RIMS-1646

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By

Makoto FUCHIWAKI, Michihiko FUJII, Kyoji SAITO
and Shunsuke TSUCHIOKA

November 2008

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Makoto Fuchiwaki,* Michihiko Fujii,† Kyoji Saito‡ and Shunsuke Tsuchioka§

November 7, 2008

Abstract

In this paper, we construct minimum state geodesic word acceptors for each Artin monoid of finite type with respect to the standard generator system by modifying the one with respect to the other generator system constructed by Charney. We note that the automata depend on the choices of liftings of the square free elements to the words over the standard generator system. Using the automata, we calculate examples of the explicit rational function expressions of the growth series by computer.

1 Introduction

For a finitely generated group (or a finitely generated monoid) G with a given generator system Σ, we have the notion of the growth series [M,Sc]

$$P_{G,\Sigma}(z) \ := \ \sum_{n=0}^{\infty} \gamma_n z^n,$$

where γ_n for $n \in \mathbf{Z}_{\geq 0}$ is the number of elements in the group (or in the monoid) G which are expressed by words of generators $\Sigma \cup \Sigma^{-1}$ (or Σ if G is a monoid) with length less than or equal to n. Even though there is a general method using a finite state geodesic automaton (see §3 for the definition) to determine the growth series [E,E-IF-Z], not many calculated examples are known [Ca,F-P,Ca-W]. In this sense, we are still far from understanding the growth series.

In a recent study [S1, §10] of the third author, the places of the poles of the growth series (if the growth series admits a meromorphic function expression) play an important role in describing the space of partition functions $\Omega(G, \Sigma)$ for a group (or a monoid). For this reason, he, in particular, asked [S1, §11] to calculate the growth series for Artin groups and Artin monoids with respect to the standard generator systems (see §2 for the definition). In this paper, we partly answer to the question by explicitly constructing finite state geodesic automata accepting the Artin monoids of finite type with respect to the standard generator systems.

*Application Deployment Department, Hitachi Ltd., Yokohama 244-8555, Japan. (makoto.fuchiwaki.ob@hitachi.com)

†Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan. (mfujii@math.kyoto-u.ac.jp)

‡Institute for the Physics and Mathematics of the Universe, The University of Tokyo, Chiba 277-8568, Japan. (kyoji.saito@ipmu.jp)

§Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan. (tshun@kurims.kyoto-u.ac.jp)

Artin groups and Artin monoids were introduced by Brieskorn and Saito [B-S] more than three decades ago. An early example of calculating their growth series is due to Xu [X], who showed that the growth series of the braid monoid of n-strings (i.e., the Artin monoid of type A_{n-1}) with respect to the standard generator system is a rational function. He also gave explicit rational function expressions for three and four strings cases. Then, Charney [C2] has given the growth series for each Artin group of finite type but with respect to a generator system consisting of all square free elements (see §2 for the definition). More precisely, based on the solution of the word problem given in [B-S,D], a geodesic automaton over that generator system is constructed in [C2]. Recently, Mairesse and Mathéus [M-M] obtained an explicit rational function expression of the growth series for the Artin group of type $I_2(p)$ for arbitrary $p \in \mathbf{Z}_{\geq 2}$ with respect to the standard generator system.

The purpose of the present paper is to construct minimum state geodesic word acceptors for each Artin monoid of finite type with respect to the standard generator system by modifying the one constructed by Charney. We note that the automata depend on the choices of liftings of the square free elements to the words over the standard generator system. Using the automata, we calculate examples of the explicit rational function expressions of the growth series by computer. Then we observe that the numerators of the rational functions are always equal to 1. In fact, this observation was proved independently in [D] and [S2], one by a shortening property of galaries of chambers of a simplicial arrangement and the other by a divisibility property of Artin monoids. For another observation on the automata, see Remark 6.4. in §6.

Let us explain the contents of this paper. In §2, §3 and §4, we review Artin groups and Artin monoids [B-S], geodesic automata on groups and monoids [E,E-IF-Z] and the geodesic automata by Charney [C2] respectively. In §5, we construct a minimum state deterministic automaton accepting the Artin monoid which is geodesic with respect to the standard generator system (an existence of such an automaton was mentioned in [C1]). In §6, we give explicit rational function expressions of the growth series for some Artin monoids. We also discuss the possibility that our atuomata provide invariants of Artin monoids other than the growth function.

The source codes and their manual for constructing the automata and calculating the growth series including a visualization are available in the following website:

`http://www.kurims.kyoto-u.ac.jp/~saito/FFST/index.html`

2 Artin groups and Artin monoids

In this section, we recall definitions and basic facts on Artin groups and Artin monoids from [B-S].

Let $M = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix (see [B, Chapter 3]) whose entries are indexed by a finite set I. That is, M is a symmetric integral matrix such that $m_{i,i} = 1$ for $i \in I$ and $m_{i,j} \geq 2$ for $i, j \in I$ with $i \neq j$. Associated with a Coxeter matrix M, we introduce the Artin group G_M, the Artin monoid G_M^+ and the Coxeter group $\overline{G}_M$ as follows.

First, we fix a finite set, called an *alphabet*,

$$A = \{a_i \mid i \in I\}$$

of letters indexed by I. Let A^* be the free monoid generated by the alphabet A. We call an element of A^* a *positive word*. In order to define the Artin group G_M and the Artin monoid G_M^+, we introduce a notation for $i, j \in I$ and a non-negative integer $q \in \mathbf{Z}_{\geq 0}$:

$$\langle a_i a_j \rangle^q \quad := \quad \underbrace{a_i a_j a_i \cdots}_{q \text{ letters}},$$

which is a positive word of length q starting with a_i and then a_i and a_j appearing alternately.

Definition 2.1. The *Artin group* associated with a Coxeter matrix M is a group presented by

$$G_M \ := \ \langle \ a_i \ (i \in I) \mid \langle a_i a_j \rangle^{m_{i,j}} = \langle a_j a_i \rangle^{m_{j,i}} \ \ (i, j \in I) \ \rangle.$$

Definition 2.2. The *Artin monoid* associated with a Coxeter matrix M is a monoid presented by

$$G_M^+ \ := \ \langle \ a_i \ (i \in I) \mid \langle a_i a_j \rangle^{m_{i,j}} = \langle a_j a_i \rangle^{m_{j,i}} \ \ (i, j \in I) \ \rangle^+,$$

where we mean by the right-hand side a quotient of the free monoid A^* by an equivalence relation on A^* defined as follows: (i) two positive words $\omega, \omega' \in A^*$ are elementary equivalent if there are positive words $u, v \in A^*$ and indices $i, j \in I$ such that $\omega = u\langle a_i a_j \rangle^{m_{i,j}} v$ and $\omega' = u\langle a_j a_i \rangle^{m_{j,i}} v$, and (ii) two words $\omega, \omega' \in A^*$ are equivalent if there is a sequence $\omega_0 = \omega, \omega_1, \cdots, \omega_k = \omega'$ for some $k \in \mathbf{Z}_{\geq 0}$ such that ω_i is elementary equivalent to ω_{i+1} for $i = 0, \cdots, k-1$.

Let us denote by $|u|$ the number of letters in a positive word u, called the *degree* of u. By the above Definition 2.2, positive words in an equivalent class in G_M^+ have the same degree. By associating the degree to each equivalent class, we have a homomorphism:

$$\deg \ : \ G_M^+ \ \longrightarrow \ \mathbf{Z}_{\geq 0}.$$

Definition 2.3. The *Coxeter group* associated with a Coxeter matrix M is a group presented by

$$\overline{G}_M \ := \ \langle \ a_i \ (i \in I) \mid \langle a_i a_j \rangle^{m_{i,j}} = \langle a_j a_i \rangle^{m_{j,i}} \ \ (i, j \in I) \ , \ a_i^2 = 1 \ \ (i \in I) \ \rangle.$$

Definition 2.4. We call the set A the *standard generator system* of the Artin group G_M, of the Artin monoid G_M^+ and of the Coxeter group $\overline{G}_M$.

We shall call M a Coxeter matrix *of finite type* if $\overline{G}_M$ is a finite group. It is well-known that indecomposable Coxeter matrices of finite type are classified into the following types: A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$, E_n $(6 \leq n \leq 8)$, F_4, G_2, H_n $(n = 3, 4)$ and $I_2(p)$

$(p \geq 5, p \neq 6)$ (for examples, see [B]). In the following discussion of this paper, M is always one of them.

By the Definitions 2.1, 2.2 and 2.3, there are natural homomorphisms $G_M^+ \to G_M$ and $G_M \to \overline{G}_M$. For the former homomorphisms, the following injectivity is well-known.

Theorem 2.5. (see [B-S, §5.5]) *Let M be a Coxeter matrix of finite type. Then the homomorphism $G_M^+ \to G_M$ is injective.*

In order to understand the composite homomorphism $G_M^+ \to G_M \to \overline{G}_M$, we recall the concepts of square free elements.

Definition 2.6. An element $g \in G_M^+$ is called a *square free element* if no word ω of the equivalent class g admits an expression ua_ia_iv for some $u, v \in A^*$ and some $i \in I$. We set

$$\mathrm{QFG}_M^+ := \{\mu \in G_M^+ \mid \mu \text{ is a square free element }\}.$$

Theorem 2.7.(see [B-S, §5.6]) *Let M be a Coxeter matrix of finite type. Then the restriction of the canonical map $G_M^+ \to \overline{G}_M$ to the subset QFG_M^+ is bijective.*

Finally, we review basic facts on fundamental elements.

Definition 2.8. We say that $\omega \in G_M^+$ divides $\omega' \in G_M^+$ from the left (resp. right) and denote $\omega \mid_l \omega'$ (resp. $\omega \mid_r \omega'$), if there are words $u, v \in A^*$ such that u belongs to the equivalence class ω and uv (resp. vu) belongs to the equivalence class ω'. For an element $\omega \in G_M^+$, put

$$I_l(\omega) := \{i \in I \mid a_i \mid_l \omega\} \quad \text{and} \quad I_r(\omega) := \{i \in I \mid a_i \mid_r \omega\}.$$

Lemma-Definition 2.9. (see [B-S, §5]) Let M be a Coxeter matrix of finite type. Then, for any subset J of I, there exists an element $\Delta_J \in G_M^+$ with the following two properties:

1. For any $i \in J$, we have $a_i \mid_l \Delta_J$ and $a_i \mid_r \Delta_J$.
2. If an element $u \in G_M^+$ satisfies $a_i|_l u$ (resp. $a_i|_r u$) for any $i \in J$, then $\Delta_J|_l u$ (resp. $\Delta_J|_r u$).

The element Δ_J is unique and is called the *fundamental element for J*. The fundamental element for I is simply denoted by Δ and is called the fundamental element.

We have the following table of the degree of the fundamental elements.

M	A_n	B_n	D_n	E_6	E_7	E_8	F_4	G_2	H_3	H_4	$I_2(p)$
$\deg(\Delta)$	$n(n+1)/2$	n^2	$n(n-1)$	36	63	120	24	6	15	60	p

Remark 2.10. For any subset J of I, let us denote by $M|_J$ the Coxeter matrix obtained from M by restricting the index set from I to J. We have the following facts ([B-S]).

1. $\{$ left divisors of Δ_J in G^+_M $\}$ $=$ $\{$ left divisors of Δ_J in G^+_M $\}$ $=$ $\mathrm{QFG}^+_{M|_J}$.
2. $\deg(\Delta_J) =$ the number of reflections in the Coxeter group $\overline{G}_{M|_J}$
   $=$ the maximal length of the elements of the Coxeter group $\overline{G}_{M|_J}$.

3 Automata on groups and monoids

In this section, we briefly review some definitions concerning automata and automatic groups, referring to [E,E-IF-Z,K].

Definition 3.1. A *deterministic finite automaton* (*DFA* for short) W is a quintuple $(S, \Gamma, \tau, s_0, Y)$, where

- S is a finite set (called the *set of states*),
- Γ is a finite set (called the *alphabet*),
- $\tau : S \times \Gamma \to S$ is a map (called the *transition function*),
- $s_0 \in S$ is an element of S (called the *start state*),
- $Y \subseteq S$ is a subset of S (called the *set of accept states*).

We often call W a *DFA over* Γ to emphasize the alphabet.

Let Γ^* be the set of all strings over the alphabet Γ, which is identical to the free monoid generated by Γ. We use both of "a positive word" and "a string" to represent an element of Γ^*. We denote the empty string by ϵ. The transition function τ extends to a function

$$\tau : S \times \Gamma^* \longrightarrow S,$$

according to the following natural inductive rules

$$\begin{array}{lll} \tau(s,\epsilon) & := & s, \\ \tau(s,\omega\gamma) & := & \tau(\tau(s,\omega),\gamma) \quad (\omega \in \Gamma^*,\ \gamma \in \Gamma). \end{array}$$

The *language accepted* by W is defined as follows:

$$L(W) := \{\omega \in \Gamma^* \mid \tau(s_0,\omega) \in Y\}.$$

Let G be a finitely generated group or a finitely generated monoid with a given finite generator system Σ. We set

$$\Sigma' := \begin{cases} \Sigma \cup \Sigma^{-1} & \text{if } G \text{ is a group,} \\ \Sigma & \text{if } G \text{ is a monoid.} \end{cases}$$

Let

$$\pi : \Sigma'^* \to G$$

be the natural surjective semigroup homomorphism. For $g \in G$, the length $l_\Sigma(g)$ of g with respect to Σ is defined by

$$l_\Sigma(g) := \min\{k \in \mathbf{Z}_{\geq 0} \mid g = \pi(\sigma_1 \cdots \sigma_k) \text{ for some } \sigma_i \in \Sigma' \ (i = 1, \ldots, k)\}.$$

In the case of the Artin monoid G_M^+, we have $l_\Sigma(g) = \deg(g)$ for $g \in G_M^+$.

Definition 3.2. Let G be a group or a monoid with a given finite generator system Σ. A DFA W is called a *word acceptor* for (G, Σ') if W is a deterministic finite automaton over Σ' such that the composite $\pi|_{L(W)} : L(W) \subseteq \Sigma'^* \to G$ is bijective.

Definition 3.3. Let G be a group or monoid with a given finite generator system Σ. Let W be a word acceptor for (G, Σ') and $\pi|_{L(W)} : L(W) \simeq G$ the associated bijective map. We say that W is *geodesic* over Σ', if for any $w \in L(W)$, w is a shortest representative of $\pi(w)$ in Σ'^*.

We will produce non-deterministic finite automata in constructing a desired deterministic finite automaton. We recall their definitions as follows.

Definition 3.4. A *non-deterministic finite automaton* (*NFA* for short) W is a quintuple $(S, \Gamma, \tau, S_0, Y)$, where

- S is a finite set (called the *set of states*),
- Γ is a finite set (called the *alphabet*),
- $\tau \subseteq S \times \Gamma \times S$ (called the *set of arrows*),
- S_0 is a subseteq of S (called the *set of start states*),
- $Y \subseteq S$ is a subset of S (called the *set of accept states*).

We often call W an *NFA over* Γ to emphasize the alphabet.

A triple $(s_1, \gamma, s_2) \in \tau$ is called an *arrow* and γ is called the *label* of the arrow. The *language accepted* by W is defines as follows:

$$\begin{aligned} L(W) := \{\omega \in \Gamma^* \mid \ & \exists k \in \mathbf{Z}_{\geq 0},\ \exists (s_1, \gamma_1, s_2, \ldots, s_k, \gamma_k, s_{k+1}) \text{ s.t. } \\ & \omega = \gamma_1 \cdots \gamma_k, s_1 \in S_0, s_{k+1} \in Y \text{ and } \\ & 1 \leq \forall i \leq k, (s_i, \gamma_i, s_{i+1}) \in \tau, \gamma_i \in \Gamma\}. \end{aligned}$$

Remark 3.5. A subset $L \subseteq \Gamma^*$ is called a *regular language (over* Γ*)* if there exists a DFA X such that $L = L(X)$ (see [K, Lecture 3]) and many characterizations of regularity are known such as

by NFA: A subset $L \subseteq \Gamma^*$ is regular iff there exists an NFA X such that $L = L(X)$ (see [K, Lecture 6]).

by regular expression: A subset $L \subseteq \Gamma^*$ is regular iff there exists a *regular expression* α such that $L = L(\alpha)$ (see [K, Lecture 8]).

by some kind of grammers: A subset $L \subseteq \Gamma^*$ is regular iff there exists a *right-linear* (or equivalently *left-linear*/*strongly right-linear*/*strongly left-linear*) grammer G such that $L = L(G)$ (see [K, Homework 5]).

In this sense, regularity of a language is a rigid concept.

Remark 3.6. It is well-known that for each regular language $L \subseteq \Gamma^*$, there exists a minimum state DFA X over Γ such that $L = L(X)$ and X is unique up to isomorphism (see [K, Lecture 15]). In other words, we can speak of "the automaton" that accepts L. As a practical importance, there is a standard algorithm to obtain a minimum state DFA X' for a given NFA X such that $L(X') = L(X)$ by the following two steps. Here we just recall them briefly and give references.

The subset construction: This algorithm gives a DFA $X'' = (2^S, \Gamma, \tau'', S_0, Y'')$ for a given NFA $X = (S, \Gamma, \tau, S_0, Y)$ such that $L(X'') = L(X)$ as follows (see [K, Lecture 6]).

$$\begin{aligned}&\tau'' : 2^S \times \Gamma \longrightarrow 2^S, \quad (X, a) \longmapsto \{y \in S \mid \exists x \in X \text{ s.t. } (x, a, y) \in \tau\}\\ &Y'' = \{X \subseteq S \mid Y \cap X \neq \emptyset\}\end{aligned}$$

The minimization algorithm: This algorithm gives a minimum state DFA $X' = (S', \Gamma, \tau', s', Y')$ for a given DFA $X'' = (S'', \Gamma, \tau'', s'', Y'')$ such that $L(X') = L(X'')$ as the following steps (see [K, Lecture 14]).

1. Get rid of inaccessible states; that is, states q for which there exists no $x \in \Gamma^*$ such that $\tau''(s'', x) = q$. By this we obtain a subset $S''' \subseteq S''$ and a DFA X'''

$$\begin{aligned}&S''' = S'' \setminus \{\text{inaccessible states}\}\\ &X''' = (S''', \Gamma, \tau''|_{S'''\times\Gamma}, s'', Y'' \cap S''') =: (S''', \Gamma, \tau''', s''', Y''')\end{aligned}$$

   such that $L(X''') = L(X'')$ and X''' has no inaccesible state.

2. Collapse "equivalent" states. We define an equivalence relation $\approx$ on S''' by

$$p \approx q \overset{\text{def}}{\Longleftrightarrow} \forall x \in \Gamma^* (\tau'''(p, x) \in Y''' \Leftrightarrow \tau'''(q, x) \in Y''')$$

   for $p, q \in S'''$. An essential part of the minimization algorithm is a calculation of the equivalence relation $\approx$ by a variant of greedy method which we don't explain in this paper (see [K, Lecture 14]). After a calculation of the equivalence relation $\approx$, the desired X' is given by

$$\begin{aligned}&S' = S'''/\approx, \quad s' = [s'''], \quad Y' = Y'''/\approx\\ &\tau' : S' \times \Gamma \longrightarrow S', ([p], a) \longmapsto [\tau'''(p, a)]\end{aligned}$$

   where $[p]$ for $p \in S'''$ means an equivalent class of p. Note Y' and τ' are well-defined.

4 A geodesic word acceptor for an Artin group by Charney

In this section, we review a geodesic word acceptor for an Artin group over a generator system consisting of all square free elements that is constructed by Charney [C2].

Since M is a Coxeter matrix of finite type, G_M^+ is regarded as a subset of G_M by Theorem 2.5. Let

$$\Lambda_M := \mathrm{QFG}_M^+ \setminus \{e\} (\subseteq G_M^+ \subseteq G_M),$$

where e denotes the identity element of the Artin monoid G_M^+. Since the standard generator system A is a subset of Λ_M, Λ_M is a generator system of the Artin group G_M. We denote the natural surjection by ξ

$$\xi : (\Lambda_M \sqcup \Lambda_M^{-1})^* \to G_M.$$

We recall the Charney's geodesic word acceptor as follows.

Definition 4.1. For each square free element $\lambda \in \Lambda_M$, set

$$\begin{array}{rcl} \mathcal{S}(\lambda) & := & \{a_i \in A \mid i \in I_l(\lambda)\}, \\ \mathcal{E}(\lambda) & := & \{a_i \in A \mid i \in I_r(\lambda)\}. \end{array}$$

We call $\mathcal{S}(\lambda)$ and $\mathcal{E}(\lambda)$ the *start set* and the *end set* of λ, respectively.

Define a DFA U by

$$U := (S, \Lambda_M \sqcup \Lambda_M^{-1}, \tau, e, Y),$$

where S, τ and Y are defined by

$$\begin{array}{rcl} S := & & (2^A \setminus \{\phi\}) \sqcup (2^{A^{-1}} \setminus \{\phi\}) \sqcup \{\mathsf{fail}, e\}, \\ \tau & : & S \times (\Lambda_M \sqcup \Lambda_M^{-1}) \to S; \text{ the transition function defined by} \\ & & \tau(T, \lambda^{\mathrm{sgn}}) := \begin{cases} \mathcal{E}(\lambda) & \text{if sgn} = +1 \text{ and } T \subseteq \mathcal{S}(\lambda) \subseteq A, \\ \mathcal{S}(\lambda)^{-1} & \text{if sgn} = -1 \text{ and } \mathcal{E}(\lambda)^{-1} \subseteq T \subseteq A^{-1}, \\ \mathcal{S}(\lambda)^{-1} & \text{if sgn} = -1 \text{ and } T \subseteq A \setminus \mathcal{E}(\lambda), \\ \mathcal{E}(\lambda) & \text{if sgn} = +1 \text{ and } T = e, \\ \mathcal{S}(\lambda)^{-1} & \text{if sgn} = -1 \text{ and } T = e, \\ \mathsf{fail} & \text{if else,} \end{cases} \\ Y := & & (2^A \setminus \{\phi\}) \sqcup (2^{A^{-1}} \setminus \{\phi\}) \sqcup \{e\}. \end{array}$$

Here, fail denotes a "failure state" (no edges emanate from fail to any other state).

Theorem 4.2.(see [C2]) *Let M be a Coxeter matrix of finite type. Then U is a geodesic word acceptor for $(G_M, \Lambda_M \sqcup \Lambda_M^{-1})$.*

We remark that Theorem 4.2 is a direct consequence of the theorem below.

Theorem 4.3.(see [C2]) *Let M be a Coxeter matrix of finite type. Then, for any $g \in G_M$, there exists a unique string $\lambda_1 \cdots \lambda_j \lambda_{j+1}^{-1} \cdots \lambda_{j+k}^{-1} \in (\Lambda_M \sqcup \Lambda_M^{-1})^*$ such that*

$$\begin{cases} \lambda_1, \ldots, \lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+k} \in \Lambda_M, \\ g = \xi(\lambda_1 \cdots \lambda_j \lambda_{j+1}^{-1} \cdots \lambda_{j+k}^{-1}), \\ \mathcal{E}(\lambda_i) \subseteq \mathcal{S}(\lambda_{i+1}) \quad (i = 1, \ldots, j-1), \\ \mathcal{E}(\lambda_{j+i+1}) \subseteq \mathcal{S}(\lambda_{j+i}) \quad (i = 1, \ldots, k-1), \\ \mathcal{E}(\lambda_j) \subseteq A \setminus \mathcal{E}(\lambda_{j+1}). \end{cases}$$

Definition 4.4. We call the string $\lambda_1 \cdots \lambda_j \lambda_{j+1}^{-1} \cdots \lambda_{j+k}^{-1}$ in the theorem above the *normal form* for $g \in G_M$. Note that the normal form given here is different from "the normal form" considered in [B-S, §6].

5 A geodesic word acceptor for an Artin monoid

Let M be a Coxeter matrix of finite type. In this section, we construct a word acceptor for G_M^+ which is geodesic over A.

5.1 An NFA over Λ_M

Let M be a Coxeter matrix of finite type. Let $U = (S, \Lambda_M \sqcup \Lambda_M^{-1}, \tau, e, Y)$ be the word acceptor given in Section 4. We consider the following DFA over Λ_M, that is a half of U.

$$U^+ = (S, \Lambda_M, \tau^+ := \tau|_{S \times \Lambda_M}, e, Y).$$

Lemma 5.1. $L(U) \cap \Lambda_M^* = L(U^+)$.

Proof. Take an element $\omega \in L(U) \cap \Lambda_M^*$. Then ω is a string on Λ_M and an element of the domain of the transition function of U^+. We have also that $\tau^+(e, \omega) = \tau(e, \omega) \in Y$. Thus $\omega \in L(U^+)$. Hence $L(U) \cap \Lambda_M^* \subseteq L(U^+)$. It is obvious that the opposite inclusion holds. $\square$

Next, consider a subset of S defined by

$$S_0 := \{\tau^+(e, \lambda) \mid \lambda \in \Lambda_M \}.$$

Then consider the following NFA over Λ_M

$$\widetilde{U}^+ := (S, \Lambda_M, \widetilde{\tau}^+, S_0, Y),$$

where

$$\widetilde{\tau}^+ := \{(s_1, \lambda, s_2) \in S \times \Lambda_M \times S \mid \tau^+(s_1, \lambda) = s_2\}.$$

The language accepted by $\widetilde{U}^+$ is

$$L(\widetilde{U}^+) = \{\omega \in \Lambda_M^* \mid \quad {}^\exists k \in \mathbf{Z}_{\geq 0},\ {}^\exists(s_1, \lambda_1, s_2, \ldots, s_k, \lambda_k, s_{k+1}) \text{ s.t. } s_1 \in S_0,\ s_{k+1} \in Y,\ \lambda_i \in \Lambda_M\ (i = 1, \ldots, k),\ \omega = \lambda_1 \cdots \lambda_k\}.$$

Lemma 5.2. $L(\widetilde{U}^+) = L(U^+)$.

The following lemma described in [C2, §2] is necessary to show Lemma 5.2.

Lemma 5.3. (see [C2, §2]) A word $\lambda_1 \lambda_2 \cdots \lambda_k \in (\Lambda_M \sqcup \Lambda_M^{-1})^*$ is a normal form if and only if $\lambda_i \lambda_{i+1}$ is a normal form for $i \in \{1, \ldots, k-1\}$.

Proof of Lemma 5.2. Take an element $\lambda_1\lambda_2\cdots\lambda_k \in L(U^+)$. Let λ is an arbitrary element in $\mathcal{S}(\lambda_1)$. Then, by $\lambda \in A$, we have $\mathcal{E}(\lambda) = \{\lambda\} \subseteq \mathcal{S}(\lambda_1)$. Then $\lambda\lambda_1$ is a normal form. By Lemma 5.3, $\lambda_1\lambda_2, \lambda_2\lambda_3, \ldots, \lambda_{k-1}\lambda_k$ are normal forms. Again by Lemma 5.3, $\lambda\lambda_1\lambda_2\cdots\lambda_k$ is a normal form. Thus $\lambda\lambda_1\lambda_2\cdots\lambda_k \in L = L(U)$. Then, by Lemma 5.1, we have $\lambda\lambda_1\lambda_2\cdots\lambda_k \in L(U^+)$. Hence, by $\lambda \in \Lambda_M$, we have $\lambda_1\lambda_2\cdots\lambda_k \in L(\widetilde{U}^+)$. Therefore $L(U^+) \subseteq L(\widetilde{U}^+)$.

Conversely, for any element $\lambda_1\cdots\lambda_k \in L(\widetilde{U}^+)$, there exists an element $\lambda \in \Lambda_M$ such that $\lambda\lambda_1\cdots\lambda_k \in L(U^+)$. Then, by Lemma 5.3, $\lambda_1\cdots\lambda_k$ is a normal form. Thus $\lambda_1\cdots\lambda_k \in L(U^+)$. Hence $L(\widetilde{U}^+) \subseteq L(U^+)$. □

By Theorem 2.5, G_M^+ is regarded as a subset of G_M. Let η be the natural surjection.

$$\eta \;:\; \Lambda_M^* \longrightarrow G_M^+ \subseteq G_M.$$

Lemma 5.4. $\eta|_{L(U^+)}$ $(= \eta|_{L(\widetilde{U}^+)})$ *is bijective.*

Proof. We have $\eta = \xi|_{\Lambda_M^*}$ and, by Lemma 5.1, $L(U^+) = L(U) \cap \Lambda_M^*$. We obtain

$$\eta|_{L(U^+)} = \eta|_{L(U)\cap\Lambda_M^*} = (\xi|_{\Lambda_M^*})|_{L(U)\cap\Lambda_M^*} = \xi|_{L(U)\cap\Lambda_M^*} = (\xi|_{L(U)})|_{\Lambda_M^*}.$$

Thus $\eta|_{L(U^+)} = (\xi|_{L(U)})|_{\Lambda_M^*}$ is injective, because $(\xi|_{L(U)})$ is bijective. For any $g \in G_M^+ = G_M \cap \eta(\Lambda_M^*)$, by Theorem 4.3, there exists $\lambda_1, \ldots, \lambda_k \in \Lambda_M$ such that $\lambda_1\cdots\lambda_k$ is the normal form for g. Thus we have $\lambda_1\cdots\lambda_k \in L(U) \cap \Lambda_M^*$ and $\eta(\lambda_1\cdots\lambda_k) = g$. Therefore, $\eta|_{L(U^+)}$ is surjective. □

5.2 A generalized finite automaton over A

We construct a generalized finite automaton for the Artin monoid G_M^+ whose alphabet is the standard generator system A from the NFA $\widetilde{U}^+$. A generalized finite automaton in this section means a generalization of NFA whose difference is that its arrows are labeled by strings not just letters.

First of all, choose a lift $\iota : \Lambda_M \to A^*$ so that $\pi \circ \iota = 1_{\Lambda_M}$, where $\pi : A^* \to G_M^+$ is the quotient semigroup homomorphism.

$$\begin{tikzcd} A^* \arrow[dr, "\pi"] & \\ \Lambda_M \arrow[u, "\iota"] \arrow[r, "\text{inclusion}"'] & G_M^+ \end{tikzcd}$$

Define a semigroup homomorphism $\bar{\iota} : \Lambda_M^* \to A^*$ by

$$\bar{\iota}(\lambda_1\cdots\lambda_k) := \iota(\lambda_1)\cdots\iota(\lambda_k), \qquad \lambda_1, \ldots, \lambda_k \in \Lambda_M.$$

Then we obtain a generalized finite automaton V on A defined by

$$V := (S, A, \psi, S_0, Y),$$

where

$$\psi := \{(s_1, \iota(\lambda), s_2) \in S \times A^* \times S \mid \lambda \in \Lambda_M,\ (s_1, \lambda, s_2) \in \widetilde{\tau}^+\}.$$

The automaton V depends on the choice of the lift ι. The language accepted by V is defined similarly in the case of NFA as follows

$$\begin{aligned} L(V) = \{\omega \in A^* \mid \quad & {}^{\exists}k \in \mathbf{Z}_{\geq 0},\ {}^{\exists}(s_1, \iota(\lambda_1), s_2, \ldots, s_k, \iota(\lambda_k), s_{k+1}) \text{ s.t.} \\ & s_1 \in S_0, s_{k+1} \in Y, \omega = \iota(\lambda_1) \cdots \iota(\lambda_k) \text{ and} \\ & 1 \leq \forall i \leq k, \lambda_i \in \Lambda_M, (s_i, \iota(\lambda_i), s_{i+1}) \in \psi\}. \end{aligned}$$

Lemma 5.5. $\bar{\iota}(L(\widetilde{U}^+)) = L(V)$.

Proof. For any $\lambda_1, \ldots, \lambda_k \in \Lambda_M$,

$$\begin{aligned} \lambda_1 \cdots \lambda_k \in L(\widetilde{U}^+) \iff & {}^{\exists}s_1, \ldots, s_{k+1} \in S \\ & \text{s.t. } (s_i, \lambda_i, s_{i+1}) \in \widetilde{\tau}^+ \ (i = 1, \ldots, k),\ s_1 \in S_0,\ s_{k+1} \in Y, \\ \Longrightarrow & {}^{\exists}s_1, \ldots, s_{k+1} \in S \\ & \text{s.t. } (s_i, \iota(\lambda_i), s_{i+1}) \in \psi \ (i = 1, \ldots, k), s_1 \in S_0,\ s_{k+1} \in Y, \\ \Longrightarrow & \iota(\lambda_1) \cdots \iota(\lambda_k) = \bar{\iota}(\lambda_1 \cdots \lambda_k) \in L(V). \end{aligned}$$

Therefore we have $\bar{\iota}(L(\widetilde{U}^+)) \subseteq L(V)$. Conversely, for any $b_1, \ldots, b_k \in A$,

$$\begin{aligned} b_1 \cdots b_k \in L(V) \iff & {}^{\exists}(s_i, \iota(\lambda_i), s_{i+1}) \in \psi \ (i = 1, \ldots, k), \\ & \text{s.t. } \begin{cases} s_1 \in S_0,\ s_{k+1} \in Y, \\ b_1 \cdots b_k \in L(\iota(\lambda_1) \cdots \iota(\lambda_k)), \end{cases} \\ \Longrightarrow & {}^{\exists}\widetilde{\lambda}_i \in \iota^{-1}(\iota(\lambda_i)) \ (i = 1, \ldots, k), {}^{\exists}s_1, \ldots, s_{k+1} \in S, \\ & \text{s.t. } \begin{cases} (s_i, \widetilde{\lambda}_i, s_{i+1}) \in \widetilde{\tau}^+ \ (i = 1, \ldots, k), \\ s_1 \in S_0,\ s_{k+1} \in Y, \\ \bar{\iota}(\widetilde{\lambda}_1 \cdots \widetilde{\lambda}_k) = b_1 \cdots b_k, \end{cases} \\ \Longrightarrow & {}^{\exists}\widetilde{\lambda}_i \in \Lambda_M \text{ s.t. } \widetilde{\lambda}_1 \cdots \widetilde{\lambda}_k \in L(\widetilde{U}^+),\ \bar{\iota}(\widetilde{\lambda}_1 \cdots \widetilde{\lambda}_k) = b_1 \cdots b_k, \\ \iff & b_1 \cdots b_k \in \bar{\iota}(L(\widetilde{U}^+)). \end{aligned}$$

Then we have $L(V) \subseteq \bar{\iota}(L(\widetilde{U}^+))$. $\square$

Now we have the following commutative diagram.

$$\begin{tikzcd}
A^* \arrow[r, phantom, "\supseteq"] & L(V) \arrow[dr, "\pi|_{L(V)}"] & \\
\Lambda_M^* \arrow[u, "\bar{\iota}"] \arrow[r, phantom, "\supseteq"] & L(\widetilde{U}^+) \arrow[u, "\bar{\iota}|_{L(\widetilde{U}^+)}"'] \arrow[r, "\simeq", "\eta|_{L(\widetilde{U}^+)}"'] & G_M^+
\end{tikzcd}$$

By Lemma 5.4, $\eta|_{L(\widetilde{U}^+)}$ is bijective. Then $\bar{\iota}|_{L(\widetilde{U}^+)}$ is injective. Thus, by Lemma 5.5, $\bar{\iota}|_{L(\widetilde{U}^+)}$ is bijective. Therefore we have $\pi|_{L(V)} : L(V) \simeq G_M^+$.

5.3 An NFA over A

We produce an NFA over A by dividing each arrows of the automaton V as follows.

Take an arrow $t = (s_1, b_0 b_1 \cdots b_{n_t-1}, s_2) \in \psi$ ($\subseteq S \times A^* \times S$) of the generalized finite automaton V. Divide t into n_t pieces between b_{k-1} and b_k ($k \in \{1, \ldots, n_t - 1\}$). Denote these dividing points by $s'_{t,k}$ and set $s'_{t,0} := s_1$, $s'_{t,n_t} := s_2$ (see Figure 1).

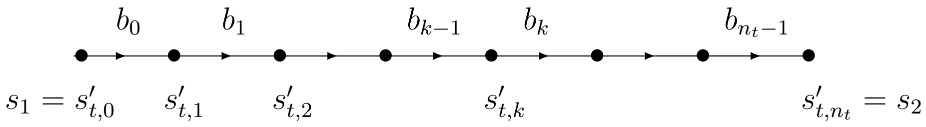


Figure 1

Define

$$S' := \{s'_{t,k} \mid t = (s_1, b_0 \cdots b_{n_t-1}, s_2) \in \psi, k \in \{0, \ldots, n_t\}\}.$$

Then define

$$\psi' := \{(s'_{t,k}, b_k, s'_{t,k+1}) \in S' \times A \times S' \mid t = (s_1, b_0 \cdots b_{n_t-1}, s_2) \in \psi, k \in \{0, \ldots, n_t - 1\}\}.$$

It is obvious that $S_0 \subseteq S'$ and $Y \subseteq S'$. Then we define the following an NFA over A

$$V' := (S', A, \psi', S_0, Y).$$

The language accepted by V' is described as

$$\begin{aligned} L(V') = \{\omega \in A^* \mid \quad & {}^{\exists}k \in \mathbf{Z}_{\geq 0},\ {}^{\exists}(s'_1, b_1, s'_2, \ldots, s'_k, b_k, s'_{k+1}) \text{ s.t.} \\ & s'_1 \in S_0, s'_{k+1} \in Y, \omega = b_1 \cdots b_k \text{ and} \\ & 1 \leq \forall i \leq k, b_i \in A, (s'_i, b_i, s'_{i+1}) \in \psi'\}. \end{aligned}$$

By considering the construction of V', it can be seen that $L(V') = L(V)$.

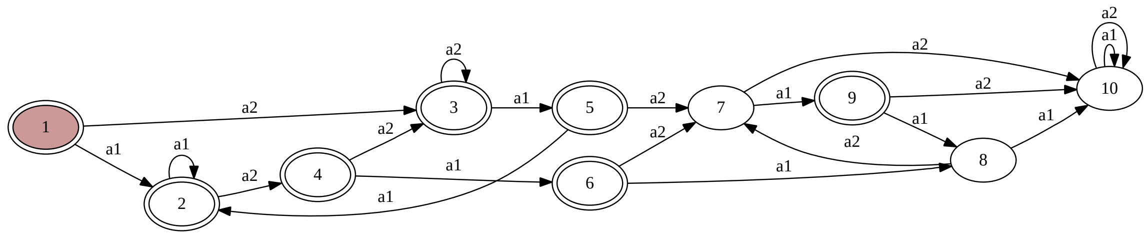
5.4 A minimum state DFA for an Artin monoid over A

By the algorithms reviewed in Remark 3.6, we obtain a unique (up to isomorphism) minimum state DFA $\mathbf{W}$ such that $L(\mathbf{W}) = L(V')$ and clearly $\mathbf{W}$ is geodesic over A. Note that the minimum state DFA $\mathbf{W}$ depends on the choice of a lift $\iota : \Lambda_M \to A^*$. If we choose another lift ι', we may obtain a minimum state DFA $\mathbf{W}'$ that is not isomorphic to $\mathbf{W}$. For example, even the number of stetes of $\mathbf{W}$ and that of $\mathbf{W}'$ are not necessarly the same (see Remark 6.4).

On the website given in §1, we attach source codes that calculate a minimum state DFA $\mathbf{W}$ for a given lift ι. For example, the following is a visualization of the minimum state word acceptor for the Artin monoid associated with the Coxeter matrix $M = A_2$ for a lift ι defined as follows

$$\begin{aligned}\iota : \Lambda_M &\longrightarrow A^*, \\ &e \mapsto \epsilon, \quad a_1 \mapsto a_1, \quad a_2 \mapsto a_2, \quad a_1a_2 \mapsto a_1a_2, \\ &a_2a_1 \mapsto a_2a_1, \quad a_1a_2a_1 (= a_2a_1a_2) \mapsto a_1a_2a_1.\end{aligned}$$

Here the filled state is the start state and the double-circled states are the accept states. In other words, the start state is 1 and the accept states are 1,2,3,4,5,6, and 9. Since we choose the string $a_1a_2a_1 \in A^*$ for a lift of $a_1a_2a_1 = a_2a_1a_2 \in \overline{G}_M$, the string $a_1a_2a_1 \in A^*$ is accepted but the string $a_2a_1a_2 \in A^*$ is rejected.



We further explain by length 4 strings. There are 16 strings of length 4 and acceptance/rejection by this automaton is given as follows. Note that in $G^+_{A_3}$ the following holds.

$$a_2a_2a_1a_2 = a_2a_1a_2a_1 = a_1a_2a_1a_1, \quad a_1a_1a_2a_1 = a_1a_2a_1a_2 = a_2a_1a_2a_2.$$

From this, one easily see that this automaton accepts only one representative for each degree 4 element in $G^+_{A_3}$.

string	$a_1a_1a_1a_1$	$a_1a_1a_1a_2$	$a_1a_1a_2a_1$	$a_1a_1a_2a_2$	$a_1a_2a_1a_1$	$a_1a_2a_1a_2$	$a_1a_2a_2a_1$	$a_1a_2a_2a_2$
final state	accept	accept	accept	accept	reject	reject	accept	accept

string	$a_2a_1a_1a_1$	$a_2a_1a_1a_2$	$a_2a_1a_2a_1$	$a_2a_1a_2a_2$	$a_2a_2a_1a_1$	$a_2a_2a_1a_2$	$a_2a_2a_2a_1$	$a_2a_2a_2a_2$
final state	accept	accept	accept	reject	accept	reject	accept	accept

6 Growth functions for Artin monoids

In this section, we consider growth functions and some possible invariants for Artin monoids through the geodesic automatic structures constructed in the previous section.

Let M be a Coxeter matrix of finite type. Then the *growth series* and the *spherical growth series* for the Artin monoid G^+_M are defined by the formal power series

$$P_{G^+_M}(z) := \sum_{k=0}^{\infty} \#\{g \in G^+_M \mid \deg(g) \leq k\}\ z^k \quad \text{and} \quad \dot{P}_{G^+_M}(z) := \sum_{k=0}^{\infty} \#(\deg^{-1}(\mathrm{k}))\ z^k,$$

respectively. We clearly have $(1-z)P_{G^+_M}(z) = \dot{P}_{G^+_M}(z)$. So studying one of these series is equivalent to studying the other. In this paper we study the spherical growth series. The spherical growth series is a holomorphic function near 0, since the number of the generators in A is finite. Thus its radius of convergence is at least $\frac{1}{\#(A)-1}$ (see [E2, Lemma 1.2]). We call the spherical growth series the *spherical growth function*.

We shall see that the spherical growth function of G_M^+ is a rational function by making use of the geodesic word acceptor $\mathbf{W}$ as follows. Let us put $\mathbf{W}$ as $\mathbf{W} = (\mathbf{S}, A, \mathbf{\Psi}, \mathbf{S}_0, \mathbf{Y})$. Let $m := \#\ \mathbf{S}$. Order all the states of $\mathbf{W}$ arbitrarily and write them as $\mathbf{S} = \{x_1, \ldots, x_m\}$. The *transition matrix* $T = (t_{i,j}) \in M(m, \mathbf{Z})$ of $\mathbf{W}$ is defined by

$$t_{i,j} := \#\{a \in A \mid \mathbf{\Psi}(x_i, a) = x_j\} \quad \text{for } x_i, x_j \in \mathbf{S}.$$

Let $\mathbf{v} := (v_i)_{x_i \in \mathbf{S}}$ and $\mathbf{u} := (u_i)_{x_i \in \mathbf{S}}$ be the characteristic functions of $\mathbf{S}_0$ and $\mathbf{Y}$, respectively. That is,

$$v_i := \begin{cases} 1 & (x_i = \mathbf{S}_0), \\ 0 & (x_i \neq \mathbf{S}_0), \end{cases} \qquad \text{and} \quad u_i := \begin{cases} 1 & (x_i \in \mathbf{Y}), \\ 0 & (x_i \notin \mathbf{Y}). \end{cases}$$

Then we have

$${}^t\mathbf{v}\ T^k\ \mathbf{u} = \#\mathrm{deg}^{-1}(k),$$

since $\mathbf{W}$ is geodesic. From this equality, we have

$$\dot{P}_{G_M^+}(z) = \sum_{k=0}^{\infty} ({}^t\mathbf{v}\ T^k\ \mathbf{u})\ z^k = {}^t\mathbf{v}\ (E - zT)^{-1}\ \mathbf{u},$$

where E is the identity matrix of size $m \times m$. The right-hand side is a rational function whose denominator is a factor of $\det(E - zT)$.

On the website given in §1, we attach the source codes for describing the transition matrix T for $\mathbf{W}$. As explained in the website, we obtain the exact forms of the growth functions $\dot{P}_{G_M^+}(z) = {}^t\mathbf{v}\ (E - zT)^{-1}\ \mathbf{u}$ for several Coxeter matricies M of finite type by combining other softwares such as Mathematica, Maple and REDUCE which afford symbolic linear algebra as follows:

$$\begin{aligned}
\dot{P}_{G_{A_3}^+}(z) &= \frac{1}{1 - 3z + z^2 + 2z^3 - z^6}, \\
\dot{P}_{G_{A_4}^+}(z) &= \frac{1}{1 - 4z + 3z^2 + 3z^3 - 2z^4 - 2z^6 + z^{10}}, \\
\dot{P}_{G_{A_5}^+}(z) &= \frac{1}{1 - 5z + 6z^2 + 3z^3 - 6z^4 - 2z^6 + 2z^7 + 2z^{10} - z^{15}}, \\
\dot{P}_{G_{B_3}^+}(z) &= \frac{1}{1 - 3z + z^2 + z^3 + z^4 - z^9}, \\
\dot{P}_{G_{B_4}^+}(z) &= \frac{1}{1 - 4z + 3z^2 + 2z^3 - z^5 - z^6 - z^9 + z^{16}}, \\
\dot{P}_{G_{B_5}^+}(z) &= \frac{1}{1 - 5z + 6z^2 + 2z^3 - 3z^4 - 2z^5 - 2z^6 + 2z^7 - z^9 + 2z^{10} + z^{16} - z^{25}}, \\
\dot{P}_{G_{D_4}^+}(z) &= \frac{1}{1 - 4z + 3z^2 + 2z^3 - 3z^6 + z^{12}}, \\
\dot{P}_{G_{D_5}^+}(z) &= \frac{1}{1 - 5z + 6z^2 + 2z^3 - 4z^4 + z^5 - 4z^6 + z^7 + 2z^{10} + z^{12} - z^{20}}, \\
\dot{P}_{G_{F_4}^+}(z) &= \frac{1}{1 - 4z + 3z^2 + 2z^3 - z^4 - 2z^9 + z^{24}}, \\
\dot{P}_{G_{H_3}^+}(z) &= \frac{1}{1 - 3z + z^2 + z^3 + z^5 - z^{15}}, \\
\dot{P}_{G_{I_2(p)}^+}(z) &= \frac{1}{1 - 2z + z^p} \quad \text{(for many } p).
\end{aligned}$$

Remark 6.2. Saito, the third author of the paper, and Deligne showed independently the following theorem describing the growth functions of Artin monoids in terms of fundamental elements. Here the expression of the polynomial $N_M(z)$ is taken from [S2].

Theorem 6.3.(see [D,S2]) *Let M be a Coxeter matrix of finite type. Then the growth function for the Artin monoid G^+_M over A has the form*

$$\dot{P}_{G^+_M}(z) = \frac{1}{N_M(z)},$$

where $N_M(z)$ is the polynomial defined by

$$N_M(z) := \sum_{J \subseteq I} (-1)^{\#J} \; z^{\deg(\Delta_J)}.$$

Remark 6.4. Recall that the minimal word acceptor $\mathbf{W}$ constructed in §5 and its transition matrix T given in §6 depend on the choice of a lift ι (see 5.2 and 5.4) of the square free elements. For example, for type A_3 there are possible 12441600 lifts of the square free elements. The number of states $s = s(\iota)$ of $\mathbf{W} = \mathbf{W}(\iota)$, i.e., the size of the matrix $T = T(\iota)$ varies as the following table while a lift ι varies among all 12441600 lifts. Here we put

$$c_N := \#\{\iota : \Lambda_M \to A^* | \pi \circ \iota = 1_{\Lambda_M} \text{ and } s(\iota) = N\},$$

where $\pi : A^* \to G^+_M$ is the quotient semigroup homomorphism (see §5.2).

N	36	37	38	39	40	41	42	43	44	45	46	47
c_N	6	16	106	242	472	892	1736	2772	5446	8748	14434	21514
N	48	49	50	51	52	53	54	55	56	57	58	59
c_N	33144	45144	64910	88382	120198	164206	216044	275780	339668	405926	467864	538824
N	60	61	62	63	64	65	66	67	68	69	70	71
c_N	611852	686414	746652	783874	793324	767584	728598	678768	623716	565474	501206	435064
N	72	73	74	75	76	77	78	79	80	81	82	83
c_N	365850	301480	243434	194766	151450	116394	87590	65686	48990	35698	25842	19148
N	84	85	86	87	88	89	90	91	92	93	94	95
c_N	13740	9714	6762	4766	3420	2456	1754	1114	840	634	360	246
N	96	97	98	99	100	101	102	103	104	105	106	107
c_N	124	100	94	56	36	24	8	8	8	2	2	4

However, the denominator $N_M(z)$ of the growth function $\dot{P}_{G^+_M}(z)$, which is a factor of $\det(E - zT)$, is, of course, an invariant of the Artin monoid independent of the choice of the lift. Therefore, it should be interesting if one could find further invariants of the Artin monoids from the minimal geodesic word acceptor $\mathbf{W}$ or from the associated transition matrix T. For the type A_3, we have established the following fact by checking all lifts by computer.

Fact. *For the type A_3, the following 5 irreducible polynomials*

$$\begin{aligned} &f_1^{A_3}(z) := z^5 N_{A_3}(z^{-1})/(1 - z^{-1}) = z^5 - 2z^4 - z^3 + z^2 + z + 1, \\ &f_2^{A_3}(z) := z^5 + z^2 - 1, \quad f_3^{A_3}(z) := z^5 - z^2 + 1, \\ &f_4^{A_3}(z) := z^3 + z^2 - 1, \quad f_5^{A_3}(z) := z^3 - z^2 + 1. \end{aligned}$$

are factors of the characteristic polynomial $\det(zE - T)(= z^{s(\iota)} \det(E - z^{-1}T))$ *for any lift* ι.

Varying several (but not all) lifts for other types, we have a similar observation as follows. As in our website given in §1, we further have a similar observation for other types

$$A_5, A_6, B_4, B_5, B_6, D_5, D_6, H_4, F_4.$$

The meaning of this observation is obscure.

Observation. *For the types* $M = (m_{ij})_{i,j\in I} \in \{A_4, B_3, D_4, H_3\}$*, consider some number of irreducible polynomials* $f_i^M(z) \in \mathbf{Z}[z]$ $i = 1, 2, \cdots$ *given in the following table, where we put*[1] $f_1^M(z) := z^{\deg(\Delta_I)-1} N_M(z^{-1})/(1 - z^{-1})$. *Then each* $f_i^M(z)$ $i = 1, 2, \cdots$ *is a factor of the characteristic polynomial* $\det(zE - T)$ *for any lift* ι.

M	$f_1^M(z), f_2^M(z), \cdots$
A_4	$z^9 - 3z^8 + 3z^6 + z^5 + z^4 - z^3 - z^2 - z - 1, f_2^{A_4}, f_3^{A_4}$
B_3	$z^8 - 2z^7 - z^6 + z^4 + z^3 + z^2 + z + 1, f_2^{B_3}$
D_4	$z^{11} - 3z^{10} + 2z^8 + 2z^7 + 2z^6 - z^5 - z^4 - z^3 - z^2 - z - 1, f_2^{D_4}, f_3^{D_4}$
H_3	$z^{14} - 2z^{13} - z^{12} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1, f_2^{H_3}$

$f_2^{A_4} := z^{45} + z^{42} + z^{41} + 2z^{40} - z^{39} + z^{38} + z^{37} + z^{36} - 2z^{35} - 2z^{34} + 4z^{33} - 7z^{32} - z^{31} - 5z^{29} + 7z^{28} - 15z^{27} + 3z^{26} + 8z^{25} - 15z^{24} + 14z^{23} - 6z^{22} + z^{21} + 16z^{20} - 19z^{19} + 13z^{18} + 8z^{17} - 14z^{16} + 9z^{15} - 7z^{14} + 2z^{13} + 6z^{12} - 12z^{11} + 9z^{10} - z^9 - 11z^8 + 8z^7 + 2z^6 - 2z^5 - 2z^3 + 2z^2 + z - 1$
$f_3^{A_4} := z^{43} - 2z^{42} + z^{41} + z^{40} - 2z^{39} + z^{38} - z^{37} + 4z^{36} - 4z^{35} + 3z^{34} - 9z^{32} + 10z^{31} + 2z^{30} - 13z^{29} + 6z^{28} + 10z^{27} + 3z^{26} - 15z^{25} - 2z^{24} + 5z^{23} + 5z^{22} - z^{21} - 11z^{20} + 4z^{19} + 17z^{18} - 5z^{17} - 6z^{16} - 9z^{15} - 6z^{14} + 22z^{13} + 3z^{12} - 14z^{11} - 7z^{10} + 10z^9 + 12z^8 - 11z^7 - 9z^6 + 5z^5 + 5z^4 + z^3 - 3z^2 - z + 1$
$f_2^{B_3} := z^{30} + z^{28} - z^{24} - 7z^{22} - 7z^{20} - 5z^{18} - 6z^{16} + 2z^{14} + 4z^{12} + 3z^{10} + 4z^8 + z^6 - z^4 - 1$
$f_2^{D_4} := z^{37} + z^{36} + 2z^{35} + 2z^{34} + 4z^{33} + z^{32} + z^{31} - z^{30} + 2z^{29} - 3z^{28} - 4z^{26} - 3z^{25} - z^{23} + 2z^{20} + z^{19} - z^{18} - 4z^{17} + 3z^{16} + 2z^{15} + z^{14} - z^{13} + 2z^{11} - 3z^{10} + z^9 + z^7 + 3z^6 - 4z^5 + z^3 + z - 1$
$f_3^{D_4} := z^{28} - z^{27} + 2z^{24} + z^{23} - 2z^{22} + z^{20} + 2z^{19} - 2z^{17} + z^{16} + 3z^{15} + 2z^{14} - 4z^{13} - 5z^{12} + 2z^{11} + 7z^{10} + z^9 - 5z^8 - 5z^7 + z^6 + 5z^5 - 2z^3 - z^2 + 1$
$f_2^{H_3} := z^{56} + z^{52} + 2z^{48} + 4z^{46} + z^{44} - 8z^{42} + 2z^{40} + z^{36} + z^{34} - 11z^{32} - 3z^{30} + 10z^{28} + z^{24} - 3z^{22} - 4z^{20} + 8z^{18} + 3z^{16} - 5z^{14} - z^{10} + z^8 + 2z^6 - 2z^4 - z^2 + 1$

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