Bounded fractionality of the multiflow feasibility problem for demand graph $K_3 + K_3$ and related maximization problems

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October 2008,
August 2010 (revised)

Abstract

We consider the multiflow feasibility problem whose demand graph is the vertex-disjoint union of two triangles. We show that this problem has a 1/12-integral solution whenever it is feasible and satisfies the Euler condition. This solves a conjecture raised by Karzanov, and completes the classification of the demand graphs having bounded fractionality. We reduce this problem to the multiflow maximization problem whose terminal weight is the graph metric of the complete bipartite graph, and show that it always has a 1/12-integral optimal multiflow for every inner Eulerian graph.

1 Introduction

Let G be an undirected graph with node set VG, edge set EG, and nonnegative edge capacity $c : EG \to \mathbf{R}_+$. Let $S \subseteq VG$ be a set of *terminals*. An *S-path* is a path connecting distinct terminals in S. A *multiflow* $f = (\mathcal{P}, \lambda)$ is a pair of a set $\mathcal{P}$ of S-paths and its nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying the capacity constraint:

$$\sum \{\lambda(P) \mid P \in \mathcal{P} : P \text{ contains } e\} \leq c(e) \quad (e \in EG).$$

We are given another (simple) graph $H = (S, R)$, called a *demand graph*, and a demand function $q : R \to \mathbf{R}_+$. The *multiflow feasibility problem* is formulated as follows:

(1.1) Find a multiflow f satisfying the demand requirement

$$\sum \{\lambda(P) \mid P \in \mathcal{P} : P \text{ connects } s \text{ and } t\} = q(st) \quad (st \in R),$$

or establish that there is no such a multiflow.

The multiflow feasibility problem (1.1) is said to be *feasible* if it has a multiflow satisfying the demand requirement, which we call a *feasible multiflow*. A multiflow $f = (\mathcal{P}, \lambda)$ is said to be *integral*, *half-integral*, and *$1/k$-integral* if λ is integer-valued, 2λ is integer-valued, and $k\lambda$ is integer-valued, respectively.

The max-flow min-cut theorem, due to Ford-Fulkerson [3], says that if H is K_2 (one edge), both c and q are integral, and the problem is feasible, then there exists an integral feasible multiflow. Hu [8] extended this result to two-commodity flows, saying that if $H = K_2 + K_2$ (a matching of size 2), both c and q are integral, and the problem is feasible, then there exists a half-integral feasible multiflow. On the other hand, the 3-commodity flow problem, that corresponds to $H = K_2 + K_2 + K_2$ (a matching of size 3), does not have such a property. Lomonosov [16] gave an infinite series of the feasible 3-commodity flow problems with integer capacity and demand in which there is no fixed integer k such that all these problems have a $1/k$-integral feasible multiflow; see [18, Chapter 70, p.1232].

Motivated by these examples, following [11], we define the *fractionality* of a simple graph H by the least positive integer k with the property that the multiflow feasibility problem (1.1) for every integer-capacitated graph G and demand graph H with every integer demand has a $1/k$-integral feasible multiflow whenever the problem is feasible. If such an integer k does not exist, we define the fractionality to be the infinity. Karzanov raised the following problem:

Classify the demand graph H having bounded fractionality.

Lomonosov's 3-commodity example implies that if H has a matching of size 3, then the fractionality of H is infinity. Therefore we may restrict ourselves to considering demand graphs without a matching of size 3. Such a graph falls into one of the following three classes:

(i) K_4, C_5, or the union of two stars.

(ii) K_5 or the union of a star and a triangle K_3.

(iii) $K_3 + K_3$, i.e., the vertex-disjoint sum of two triangles.

The works by Rothschild and Winston [17], Seymour [19] and Lomonosov [16] established the (half-)integrality for the class (i). Here we say " (1.1) *satisfies the Euler condition*" if both c and q are integer-valued and for each node x the sum of $c(e)$ and $q(e)$ over all edges e incident to x is even.

Theorem 1.1 ([16, 17, 19])**.** *Suppose that H is K_4, C_5, or the union of two stars. If* (1.1) *is feasible and satisfies the Euler condition, then there exists an integral feasible multiflow.*

In particular, the graphs of the class (i) (except one star having fractionality 1) have fractionality 2. Karzanov [10] showed that the same result holds for the class (ii).

Theorem 1.2 ([10])**.** *Suppose that H is K_5 or the union of a star and a triangle. If* (1.1) *is feasible and satisfies the Euler condition, then there exists an integral feasible multiflow.*

For the remaining last class (iii): $H = K_3 + K_3$, it is known that the fractionality is at least 4; see [18, p. 1275]. Karzanov [12] conjectured that $K_3 + K_3$ also has bounded fractionality, and also conjectured, more strongly, that the feasibility and the Euler condition imply the existence of a half-integral feasible multiflow, and in particular the fractionality of $K_3 + K_3$ equals the lower bound 4. These two conjectures are also raised as Problem 52 and Problem 51 in Schrijver's book [18]; also see p. 1274. The main result of this paper solves the weaker conjecture (Problem 52) affirmatively as follows:

Theorem 1.3. *Suppose that H is $K_3 + K_3$. If* (1.1) *is feasible and satisfies the Euler condition, then there exists a* $1/12$*-integral feasible multiflow.*

This result completes the classification of the demand graphs having bounded fractionality. In particular, the fractionality of $H = K_3 + K_3$ is one of $4, 8, 12, 24$. We however do not know whether the constant 12 is tight.

$K_{n,m}$-metric-weighted maximum multiflow problem. In fact, the multiflow feasibility problem for $H = K_3 + K_3$ reduces to a certain maximization problem. As above, let G be an undirected graph with nonnegative edge-capacity c and terminal set $S \subseteq VG$. An *inner node* is a node that is not a terminal. G is said to be *inner Eulerian* (with respect to S) if c is integer-valued and for each inner node x the sum of capacity $c(e)$ over all edges e incident to x is even. Let $K_{n,m}$ be the complete bipartite graph on S. Consider the following multiflow maximization problem (*$K_{n,m}$-metric-weighted maximum multiflow problem*):

$$\text{(1.2)} \qquad \text{Maximize} \sum_{P \in \mathcal{P}} \mathrm{dist}_{K_{n,m}}(s_P, t_P)\lambda(P) \text{ over all multiflows } f = (\mathcal{P}, \lambda),$$

where s_P and t_P denote the ends of P, and $\mathrm{dist}_{K_{n,m}}$ denotes the graph metric induced by $K_{n,m}$. Suppose that the bipartition of $K_{n,m}$ is $\{A, B\}$. If a path $P \in \mathcal{P}$ is an A-path or a B-path, then P contributes $2\lambda(P)$ for the objective value of (1.2). If P connects A and B, then P contributes $\lambda(P)$.

For the case of $\min(n, m) = 2$, Karzanov and Mannoussakis [15] showed that (1.2) has an integral optimal multiflow for every inner Eulerian graph. For the case of $\min(n, m) \geq 3$, however, such an integrality result does not hold. For example, S is a six-set having $K_{3,3}$, G is a star having S as the leafs, with unit capacity. Then there is no integral optimal multiflow. We will derive the main theorem (Theorem 1.3) from:

Theorem 1.4. *There exists a* $1/12$*-integral optimal multiflow in* (1.2) *for every inner Eulerian graph.*

For any terminal weight $\mu : S \times S \to \mathbf{R}_+$, we can also define the *fractionality* of μ in a similar way. Hence the fractionality of $\mathrm{dist}_{K_{n,m}}$ is bounded. This result is an important step toward the classification of the terminal weights having bounded fractionality; see [5, 6, 7].

This paper is organized as follows. In Section 2, we describe a combinatorial duality relation (Theorem 2.1) for (1.2) due to Karzanov [13, 14], and its two optimality criterions: the first one (Lemma 2.2) is well-known and the second one (Proposition 2.3) is new. We explain a reduction of the feasibility problem for $H = K_3 + K_3$ to the maximization problem for $K_{3,3}$ in Section 2.2. The proof of the combinatorial duality relation together with the second optimality criterion is given in Section 2.3. Our proof of Theorem 1.4 is a kind of a primal-dual algorithm involving fractional splitting-off operations and dual updates, which we call *SPUP* (*Splitting-off with Potential UPdate*). In Section 3, we describe a basic idea of SPUP to get an optimal multiflow with a small denominator, and then prove the main theorem in subsequent subsections. Section 4 gives some concluding remarks.

Notation. $\mathbf{R}$ and $\mathbf{R}_+$ denote the sets of reals and nonnegative reals, respectively. Similarly, $\mathbf{Z}$ and $\mathbf{Z}_+$ denote the sets of integers and nonnegative integers, respectively. The set of functions from a set V to $\mathbf{R}$ (resp. $\mathbf{R}_+$) is denoted by $\mathbf{R}^V$ (resp. $\mathbf{R}_+^V$).

In this paper, by a graph we mean an undirected graph with possible parallel edges and loops. For a graph G, the set of vertices is denoted by VG, and the set of edges is denoted by EG. An edge e joining vertices x, y is denoted by xy. We will treat two types of graphs: one is a supply graph G in which multiflows flow, and the other one is a simple graph Γ that represents dual variables (*potentials*). To distinguish the roles of G and Γ, a vertex of a supply graph G is particularly called a *node*. We assume that a supply graph G is always endowed with a nonnegative edge-capacity c, i.e., $G = (VG, EG; c)$. The *degree* of node $x \in VG$ is the sum of $c(e)$ over all edges e incident to x. For a positive integer k, let kG denote the graph obtained from G by multiply capacity c by k, i.e., $kG = (VG, EG; kc)$.

Without noted, a path P means a simple path, i.e., there are no repeated nodes and edges in P. For subsets $A_1, A_2, \ldots A_m$ of nodes, a path P passing $A_1, A_2, \ldots, A_m$ in order is called an $(A_1, A_2, \ldots, A_m)$-path. We denote singleton set $\{a\}$ simply by a. In our problem, the terminal set S is partitioned into two sets A and B. For $a \in A$ and $b \in B$, $A \setminus a$ and $B \setminus b$ are simply denoted by $\bar{a}$ and $\bar{b}$, respectively. For a path (or a cycle) P and a function d on edges set EG, $d(P)$ denotes the sum of $d(e)$ over edges e in P.

We always consider multiflows in graphs with rational capacity and rational demand. Therefore, by allowing $\mathcal{P}$ to be a *multiset*, we can represent $f = (\mathcal{P}, \lambda)$ by a pair of a multiset $\mathcal{P}$ of S-paths and a *uniform* flow-value function $\lambda(P) = \delta$ $(P \in \mathcal{P})$ for some positive rational δ. We shall adopt this expression, denoted by $f = (\mathcal{P}; \delta)$. For an edge e, the subset of paths in $\mathcal{P}$ passing e is denoted by $\mathcal{P}(e)$, and the total sum of its flow-values is denoted by f^e, i.e., $f^e = \delta|\mathcal{P}(e)|$. Similarly, for two edges e, e', the subset of paths in $\mathcal{P}$ passing both e and e' is denoted by $\mathcal{P}(e, e')$, and the total sum of its flow-values is denoted by $f^{e,e'}$.

By a metric d on a set S we mean a function defined on $S \times S$ satisfying $d(s,t) = d(t,s) \geq d(t,t) = 0$ and the triangle inequalities $d(s,t) + d(t,u) \geq d(s,u)$ for $s,t,u \in S$. We often regard a metric d on VG of a graph G as $d : EG \to \mathbf{R}_+$ by $d(e) = d(x,y)$ for $e = xy$. For a graph $\mathit{\Gamma}$ and positive real $\gamma > 0$, let $\mathrm{dist}_{\mathit{\Gamma},\gamma}$ denote the shortest path metric on $V\mathit{\Gamma}$ by $\mathit{\Gamma}$ with respect to uniform edge-length γ. If $\gamma = 1$, then we denote $\mathrm{dist}_{\mathit{\Gamma},1}$ simply by $\mathrm{dist}_{\mathit{\Gamma}}$.

2 $K_{n,m}$-metric-weighted maximum multiflow problem

Let G be a graph with edge-capacity c and terminal set $S \subseteq VG$. Suppose that S is partitioned into two sets A and B with $\min\{|A|, |B|\} \geq 3$. Let $\mu_{A,B}$ be the metric on S defined by

$$\mu_{A,B}(s,t) = \begin{cases} 4 & \text{if } s \neq t,\ s,t \in A \text{ or } s,t \in B, \\ 2 & \text{if } (s,t) \in A \times B \text{ or } (t,s) \in A \times B, \\ 0 & \text{if } s = t, \end{cases} \qquad (s,t \in S).$$

Namely $\mu_{A,B}$ is twice the graph metric of the complete bipartite graph with bipartition $\{A, B\}$. For a multiflow $f = (\mathcal{P}; \delta)$, let $\mu_{A,B} \circ f := \sum_{P \in \mathcal{P}} \mu_{A,B}(s_P, t_P)\delta$. Instead of (1.2) we may consider the following scaled version:

(2.1) $$\text{Maximize } \mu_{A,B} \circ f \text{ over all multiflows } f.$$

The maximum value is denoted by $\mathrm{opt}(G)$.

A combinatorial duality relation. First we describe a combinatorial duality relation for (2.1). Let $\mathit{\Gamma}$ be a simple graph whose vertices $V\mathit{\Gamma}$ are

$$p^O,\ p^a,\ p^b,\ p^{ab} \quad ((a,b) \in A \times B),$$

and edges $E\mathit{\Gamma}$ are

$$p^O p^{ab},\ p^a p^{ab},\ p^b p^{ab} \quad ((a,b) \in A \times B).$$

Namely, $\mathit{\Gamma}$ is the graph obtained by subdividing the complete bipartite graph with bipartition $\{\{p^a\}_{a \in A}, \{p^b\}_{b \in B}\}$ and joining a new point p^O and each subdivided point p^{ab}. See Figure 1. Note that $\mathit{\Gamma}$ has $\mu_{A,B}$ as a submetric, i.e.,

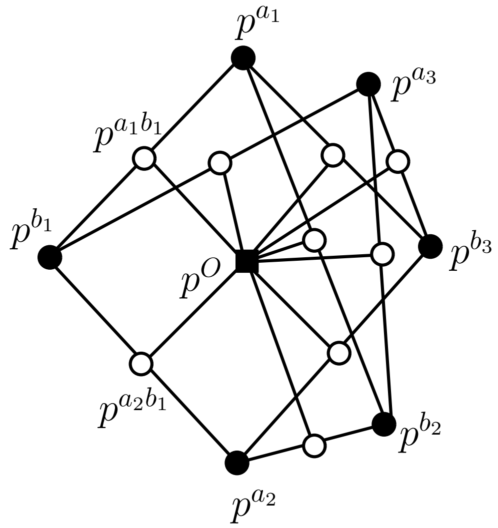

Figure 1: Graph Γ for $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$

$$\mu_{A,B}(s,t) = \mathrm{dist}_\Gamma(p^s, p^t) \quad (s, t \in S). \tag{2.2}$$

Consider the following discrete location problem on Γ:

$$\begin{array}{lll} \text{Minimize} & \displaystyle\sum_{e=xy \in EG} c(e)\mathrm{dist}_\Gamma(\rho(x), \rho(y)) \\ \text{subject to} & \rho : VG \to V\Gamma, \\ & \rho(s) = p^s \ (s \in S = A \cup B). \end{array} \tag{2.3}$$

Karzanov [13, 14] proved the following combinatorial min-max relation (see [14, p. 241]):

Theorem 2.1 ([13, 14])**.** *The maximum value of* (2.1) *is equal to the minimum value of* (2.3)*.*

We give a proof in Section 2.3; the proof technique is important for us. We call a feasible solution ρ of (2.3) a *potential*. For a potential ρ, a metric d^ρ on VG is defined by

$$d^\rho(x,y) = \mathrm{dist}_\Gamma(\rho(x), \rho(y)) \quad (x, y \in VG),$$

and the corresponding objective value $\sum_{e \in EG} c(e)d^\rho(e)$ is denoted by $d^\rho(G)$.

Optimality criterion I. Second we describe the optimality criterion of primal-dual type. For a multiflow $f = (\mathcal{P}; \delta)$ and a potential ρ, the weak duality implies

$$\mu_{A,B} \circ f \leq d^\rho(G).$$

The duality gap $d^\rho(G) - \mu_{A,B} \circ f$ is given by

$$\sum_{e \in EG} d^\rho(e)(c(e) - f^e) + \sum_{P \in \mathcal{P}} \{d^\rho(P) - \mu_{A,B}(s_P, t_P)\}\delta. \tag{2.4}$$

Note that the second term is nonnegative by (2.2). Thus we obtain an optimality criterion:

$(\rho(u),\rho(v))$	$\mathcal{P}(uv)$ consists of
(p^a,p^O)	$(a,u,v,\bar{a})$-paths
$(p^{ab},p^{a'b})$	(a,u,v,a')-paths
$(p^{ab},q^{a'b'})$	(a,u,v,a')-paths and (b,u,v,b')-paths
$(p^a,q^{a'b})$	(a,u,v,a')-paths
$(p^a,p^{a'})$	(a,u,v,a')-paths

Table 1: Types of paths in $\mathcal{P}(uv)$

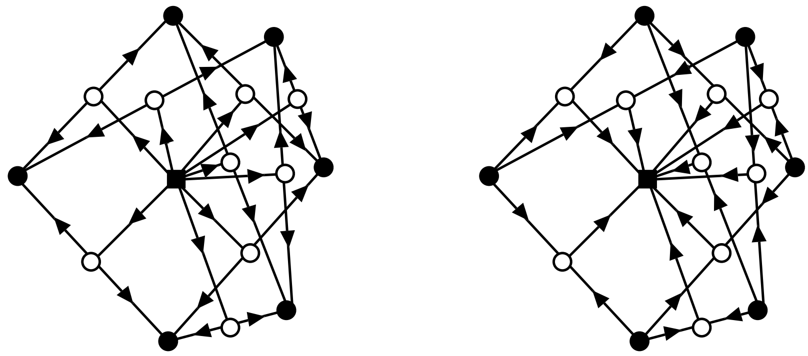
(a) forward (b) backward

Figure 2: (a) forward orientation and (b) backward orientation

Lemma 2.2. *A multiflow $f=(\mathcal{P};\delta)$ and a potential ρ are both optimal if and only if*

$$\forall e \in EG : d^\rho(e) > 0 \;\Rightarrow\; f^e = c(e),$$
$$\forall P \in \mathcal{P} \;\Rightarrow\; d^\rho(P) = \mu_{A,B}(s_P,t_P).$$

Let $f=(\mathcal{P};\delta)$ and ρ be an optimal multiflow and an optimal potential, respectively. Let $e=uv$ be an edge and P an (s,u,v,t)-path in $\mathcal{P}(e)$. The second condition in the previous lemma says that P is mapped to a shortest path connecting p^s and p^t in Γ by ρ. Therefore the ends s and t of P must satisfy

$$\mathrm{dist}_\Gamma(p^s,p^t) = \mathrm{dist}_\Gamma(p^s,\rho(u)) + \mathrm{dist}_\Gamma(\rho(u),\rho(v)) + \mathrm{dist}_\Gamma(\rho(v),p^t). \tag{2.5}$$

From this relation we can (sometime completely) determine the ends of paths in $\mathcal{P}(e)$. Some of them are summarized in Table 1.

Optimality criterion II. Third we describe another optimality criterion involving potentials only. We endow Γ with two orientations. The *forward orientation* of Γ is an orientation such that p^s are sinks and p^O is the unique source. The *backward orientation* of Γ is the reverse of the forward orientation. See Figure 2. For a potential ρ, a potential ρ' is called a *forward neighbor* of ρ if for each $x\in VG$ with $\rho(x)\neq\rho'(x)$, $\overrightarrow{\rho(x)\rho'(x)}$ is an edge of the forward orientation, or $(\rho(x),\rho'(x))=(p^O,p^s)$ for some $s\in S$. Similarly, a potential ρ' is called a *backward neighbor* of ρ if for each $x\in VG$ with $\rho(x)\neq\rho'(x)$, $\overrightarrow{\rho(x)\rho'(x)}$ is an edge of the backward orientation, or $(\rho(x),\rho'(x))=(p^s,p^O)$ for some $s\in S$. A forward or backward neighbor is also called a *neighbor*.

Proposition 2.3. *A potential ρ is not optimal if and only if there exists a neighbor ρ' of ρ with $d^{\rho'}(G) < d^\rho(G)$.*

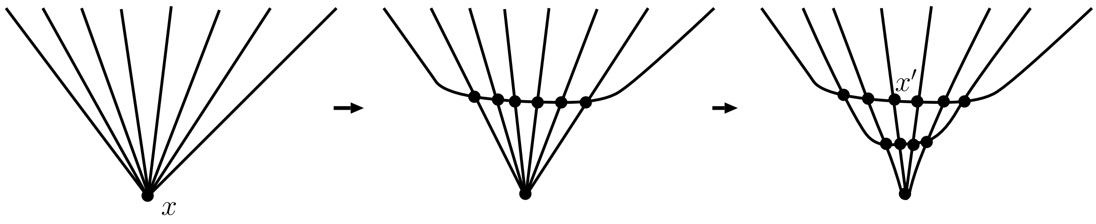

Figure 3: Reduction of an inner node

Namely we can check the optimality of a given potential ρ by evaluating $d^{\rho'}(G)$ only for neighbors ρ' of ρ. The proof is given in Section 2.3.

Uncrossing lemma. For an optimal potential ρ, let C_ρ denote the set of nodes $y \in VG$ with $\rho(y) = p^O$. We will see that nodes in C_ρ have difficulty for our splitting-off procedure. For two optimal potentials ρ_1, ρ_2, the following lemma, called *uncrossing lemma*, produces third optimal potential ρ decreasing nodes in C_{ρ_1}.

Lemma 2.4. *For two optimal potentials ρ_1, ρ_2, there exists an optimal forward neighbor ρ of ρ_1 with $C_\rho \subseteq C_{\rho_1} \cap C_{\rho_2}$.*

This plays a key role in our splitting-off procedure. The proof is given in Section 2.3.

2.1 Euler condition and degree reduction

Recall that graph G is called inner Eulerian if capacity c is integral and the degree of each inner node is even.

Lemma 2.5. *Suppose that G is inner Eulerian. For two potentials ρ, ρ', difference $d^{\rho'}(G) - d^{\rho}(G)$ is an even integer.*

Proof. Since G is inner Eulerian, there are cycles $C_1, C_2, \ldots, C_k$ and S-paths $P_1, P_2, \ldots, P_l$ such that

$$(2.6) \qquad d^{\rho'}(G) - d^{\rho}(G) = \sum_{i=1}^{k}\{d^{\rho'}(C_i) - d^{\rho}(C_i)\} + \sum_{j=1}^{l}\{d^{\rho'}(P_j) - d^{\rho}(P_j)\}.$$

Since Γ is bipartite, both $d^{\rho'}(C_i) - d^{\rho}(C_i)$ and $d^{\rho'}(P_j) - d^{\rho}(P_j)$ are even. □

There is a standard method reducing (2.1) to the problem on a graph with small-degree; see [4, p. 50] for example. Suppose that G is inner Eulerian. By multiplying edges, we can make each edge have unit capacity. Take an inner node $x \in VG$ of degree greater than 4. Transform G into G' by changing the incidence at x as in Figure 3.

Then we can easily see that any $1/k$-integral multiflow in G' can be transformed into a $1/k$-integral multiflow in G having the same objective value, and any $1/k$-integral multiflow in G can also be transformed into a $1/k$-integral multiflow in G' having the same objective value. Furthermore,

(2.7) any optimal potential ρ for G is extended to an optimal potential ρ for G' by setting $\rho(x') := \rho(x)$ for each new node x' in G',

which is an easy consequence of the optimality criterion I (Lemma 2.2).

2.2 Reducing the feasibility problem for $K_3 + K_3$ to the maximization problem for $K_{3,3}$

Here we show that $1/k$-integrality of (2.1) implies $1/k$-integrality of (1.1). Let G be a graph with capacity c and terminal set $S = \{s_1, s_2, s_3, t_1, t_2, t_3\}$. Let $H = (S, R)$ be a demand graph with $R = \{s_i s_j\}_{1 \leq i < j \leq 3} \cup \{t_i t_j\}_{1 \leq i < j \leq 3}$ and let q be a demand function on R. Construct a new graph G' from G by adding new terminal set $S' = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ with $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$ and by adding edge $a_i s_i$ of capacity $q(s_i s_j) + q(s_i s_k)$ and edge $b_i t_i$ of capacity $q(t_i t_j) + q(t_i t_k)$ for distinct i, j, k. Then G' is inner Eulerian (with respect to S') if (1.1) satisfies the Euler condition for (G, H, q). Consider maximization problem (2.1) for G', S'.

Suppose that (1.1) is feasible. We first prove that the following potential $\rho : VG' \to V\Gamma$ is optimal to (2.1):

$$\rho(x) = \begin{cases} p^x & \text{if } x \in S', \\ p^O & \text{otherwise}, \end{cases} \qquad (x \in VG').$$

Indeed, take any feasible multiflow f in (1.1). For each path P in f, if P connects s_i and s_j (resp. t_i and t_j), then extend P by adding edges $a_i s_i$ and $s_j a_j$ (resp. $b_i t_i$ and $t_j b_j$). Let f' be the resulting multiflow for G', S'. By construction, (f', ρ) fulfills the optimality criterion (Lemma 2.2).

Next suppose that there is a $1/k$-integral optimal multiflow $f^* = (\mathcal{P}; 1/k)$ in (2.1). Since ρ, f^* are optimal, by Lemma 2.2 we have $(f^*)^{a_i s_i} = c(a_i s_i) = q(s_i s_j) + q(s_i s_k)$ and $(f^*)^{b_i t_i} = c(b_i t_i) = q(t_i t_j) + q(t_i t_k)$ for distinct i, j, k. By Table 1, $\mathcal{P}(a_i s_i)$ consists of $(a_i, \{a_j, a_k\})$-paths, and $\mathcal{P}(b_i t_i)$ consists of $(b_i, \{b_j, b_k\})$-paths. Consequently f^* consists of (a_i, a_j)-paths of the total flow-value $q(s_i s_j)$ for $1 \leq i < j \leq 3$ and (b_i, b_j)-paths of the total flow-value $q(t_i t_j)$ for $1 \leq i < j \leq 3$. Restricting f^* to G, we get a $1/k$-integral feasible multiflow for (1.1). Hence Theorem 1.4 implies Theorem 1.3.

2.3 Proof

The goal of this section is to prove Theorem 2.1, Proposition 2.3, and Lemma 2.4. As is well-known in the multiflow theory [16], the LP-dual to (2.1) is given by:

$$(2.8) \qquad \begin{array}{rl} \text{Minimize} & \displaystyle\sum_{e \in EG} c(e) d(e) \\ \text{subject to} & d\text{: metric on } VG, \\ & d(s,t) = \mu_{A,B}(s,t) \quad (s, t \in S). \end{array}$$

We are going to show that every extreme solution d of this LP can be represented as $d = d^\rho$ for some potential ρ in (2.3).

Metrized polyhedral complex $\mathcal{T}_{A,B}$. As in [13, 14], we construct a metric space $\mathcal{T}_{A,B}$ with property that every minimal feasible solution in (2.8) is isometrically embedded into it. This metric space is nothing but the *tight span* of $\mu_{A,B}$, introduced independently by Isbell [9] and Dress [2].

Let $\mathcal{T}_{A,B}$ be the polyhedral subset in $\mathbf{R}_+^{A \cup B}$ consisting of points q with $\sum_{s \in A \cup B} q(s) \leq 2$ and $\{s \in A \cup B \mid q(s) > 0\} \subseteq \{a, b\}$ for some $(a, b) \in A \times B$. For $s \in A \cup B$ let q^s be the point defined by $q^s(u) = 2$ if $s = u$ and $q^s(u) = 0$ otherwise, and let q^O be the origin ($q^O(u) = 0$ for $u \in S$). Let $V\mathcal{T}_{A,B} := \{q^O\} \cup \{q^a\}_{a \in A} \cup \{q^b\}_{b \in B}$. Let σ_{ab} denote the convex hull of q^O, q^a, q^b for $(a, b) \in A \times B$. Then $\mathcal{T}_{A,B}$ is the union of 2-dimensional cell (2-cell) σ_{ab} over all $(a, b) \in A \times B$. Combinatorially speaking, $\mathcal{T}_{A,B}$ is the join of one point

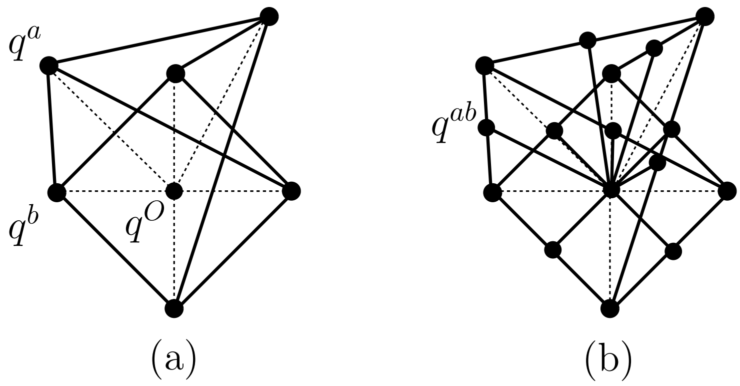

Figure 4: (a) $\mathcal{T}_{A,B}$ and (b) Γ^1

and the complete bipartite graph with bipartition $\{A, B\}$. See Figure 4 (a). We endow $\mathcal{T}_{A,B}$ with metric $d_{\mathcal{T}_{A,B}}$ by the following way. For a path P (one-dimensional curve) in $\mathcal{T}_{A,B}$, its length is measured by the l_∞-distance on $\mathbf{R}^{A \cup B}$, where the l_∞-distance of two points p, q is defined by $\|p - q\|_\infty = \max_{s \in A \cup B} |p(s) - q(s)|$. For two points $p, q \in \mathcal{T}_{A,B}$, the metric $d_{\mathcal{T}_{A,B}}(p, q)$ is defined by the infimum of the length of paths connecting p and q in $\mathcal{T}_{A,B}$. Note that this metric is *not* equal to the restriction of $(\mathbf{R}^{A \cup B}, l_\infty)$ to $\mathcal{T}_{A,B}$.

For distinct $a, a' \in A, b, b' \in B$, the union $\sigma_{ab} \cup \sigma_{ab'} \cup \sigma_{a'b} \cup \sigma_{a'b'}$ is called an *apartment*, which is isometric to the square $\{(x_1, x_2) \in \mathbf{R}^2 \mid -2 \leq x_1 \pm x_2 \leq 2\}$ in $(\mathbf{R}^2, l_\infty)$. Recall that the l_∞-plane is isometric to the l_1-plane. So an apartment is also isometric to the square $\{(x_1, x_2) \in \mathbf{R}^2 \mid -1 \leq x_1, x_2 \leq 1\}$ in $(\mathbf{R}^2, l_1)$.

Every pair of points p, q is joined by a shortest path within some apartment. From this, we see

$$d_{\mathcal{T}_{A,B}}(q^s, q^t) = \mu_{A,B}(s, t) \quad (s, t \in S). \tag{2.9}$$

Namely $\mu_{A,B}$ is isometrically embedded into $\mathcal{T}_{A,B}$ by $s \mapsto q^s$.

Let us return back to the study of LP (2.8). For a map $\rho : VG \to \mathcal{T}_{A,B}$, let d^ρ be the metric on VG defined by $d^\rho(x, y) = d_{\mathcal{T}_{A,B}}(\rho(x), \rho(y))$ for $x, y \in VG$. By (2.9) we have:

(2.10) For any map $\rho : VG \to \mathcal{T}_{A,B}$ satisfying $\rho(s) = q^s$ for $s \in S$, the metric d^ρ is feasible to (2.8).

Conversely every minimal solution in (2.8) can be represented in this way:

(2.11) For any metric d feasible to (2.8), there exists a map $\rho : VG \to \mathcal{T}_{A,B}$ such that $\rho(s) = q^s$ $(s \in S)$ and $d^\rho(x, y) \leq d(x, y)$ $(x, y \in VG)$.

This is a special case of [13, Theorem 4.2]. We give a short proof. For a point $q \in \mathcal{T}_{A,B}$ and a nonnegative real $r \geq 0$, let $B(q, r)$ be the set of points q' with $d_{\mathcal{T}_{A,B}}(q', q) \leq r$, i.e., it is the *ball* with center q and radius r. Here we claim:

(2.12) The collection of balls in $\mathcal{T}_{A,B}$ has the Helly property.

Assuming this property, we prove (2.11). Let d be a metric feasible to (2.8). Let

$VG = \{x_1, x_2, \ldots, x_n\}$ and $S = \{x_1, x_2, \ldots, x_k\}$. Define $\rho : VG \to \mathcal{T}_{A,B}$ recursively by

$$\rho(x_i) := \begin{cases} q^{x_i} & \text{if } i \le k, \\ \text{an arbitrary point in } \bigcap_{j=1}^{i-1} B(\rho(x_j), d(x_j, x_i)) & \text{if } k < i \le n, \end{cases} \quad (i = 1, 2, \ldots, n).$$

By (2.9), we have $d^\rho(x_i, x_j) = d(x_i, x_j) = \mu_{A,B}(x_i, x_j)$ for $1 \le i < j \le k$. We prove by induction that $\bigcap_{j=1}^{i-1} B(\rho(x_j), d(x_j, x_i))$ is nonempty for $k < i \le n$. If true, then $\rho(x_i) \in B(\rho(x_j), d(x_j, x_i))$ implies $d^\rho(x_j, x_i) \le d(x_j, x_i)$, as required. By the Helly property (2.12), it suffices to verify pairwise nonempty intersection $B(\rho(x_{j'}), d(x_{j'}, x_i)) \cap B(\rho(x_j), d(x_j, x_i)) \neq \emptyset$ for $j' < j < i$. Here two balls $B(q, r)$ and $B(q', r')$ intersect if and only if $d_{\mathcal{T}_{A,B}}(q, q') \le r + r'$. Therefore the nonemptyness of $B(\rho(x_{j'}), d(x_{j'}, x_i)) \cap B(\rho(x_j), d(x_j, x_i))$ follows from $d(x_{j'}, x_i) + d(x_j, x_i) \ge d(x_{j'}, x_j) \ge d_{\mathcal{T}_{A,B}}(\rho(x_{j'}), \rho(x_j))$, where the last inequality follows from the induction. Now the proof of (2.11) is complete.

Sketch of the proof of (2.12). The Helly property (2.12) was shown by Chepoi [1, Section 7] for a more general class of metrized complexes; also see [6]. So we sketch it. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be a collection of balls having pairwise nonempty intersection $B_i \cap B_j \neq \emptyset$. Consider the intersection $B_i \cap \mathcal{A}$ of ball B_i and an apartment $\mathcal{A}$. Regard $\mathcal{A}$ as a square $\{(x_1, x_2) \in \mathbf{R}^2 \mid -2 \le x_1 \pm x_2 \le 2\}$ in the l_∞-plane. Then we see that if $B_i \cap \mathcal{A} \neq \emptyset$, then $B_i \cap \mathcal{A} = R \cap \mathcal{A}$ for some rectangle $R \subseteq \mathbf{R}^2$ each of whose edge is parallel to a coordinate axis. From this property, we see that $(*)$ if B_i meets both σ_{ab} and $\sigma_{a'b'}$ for distinct a, a', b, b', then B_i includes q^O. For $s \in S$, let T^s be the union of 2-cells containing q^s. By $(*)$, we see that there is $(a, b) \in A \times B$ such that each ball not containing q^O is contained by T^a or T^b. Consequently, for some 2-cell σ $(\subseteq T^a \cup T^b)$, each pair of balls in $\mathcal{B}$ intersects at σ. So checking Helly property of $\mathcal{B}$ reduces to that of $\{B_i \cap \sigma\}_{i=1}^m$; it is easy to verify $\bigcap_{i=1}^m (B_i \cap \sigma) \neq \emptyset$. □

Drawing grids on $\mathcal{T}_{A,B}$ and constructing convex combination. For a positive integer k, a point $q \in \mathcal{T}_{A,B}$ is said to be *$1/k$-integral* if $q(s) + q(t) \in 2\mathbf{Z}/k$ for all $s, t \in S$ (possibly $s = t$). The set of $1/k$-integral points is denoted by $V^k\mathcal{T}_{A,B}$. We are going to show that *the metric on $1/k$-integral points can be decomposed into a convex combination of the metrics on integral points.*

Let Γ^k be the graph on vertex set $V^k\mathcal{T}_{A,B}$ with edge set $\{pq \mid d_{\mathcal{T}_{A,B}}(p, q) = 1/k\}$. In particular $V^1\mathcal{T}_{A,B}$ consists of q^O, q^a, q^b and the midpoint q^{ab} of q^a and q^b. Thus Γ^1 is isomorphic to Γ by $q^u \mapsto p^u$. Graph Γ^k is drawn in $\mathcal{T}_{A,B}$ so that its restriction to each apartment is a grid graph each of whose edge is parallel to a coordinate axis in the local l_1-plane; see Figure 5. For any pair of $1/k$-integral points p, q, we can take a shortest path P connecting p, q such that P lies on the edges of Γ^k (as in the proof of [5, Proposition 4.2]). So we have

$$d_{\mathcal{T}_{A,B}}(p, q) = \mathrm{dist}_{\Gamma^k, 1/k}(p, q) \quad (p, q \in V^k\mathcal{T}_{A,B} = V\Gamma^k).$$

Next we introduce the notion of orbits [13] to decompose dist_{Γ^k}. Two edges $e, e' \in E\Gamma^k$ are called *mates* if there is a 4-cycle containing e and e' as a nonadjacent pair. Two edges $e, e' \in E\Gamma^k$ are called *projective* if there is a sequence of edges $e = e_1, e_2, \ldots, e_m = e'$ such that e_i and e_{i+1} are mates. The projectiveness defines an equivalence relation on $E\Gamma^k$. An equivalence class is called an *orbit*. Γ^k has k orbits $\{O_1, O_2, \ldots, O_k\}$. Order $O_1, O_2, \ldots, O_k$ so that $i < j$ if and only if O_i is closer to q^O than O_j. For an orbit O_i, the *orbit graph* Γ_i^k is the graph obtained by contracting all edges not in O_i and deleting multiple edges and loops appearing. Then the orbit graph Γ_i^k is isomorphic to $\Gamma^1 = \Gamma$.

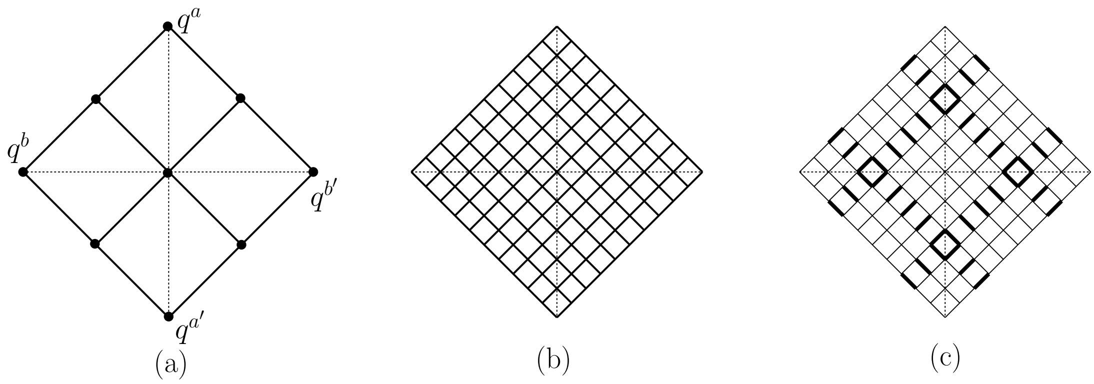

Figure 5: (a) apartment with Γ^1, (b) apartment with Γ^5, and (c) orbit O_3

By construction we obtain a unique map $\phi_i : V\Gamma^k \to V\Gamma$ with property that $\phi_i(q^s) = p^s$ ($s \in S$) and $\phi_i(p)$ is the contracted vertex. Then the following decomposition property holds:

$$\mathrm{dist}_{\Gamma^k,1/k}(p,q) = \frac{1}{k}\sum_{i=1}^{k} \mathrm{dist}_{\Gamma^1}(\phi_i(p),\phi_i(q)) \quad (p,q \in V\Gamma^k). \tag{2.13}$$

Indeed, consider zigzag shortest path P within some apartment, and consider $\phi_i(P)$ ($i = 1,2,\ldots,k$), which is also shortest in Γ; this is a special case of [14, Statement 2.2]. Now we are ready to prove Theorem 2.1, Proposition 2.3 and Lemma 2.4.

Proof of Theorem 2.1. Take a rational metric d feasible to LP (2.8). It suffices to show the existence of a potential ρ^* in (2.3) with $d^{\rho^*} \leq d$. By (2.11), there is a map $\rho : VG \to \mathcal{T}_{A,B}$ such that $\rho(s) = q^s$ ($s \in S$) and $d^\rho \leq d$. By rationality of d and the construction of ρ, we can take such a map ρ with $\rho(VG) \subseteq V^k\mathcal{T}_{A,B}$ for $k > 0$. Now we can regard ρ as $VG \to V\Gamma^k$. Consider orbits in Γ^k, and maps ϕ_i. By (2.13), we have $d^\rho = (1/k)\sum_{i=1}^k d^{\phi_i\circ\rho}$. So there is an index i with $d^{\phi_i\circ\rho} \leq d^\rho$. Then $\phi_i \circ \rho : VG \to V\Gamma$ is a desired potential.

Proof of Proposition 2.3. It suffices to show the only-if part. Take a potential $\rho : VG \to V\Gamma$. Regard ρ as $VG \to V^1\mathcal{T}_{A,B}$. Suppose that a potential ρ is not optimal. This implies that d^ρ is not optimal to (2.8). By convexity, for a sufficiently small $0 \leq \epsilon < 1/2$, there is a rational metric d feasible to (2.8) such that $\sum_{e\in EG} c(e)d(e) < \sum_{e\in EG} c(e)d^\rho(e)$, and $|d(x,y) - d^\rho(x,y)| \leq \epsilon$ for $x,y \in VG$. By (2.11) there is a map $\rho^* : VG \to V\mathcal{T}_{A,B}$ such that $\rho^*(s) = q^s$ ($s \in S$) and $d^{\rho^*} \leq d$. Here we claim

$$d_{\mathcal{T}_{A,B}}(\rho(x),\rho^*(x)) \leq \epsilon \quad (x \in VG). \tag{2.14}$$

Indeed, suppose $\rho(x) \neq \rho^*(x)$. Then, for some $s \in S$ we have $d_{\mathcal{T}_{A,B}}(q^s,\rho^*(x)) > d_{\mathcal{T}_{A,B}}(q^s,\rho(x))$. From this $d_{\mathcal{T}_{A,B}}(\rho(x),\rho^*(x)) \leq d_{\mathcal{T}_{A,B}}(q^s,\rho^*(x)) - d_{\mathcal{T}_{A,B}}(q^s,\rho(x)) = (d(s,x) - d^\rho(s,x)) + (d^{\rho^*}(s,x) - d(s,x)) \leq \epsilon$. Again we may assume that $\rho^*(VG) \subseteq V^k\mathcal{T}_{A,B} = V\Gamma^k$. Consider the orbits in Γ^k and the maps ϕ_i. Then we have $d^{\rho^*} = (1/k)\sum_{i=1}^k d^{\phi_i\circ\rho^*}$, and there is an index i with $d^{\phi_i\circ\rho^*}(G) \leq d^\rho(G)$. Then $(\phi_i\circ\rho^*)(x) \neq \rho(x)$ if and only if a shortest path between $\rho^*(x)$ and $\rho(x)$ crosses O_i. The balls $B(q,\epsilon)$ ($q \in V^1\mathcal{T}_{A,B}$) are pairwise disjoint by $\epsilon < 1/2$, and each $\rho^*(x)$ belongs to $B(\rho(x),\epsilon)$ by (2.14). Suppose $1 \leq i \leq k/2$. Then O_i does not meet $B(p^s,\epsilon)$, and the change $\rho(x) \to (\phi_i\circ\rho^*)(x)$ is one of $p^O \to p^{ab}$, $p^O \to p^a$, $p^O \to p^b$, $p^{ab} \to p^a$, and $p^{ab} \to p^b$.

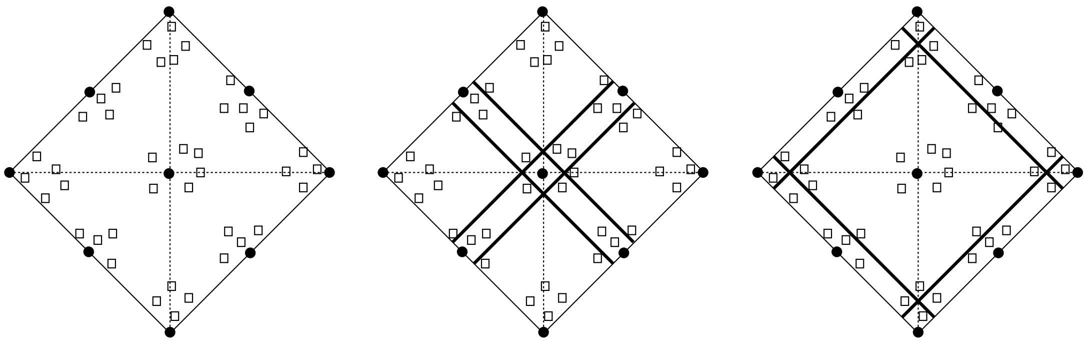

Figure 6: Perturbing d^ρ

This implies that $\phi_i \circ \rho^*$ is a forward neighbor. Suppose $k/2 < i \leq k$. Then O_i does not meet $B(p^O, \epsilon)$, and the change occurs in the reverse way. This implies that $\phi_i \circ \rho^*$ is a backward neighbor. Figure 6 illustrates this situation restricted to some apartment. In this figure, a small square box represents $\rho'(x)$, which belongs to the ball with center $\rho(x)$ (black dot point) and radius $\epsilon < 1/2$. Consider orbit O_i, which is represented by bold lines in (b) for $1 \leq i \leq k/2$ and in (c) for $k/2 < i \leq k$.

Proof of Lemma 2.4. Let ρ_1, ρ_2 be optimal potentials. We use a similar perturbation idea. Take a sufficiently small rational $\epsilon > 0$. Let $d := (1-\epsilon)d^{\rho_1} + \epsilon d^{\rho_2}$. Then d is optimal to (2.8). According to (2.11), we can take $\rho : VG \to \mathcal{T}_{A,B}$ such that $\rho(s) = q^s$ ($s \in S$) and $d^\rho \leq d$. We may assume $\rho(VG) \subseteq V^k\mathcal{T}_{A,B}$. Of course d^ρ is optimal to (2.8). Consider orbits and maps ϕ_i as above. Decompose ρ into $\phi_i \circ \rho$ ($i = 1, 2, \ldots, k$). They are all optimal to (2.3). We show that $\phi_1 \circ \rho$ is a required neighbor.

Since $|d^{\rho_1}(x,y) - d(x,y)|$ is sufficiently small ($< 1/2$), by the same argument as above, $\phi_1 \circ \rho$ is a forward neighbor of ρ_1, and hence $C_{\phi_1 \circ \rho} \subseteq C_{\rho_1}$. Take $x \in VG$ with $\rho_1(x) = p^O$ and $\rho_2(x) \neq p^O$. It suffices to show $\rho(x) \neq q^O$; this implies $\phi_1 \circ \rho(x) \neq p^O$. We may assume $\rho_2(x) = p^a$ or p^{ab}. Since $d^{\rho_1}(a,x) = 2$ and $d^{\rho_2}(a,x) \in \{0,1\}$, we have $d(a,x) = 2 - \epsilon(2 - d^{\rho_2}(a,x)) < 2$. Consider the ball $B(q^a, d(a,x))$, which does not contain q^O by $d_{\mathcal{T}_{A,B}}(q^a, q^O) = 2$. On the other hand, $d_{\mathcal{T}_{A,B}}(q^a, \rho(x)) = d^\rho(a,x) \leq d(a,x)$ implies that $B(q^a, d(a,x))$ includes $\rho(x)$. Thus $q^O \neq \rho(x)$.

3 Fractional splitting-off

Let G be an graph with terminal set S and unit edge-capacity (allowing multiple edges and loops). Let us introduce the fractional splitting-off operation. For two consecutive edges e and e' incident to y, a triple (e, y, e') is called a *fork*. For a fork $\tau = (e, y, e')$ and $\alpha \in [0, 2]$, the *fractional splitting-off* operation is to add a new node y^τ, reconnect e and e' to y^τ, and join y and y^τ by a new edge $e^\tau = yy^\tau$ of capacity $c(e^\tau) = 2 - \alpha$. The resulting graph is denoted by $G^{\tau,\alpha}$; see Figure 7. We obtain a multiflow in G from any multiflow in $G^{\tau,\alpha}$ by contracting edge e^τ. Conversely we obtain a multiflow in $G^{\tau,0}$ from any multiflow in G, since the amount of flows coming from e, e' is at most 2. In particular, $\mathrm{opt}(G^{\tau,\alpha}) \leq \mathrm{opt}(G^{\tau,0}) = \mathrm{opt}(G)$. The maximum possible $\alpha \in [0,2]$ with $\mathrm{opt}(G) = \mathrm{opt}(G^{\tau,\alpha})$ is denoted by $\alpha_\tau = \alpha_\tau(G)$, and is called the *splitting capacity*. If $\alpha_\tau = 2$, then we say "*τ is splittable*" and we simply let $G^{\tau,2}$ be the graph obtained

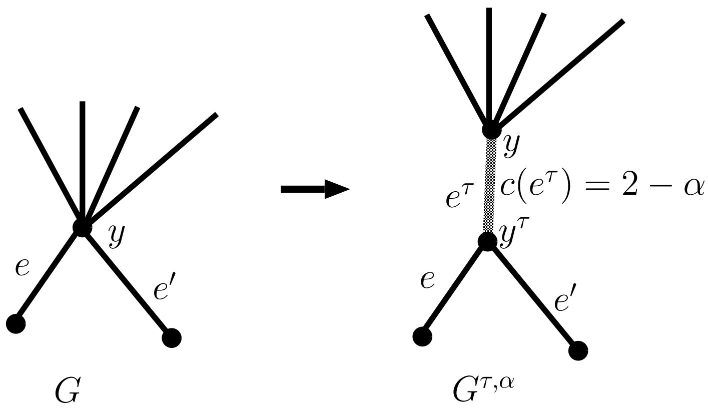

Figure 7: Fractional splitting-off

by deleting edge e^τ from $G^{\tau,0}$. If $\alpha_\tau < 2$, then we say "*τ is unsplittable*". Two forks $(e_1, y, e_2), (e_1', y', e_2')$ are said to be *disjoint* if $y \neq y'$ or all e_1, e_2, e_1', e_2' are distinct.

Our proof scheme is to choose pairwise disjoint forks $\tau_0, \tau_1, \ldots, \tau_{m-1}$ in G and to produce graphs $G = G_0, G_1, G_2, \ldots, G_m$ such that

(i) $G_{i+1} = (G_i)^{\tau_i, \alpha_i}$ for $\alpha_i = \alpha_{\tau_i}(G_i)$, and

(ii) kG_m has an integral optimal multiflow for an integer $k > 0$.

Recall that kG_m denotes the graph obtained from G_m by multiplying capacity by k. Since τ_i, τ_j $(i < j)$ are disjoint, fork τ_j is well-defined in G_{j-1}. From a $1/k$-integral optimum f in G_m, by reversing the operations we obtain a $1/k$-integral optimal multiflow in the initial graph $G = G_0$. How can we guarantee condition (ii) ? A particular lucky situation is: τ_i is splittable in G_i for each i, and G_m has no inner node of degree greater than 2. Then G_m has an integral optimal multiflow. Indeed G_m is the union of cycles and S-paths so that they are edge-disjoint and node-disjoint at $VG_m \setminus S$, each cycle meets at most one terminal, and each S-path meets exactly two terminals. Since $\mu_{A,B}$ is a metric, a multiflow consisting of these S-paths with unit flow-value is obviously optimal.

However we cannot expect such a lucky situation since our problem admits no integral optimal multiflow in general. We will see that an optimal potential can be used as a powerful *certificate* for condition (ii). We will keep an optimal potential during the splitting-off process, according to the following property:

(3.1) Let ρ be an optimal potential for G, τ a fork at node y, and $\alpha \in [0, \alpha_\tau]$. Extend ρ to $VG^{\tau,\alpha} \to V\Gamma$ by setting $\rho(y^\tau) := \rho(y)$. Then the resulting ρ is optimal for $G^{\tau,\alpha}$.

This follows from $\mathrm{opt}(G) = \mathrm{opt}(G^{\tau,\alpha}) \leq d^\rho(G^{\tau,\alpha}) = d^\rho(G) = \mathrm{opt}(G)$ (since $d^\rho(e^\tau) = 0$). In particular we can always extend an optimal potential for G to an optimal potential for $G^{\tau,0}$. The starting point of our scheme is a formula of α_τ in terms of neighbors.

Proposition 3.1. *Let τ be an unsplittable fork and ρ an optimal potential. Then we have the following.*

$$\alpha_\tau = \min \left\{ (d^{\rho'}(G^{\tau,0}) - d^\rho(G^{\tau,0}))/d^{\rho'}(e^\tau) \;\middle|\; \rho' \text{: neighbor of } \rho \text{ with } d^{\rho'}(e^\tau) > 0 \right\},$$

In particular, if G is inner Eulerian, then we have

$$\alpha_\tau \in \left\{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}\right\}.$$

In the formula, ρ is extended to $VG^{\tau,0} \to V\Gamma$ according to (3.1). A neighbor ρ' attaining α_τ is called *critical.* Note that both ρ and ρ' are optimal for G^{τ,α_τ}.

Proof. Note that $\mathrm{opt}(G) = d^\rho(G) = d^\rho(G^{\tau,\alpha})$ for every $\alpha \in [0,2]$. Therefore Proposition 2.3 implies that $\mathrm{opt}(G^{\tau,\alpha}) = \mathrm{opt}(G)$ if and only if $d^\rho(G^{\tau,\alpha}) \leq d^{\rho'}(G^{\tau,\alpha})$ holds for every neighbor ρ' of ρ. From this fact together with $d^{\rho'}(G^{\tau,\alpha}) - d^\rho(G^{\tau,\alpha}) = d^{\rho'}(G^{\tau,0}) - d^\rho(G^{\tau,0}) - \alpha d^{\rho'}(e^\tau)$, we obtain the desired formula. The latter part immediately follows from $\mathrm{dist}_\Gamma(p,q) \in \{0,1,2,3,4\}$ and $d^{\rho'}(G^{\tau,0}) - d^\rho(G^{\tau,0}) \in 2\mathbf{Z}$ (Lemma 2.5). □

An overview of the proof. Here we describe an overview of the proof of the main theorem (Theorem 1.4). Suppose that we are given an inner Eulerian graph G with unit capacity and an optimal potential ρ. For a fork τ at inner node y, the denominator of α_τ crucially depends on the position $\rho(y)$ in Γ; if $\rho(y) \in \{p^a, p^b, p^{ab}\}$ then $d^{\rho'}(e^\tau)$ takes $\{1,2\}$, and if $\rho(y) = p^O$ then $d^{\rho'}(e^\tau)$ takes $\{1,2,3,4\}$. Motivated by this fact, we partition VG into the following three sets, according to ρ:

$$\begin{aligned} S_\rho &= \{y \in VG \mid \rho(y) = p^s \text{ for some } s \in S\}, \\ M_\rho &= \{y \in VG \mid \rho(y) = p^{ab} \text{ for some } (a,b) \in A \times B\}, \\ C_\rho &= \{y \in VG \mid \rho(y) = p^O\}. \end{aligned} \tag{3.2}$$

Nodes in S_ρ have a particular nice property, which we will show in Section 3.2, that *if $y \in S_\rho$, then y has a splittable fork* (Proposition 3.6). An immediate corollary is a powerful certificate for the existence of an integral optimal multiflow:

(3.3) If $M_\rho \cup C_\rho = \emptyset$, then there exists an integral optimal multiflow.

We will try to decrease nodes in $M_\rho \cup C_\rho$ by the following way. Take a node $y \in C_\rho$ and a fork τ at y. Take a critical neighbor ρ' of ρ with respect to τ. Then ρ' is forward since any backward neighbor ρ'' fulfills $d^{\rho''}(e^\tau) = 0$. Suppose $\alpha_\tau = 3/2$ (say). Then $d^{\rho'}(e^\tau) = 4$ and thus $(\rho'(y), \rho'(y^\tau)) = (p^a, p^{a'})$ (or $(p^b, p^{b'})$). Update $G \leftarrow G^{\tau,\alpha_\tau}$ and $\rho \leftarrow \rho'$. Then *the cardinality of $M_\rho \cup C_\rho$ strictly decreases.* Therefore, if $M_\rho \cup C_\rho = \emptyset$ (luckily), then $4G$ has an integral optimal multiflow by (3.3), and thus the initial graph has a 1/4-integral optimal multiflow. So consider the case where there still exists a node $x \in C_\rho$; $x \neq y$. Again, take a fork τ' at x, and consider $\alpha_{\tau'}$. Now G has edge e^τ of capacity 1/2, and hence is not inner Eulerian. Nevertheless $\alpha_{\tau'}$ still takes values $0, 1/2, 2/3, 1, 4/3, 3/2$. Why ? Consider a critical neighbor ρ' of ρ with respect to τ', and compare ρ' with ρ. Since ρ' is forward, $(\rho(y^\tau), \rho(y)) = (\rho'(y^\tau), \rho'(y)) = (p^a, p^{a'})$ holds. Therefore $c(e^\tau)d^\rho(e^\tau)$ and $c(e^\tau)d^{\rho'}(e^\tau)$ cancel out in $d^{\rho'}(G^{\tau',0}) - d^\rho(G^{\tau',0})$. Since the deletion of e^τ makes G inner Eulerian, *the difference $d^{\rho'}(G^{\tau',0}) - d^\rho(G^{\tau',0})$ is an even integer,* and thus α_τ is half- or 2/3-integral. This observation suggests a possibility to repeat such a procedure until $M_\rho \cup C_\rho = \emptyset$ with bounding the denominator of $c(e^\tau)$ of edges e^τ produced.

Our proof is based on this idea. We always keep a graph G together with its optimal potential ρ; we denote it by $(G;\rho)$. We will pick a node $x \in M_\rho \cup C_\rho$, and a fork τ at x (of degree at least four). If τ is splittable, then update $G \leftarrow G^{\tau,2}$, and keep ρ, that is also optimal to the new graph. Suppose that τ is unsplittable. Then take a critical neighbor ρ', and update the graph together with the optimal potential $(G;\rho) \leftarrow (G^{\tau,\alpha_\tau};\rho')$. We call this operation the *SPUP (Splitting-off with Potential-UPdate)* at τ with respect to a critical neighbor ρ'; we also call it α-*SPUP* if $\alpha = \alpha_\tau$. In particular, if ρ' is forward, the corresponding SPUP is said to be *forward*. In the sequential forward SPUP operations, C_ρ is nonincreasing, and M_ρ is nonincreasing if $C_\rho = \emptyset$. We will try to repeat the forward SPUP operations until $M_\rho \cup C_\rho = \emptyset$ with keeping kG inner Eulerian for constant k.

The remaining of this section is organized as follows. Section 3.1 describes several basic properties of the fractional splitting-off. Section 3.2 proves (3.3). Section 3.3 investigates the splitting properties at nodes in M_ρ, and shows that if C_ρ is empty, then there exists a half-integral optimal multiflow (Corollary 3.9). The final Section 3.4 completes the proof of the main theorem (Theorem 1.4) by showing that the forward SPUP operations at C_ρ succeed with keeping $12G$ inner Eulerian until $M_\rho \cup C_\rho = \emptyset$.

3.1 Basic properties

In this section, we list several basic properties of SPUP. We only consider forks $\tau = (e, y, e')$ with $c(e) = c(e') = 1$, although c may not be integral. As mentioned, we identify multiflows in G and multiflows in $G^{\tau,0}$. In particular, for a multiflow $f = (\mathcal{P}; \delta)$ and a fork $\tau = (e, y, e')$, we often use the following relations:

$$\mathcal{P}(e^\tau) = \mathcal{P}(e) \cup \mathcal{P}(e') \setminus \mathcal{P}(e, e') \text{ and } f^{e^\tau} = f^e + f^{e'} - 2f^{e,e'}.$$

We start with relationship among optimal potentials, splitting capacity, and saturation by optimal multiflows.

Lemma 3.2. (1) *For a fork $\tau = (e, y, e')$ and an optimal multiflow f, we have*

$$\alpha_\tau \geq 2 - f^{e^\tau} \geq 2f^{e,e'}.$$

(2) *For disjoint forks τ, τ', we have $\alpha_{\tau'}(G^{\tau,\alpha_\tau(G)}) \leq \alpha_{\tau'}(G)$.*

(3) *Let ρ be an optimal potential, and let e be an edge with $d^\rho(e) = 0$. If $c(e) = f^e$ for every optimal multiflow f, then there is a neighbor ρ' of ρ such that $d^{\rho'}(e) > 0$ and ρ' is optimal.*

Proof. (1). Obviously f is a multiflow in $G^{\tau,f^{e^\tau}}$. This means $\mu_{A,B} \circ f \leq \mathrm{opt}(G^{\tau,f^{e^\tau}}) \leq \mathrm{opt}(G) = \mu_{A,B} \circ f$. The second inequality follows from $2 - f^{e^\tau} = (1 - f^e) + (1 - f^{e'}) + 2f^{e,e'} \geq 2f^{e,e'}$.

(2). Since τ and τ' are disjoint, $(G^{\tau,\alpha_\tau})^{\tau',\alpha_{\tau'}}$ is well-defined for $\alpha_\tau = \alpha_\tau(G)$ and $\alpha_{\tau'} = \alpha_{\tau'}(G^{\tau,\alpha_\tau})$, and $\mathrm{opt}((G^{\tau,\alpha_\tau})^{\tau',\alpha_{\tau'}}) = \mathrm{opt}(G)$. Take an optimal multiflow f in $(G^{\tau,\alpha_\tau})^{\tau',\alpha_{\tau'}}$. By contracting e^τ and $e^{\tau'}$, we obtain an optimal flow f in G. Then $f^{e^{\tau'}} = 2 - \alpha_{\tau'}(G^{\tau,\alpha_\tau})$. Thus (1) implies the desired inequality.

(3). Decrease $c(e)$ by $\beta \geq 0$. The resulting graph is denoted by $G^{e,\beta}$. Obviously $\mathrm{opt}(G^{e,\beta}) \leq \mathrm{opt}(G)$. By the same argument as in the proof of Proposition 3.1, the maximum possible $\beta \geq 0$ with $\mathrm{opt}(G^{e,\beta}) = \mathrm{opt}(G)$ is the minimum of $\{d^{\rho'}(G) - d^\rho(G)\}/d^{\rho'}(e)$ over all neighbors ρ' of ρ with $d^{\rho'}(e) > 0$. By the hypothesis (and (1)), this must be zero. Any neighbor ρ' attaining the maximum β is a required optimal neighbor. $\square$

Exchange/anti-exchange operations and homogeneity. We will often use two simple flow-rearrangements at an edge $e = xy$. Let $f = (\mathcal{P}; \delta)$ be a multiflow. Take two paths P_1 and P_2 from $\mathcal{P}(e)$. The *exchange operation* of P_1 and P_2 at e is the following. For $i = 1, 2$, split P_i at x into two paths P_i^1 and P_i^2 so that P_i^2 contains y. Reconnect P_1^1 and P_2^2 at x, and reconnect P_2^1 and P_1^2 at x. If the resulting multiflow has nonsimple paths (resp. cycles), then simplify (resp. delete) them.

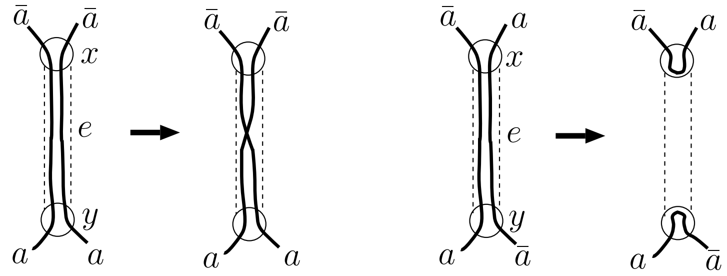

Figure 8: Exchange and anti-exchange operations (keeping flow-value)

There is a reverse operation. Reconnect P_1^1 and P_2^1 at x, and reconnect P_2^2 ant P_1^2 at x. Then the resulting multiflow contains nonsimple paths. So simplify them, and delete cycles appearing. This operation is called the *anti-exchange operation* at e. See Figure 8 for exchange and anti-exchange operations (keeping flow-value $\mu_{A,B} \circ f$).

Consider the case where a subset $\mathcal{Q} \subseteq \mathcal{P}(e)$ consists of either $(a, x, y, \bar{a})$-paths or $(b, x, y, \bar{b})$-paths. Then the exchange operation at e of every pair of paths in $\mathcal{Q}$ does not decrease the flow-value. Namely f is *single-commodity flow* at e. In this case, we say "*$\mathcal{Q}$ is homogeneous*", and also say "*f is homogeneous at e*" if $\mathcal{P}(e)$ is homogeneous.

Splitting-off at an inner node of degree four. As seen in Section 2.1, we may consider the problem (2.1) for an inner Eulerian graph with unit capacity all of whose inner node have at most degree four. Here we study splitting properties at an inner node of degree four. Suppose that inner node y is incident to four edges e, e_1, e_2, e_3 with unit capacity. Let $e = xy, e_1 = x_1y, e_2 = x_2y, e_3 = x_3y$; some of nodes x, x_1, x_2, x_3 may coincide.

Lemma 3.3. *If y has multiple edges e, e_1 $(x = x_1)$, then fork (e, y, e_2) is splittable.*

Proof. Split (e, y, e_2) off. This produces two pairs $\{e, e_2\}, \{e_1, e_3\}$ of series edges. Replace e, e_2 by one edge xx_1 and replace e_1, e_3 by one edge xx_3. The resulting graph is isomorphic to the graph G' obtained by contracting two edges e, e_1 in the original graph G. Obviously $\mathrm{opt}(G') \geq \mathrm{opt}(G)$. This means that (e, y, e_2) is splittable. $\square$

We assume that x, x_1, x_2, x_3 are all distinct. There is a useful symmetry:

$$G^{(e,y,e_i),\alpha} \simeq G^{(e_j,y,e_k),\alpha} \quad (\text{distinct } i, j, k).$$

So it suffices to consider three forks $\tau_i := (e, y, e_i)$ $(i = 1, 2, 3)$. The splitting capacity α_{τ_i} is simply denoted by α_i $(i = 1, 2, 3)$.

Lemma 3.4. (1) $\alpha_1 + \alpha_2 + \alpha_3 \geq 2$.

(2) $2\alpha_1 + \alpha_2 + \alpha_3 \geq 4$ *if there is an optimal multiflow for G^{τ_1,α_1} being homogeneous at e^{τ_1}.*

Proof. Take an optimal multiflow f for G^{τ_1,α_1} and regard it as an optimum for G by contracting e^{τ_1}. By Lemma 2.2, we have $f^{e^{\tau_1}} = 2 - \alpha_1$. By Lemma 3.2 (1) and symmetry, we have $\alpha_2 + \alpha_3 \geq (f^{e,e_2} + f^{e_1,e_3}) + (f^{e,e_3} + f^{e_1,e_2}) = f^{e^{\tau_1}} = 2 - \alpha_1$, implying (1). Next suppose that f is homogeneous at e^{τ_1}. Since $f^{e^{\tau_1}} = f^{e^{\tau_1},e} + f^{e^{\tau_1},e_1} = f^{e^{\tau_1},e_3} + f^{e^{\tau_1},e_2}$, we may assume $f^{e^{\tau_1},e_2} \geq f^{e^{\tau_1},e} \geq 1 - \alpha_1/2 \geq f^{e^{\tau_1},e_1} \geq f^{e^{\tau_1},e_3}$ by relabeling e, e_1, e_2, e_3 if necessarily. Here $f^{e^{\tau_1},e_2} \geq f^{e^{\tau_1},e} \geq f^{e^{\tau_1},e_1}$ and f is homogeneous at e^{τ_1}. By repeating exchange operations at e^{τ_1} for two paths, one in $\mathcal{P}(e, e_3)$ and one in $\mathcal{P}(e_1, e_2)$, we can make f fulfill $f^{e,e_2} = f^{e^{\tau_1},e}$. Similarly we can also make f fulfill $f^{e_1,e_2} = f^{e^{\tau_1},e_1}$. So $\alpha_2 + \alpha_3 \geq 2(f^{e^{\tau_1},e} + f^{e^{\tau_1},e_1}) = 2f^{e^{\tau_1}} = 2(2 - \alpha_1)$. Thus we have (2). $\square$

A key lemma. Let $f = (\mathcal{P}; \delta)$ be an optimal multiflow and ρ an optimal potential. Let y be an inner node and $\tau = (e, y, e')$ an unsplittable fork at y. Let ρ' be a critical neighbor with respect to τ. Suppose that $\alpha_\tau = 2 - f^{e^\tau}$ (or $2f^{e,e'}$) holds (see Lemma 3.2 (1)). Then f can also be regarded as an optimal multiflow for G^{τ,α_τ}. By optimality criterion (Lemma 2.2) for (f, ρ'), each path P in $\mathcal{P}(e^\tau)$ fulfills

$$d^{\rho'}(s_P, y^\tau) + d^{\rho'}(y^\tau, y) + d^{\rho'}(y, t_P) = \mu_{A,B}(s_P, t_P), \tag{3.4}$$

where P is supposed to be an (s_P, y^τ, y, t_P)-path. From the position $\{\rho'(y^\tau), \rho'(y)\}$ together with Table 1, we can determine (sometime completely) the ends of paths in $\mathcal{P}(e^\tau)$. This is a lucky case.

We need to analyze $\mathcal{P}(e^\tau)$ for general case $\alpha_\tau \geq 2 - f^{e^\tau} \geq 2f^{e,e'}$ with possibly strict inequality. Let $\mathcal{P}(e^\tau; \rho')$ be the set of paths P in $\mathcal{P}(e^\tau)$ satisfying (3.4). Its flow-value is denoted by $f^{e^\tau;\rho'}$. We can estimate $f^{e^\tau;\rho'}$ by the following formulas:

Lemma 3.5. (1) $d^{\rho'}(e^\tau) f^{e^\tau;\rho'} + (d^{\rho'}(e^\tau) - 2)(f^{e^\tau} - f^{e^\tau;\rho'}) \geq d^{\rho'}(e^\tau)(2 - \alpha_\tau)$.

(2) *If* $d^{\rho'}(e^\tau) \geq 2$, *then* $f^{e^\tau;\rho'} \geq 2 + (d^{\rho'}(e^\tau) - 2) f^{e,e'} - \alpha_\tau d^{\rho'}(e^\tau)/2$.

Proof. We utilize the formula (2.4) of the duality gap. Let f' be a (non-optimal) multiflow for G^{τ,α_τ} obtained by deleting all paths in $\mathcal{P}(e^\tau)$ from f. Since ρ' is optimal for G^{τ,α_τ} and $\mathrm{opt}(G) = \mathrm{opt}(G^{\tau,\alpha_\tau})$, the duality gap between ρ' and f' is equal to $\sum_{P \in \mathcal{P}(e^\tau)} \mu_{A,B}(s_P, t_P)\delta$. We next estimate the first term $\varDelta_1 := \sum_{e \in EG^{\tau,\alpha_\tau}} d^{\rho'}(e)(c(e) - (f')^e)$ in the RHS of (2.4). Then we have

$$\begin{aligned} \varDelta_1 &= d^{\rho'}(e^\tau)(2 - \alpha_\tau) + \sum_{P \in \mathcal{P}(e^\tau)} d^{\rho'}(P \setminus e^\tau)\delta + \sum_{e \in EG^{\tau,\alpha_\tau}, e \neq e^\tau} d^{\rho'}(e)(c(e) - f^e) \\ &\geq d^{\rho'}(e^\tau)(2 - \alpha_\tau) + \sum_{P \in \mathcal{P}(e^\tau)} \{d^{\rho'}(s_P, y^\tau) + d^{\rho'}(y, t_P)\}\delta. \end{aligned}$$

Therefore we have

$$\sum_{P \in \mathcal{P}(e^\tau)} \mu_{A,B}(s_P, t_P)\delta \geq \varDelta_1 \geq d^{\rho'}(e^\tau)(2 - \alpha_\tau) + \sum_{P \in \mathcal{P}(e^\tau)} \{d^{\rho'}(s_P, y^\tau) + d^{\rho'}(y, t_P)\}\delta.$$

From this we obtain

$$\begin{aligned} &\sum_{P \in \mathcal{P}(e^\tau)} \left[d^{\rho'}(e^\tau) - \left\{ d^{\rho'}(s_P, y^\tau) + d^{\rho'}(e^\tau) + d^{\rho'}(y, t_P) - \mu_{A,B}(s_P, t_P) \right\} \right] \delta \\ &\geq d^{\rho'}(e^\tau)(2 - \alpha_\tau). \end{aligned}$$

Here $d^{\rho'}(s_P, y^\tau) + d^{\rho'}(e^\tau) + d^{\rho'}(y, t_P) - \mu_{A,B}(s_P, t_P)$ is a nonnegative even integer, since $\mathrm{dist}_\varGamma(p^s, p^t) = \mu_{A,B}(s, t)$ and $\varGamma$ is bipartite. Therefore we obtain the first inequality. The second follows from substituting $f^{e^\tau} = f^e + f^{e'} - 2f^{e,e'} \leq 2 - 2f^{e,e'}$ to the first. $\square$

3.2 Splitting-off at S_ρ

Recall the partition $\{S_\rho, M_\rho, C_\rho\}$ of VG defined by (3.2). The goal of this subsection is to prove the following:

Proposition 3.6. *Let G be an inner Eulerian graph and ρ an optimal potential. For any inner node $y \in S_\rho$ of degree four, there exists a splittable fork at y.*

Corollary 3.7. *Let G be an inner Eulerian graph, and ρ an optimal potential. If $M_\rho \cup C_\rho = \emptyset$, then there exists an integral optimal multiflow.*

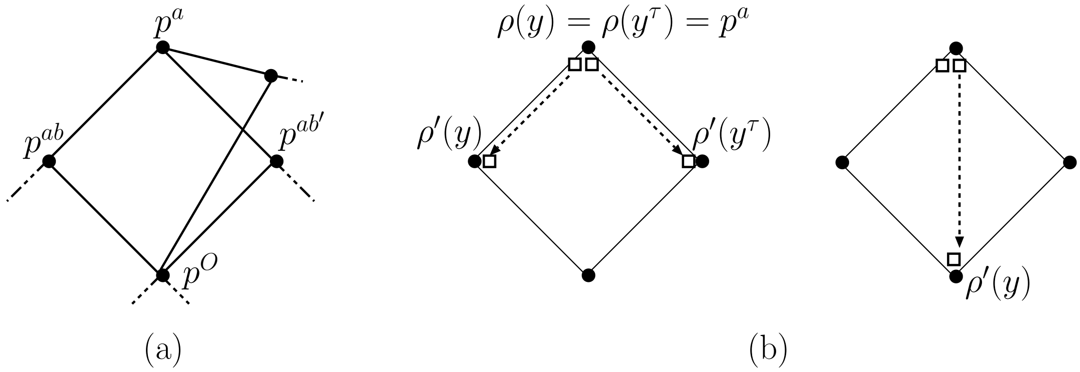

Figure 9: (a) the graph structure around p^a and (b) behavior of neighbors

Proof. Make each inner node in S_ρ have degree four by the method in Section 2.1 together with (2.7). By the repeated applications of the previous proposition, we can split all inner nodes off until there is no inner node of degree at least four. $\square$

Let us start the proof of Proposition 3.6. We may assume $\rho(y) = p^a$ for $a \in A$. By Lemma 3.3, we may assume that y is incident to four distinct nodes x, x_1, x_2, x_3; we use the notation in Section 3.1. Figure 9 (a) illustrates the graph structure of Γ around p^a; this is the complete bipartite graph $K_{2,m}$. Then we can combine the idea of neighbors (Proposition 3.1) and Karzanov's splitting-off technique used in [10, 15].

For any unsplittable fork τ at y with $\alpha_\tau > 0$, its critical neighbor ρ' of ρ is necessarily backward, and $\{\rho'(y^\tau), \rho'(y)\}$ is

(i) $\{p^O, p^a\}$ or (ii) $\{p^{ab}, p^{ab'}\}$ for distinct $b, b' \in B$.

See Figure 9 (b). Here $\{\rho'(y^\tau), \rho'(y)\} = \{p^a, p^{ab}\}$ implies $d^\rho(e^\tau) = 1$ and $\alpha_\tau = 0$. In both cases (i-ii), $d^\rho(e^\tau) = 2$ and $\alpha_\tau \in \{0, 1\}$.

Consider three forks $\tau_i = (e, y, e_i)$ $(i = 1, 2, 3)$. We may assume that τ_1 is unsplittable with $\alpha_1 > 0$ (by Lemma 3.4 (1)). Take a critical neighbor ρ_1 of ρ with respect to τ_1. Then the position $\{\rho_1(y^{\tau_1}), \rho_1(y)\}$ is of case (i) or (ii), and $\alpha_1 = 1$. Take an optimal multiflow $f = (\mathcal{P}; \delta)$ for G^{τ_1, α_1}, and regard it as an optimal multiflow for G. Here consult Table 1. In both cases (i-ii), f is homogeneous at e^{τ_1}; $\mathcal{P}(e^{\tau_1})$ consists of $(a, \bar{a})$-paths for (i) and consists of (b, b')-paths for (ii). Now $f^{e^{\tau_1}} = 1$. By relabeling and exchange operations at e^{τ_1} (as in the proof of Lemma 3.4), we can make f (with keeping the optimality) so that $f^{e,e_2} \geq 1/2 \geq f^{e_1,e_2} + f^{e_1,e_3}$ and $f^{e,e_3} = 0$. If $f^{e,e_2} > 1/2$, then $\alpha_2 > 1$ (by Lemma 3.2 (1)), and τ_2 is splittable (since $\alpha_2 \in \{0, 1, 2\}$). So we assume that $f^{e,e_2} = 1/2$ and τ_2 is unsplittable with $\alpha_2 = 1$. Consider a critical neighbor ρ_2 for τ_2, which is also of case (i) or (ii). By $f^{e,e_2} = 1/2$, f can also be regarded as an optimal multiflow for G^{τ_2, α_2}. So f is also homogeneous at e^{τ_2} and $f^{e^{\tau_2}} = f^{e_2,e_3} + f^{e_1,e_2} + f^{e,e_1} = 1$. Then we have $f^{e,e_1} = 1/2$ and $f^{e_2,e_3} + f^{e_1,e_2} = 1/2$ (since $f^{e,e_1} \leq c(e) - f^{e,e_2} = 1/2$ and $f^{e_2,e_3} + f^{e_1,e_2} \leq c(e_2) - f^{e,e_2} = 1/2$). Suppose $f^{e_2,e_3} > 0$ and $f^{e_1,e_2} < 1/2$. Then $f^{e_1,e_3} > 0$ (by $f^{e_1,e_2} + f^{e_1,e_3} = 1/2$). Here both $\mathcal{P}(e_2, e_3) \cup \mathcal{P}(e, e_1) \subseteq \mathcal{P}(e^{\tau_2})$ are $\mathcal{P}(e, e_2) \cup \mathcal{P}(e_1, e_3) \subseteq \mathcal{P}(e^{\tau_1})$ are homogeneous. By exchanging paths at e^{τ_2}, one in $\mathcal{P}(e_2, e_3)$ and one in $\mathcal{P}(e, e_1)$, and by exchanging paths at e^{τ_1}, one in $\mathcal{P}(e, e_2)$ and one in $\mathcal{P}(e_1, e_3)$, we can make f (with keeping optimality) so that $f^{e_1,e_2} > 1/2$; see Figure 10. Then $\alpha_3 > 1$ and τ_3 is splittable. So we suppose

$$f^{e,e_1} = f^{e,e_2} = f^{e_1,e_2} = 1/2 \text{ and } f^{e_3} = 0. \tag{3.5}$$

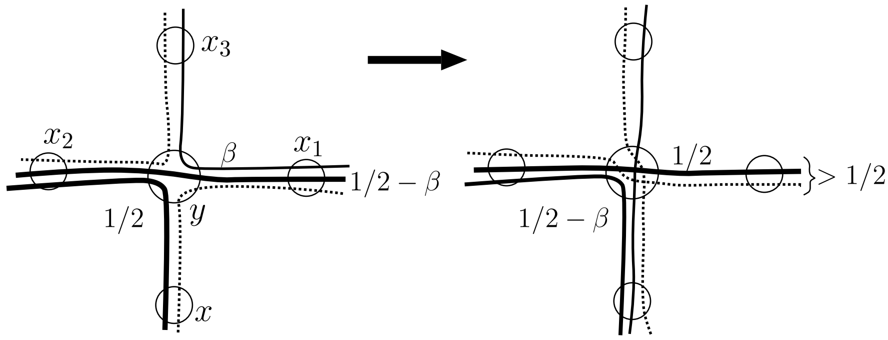

Figure 10: flow modification

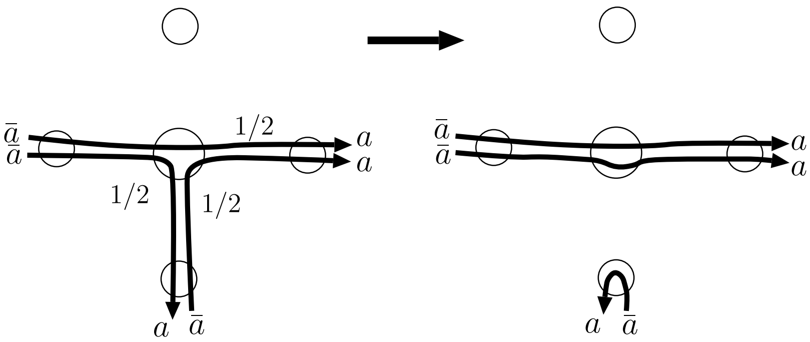

Figure 11: flow modification

Now f can be regarded as an optimal multiflow for both G^{τ_1,α_1} and G^{τ_2,α_2}. Suppose that a path P in $\mathcal{P}(e_1, e_2)$ is an (s, x_1, y, x_2, t)-path. Then P can be regarded as an (s, y^{τ_1}, y, t)-path and as an (s, y, y^{τ_2}, t)-path. Therefore (s, t) fulfills relation (2.5) for both (ρ_1, e^{τ_1}) and (ρ_2, e^{τ_2}). This implies the existence of two shortest paths in Γ: one passing $p^s \to \rho_1(y^{\tau_1}) \to \rho_1(y) \to p^t$ and another passing $p^s \to \rho_2(y) \to \rho_2(y^{\tau_2}) \to p^t$. This determines $(\rho_1(y^{\tau_1}), \rho_1(y)) = (\rho_2(y), \rho_2(y^{\tau_2}))$. Consider the case (i) with $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^a, p^O)$. Then $\mathcal{P}(e, e_2)$ consists of $(a, x, y, x_2, \bar{a})$-paths and $\mathcal{P}(e, e_1)$ consists of $(\bar{a}, x, y, x_1, a)$-paths. Therefore the anti-exchange operation at e for two paths, one in $\mathcal{P}(e, e_2)$ and one in $\mathcal{P}(e, e_1)$, keeps the optimality, and makes f fulfill $f^{e_1,e_2} > 1/2$; see Figure 11. Then $\alpha_3 > 1$, and τ_3 is splittable. Also for $(\rho_1(y^{\tau_1}), \rho_1(y)) = (p^O, p^a)$ or $(p^{ab}, p^{ab'})$, the same anti-change exchange operation at e works, and τ_3 is splittable. This completes the proof of Proposition 3.6.

3.3 Splitting-off at M_ρ

Graph with optimal potential $(G; \rho)$ is called *restricted Eulerian* if capacity c is integral and each node in $M_\rho \cup C_\rho$ has even degree.

Proposition 3.8. *Suppose that $(G; \rho)$ is restricted Eulerian. For any inner node $y \in M_\rho$ of degree four with $\rho(y) = p^{ab}$, at least one of the following holds:*

(0) *There exists a splittable fork at y.*

(1) *There exists an optimal forward neighbor ρ' of ρ with $\rho'(y) \neq \rho(y)$.*

(2) *There exist a fork τ at y and its critical forward neighbor ρ' of ρ such that $\alpha_\tau = 1$,*

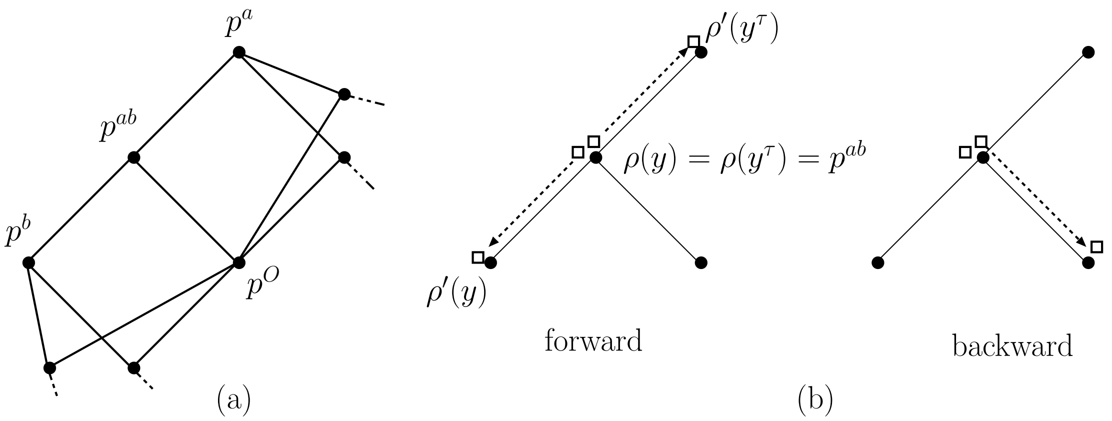

Figure 12: (a) the graph structure around p^{ab} and (b) behavior of neighbors

$\{\rho'(y^\tau), \rho'(y)\} = \{p^a, p^b\}$, and thus the corresponding 1-SPUP keeps $(G;\rho)$ restricted Eulerian.

Corollary 3.9. *Suppose that $(G;\rho)$ is restricted Eulerian and $C_\rho = \emptyset$. Then there exists a half-integral optimal multiflow.*

Proof. Make each node in M_ρ have degree four. According to the previous proposition, for each node in M_ρ, apply the splitting-off (with replacing series edges by one edge), the forward 1-SPUP, or replace ρ by its optimal forward neighbor, which keeps $(G;\rho)$ restricted Eulerian. Since C_ρ is empty, the set M_ρ strictly decreases. Repeat this process until $M_\rho \cup C_\rho$ is empty. Now $2G$ is inner Eulerian. By Corollary 3.7, G has a half-integral optimal multiflow, and so does the original graph. $\square$

Let us start the proof of Proposition 3.8. Suppose that y is incident to four distinct nodes x, x_1, x_2, x_3. Suppose further that all three forks τ_1, τ_2, τ_3 are unsplittable. Take a critical neighbor ρ_i of ρ at τ_i for $i = 1, 2, 3$. Then the following holds (for $i = 1, 2, 3$):

(3.6) (i) If ρ_i is backward, then $\alpha_i \in \{0, 1\}$ and $\{\rho_i(y^{\tau_i}), \rho_i(y)\} = \{p^O, p^{ab}\}$.

(ii) If ρ_i is forward, then $\alpha_i = 0$, or $\alpha_i = 1$ and $\{\rho_i(y^{\tau_i}), \rho_i(y)\} = \{p^a, p^b\}$.

Proof. See Figure 12 for graph structure around p^{ab} and the behavior of backward/forward neighbor. Suppose that ρ_i is backward. Then $d^{\rho_i}(e^{\tau_i}) = 1$ and $d^{\rho}(G^{\tau_i,0}) - d^{\rho}(G^{\tau_i,0}) \in \mathbf{Z}$ (c is integral). Suppose that ρ_i is forward. Since G is inner Eulerian with respect to S_ρ, there are cycles C_i and S_ρ-paths P_j such that (2.6) holds. Since ρ_i is forward, we have $\rho(x) = \rho_i(x)$ for $x \in S_\rho$. Therefore the potential of ends of any S_ρ-path are fixed. Consequently $d^{\rho_i}(G^{\tau_i,0}) - d^{\rho_i}(G^{\tau_i,0})$ is even. Since $\{\rho_i(y^{\tau_i}), \rho_i(y)\} \subseteq \{p^a, p^{ab}, p^b\}$, we have $d^{\rho_i}(e^{\tau_i}) \in \{1, 2\}$, $\alpha_i \in \{0, 1\}$, and $\alpha_i = 1$ if and only if $\{\rho(y^{\tau_i}), \rho(y)\} = \{p^a, p^b\}$. $\square$

Our goal is to prove the following statement:

(3.7) At least one of ρ_1, ρ_2, ρ_3 is forward.

Suppose that ρ_1 is forward (say). Then $\alpha_1 \in \{0, 1\}$. If $\alpha_1 = 1$, then (2) in Proposition 3.8 occurs. Suppose $\alpha_1 = 0$. In $(G^{\tau_1,0}; \rho_1)$, at least one of y and y^{τ_1}, say y, falls into S_ρ. Then y^{τ_1} has three neighbors y, x, x_1 with $c(e^{\tau_1}) = 2$. Multiply e^{τ_1} by two edges e', e'' of unit capacity. By Lemma 3.3, fork (e', y^{τ_1}, e) is splittable, and split it off. Replace series edges by one edge. The resulting graph is isomorphic to the original graph G. So ρ_1 can be regarded as an optimal forward neighbor for the original graph; namely (1) holds.

Proof of (3.7). Suppose indirectly that ρ_1, ρ_2, ρ_3 are all backward. By relabeling and symmetry, we may assume that

(3.8) (i) $(\rho_i(y^{\tau_i}), \rho_i(y)) = (p^O, p^{ab})$ for $i = 1, 2, 3$, or

(ii) $(\rho_i(y^{\tau_i}), \rho_i(y)) = (p^{ab}, p^O)$ for $i = 1, 2, 3$.

Take an optimal multiflow f for G. By Lemma 3.5 (1) with $d^{\rho_i}(e^{\tau_i}) = 1$, we have

$$f^{e^{\tau_i};\rho_i} - (f^{e^{\tau_i}} - f^{e^{\tau_i};\rho_i}) \geq 2 - \alpha_i \quad (i = 1, 2, 3). \tag{3.9}$$

Here we claim

$$\mathcal{P}(e^{\tau_i}; \rho_i) \cap \mathcal{P}(e^{\tau_j}; \rho_j) \cap \mathcal{P}(e_i, e_j) = \emptyset \quad (\text{distinct } i, j, k). \tag{3.10}$$

Suppose not. Take $P \in \mathcal{P}(e^{\tau_i}; \rho_i) \cap \mathcal{P}(e^{\tau_j}; \rho_j) \cap \mathcal{P}(e_i, e_j)$. Suppose that P is an (s, x_i, y, x_j, t)-path, which can also be regarded as an (s, y^{τ_i}, y, t)-path and as an (s, y, y^{τ_j}, t)-path. Therefore (s, t) fulfills (3.4) for (ρ_i, e^{τ_i}) and for (ρ_j, e^{τ_j}). By (3.8), this means the existence of two shortest paths in Γ: one passing $p^s \to p^O \to p^{ab} \to p^t$ and another passing $p^s \to p^{ab} \to p^O \to p^t$. This is impossible. So we have (3.10).

Here $\mathcal{P}(e^{\tau_i}; \rho_i)$ is the disjoint union of three sets $\mathcal{P}(e^{\tau_i}; \rho_i) \cap \mathcal{P}(e)$, $\mathcal{P}(e^{\tau_i}; \rho_i) \cap \mathcal{P}(e_i, e_j)$, and $\mathcal{P}(e^{\tau_i}; \rho_i) \cap \mathcal{P}(e_i, e_k)$. The corresponding flow-values are denoted by $f_0^{e^{\tau_i};\rho_i}$, $f_{ij}^{e^{\tau_i};\rho_i}$, and $f_{ik}^{e^{\tau_i};\rho_i}$, respectively. Then we have

$$f^{e^{\tau_i};\rho_i} = f_0^{e^{\tau_i};\rho_i} + f_{ij}^{e^{\tau_i};\rho_i} + f_{ik}^{e^{\tau_i};\rho_i} \quad (\text{distinct } i, j, k).$$

By (3.10) and $\mathcal{P}(e^{\tau_i}) \supseteq (\mathcal{P}(e_i, e_j) \cap \mathcal{P}(e^{\tau_j}; \rho_j)) \cup (\mathcal{P}(e_i, e_k) \cap \mathcal{P}(e^{\tau_k}; \rho_k))$ (disjoint union), we have

$$f^{e^{\tau_i}} - f^{e^{\tau_i};\rho_i} \geq f_{ij}^{e^{\tau_j};\rho_j} + f_{ik}^{e^{\tau_k};\rho_k} \quad (\text{distinct } i, j, k).$$

Substituting these two relations to (3.9), we get

$$f_0^{e^{\tau_i};\rho_i} + f_{ij}^{e^{\tau_i};\rho_i} + f_{ik}^{e^{\tau_i};\rho_i} - f_{ij}^{e^{\tau_j};\rho_j} - f_{ik}^{e^{\tau_k};\rho_k} \geq 2 - \alpha_i \quad (\text{distinct } i, j, k).$$

Summing these three inequalities yields

$$f_0^{e^{\tau_1};\rho_1} + f_0^{e^{\tau_2};\rho_2} + f_0^{e^{\tau_3};\rho_3} \geq 6 - \alpha_1 - \alpha_2 - \alpha_3.$$

Since $f^{e,e_j} + f^{e,e_k} \geq f_0^{e^{\tau_i};\rho_i}$, we have $2f^e = 2(f^{e,e_1} + f^{e,e_2} + f^{e,e_3}) \geq 6 - \alpha_1 - \alpha_2 - \alpha_3$. By $1 = c(e) \geq f^e$, we obtain $\alpha_1 + \alpha_2 + \alpha_3 \geq 4$. However this contradicts $\alpha_i \in \{0, 1\}$; see (3.6) (i). □

3.4 Splitting-off at C_ρ

Finally we show that the forward SPUP at C_ρ successfully achieves $C_\rho = \emptyset$ keeping $(6G; \rho)$ restricted Eulerian. This completes the proof of Theorem 1.4.

	$\{\rho'(y), \rho'(y^\tau)\}$	$d^{\rho'}(e^\tau)$	α_τ, G admissible	α_τ, $3G$ admissible
(1a)	$\{p^{ab}, p^O\}$	1	0	$0, 2/3, 4/3$
(1b)	$\{p^{ab}, p^a\}$, $\{p^{ab}, p^b\}$	1	0	$0, 2/3, 4/3$
(2a)*	$\{p^O, p^a\}$, $\{p^O, p^b\}$	2	$0, 1$	$0, 1/3, 2/3, 1, 4/3, 5/3$
(2b)*	$\{p^{ab}, q^{ab'}\}$, $\{p^{ab}, q^{a'b}\}$	2	$0, 1$	$0, 1/3, 2/3, 1, 4/3, 5/3$
(2c)	$\{p^{ab}, q^{a'b'}\}$	2	$0, 1$	$0, 1/3, 2/3, 1, 4/3, 5/3$
(2d)	$\{p^a, p^b\}$	2	$0, 1$	$0, 1/3, 2/3, 1, 4/3, 5/3$
(3)*	$\{p^{ab}, p^{b'}\}$, $\{p^{ab}, p^{a'}\}$	3	$0, 2/3, 4/3$	$2m/9$ $(0 \leq m \leq 8)$
(4)*	$\{p^a, p^{a'}\}$, $\{p^b, p^{b'}\}$	4	$0, 1/2, 1, 3/2$	$m/6$ $(0 \leq m \leq 11)$

Table 2: Classification of $\{\rho'(y), \rho'(y^\tau)\}$

Setting up. We are given a graph G with an optimal potential ρ. For an unsplittable fork τ at node $y \in C_\rho$, its critical neighbor ρ' is always forward (since any backward neighbor ρ' fulfills $d^{\rho'}(e^\tau) = 0$). As seen in Section 3.3, we can sometimes keep α_τ half- and 2/3-integral without inner Eulerian condition. We begin with introducing such a condition sharpening the restricted Eulerian condition.

An edge $e = xy \in EG$ with $(\rho(x), \rho(y)) = (p^{ab}, p^{a'b'})$ for distinct $a, a' \in A, b, b' \in B$ is said to be *mixed* in $(G; \rho)$ if every optimal flow f is *not* homogeneous at e, i.e., f contains both A-paths and B-paths at e. Otherwise e is said be *unmixed*. Let E_{S_ρ} denote the set of edges with both ends belonging to S_ρ. $(G; \rho)$ is called *admissible* if for some set $\tilde{E}$ of mixed edges, $(G - E_{S_\rho} - \tilde{E}; \rho)$ is restricted Eulerian.

(3.11) Suppose that $(G; \rho)$ is admissible. For an unsplittable fork τ at $y \in C_\rho$ and its critical neighbor ρ' of ρ, we have $d^{\rho'}(G^{\tau,0}) - d^\rho(G^{\tau,0}) \in 2\mathbf{Z}$.

Proof. By definition there is a set $\tilde{E}$ of mixed edges such that $(G - E_{S_\rho} - \tilde{E}; \rho)$ is restricted Eulerian. Let $\bar{G} = G - E_{S_\rho} - \tilde{E}$. Then $d^{\rho'}(G^{\tau,0}) - d^\rho(G^{\tau,0})$ is equal to

$$d^{\rho'}(\bar{G}^{\tau,0}) - d^\rho(\bar{G}^{\tau,0}) + \sum_{e \in E_{S_\rho}} c(e)(d^{\rho'}(e) - d^\rho(e)) + \sum_{e \in \tilde{E}} c(e)(d^{\rho'}(e) - d^\rho(e)).$$

Here ρ' is necessarily forward, and thus the second term vanishes. Also $(\bar{G}; \rho)$ is restricted Eulerian, and thus the first term is even; see the proof of (3.6). We show $d^{\rho'}(e) = d^\rho(e) = 2$ for any mixed edge $e = xy$. Suppose $(\rho(x), \rho(y)) = (p^{ab}, p^{a'b'})$. Since ρ' is forward, the possible positions of $(\rho'(x), \rho'(y))$ are $(p^{ab}, p^{a'b'})$, $(p^a, p^{b'})$, $(p^b, p^{a'})$ $(p^a, p^{a'})$, $(p^b, p^{b'})$, $(p^a, p^{a'b'})$, $(p^b, p^{a'b'})$, $(p^{ab}, p^{a'})$, and $(p^{ab}, p^{b'})$. Consult Table 1. Then the last six cases are all impossible since every optimal flow f in G^{τ,α_τ} is homogeneous at e^τ, and can also be regarded as an optimal flow in G. $\Box$

Let us start the SPUP procedure. Suppose that the initial graph G is an inner Eulerian graph with unit capacity. Also we may assume that G has no splittable fork, and each inner node has degree four. Take an optimal potential ρ. $(G; \rho)$ is trivially restricted Eulerian and admissible. For a fork τ at $y \in C_\rho$ and its critical (forward) neighbor ρ' of ρ, the possible cases of $\{\rho'(y), \rho'(y^\tau)\}$ with $(d^\rho(e^\tau), \alpha_\tau)$ are summarized in Table 2, where the asterisk $*$ means that every optimal multiflow f for G^{τ,α_τ} is homogeneous at e^τ; see Table 1.

We apply forward SPUP at a fork having maximum α_τ at first three stages. Then by Lemma 3.2 (2) the maximum value of α_τ over forks τ at C_ρ decreases. When α_τ becomes close to 1, the estimation by Lemmas 3.4 and 3.5 becomes effective.

3/2-**SPUP.** By examining all forks at C_ρ, take a fork τ at $y \in C_\rho$ with $\alpha_\tau = 3/2$. Take a critical neighbor ρ' of ρ with respect to τ. Then $d^{\rho'}(e^\tau) = 4$, and thus $\{\rho'(y), \rho'(y^\tau)\}$ is of case (4) in Table 2. Apply 3/2-SPUP $(G;\rho) \leftarrow (G^{\tau,\alpha_\tau};\rho')$. Both y, y^τ fall into S_ρ. Then $(G;\rho)$ is admissible and $(2G;\rho)$ is restricted Eulerian. Repeat this process until there is no fork τ at C_ρ with $\alpha_\tau = 3/2$. After that, the possible values of α_τ are $0, 2/3, 1, 4/3$. Note that $1/2$ never occurs since $\alpha_\tau = 1/2$ and Lemma 3.4 (2) imply the existence of another fork τ' with $\alpha_{\tau'} = 3/2$.

4/3-**SPUP and** 7/6-**SPUP.** By examining all forks at C_ρ, take a fork τ at y in C_ρ with $\alpha_\tau = 4/3$ (case (3) in Table 2). Apply 4/3-SPUP $(G;\rho) \leftarrow (G^{\tau,\alpha_\tau};\rho')$; both y, y^τ go out from C_ρ, one falls into M_ρ, the other one falls into S_ρ, and e^τ has capacity $2/3$. Therefore

(3.12) $\quad (3G;\rho)$ is admissible and $(6G;\rho)$ is restricted Eulerian.

From now on we keep this condition (3.12). In the next forward SPUP, α_τ takes a value in $1/3(2\mathbf{Z}_+/3 \cup \mathbf{Z}_+/2)$; see the fifth column in Table 2. Note that $\alpha_\tau > 4/3$ is impossible by Lemma 3.2 (2). By this fact together with Lemma 3.4 (2), $\alpha_\tau \in \{1/6, 2/9, 4/9\}$ is also impossible. So the possible values of α_τ are $0, 1/3, 2/3, 5/6, 8/9, 1, 10/9, 7/6, 4/3$. In the subsequent 4/3-SPUP, we use the following hidden property:

(3.13) $\quad$ For a fork τ at node y in C_ρ with $\alpha_\tau = 4/3$, there is a critical neighbor ρ^* of ρ with $y, y^\tau \notin C_{\rho^*}$.

Proof. Let $(\tilde{G};\tilde{\rho})$ be the graph with optimal potential after the last 3/2-SPUP. By Lemma 3.2 (2), we have $4/3 = \alpha_\tau(G) \leq \alpha_\tau(\tilde{G}) \leq 4/3$. Thus $\alpha_\tau(\tilde{G}) = 4/3$. This means that *we could choose this fork τ in the first 4/3-SPUP* for $(\tilde{G};\tilde{\rho})$. Take a critical neighbor ρ' of $\tilde{\rho}$ for this fork τ at $\tilde{G}$. Since $(\tilde{G};\tilde{\rho})$ is admissible, $\{\rho'(y), \rho'(y^\tau)\}$ is of case (3); both $\rho'(y), \rho'(y^\tau)$ are in the outside of $C_{\rho'}$. Then ρ' can be extended as an optimal potential for G^{τ,α_τ} by setting $\rho'(z^{\tau'}) := \rho'(z)$ for each fork τ' at node z processed after the last 3/2-SPUP (τ' and τ are disjoint). Now we obtain two optimal potentials ρ, ρ' for G^{τ,α_τ} with $y, y^\tau \notin C_{\rho'}$. By uncrossing lemma (Lemma 2.4), we can take an optimal forward neighbor ρ^* of ρ with $y^\tau, y \notin C_{\rho^*}$, which is also a critical neighbor of ρ for τ, and is a required one (the construction of ρ^* in the proof of the uncrossing lemma implies $d^{\rho^*}(e^\tau) > 0$). $\square$

Therefore we can repeat 4/3-SPUP so that split nodes y, y^τ always go out from C_ρ; in particular C_ρ strictly decreases, and every node in C_ρ remains to have degree four. Repeat it until there is no fork τ with $\alpha_\tau = 4/3$. After the procedure, the possible values of α_τ are $0, 1/3, 2/3, 5/6, 8/9, 1, 10/9, 7/6$. Next apply SPUP for a fork τ at $y \in C_\rho$ with $\alpha_\tau = 7/6$ (as long as exists). In this case, its critical neighbor ρ' is of case (4); both y and y^τ fall into S_ρ. So 7/6-SPUP keeps (3.12). After the procedure, the possible values of α_τ are $0, 1/3, 2/3, 8/9, 1, 10/9$; Lemma 3.4 (2) and $\alpha_\tau < 7/6$ exclude $\alpha_\tau = 5/6$.

1-**SPUP.** Take any fork τ at $y \in C_\rho$, and take a critical neighbor ρ' of ρ at τ. The possible cases of (α_τ, ρ') are $\alpha_\tau = 1/3$ in (2c, 2d), $\alpha_\tau = 2/3$ in (1a, 1b, 2c, 2d), $\alpha_\tau = 8/9$ in (3), $\alpha_\tau = 1$ in (2a, 2b, 2c, 2d, 4), and $\alpha_\tau = 10/9$ in (3). Note that Lemma 3.4 (2) and $\alpha_\tau < 4/3$ exclude $\alpha_\tau \in \{1/3, 2/3\}$ in (2a, 2b, 3, 4).

The main obstruction to keep (3.12) is an occurrence of $\alpha_\tau = 10/9$ in (3), or $\alpha_\tau = 1$ in (2c) with e^τ unmixed in $(G^{\tau,\alpha_\tau};\rho')$. We can avoid such an SPUP by examining all three forks at y; recall that y has degree four. Suppose that y is incident to four distinct nodes x, x_1, x_2, x_3 by edges $e = xy, e_i = x_i y$ $(i = 1, 2, 3)$. For a fork $\tau_i = (e, y, e_i)$, let ρ_i be a critical neighbor of ρ with respect to τ_i $(i = 1, 2, 3)$. The main claim here is the following:

(3.14) Suppose that both ρ_2 and ρ_3 are neither of case (2d) nor case (4).

(i) If ρ_1 is of case (3) with $\alpha_1 = 10/9$ or (2b) with $\alpha_1 = 1$, then ρ_2 or ρ_3 is of case (2c).

(ii) If ρ_1 is of case (2c) with e^{τ_1} unmixed in $(G^{\tau_1,\alpha_1}; \rho_1)$, then both ρ_2 and ρ_3 are of case (2c), and $\{\rho_2(y), \rho_3(y), \rho_2(y^{\tau_2}), \rho_3(y^{\tau_3})\} = \{p^{ab}, p^{ab'}, p^{a'b'}, p^{a'b}\}$ for distinct $a, a' \in A$, $b, b' \in B$.

Proof. Suppose $\alpha_1 = 10/9$ in (3), $\alpha_1 = 1$ in (2b), or $\alpha_1 = 1$ in (2c) with e^{τ_1} unmixed in $(G^{\tau_1,\alpha_1}; \rho_1)$. In all cases, we can take an optimal multiflow $f = (\mathcal{P}; \delta)$ for G^{τ_1,α_1} such that $\mathcal{P}(e^{\tau_1})$ consists of (a, y, y^{τ_1}, a')-paths for distinct $a, a' \in A$ or consists of (b, y, y^{τ_1}, b')-paths for distinct $b, b' \in B$. We may assume the former case. Take such an optimal multiflow f such that $\sum_{e \in EG} f^e$ is minimum; it exists within rational multiflows. Then f is homogeneous at e^{τ_1}, and $f^{e^{\tau_1}} = 2 - \alpha_1$. Since $f^{e, e^{\tau_1}} + f^{e_1, e^{\tau_1}} = f^{e_2, e^{\tau_1}} + f^{e_3, e^{\tau_1}} = f^{e^{\tau_1}}$, by relabeling (fixing $\{\{e, e_1\}, \{e_2, e_3\}\}$) we may assume $f^{e_2, e^{\tau_1}} \geq f^{e, e^{\tau_1}} \geq 1 - \alpha_1/2 \geq f^{e_1, e^{\tau_1}} \geq f^{e_3, e^{\tau_1}}$. Let $f^{e, e^{\tau_1}} = 1 - \alpha_1/2 + \epsilon$ for $\epsilon \geq 0$. Since $\mathcal{P}(e^{\tau_1})$ is homogeneous, by exchange operations at e^{τ_1} we can make f fulfill $f^{e, e_2} = f^{e, e^{\tau_1}} = 1 - \alpha_1/2 + \epsilon \geq 4/9 + \epsilon$. Consider τ_2 and its critical neighbors ρ_2. Then $\alpha_2 \in \{8/9, 1, 10/9\}$ (Lemma 3.2 (1)) and ρ_2 is of case (2a), (2b), (2c), or (3) (by the assumption). In particular $\epsilon \leq 1/9$. We show that all cases except (2c) are impossible. By Lemma 3.5 (2) with $f^{e, e_2} = 1 - \alpha_1/2 + \epsilon$ we have

$$f^{e^{\tau_2}; \rho_2} \geq 2 + (d^{\rho_2}(e^{\tau_2}) - 2) f^{e, e_2} - \frac{\alpha_2 d^{\rho_2}(e^{\tau_2})}{2} = \begin{cases} 7/9 + \epsilon & \text{if } \alpha_1 = \alpha_2 = 10/9, \\ 5/6 + \epsilon & \text{if } \alpha_1 = 1, \alpha_2 = 10/9, \\ 1 & \text{if } \alpha_2 = 1, \\ 10/9 & \text{if } \alpha_1 = 10/9, \alpha_2 = 8/9. \end{cases}$$

Suppose to the contrary that ρ_2 is not of case (2c). Then $\mathcal{P}(e^{\tau_2}; \rho_2)$ is homogeneous. Suppose $\mathcal{P}(e^{\tau_2}; \rho_2) \cap \mathcal{P}(e_1, e_2) \neq \emptyset$. Since $\mathcal{P}(e_1, e_2)$ consists of A-paths, $\mathcal{P}(e^{\tau_2}; \rho_2)$ also consists of A-paths of flow-value at least $7/9 + \epsilon$. Then $\mathcal{P}(e^{\tau_2}; \rho_2) \cap \mathcal{P}(e, e_1) \neq \emptyset$ since $\mathcal{P}(e^{\tau_2}; \rho_2) \cap \mathcal{P}(e_2)$ has flow-value at most $1 - f^{e, e_2} \leq 5/9 - \epsilon$. In particular $\mathcal{P}(e, e_1)$ has an $(\bar{a}', x, y, x_1, a')$-path or an $(a, x, y, x_1, \bar{a})$-path. However the anti-exchange operation at e for two paths, one in $\mathcal{P}(e, e_2)$ and one in $\mathcal{P}(e, e_1)$, decreases the flow support $\sum_{e \in EG} f^e$ with keeping optimality; a contradiction to the minimality assumption. So suppose $\mathcal{P}(e^{\tau_2}; \rho_2) \cap \mathcal{P}(e_1, e_2) = \emptyset$. Then $\mathcal{P}(e^{\tau_2}; \rho_2) \subseteq \mathcal{P}(e, e_1) \cup \mathcal{P}(e_2, e_3)$, and therefore both $\mathcal{P}(e, e_1)$ and $\mathcal{P}(e_2, e_3)$ have flow-values at least $7/9 + \epsilon - (1 - f^{e, e_2}) \geq 2/9 + 2\epsilon$. By exchange operations (at e^{τ_1} and at e^{τ_2}) we can rearrange f so that $f^{e_1, e_2} \geq 4/9 - \epsilon + 2/9 + 2\epsilon = 2/3 + \epsilon$ as in Figure 13. Then $\alpha_3 \geq 4/3$; a contradiction. Thus ρ_2 is of case (2c). This proves (i).

Next we prove (ii). In the argument above, suppose that ρ_1 is of case (2c) with e^{τ_1} unmixed in $(G^{\tau_1,\alpha_1}; \rho_1)$. Then ρ_2 is of case (2c) with $\alpha_2 = 1$. Necessarily $f^{e, e_2} = 1/2$ and $f^{e_1, e_2} + f^{e_1, e_3} = 1/2$. So we can change the roles of e and e_1. Thus ρ_3 is also of case (2c) with $\alpha_3 = 1$. As above, $\mathcal{P}(e, e_1)$ cannot have A-paths. So we may assume that $\mathcal{P}(e, e_1)$ has (b', x, y, x_1, b)-paths for distinct $b, b' \in B$. By exchange operations f can be regarded as an optimal flow for G^{τ_2,α_2} and for G^{τ_3,α_3}. $\mathcal{P}(e^{\tau_2})$ includes both (a, y^{τ_2}, y, a')-paths and (b', y^{τ_2}, y, b)-paths, which determines $(\rho_2(y^{\tau_2}), \rho_2(y)) = (p^{ab'}, p^{a'b})$. Simi-

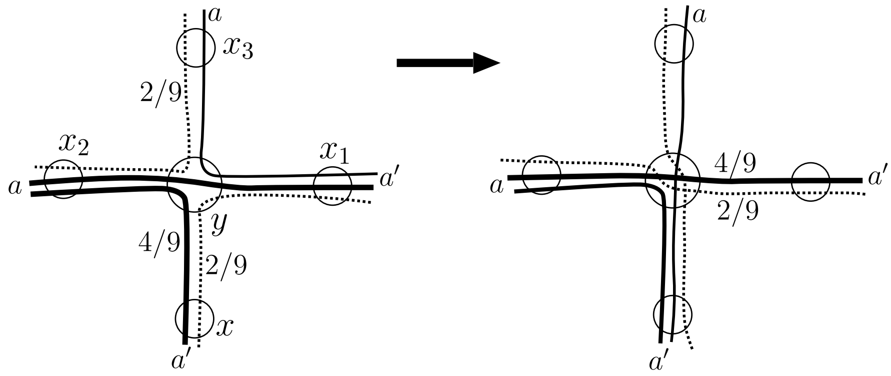

Figure 13: flow modification

larly $\mathcal{P}(e^{\tau_3})$ includes both (a', y^{τ_3}, y, a)-paths and (b', y^{τ_3}, y, b)-paths, which determines $(\rho_3(y^{\tau_3}), \rho_3(y)) = (p^{a'b'}, p^{ab})$. $\square$

Let us proceed, assuming (3.14). If some ρ_i is of case (2d) or (4) with $\alpha_i = 1$, then apply SPUP for τ_i, which keeps (3.12) and sweeps y, y^{τ_i} from C_ρ into S_ρ. So suppose that neither (2d) nor (4) occurs. Suppose $\alpha_i = 10/9$ in (3) or $\alpha_i = 1$ in (2b). By (3.14), for $j \neq i$, ρ_j is of case (2c) with $\alpha_j = 1$ by (i), and also e^{τ_j} is guaranteed to be mixed in $(G^{\tau_j, \alpha_j}; \rho_j)$ by (ii). Apply 1-SPUP for τ_j, which sweeps y, y^{τ_i} from C_ρ into M_ρ. Add e^{τ_j} to $\tilde{E}$, which keeps (3.12). If $\alpha_i = 8/9$, then $\alpha_j = \alpha_k = 10/9$ by Lemma 3.4 (2). However this is impossible by (i). So suppose (say) $\alpha_1 = 1$ with (2c). If (ρ_2, ρ_3) violates the configuration in (ii), then e^{τ_1} is guaranteed to be mixed in $(G^{\tau_1, \alpha_1}; \rho_1)$, and apply 1-SPUP for τ_1 with adding e^{τ_1} to $\tilde{E}$. Suppose that (ρ_2, ρ_3) fulfills the configuration in (ii) (for ρ_1); say $\{\rho_2(y^{\tau_2}), \rho_2(y)\} = \{p^{ab}, p^{a'b'}\}$ and $\{\rho_3(y^{\tau_3}), \rho_3(y)\} = \{p^{a'b}, p^{ab'}\}$. Consider condition (ii) for ρ_2 (by changing roles of ρ_1 and ρ_2). Suppose that (ρ_1, ρ_3) fulfills the configuration in (ii); otherwise apply 1-SPUP for τ_2, ρ_2 as above. Then we have $\{\rho_1(y), \rho_1(y^{\tau_1})\} = \{p^{ab}, p^{a'b'}\} = \{\rho_2(y), \rho_2(y^{\tau_2})\}$, which violates the configuration in (ii) for ρ_3. Thus e^{τ_3} is guaranteed to be mixed in $(G^{\tau_3, \alpha_3}; \rho_3)$, and apply 1-SPUP for τ_3, ρ_3.

Apply such a 1-SPUP as long as possible, which keeps (3.12). Suppose that C_ρ still has a node y (incident to e, e_1, e_2, e_3); otherwise we arrive the goal where $(6G; \rho)$ is restricted Eulerian with $C_\rho = \emptyset$. Again consider critical neighbors ρ_1, ρ_2, ρ_3 for three forks τ_1, τ_2, τ_3. Then $\alpha_i = 1$ in (2a), or $\alpha_i = 1/3, 2/3$. Suppose $\alpha_1 = 1$ in (2a). Then necessarily $\alpha_2 = \alpha_3 = 1$ (by Lemma 3.4 (2)). So both ρ_2 and ρ_3 are of case (2a). Let f be an *arbitrary* optimal multiflow f. By Lemma 3.5 (2), $f^{e^{\tau_i}; \rho_i} \geq 1$. Also $\mathcal{P}(e^{\tau_i}; \rho_i)$ is homogeneous. So we can apply the same argument for (3.5) in Section 3.2 (with replacing $\mathcal{P}(e^{\tau_i})$ by $\mathcal{P}(e^{\tau_i}; \rho_i)$). Then, by appropriate relabeling of e, e_1, e_2, e_3, we have

$$f^{e,e_1} = f^{e_1,e_2} = f^{e,e_2} = 1/2 \text{ and } f^{e_3} = 0. \tag{3.15}$$

Now f is arbitrary. For other optimal multiflow f', one of edge $e' \in \{e, e_1, e_2, e_3\}$ incident to y has no flow. Necessarily $e' = e_3$ holds (otherwise $(f + f')/2$ is optimal and never fulfills (3.15)). So e, e_1, e_2 are saturated by every optimal multiflow. According to Lemma 3.2 (3), we may assume that x, x_1, x_2 are in the outside of C_ρ (by replacing ρ by its forward neighbor if necessarily). e_3 has no flow. So x_3 belongs to C_ρ (Lemma 2.2). Again any fork τ at x_3 fulfills $\alpha_\tau = 1$ (by $f^{e^\tau} \leq 1$). So the flow configuration around y, x_3 is given as in Figure 14.

Consider the case where $\alpha_i = 1/3$ or $2/3$ for $i = 1, 2, 3$. By Lemma 3.4 (1), we have

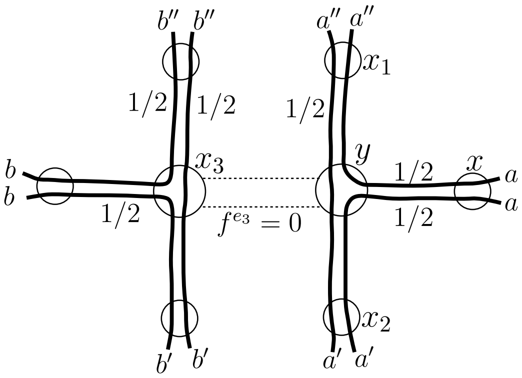

Figure 14: flow configuration around y, x_3

$\alpha_1 = \alpha_2 = \alpha_3 = 2/3$. From Lemma 3.2 (1) every optimal multiflow f necessarily satisfies

$$f^{e,e_1} = f^{e,e_2} = f^{e,e_3} = f^{e_1,e_2} = f^{e_1,e_3} = f^{e_2,e_3} = 1/3. \tag{3.16}$$

Consequently all edges incident to y are saturated by every optimal multiflow. Again, according to Lemma 3.2 (3), we may assume that x, x_1, x_2, x_3 are in the outside of C_ρ.

Let $G \leftarrow 6G$. Then the capacity of G is integer-valued, and $(G; \rho)$ is restricted Eulerian. By multiplying edges, make G have unit edge-capacity. By (3.15) and (3.16), there is an optimal multiflow f with property that $f^{e,e'} \in \{0, 1\}$ for each fork $\tau = (e, y, e')$ at $y \in C_\rho$. This means that all nodes in C_ρ are completely splittable. Splitting them off, and replace series edges by one edge. Then $(G; \rho)$ is restricted Eulerian with $C_\rho = \emptyset$. By Corollary 3.9, there exists half-integral optimal multiflow for the current graph, and thus there exists a 1/12-integral optimal multiflow for the initial graph. This completes the proof of the main theorem (Theorem 1.4).

4 Concluding remarks

In this paper, we proved that the multiflow feasibility problems for demand graph K_3+K_3 and $K_{n,m}$-metric weighted maximum multiflow problems have bounded fractionality. However, we do not know whether the constant $k = 12$ is tight. The main obstruction is an occurrence of 4/3-SPUP and 3/2-SPUP. If one could avoid such an SPUP, then one would get a half-integral optimal multiflow. Unfortunately, we could not do it.

Our approach is applicable to prove the existence of an 1/12-integral optimal multiflow for a larger class of maximum multiflow problems and also provides a polynomial time algorithm to find it; see [6, 7] for detail.

Acknowledgments

We thank the referee for helpful comments and suggestions. This work is supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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