epsilon$. We can rearrange f so that $f^{e_1,e_2} \geq 1/3+2\epsilon+1/2-\epsilon = 5/6+\epsilon$ as in Figure 20. Then $\alpha(\tau_3) \geq 5/3$. A contradiction.

Suppose $\alpha(\tau_2) = 1$. Then $f^{e,e_2} = \alpha(\tau_2)/2 = 1/2$ and f is also optimal for $(G^{\tau_2}, c - \chi_{e^{\tau_2}})$. By (3.10) and (3.39), $\mathcal{P}(e, e_1)$ consists of B-paths of flow-values $1/2$, and $\mathcal{P}(e_2, e_3)$ consists B-paths of $1/2 - f^{e_1,e_2}$. Put $g = f^{e_1,e_2}$. We can rearrange f so that $f^{e_1,e_2} = 1/2 + 1/2 - g = 1 - g$. Therefore $4/9 \leq g \leq 1/2$. Let $\tau_3 = (e, y, e_3)$. $\mathcal{P}(e^{\tau_3})$ consists of A-paths of flow-value g and B-paths of flow-value g. Suppose $\alpha(\tau_3) = 10/9$, i.e., the case (3). Applying the inequality (3.11) in Lemma 3.11 for (τ_3, ρ_3), we obtain

$$2 \geq 2g + 2g \geq 2f^{e^{\tau_3}:\rho_3} + f^{e^{\tau_3}} \geq 3(2 - 10/9) = 8/3 > 2,$$

where $f^{e^{\tau_3}:\rho_3} \leq g$, $f^{e^{\tau_3}} = 2g$, and $g \leq 1/2$. A contradiction.

Therefore $\alpha(\tau_3) = 1$, and $g = f^{e_1,e_2} = 1/2$ ($g < 1/2$ implies $f^{e_1,e_2} > 1/2$ and $\alpha(\tau_3) > 1$). Then both a critical neighbor ρ_2 w.r.t. τ_2 and a critical neighbor ρ_3 w.r.t. τ_3 are necessarily the case (2-3) since both (2.1) and (2.2) implies that $\mathcal{P}(e^{\tau_2})$ (or $\mathcal{P}(e^{\tau_3})$) is single commodity paths. We may assume that $(\rho_2(y^{\tau_2}), \rho_2(y)) = (p^{a\tilde{b}'}, p^{a'\tilde{b}})$ with $\tilde{b} \neq \tilde{b}'$. Then $\mathcal{P}(e, e_1)$ consists of $(\tilde{b}, \tilde{b}')$-paths of flow-value $1/2$. See Figure 22. This implies $(\rho_3(y^{\tau_3}), \rho_3(y)) = (p^{a'\tilde{b}'}, p^{a\tilde{b}})$ (by regarding f as an

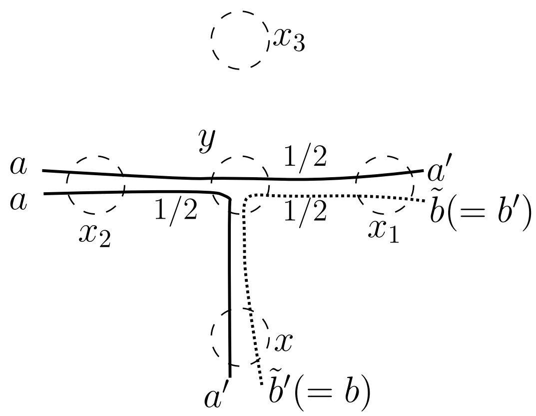


Figure 22: Flow configuration of f at y

optimal multiflow to $(G^{\tau_3}, c-\chi_{e^{\tau_3}})$). By the assumption that e^{τ_2} is not mixed in $(G^{\tau_2}, c-\chi_{e^{\tau_2}};\rho_2)$, we can take an optimal multiflow $f_2 = (\mathcal{P}_2, \lambda_2)$ for $(G^{\tau_2}, c-\chi_{e^{\tau_2}})$ such that $\mathcal{P}_2(e^{\tau_2})$ consists of either (a,a')-paths or $(\tilde{b},\tilde{b}')$-paths. We take such f_2 with $\sum_{e\in EG^{\tau_2}}(f_2)^e$ minimum. By the same arguments above, the possible flow configurations of f_2 at y are classified into the eight patterns in Figure 23, where the bold lines and the broken lines represent A-paths and B-paths of flow-value 1/2, respectively, and the positions of y, x, x_1, x_2, x_3 are the same as in Figure 22. f_2 can be regarded as an optimal multiflow for $(G^{\tau_1}, c-\chi_{e^{\tau_1}})$ and $(G^{\tau_3}, c-\chi_{e^{\tau_3}})$. $(\rho_3(y^{\tau_3}), \rho_3(y)) = (p^{a'\tilde{b}'}, p^{a\tilde{b}})$ implies that $\mathcal{P}_2(e^{\tau_3})$ consists of (a', y^{τ_3}, y, a)-paths and $(\tilde{b}', y^{\tau_3}, y, \tilde{b})$-paths. Here, recall the relation (3.17). Then (a-2), (a-4), (b-1), and (b-3) are impossible. Similarly, (a-1) contradicts to $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^{a'b'}, p^{ab})$. For (b-4), $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^{a'b'}, p^{ab})$ implies that $\mathcal{P}_2(e_1, e_3)$ consists of (a', x_1, y, x_3, a)-paths. On the other hand, $(\rho_3(y^{\tau_3}), \rho_3(y)) = (p^{a'\tilde{b}'}, p^{a\tilde{b}})$ implies that $\mathcal{P}_2(e_1, e_3)$ consists of (a, x_1, y, x_3, a')-paths. A contradiction. Therefore, f_2 is necessarily (b-2) or (a-3). If f_2 is (b-2), then $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^{a'b'}, p^{ab})$ implies that $\mathcal{P}_2(e_1, e_2)$ consists of (b', x_1, y, x_2, b)-paths, and thus $(\tilde{b}, \tilde{b}') = (b', b)$. Similarly, if f_2 is (a-3), then $\mathcal{P}_2(e_1, e_3)$ consists of (b', x_1, y, x_3, b)-paths by $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^{a'b'}, p^{ab})$ and also consists of $(\tilde{b}, x_1, y, x_3, \tilde{b}')$-paths by $(\rho_3(y^{\tau_3}), \rho_3(y)) = (p^{a'\tilde{b}'}, p^{a\tilde{b}})$, and thus $(\tilde{b}, \tilde{b}') = (b', b)$. Consequently, in the both cases, we have

$$(\tilde{b}, \tilde{b}') = (b', b).$$

In particular, $\mathcal{P}(e, e_1)$ consists of (b, x, y, x_1, b')-paths (see Figure 22).

Again we can take an optimal multiflow f_3 for $(G^{\tau_3}, c-\chi_{e^{\tau_3}})$ such that e^{τ_3} are saturated by either (a,a')-paths or $(\tilde{b},\tilde{b}')$-paths ((b',b)-paths). We take such $f_3 = (\mathcal{P}_3, \lambda_3)$ with $\sum_{e\in EG^{\tau_3}}(f_3)^e$ minimum. By the same argument above, the possible flow configurations of f_3 at y are classified into the eight patterns in Figure 24. Again (a-1),(a-4), (b-2), and (b-3) contradict to $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^{a'b'}, p^{ab})$, and (b-1) and (a-2) contradict to $(\rho_2(y^{\tau_2}), \rho_2(y)) = (p^{ab}, p^{a'b'})$. For (b-4), $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^{a'b'}, p^{ab})$ implies that $\mathcal{P}_3(e, e_3)$ consists of (a', x, y, x_3, a)-paths. However, $(\rho_2(y^{\tau_2}), \rho_2(y)) = (p^{ab}, p^{a'b'})$ implies that $\mathcal{P}_3(e, e_3)$ consists of (a, x, y, x_3, a')-paths. A contradiction. For (a-3), $(\rho_2(y^{\tau_2}), \rho_2(y)) = (p^{ab}, p^{a'b'})$ implies that $\mathcal{P}_3(e, e_3)$ consists of (b, x, y, x_3, b')-paths, and $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^{a'b'}, p^{ab})$

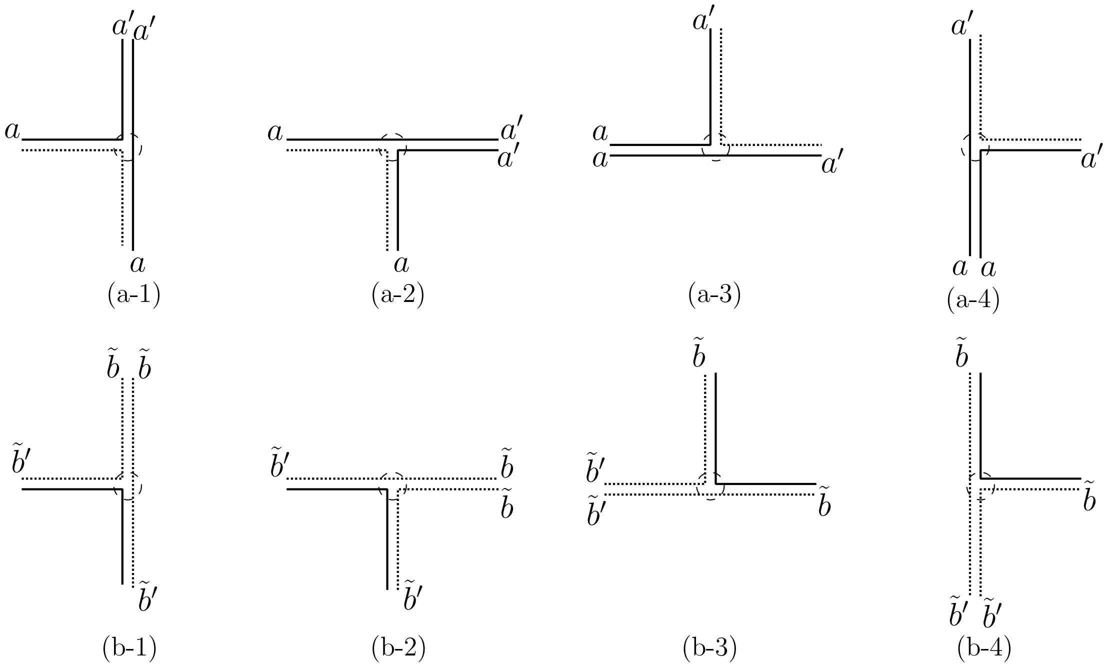


Figure 23: The possible flow configurations of f_2 at y

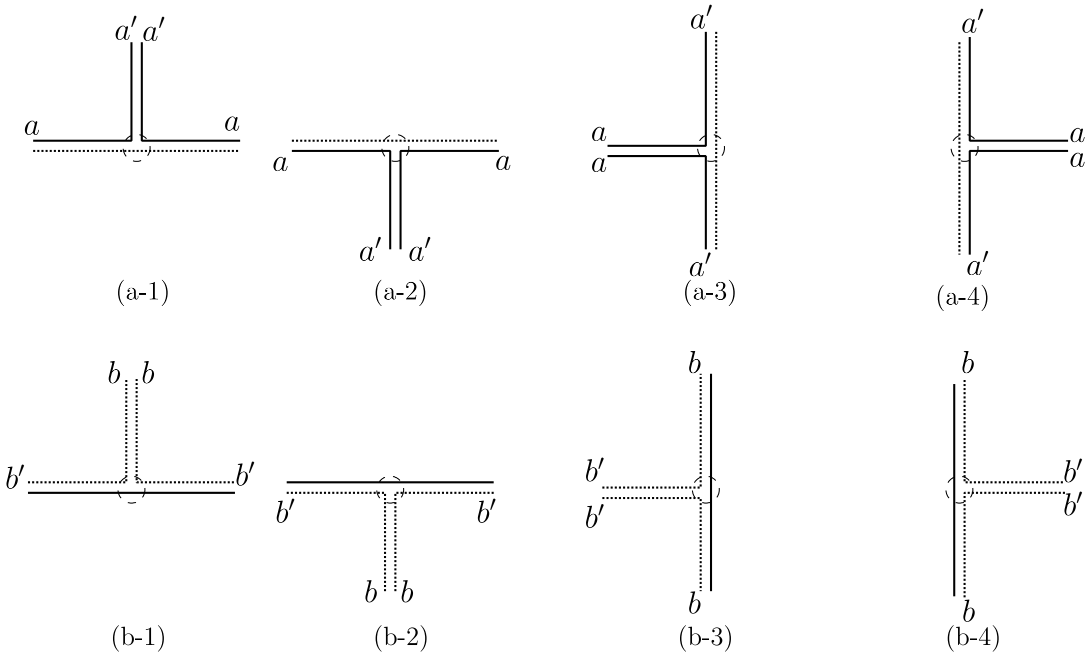


Figure 24: The possible flow configurations of f_3 at y

implies that $\mathcal{P}_3(e, e_3)$ consists of (b', x, y, x_3, b)-paths. A contradiction.

The cases (2-1). Second, we show that (2-1) never occurs. Suppose to the contrary that the case (2-1) occurs for $\tau_1 = (e, y, e_1)$. Take a critical neighbor ρ_1 to ρ w.r.t. τ_1. We may assume that $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^{ab}, p^{a'b})$. Take an optimal multiflow f for $(G^{\tau_1}, c - \chi_{e^{\tau_1}})$. Then $\mathcal{P}(e^{\tau_1})$ consists of (a', a)-paths. The situation is exactly the same as the case (2-2). Therefore we can apply the same argument above (more easy).

The case (3). Third, we show that (3) never occurs. Suppose to the contrary that the case (3) occurs for $\tau_1 = (e, y, e_1)$. Let ρ_1 be a critical neighbor to ρ w.r.t. τ_1. Then we may assume that $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^{a'}, p^{ab})$ or $(p^{a'b}, p^a)$. Take an optimal multiflow f for $(G^{\tau_1}, c - \alpha(\tau_1)\chi_{e^{\tau_1}})$ with $\sum_{e \in EG^{\tau_1}} f^e$ minimum. Then $\mathcal{P}(e^{\tau_1})$ consists of (a', y^{τ_1}, y, a)-paths (by (2.8)). Again, by the minimality,

(3.40) $\qquad \mathcal{P}(e, e_1)$ cannot contain A-paths.

Case 1: $\alpha(\tau_1) = 8/9$. Then $f^{e^{\tau_1}} = 10/9$. We may assume that $f^{e,e_1} = 5/9$ and $f^{e,e_3} = 0$. Then $\alpha(\tau_2) = 2f^{e,e_1} = 10/9$, and f is also optimal to $(G^{\tau_2}, c - \alpha(\tau_2)\chi_{e^{\tau_2}})$. Then $f^{e^{\tau_2}} = 8/9$. If a critical neighbor ρ_2 w.r.t. τ_2 satisfies $\{\rho_2(y^{\tau_2}), \rho_2(y)\} = \{p^{a''b''}, p^{a'''}\}$, then $\mathcal{P}(e, e_1)$ is saturated by A-paths, and this is a contradiction to (3.40). Therefore ρ_2 satisfies $\{\rho_2(y^{\tau_2}), \rho_2(y)\} = \{p^{a''b''}, p^{b'''}\}$, and f^{e_1,e_2} is necessarily zero. Then we can rearrange f so that $f^{e,e_3} = 1$ as in Figure 13, and thus (e, y, e_3) is splittable. A contradiction.
Case 2: $\alpha(\tau_1) = 10/9$. Then $f^{e^{\tau_1}} = 8/9$. Therefore we may assume that $f^{e,e_2} \geq 4/9$ and $f^{e,e_3} = 0$. Then $\alpha(\tau_2) \in \{8/9, 1, 10/9\}$. The case $\alpha(\tau_2) = 8/9$ reduces to Case 1 above.
Case 2-1: $\alpha(\tau_2) = 1$. Take a critical neighbor ρ_2 w.r.t. τ_2. Then $\{\rho_2(y^{\tau}), \rho_2(y)\} = \{p^O, p^s\}$ for some $s \in A \cup B$ (case (2.2)). If $s \in A$, then, by Lemma 3.8, $\mathcal{P}(e^{\tau_2} : \rho_2)$ consists of A-paths of flow-value at least 1, and thus $\mathcal{P}(e, e_1)$ has A-paths, which contradicts to (3.40). Therefore $s = b \in B$. Then $\mathcal{P}(e^{\tau_2} : \rho_2)$ consists of B-paths of flow-value at least 1. Since $\mathcal{P}(e_1, e_2)$ consists of A-paths (if nonempty), $\mathcal{P}(e^{\tau_2} : \rho_2) \subseteq \mathcal{P}(e, e_1) \cup \mathcal{P}(e_2, e_3)$ necessarily holds. Put $\epsilon = f^{e,e_2} - 4/9 \geq 0$. Both $\mathcal{P}(e, e_1)$ and $\mathcal{P}(e_2, e_3)$ contain B-paths of flow-value at least $1 - (1 - f^{e,e_2}) = 1 - 5/9 + \epsilon = 4/9 + \epsilon$. Then we can rearrange f so that $f^{e_1,e_2} \geq 4/9 - \epsilon + 4/9 + \epsilon > 2/3$; see Figure 25. Thus $\alpha(\tau_3) > 4/3$. A contradiction.
Case 2-2: $\alpha(\tau_2) = 10/9$. Take a critical neighbor ρ_2 w.r.t. τ_2. Then we have:

(i) $\{\rho_2(y^{\tau_2}), \rho_2(y)\} = \{p^{a''b'}, p^{a'''}\}$ for distinct $a'', a''' \in A$ and $b' \in B$, or

(ii) $\{\rho_2(y^{\tau_2}), \rho_2(y)\} = \{p^{a''b'}, p^{b''}\}$ for distinct $b', b'' \in B$ and $a'' \in A$.

Put $\epsilon = f^{e,e_2} - 4/9 \geq 0$. By Lemma 3.8, we have

$$f^{e^{\tau_2} : \rho_2} \geq 2 - 2(4/9 + \epsilon) - (3/2)(10/9 - 2(4/9 + \epsilon)) = 7/9 + \epsilon.$$

If (i) occurs, then $\mathcal{P}(e^{\tau_2} : \rho_2)$ consists of A-paths of flow-value at least $7/9 + \epsilon$, and consequently $\mathcal{P}(e, e_1)$ must have A-paths (of flow-value at least $7/9 + \epsilon - (1 - f^{e,e_2}) = 2/9 + 2\epsilon$) and this contradicts to (3.40). Therefore (ii) occurs. By (3.40), $\mathcal{P}(e^{\tau_2} : \rho_2)$

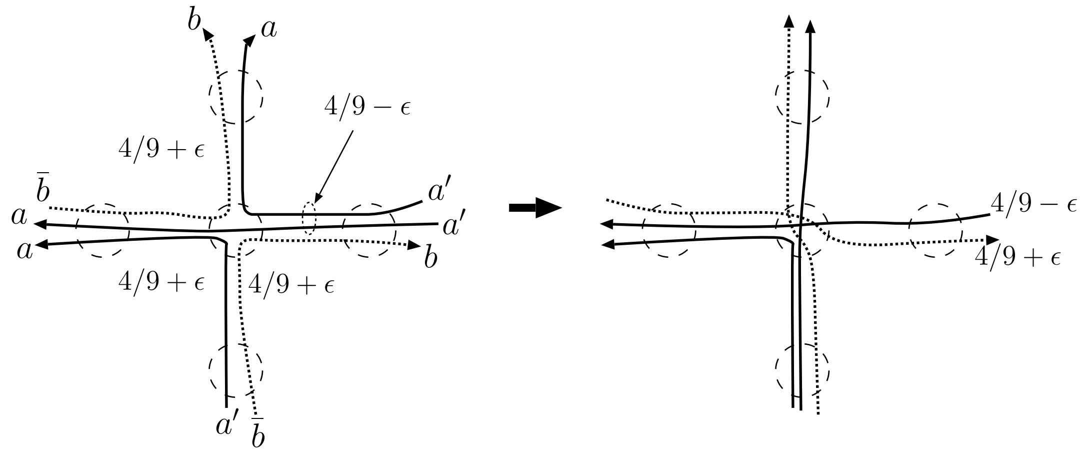


Figure 25: Flow modification

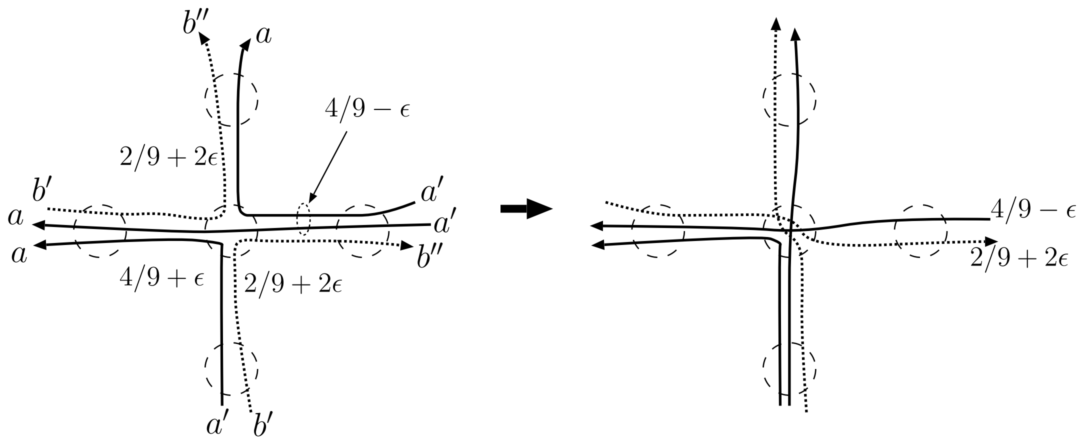


Figure 26: Flow modification

consists of B-paths of flow-value at least $7/9+\epsilon$. Since $\mathcal{P}(e_1, e_2)$ consists of A-paths (if nonempty), we have $\mathcal{P}(e^{\tau_2} : \rho_2) \subseteq \mathcal{P}(e, e_1) \cup \mathcal{P}(e_2, e_3)$. Then both $\mathcal{P}(e, e_1)$ and $\mathcal{P}(e_2, e_3)$ have B-paths of flow-values at least $7/9 + \epsilon - (1 - f^{e,e_2}) = 2/9 + 2\epsilon$. We can rearrange f so that $f^{e_1,e_2} \geq 4/9 - \epsilon + 2/9 + 2\epsilon = 2/3 + \epsilon$; see Figure 26. Then $\alpha(\tau_3) \geq 4/3$. A contradiction.

The case (2-2). Therefore, for any fork τ at any untouched node $y \in C_\rho$, any critical neighbor ρ' to ρ is of type (2-2):

(2-2) $\{\rho'(y^\tau), \rho'(y)\} = \{p^O, p^s\}$, $d^{\rho'}(e^\tau) = 2$, and $\alpha(\tau) = 1$.

By symmetry (3.16) and by changing roles of A and B, we may assume that $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^O, p^a)$ for a critical neighbor ρ_1 to ρ w.r.t. τ_1. Take an *arbitrary* optimal multiflow $f = (\mathcal{P}, \lambda)$ for (G, c). By Lemma 3.8, $\mathcal{P}(e^{\tau_1} : \rho_1)$ consists of $(\bar{a}, y^{\tau_1}, y, a)$-paths of flow-value at least 1. By rearranging A-paths in $\mathcal{P}(e^{\tau_1})$, we have $f^{e,e_2} \geq 1/2$. Therefore $f^{e,e_2} = 1/2$ since $f^{e,e_2} > 1/2$ implies that τ_2 is splittable. Consider the fork $\tau_2 = (e, y, e_2)$. Then $\alpha(\tau_2) = 2f^{e,e_2} = 1$. f is also optimal for $(G^{\tau_2}, c - \chi_{e^{\tau_2}})$. Then both e_2 and e are saturated. If both $\mathcal{P}(e, e_1)$ and $\mathcal{P}(e_2, e_3)$ are nonempty, then we can rearrange f so that $f^{e,e_3} > 1/2$ and $\alpha(\tau_3) > 1$,

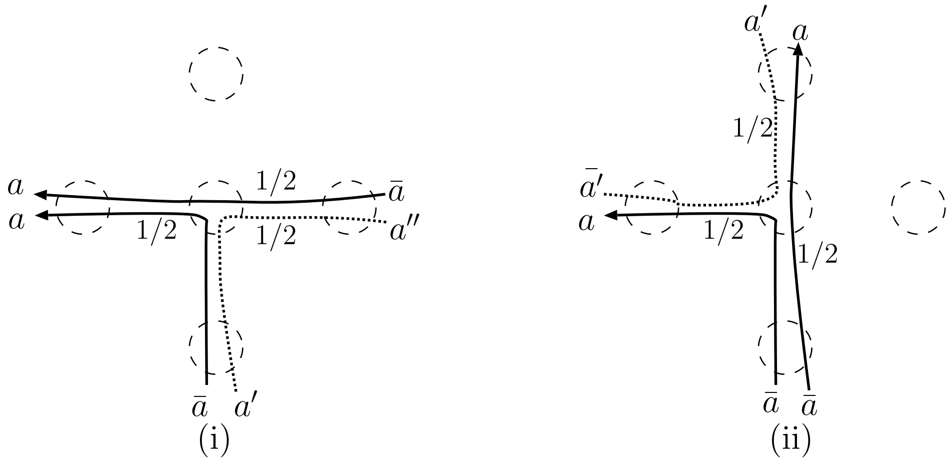


Figure 27: Flow configurations at y

and this is a contradiction. Therefore one of $\mathcal{P}(e, e_1)$ and $\mathcal{P}(e_2, e_3)$ is empty. Then the possible flow configurations at y are:

(i) $f^{e,e_2} = f^{e_1,e_2} = f^{e,e_1} = 1/2$ and $f^{e_3} = 0$, or

(ii) $f^{e,e_2} = f^{e,e_3} = f^{e_2,e_3} = 1/2$ and $f^{e_1} = 0$.

See Figure 27. In fact, (ii) is impossible. Indeed, consider a critical neighbor ρ_2 w.r.t. τ_2. Then it satisfies $\{\rho_2(y^{\tau_2}), \rho_2(y)\} = \{p^O, p^{\tilde{a}}\}$ for $\tilde{a} \in A$ since $\mathcal{P}(e^{\tau_2})$ contains A-paths. Therefore, $\mathcal{P}(e_2, e_3)$ consists of A-paths. By a rearrangement similar to Figure 12, we can modify f so that $f^{e,e_2} > 1/2$ or $f^{e,e_3} > 1/2$, which yields a contradiction. Suppose the case (i). Then $\mathcal{P}(e, e_1)$ consists of (a', x, y, x_1, a'')-paths for distinct $a', a'' \in A$ with $a' \neq a$ and $a'' \neq a$. Otherwise, we can modify f so that $f^{e,e_2} > 1/2$ or $f^{e_1,e_2} > 1/2$ by a rearrangement similar to Figure 12. Similarly, $\mathcal{P}(e, e_2)$ consists of (a', x, y, x_2, a)-paths and $\mathcal{P}(e_1, e_2)$ consists of (a'', x_1, y, x_2, a)-paths. Recall that f is an arbitrary optimal multiflow. Then y^{τ_1} is a tri-fixed in $(G^\tau, c - \chi_{\tau_1}; \rho_1)$; recall the definition of tri-fixed nodes given in Section 3.4. Therefore the SPUP succeeds at each nodes in C_ρ.

We now arrive at the goal where C_ρ has no untouched nodes and $(G, c; \rho)$ keeps (3.35). Then $(G, 6c; \rho)$ satisfies the condition of Proposition 3.15. Therefore (G, c) has a 1/12-integral optimal multiflow, and so does the original graph. This completes the proof of Theorem 1.4.

4 Concluding remarks

In this paper, we prove that the multiflow feasibility problems for demand graph $K_3 + K_3$ and $K_{n,m}$-metric weighted maximum multiflow problems have bounded fractionality. However, we do not know whether the constant $k = 1/12$ (under the Euler condition) is tight. The main obstruction is an occurrence of the SPUP corresponding to $\alpha(\tau) \in \{4/3, 3/2\}$ at C_ρ in our proof, which causes the violation of the Eulerianness to $(G, c; \rho)$. If one could avoid such a SPUP, then the existence of a half-integral optimal multiflow would follow, which implies the stronger conjecture ($k = 2$). Unfortunately, we could not do it.

Our approach is applicable to prove the existence of a 1/12-integral optimal multiflow for a larger class of maximum multiflow problems [5].

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