RIMS-1647

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November 2008

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Abstract

The purpose of this paper is twofold. One is to give a survey of our study on the reductions of harmonic bundles, and the other is to explain a simple application in the study of TERP structure. In particular, we investigate the asymptotic behaviour of the "new supersymmetric index" for variation of pure polarized TERP structures.

Keywords: harmonic bundle, TERP structure, new supersymmetric index
MSC: 32L05, 14D07

1 Introduction

In our previous papers [17], [18] and [19], we studied asymptotic behaviour of tame and wild harmonic bundles. Briefly, one of the main results is the following sequence of reductions of harmonic bundles:

$$\boxed{\begin{array}{c}\text{wild}\\ \text{(irregular)}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{tame}\\ \text{(regular)}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{twistor}\\ \text{nilpotent orbit}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{twistor nilpotent orbit}\\ \text{of split type}\end{array}} \tag{1}$$

A reduced object is simpler than the original one, but it still gives a good approximation of the original one. And, a twistor nilpotent orbit of split type comes from a variation of polarized pure Hodge structures, whose asymptotic behaviour was deeply studied by E. Cattani, A. Kaplan, M. Kashiwara, T. Kawai and W. Schmid. Thus, we can say that the asymptotic behaviour of wild harmonic bundles is understood pretty well.

The main purpose of this paper is twofold. One is to give a survey of these reductions, and the other is to explain a simple application in the study of TERP structure.

C. Hertling [7] initiated the study of TERP structures inspired by mathematical physics and singularity theory. The study was further developed by Hertling and C. Sevenheck. For example, they investigated "nilpotent orbit" [8], asymptotic behaviour of tame variation of TERP structures and classifying spaces [9]. We refer to the above papers and a survey [10] for more details and precise.

Remark 1.1 *Their "nilpotent orbit" is called "HS-orbit" (Hertling-Sevenheck orbit) in this paper. We can consider several kinds of generalization of "nilpotent orbit" in the theory of TERP structures and twistor structures. HS-orbit is the one. Another one is twistor nilpotent orbit studied in* [18]*, which we will mainly use in this paper.* ∎

Remark 1.2 *We prefer to regard TERP structure as integrable twistor structure with a real structure and a pairing studied by C. Sabbah. It is called twistor-TERP structure in this paper.* ∎

We will give an enrichment of the sequence (1) with TERP structures or integrable twistor structures. As an application, we will study the behaviour of "new supersymmetric index" of variation of pure polarized TERP structures. Let ∇ be a meromorphic connection of $V = \mathcal{O}_{\mathbb{P}^1}^{\oplus\, r}$ admitting a pole at $\{0,\infty\}$ of at most order two. Let d be the natural connection of V. Then, we have the expression $\nabla = d + \bigl(\lambda^{-1}\cdot\mathcal{U}_1 - \mathcal{Q} - \lambda\cdot\mathcal{U}_2\bigr)\cdot d\lambda/\lambda$, where $\mathcal{U}_i, \mathcal{Q} \in \mathrm{End}(V)$. If (V,∇) is equipped with a real structure and a polarization (see Subsection 2.1.5), there is some more restriction on them. Anyway, $\mathcal{Q}$ is called the supersymmetric index of (V,∇). We set $X := \bigl\{(z_1,\ldots,z_n)\,\big|\,|z_i| < 1\bigr\}$ and $D := \bigcup_{i=1}^{n}\{z_i = 0\}$. Let $(\mathcal{V}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S}, \kappa)$ be a variation of pure polarized twistor-TERP structures of weight 0 on $\mathbb{P}^1 \times (X - D)$. (See Subsection 2.1.) It is called unramifiedly good wild (resp. tame), if the underlying harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ is so. (See Subsection 6.1.) For each point

$P \in X - D$, we have the new supersymmetric index $\mathcal{Q}_P \in \mathrm{End}(\mathcal{V}^{\triangle}_{|\mathbb{P}^1 \times P}) \simeq \mathrm{End}(E_{|P})$ of $(\mathcal{V}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle})_{|\mathbb{P}^1 \times P}$, and thus we obtain a C^{∞}-section $\mathcal{Q}$ of $\mathrm{End}(E)$. We are interested in the behaviour of $\mathcal{Q}$ around $(0, \ldots, 0)$. The result is the following:

- In the case of twistor-TERP nilpotent orbit of split type, the new supersymmetric index can be easily computed from the data of the corresponding polarized mixed twistor-TERP structure. In particular, their eigenvalues are constant. (See Section 3.)
- From a twistor-TERP nilpotent orbit $(\mathcal{V}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S}, \kappa)$, we obtain a twistor-TERP nilpotent orbit of split type $(\mathcal{V}_0, \widetilde{\mathbb{D}}^{\triangle}_0, \mathcal{S}_0, \kappa_0)$, by taking Gr with respect to the weight filtration. (Precisely, Gr is taken for the corresponding polarized mixed twistor-TERP structure.) The new supersymmetric index $\mathcal{Q}$ of $(\mathcal{V}, \widetilde{\mathbb{D}}^{\triangle})$ can be approximated by the new supersymmetric index $\mathcal{Q}_0$ of $(\mathcal{V}_0, \widetilde{\mathbb{D}}^{\triangle}_0)$ up to $O\Big(\sum(-\log|z_i|)^{-1/2}\Big)$. In particular, the eigenvalues of $\mathcal{Q}$ are constant up to $O\Big(\sum(-\log|z_i|)^{-\delta}\Big)$ for some $\delta > 0$. (See Section 4.)
- From a tame variation of polarized pure twistor-TERP-structures $(\mathcal{V}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S}, \kappa)$, we obtain a twistor-TERP nilpotent orbit $(\mathcal{V}_0, \widetilde{\mathbb{D}}^{\triangle}_0, \mathcal{S}_0, \kappa_0)$ associated to the limit mixed twistor-TERP structure which was essentially considered in [9] as an enrichment of the limit mixed twistor structure in [18]. We can approximate the new supersymmetric index $\mathcal{Q}$ of $(\mathcal{V}, \widetilde{\mathbb{D}}^{\triangle})$ by the new supersymmetric index $\mathcal{Q}_0$ of $(\mathcal{V}_0, \widetilde{\mathbb{D}}^{\triangle}_0)$ up to $O\Big(\sum|z_i|^{\epsilon}\Big)$ for some $\epsilon > 0$. In particular, the eigenvalues of $\mathcal{Q}_0$ approximate those of $\mathcal{Q}$ up to $O\Big(\sum|z_i|^{\epsilon'}\Big)$ for some $\epsilon' > 0$. (See Subsection 7.4 for more precise statements.)
- From a wild variation of polarized pure twistor-TERP structures $(\mathcal{V}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S}, \kappa)$, we obtain a tame variation of polarized pure twistor-TERP structures $(\mathcal{V}_0, \widetilde{\mathbb{D}}^{\triangle}_0, \mathcal{S}_0, \kappa_0)$, by taking Gr with respect to Stokes filtrations. We can approximate the new supersymmetric index $\mathcal{Q}$ of $(\mathcal{V}, \widetilde{\mathbb{D}}^{\triangle})$ by the new supersymmetric index $\mathcal{Q}_0$ of $(\mathcal{V}_0, \widetilde{\mathbb{D}}^{\triangle}_0)$ up to a term with exponential decay. In particular, the eigenvalues of $\mathcal{Q}_0$ approximate those of $\mathcal{Q}$ up to exponential decay. (See Subsection 7.3 for more precise statements.)

In each case, we will construct a C^{∞}-map $\mathcal{V}_0 \longrightarrow \mathcal{V}$, which does not preserve but approximate the additional structures. (More precisely, $\mathcal{V}_0$ should be twisted.) It would be interesting to clarify the precise relation between these results and the celebrated nilpotent orbit theorem for Hodge structures due to W. Schmid [23]. (See also [9].)

As a corollary, we obtain the convergence of the eigenvalues of new supersymmetric indices of wild harmonic bundles on a punctured disc. In his recent work (Section 3 of [22]), Sabbah studied the eigenvalues of new supersymmetric indices for polarized wild pure integrable twistor D-modules on curves. Since wild harmonic bundles are prolonged to polarized wild pure twistor D-modules [19], we can also deduce the above convergence in the curve case from his results.

We also show that if a TERP-structure induces an HS-orbit, then it is a mixed-TERP structure in the sense of [8] by using the reduction from wild to tame, which was conjectured by Hertling and Sevenheck.

Outline of this paper In Subsection 2.1 we recall integrable pure twistor structure and TERP structure and their variations in our convenient way, which were originally studied by Hertling, Sabbah and Sevenheck. We look at some basic examples in Subsection 2.2. In particular, we introduce the notions of integrable twistor nilpotent orbit and twistor-TERP nilpotent orbit. In Subsection 2.3, we argue a convergence of integrable pure twistor structures and new supersymmetric indices. The result will be used in many times. In Subsection 2.4, we consider a variation of polarized mixed twistor structures. In Subsection 2.4.2, we explain the reduction from polarized mixed twistor structure to polarized mixed twistor structure *of split type*. In Subsection 2.4.3, we give a C^{∞}-splitting of weight filtrations compatible with nilpotent maps, which is a preparation for Section 4.

In Section 3, we study polarized mixed twistor structure *of split type* with some additional structures. It is quite easy to handle. In Section 4, we show the correspondence between twistor nilpotent orbits and polarized mixed twistor structures. We have already established the way from twistor nilpotent orbits to polarized mixed twistor structures in [18]. The converse was also established in the curve case. The higher dimensional case is new. The correspondence is easily enriched with integrability and real structures. We also show that a twistor nilpotent orbit is approximated with a twistor nilpotent orbit *of split type*.

In Section 5, we give a review on Stokes structure and reductions for a family of meromorphic λ-flat bundles, studied in Sections 7 and 8 in [19]. We give some minor complementary results on connections along the λ-direction and pseudo-good lattices.

In Section 6, we explain the reduction from unramifiedly good wild harmonic bundles to polarized mixed twistor structures, studied in [18] and [19]. We give a review on the prolongation of harmonic bundles in Subsection 6.3. Then, in Subsection 6.4, we review the reduction from unramifiedly good wild bundles to tame harmonic bundles as the Gr with respect to Stokes filtrations, which is one of the main results in [19], and in Subsection 6.5, we review the reduction from tame harmonic bundles to polarized mixed twistor structure as the Gr with respect to KMS-structure, which is one of the main results in [18]. Together with the result in Section 4, we can regard it as the reduction to nilpotent orbits.

In Section 7, we argue an enrichment of the reductions with integrability and real structure. One of the main issues is to obtain a meromorphic extension of the connection along the λ-direction. For that purpose, we prepare some estimate in Subsection 7.1. Then, it is easy to obtain the meromorphic prolongment of variations of integrable twistor structures and the enrichment of the sequence of reductions as in (1). We also show that the reduced one gives a good approximation of the original one. In particular, we obtain the results on approximation of the new supersymmetric indices of wild or tame variation of integrable twistor structures.

In Section 8, we study the reduction of HS-orbit.

Acknowledgement This paper grew out of my effort to understand the work due to Claus Hertling, Claude Sabbah and Christian Sevenheck on TERP structure and integrable twistor structure. I am grateful to them who attracted my attention to this subject. I also thank their comments on the earlier versions of this paper. In particular, Hertling kindly sent a surprisingly detailed and careful report, which was quite helpful for improving this paper and correcting some errors in earlier versions.

I wish to express my thanks to Yoshifumi Tsuchimoto and Akira Ishii for their constant encouragement.

I gave talks on the sequence (1) at the conferences "From tQFT to tt* and integrability" in Augsburg and "New developments in Algebraic Geometry, Integrable Systems and Mirror symmetry" in Kyoto. This paper is an enhancement of the talks. I would like to express my gratitude to the organizers of the conferences on this occasion.

I am grateful to the partial financial support by Ministry of Education, Culture, Sports, Science and Technology.

2 Preliminary

2.1 Integrable twistor structure

We recall the notion of integrable twistor structures and TERP structures in our convenient way just for our understanding. See [7], [8] and [21] for the original definitions and for more details. We also recall twistor structures introduced in [27]. See also [17] and [18].

2.1.1 Some sheaves and differential operators on $\mathbb{P}^1 \times X$

Let $\mathbb{P}^1$ denote a one dimensional complex projective space. We regard it as the gluing of two complex lines $\boldsymbol{C}_\lambda$ and $\boldsymbol{C}_\mu$ by $\lambda = \mu^{-1}$. We set $\boldsymbol{C}_\lambda^* := \boldsymbol{C}_\lambda - \{0\}$.

Let X be a complex manifold. We set $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\mathcal{X}^0 := \{0\} \times X$. Let $\widetilde{\Omega}^{1,0}_{\mathcal{X}}$ be the C^∞-bundle associated to $\Omega^{1,0}_{\mathcal{X}}(\log \mathcal{X}^0) \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{X}^0)$. We put $\widetilde{\Omega}^{0,1}_{\mathcal{X}} := \Omega^{0,1}_{\mathcal{X}}$, and we define

$$\widetilde{\Omega}^1_{\mathcal{X}} := \widetilde{\Omega}^{1,0}_{\mathcal{X}} \oplus \widetilde{\Omega}^{0,1}_{\mathcal{X}}, \quad \widetilde{\Omega}^{\cdot}_{\mathcal{X}} := \bigwedge^{\cdot} \widetilde{\Omega}^1_{\mathcal{X}}$$

The associated sheaves of C^∞-sections are denoted by the same symbols. Let $\widetilde{\mathbb{D}}^f_X : \widetilde{\Omega}^{\cdot}_{\mathcal{X}} \longrightarrow \widetilde{\Omega}^{\cdot+1}_{\mathcal{X}}$ denote the differential operator induced by the exterior differential d.

Let $X^\dagger$ denote the conjugate of X. We set $\mathcal{X}^\dagger := \boldsymbol{C}_\mu \times X^\dagger$. By the same procedure, we obtain the C^∞-bundles $\widetilde{\Omega}^\cdot_{\mathcal{X}^\dagger}$ with the differential operator $\widetilde{\mathbb{D}}^{\dagger f}_X$.

Their restrictions to $\boldsymbol{C}^*_\lambda \times X = \boldsymbol{C}^*_\mu \times X^\dagger$ are naturally isomorphic:

$$\big(\widetilde{\Omega}^\cdot_{\mathcal{X}}, \widetilde{\mathbb{D}}^f_X\big)_{|\boldsymbol{C}^*_\lambda \times X} = \big(\Omega^\cdot_{\boldsymbol{C}^*_\lambda \times X}, d\big) = \big(\widetilde{\Omega}^\cdot_{\mathcal{X}^\dagger}, \widetilde{\mathbb{D}}^{\dagger f}_X\big)_{|\boldsymbol{C}^*_\mu \times X^\dagger}$$

By gluing them, we obtain the C^∞-bundles $\widetilde{\Omega}^\cdot_{\mathbb{P}^1\times X}$ with a differential operator $\widetilde{\mathbb{D}}^\triangle_X$.

Remark 2.1 *$\widetilde{\mathbb{D}}^f_X$ and $\widetilde{\mathbb{D}}^{\dagger f}_X$ are denoted also by d, if there is no risk of confusion.* ∎

We have the decomposition $\widetilde{\Omega}^1_{\mathbb{P}^1\times X} = \xi\Omega^1_X \oplus \widetilde{\Omega}^1_{\mathbb{P}^1}$ into the X-direction and the $\mathbb{P}^1$-direction. The restriction of $\widetilde{\mathbb{D}}^\triangle_X$ to the X-direction is denoted by $\mathbb{D}^\triangle_X$. The restriction to the $\mathbb{P}^1$-direction is denoted by $d_{\mathbb{P}^1}$. We have the decomposition

$$\widetilde{\Omega}^1_{\mathbb{P}^1} = \pi^*\Omega^{1,0}_{\mathbb{P}^1}(2\cdot\{0,\infty\}) \oplus \pi^*\Omega^{0,1}_{\mathbb{P}^1},$$

into the $(1,0)$-part and the $(0,1)$-part, where π denotes the projection $\mathbb{P}^1\times X \longrightarrow \mathbb{P}^1$. We have the corresponding decomposition $d_{\mathbb{P}^1} = \partial_{\mathbb{P}^1} + \overline{\partial}_{\mathbb{P}^1}$.

Let $\nu : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a diffeomorphism. Assume ν satisfies one of the following:

(A1) ν is holomorphic with $\nu(0) = 0$ and $\nu(\infty) = \infty$.

(A2) ν is anti-holomorphic with $\nu(0) = \infty$ and $\nu(\infty) = 0$.

In particular, we will often use the maps σ, γ and j:

$$\sigma([z_0 : z_1]) = [-\overline{z}_1 : \overline{z}_0], \quad \gamma([z_0 : z_1]) = [\overline{z}_1 : \overline{z}_0], \quad j([z_0 : z_1]) = [-z_0 : z_1]$$

The induced diffeomorphism $\mathbb{P}^1 \times X \longrightarrow \mathbb{P}^1 \times X$ is also denoted by ν. In the case (A1), we have the natural isomorphism $\Phi_\nu : \nu^*\widetilde{\Omega}^\cdot_{\mathbb{P}^1\times X} \simeq \widetilde{\Omega}^\cdot_{\mathbb{P}^1\times X}$ of C^∞-vector bundles given by the ordinary pull back. In the case (A2), the multiplication of C^∞-functions on $\nu^*\widetilde{\Omega}^\cdot_{\mathbb{P}^1\times X}$ is twisted as $g\cdot\nu^*(\omega) = \nu^*\big(\overline{\nu^*(g)}\cdot\omega\big)$ for a function g and a section ω of $\widetilde{\Omega}^\cdot_{\mathbb{P}^1\times X}$. Then, we have the C^∞-isomorphism $\Phi_\nu : \nu^*\widetilde{\Omega}^\cdot_{\mathbb{P}^1\times X} \simeq \widetilde{\Omega}^\cdot_{\mathbb{P}^1\times X}$ given by the complex conjugate and the ordinary pull back

$$\Phi_\nu(\nu^*\omega) = \overline{\nu^*(\omega)}.$$

It is easy to check that $\Phi_\nu \circ \nu^*(\widetilde{\mathbb{D}}^\triangle_X) = \widetilde{\mathbb{D}}^\triangle_X \circ \Phi_\nu$. Similar relations hold for $\mathbb{D}^\triangle_X$ and $d_{\mathbb{P}^1}$. If we are given an additional bundle $\mathcal{F}$, the induced isomorphism $\mathcal{F}\otimes\nu^*\big(\widetilde{\Omega}^\cdot_{\mathbb{P}^1\times X}\big) \simeq \mathcal{F}\otimes\widetilde{\Omega}^\cdot_{\mathbb{P}^1\times X}$ is also denoted by Φ_ν.

2.1.2 Definitions and some remarks

Variation of twistor structures Let V be a C^∞-vector bundle on $\mathbb{P}^1 \times X$. We use the same symbol to denote the associated sheaf of C^∞-sections. A $\mathbb{P}^1$-holomorphic structure of V is defined to be a differential operator

$$d''_{\mathbb{P}^1,V} : V \longrightarrow V \otimes \pi^*\Omega^{0,1}_{\mathbb{P}^1}$$

satisfying (i) $d''_{\mathbb{P}^1,V}(f\cdot s) = f\cdot d''_{\mathbb{P}^1,V}(s) + \overline{\partial}_{\mathbb{P}^1}(f)\cdot s$ for a C^∞-function f and a section s of V, (ii) $d''_{\mathbb{P}^1,V}\circ d''_{\mathbb{P}^1,V} = 0$. Such a tuple $(V, d''_{\mathbb{P}^1,V})$ is called a $\mathbb{P}^1$-holomorphic vector bundle.

A $T\widetilde{T}$-structure of $(V, d''_{\mathbb{P}^1,V})$ is a differential operator

$$\mathbb{D}^\triangle_V : V \longrightarrow V \otimes \xi\Omega^1_X$$

such that (i) $\mathbb{D}^\triangle_V(f\cdot s) = f\cdot\mathbb{D}^\triangle_V(s) + \mathbb{D}^\triangle_X(f)\cdot s$ for a C^∞-function f and a section s of V, (ii) $(d''_{\mathbb{P}^1,V} + \mathbb{D}^\triangle_V)^2 = 0$. Such a tuple $(V, d''_{\mathbb{P}^1,V}, \mathbb{D}^\triangle_V)$ is called *a $T\widetilde{T}$-structure* in [7], or *a variation of $\mathbb{P}^1$-holomorphic vector bundles* in [18]. In this section, we prefer to call it *variation of twistor structures.*

If X is a point, it is just a holomorphic vector bundle on $\mathbb{P}^1$.

Remark 2.2 *We will often omit to specify $d''_{\mathbb{P}^1,V}$ when we consider $\mathbb{P}^1$-holomorphic bundles or variations of twistor structures (variations of $\mathbb{P}^1$-holomorphic bundles).* ∎

Variation of integrable twistor structures A $T\widetilde{TE}$-structure of V is a differential operator

$$\widetilde{\mathbb{D}}_V^{\triangle} : V \longrightarrow V \otimes \widetilde{\Omega}^1_{\mathbb{P}^1 \times X}$$

satisfying (i) $\widetilde{\mathbb{D}}_V^{\triangle}(f \cdot s) = \widetilde{\mathbb{D}}_X^{\triangle}(f) \cdot s + f \cdot \widetilde{\mathbb{D}}_V^{\triangle}(s)$ for a C^{∞}-function f and a section s of V, (ii) $\widetilde{\mathbb{D}}_V^{\triangle} \circ \widetilde{\mathbb{D}}_V^{\triangle} = 0$. Such a tuple $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ is called a variation of integrable twistor structures.

If X is a point, it is equivalent to a holomorphic vector bundle V on $\mathbb{P}^1$ with a meromorphic connection ∇ which admits a pole at $\{0, \infty\}$ with at most order 2, i.e.,

$$\nabla(V) \subset V \otimes \Omega^1\bigl(2 \cdot \{0, \infty\}\bigr).$$

In this case, it is simply called an integrable twistor structure.

Morphisms A morphism of variation of twistor structures $F : (V_1, d''_{\mathbb{P}^1,V_1}, \mathbb{D}_{V_1}^{\triangle}) \longrightarrow (V_2, d''_{\mathbb{P}^1,V_2}, \mathbb{D}_{V_2}^{\triangle})$ is defined to be a morphism of the associated sheaves of C^{∞}-sections, compatible with the differential operators. If X is a point, it is equivalent to an $\mathcal{O}_{\mathbb{P}^1}$-morphism.

A morphism of variation of integrable twistor structures $F : (V_1, \widetilde{\mathbb{D}}_{V_1}^{\triangle}) \longrightarrow (V_2, \widetilde{\mathbb{D}}_{V_2}^{\triangle})$ is defined to be a morphism of the associated sheaves of C^{∞}-sections, compatible with the differential operators. If X is a point, it is equivalent to an $\mathcal{O}_{\mathbb{P}^1}$-morphism compatible with the meromorphic connections.

Some functoriality Let $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ be a variation of integrable twistor structures. Let $f : Y \longrightarrow X$ be a holomorphic map of complex manifolds. Then, we have the naturally induced variation of integrable twistor structures $f^*(V, \widetilde{\mathbb{D}}_V^{\triangle})$ as in the case of ordinary connections.

Let $\nu : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a diffeomorphism satisfying one of (A1) or (A2) above. Then, ν^*V is naturally equipped with a $T\widetilde{TE}$-structure $\widetilde{\mathbb{D}}_{\nu^*V}^{\triangle}$ given as follows:

$$\widetilde{\mathbb{D}}_{\nu^*V}^{\triangle}\bigl(\Phi_{\nu}(\nu^* s)\bigr) = \Phi_{\nu}\Bigl(\nu^*\bigl(\widetilde{\mathbb{D}}_V^{\triangle}(s)\bigr)\Bigr)$$

Here, s denotes a section of $V \otimes \widetilde{\Omega}^{\cdot}_X$.

We also have the pull back of variation of twistor structures via f and ν as above.

Pure and mixed Let $(V, d''_{\mathbb{P}^1,V})$ be a $\mathbb{P}^1$-holomorphic vector bundle on $\mathbb{P}^1 \times X$. It is called pure of weight w if the restrictions $V_P := (V, d''_{\mathbb{P}^1,V})_{|\mathbb{P}^1 \times \{P\}}$ are pure twistor structures of weight w for any $P \in X$, i.e., V_P are isomorphic to direct sums of $\mathcal{O}_{\mathbb{P}^1}(w)$. A variation of (integrable) twistor structures is called pure of weight w, if the underlying $\mathbb{P}^1$-holomorphic vector bundle is pure of weight w.

Let W be a filtration of V by vector subbundles indexed by integers. We say that W is $\mathbb{P}^1$-holomorphic, if each W_n are preserved by $d''_{\mathbb{P}^1,V}$. We have induced $\mathbb{P}^1$-holomorphic vector bundles $\mathrm{Gr}_n^W(V, d''_{\mathbb{P}^1,V})$. Then, $(V, d''_{\mathbb{P}^1}, W)$ is called mixed, if each $\mathrm{Gr}_n^W(V, d''_{\mathbb{P}^1,V})$ is pure of weight n. When $(V, d''_{\mathbb{P}^1})$ is equipped with $T\widetilde{T}$-structure $\mathbb{D}_V^{\triangle}$ (resp. $T\widetilde{TE}$-structure $\widetilde{\mathbb{D}}_V^{\triangle}$), we say that W is $\mathbb{D}_V^{\triangle}$-flat (resp. $\widetilde{\mathbb{D}}_V^{\triangle}$-flat) or more simply flat, if each W_n is preserved by the operator. In that case, $(V, d''_{\mathbb{P}^1,V}, \mathbb{D}_V^{\triangle}, W)$ (resp. $(V, \widetilde{\mathbb{D}}_V^{\triangle}, W)$) is called mixed, if $(V, d''_{\mathbb{P}^1}, W)$ is mixed.

New supersymmetric index Let (V, ∇) be a pure integrable twistor structure of weight 0. We have a global trivialization $V \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$, which is uniquely determined up to obvious ambiguity. Let d denote the natural connection of $\mathcal{O}_{\mathbb{P}^1}^{\oplus r}$. Then, we have the decomposition

$$\nabla = d + \Bigl(\lambda^{-1}\mathcal{U}_1 - \mathcal{Q} - \lambda \cdot \mathcal{U}_2\Bigr)\frac{d\lambda}{\lambda}, \tag{2}$$

where $\mathcal{U}_1, \mathcal{U}_2, \mathcal{Q} \in H^0\bigl(\mathbb{P}^1, \mathrm{End}(V)\bigr)$, The operator $\mathcal{Q}$ is called the new supersymmetric index. If (V, ∇) is equipped with a polarization (Subsection 2.1.4), $\mathcal{U}_2$ and $\mathcal{U}_1$ are adjoint with respect to the induced hermitian metric, as observed by Hertling and Sabbah.

If we are given a variation of polarized pure integrable twistor structures, we obtain such operators in family.

2.1.3 Simple examples

We recall some simplest examples of integrable pure twistor structures.

Example (Tate object) Let $\mathbb{T}(w)$ be a Tate object in the theory of twistor structures. (See [27] and Subsection 3.3.1 of [18].) It is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2w)$, and it is equipped with the distinguished frames

$$\mathbb{T}(w)_{|\boldsymbol{C}_\lambda} = \mathcal{O}_{\boldsymbol{C}_\lambda} \cdot t_0^{(w)}, \quad \mathbb{T}(w)_{|\boldsymbol{C}_\mu} = \mathcal{O}_{\boldsymbol{C}_\mu} \cdot t_\infty^{(w)}, \quad \mathbb{T}(w)_{|\boldsymbol{C}_\lambda^*} = \mathcal{O}_{\boldsymbol{C}_\lambda^*} \cdot t_1^{(w)}.$$

The transformation is given by

$$t_0^{(w)} = (\sqrt{-1}\lambda)^w \cdot t_1^{(w)}, \quad t_\infty^{(w)} = (-\sqrt{-1}\mu)^w \cdot t_1^{(w)}.$$

In particular, $(\sqrt{-1}\lambda)^{-2w} t_0^{(w)} = t_\infty^{(w)}$. We have the meromorphic connection $\nabla_{\mathbb{T}(w)}$ on $\mathbb{T}(w)$ determined by

$$\nabla_{\mathbb{T}(w)} t_1^{(w)} = 0, \quad \nabla_{\mathbb{T}(w)} t_0^{(w)} = t_0^{(w)} \cdot \left(w \cdot \frac{d\lambda}{\lambda} \right), \quad \nabla_{\mathbb{T}(w)} t_\infty^{(w)} = t_\infty^{(w)} \cdot \left(w \cdot \frac{d\mu}{\mu} \right).$$

In the following, the connection of $\mathbb{T}(w)$ is always given as above, and hence we often omit to specify it explicitly.

We may identify $\mathbb{T}(w)$ with $\mathcal{O}_{\mathbb{P}^1}\big(-w \cdot 0 - w \cdot \infty\big)$ by the correspondence $t_1^{(w)} \longleftrightarrow 1$, up to constant multiplication. In particular, we implicitly use the identification of $\mathbb{T}(0)$ with $\mathcal{O}_{\mathbb{P}^1}$ by $t_1^{(0)} \longleftrightarrow 1$. We will also implicitly use the identification $\mathbb{T}(m) \otimes \mathbb{T}(n) \simeq \mathbb{T}(m+n)$ given by $t_a^{(m)} \otimes t_a^{(n)} \longleftrightarrow t_a^{(m+n)}$.

Example In Subsection 3.3.2 of [18], we considered a line bundle $\mathcal{O}(p,q)$ on $\mathbb{P}^1$, which is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(p+q)$ and equipped with the distinguished frames:

$$\mathcal{O}(p,q)_{|\boldsymbol{C}_\lambda} = \mathcal{O}_{\boldsymbol{C}_\lambda} \cdot f_0^{(p,q)}, \quad \mathcal{O}(p,q)_{|\boldsymbol{C}_\mu} = \mathcal{O}_{\boldsymbol{C}_\mu} \cdot f_\infty^{(p,q)}, \quad \mathcal{O}(p,q)_{|\boldsymbol{C}_\lambda^*} = \mathcal{O}_{\boldsymbol{C}_\lambda^*} \cdot f_1^{(p,q)}.$$

The transformation is given by

$$f_0^{(p,q)} = (\sqrt{-1}\lambda)^{-p} \cdot f_1^{(p,q)}, \quad f_\infty^{(p,q)} = (-\sqrt{-1}\mu)^{-q} \cdot f_1^{(p,q)}.$$

In particular, $(\sqrt{-1}\lambda)^{p+q} f_0^{(p,q)} = f_\infty^{(p,q)}$. We have the meromorphic connection $\nabla_{\mathcal{O}(p,q)}$ on $\mathcal{O}(p,q)$ determined by

$$\nabla_{\mathcal{O}(p,q)} f_1^{(p,q)} = 0, \quad \nabla_{\mathcal{O}(p,q)} f_0^{(p,q)} = f_0^{(p,q)} \cdot \left(-p \frac{d\lambda}{\lambda} \right), \quad \nabla_{\mathcal{O}(p,q)} f_\infty^{(p,q)} = f_\infty^{(p,q)} \cdot \left(-q \frac{d\mu}{\mu} \right)$$

In the following, the connection of $\mathcal{O}(p,q)$ is always given as above, and hence we will often omit to specify it explicitly.

We may naturally identify $\mathcal{O}(p,q)$ with $\mathcal{O}_{\mathbb{P}^1}(p \cdot 0 + q \cdot \infty)$ by the correspondence $f_1^{(p,q)} \longleftrightarrow 1$, up to constant multiplication. We will implicitly use the identification $\mathcal{O}(p,q) \otimes \mathcal{O}(p',q') \simeq \mathcal{O}(p+p', q+q')$ given by $f_a^{(p,q)} \otimes f_a^{(p',q')} \longleftrightarrow f_a^{(p+p',q+q')}$. We will also implicitly identify $\mathbb{T}(w)$ with $\mathcal{O}(-w,-w)$ by $t_a^{(w)} = f_a^{(-w,-w)}$ for $a = 0, 1, \infty$.

Let X be a complex manifold. We have the pull back of $\mathbb{T}(w)$ and $\mathcal{O}(p,q)$ via the map from X to a point. They are denoted by $\mathbb{T}(w)_X$ and $\mathcal{O}(p,q)_X$, respectively. We will often omit to denote X, if there is no risk of confusion.

2.1.4 Polarization

Recall that we have the isomorphism ([18])

$$\iota_{\mathbb{T}(w)} : \sigma^* \mathbb{T}(w) \simeq \mathbb{T}(w),$$

given by the natural identification $\sigma^* \mathcal{O}\big(-w \cdot 0 - w \cdot \infty\big) \simeq \mathcal{O}\big(-w \cdot 0 - w \cdot \infty\big)$ via $\sigma^*(1) \longleftrightarrow 1$, or equivalently,

$$\sigma^* t_1^{(w)} \longleftrightarrow t_1^{(w)}, \quad \sigma^* t_0^{(w)} \longleftrightarrow (-1)^w \cdot t_0^{(w)}, \quad \sigma^* t_\infty^{(w)} \longleftrightarrow (-1)^w \cdot t_\infty^{(w)}.$$

It preserves the flat connections, i.e., $\iota_{\mathbb{T}(w)} : \sigma^*(\mathbb{T}(w), \nabla_{\mathbb{T}(w)}) \simeq (\mathbb{T}(w), \nabla_{\mathbb{T}(w)})$.

For a variation of integrable twistor structures $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ on $\mathbb{P}^1 \times X$, a morphism

$$\mathcal{S} : (V, \widetilde{\mathbb{D}}_V^{\triangle}) \otimes \sigma^*(V, \widetilde{\mathbb{D}}_V^{\triangle}) \longrightarrow \mathbb{T}(-w)_X$$

is called a pairing of weight w, if it is $(-1)^w$-symmetric in the following sense:

$$\iota_{\mathbb{T}(-w)} \circ \sigma^*\mathcal{S} = (-1)^w \mathcal{S} \circ \mathrm{exchange} : \sigma^*V \otimes V \longrightarrow \mathbb{T}(-w)_X$$

Here, exchange denotes the natural morphism $\sigma^*V \otimes V \longrightarrow V \otimes \sigma^*V$ induced by the exchange of the components. Similarly, we have the notion of pairing for variations of twistor structures.

Definition 2.3 *Let $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ be a variation of integrable pure twistor structure of weight w on $\mathbb{P}^1 \times X$. Let $\mathcal{S} : (V, \widetilde{\mathbb{D}}_V^{\triangle}) \otimes \sigma^*(V, \widetilde{\mathbb{D}}_V^{\triangle}) \longrightarrow \mathbb{T}(-w)_X$ be a pairing of weight w. We say that $\mathcal{S}$ is a polarization of $(V, \widetilde{\mathbb{D}}_V^{\triangle})$, if $\mathcal{S}_P := \mathcal{S}_{|\mathbb{P}^1 \times \{P\}}$ is a polarizations of $V_P := (V, d''_{\mathbb{P}^1})_{|\mathbb{P}^1 \times \{P\}}$ for each $P \in X$. Namely, the following holds:*

- *If $w = 0$, the induced Hermitian pairing $H^0(\mathcal{S}_P)$ of $H^0(\mathbb{P}^1, V_P)$ is positive definite.*
- *In the general case, the induced pairing $\mathcal{S}_P \otimes \mathcal{S}_{0,-w}$ of $V_P \otimes \mathcal{O}(0, -w)$ is a polarization of the pure twistor structure. (See Example 2 below for $\mathcal{S}_{0,-w}$.)*

The notion of polarization for variation of pure twistor structures is defined in a similar way. ∎

Example 1 The identification $\iota_{\mathbb{T}(w)}$ induces the flat morphism $\mathcal{S}_{\mathbb{T}(w)} : \mathbb{T}(w) \otimes \sigma^*\mathbb{T}(w) \longrightarrow \mathbb{T}(2w)$, which is a polarization of $\mathbb{T}(w)$ of weight $-2w$.

Example 2 The flat isomorphism $\iota_{(p,q)} : \sigma^*\mathcal{O}(p, q) \simeq \mathcal{O}(q, p)$ in [18] is given by

$$\sigma^* f_0^{(p,q)} \longmapsto (\sqrt{-1})^{p+q} f_\infty^{(q,p)}, \quad \sigma^* f_\infty^{(p,q)} \longmapsto (-\sqrt{-1})^{p+q} f_0^{(q,p)}, \quad \sigma^* f_1^{(p,q)} \longmapsto (\sqrt{-1})^{q-p} f_1^{(q,p)}.$$

Hence, we obtain the morphism $\mathcal{S}_{p,q} : \mathcal{O}(p, q) \otimes \sigma^*\mathcal{O}(p, q) \longrightarrow \mathbb{T}(-p-q)$, which is a polarization of weight $p+q$.

2.1.5 Real structure and twistor-TERP structure

Definition 2.4 *A real structure of a variation of integrable twistor structure $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ is defined to be an isomorphism*

$$\kappa : \gamma^*(V, \widetilde{\mathbb{D}}_V^{\triangle}) \simeq (V, \widetilde{\mathbb{D}}_V^{\triangle})$$

such that $\gamma^*(\kappa) \circ \kappa = \mathrm{id}$. ∎

We fix the real structure $\kappa_{\mathbb{T}(w)}$ of $\mathbb{T}(w)$ given by the correspondence

$$\gamma^* t_1^{(w)} \longleftrightarrow t_1^{(w)}, \quad \gamma^* t_0^{(w)} \longleftrightarrow t_\infty^{(w)}, \quad \gamma^* t_\infty^{(w)} \longleftrightarrow t_0^{(w)}.$$

Definition 2.5 *Let $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ be a variation of integrable twistor structures equipped with a pairing $\mathcal{S}$ of weight w and a real structure κ. We say that κ and $\mathcal{S}$ are compatible, if the following diagram is commutative:*

$$\begin{tikzcd}
\gamma^*V \otimes \gamma^*\sigma^*V \arrow[r, "\gamma^*\mathcal{S}"] \arrow[d, "\kappa\otimes\sigma^*\kappa"'] & \gamma^*\mathbb{T}(-w) \arrow[d, "\kappa_{\mathbb{T}(-w)}"'] \\
V \otimes \sigma^*V \arrow[r, "\mathcal{S}"] & \mathbb{T}(-w)
\end{tikzcd}$$

Namely, $\kappa_{\mathbb{T}(-w)} \circ \gamma^*\mathcal{S} = \mathcal{S} \circ (\kappa \otimes \sigma^*\kappa)$ holds. In that case, we also say that κ is a real structure of $(V, \widetilde{\mathbb{D}}_V^{\triangle}, \mathcal{S})$, or that $\mathcal{S}$ is a pairing of $(V, \widetilde{\mathbb{D}}_V^{\triangle}, \kappa)$ with weight w. ∎

Definition 2.6 *Let $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ be a variation of integrable twistor structure equipped with a pairing $\mathcal{S}$ of weight w and a real structure κ. The tuple $(V, \widetilde{\mathbb{D}}_V^{\triangle}, \mathcal{S}, \kappa, -w)$ is called a variation of twistor-TERP structure, if (i) $\mathcal{S}$ is perfect, (ii) $\mathcal{S}$ and κ are compatible.*

If X is a point, it is called a twistor-TERP structure. ∎

It is easy to observe that twistor-TERP structure is just an expression of TERP structure [7] in terms of twistor structures, which we will explain later.

Definition 2.7 *A variation of twistor-TERP structures $(V, \widetilde{\mathbb{D}}_V^{\triangle}, \mathcal{S}, \kappa, -w)$ is called pure, if $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ is pure with weight w. It is called polarized, if $(V, \widetilde{\mathbb{D}}_V^{\triangle}, \mathcal{S})$ is polarized.* ∎

Remark 2.8 *If a variation of twistor-TERP structure $(V, \widetilde{\mathbb{D}}_V^{\triangle}, \mathcal{S}, \kappa, -w)$ is pure, we also say that "$(V, \widetilde{\mathbb{D}}_V^{\triangle}, \mathcal{S}, \kappa)$ is a variation of pure twistor-TERP structure of weight w."* ∎

Example A Tate object $\big(\mathbb{T}(w), \nabla_{\mathbb{T}(w)}, \mathcal{S}_{\mathbb{T}(w)}, \kappa_{\mathbb{T}(w)}, 2w\big)$ is a pure polarized twistor-TERP structure.

2.1.6 Gluing construction

Variation of integrable twistor structures We can describe a variation of integrable twistor structures as gluing. We set $\mathcal{X} := \boldsymbol{C}_\lambda \times X$, $\mathcal{X}^0 := \{0\} \times X$, $\mathcal{X}^\dagger := \boldsymbol{C}_\mu \times X^\dagger$ and $\mathcal{X}^{\dagger 0} := \{0\} \times X^\dagger$.

Let V_0 be a holomorphic vector bundle on $\mathcal{X}$ with a meromorphic flat connection (TE-structure [7])

$$\nabla_{V_0} : V_0 \longrightarrow V_0 \otimes \Omega^{1,0}_{\mathcal{X}}(\log \mathcal{X}^0) \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{X}^0).$$

We use the same symbol to denote the associated differential operator $V_0 \longrightarrow V_0 \otimes \widetilde{\Omega}^1_{\mathcal{X}}$ in the C^∞-category. (The holomorphic structure d''_{V_0} is also included.) Let V_∞ be a holomorphic vector bundle on $\mathcal{X}^\dagger$ with a meromorphic flat connection ($\widetilde{T}E$-structure [7])

$$\nabla_{V_\infty} : V_\infty \longrightarrow V_\infty \otimes \Omega^{1,0}_{\mathcal{X}^\dagger}(\log \mathcal{X}^{\dagger\, 0}) \otimes \mathcal{O}_{\mathcal{X}^\dagger}(\mathcal{X}^{\dagger\, 0}).$$

We use the same symbol to denote the associated differential operator $V_\infty \longrightarrow V_\infty \otimes \widetilde{\Omega}^1_{\mathcal{X}^\dagger}$ in the C^∞-category. Assume that we are given an isomorphism Φ of C^∞-flat bundles:

$$\Phi : (V_0, \nabla_{V_0})_{|\boldsymbol{C}^*_\lambda \times X} \simeq (V_\infty, \nabla_{V_\infty})_{|\boldsymbol{C}^*_\mu \times X^\dagger}$$

We obtain the C^∞-vector bundle V on $\mathbb{P}^1 \times X$ by gluing V_0 and V_∞ via Φ. Since Φ is flat, ∇_{V_0} and ∇_{V_∞} induce the $T\widetilde{T}E$-structure $\widetilde{\mathbb{D}}_V^{\triangle} : V \longrightarrow V \otimes \widetilde{\Omega}^1_{\mathbb{P}^1 \times X}$. Thus, we obtain a variation of integrable twistor structures $(V, \widetilde{\mathbb{D}}_V^{\triangle})$.

Conversely, we naturally obtain a tuple of (V_0, ∇_{V_0}), $(V_\infty, \nabla_{V_\infty})$ and Φ as above from a variation of integrable twistor structures $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ as the restriction to $\mathcal{X}$ and $\mathcal{X}^\dagger$, respectively. In this situation, we set

$$\mathrm{Glue}\big((V_0, \nabla_{V_0}), (V_\infty, \nabla_{V_\infty}), \Phi\big) := (V, \widetilde{\mathbb{D}}_V^{\triangle}).$$

Pairing and real structure Note that we have the natural isomorphisms $\nu^* \widetilde{\Omega}^1_{\mathcal{X}^\dagger} \simeq \widetilde{\Omega}^1_{\mathcal{X}}$ and $\nu^* \widetilde{\Omega}^1_{\mathcal{X}} \simeq \widetilde{\Omega}^1_{\mathcal{X}^\dagger}$ for anti-holomorphic diffeomorphism $\nu : \boldsymbol{C}_\lambda \longrightarrow \boldsymbol{C}_\mu$ or $\boldsymbol{C}_\mu \longrightarrow \boldsymbol{C}_\lambda$, as in the case of $\widetilde{\Omega}^1_{\mathbb{P}^1 \times X}$. Let V_0 be a holomorphic vector bundle on $\mathcal{X}$ with a TE-structure ∇_{V_0}. By the above isomorphisms, $\gamma^* V_0$ and $\sigma^* V_0$ are naturally equipped with $\widetilde{T}E$-structure $\nabla_{\gamma^* V_0}$ and $\nabla_{\sigma^* V_0}$. Similarly, if we are given a holomorphic vector bundle V_∞ on $\mathcal{X}^\dagger$ with $\widetilde{T}E$-structure, $\sigma^* V_\infty$ and $\gamma^* V_\infty$ are naturally equipped with TE-structures. We remark that there exist the natural isomorphisms:

$$\mathrm{Glue}\big(\gamma^*(V_\infty, \nabla_{V_\infty}), \gamma^*(V_0, \nabla_{V_0}), \gamma^* \Phi^{-1}\big) \simeq \gamma^* \, \mathrm{Glue}\big((V_0, \nabla_{V_0}), (V_\infty, \nabla_{V_\infty}), \Phi\big)$$

$$\mathrm{Glue}\big(\sigma^*(V_\infty, \nabla_{V_\infty}), \sigma^*(V_0, \nabla_{V_0}), \sigma^* \Phi^{-1}\big) \simeq \sigma^* \, \mathrm{Glue}\big((V_0, \nabla_{V_0}), (V_\infty, \nabla_{V_\infty}), \Phi\big)$$

A real structure of variation of integrable twistor structure corresponds to a pair of isomorphisms

$$\kappa_0 : \gamma^*(V_\infty, \nabla_{V_\infty}) \simeq (V_0, \nabla_{V_0}), \quad \kappa_\infty : \gamma^*(V_0, \nabla_{V_0}) \simeq (V_\infty, \nabla_{V_\infty})$$

such that (i) $\gamma^*\kappa_0 = \kappa_\infty^{-1}$, (ii) the following commutativity holds on $\boldsymbol{C}_\lambda^* \times X$:

$$\begin{array}{ccc} \gamma^* V_\infty & \xrightarrow{\kappa_0} & V_0 \\ {\scriptstyle \gamma^*\Phi^{-1}}\big\downarrow & & {\scriptstyle \Phi}\big\downarrow \\ \gamma^* V_0 & \xrightarrow{\kappa_\infty} & V_\infty \end{array}$$

A pairing of weight w corresponds to

$$\mathcal{S}_0 : (V_0, \nabla_{V_0}) \otimes \sigma^*(V_\infty, \nabla_{V_\infty}) \longrightarrow \mathbb{T}(-w)_{|\mathcal{X}}, \quad \mathcal{S}_\infty : (V_\infty, \nabla_{V_\infty}) \otimes \sigma^*(V_0, \nabla_{V_0}) \longrightarrow \mathbb{T}(-w)_{|\mathcal{X}^\dagger}$$

such that (i) $\iota_{\mathbb{T}(-w)} \circ \sigma^*\mathcal{S}_\infty = (-1)^w \mathcal{S}_0 \circ \text{exchange}$, (ii) it is compatible with the gluing. Compatibility of $\mathcal{S}$ and κ is $\kappa_{\mathbb{T}(-w)} \circ \gamma^*\mathcal{S}_\infty = \mathcal{S}_0 \circ \big(\kappa_0 \otimes \sigma^*\kappa_\infty\big)$.

Variation of twistor structures The above gluing description is essentially the same as that for a variation of twistor structures in [27], which we recall in the following. See also [18]. We have the decomposition $\widetilde{\Omega}^1_{\mathcal{X}} = \xi\widetilde{\Omega}^1_{X|\mathcal{X}} \oplus \widetilde{\Omega}_{\boldsymbol{C}_\lambda}$ into the X-direction and the $\boldsymbol{C}_\lambda$-direction. Let d_X denote the restriction of the exterior differential to the X-direction. Similarly, we have the decomposition $\widetilde{\Omega}^1_{\mathcal{X}^\dagger} = \xi\widetilde{\Omega}^1_{X|\mathcal{X}^\dagger} \oplus \widetilde{\Omega}_{\boldsymbol{C}_\mu}$, and the restriction of $\widetilde{\mathbb{D}}^{\dagger f}_X$ to the X-direction is denoted by $d_{X^\dagger}$. The notions of $\boldsymbol{C}_\lambda$-holomorphic bundles or $\boldsymbol{C}_\mu$-holomorphic bundles are defined as in the case of $\mathbb{P}^1$-holomorphic bundles.

Let $(V_0, d''_{\boldsymbol{C}_\lambda, V_0})$ be a $\boldsymbol{C}_\lambda$-holomorphic bundle on $\mathcal{X}$. A T-structure [7] of V_0 is a differential operator

$$\mathbb{D}^f_{V_0} : V_0 \longrightarrow V_0 \otimes \xi\Omega^1_{X|\mathcal{X}}$$

satisfying (i) $\mathbb{D}^f_{V_0}(f \cdot s) = d_X f \cdot s + f \cdot \mathbb{D}^f_{V_0}(s)$ for a function f and a section s of V, (ii) $\big(d''_{\boldsymbol{C}_\lambda, V_0} + \mathbb{D}^f_{V_0}\big)^2 = 0$.

Let $(V_\infty, d''_{\boldsymbol{C}_\mu, V_\infty})$ be a $\boldsymbol{C}_\mu$-holomorphic vector bundle on $\mathcal{X}^\dagger$. A $\widetilde{T}$-structure [7] is defined to be a differential operator

$$\mathbb{D}^{\dagger f}_{V_\infty} : V_\infty \longrightarrow V_\infty \otimes \xi\Omega^1_{X|\mathcal{X}^\dagger}$$

satisfying conditions similar to (i) and (ii) above.

Assume that we are given an isomorphism Φ:

$$\Phi : (V_0, d''_{\boldsymbol{C}_\lambda, V_0}, \mathbb{D}^f_{V_0})_{|\boldsymbol{C}_\lambda^* \times X} \simeq (V_\infty, d''_{\boldsymbol{C}_\mu, V_\infty}, \mathbb{D}^{\dagger f}_{V_\infty})_{|\boldsymbol{C}_\mu^* \times X^\dagger} \tag{3}$$

We obtain the C^∞-vector bundle V on $\mathbb{P}^1 \times X$ by gluing V_0 and V_∞ via Φ. By the condition (3), $d''_{\boldsymbol{C}_\lambda, V_0}$ and $d''_{\boldsymbol{C}_\mu, V_\infty}$ give $\mathbb{P}^1$-holomorphic structure $d''_{\mathbb{P}^1, V}$, and $\mathbb{D}^f_{V_0}$ and $\mathbb{D}^{\dagger f}_{V_\infty}$ induce the $T\widetilde{T}$-structure $\mathbb{D}^\triangle_V$. Thus, we obtain a variation of twistor structures $(V, d''_{\mathbb{P}^1, V}, \mathbb{D}^\triangle_V)$.

Conversely, we naturally obtain such a tuple of $(V_0, d''_{\boldsymbol{C}_\lambda, V_0}, \mathbb{D}^f_{V_0})$, $(V_\infty, d''_{\boldsymbol{C}_\mu, V_\infty}, \mathbb{D}^{\dagger f}_{V_\infty})$ and Φ from a variation of twistor structures $(V, d''_{\mathbb{P}^1, V}, \mathbb{D}^\triangle_V)$ as the restriction to $\mathcal{X}$ and $\mathcal{X}^\dagger$, respectively. In this situation, we set

$$\mathrm{Glue}\big((V_0, d''_{\boldsymbol{C}_{\lambda,}, V_0}, \mathbb{D}^f_{V_0}), (V_\infty, d''_{\boldsymbol{C}_\mu, V_\infty}, \mathbb{D}^{\dagger f}_{V_\infty}), \Phi\big) := (V, \widetilde{\mathbb{D}}^\triangle_V)$$

Remark 2.9 *Let p_λ be the projection $\mathcal{X} \longrightarrow X$. Under the natural isomorphism*

$$\xi\Omega^1_{X|\mathcal{X}} = \lambda^{-1} \cdot p_\lambda^{-1}\Omega^{1,0}_X \oplus p_\lambda^{-1}\Omega^{0,1}_X \simeq p_\lambda^{-1}\Omega^{1,0}_X \oplus p_\lambda^{-1}\Omega^{0,1}_X = p_\lambda^{-1}\Omega^1_X,$$

a T-structure $\mathbb{D}^f_{V_0}$ induces a holomorphic family of flat λ-connections $\mathbb{D}_{V_0}$. Similarly, a $\widetilde{T}$-structure of $\mathbb{D}^{\dagger f}_{V_\infty}$ naturally induces a holomorphic family of flat μ-connections $\mathbb{D}^\dagger_{V_\infty}$. Hence, a variation of twistor structure is regarded as the gluing of families of λ-flat bundles and μ-flat bundles. ∎

2.1.7 Relation with harmonic bundles

We recall a fundamental equivalence due to Hertling and Sabbah. Let X be a complex manifold. Let $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$ be a variation of pure polarized integrable twistor structures of weight 0 on $\mathbb{P}^1 \times X$. By the equivalence between harmonic bundles and variations of pure polarized twistor structures due to Simpson, we have the underlying harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on X. Moreover, it is equipped with C^∞-sections $\mathcal{U}$ and $\mathcal{Q}$ of $\mathrm{End}(E)$ satisfying the following equations:

$$\overline{\partial}_E \mathcal{U} = 0, \quad [\mathcal{U}, \theta] = 0, \quad \mathcal{Q} = \mathcal{Q}^\dagger \tag{4}$$

$$\partial_E \mathcal{U} - [\theta, \mathcal{Q}] + \theta = 0, \quad \partial_E \mathcal{Q} + [\theta, \mathcal{U}^\dagger] = 0 \tag{5}$$

Here, $\mathcal{U}_{|Q}$ and $\mathcal{Q}_{|Q}$ $(Q \in X)$ are obtained as in (2), and $\mathcal{U}^\dagger$ and $\mathcal{Q}^\dagger$ denote the adjoint of $\mathcal{U}$ and $\mathcal{Q}$ with respect to h, respectively. Conversely, we obtain a variation of polarized pure integrable twistor structures $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$ from a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ with $\mathcal{U}$ and $\mathcal{Q}$ satisfying (4) and (5). Let $p : \mathbb{P}^1 \times X \longrightarrow X$ be the projection. We set $\mathcal{E}^{\triangle} := p^{-1}E$ on which we have the natural connection $d_{\mathbb{P}^1}$ along the $\mathbb{P}^1$-direction. We set

$$\nabla_\lambda := d_{\mathbb{P}^1} + \bigl(\lambda^{-1} \cdot \mathcal{U} - \mathcal{Q} - \lambda \cdot \mathcal{U}^\dagger\bigr) \frac{d\lambda}{\lambda}$$

It gives a flat connection of $\mathcal{E}^{\triangle}$ along the $\mathbb{P}^1$-direction. Then, we obtain a $T\widetilde{T}E$-structure

$$\widetilde{\mathbb{D}}^{\triangle} := \bigl(\overline{\partial}_E + \lambda\theta^\dagger\bigr) + \bigl(\partial_E + \lambda^{-1}\theta\bigr) + \nabla_\lambda : \mathcal{E}^{\triangle} \longrightarrow \mathcal{E}^{\triangle} \otimes \widetilde{\Omega}^1_{\mathbb{P}^1 \times X}.$$

The pairing $\mathcal{S}$ is induced by $\mathcal{S}(u \otimes \sigma^* v) = h(u, \sigma^* v)$.

Let us also see the gluing construction of the above $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$. Let $(E, \overline{\partial}_E, \theta, h, \mathcal{U}, \mathcal{Q})$ be as above. Let p_λ be the projection $\mathcal{X} \longrightarrow X$. Let $\mathcal{E}$ be the holomorphic vector bundle $\bigl(p_\lambda^{-1}E,\, \overline{\partial}_E + \lambda\theta^\dagger + \overline{\partial}_\lambda\bigr)$, where $\overline{\partial}_\lambda$ denotes the natural λ-holomorphic structure of $\mathcal{E}$. We have the family of flat λ-connections $\mathbb{D} = \overline{\partial}_E + \lambda\theta^\dagger + \lambda\partial_E + \theta$ of $\mathcal{E}$. The associated family of flat connections is given by $\mathbb{D}^f = \overline{\partial}_E + \lambda\theta^\dagger + \partial_E + \lambda^{-1}\theta$. Then, $\widetilde{\mathbb{D}}^f := \mathbb{D}^f + \nabla_\lambda$ gives a meromorphic flat connection of $\mathcal{E}$.

Let p_μ be the projection $\mathcal{X}^\dagger \longrightarrow X^\dagger$. Let $\mathcal{E}^\dagger$ be the holomorphic vector bundle $\bigl(p_\mu^{-1}E,\, \partial_E + \mu\theta + \overline{\partial}_\mu\bigr)$, where $\overline{\partial}_\mu$ denotes the natural μ-holomorphic structure of $\mathcal{E}^\dagger$. We have the family of flat μ-connections $\mathbb{D}^\dagger = \partial_E + \mu\theta + \mu\overline{\partial}_E + \theta^\dagger$ of $\mathcal{E}^\dagger$. The associated family of flat connections is given by $\mathbb{D}^{\dagger\, f} = \partial_E + \mu\theta + \overline{\partial}_E + \mu^{-1}\theta^\dagger$. Then, $\widetilde{\mathbb{D}}^{\dagger\, f} := \mathbb{D}^{\dagger\, f} + \nabla_\lambda$ gives a meromorphic flat connection of $\mathcal{E}^\dagger$.

We have the induced pairings $\mathcal{S}_0 : \mathcal{E} \otimes \sigma^*\mathcal{E}^\dagger \longrightarrow \mathcal{O}_{\mathcal{X}}$ and $\mathcal{S}_\infty : \mathcal{E}^\dagger \otimes \sigma^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathcal{X}^\dagger}$ induced by h. Then, $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$ is obtained as the gluing of $(\mathcal{E}, \widetilde{\mathbb{D}}^f)$, $(\mathcal{E}^\dagger, \widetilde{\mathbb{D}}^{\dagger\, f})$ and $(\mathcal{S}_0, \mathcal{S}_\infty)$ by the procedure in Subsection 2.1.6.

2.1.8 TERP and twistor-TERP

Let us observe that the notions of TERP-structure and twistor-TERP structure are equivalent. First, let us introduce a pairing P induced by κ and $\mathcal{S}$. Then, we argue the equivalence in the case that X is a point, for simplicity. We give a remark for the family case in the end.

The induced pairing P We set $j := \gamma \circ \sigma = \sigma \circ \gamma$, which is a holomorphic involution of $\mathbb{P}^1$. We have the induced isomorphisms

$$\sigma^*\kappa : j^*\mathbb{T}(w) \simeq \sigma^*\mathbb{T}(w), \quad j^*\kappa : \sigma^*\mathbb{T}(w) \simeq j^*\mathbb{T}(w).$$

We have the following equality:

$$\sigma^*\kappa \circ j^*\kappa = j^*\bigl(\gamma^*\kappa \circ \kappa\bigr) = j^*(\mathrm{id}) = \mathrm{id}$$

We will use similar relations implicitly. We also remark the commutativity of the following diagram, which can be checked by a direct calculation:

$$\begin{array}{ccc} j^*\mathbb{T}(w) & \xrightarrow{\gamma^*\iota_{\mathbb{T}(w)}} & \gamma^*\mathbb{T}(w) \\ {\scriptstyle \sigma^*\kappa_{\mathbb{T}(w)}}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_{\mathbb{T}(w)}} \\ \sigma^*\mathbb{T}(w) & \xrightarrow{\iota_{\mathbb{T}(w)}} & \mathbb{T}(w) \end{array}$$

The composite $j^*\mathbb{T}(w) \longrightarrow \mathbb{T}(w)$ is denoted by $\rho_{\mathbb{T}(w)}$.

Let $(V, \widetilde{\mathbb{D}}^{\triangle}_V, \mathcal{S}, \kappa, -w)$ be a variation of twistor-TERP structure. We define a pairing $P : V \otimes j^*V \longrightarrow \mathbb{T}(-w)$ by

$$P := (\sqrt{-1})^w \cdot \mathcal{S} \circ (1 \otimes \sigma^* \kappa). \tag{6}$$

Lemma 2.10 *P is $(-1)^w$-symmetric in the sense that the following diagram is commutative:*

$$\begin{array}{ccc} j^*V \otimes V & \xrightarrow{j^*P} & j^*\mathbb{T}(-w) \\ {\scriptstyle \text{exchange}}\downarrow & & \downarrow{\scriptstyle \rho_{\mathbb{T}(-w)}} \\ V \otimes j^*V & \xrightarrow{(-1)^w P} & \mathbb{T}(-w) \end{array}$$

Namely, $\rho_{\mathbb{T}(-w)} \circ j^*P = (-1)^w \cdot P \circ \mathrm{exchange}$. *Here,* exchange *denotes the natural morphism exchanging the components.*

Proof We have the following equality:

$$\rho_{\mathbb{T}(-w)} \circ j^*P = (\sqrt{-1})^w \kappa_{\mathbb{T}(-w)} \circ \gamma^* \iota_{\mathbb{T}(-w)} \circ j^*\mathcal{S} \circ \bigl(1 \otimes j^*\sigma^*\kappa\bigr)$$
$$= (\sqrt{-1})^w \kappa_{\mathbb{T}(-w)} \circ \gamma^* \iota_{\mathbb{T}(-w)} \circ \bigl(\gamma^*\sigma^*\mathcal{S}\bigr) \circ \bigl(1 \otimes \gamma^*\kappa\bigr) = (\sqrt{-1})^w \kappa_{\mathbb{T}(-w)} \circ \gamma^* \Bigl(\iota_{\mathbb{T}(-w)} \circ \sigma^*\mathcal{S}\Bigr) \circ \bigl(1 \otimes \gamma^*\kappa\bigr) \tag{7}$$

By using the compatibility of $\mathcal{S}$ and κ, we obtain

$$(-1)^w P \circ \mathrm{exchange} = (\sqrt{-1})^w (-1)^w \mathcal{S} \circ (1 \otimes \sigma^*\kappa) \circ \mathrm{exchange} = (\sqrt{-1})^w (-1)^w \mathcal{S} \circ (\kappa \otimes \sigma^*\kappa) \circ (\gamma^*\kappa \otimes 1) \circ \mathrm{exchange}$$
$$= (\sqrt{-1})^w \kappa_{\mathbb{T}(-w)} \circ \gamma^* \Bigl((-1)^w \mathcal{S} \circ \mathrm{exchange}\Bigr) \circ \bigl(1 \otimes \gamma^*\kappa\bigr) \tag{8}$$

Thus, we are done. ∎

Lemma 2.11 *The following diagram is commutative:*

$$\begin{array}{ccc} \gamma^*V \otimes \sigma^*V & \xrightarrow{\gamma^*P} & \gamma^*\mathbb{T}(-w) \\ {\scriptstyle \kappa \otimes j^*\kappa}\downarrow & & \downarrow{\scriptstyle \kappa_{\mathbb{T}(-w)}} \\ V \otimes j^*V & \xrightarrow{(-1)^w P} & \mathbb{T}(-w) \end{array}$$

Namely, $(-1)^w P \circ \bigl(\kappa \otimes j^*\kappa\bigr) = \kappa_{\mathbb{T}(-w)} \circ \gamma^*P$.

Proof We have the following equalities:

$$(\sqrt{-1})^{-w} P \circ (\kappa \otimes j^*\kappa) = \mathcal{S} \circ (1 \otimes \sigma^*\kappa) \circ (\kappa \otimes j^*\kappa) = \mathcal{S} \circ (\kappa \otimes \sigma^*\kappa) \circ (1 \otimes j^*\kappa) \tag{9}$$

$$\kappa_{\mathbb{T}(-w)} \circ \gamma^*\bigl((\sqrt{-1})^{-w} P\bigr) = \kappa_{\mathbb{T}(-w)} \circ \gamma^*\bigl(\mathcal{S} \circ (1 \otimes \sigma^*\kappa)\bigr) = \kappa_{\mathbb{T}(-w)} \circ (\gamma^*\mathcal{S}) \circ (1 \otimes j^*\kappa) \tag{10}$$

Then, the claim of the lemma follows from the compatibility of $\mathcal{S}$ and κ. ∎

From twistor-TERP to TERP Let $(V, \nabla, \mathcal{S}, \kappa, -w)$ be a twistor-TERP structure. Let us explain how to associate a TERP structure $(H, H'_{\boldsymbol{R}}, \nabla, P', -w)$ in the sense of Hertling. We set $H := V_{|\boldsymbol{C}_\lambda}$ and $H' := V_{|\boldsymbol{C}^*_\lambda}$. In general, for a $\boldsymbol{C}$-vector bundle U, let $\overline{U}$ denote the conjugate of U, i.e., $\overline{U} = U$ as an $\boldsymbol{R}$-vector bundle, and the multiplication of $\sqrt{-1}$ on $\overline{U}$ is given by the multiplication of $-\sqrt{-1}$ on U. Note that $\gamma^*(H)_{|\lambda}$ for $\lambda \neq 0$ is naturally identified with $\overline{H}_{|\overline{\lambda}^{-1}}$.

The following diagram for $\lambda \neq 0$ is commutative by the flatness of κ:

$$\begin{array}{ccc} \overline{H}_{|\lambda} & \xrightarrow{\Pi_\lambda} & \overline{H}_{|\overline{\lambda}^{-1}} \\ {\scriptstyle \kappa_{|\overline{\lambda}^{-1}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \kappa_{|\lambda}} \\ H_{|\overline{\lambda}^{-1}} & \xrightarrow{\Pi_\lambda^{-1}} & H_{|\lambda} \end{array} \tag{11}$$

Here, Π_λ denotes the parallel transform along the segment connecting λ and $\overline{\lambda}^{-1}$, as often used in [7]. A flat isomorphism $\kappa' : \overline{H}_{|\boldsymbol{C}^*_\lambda} \simeq H_{|\boldsymbol{C}^*_\lambda}$ is given by the composite of the morphisms, i.e., $\kappa'_{|\lambda} := \kappa_{|\lambda} \circ \Pi_\lambda$. Because $\gamma^*\kappa \circ \kappa = \mathrm{id}$, the composite

$$\overline{H}_{|\lambda} \xrightarrow{\kappa_{|\overline{\lambda}^{-1}}} H_{|\overline{\lambda}^{-1}} \xrightarrow{\kappa_\lambda} \overline{H}_{|\lambda}$$

is the identity. Let us check $\kappa' \circ \kappa' = \mathrm{id}$ by using the commutativity (11):

$$\kappa'_\lambda \circ \kappa'_\lambda = \big(\kappa_\lambda \circ \Pi_\lambda\big) \circ \big(\Pi_\lambda^{-1} \circ \kappa_{|\overline{\lambda}^{-1}}\big) = \kappa_\lambda \circ \kappa_{|\overline{\lambda}^{-1}} = \mathrm{id}$$

Hence, κ' gives a flat real structure of H'. Thus, we obtain a real flat subbundle $H'_{\boldsymbol{R}}$ of $H_{|\boldsymbol{C}^*_\lambda}$. By restricting P, we obtain a pairing:

$$P_{|\boldsymbol{C}_\lambda} : H \otimes j^*H \longrightarrow \mathbb{T}(-w)_{|\boldsymbol{C}_\lambda} = \mathcal{O}_{\boldsymbol{C}_\lambda} \cdot (\sqrt{-1}\lambda)^{-w} t_1^{(-w)}$$

By taking the coefficients of $t_1^{(-w)}$, we obtain a flat morphism

$$P' : H' \otimes j^*H' \longrightarrow \mathcal{O}_{\boldsymbol{C}^*_\lambda}$$

such that $\lambda^w \cdot P'$ induces a perfect pairing $H \otimes j^*H \longrightarrow \mathcal{O}_{\boldsymbol{C}_\lambda}$. By Lemma 2.10, P' is $(-1)^w$-symmetric.

Lemma 2.12 $P'\big(H'_{\boldsymbol{R}} \otimes_{\boldsymbol{R}} j^*H'_{\boldsymbol{R}}\big) \subset (\sqrt{-1})^w \boldsymbol{R}$.

Proof Note that κ gives real structures $\kappa_{|a} : \overline{H_{|a}} \simeq H_{|a}$ for $a = 1, -1$. By Lemma 2.11, we have

$$(\sqrt{-1})^w \cdot P_{|1} \circ \big(\kappa_{|1} \otimes \kappa_{|-1}\big) = (\kappa_{\mathbb{T}(-w)})_{|1} \circ \big((\sqrt{-1})^w P_{|1}\big). \tag{12}$$

We obtain $P'_{|1}\big(H_{|1} \otimes H_{|-1}\big) \subset (\sqrt{-1})^w \boldsymbol{R}$. Then, the claim of the lemma follows from the flatness of P'. ∎

Thus, we obtain a TERP-structure $(H, H'_{\boldsymbol{R}}, \nabla, P', -w)$.

From TERP to twistor-TERP Conversely, we obtain a twistor-TERP structure $(V, \nabla, \kappa, \mathcal{S}, -w)$ from a TERP structure $(H, H'_{\boldsymbol{R}}, \nabla, P', -w)$. We set $V_0 := H$ and $V_\infty := \gamma^*H$. We have the flat isomorphism

$$\tau_{\mathrm{real}} : H_{|\boldsymbol{C}^*_\lambda} \simeq \gamma^*(H_{|\boldsymbol{C}^*_\lambda}),$$

obtained as the composite of the conjugate with respect to the real structure and the parallel transform along the segment connecting λ and $\overline{\lambda}^{-1}$. By gluing (H, ∇) and $\gamma^*(H, \nabla)$ via τ_{real}, we obtain an integrable twistor structure (V, ∇).

By construction, we have $\gamma^*(\tau_{\mathrm{real}}) = \tau_{\mathrm{real}}^{-1}$, and the following diagram is commutative:

$$\begin{array}{ccc} H_{|\boldsymbol{C}^*_\lambda} & \xrightarrow{\tau_{\mathrm{real}}} & \gamma^*(H_{|\boldsymbol{C}^*_\lambda}) \\ =\Big\downarrow & & =\Big\downarrow \\ \gamma^*\big(\gamma^* H_{|\boldsymbol{C}^*_\lambda}\big) & \xrightarrow{\gamma^*\tau_{\mathrm{real}}^{-1}} & \gamma^* H_{|\boldsymbol{C}^*_\lambda} \end{array}$$

Hence, a morphism $\kappa : \gamma^*(V, \nabla) \simeq (V, \nabla)$ is given by the gluing of $\gamma^*V_\infty \simeq V_0$ and $\gamma^*V_0 \simeq V_\infty$ induced by the identity. Clearly it satisfies $\gamma^*\kappa \circ \kappa = \mathrm{id}$. The restriction $\kappa_{|\boldsymbol{C}^*_\lambda} : \gamma^*(V)_{|\boldsymbol{C}^*_\lambda} \longrightarrow V_{|\boldsymbol{C}^*_\lambda}$ is identified with $\tau_{\mathrm{real}}^{-1} : \gamma^*H_{|\boldsymbol{C}^*_\lambda} \simeq H_{|\boldsymbol{C}^*_\lambda}$.

Let $P_0 : V_0 \otimes j^* V_0 \longrightarrow \mathcal{O}_{\boldsymbol{C}_\lambda} \cdot t_0^{(-w)}$ be given by

$$P_0 = P' \cdot t_1^{(-w)} = P' \cdot \big(\sqrt{-1}\lambda\big)^w \cdot t_0^{(-w)}.$$

We have the induced morphism

$$\kappa_{\mathbb{T}(-w)} \circ \gamma^* P_0 : V_\infty \otimes j^* V_\infty \longrightarrow \mathcal{O}_{\boldsymbol{C}_\mu} \cdot t_\infty^{(-w)}.$$

We obtain the following equalities for linear maps $\overline{H_{|1}} \otimes \overline{H_{|-1}} \longrightarrow \mathbb{T}(-w)_{|1}$ from $P'\big(H'_{\boldsymbol{R}} \otimes_{\boldsymbol{R}} j^* H'_{\boldsymbol{R}}\big) \subset (\sqrt{-1})^w \boldsymbol{R}$:

$$(\sqrt{-1})^w \cdot P_{0|1} \circ \big(\kappa_{|1} \otimes \kappa_{|-1}\big) = (\kappa_{\mathbb{T}(-w)})_{|1} \circ \big((\sqrt{-1})^w P_{0|1}\big) = (-\sqrt{-1})^w (\kappa_{\mathbb{T}(-w)})_{|1} \circ \big(\gamma^* P_{0|1}\big)$$

Here, we have used the natural identification $P_{0|1} = (\gamma^* P_0)_{|1}$. The first and third terms are obtained as the restrictions of morphisms $(V_\infty \otimes j^* V_\infty)_{|\boldsymbol{C}^*_\lambda} \longrightarrow \mathcal{O}_{\boldsymbol{C}^*_\lambda} \cdot t_1^{(-w)}$ to the fiber over 1. By flatness, we obtain the following equality on $\boldsymbol{C}^*_\lambda$:

$$(-1)^w \cdot P_0 \circ \big(\kappa \otimes j^* \kappa\big) = \kappa_{\mathbb{T}(-w)} \circ \gamma^* P_0 \tag{13}$$

Hence, the pairings P_0 and $(-1)^w \kappa_{\mathbb{T}(-w)} \circ \gamma^* P_0$ induce $P : V \otimes j^* V \longrightarrow \mathbb{T}(-w)$. Since P' is $(-1)^w$-symmetric, P is also $(-1)^w$-symmetric in the sense of Lemma 2.10. From (13), we obtain

$$(-1)^w \cdot P \circ \big(\kappa \otimes j^* \kappa\big) = \kappa_{\mathbb{T}(-w)} \circ \gamma^* P. \tag{14}$$

The pairing $\mathcal{S}$ is constructed from P and κ by the relation (6). The compatibility of κ and $\mathcal{S}$ follows from (9), (10) and (14). The pairing $\mathcal{S}$ is $(-1)^w$-symmetric, which follows from (7), (8) and the compatibility with κ. Thus, we obtain a twistor-TERP structure $(V, \nabla, \mathcal{S}, \kappa, -w)$.

Hertling's vector bundle Let $(H, H'_{\boldsymbol{R}}, \nabla, P, -w)$ be a TERP-structure corresponding to a twistor-TERP structure $(V, \nabla, \mathcal{S}, \kappa, -w)$. Recall that Hertling constructed an integrable twistor structure $(\widehat{H}, \nabla)$ from a TERP-structure $(H, H'_{\boldsymbol{R}}, \nabla, P, -w)$ by gluing H and $\gamma^* H$ via a map τ. (See [7].) We do not recall τ and his construction here, but $\widehat{H}$ is naturally isomorphic to $V \otimes \mathcal{O}(0, -w)$ by the following correspondence:

$$H = V_0 \longleftrightarrow V_0 \otimes \mathcal{O}(0, -w)_0, \quad a \longleftrightarrow a \otimes f_0^{(0,-w)}$$

$$\gamma^* H \longleftrightarrow \gamma^* V_0 \otimes \mathcal{O}(0, -w)_\infty, \quad \gamma^* b \longleftrightarrow \gamma^* b \otimes (\sqrt{-1})^w f_\infty^{(0,-w)}$$

According to [7] and [8], $(H, H'_{\boldsymbol{R}}, \nabla, P, -w)$ is defined to be pure if $(\widehat{H}, \nabla)$ is pure of weight 0. They consider the hermitian pairing h of $H^0(\mathbb{P}^1, \widehat{H})$ given by $\lambda^w \cdot P' \circ (1 \otimes \tau)$, and $(H, H'_{\boldsymbol{R}}, \nabla, P, -w)$ is defined to be polarized if h is positive definite.

Lemma 2.13 *$(H, H'_{\boldsymbol{R}}, \nabla, P, -w)$ is pure (polarized), if and only if $(V, \nabla, \mathcal{S}, \kappa, -w)$ is pure (polarized).*

Proof The claim for purity is obvious. Let us consider polarizability. We have only to show that h is the hermitian pairing induced by $\widetilde{\mathcal{S}} := \mathcal{S} \otimes \mathcal{S}_{0,-w}$, under the identification of $\widehat{H}$ and $V \otimes \mathcal{O}(0, -w)$.

Let $\widehat{a}, \widehat{b} \in H^0(\mathbb{P}^1, \widehat{H})$. Under the identification $\widehat{H}_{|\boldsymbol{C}_\lambda} = H$ and $\widehat{H}_{|\boldsymbol{C}_\mu} = \gamma^* H$, the sections a and b of H are determined by $a := \widehat{a}_{|\boldsymbol{C}_\lambda}$ and $\gamma^* b := \widehat{b}_{|\boldsymbol{C}_\mu}$. By definition, we have

$$h(\widehat{a}, \widehat{b}) = \lambda^w P'(a, j^* b)$$

Let us look at $\widetilde{\mathcal{S}}_{|\boldsymbol{C}_\lambda}$. Under the above identification, the pairing of $\widehat{a}$ and $\widehat{b}$ is given by

$$\widetilde{\mathcal{S}}\Big(a \otimes f_0^{(0,-w)}, \sigma^*\big(\gamma^* b \otimes (\sqrt{-1})^w f_\infty^{(0,-w)}\big)\Big) = \mathcal{S}\big(a, \sigma^*(\gamma^* b)\big) \cdot t_0^{(w)} = \mathcal{S}\big(a, j^* b\big) \cdot t_0^{(w)} =: \mathcal{S}_0\big(a, j^* b\big)$$

Let us compare $\lambda^w P'(a, j^* b)$ and $\mathcal{S}_0\big(a, j^* b\big)$. Since $\kappa_{|\boldsymbol{C}_\mu}$ is the same as the identity $V_\infty = \gamma^* H \longrightarrow \gamma^* V_0 = \gamma^* H$, we have

$$P_{|\boldsymbol{C}_\lambda} = (\sqrt{-1})^w \mathcal{S} \circ (1 \otimes \sigma^* \kappa)_{|\boldsymbol{C}_\lambda} = (\sqrt{-1})^w \mathcal{S}_{|\boldsymbol{C}_\lambda}$$

Hence, we have the following equality:

$$P'(a, j^* b) \cdot t_1^{(-w)} = P\big(a, j^* b\big) = \mathcal{S}\big(a, j^* b\big) \cdot (\sqrt{-1})^w = (\sqrt{-1})^w \cdot \mathcal{S}_0\big(a, j^* b\big) \cdot t_0^{(-w)} = \lambda^{-w} \mathcal{S}_0\big(a, j^* b\big) \cdot t_1^{(-w)}$$

Thus, we obtain $\lambda^w \cdot P'(a, j^* b) = \mathcal{S}_0(a, j^* b)$. Therefore, $\widetilde{\mathcal{S}}$ induces h. ∎

Family version The correspondence is generalized in the family case. We set $H := V_{|\boldsymbol{C}_\lambda \times X}$. It is equipped with TE-structure ∇ obtained as the restriction of $\widetilde{\mathbb{D}}_V^{\triangle}$. As in the previous case, we obtain a flat $\boldsymbol{C}$-anti-linear isomorphism $\kappa' : H_{|\boldsymbol{C}_\lambda^* \times X} \simeq H_{|\boldsymbol{C}_\lambda^* \times X}$ and a flat pairing $P : H' \otimes j^* H' \longrightarrow \mathcal{O}_{\boldsymbol{C}^* \times X}$. It is easy to check that $(H, H'_{\boldsymbol{R}}, \nabla, P, -w)$ is a variation of TERP structures. The converse can be constructed similarly. The correspondence preserves "pure" and "polarized", for which we have only to check the case in which X is a point.

2.2 Basic examples

2.2.1 Example associated to a holomorphic function

Let $\mathfrak{a}$ be a holomorphic function on a complex manifold X. We set

$$V_0 := \mathcal{O}_{\boldsymbol{C}_\lambda \times X} \cdot e, \quad \nabla_{V_0}(e) = e \cdot d\big(\lambda^{-1} \cdot \mathfrak{a}\big),$$

$$V_\infty := \mathcal{O}_{\boldsymbol{C}_\mu \times X^\dagger} \cdot e^\dagger, \quad \nabla_{V_\infty}(e^\dagger) = e^\dagger \cdot d\big(\mu^{-1} \cdot \overline{\mathfrak{a}}\big).$$

We put $s := \exp\big(-\lambda^{-1}\mathfrak{a}\big) \cdot e$ and $s^\dagger := \exp\big(-\mu^{-1}\overline{\mathfrak{a}}\big) \cdot e^\dagger$, which are flat sections of $V_{0|\boldsymbol{C}_\lambda^* \times X}$ and $V_{\infty|\boldsymbol{C}_\mu^* \times X^\dagger}$, respectively. A gluing $\Phi : V_{0|\boldsymbol{C}_\lambda^* \times X} \simeq V_{\infty|\boldsymbol{C}_\mu^* \times X^\dagger}$ is given by $\Phi(s) = s^\dagger$, in other words,

$$\Phi(e) = \exp\big(\lambda^{-1}\mathfrak{a} - \mu^{-1}\overline{\mathfrak{a}}\big) \cdot e^\dagger.$$

Let V be the C^∞-bundle obtained as the gluing of V_0 and V_∞ via Φ, which is equipped with $T\widetilde{T}E$-structure. For each point $P \in X$, the restriction $V_{|\mathbb{P}^1 \times \{P\}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}$, and hence $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ is pure of weight 0. A real structure κ is given by $\kappa(\gamma^* e^\dagger) = e$ and $\kappa(\gamma^* e) = e^\dagger$. We can check that κ actually gives a flat isomorphism $\gamma^* V \simeq V$. A pairing $\mathcal{S}$ of V with weight 0 is given by $e \otimes \sigma^* e^\dagger \longmapsto t_0^{(0)}$ and $e^\dagger \otimes \sigma^* e \longmapsto t_\infty^{(0)}$. It is easy to check that $\mathcal{S}$ actually gives a symmetric flat pairing $V \otimes \sigma^* V \longrightarrow \mathbb{T}(0)_X$. The compatibility of $\mathcal{S}$ and κ can be checked by a direct calculation:

$$\kappa_{\mathbb{T}(0)} \circ \gamma^* \mathcal{S}(\gamma^* e^\dagger \otimes \gamma^* \sigma^* e) = \kappa_{\mathbb{T}(0)}\big(\gamma^*\big(\mathcal{S}(e^\dagger \otimes \sigma^* e)\big)\big) = \kappa_{\mathbb{T}(0)} \gamma^* t_\infty^{(0)} = t_0^{(0)}$$

$$\mathcal{S} \circ (\kappa \otimes \sigma^* \kappa)(\gamma^* e^\dagger \otimes \sigma^* \gamma^* e) = \mathcal{S}\big(e \otimes \sigma^* e^\dagger\big) = t_0^{(0)}$$

Hence, we obtain a variation of twistor-TERP structure denoted by $L(\mathfrak{a})$. It is polarized. The underlying harmonic bundle is given by the line bundle $\mathcal{O}_X \cdot v$ with the Higgs field $\theta \cdot v = v \cdot d\mathfrak{a}$ and the hermitian metric $h(v, v) = 1$, where $v := e_{|\{0\} \times X}$. The operators $\mathcal{U}$ and $\mathcal{Q}$ are $\mathcal{U} = -\mathfrak{a}$ and $\mathcal{Q} = 0$.

2.2.2 Example associated to unitary flat bundles of rank one

In general, a variation of pure polarized Hodge structures provides us with an example of variation of pure polarized integrable twistor structures. Any unitary flat bundle naturally gives a variation of pure polarized Hodge structures, and hence an integrable variation of pure polarized integrable twistor structure.

In particular, we will use the following example. Let $X := \boldsymbol{C}^n$ and $D := \bigcup_{i=1}^{\ell} \{z_i = 0\}$. For any $\boldsymbol{a} \in \boldsymbol{R}^\ell$, we have the unitary flat bundle

$$\mathcal{O}_{X-D} \cdot e, \quad \nabla e = e \cdot \Big(-\sum_{i=1}^{\ell} a_i \cdot \frac{dz_i}{z_i}\Big)$$

The associated variation of integrable polarized pure integrable structures is denoted by $L(\boldsymbol{a})$.

More specifically, it is obtained as the gluing of the following meromorphic flat bundles:

$$V_0 = \mathcal{O}_{\boldsymbol{C}_\lambda \times (X-D)} \cdot e, \quad \nabla_{V_0} e = e \cdot \Big(-\sum_{i=1}^{\ell} a_i \cdot \frac{dz_i}{z_i}\Big)$$

$$V_\infty = \mathcal{O}_{\boldsymbol{C}_\mu \times (X^\dagger - D^\dagger)} \cdot e^\dagger, \quad \nabla_{V_\infty} e^\dagger = e^\dagger \cdot \Big(\sum_{i=1}^{\ell} a_i \cdot \frac{d\overline{z}_i}{\overline{z}_i}\Big)$$

The gluing is given by $\Phi(e) = \prod_{i=1}^{\ell} |z_i|^{-2a_i} \cdot e^{\dagger}$. The pairing is given by $\mathcal{S}(e, \sigma^* e^{\dagger}) = 1$. The underlying harmonic bundle is the line bundle $\mathcal{O}_{X-D} \cdot v$ with $\theta \cdot v = 0$ and $h(v,v) = \prod_{i=1}^{\ell} |z_i|^{-2a_i}$, where $v = e_{|\{0\}\times(X-D)}$. The operators $\mathcal{U}$ and $\mathcal{Q}$ are 0.

2.2.3 Example induced by nilpotent maps

Let Y be a complex manifold. We set $X := \boldsymbol{C}^{\ell} \times Y$, $D = \bigcup_{i=1}^{\ell} \{z_i = 0\} \times Y$. We put $\mathcal{X} := \boldsymbol{C}_{\lambda} \times X$ and $\mathcal{X}^{\dagger} := \boldsymbol{C}_{\mu} \times X^{\dagger}$. We use the symbols $\mathcal{D}$, $\mathcal{Y}$, $\mathcal{D}^{\dagger}$ and $\mathcal{Y}^{\dagger}$ in similar meanings. Let $q_0 : \mathcal{X} \longrightarrow \mathcal{Y}$ and $q_{\infty} : \mathcal{X}^{\dagger} \longrightarrow \mathcal{Y}^{\dagger}$ denote the naturally defined projections.

Let $(V, \mathbb{D}^{\triangle})$ be a variation of $\mathbb{P}^1$-holomorphic vector bundles on $\mathbb{P}^1 \times Y$ with a tuple $\boldsymbol{f}$ of nilpotent morphisms

$$f_i : V \longrightarrow V \otimes \mathbb{T}(-1), \quad i = 1, \ldots, \ell$$

such that (i) $[f_i, f_j] = 0$, (ii) they are $\mathbb{P}^1$-holomorphic and $\mathbb{D}^{\triangle}$-flat. We recall a construction of the variation of $\mathbb{P}^1$-holomorphic vector bundles on $\mathbb{P}^1 \times (X - D)$ associated to $(V, \boldsymbol{f})$ given in Subsection 3.5.3 of [18] with a minor generalization. (We considered the case in which Y is a point in [18].)

We regard $(V, \mathbb{D}_V^{\triangle})$ as the gluing of a family of λ-flat bundles $(V_0, \mathbb{D}_{V_0})$ on $\mathcal{Y}$, and a family of μ-flat bundles $(V_{\infty}, \mathbb{D}^{\dagger}_{V_{\infty}})$ on $\mathcal{Y}^{\dagger}$. We obtain a holomorphic vector bundle $\mathcal{V}_0 := q_0^* V_0$ on $\mathcal{X} - \mathcal{D}$ with a family of flat λ-connections $q_0^* \mathbb{D}_{V_0}$. We naturally identify $\mathbb{T}(0)_{|\mathcal{X}-\mathcal{D}} \simeq \mathcal{O}_{\mathcal{X}-\mathcal{D}}$ by the trivialization $t_0^{(0)}$. We also use the natural identification $\mathbb{T}(-1) \otimes \mathbb{T}(1) \simeq \mathbb{T}(0)$. We have the $q_0^* \mathbb{D}_{V_0}$-flat endomorphisms $q_0^* f_i \otimes t_0^{(1)} \in \mathrm{End}(\mathcal{V}_0)$. We obtain the family of flat λ-connections on $\mathcal{V}_0$ given as follows:

$$\mathbb{D}_{\mathcal{V}_0} := q_0^* \mathbb{D}_{V_0} + \sum_{i=1}^{\ell} q_0^* f_i \otimes t_0^{(1)} \frac{dz_i}{z_i}$$

Similarly, we obtain a holomorphic vector bundle $\mathcal{V}_{\infty} := q_{\infty}^* V_{\infty}$ on $\mathcal{X}^{\dagger} - \mathcal{D}^{\dagger}$ with a family of flat μ-connections $q_{\infty}^* \mathbb{D}^{\dagger}_{V_{\infty}}$. We have the $q_{\infty}^* \mathbb{D}^{\dagger}_{V_{\infty}}$-flat endomorphisms $q_{\infty}^* f_i \otimes t_{\infty}^{(1)} \in \mathrm{End}(\mathcal{V}_{\infty})$. Hence, we obtain the following family of flat μ-connections:

$$\mathbb{D}^{\dagger}_{\mathcal{V}_{\infty}} := q_{\infty}^* \mathbb{D}^{\dagger}_{V_{\infty}} + \sum_{i=1}^{\ell} q_{\infty}^* f_i \otimes t_{\infty}^{(1)} \frac{d\overline{z}_i}{\overline{z}_i}$$

Let $\Psi_V : V_{0|\boldsymbol{C}_{\lambda}^* \times Y} \simeq V_{\infty|\boldsymbol{C}_{\mu}^* \times Y}$ denote the gluing. An isomorphism $\Psi : \mathcal{V}_{0|\boldsymbol{C}_{\lambda}^* \times (X-D)} \longrightarrow \mathcal{V}_{\infty|\boldsymbol{C}_{\mu}^* \times (X^{\dagger}-D^{\dagger})}$ is given as follows:

$$\Psi := \Psi_V \circ \exp\Bigl(\sum_{i=1}^{\ell} \log|z_i|^2 \cdot q_0^* f_i \otimes \sqrt{-1} t_1^{(1)}\Bigr) \tag{15}$$

By construction, Ψ is holomorphic with respect to λ.

Lemma 2.14 $\Psi \circ \mathbb{D}^f_{\mathcal{V}_0} = \mathbb{D}^{\dagger f}_{\mathcal{V}_{\infty}} \circ \Psi$.

Proof We have the following expressions:

$$\mathbb{D}^f_{\mathcal{V}_0} = q_0^* \mathbb{D}^f_{V_0} + \sum_{i=1}^{\ell} q_0^* f_i \otimes (\sqrt{-1} t_1^{(1)}) \frac{dz_i}{z_i}, \quad \mathbb{D}^{\dagger f}_{\mathcal{V}_{\infty}} = q_{\infty}^* \mathbb{D}^{\dagger f}_{V_{\infty}} + \sum_{i=1}^{\ell} q_{\infty}^* f_i \otimes (-\sqrt{-1} t_1^{(1)}) \frac{d\overline{z}_i}{\overline{z}_i} \tag{16}$$

Because $\Psi_V \circ \mathbb{D}^f_{V_0} = \mathbb{D}^{\dagger f}_{V_{\infty}} \circ \Psi_V$, we have the following:

$$q_{\infty}^* \mathbb{D}^{\dagger f}_{V_{\infty}} \circ \Psi - \Psi \circ q_0^* \mathbb{D}^f_{V_0} = \Psi_V \circ q_0^* \mathbb{D}^f_{V_0} \Bigl(\exp\Bigl(\sum_{i=1}^{\ell} \log|z_i|^2 \cdot q_0^* f_i \otimes \sqrt{-1} t_1^{(1)}\Bigr)\Bigr)$$
$$= \Psi \circ \Bigl(\sum_{i=1}^{\ell} \Bigl(\frac{dz_i}{z_i} + \frac{d\overline{z}_i}{\overline{z}_i}\Bigr) \cdot q_0^* f_i \otimes \sqrt{-1} t_1^{(1)}\Bigr)$$

Then, the claim of the lemma follows. ∎

Let $\mathrm{TNIL}(V,\mathbb{D}_V^{\triangle},\boldsymbol{f})$ denote the variation of $\mathbb{P}^1$-holomorphic bundles on $(X-D)\times\mathbb{P}^1$ obtained as the gluing of $(\mathcal{V}_0,\mathbb{D}_{\mathcal{V}_0})$ and $(\mathcal{V}_\infty,\mathbb{D}^{\dagger}_{\mathcal{V}_\infty})$ via Ψ.

Assume that $(V,\mathbb{D}_V^{\triangle})$ is equipped with a $(-1)^w$-symmetric pairing $\mathcal{S}:(V,\mathbb{D}_V^{\triangle})\otimes\sigma^*(V,\mathbb{D}_V^{\triangle})\longrightarrow\mathbb{T}(-w)$ such that $\mathcal{S}(f_i\otimes\mathrm{id})+\mathcal{S}(\mathrm{id}\otimes\sigma^*f_i)=0$ for any i. Then, we have the induced $(-1)^w$-symmetric pairing

$$\mathrm{TNIL}(\mathcal{S}):\mathrm{TNIL}(V,\mathbb{D}_V^{\triangle},\boldsymbol{f})\otimes\sigma^*\,\mathrm{TNIL}(V,\mathbb{D}_V^{\triangle},\boldsymbol{f})\longrightarrow\mathbb{T}(-w).$$

It is obtained as the gluing of the pairings

$$\mathcal{S}_0:\mathcal{V}_0\otimes\sigma^*\mathcal{V}_\infty\longrightarrow\mathbb{T}(-w)_{|\mathcal{X}-\mathcal{D}},\quad \mathcal{S}_\infty:\mathcal{V}_\infty\otimes\sigma^*\mathcal{V}_0\longrightarrow\mathbb{T}(-w)_{|\mathcal{X}^\dagger-\mathcal{D}^\dagger},$$

which are the pull backs of $V_0\otimes\sigma^*V_\infty\longrightarrow\mathbb{T}(-w)_{|\boldsymbol{C}_\lambda}$ and $V_\infty\otimes\sigma^*V_0\longrightarrow\mathbb{T}(-w)_{|\boldsymbol{C}_\mu}$. (See Subsection 3.6.1 of [18].)

Enrichment Assume that $(V,\mathbb{D}_V^{\triangle})$ is enriched to a variation of integrable twistor structures $(V,\widetilde{\mathbb{D}}_V^{\triangle})$ such that f_j are $\widetilde{\mathbb{D}}_V^{\triangle}$-flat, which is obtained as the gluing of (V_0,∇_{V_0}) and $(V_\infty,\nabla_{V_\infty})$ via Ψ_V. Then $\mathrm{TNIL}(V,\mathbb{D}_V^{\triangle},\boldsymbol{f})$ is also enriched to the variation of integrable twistor structures $\mathrm{TNIL}(V,\widetilde{\mathbb{D}}_V^{\triangle},\boldsymbol{f})$, which can be checked by an obvious enrichment of the argument in the proof of Lemma 2.14. The TE-structure $\nabla_{\mathcal{V}_0}$ and the $\widetilde{T}E$-structure $\nabla_{\mathcal{V}_\infty}$ are given by essentially the same formula as (16):

$$\nabla_{\mathcal{V}_0}=q_0^*\nabla_{V_0}+\sum_{i=1}^{\ell}q_0^*f_i\otimes(\sqrt{-1}t_1^{(1)})\frac{dz_i}{z_i},\quad \nabla_{\mathcal{V}_\infty}=q_\infty^*\nabla_{V_\infty}+\sum_{i=1}^{\ell}q_\infty^*f_i\otimes(-\sqrt{-1}t_1^{(1)})\frac{d\overline{z}_i}{\overline{z}_i}$$

If we are given a pairing $\mathcal{S}$ of $(V,\widetilde{\mathbb{D}}_V^{\triangle})$ with weight w such that $\mathcal{S}\circ(f_j\otimes\mathrm{id})+\mathcal{S}\circ(\mathrm{id}\otimes\sigma^*f_j)=0$, we have a naturally induced pairing $\mathrm{TNIL}(\mathcal{S})$ of $\mathrm{TNIL}(V,\widetilde{\mathbb{D}}_V^{\triangle},\boldsymbol{f})$ with weight w. Assume that we are given a real structure κ of $(V,\widetilde{\mathbb{D}}_V^{\triangle},\mathcal{S})$ such that $\kappa\circ\gamma^*f_i=f_i\circ\kappa$. Because $\kappa_0\circ\gamma^*(f_i\otimes t_1^{(1)})=(f_i\otimes t_1^{(1)})\circ\kappa_0$, we obtain isomorphisms:

$$\kappa_0:\gamma^*(\mathcal{V}_\infty,\nabla_{\mathcal{V}_\infty})\simeq(\mathcal{V}_0,\nabla_{\mathcal{V}_0}),\quad \kappa_\infty:\gamma^*(\mathcal{V}_0,\nabla_{\mathcal{V}_0})\simeq(\mathcal{V}_\infty,\nabla_{\mathcal{V}_0})$$

The following diagram on $(X-D)\times\boldsymbol{C}_\lambda^*$ is commutative:

$$\begin{array}{ccc}\gamma^*\mathcal{V}_\infty & \xrightarrow{\kappa_0} & \mathcal{V}_0\\ {\scriptstyle\gamma^*\Psi^{-1}}\big\downarrow & & {\scriptstyle\Psi}\big\downarrow\\ \gamma^*\mathcal{V}_0 & \xrightarrow{\kappa_\infty} & \mathcal{V}_\infty\end{array}$$

To see it, we have only to remark

$$\begin{aligned}\Psi\circ\kappa&=\Psi_V\circ\exp\Big(\sum_{i=1}^{n}\log|z_i(P)|^2\cdot f_i\otimes\sqrt{-1}t_1^{(1)}\Big)\circ\kappa\\ &=\kappa\circ\gamma^*\Psi_V^{-1}\circ\exp\Big(-\sum_{i=1}^{n}\log|z_i(P)|^2\cdot\gamma^*\big(f_i\otimes\sqrt{-1}t_1^{(1)}\big)\Big)=\kappa\circ\gamma^*\Psi^{-1}\end{aligned}\qquad(17)$$

Hence, we obtain the isomorphism $\mathrm{TNIL}(\kappa):\gamma^*\,\mathrm{TNIL}(V,\widetilde{\mathbb{D}}_V^{\triangle},\boldsymbol{f})\simeq\mathrm{TNIL}(V,\widetilde{\mathbb{D}}_V^{\triangle},\boldsymbol{f})$. By construction, it is easy to check

$$\gamma^*\,\mathrm{TNIL}(\kappa)\circ\mathrm{TNIL}(\kappa)=\mathrm{id}\,.$$

It is also easy to check the compatibility condition, if the original $\mathcal{S}$ and κ are compatible. Therefore, we obtain a variation of twistor-TERP structures $\mathrm{TNIL}(V,\widetilde{\mathbb{D}}_V^{\triangle},\boldsymbol{f},\mathcal{S},\kappa,-w)$ on $X-D$ from a variation of twistor-TERP structures $(V,\widetilde{\mathbb{D}}_V^{\triangle},\mathcal{S},\kappa,-w)$ with $\boldsymbol{f}=(f_i)$ as above.

Definition 2.15 *Let $(V, \mathbb{D}_V^{\triangle}, \boldsymbol{f}, \mathcal{S})$ be as above. We set $X^*(R) := Y \times \{(z_1, \ldots, z_n) \,|\, 0 < |z_i| < R\}$ for $R > 0$.*

- *If there exists $R > 0$ such that $\mathrm{TNIL}(V, \mathbb{D}_V^{\triangle}, \boldsymbol{f}, \mathcal{S})_{|\mathbb{P}^1 \times X^*(R)}$ is pure and polarized, it is called a twistor nilpotent orbit of weight w.*
- *If moreover $(V, \mathbb{D}_V^{\triangle})$ is enriched to a variation of integrable twistor structures $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ such that f_j and $\mathcal{S}$ are $\widetilde{\mathbb{D}}_V^{\triangle}$-flat, $\mathrm{TNIL}(V, \widetilde{\mathbb{D}}_V^{\triangle}, \boldsymbol{f}, \mathcal{S})_{|\mathbb{P}^1 \times X^*(R)}$ is called an integrable twistor nilpotent orbit of weight w. (We often omit to distinguish "integrable" if there is no risk of confusion.)*
- *If moreover $(V, \widetilde{\mathbb{D}}_V^{\triangle}, \mathcal{S})$ is equipped with a real structure κ such that $\kappa \circ \gamma^* f_i = f_i \circ \kappa$, the variation $\mathrm{TNIL}(V, \widetilde{\mathbb{D}}_V^{\triangle}, \mathcal{S}, \kappa, -w)_{|\mathbb{P}^1 \times X^*(R)}$ is called a twistor-TERP nilpotent orbit.* ∎

Remark 2.16 *The notion of a twistor-TERP nilpotent orbit is different from "nilpotent orbit" defined by Hertling and Sevenheck. Their "nilpotent orbit" is called HS-orbit in this paper.* ∎

2.3 Convergence

2.3.1 Complement on convergence of pure polarized twistor structures

Let $(V^{(i)}, \mathcal{S}^{(i)})$ $(i = 0, 1)$ be polarized pure twistor structures with weight 0 of rank r. Let $h^{(i)}$ be the hermitian metrics of $V^{(i)}$ corresponding to $\mathcal{S}^{(i)}$, and let $d^{(i)}$ denote the associated flat unitary connections of $V^{(i)}$, which are equal to the natural connection given by holomorphic trivializations $V^{(i)} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$. Let $\overline{\partial}^{(i)}$ denote the $(0,1)$-part of $d^{(i)}$, which is the same as the holomorphic structures of $V^{(i)}$. We fix a hermitian metric g of $\Omega_{\mathbb{P}^1}^{1,0} \oplus \Omega_{\mathbb{P}^1}^{0,1}$. Let $\Phi : V^{(0)} \longrightarrow V^{(1)}$ be a C^∞-isomorphism such that the following holds for some $\epsilon > 0$:

(A1) $\bigl|\Phi^* \overline{\partial}^{(1)} - \overline{\partial}^{(0)}\bigr|_{h^{(0)}, g} \leq \epsilon$ as a C^∞-section of $\mathrm{End}(V^{(0)}) \otimes \Omega^{0,1}$.

(A2) $\bigl|\Phi^* \mathcal{S}^{(1)} - \mathcal{S}^{(0)}\bigr|_{h^{(0)}} \leq \epsilon$ as a C^∞-section of $\mathrm{Hom}\bigl(V^{(0)} \otimes \sigma^* V^{(0)}, \mathbb{T}(0)\bigr)$.

(A3) $\bigl|\overline{\partial}^{(0)}\bigl(\Phi^* \mathcal{S}^{(1)} - \mathcal{S}^{(0)}\bigr)\bigr|_{h^{(0)}, g} = \bigl|\overline{\partial}^{(0)} \Phi^* \mathcal{S}^{(1)}\bigr|_{h^{(0)}, g} \leq \epsilon$ as a C^∞-section of $\mathcal{H}om\bigl(V^{(0)} \otimes \sigma^* V^{(0)}, \mathbb{T}(0)\bigr) \otimes \Omega_{\mathbb{P}^1}^{0,1}$, where $\overline{\partial}^{(0)}$ denotes the induced holomorphic structure on $\mathcal{H}om\bigl(V^{(0)} \otimes \sigma^* V^{(0)}, \mathbb{T}(0)\bigr)$.

Lemma 2.17 *There exists a constant $C_0 > 0$, which is independent of ϵ, with the following property:*

- *If $B^{-1} \cdot \Phi^* h^{(1)} \leq h^{(0)} \leq B \cdot \Phi^* h^{(1)}$ for some $B > 1$, the following holds:*

$$\bigl|\Phi^* d^{(1)} - d^{(0)}\bigr|_{h^{(0)}, g} \leq C_0 \cdot B^2 \cdot \epsilon \tag{18}$$

Proof In the following argument, C_i denote positive constants independent of ϵ. Let $\partial_{h^{(i)}}^{(i)}$ denote the $(1,0)$-part of $d^{(i)}$, which are determined by $h^{(i)}$ and $\overline{\partial}^{(i)}$. To show (18), we have only to estimate $\bigl|\partial_{h^{(0)}}^{(0)} - \Phi^* \partial_{h^{(1)}}^{(1)}\bigr|_{h^{(0)}}$.

Let $e_1, \ldots, e_r$ be an orthogonal frame of $V^{(0)}$ with respect to $h^{(0)}$. Because $\Phi^* h^{(1)}(e_i, e_j) = \Phi^* \mathcal{S}^{(1)}\bigl(e_i \otimes \sigma^* e_j\bigr)$, we have the following estimate for any i, j:

$$\Bigl|\overline{\partial}\bigl(\Phi^* h^{(1)}(e_i, e_j)\bigr)\Bigr|_g = \Bigl|\overline{\partial}^{(0)}(\Phi^* \mathcal{S}^{(1)})(e_i \otimes \sigma^* e_j)\Bigr|_g \leq C_1 \cdot \epsilon$$

Hence, we obtain

$$\Bigl|\partial\bigl(\Phi^* h^{(1)}(e_i, e_j)\bigr)\Bigr|_g \leq C_1 \cdot \epsilon \tag{19}$$

Let $\partial_{h^{(1)}}^{(0)}$ denote the $(1,0)$-operator determined by $\Phi^* h^{(1)}$ and $\overline{\partial}^{(0)}$. From $B^{-1} \cdot \Phi^* h^{(1)} \leq h^{(0)} \leq B \cdot \Phi^* h^{(1)}$ and (19), we obtain

$$\bigl|\partial_{h^{(1)}}^{(0)} - \partial_{h^{(0)}}^{(0)}\bigr|_{h^{(0)}, g} \leq C_2 \cdot B \cdot \epsilon$$

By $\left|\Phi^*\overline{\partial}^{(1)}-\overline{\partial}^{(0)}\right|_{h^{(0)}}\leq C_3\cdot\epsilon$ and $B^{-1}\cdot\Phi^*h^{(1)}\leq h^{(0)}\leq B\cdot\Phi^*h^{(1)}$, we have

$$\left|\Phi^*\overline{\partial}^{(1)}-\overline{\partial}^{(0)}\right|_{\Phi^*h^{(1)}}\leq C_4\cdot B\cdot\epsilon$$

Hence, we obtain $\left|\Phi^*\partial^{(1)}_{h^{(1)}}-\partial^{(0)}_{h^{(1)}}\right|_{\Phi^*h^{(1)}}\leq C_4\cdot B\cdot\epsilon$, which implies

$$\left|\Phi^*\partial^{(1)}_{h^{(1)}}-\partial^{(0)}_{h^{(1)}}\right|_{h^{(0)}}\leq C_5\cdot B^2\cdot\epsilon$$

Thus, we obtain (18). ∎

Lemma 2.18 *There exist constants $\epsilon_0>0$, $C_{10}>0$ and $C_{11}>0$ such that the following holds if $\epsilon\leq\epsilon_0$:*

$$\left|\Phi^*h^{(1)}-h^{(0)}\right|_{h^{(0)}}\leq C_{10}\cdot\epsilon \tag{20}$$

$$\left|\Phi^*d^{(1)}-d^{(0)}\right|_{h^{(0)}}\leq C_{11}\cdot\epsilon \tag{21}$$

Proof According to the result in Subsection 2.8 of [19], if ϵ_0 is sufficiently small, (20) holds for some C_{10}. Then, we obtain (21) from Lemma 2.17. ∎

2.3.2 Approximation of pure polarized integrable twistor structures

Let $(V^{(i)},\nabla^{(i)},\mathcal{S}^{(i)})$ $(i=1,2)$ be integrable polarized pure twistor structures. Let $h^{(i)}$ be the hermitian metrics of $V^{(i)}$ corresponding to $\mathcal{S}^{(i)}$. We fix a hermitian metric $\widetilde{g}$ of $\Omega^{1,0}_{\mathbb{P}^1}\big(2\cdot 0+2\cdot\infty\big)\oplus\Omega^{0,1}_{\mathbb{P}^1}$. Let $\Phi:V^{(0)}\longrightarrow V^{(1)}$ be a C^∞-isomorphism such that the following holds for some $\epsilon>0$:

(B1) $\left|\Phi^*\nabla^{(1)}-\nabla^{(0)}\right|_{h^{(0)},\widetilde{g}}\leq\epsilon$ as a C^∞-section of $\mathrm{End}(V^{(0)})\otimes\Big(\Omega^{1,0}_{\mathbb{P}^1}\big(2\cdot 0+2\cdot\infty\big)\oplus\Omega^{0,1}_{\mathbb{P}^1}\Big)$. Note that it implies (A1) in Subsection 2.3.1.

(B2) Conditions (A2) and (A3) are satisfied.

Lemma 2.19 *There exists a constant $C_{20}>0$, which is independent of ϵ, with the following property:*

- *If $B^{-1}\cdot\Phi^*h^{(1)}\leq h^{(0)}\leq B\cdot\Phi^*h^{(1)}$ for some $B>1$, the following holds:*

$$\left|\Phi^*\mathcal{U}^{(1)}-\mathcal{U}^{(0)}\right|_{h^{(0)}}\leq C_{20}\cdot B^2\cdot\epsilon,\qquad \left|\Phi^*\mathcal{Q}^{(1)}-\mathcal{Q}^{(0)}\right|_{h^{(0)}}\leq C_{20}\cdot B^2\cdot\epsilon.$$

Proof In the following argument, C_i denote positive constants independent of ϵ. By Lemma 2.18, we have $\left|\Phi^*d^{(1)}-d^{(0)}\right|_{h^{(0)},\widetilde{g}}\leq C_{21}\cdot B^2\cdot\epsilon$. We obtain the following estimate:

$$\left|\Big(\lambda^{-1}\cdot(\Phi^*\mathcal{U}^{(1)}-\mathcal{U}^{(0)})-(\Phi^*\mathcal{Q}^{(1)}-\mathcal{Q}^{(0)})-\lambda\cdot(\Phi^*\mathcal{U}^{(1)\dagger}-\mathcal{U}^{(0)\dagger})\Big)\cdot d\lambda/\lambda\right|_{h^{(0)},\widetilde{g}}\leq C_{22}\cdot B^2\cdot\epsilon$$

Then, the claim of the lemma follows. ∎

Lemma 2.20 *There exist constants $\epsilon_0>0$ and C_{30}, such that the following holds for any $0<\epsilon\leq\epsilon_0$:*

$$\left|\Phi^*h^{(1)}-h^{(0)}\right|_{h^{(0)}}\leq C_{30}\cdot\epsilon,\quad \left|\Phi^*\mathcal{U}^{(1)}-\mathcal{U}^{(0)}\right|_{h^{(0)}}\leq C_{30}\cdot\epsilon,\qquad \left|\Phi^*\mathcal{Q}^{(1)}-\mathcal{Q}^{(0)}\right|_{h^{(0)}}\leq C_{30}\cdot\epsilon.$$

Proof It can be shown by the argument in the proof of Lemma 2.19. ∎

2.4 Variation of polarized mixed twistor structures and its enrichment

2.4.1 Definitions

Variation of polarized mixed twistor structures Let X be a complex manifold. Let $(V, \mathbb{D}^{\triangle})$ be a variation of $\mathbb{P}^1$-holomorphic vector bundles on $\mathbb{P}^1 \times X$ equipped with an increasing filtration W indexed by $\mathbb{Z}$ in the category of vector bundles, which is $\mathbb{P}^1$-holomorphic and $\mathbb{D}^{\triangle}$-flat. If each $\mathrm{Gr}^W_n(V)$ is a variation of pure twistor structure of weight n, $(V, W, \mathbb{D}^{\triangle})$ is called a variation of mixed twistor structures. Assume we are given the following data on $(V, W, \mathbb{D}^{\triangle})$, which are $\mathbb{P}^1$-holomorphic and $\mathbb{D}^{\triangle}$-flat:

- A tuple $\boldsymbol{f}$ of nilpotent morphisms $f_j : V \longrightarrow V \otimes \mathbb{T}(-1)$ $(j = 1, \ldots, n)$, which are mutually commutative.
- A $(-1)^w$-symmetric pairing $\mathcal{S} : V \otimes \sigma^* V \longrightarrow \mathbb{T}(-w)$.
- For each $P \in X$, the restriction $(V, W, \boldsymbol{f}, \mathcal{S})_{|\mathbb{P}^1 \times \{P\}}$ is a polarized mixed twistor structure of weight w in n-variables. (See Subsection 3.48 of [18].)

Then, such a tuple $(V, \mathbb{D}^{\triangle}, W, \boldsymbol{f}, \mathcal{S})$ is called a variation of polarized mixed twistor structures. Since W is determined by $\boldsymbol{f}$ as the weight filtration of $f(\underline{n}) := \sum_{j=1}^n f_j$ up to shift by w, we sometimes omit to denote W.

Enrichment If $\mathbb{D}^{\triangle}$ and the $\mathbb{P}^1$-holomorphic structure are extended to $T\widetilde{T}E$-structure $\widetilde{\mathbb{D}}^{\triangle}$ for which $\boldsymbol{f}$ and $\mathcal{S}$ are flat, $(V, \widetilde{\mathbb{D}}^{\triangle}, W, \boldsymbol{f}, \mathcal{S})$ is called a variation of polarized mixed integrable twistor structures of weight w in n-variables. Note that W is automatically $\widetilde{\mathbb{D}}^{\triangle}$-flat.

If moreover $(V, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$ is equipped with real structure κ such that $\kappa \circ \gamma^* f_j = f_j \circ \kappa$, then such a tuple $(V, \widetilde{\mathbb{D}}^{\triangle}, W, \boldsymbol{f}, \mathcal{S}, \kappa, -w)$ is called a variation of polarized mixed twistor-TERP structures in n-variables.

Remark 2.21 *The notion of polarized mixed twister-TERP structure is different from "mixed TERP structure" defined by Hertling and Sevenheck (Section* 9 *of* [8]*).* ∎

Split type Let $(V, W, \mathbb{D}^{\triangle})$ be a variation of mixed twistor structures. It is called of split type, if it is equipped with a grading $V = \bigoplus V_m$ such that (i) it is $\mathbb{P}^1$-holomorphic and $\mathbb{D}^{\triangle}$-flat, (ii) $W_m = \bigoplus_{p \leq m} V_p$. Each $(V_m, \mathbb{D}^{\triangle})$ is a variation of pure twistor structures of weight m.

A variation of polarized mixed twistor structures of weight w in n-variables $(V, W, \mathbb{D}^{\triangle}, \boldsymbol{f}, \mathcal{S})$ is called of split type, if the underlying variation of mixed twistor structures $(V, W, \mathbb{D}^{\triangle})$ is of split type with a grading $V = \bigoplus V_m$. By using $H^0\bigl(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)\bigr) = 0$ for any $m < 0$, we can show that the following:

- $f_j(V_p) \subset V_{p-2} \otimes \mathbb{T}(-1)$.
- The restriction of $\mathcal{S}$ to $V_p \otimes \sigma^* V_q$ is 0 unless $p + q = 2w$.

Similarly, a variation of polarized mixed integrable twistor structures of weight w in n-variables $(V, W, \widetilde{\mathbb{D}}^{\triangle}, \boldsymbol{f}, \mathcal{S})$ is called of split type, if the underlying variation of polarized mixed twistor structure is of split type with a $\widetilde{\mathbb{D}}^{\triangle}$-flat grading.

A polarized mixed twistor-TERP structures $(V, W, \nabla, \boldsymbol{f}, \mathcal{S}, \kappa, -w)$ in n-variables is called of split type, if the underlying variation of mixed integrable twistor structures is of split type with a grading $V = \bigoplus V_m$ such that $\kappa(\gamma^* V_m) = V_m$.

2.4.2 Reduction

Let $(V, W, \mathbb{D}^{\triangle}, \boldsymbol{f}, \mathcal{S})$ be a variation of polarized mixed twistor structures of weight w in n-variables. We obtain a variation of $\mathbb{P}^1$-holomorphic vector bundles $(V^{(0)}, \mathbb{D}^{(0)\triangle}) := \mathrm{Gr}^W(V, \mathbb{D}^{\triangle})$. It is naturally equipped with a grading $V^{(0)} = \bigoplus \mathrm{Gr}^W_m(V)$ and a filtration $W^{(0)}_m = \bigoplus_{p \leq m} \mathrm{Gr}^W_p(V)$. We have induced morphisms $f_j^{(0)} : \mathrm{Gr}^W_m(V) \longrightarrow \mathrm{Gr}^W_{m-2}(V) \otimes \mathbb{T}(-1)$, and hence $f_j^{(0)} : V^{(0)} \longrightarrow V^{(0)} \otimes \mathbb{T}(-1)$. We also obtain induced morphisms $\mathcal{S}^{(0)} : \mathrm{Gr}^W_{w-m}(V) \otimes \sigma^* \mathrm{Gr}^W_{w+m}(V) \longrightarrow \mathbb{T}(-w)$, and hence $\mathcal{S}^{(0)} : V^{(0)} \otimes \sigma^*(V^{(0)}) \longrightarrow \mathbb{T}(-w)$. It is known that $(V^{(0)}, W^{(0)}, \boldsymbol{f}^{(0)}, \mathcal{S}^{(0)})_{|\mathbb{P}^1 \times \{P\}}$ are polarized mixed twistor structures of split type with weight w in n-variables.

(See [18]. It can be shown directly and easily.) Hence, $(V^{(0)},W^{(0)},\mathbb{D}^{(0)\triangle},\boldsymbol{f}^{(0)},\mathcal{S}^{(0)})$ is a variation of polarized mixed twistor structures of split type with weight w in n-variables. It is denoted by $\mathrm{Gr}^W(V,W,\mathbb{D}^{\triangle},\boldsymbol{f},\mathcal{S})$.

If $(V,W,\mathbb{D}^{\triangle},\boldsymbol{f},\mathcal{S})$ is enriched to a variation of polarized mixed integrable twistor structures with weight w in n-variables, $\mathrm{Gr}^W(V,W,\mathbb{D}^{\triangle},\boldsymbol{f},\mathcal{S})$ is also integrable. If moreover the variation of polarized mixed integrable twistor structures is enriched to a variation of polarized mixed twistor-TERP structures, Gr^W is also enriched to a variation of polarized mixed twistor-TERP structures of split type.

2.4.3 Splittings

Preliminary Let $(V_i,W,\mathbb{D}_i^{\triangle})$ $(i=1,2)$ be variations of mixed twistor structures on $\mathbb{P}^1\times X$ with a morphism $F:(V_1,W,\mathbb{D}_1^{\triangle})\longrightarrow(V_2,W,\mathbb{D}_2^{\triangle})$. We set $(V_i^{(0)},\mathbb{D}_i^{(0)\triangle}):=\mathrm{Gr}^W(V_i,\mathbb{D}_i^{\triangle})$ on which we have the naturally induced filtrations $W^{(0)}$. We also obtain induced morphism $F^{(0)}:(V_1^{(0)},W^{(0)},\mathbb{D}_1^{(0)\triangle})\longrightarrow(V_2^{(0)},W^{(0)},\mathbb{D}_2^{(0)\triangle})$. The following lemma is standard.

Lemma 2.22 *The rank of $F_{|(\lambda,P)}$ is independent of $(\lambda,P)\in\mathbb{P}^1\times X$. The morphism F is strict with respect to the weight filtration. Hence, $\operatorname{Ker}F$ with the induced filtration $W(\operatorname{Ker}F)$ is a mixed twistor structure, and we have the isomorphism $\operatorname{Ker}F^{(0)}\simeq\mathrm{Gr}^W\bigl(\operatorname{Ker}F\bigr)$.*

Proof If X is a point, the claims are well known and easy to show. Namely, it is shown in Lemma 2.20 of [17] that (i) $\operatorname{Ker}(F)$ is a subbundle of V_1, (ii) F is strict with respect to the weight filtrations, i.e., $F(W_l(V_1))=F(V_1)\cap W_l(V_2)$, (iii) $\operatorname{Ker}(F)$ with the induced weight filtration is a mixed twistor structure. We obtain the isomorphism $\operatorname{Ker}F^{(0)}\simeq\mathrm{Gr}^W\bigl(\operatorname{Ker}F\bigr)$ from the strictness.

Let us consider the general case. By using the flatness, it is easy to show that $\operatorname{rank}F_{|(1,P)}$ and $\operatorname{rank}F^{(0)}_{|(1,P)}$ are independent of the choice of a point $P\in X$. Then, the claim of the lemma follows. ∎

Corollary 2.23 *Let $(V_i,W,\mathbb{D}_i^{\triangle})$ $(i=0,1,\ldots,m)$ be variations of mixed twistor structures with morphisms $F_i:(V_0,W,\mathbb{D}_0^{\triangle})\longrightarrow(V_i,W,\mathbb{D}_i^{\triangle})$ $(i=1,\ldots,m)$. Then, we have the following natural isomorphism of variations of mixed twistor structures:*

$$\mathrm{Gr}^W\Bigl(\bigcap_{i=1}^m\operatorname{Ker}F_i\Bigr)\simeq\bigcap_{i=1}^m\operatorname{Ker}F_i^{(0)}$$

Here, $F_i^{(0)}$ denote induced morphisms $(V_0^{(0)},W^{(0)},\mathbb{D}_0^{(0)\triangle})\longrightarrow(V_i^{(0)},W^{(0)},\mathbb{D}_i^{(0)\triangle})$. ∎

Local splitting Let $(V,W,\mathbb{D}^{\triangle})$ be a variation of mixed twistor structures. Let $\boldsymbol{N}=(N_j\,|\,j=1,\ldots,\ell)$ be a tuple of morphisms $N_j:(V,W,\mathbb{D}^{\triangle})\longrightarrow(V,W,\mathbb{D}^{\triangle})\otimes\mathbb{T}(-1)$ which are mutually commutative. Let $(V^{(0)},W^{(0)},\mathbb{D}^{(0)\triangle})$ be as above. Let $\boldsymbol{N}^{(0)}=(N_j^{(0)}\,|\,j=1,\ldots,\ell)$ be the induced commuting tuple of morphisms $N_j^{(0)}:(V^{(0)},W^{(0)},\mathbb{D}^{(0)\triangle})\longrightarrow(V^{(0)},W^{(0)},\mathbb{D}^{(0)\triangle})\otimes\mathbb{T}(-1)$.

We set $\overline{V}:=\mathrm{Hom}(V^{(0)},V)$, which is naturally equipped with the operator $\overline{\mathbb{D}}^{\triangle}$ and an induced filtration $\overline{W}$. Let $\overline{N}_j:(\overline{V},\overline{W},\overline{\mathbb{D}})\longrightarrow(\overline{V},\overline{W},\overline{\mathbb{D}})\otimes\mathbb{T}(-1)$ be the morphisms of mixed twistor structures given by $\overline{N}_j(f)=N_j\circ f-f\circ N_j^{(0)}$. Similarly, we set $\overline{V}^{(0)}:=\mathrm{Hom}(V^{(0)},V^{(0)})$ on which we have the naturally induced operator $\overline{\mathbb{D}}^{(0)\triangle}$, filtration $\overline{W}^{(0)}$ and morphisms of mixed twistor structures $\overline{N}_j^{(0)}:(\overline{V}^{(0)},\overline{W}^{(0)},\overline{\mathbb{D}}^{(0)})\longrightarrow(\overline{V}^{(0)},\overline{W}^{(0)},\overline{\mathbb{D}}^{(0)})\otimes\mathbb{T}(-1)$.

We have the natural isomorphism $\mathrm{Gr}^{\overline{W}}(\overline{V})\simeq\overline{V}^{(0)}$. The induced filtrations and the morphisms coincide. According to Corollary 2.23, we have the following isomorphism of variations of mixed twistor structures:

$$\mathrm{Gr}^{\overline{W}}\Bigl(\bigcap\operatorname{Ker}\overline{N}_j\Bigr)\simeq\bigcap\operatorname{Ker}\overline{N}_j^{(0)}.$$

Then, we obtain the following corollary.

Corollary 2.24 *Let (λ,P) be any point of $\boldsymbol{C}_\lambda\times X$, and let U be a small neighbourhood of (λ,P). There exists a C^∞-morphism $F:V^{(0)}_{|U}\longrightarrow V_{|U}$ with the following property:*

- *It preserves the weight filtration, and the induced morphism on* $\mathrm{Gr}^W(V^{(0)}_{|U}) \longrightarrow \mathrm{Gr}^W(V_{|U})$ *is the identity.*
- $F \circ N_j^{(0)} = N_j \circ F$ *for* $j = 1, \dots, \ell$. ∎

C^∞-splitting Let $(V, W, \mathbb{D}^\triangle, \boldsymbol{N})$ and $(V^{(0)}, W^{(0)}, \mathbb{D}^{(0)\triangle}, \boldsymbol{N}^{(0)})$ be as above.

Lemma 2.25 *There exists a* C^∞*-isomorphism* $\Phi : V^{(0)} \longrightarrow V$ *with the following property:*

- Φ *preserves the weight filtration* W*, and* $\mathrm{Gr}^W \Phi$ *is the identity* $\mathrm{Gr}^W(V^{(0)}) = \mathrm{Gr}^W(V)$.
- $\Phi \circ N_j^{(0)} = N_j \circ \Phi$ *for* $j = 1, \dots, \ell$.

Proof Let $U \subset \boldsymbol{C}_\lambda$ be a compact region with $U \cup \sigma(U) = \mathbb{P}^1$. We take a locally finite open covering $U \times X \subset \bigcup_{p \in I} \mathcal{U}_p$ such that we have C^∞-isomorphisms $\Phi_{\mathcal{U}_p} : V^{(0)}_{|\mathcal{U}_p} \longrightarrow V_{|\mathcal{U}_p}$ as in Corollary 2.24, i.e., $\Phi_{\mathcal{U}_p} \circ N_j^{(0)} = N_j \circ \Phi_{\mathcal{U}_p}$ for any j. Similarly, we take a locally finite open covering $\sigma(U) \times X^\dagger \subset \bigcup_{q \in J} \mathcal{U}_q^\dagger$ such that we have C^∞-isomorphisms $\Phi_{\mathcal{U}_q^\dagger} : V^{(0)}_{|\mathcal{U}_q^\dagger} \simeq V_{|\mathcal{U}_q^\dagger}$ as in Corollary 2.24. We take a partition of unity $\{\chi_{\mathcal{U}_p}, \chi_{\mathcal{U}_q^\dagger} \,|\, p \in I,\ q \in J\}$ subordinated to the covering $\{\mathcal{U}_p, \mathcal{U}_q^\dagger \,|\, p \in I,\ q \in J\}$ of $\mathbb{P}^1 \times X$. We obtain the C^∞-isomorphism

$$\Phi := \sum_{p \in I} \chi_{\mathcal{U}_p} \cdot \Phi_{\mathcal{U}_p} + \sum_{q \in J} \chi_{\mathcal{U}_q^\dagger} \cdot \Phi_{\mathcal{U}_q^\dagger} : V^{(0)} \longrightarrow V.$$

By construction, it has the desired property. ∎

3 Polarized mixed integrable twistor structure of split type

3.1 Basic examples in one variable

3.1.1 Rank two

Let us recall a basic example studied in Subsection 3.7.2 of [18] with a minor enrichment. We set $V^{[2]} := \mathcal{O}(0, -1) \oplus \mathcal{O}(1, 0)$. (See Subsection 2.1.3 for $\mathcal{O}(p, q)$.) It is naturally equipped with a meromorphic connection $\nabla^{[2]}$, and $(V^{[2]}, \nabla^{[2]})$ is an integrable twistor structure. We put

$$W_{-2}(V^{[2]}) := 0, \quad W_{-1}(V^{[2]}) = W_0(V^{[2]}) := \mathcal{O}(0, -1), \quad W_1(V^{[2]}) := V^{[2]}.$$

Let $F^{[2]} : V^{[2]} \longrightarrow V^{[2]} \otimes \mathbb{T}(-1)$ be given by

$$f_a^{(1,0)} \longmapsto f_a^{(0,-1)} \otimes t_a^{(-1)}, \quad (a = 0, 1, \infty), \quad f_a^{(0,-1)} \longmapsto 0.$$

A flat morphism $S^{[2]} : V^{[2]} \otimes \sigma^* V^{[2]} \longrightarrow \mathbb{T}(0)$ is given by the following correspondence:

$$S^{[2]}\big(f_1^{(1,0)} \otimes \sigma^* f_1^{(0,-1)}\big) = \sqrt{-1} t_1^{(0)}, \quad S^{[2]}\big(f_1^{(0,-1)} \otimes \sigma^* f_1^{(1,0)}\big) = -\sqrt{-1} t_1^{(0)},$$

$$S^{[2]}\big(f_1^{(1,0)} \otimes \sigma^* f_1^{(1,0)}\big) = 0, \quad S^{[2]}\big(f_1^{(0,-1)} \otimes \sigma^* f_1^{(0,-1)}\big) = 0.$$

Recall that $(V^{[2]}, W, F^{[2]}, S^{[2]})$ is a polarized mixed twistor structure of split type in one variable with weight 0 (Lemma 3.90 of [18]). Hence, $(V^{[2]}, W, \nabla^{[2]}, F^{[2]}, S^{[2]})$ is a polarized mixed integrable twistor structure of split type.

3.1.2 Twist

The bundle $V^{[2]}$ is obtained as the gluing of $V_0^{[2]} := V^{[2]}_{|\boldsymbol{C}_\lambda}$ and $V_\infty^{[2]} := V^{[2]}_{|\boldsymbol{C}_\mu}$. We would like to explain a twist of the gluing given in Subsection 3.7.2 of [18], related with the construction in Subsection 2.2.3. Let $N := F^{[2]} \otimes t_1^{(1)}$. Let $v \in V^{[2]}_{|\lambda}$ for $\lambda \neq 0, \infty$. The induced elements of $V^{[2]}_{0|\lambda}$ and $V^{[2]}_{\infty|\mu}$ are denoted by v and $v^\dagger$, respectively. The gluing for $V^{[2]}$ is given by $v = v^\dagger$. For $y \in \boldsymbol{C}$, a vector bundle $\widetilde{V}_y^{[2]}$ is given by the following twisted gluing:

$$\exp\bigl(\sqrt{-1}y \cdot N\bigr) \cdot v = v^\dagger$$

Since N is flat, we have the naturally induced flat connection $\nabla_y^{[2]}$ of $\widetilde{V}_y^{[2]}$. We also have the induced pairing $\widetilde{S}_y^{[2]}$ of $(\widetilde{V}_y^{[2]}, \nabla_y^{[2]})$ of weight 0.

For $y \neq 0$, we have a frame of $\widetilde{V}_y^{[2]}$ given as follows:

$$\widetilde{s}_1 := \sqrt{-1}\lambda \cdot f_0^{(1,0)} + \sqrt{-1}y \cdot f_0^{(0,-1)} = f_\infty^{(1,0)}$$

$$\widetilde{s}_2 := f_0^{(1,0)} = -\sqrt{-1}\mu \cdot f_\infty^{(1,0)} - \sqrt{-1}y \cdot f_\infty^{(0,-1)}$$

In particular, $(\widetilde{V}_y^{[2]}, \nabla_y^{[2]})$ is a pure integrable twistor structure of weight 0 for any $y \neq 0$. If y is a positive real number, $\widetilde{S}_y^{[2]}$ gives a polarization of $(\widetilde{V}_y^{[2]}, \nabla_y^{[2]})$ (Lemma 3.91 of [18]). Actually, $\widetilde{s}_i$ $(i = 1, 2)$ give an orthogonal frame:

$$\widetilde{S}_y^{[2]}(\widetilde{s}_i, \sigma^*\widetilde{s}_i) = y \quad (i = 1, 2), \qquad \widetilde{S}_y^{[2]}(\widetilde{s}_1, \sigma^*\widetilde{s}_2) = 0.$$

Note that $\nabla_y^{[2]}$ is logarithmic with respect to the lattice $\widetilde{V}_y^{[2]}$. For any $y \neq 0$, we have the decomposition

$$\nabla_y^{[2]} = d_y^{[2]} - \mathcal{Q}_y^{[2]} \frac{d\lambda}{\lambda}$$

Here, $d_y^{[2]}$ is a natural flat connection of $V_y^{[2]} \simeq \mathcal{O}_{\mathbb{P}^1}(0)^{\oplus 2}$. Let us calculate $\mathcal{Q}^{[2]}$. By easy calculations,

$$\nabla_y^{[2]}\widetilde{s}_1 = 0, \quad \nabla_y^{[2]}\widetilde{s}_2 = \widetilde{s}_2 \cdot \left(-\frac{d\lambda}{\lambda}\right).$$

Hence, $\mathcal{Q}^{[2]}$ is expressed by the following matrix with respect to the frame $\widetilde{s}_1, \widetilde{s}_2$:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

In particular, the eigenvalues are independent of y.

Remark 3.1 *For our application, we essentially need only the case in which y is a positive real number. Recall that we have considered a twisted isomorphism* (15)*. We will use the above consideration by setting* $y = -\sum_{i=1}^{\ell} \log |z_i|^2$. ∎

3.1.3 Rank ℓ

For any positive integer ℓ, we set $(V^{[\ell]}, \nabla^{[\ell]}) := \mathrm{Sym}^{\ell-1}(V^{[2]}, \nabla^{[2]})$, equipped with a morphism $F^{[\ell]} : V^{[\ell]} \longrightarrow V^{[\ell]} \otimes \mathbb{T}(-1)$ and a pairing $S^{[\ell]} : V^{[\ell]} \otimes \sigma^* V^{[\ell]} \longrightarrow \mathbb{T}(0)$. For any $y \in \boldsymbol{C}$, we obtain an integrable twistor structure $(\widetilde{V}_y^{[\ell]}, \nabla_y^{[\ell]}, \widetilde{S}_y^{[\ell]})$ with a pairing of weight 0, by the procedure in Subsection 3.1.2. It is also obtained as the $(\ell-1)$-th symmetric product of $(\widetilde{V}_y^{[2]}, \nabla_y^{[2]}, \widetilde{S}_y^{[2]})$. Hence, $(\widetilde{V}_y^{[\ell]}, \nabla_y^{[\ell]})$ is pure with weight 0 for each $y \neq 0$, and $\widetilde{S}_y^{[\ell]}$ gives a polarization for each $y > 0$. We have the decomposition

$$\nabla_y^{[\ell]} = d_y^{[\ell]} - \mathcal{Q}^{[\ell]} \frac{d\lambda}{\lambda}$$

Let $y \neq 0$. A frame of $\widetilde{V}_y^{[\ell]}$ is given by symmetric products $\widetilde{s}_p^{[\ell]} := \widetilde{s}_1^{\ell-1-p} \cdot \widetilde{s}_2^{p}$ $(p = 0, 1, \ldots, \ell-1)$, for which $\mathcal{Q}^{[\ell]}$ is expressed by the diagonal matrix whose p-th entry is p $(p = 0, 1, \ldots, \ell-1)$. In particular, the eigenvalues are independent of y.

3.2 Twistor nilpotent orbits of split type and their new supersymmetric indices

3.2.1 One variable case

Let Y be a complex manifold. Let $(V, W, \mathbb{D}^{\triangle}, N, S)$ be a variation of polarized mixed twistor structures of split type with weight 0 in one variable on $\mathbb{P}^1 \times Y$. The following lemma is essentially the same as Corollary 3.97 of [18].

Proposition 3.2 *There exist variations of polarized pure twistor structures $(U_\ell, \mathbb{D}^{\triangle}_\ell, S_\ell)$ of weight 0 on $\mathbb{P}^1 \times Y$ for $\ell \geq 1$, such that (i) $(V, \mathbb{D}^{\triangle}) \simeq \bigoplus_{\ell \geq 1} (U_\ell, \mathbb{D}^{\triangle}_\ell) \otimes V^{[\ell]}$, (ii) $N = \bigoplus \mathrm{id}_{U_\ell} \otimes F^{[\ell]}$ and $S = \bigoplus S_\ell \otimes S^{[\ell]}$ under the isomorphism.*

If $(V, W, \mathbb{D}^{\triangle}, N, S)$ is enriched to be integrable, $(U_\ell, \mathbb{D}^{\triangle}, S_\ell)$ are also integrable.

Proof We have the grading $V = \bigoplus_{j \in \mathbb{Z}} V_j$. For each $j \geq 0$, we set $PV_j := \mathrm{Ker}\Big(N^{j+1} : V_j \longrightarrow V_{-j-2} \otimes \mathbb{T}(j+1)\Big)$. It is a variation of pure twistor structure of weight j, and equipped with the induced polarization S_j. For $\ell \geq 1$, we set

$$U_\ell := PV_{\ell-1} \otimes \mathcal{O}(0, -\ell+1).$$

which are naturally variations of polarized pure twistor structures. Then, it is easy to observe that V has the desired decomposition. The integrable case is also easy. ∎

Let $q : Y \times \boldsymbol{C}^* \longrightarrow Y$ denote the projection. We obtain the variations of polarized pure twistor structures on $\mathbb{P}^1 \times (Y \times \boldsymbol{C}^*)$ obtained as the pull back of $(U_\ell, \mathbb{D}^{\triangle}_\ell, S_\ell)$, denoted by $q^*(U_\ell, \mathbb{D}^{\triangle}_\ell, S_\ell)$. Recall the construction in Subsection 2.2.3. We obtain the following isomorphism from Proposition 3.2:

$$\mathrm{TNIL}(V, \mathbb{D}^{\triangle}, N, S) \simeq \bigoplus_{\ell} q^*(U_\ell, \mathbb{D}^{\triangle}_\ell, S_\ell) \otimes \mathrm{TNIL}(V^{[\ell]}, \nabla^{[\ell]}, F^{[\ell]}, S^{[\ell]}) \tag{22}$$

By using the result in Subsection 3.1, we can conclude the following:

Proposition 3.3 *We set $X_+ := Y \times \big\{z \in \boldsymbol{C} \,\big|\, 0 < |z| < 1\big\}$ and $X_- := Y \times \big\{z \in \boldsymbol{C} \,\big|\, |z| > 1\big\}$. Then, $\mathrm{TNIL}(V, \mathbb{D}^{\triangle}, N)$ is a variation of pure integrable twistor structures on $\mathbb{P}^1 \times (X_+ \cup X_-)$, and the restriction $\mathrm{TNIL}(V, \mathbb{D}^{\triangle}, N, S)_{|\mathbb{P}^1 \times X_+}$ is a twistor nilpotent orbit.* ∎

Assume that $(V, \mathbb{D}^{\triangle})$ is enriched to integrable $(V, \widetilde{\mathbb{D}}^{\triangle})$ such that S and N are $\widetilde{\mathbb{D}}^{\triangle}$-flat. Let $\mathcal{Q}$ and $\mathcal{Q}^{[\ell]}$ be the new supersymmetric indices of $\mathrm{TNIL}(V, \widetilde{\mathbb{D}}^{\triangle}, N)$ and $\mathrm{TNIL}(V^{[\ell]}, \nabla^{[\ell]}, F^{[\ell]})$, respectively. We also have the new supersymmetric index $\mathcal{Q}_\ell$ of $(U_\ell, \widetilde{\mathbb{D}}^{\triangle}_\ell)$. By construction, we have the following equality, under the isomorphism (22):

$$\mathcal{Q} = \bigoplus_{\ell \geq 1} \big(\mathcal{Q}_\ell \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{Q}^{[\ell]}\big)$$

The eigenvalues of $\mathcal{Q}$ are easily calculable, once we know those of $\mathcal{Q}_\ell$. In particular, we obtain the following.

Corollary 3.4 *The eigenvalues of $\mathcal{Q}_{|q^{-1}(y)}$ are constant for any $y \in Y$, where $q : (X_+ \cup X_-) \longrightarrow Y$ denotes the projection.* ∎

3.2.2 Several variable case

Let $(V, W, \mathbb{D}^{\triangle}_V, \boldsymbol{N}, S)$ be a variation of polarized mixed twistor structures of split type with weight 0 in n-variables on $\mathbb{P}^1 \times Y$. We have the associated variation of twistor structures $\mathrm{TNIL}(V, \mathbb{D}^{\triangle}_V, \boldsymbol{N}, S)$ with a pairing of weight 0 on $(\boldsymbol{C}^*)^n \times Y$. We set $X^* = \big\{(z_1, \ldots, z_n) \in \boldsymbol{C}^n \,\big|\, 0 < |z_i| < 1\big\} \times Y$.

Proposition 3.5 $\mathrm{TNIL}(V, \mathbb{D}^{\triangle}_V, \boldsymbol{N}, S)_{|\mathbb{P}^1 \times X^*}$ *is a twistor nilpotent orbit.*

Proof For any $\boldsymbol{a} \in \boldsymbol{R}^n_{>0}$, we set $N(\boldsymbol{a}) := \sum_{i=1}^n a_i \cdot N_i$. We obtain a variation of mixed polarized twistor structures $(V, W, \mathbb{D}^{\triangle}, N(\boldsymbol{a}), S)$ of split type with weight 0 in one variable on $\mathbb{P}^1 \times Y$. Applying the result in Subsection 3.1.3 to $(V, W, \mathbb{D}^{\triangle}_V, N(\boldsymbol{a}), S)$, we obtain the desired property of $(V, W, \mathbb{D}^{\triangle}_V, \boldsymbol{N}, S)$. ∎

Definition 3.6 *An (integrable) twistor nilpotent orbit is called of split type, if it is associated to (integrable) polarized mixed twistor structures of split type.* ∎

If $(V, W, \mathbb{D}_V^\triangle, \boldsymbol{N}, S)$ is enriched to integrable $(V, W, \widetilde{\mathbb{D}}_V^\triangle, \boldsymbol{N}, S)$, the associated twistor nilpotent orbit is also enriched to integrable $\mathrm{TNIL}(V, \widetilde{\mathbb{D}}_V^\triangle, \boldsymbol{N}, S)$. Let us consider its new supersymmetric index $\mathcal{Q}$. For any $\boldsymbol{a} \in \boldsymbol{R}_{>0}^n$, we set $N(\boldsymbol{a}) := \sum_{i=1}^n a_i \cdot N_i$. According to Proposition 3.2, there exist variations of polarized pure integrable twistor structures $(U_{\boldsymbol{a},\ell}, \widetilde{\mathbb{D}}_{\boldsymbol{a},\ell}^\triangle)$ for $\ell \geq 1$ such that

$$\big(V, \widetilde{\mathbb{D}}_V^\triangle, N(\boldsymbol{a})\big) \simeq \bigoplus_{\ell \geq 1} (U_{\boldsymbol{a},\ell}, \widetilde{\mathbb{D}}_{\boldsymbol{a},\ell}^\triangle) \otimes \big(V^{[\ell]}, \nabla^{[\ell]}, F^{[\ell]}\big).$$

Lemma 3.7 *For any $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}_{>0}^n$, we have an isomorphism $\big(U_{\boldsymbol{a},\ell}, \widetilde{\mathbb{D}}_{\boldsymbol{a},\ell}^\triangle\big) \simeq \big(U_{\boldsymbol{b},\ell}, \widetilde{\mathbb{D}}_{\boldsymbol{b},\ell}^\triangle\big)$.*

Proof Let $V = \bigoplus V_j$ be the splitting. For any $\boldsymbol{a} \in \boldsymbol{R}_{>0}^n$ and $j \geq 0$, we set

$$(PV_{j,\boldsymbol{a}}, \widetilde{\mathbb{D}}^\triangle) := \mathrm{Ker}\Big(N(\boldsymbol{a})^{j+1} : (V_j, \widetilde{\mathbb{D}}^\triangle) \longrightarrow (V_{-j-2}, \widetilde{\mathbb{D}}^\triangle) \otimes \mathbb{T}(-j-1)\Big)$$

We have only to show that $(PV_{j,\boldsymbol{a}}, \widetilde{\mathbb{D}}^\triangle)$ and $(PV_{j,\boldsymbol{b}}, \widetilde{\mathbb{D}}^\triangle)$ are isomorphic, if $\boldsymbol{b}$ is sufficiently close to $\boldsymbol{a}$.

We set $(Y_{j,\boldsymbol{a}}, \widetilde{\mathbb{D}}^\triangle) = \mathrm{Im}\Big(N(\boldsymbol{a}) : (V_{j+2}, \widetilde{\mathbb{D}}^\triangle) \otimes \mathbb{T}(1) \longrightarrow (V_j, \widetilde{\mathbb{D}}^\triangle)\Big)$. Then, we obtain the flat splittings $(V_j, \widetilde{\mathbb{D}}^\triangle) = (PV_{j,\boldsymbol{a}}, \widetilde{\mathbb{D}}^\triangle) \oplus (Y_{j,\boldsymbol{a}}, \widetilde{\mathbb{D}}^\triangle)$. If $\boldsymbol{b}$ is sufficiently close to $\boldsymbol{a}$, flat isomorphisms $PV_{j,\boldsymbol{a}} \longrightarrow PV_{j,\boldsymbol{b}}$ are induced by inclusions and projections. Thus, we are done. ∎

By Lemma 3.7 and the result in Subsection 3.2.1, the eigenvalues of $\mathcal{Q}$ are easily calculable once we know the new supersymmetric indices of $(U_{\boldsymbol{a},\ell}, \widetilde{\mathbb{D}}_{\boldsymbol{a},\ell}^\triangle)$ for $\boldsymbol{a} \in \boldsymbol{R}_{>0}^\ell$ and $\ell \geq 1$. In particular, we obtain the following.

Corollary 3.8 *The eigenvalues of $\mathcal{Q}_{|q^{-1}(y)}$ are constant for any $y \in Y$, where $q : X^* \longrightarrow Y$ denotes the natural projection.* ∎

4 Integrable twistor nilpotent orbit

4.1 Statements

4.1.1 Twistor nilpotent orbits and polarized mixed twistor structures

Let Y be a complex manifold. Let $(V, \mathbb{D}_V^\triangle)$ be a variation of $\mathbb{P}^1$-holomorphic vector bundles on $\mathbb{P}^1 \times Y$ equipped with the following $\mathbb{P}^1$-holomorphic $\mathbb{D}_V^\triangle$-flat data:

- A $(-1)^w$-symmetric pairing $S : V \otimes \sigma^* V \longrightarrow \mathbb{T}(-w)$.
- A tuple $\boldsymbol{N}$ of nilpotent morphisms $N_j : V \longrightarrow V \otimes \mathbb{T}(-1)$ $(j = 1, \ldots, n)$, which are mutually commutative.
- $S(N_j \otimes \mathrm{id}) + S(\mathrm{id} \otimes \sigma^* N_j) = 0$ for $j = 1, \ldots, n$.

For simplicity of the statement, we assume the following:

- Y is contained in another complex manifold Y' as a relatively compact subset, and $(V, \mathbb{D}_V^\triangle, S, \boldsymbol{N})$ is extended on Y'.

We set $X^*(R) := \big\{(z_1, \ldots, z_n) \,\big|\, |z_i| < R\big\} \times Y$.

Theorem 4.1 $(V, \mathbb{D}_V^\triangle, \boldsymbol{N}, S)$ *is a variation of polarized mixed twistor structures with weight w in n-variables, if and only if* $\mathrm{TNIL}(V, \mathbb{D}_V^\triangle, \boldsymbol{N}, S)_{|\mathbb{P}^1 \times X^*(R)}$ *is a twistor nilpotent orbit with weight w for some $R > 0$.*

Note that the "if" part follows from Theorem 12.22 of [18]. The "only if" part immediately follows from Proposition 4.4 below and a result in Subsection 2.8 of [19]. (We apply Proposition 4.4 to each point of Y'.) The one dimensional case was proved in Proposition 3.105 of [18]. Such an equivalence for Hodge structure was established by Cattani-Kaplan-Schmid and Kashiwara-Kawai.

Corollary 4.2 *Let $(V, \mathbb{D}_V^{\triangle}, \boldsymbol{N}, S)$ be as above.*

- *Assume that $(V, \mathbb{D}_V^{\triangle})$ is enriched to integrable $(V, \widetilde{\mathbb{D}}_V^{\triangle})$ such that $\boldsymbol{N}$ and S are flat with respect to $\widetilde{\mathbb{D}}_V^{\triangle}$. Then, $(V, \widetilde{\mathbb{D}}_V^{\triangle}, \boldsymbol{N}, S)$ is a variation of polarized mixed integrable twistor structures with weight w in n variables, if and only if $\mathrm{TNIL}(V, \widetilde{\mathbb{D}}_V^{\triangle}, \boldsymbol{N}, S)_{|\mathbb{P}^1 \times X^*(R)}$ is integrable twistor nilpotent orbit for some $R > 0$.*
- *Assume moreover that $(V, \widetilde{\mathbb{D}}_V^{\triangle}, S)$ is equipped with a real structure κ which is compatible with $\boldsymbol{N}$. Then, $(V, \widetilde{\mathbb{D}}_V^{\triangle}, \boldsymbol{N}, S, \kappa, -w)$ is a variation of polarized mixed twistor-TERP structures if and only if the associated $\mathrm{TNIL}(V, \widetilde{\mathbb{D}}_V^{\triangle}, \boldsymbol{N}, \mathcal{S}, \kappa, -w)$ is a twistor-TERP nilpotent orbit on $X^*(R)$ for some $R > 0$.* ∎

Remark 4.3 *As the one variable case of Corollary* 4.2*, we obtain the correspondence between twistor-TERP nilpotent orbits and polarized mixed twistor-TERP structures. This is different from the correspondence between mixed TERP structure and HS-orbit in the regular singular case established by Hertling and Sevenheck (*[7] and [8]*).* ∎

4.1.2 Construction of an approximating C^∞-isomorphism

Let $(V, W, \mathbb{D}^{\triangle}, \boldsymbol{N}, S)$ be a variation of polarized mixed twistor structures of weight 0 in n-variables on $\mathbb{P}^1 \times Y$. As explained in Subsection 2.4.2, we obtain a variation of polarized mixed twistor structure of split type $\big(V^{(0)}, W^{(0)}, \mathbb{D}^{(0)\triangle}, \boldsymbol{N}^{(0)}, S^{(0)}\big)$ by taking Gr with respect to the weight filtration. We obtain the families of $\mathbb{P}^1$-holomorphic vector bundles $(\mathcal{V}^{\triangle}, \mathbb{D}^{\triangle}) := \mathrm{TNIL}(V, \mathbb{D}^{\triangle}, \boldsymbol{N})$ and $(\mathcal{V}^{(0)\triangle}, \mathbb{D}^{(0)\triangle}) := \mathrm{TNIL}(V^{(0)}, \mathbb{D}^{(0)\triangle}, \boldsymbol{N}^{(0)})$ on $(\boldsymbol{C}^*)^n \times Y$. They are equipped with the induced pairings $\mathcal{S}$ and $\mathcal{S}^{(0)}$. By the result in Subsection 3.2.2, $(\mathcal{V}^{(0)}, \mathbb{D}^{(0)\triangle}, \mathcal{S}^{(0)})$ is a variation of polarized pure twistor structure on $\mathbb{P}^1 \times X^*(1)$. Let $h^{(0)}$ be the corresponding pluri-harmonic metric.

We take a C^∞-isomorphism $\Phi : V^{(0)} \longrightarrow V$ as in Lemma 2.25, i.e., it satisfies (i) $\Phi \circ N_i^{(0)} = N_i \circ \Phi$ for $i = 1, \dots, n$, (ii) Φ preserves the weight filtration W, and $\mathrm{Gr}^W \Phi$ is the identity of $\mathrm{Gr}^W(V^{(0)}) = \mathrm{Gr}^W(V)$. By the property (i) for Φ and the construction of $\mathcal{V}^{\triangle}$ and $\mathcal{V}^{(0)\triangle}$, we obtain a naturally induced C^∞-isomorphism $\widetilde{\Phi} : \mathcal{V}^{(0)\triangle} \longrightarrow \mathcal{V}^{\triangle}$.

Let $\overline{\partial}_{\mathcal{V}^{\triangle}, \mathbb{P}^1}$ denote the $\mathbb{P}^1$-holomorphic structure of $\mathcal{V}^{\triangle}$. We use the symbol $\overline{\partial}_{\mathcal{V}^{(0)\triangle}, \mathbb{P}^1}$ in a similar meaning. We obtain the following C^∞-section of $\mathrm{End}(\mathcal{V}^{(0)\triangle}) \otimes \Omega^{0,1}_{\mathbb{P}^1}$ on $\mathbb{P}^1 \times X^*(1)$:

$$F := \overline{\partial}_{\mathcal{V}^{(0)\triangle}, \mathbb{P}^1} - \widetilde{\Phi}^*\big(\overline{\partial}_{\mathcal{V}^{\triangle}, \mathbb{P}^1}\big)$$

We also obtain the following C^∞-morphism:

$$G := \mathcal{S}^{(0)} - \widetilde{\Phi}^* \mathcal{S} : \mathcal{V}^{(0)\triangle} \otimes \sigma^* \mathcal{V}^{(0)\triangle} \longrightarrow \mathbb{T}(0)$$

We fix a Kähler metric g of $\mathbb{P}^1$. Although the following proposition looks rather auxiliary, it means that $(\mathcal{V}^{(0)\triangle}, \mathbb{D}^{(0)\triangle}, \mathcal{S}^{(0)})$ approximates $(\mathcal{V}^{\triangle}, \mathbb{D}^{\triangle}, \mathcal{S})$ via $\widetilde{\Phi}$ around $\mathbb{P}^1 \times \{0\} \times Y$. We will prove it in Subsection 4.2.1.

Proposition 4.4 *For any $P \in Y$, there exist a positive constant $R_P > 0$ and a neighbourhood U_P of P in Y such that the following estimate holds $\mathbb{P}^1 \times \big\{(z_1, \dots, z_n) \,\big|\, 0 < |z_j| < R_P\big\} \times U_P$:*

$$\big|F\big|_{h^{(0)}, g} = O\Big(\sum_{j=1}^n \big(-\log|z_j|\big)^{-1/2}\Big)$$

$$\big|G\big|_{h^{(0)}} = O\Big(\sum_{j=1}^n \big(-\log|z_j|\big)^{-1/2}\Big), \quad \big|\overline{\partial}_{\mathcal{V}^{(0)\triangle}, \mathbb{P}^1} G\big|_{h^{(0)}, g} = O\Big(\sum_{j=1}^n \big(-\log|z_j|\big)^{-1/2}\Big)$$

4.1.3 Estimate of the new supersymmetric index

Assume that $(V, \mathbb{D}_V^{\triangle}, \boldsymbol{N}, S)$ is enriched to integrable $(V, \widetilde{\mathbb{D}}_V^{\triangle}, \boldsymbol{N}, S)$. By taking Gr with respect to the weight filtration, we obtain a polarized mixed integrable twistor structure of split type $\bigl(V^{(0)}, W^{(0)}, \mathbb{D}_V^{(0)\triangle}, \boldsymbol{N}^{(0)}, S^{(0)}\bigr)$. Let $(\mathcal{V}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S}) = \mathrm{TNIL}(V, \widetilde{\mathbb{D}}_V^{\triangle}, \boldsymbol{N}, S)_{|\mathbb{P}^1 \times X^*(R)}$ and $(\mathcal{V}^{(0)}, \widetilde{\mathbb{D}}^{(0)\triangle}, \mathcal{S}^{(0)}) = \mathrm{TNIL}(V^{(0)}, \widetilde{\mathbb{D}}_V^{(0)}, \boldsymbol{N}^{(0)}, S^{(0)})_{|\mathbb{P}^1 \times X^*(R)}$ be the associated nilpotent orbits (Corollary 4.2). Let $\mathcal{Q}$ and h (resp. $h^{(0)}$ and $\mathcal{Q}^{(0)}$) denote the new supersymmetric index and the pluri-harmonic metric of $(\mathcal{V}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$ (resp. $(\mathcal{V}^{(0)}, \widetilde{\mathbb{D}}^{(0)\triangle}, \mathcal{S}^{(0)})$). We will prove the following proposition in Subsection 4.2.2.

Proposition 4.5 *Let $\widetilde{\Phi} : \mathcal{V}^{(0)} \longrightarrow \mathcal{V}$ be a C^∞-isomorphism constructed in Subsection 4.1.2. For any $P \in Y$, there exist $R > 0$ and a neighbourhood U_P of P in Y such that the following estimate holds with respect to $h^{(0)}$ on $\mathbb{P}^1 \times \bigl\{(z_1, \ldots, z_n) \bigm| 0 < |z_j| < R\bigr\} \times U_P$:*

$$\widetilde{\Phi}^* h - h^{(0)} = O\Bigl(\sum_{i=1}^{n} \bigl(-\log|z_i|\bigr)^{-1/2}\Bigr), \quad \widetilde{\Phi}^* \mathcal{Q} - \mathcal{Q}^{(0)} = O\Bigl(\sum_{i=1}^{n} \bigl(-\log|z_i|\bigr)^{-1/2}\Bigr)$$

In particular, the eigenvalues of $\mathcal{Q}_{|q^{-1}(y)}$ are constant up to $O\Bigl(\sum_{i=1}^{n} \bigl(-\log|z_i|\bigr)^{-\delta}\Bigr)$ for some $\delta > 0$, where $q : X^*(1) \longrightarrow Y$ denotes the natural projection.

4.2 Proof

4.2.1 Proof of Proposition 4.4

Let $C > 0$. Fix $P \in Y$. In the following, we will shrink Y instead of taking a neighbourhood U_P, for simplicity of description. We set

$$Z(C) := \Bigl\{(z_1, \ldots, z_n) \in (\boldsymbol{C}^*)^n \Bigm| \bigl|z_i\bigr|^C \leq \bigl|z_{i+1}\bigr| < 1, \ \ i = 1, \ldots, n-1\Bigr\} \times Y.$$

It is easy to observe that we have only to estimate F, G and $\overline{\partial}_{\mathcal{V}^{(0)\triangle}, \mathbb{P}^1} G$ on $\mathbb{P}^1 \times Z(C)$. For $m = 1, \ldots, n$, we put $N^{(0)}(\underline{m}) := \sum_{i \leq m} N_i^{(0)}$. Let $W(\underline{m})$ denote the weight filtration of $V^{(0)}$ induced by $N^{(0)}(\underline{m})$. Recall that the filtrations $W(\underline{1}), W(\underline{2}), \ldots, W(\underline{n})$ are compatible (Lemma 3.116 of [18]).

We take a compact region $\mathcal{U} \subset \boldsymbol{C}_\lambda$ such that the union of the interior parts of $\mathcal{U}$ and $\sigma(\mathcal{U})$ cover $\mathbb{P}^1$. Let $\boldsymbol{v} = (v_i)$ be a frame of $V^{(0)}_{|\mathcal{U} \times Y}$ compatible with $W(\underline{1}), W(\underline{2}), \ldots, W(\underline{n})$. For $m = 1, \ldots, n$, we set

$$k_m(v_i) := \frac{1}{2} \deg^{W(\underline{m})}(v_i).$$

We formally put $k_0(v_i) = 0$.

Lemma 4.6 *Let A be determined by $\bigl(\Phi^{-1} \circ \overline{\partial}\Phi\bigr)\boldsymbol{v} = \boldsymbol{v} \cdot A$. Then, $A_{i,j} = 0$ unless $k_m(v_i) \leq k_m(v_j)$ $(m = 1, \ldots, n-1)$ and $k_n(v_i) < k_n(v_j)$.*

Proof Because of our choice of Φ, it preserves the filtrations $W(\underline{m})$ $(m = 1, \ldots, n)$, and $\mathrm{Gr}^{W(\underline{n})} \Phi$ is holomorphic. Then, the claim of Lemma 4.6 immediately follows. ∎

Let $q_0 : \boldsymbol{C}_\lambda \times X^*(1) \longrightarrow \boldsymbol{C}_\lambda \times Y$ be the projection. Recall $\mathcal{V}^{(0)\triangle}_{|\boldsymbol{C}_\lambda \times X^*(1)} = q_0^* V_0$. Let $\widetilde{v}_i$ be the section of $\mathcal{V}^{(0)\triangle}_{|\mathcal{U} \times X^*(1)}$ induced by v_i, and we put

$$v_j' := \widetilde{v}_j \cdot \prod_{m=1}^{n} \bigl(-\log|z_m|\bigr)^{-k_m(v_j) + k_{m-1}(v_j)} = \widetilde{v}_j \cdot \prod_{m=1}^{n-1} \left(\frac{-\log|z_m|}{-\log|z_{m+1}|}\right)^{-k_m(v_j)} \cdot \bigl(-\log|z_n|\bigr)^{-k_n(v_j)}.$$

Due to the norm estimate for tame harmonic bundles (Theorem 13.25 of [18]), the C^∞-frame $\boldsymbol{v}' = (v'_j)$ is adapted to the metric $h^{(0)}$ on $Z(C)$, i.e., the hermitian matrix-valued functions $H = \big(h(v'_i, v'_j)\big)$ and H^{-1} are bounded on $Z(C)$. Let A' be the matrix-valued function determined by $F\boldsymbol{v}' = \boldsymbol{v}' \cdot A'$. Then, we have

$$A'_{i,j} = A_{i,j} \cdot \prod_{m=1}^{n-1} \left(\frac{-\log|z_m|}{-\log|z_{m+1}|}\right)^{k_m(v_i)-k_m(v_j)} \cdot \big(-\log|z_n|\big)^{k_n(v_i)-k_n(v_j)}.$$

Hence, we obtain $A'_{i,j} = O\Big(\big(-\log|z_n|\big)^{-1/2}\Big)$. It implies the desired estimate for F on $\mathcal{U} \times Z(C)$. Similarly, we obtain the estimate on $\sigma(\mathcal{U}) \times Z(C)$, and thus on $\mathbb{P}^1 \times Z(C)$.

Let $\boldsymbol{w}$ be a frame of $V_{|\sigma(\mathcal{U})\times Y^\dagger}$ compatible with the filtrations $W(\underline{1}), W(\underline{2}), \ldots, W(\underline{n})$. For $m = 1, \ldots, n$, we set

$$k_m(w_i) := \frac{1}{2} \deg^{W(\underline{m})}(w_i).$$

We formally put $k_0(w_i) = 0$. We set $G_0 := S^{(0)} - \Phi^* S : V^{(0)} \otimes \sigma^* V^{(0)} \longrightarrow \mathbb{T}(0)$.

Lemma 4.7 *$G_0(v_i, \sigma^* w_j) = 0$ unless the following holds:*

$$k_m(v_i) + k_m(w_j) \geq 0 \quad (m = 1, \ldots, n-1), \quad k_n(v_i) + k_n(w_j) > 0.$$

Proof By the relation $S(N_i \otimes \mathrm{id}) + S\big(\mathrm{id} \otimes \sigma^*(N_i)\big) = 0$, we have $S\big(W_p(\underline{m}) \otimes \sigma^* W_q(\underline{m})\big) = 0$ unless $p + q \geq 0$. We have similar vanishings for $S^{(0)}$. Note that Φ preserves the filtrations $W(\underline{m})$ for $m = 1, \ldots, n$, and $\mathrm{Gr}^{W(\underline{n})} \Phi$ is compatible with S and $S^{(0)}$. Thus, we obtain the claim of Lemma 4.7. ∎

Let $q_\infty : \boldsymbol{C}_\mu \times X^*(1)^\dagger \longrightarrow \boldsymbol{C}_\mu \times Y^\dagger$ be the projection. Recall $\mathcal{V}^{(0)\triangle}_{|\boldsymbol{C}_\mu \times X^*(1)^\dagger} = q_\infty^* V_\infty$. Let $\widetilde{w}_j$ be the section of $\mathcal{V}^{(0)\triangle}_{|\sigma(\mathcal{U})\times X^*(1)^\dagger}$ induced by w_j, and we put

$$w'_j := \widetilde{w}_j \cdot \prod_{m=1}^{n-1} \left(\frac{-\log|z_m|}{-\log|z_{m+1}|}\right)^{-k_m(w_j)} \cdot \big(-\log|z_n|\big)^{-k_n(w_j)}.$$

Note the following:

$$G(v'_i, \sigma^* w'_j) = G_0(v_i, \sigma^* w_j) \times \prod_{m=1}^{n-1} \left(\frac{-\log|z_m|}{-\log|z_{m+1}|}\right)^{-k_m(v_i)-k_m(w_j)} \cdot \big(-\log|z_n|\big)^{-k_n(v_i)-k_n(v_j)}$$

Hence, we obtain $\big|G\big|_{h^{(0)}} = O\Big(\big(-\log|z_n|\big)^{-1/2}\Big)$. Similarly, we obtain the estimate for $\big|\overline{\partial}_{\mathcal{V}^{(0)},\mathbb{P}^1} G\big|$. Thus, the proof of Proposition 4.4 is finished. The proof of Theorem 4.1 is also finished.

4.2.2 Proof of Proposition 4.5

We have the decompositions $\widetilde{\mathbb{D}}^\triangle = \mathbb{D}^\triangle_{\mathcal{V}_0} + \nabla_\lambda$ and $\widetilde{\mathbb{D}}^{(0)\triangle} = \mathbb{D}^{(0)\triangle}_{\mathcal{V}_0} + \nabla^{(0)}_\lambda$. By an argument used in the proof of Proposition 4.4, we obtain the following estimate with respect to $h^{(0)}$:

$$\widetilde{\Phi}^* \nabla_\lambda - \nabla^{(0)}_\lambda = O\Big(\sum_{i=1}^n \big(-\log|z_i|\big)^{-1/2}\Big)$$

Then, Proposition 4.5 follows from Lemma 2.20 with Proposition 4.4. ∎

5 Family of meromorphic λ-flat bundles

We will review some results on family of meromorphic λ-flat bundles mainly explained in Sections 7 and 8 of [19]. See also [16] and [20] for the earlier works on asymptotic analysis of meromorphic flat bundles.

5.1 Good lattice in the level $\boldsymbol{m}$

5.1.1 Preliminary

Good set of irregular values in the level $\boldsymbol{m}$ Let $\Delta^\ell := \bigl\{(z_1,\ldots,z_\ell)\,\big|\,|z_i|<1,\ i=1,\ldots,\ell\bigr\}$ denote the ℓ-dimensional multi-disc. Let $X := \Delta^\ell\times Y$ for some complex manifold Y. Let $D_i := \{z_i=0\}$ and $D := \bigcup_{i=1}^{\ell} D_i$ be hypersurfaces of X. Let $M(X,D)$ (resp. $H(X)$) denote the space of meromorphic (resp. holomorphic) functions on X whose poles are contained in D. For $\boldsymbol{m}=(m_1,\ldots,m_\ell)\in\mathbb{Z}^\ell$, we put $\boldsymbol{z}^{\boldsymbol{m}} := \prod_{i=1}^{\ell} z_i^{m_i}$.

Let $\boldsymbol{m}\in\mathbb{Z}^{\ell}_{\leq 0}-\{\boldsymbol{0}\}$. A finite set $\mathcal{I}$ of meromorphic functions $\bigl\{\mathfrak{a}=\mathfrak{a}_{\boldsymbol{m}}\cdot\boldsymbol{z}^{\boldsymbol{m}}\bigr\}\subset M(X,D)$ is called a good set of irregular values on (X,D) in the level $\boldsymbol{m}$, if the following holds:

- $\mathfrak{a}_{\boldsymbol{m}}$ are holomorphic functions on X.
- $\mathfrak{a}_{\boldsymbol{m}}-\mathfrak{b}_{\boldsymbol{m}}$ are nowhere vanishing holomorphic functions on X for any two distinct $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$.

Let $i(0)$ be the integer such that $m_{i(0)}<0$. If moreover the following condition holds, $\mathcal{I}$ is called a good set of irregular values on (X,D) in the level $(\boldsymbol{m},i(0))$.

- $\mathfrak{a}_{\boldsymbol{m}}$ are independent of the variable $z_{i(0)}$ for any $\mathfrak{a}\in\mathcal{I}$.

Remark 5.1 *The first condition is not essential. If we do not impose it, the third condition should be replaced with that $\mathfrak{a}_{\boldsymbol{m}}-\mathfrak{b}_{\boldsymbol{m}}$ are independent of $z_{i(0)}$ for any $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$.* ∎

Multi-sectors and orders on good sets of irregular values in the level $\boldsymbol{m}$ Let $X:=\Delta^\ell\times Y$ for some complex manifold Y. Let $D_i:=\{z_i=0\}$ and $D:=\bigcup_{i=1}^{\ell}D_i$ be hypersurfaces of X. Let $\mathcal{K}$ be a region of $\boldsymbol{C}_\lambda$ or a point in $\boldsymbol{C}_\lambda^*$. (For Definition 5.4, we may admit $\mathcal{K}=\{0\}$. Since we do not have to consider Stokes structure in this case, we exclude it in the following.) The product $\mathcal{K}\times X$ is expressed by $\mathcal{X}$. We use the symbols like $\mathcal{Y}$ and $\mathcal{D}$ in similar meanings. We put $W:=\mathcal{D}\cup\bigl(\{0\}\times X\bigr)$ in the case $0\in\mathcal{K}$, and $W:=\mathcal{D}$ otherwise. Let $\pi:\widetilde{\mathcal{X}}(W)\longrightarrow\mathcal{X}$ denote the real blow up of $\mathcal{X}$ along W.

In this paper, a sector of a punctured disc Δ^* means a subset of the form $\bigl\{z\,\big|\,0<|z|<R,\ \theta_0\leq\arg(z)\leq\theta_1\bigr\}$ for some $\theta_0<\theta_1$. It may be standard to admit the case $|\theta_1-\theta_0|\geq 2\pi$, but we do not care about it.

By a "multi-sector of $\mathcal{X}-W$", we mean a subset of the following form

$$U\times\prod_{i=1}^{\ell}S_i\times V,\quad\text{or}\quad S_\lambda\times\prod_{i=1}^{\ell}S_i\times V.$$

- U denotes a compact region in $\mathcal{K}$. (If $\mathcal{K}$ is a point, $U=\mathcal{K}$.)
- S_λ denotes a sector of $\mathcal{K}-\{0\}$. (If $0\notin\mathcal{K}$, we do not consider the subsets of the second type.)
- S_i denote sectors of $\Delta^*_{z_i}$.
- V denotes a compact region in Y.

For a multi-sector S, let $\overline{S}$ denote the closure of S in $\widetilde{\mathcal{X}}(W)$.

Notation 5.2 *Let $\mathcal{MS}(\mathcal{X}-W)$ denote the set of multi-sectors in $\widetilde{\mathcal{X}}(W)$. For any point $P\in\widetilde{\mathcal{X}}(W)$, let $\mathcal{MS}(P,\mathcal{X}-W)$ denote the set of multi-sectors S such that P is contained in the interior part of $\overline{S}$.* ∎

Let $\mathcal{I}$ be a good set of irregular values on $(\mathcal{X},\mathcal{D})$ in the level $\boldsymbol{m}$. We put $F_{\mathfrak{a},\mathfrak{b}}:=-\operatorname{Re}\bigl(\lambda^{-1}\cdot(\mathfrak{a}-\mathfrak{b})\bigr)\cdot|\lambda|\cdot|\boldsymbol{z}^{-\boldsymbol{m}}|$ for any distinct $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$. They determine the C^∞-functions on $\widetilde{\mathcal{X}}(W)$.

Notation 5.3 *Let A be any subset of $\widetilde{\mathcal{X}}(W)$. We say $\mathfrak{a}<_A\mathfrak{b}$ for $(\mathfrak{a},\mathfrak{b})\in\mathcal{I}^2$ if $F_{\mathfrak{a},\mathfrak{b}}(Q)<0$ for any $Q\in A$. We say $\mathfrak{a}\leq_A\mathfrak{b}$ for $(\mathfrak{a},\mathfrak{b})\in\mathcal{I}^2$ if either $\mathfrak{a}<_A\mathfrak{b}$ or $\mathfrak{a}=\mathfrak{b}$ holds. The relation $\leq_A$ gives the partial order of $\mathcal{I}$.*

We use the symbol $\leq_P$ in the case $A=\{P\}$. For a multi-sector S, we prefer the symbol $\leq_S$ to $\leq_{\overline{S}}$. We also use $\leq^\lambda_S$ and $\leq^\lambda_P$ when we emphasize the twist by λ^{-1}. ∎

For any point $P\in\pi^{-1}(W)$, there exists $S_P\in\mathcal{MS}(P,\mathcal{X}-W)$ such that the relations $\leq_P$ and $\leq_{S_P}$ coincide. Let $\mathcal{MS}(P,\mathcal{X}-W,\mathcal{I})$ denote the set of such S_P. (The definitions of $\mathcal{MS}(P,\mathcal{X}-W,\mathcal{I})$ is slightly different from that in [19].)

5.1.2 Good lattice in the level $\boldsymbol{m}$

Let Y be a complex manifold with a simple normal crossing divisor D_Y'. Let $X := \Delta_z^k \times Y$, $D_{z,i} := \{z_i = 0\}$ and $D_z := \bigcup_{i=1}^k D_{z,i}$. We also put $D_Y := \Delta_z^k \times D_Y'$ and $D := D_z \cup D_Y$. Let $\mathcal{K}$ be a point of $\boldsymbol{C}_\lambda^*$ or a compact region in $\boldsymbol{C}_\lambda$. We put $\mathcal{X} := \mathcal{K} \times X$. We use the symbols $\mathcal{Y}$, $\mathcal{D}_z$, $\mathcal{D}$ in similar meanings. Let p_λ denote the projection forgetting the $\mathcal{K}$-component. The completion of $\mathcal{X}$ along $\mathcal{D}_z$ is denoted by $\widehat{\mathcal{D}}_z$. (See [1], [2] and [15] for completion of complex analytic spaces.) We use the symbol $\widehat{\mathcal{D}}$ in a similar meaning. Let d_X denote the restriction of the exterior derivative to the X-direction.

Let E be a locally free $\mathcal{O}_{\mathcal{X}}$-module with a family of meromorphic flat λ-connections $\mathbb{D} : E \longrightarrow E \otimes p_\lambda^* \Omega_X^1(*D)$. Let $\boldsymbol{m} \in \mathbb{Z}_{<0}^k$ and $i(0) \in [1, k] = \{1, \dots, k\}$. We put $\boldsymbol{m}(1) := \boldsymbol{m} + \boldsymbol{\delta}_{i(0)}$.

Definition 5.4 *We say that $(E, \mathbb{D})$ is an unramifiedly good lattice of a family of meromorphic λ-flat bundles in the level $(\boldsymbol{m}, i(0))$, if there exists a good set of irregular values $\mathcal{I}$ in the level $(\boldsymbol{m}, i(0))$ on $(\mathcal{X}, \mathcal{D}_z)$, and a decomposition*

$$(E, \mathbb{D})_{|\widehat{\mathcal{D}}_z} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (\widehat{E}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}}) \tag{23}$$

with $\mathrm{ord}(\widehat{\mathbb{D}}_{\mathfrak{a}} - d_X \mathfrak{a}) \geq \boldsymbol{m}(1)$ *in the sense* $(\widehat{\mathbb{D}}_{\mathfrak{a}} - d_X \mathfrak{a}) \widehat{E}_{\mathfrak{a}} \subset \boldsymbol{z}^{\boldsymbol{m}(1)} \cdot \widehat{E}_{\mathfrak{a}} \otimes p_\lambda^* \Omega_X^1 \bigl(\log D\bigr)$.

The decomposition (23) *is called the irregular decomposition in the level* $(\boldsymbol{m}, i(0))$, *(or simply $\boldsymbol{m}$). We also often say that $(E, \mathbb{D})$ is a good lattice in the level $(\boldsymbol{m}, i(0))$ for simplicity.* ∎

In the case $0 \in \mathcal{K}$, we put $\mathcal{X}^0 := \{0\} \times X$ and $\mathcal{D}_z^0 := \{0\} \times D_z$. By shrinking X, we obtain the irregular decomposition $(E, \mathbb{D})_{|\mathcal{X}^0} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (E_{\mathfrak{a}, \mathcal{X}^0}, \mathbb{D}_{\mathfrak{a}}^0)$ whose completion along $\mathcal{D}_z^0$ is equal to the one induced by (23). It is uniquely extended to the $\mathbb{D}$-flat decomposition on the completion $\widehat{\mathcal{X}}^0$ of $\mathcal{X}$ along $\mathcal{X}^0$:

$$(E, \mathbb{D})_{|\widehat{\mathcal{X}}^0} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (\widehat{E}_{\mathfrak{a}, \widehat{\mathcal{X}}^0}, \widehat{\mathbb{D}}_{\mathfrak{a}})$$

We put $W := \mathcal{X}^0 \cup \mathcal{D}_z$. Let $\widehat{W}$ denote the completion along W. We obtain the decomposition:

$$(E, \mathbb{D})_{|\widehat{W}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (\widehat{E}_{\mathfrak{a}, \widehat{W}}, \widehat{\mathbb{D}}_{\mathfrak{a}}) \tag{24}$$

The decomposition (24) is also called the irregular decomposition in the level $(\boldsymbol{m}, i(0))$ if $0 \in \mathcal{K}$.

In the following, we formally set $W := \mathcal{D}_z$ if $0 \notin \mathcal{K}$. Let $\pi : \widetilde{\mathcal{X}}(W) \longrightarrow \mathcal{X}$ denote the real blow up of $\mathcal{X}$ along W. Let O_z be the origin of Δ_z^k, and we put $\mathfrak{Z} := \pi^{-1}(O_z \times \mathcal{Y})$. We consider the case that $Y = \Delta_\zeta^n$ and $D_Y' := \bigcup_{j=1}^\ell D_{\zeta,j}$, where $D_{\zeta,j} := \{\zeta_j = 0\}$. The restriction of $\mathbb{D}$ to the Δ_z^k-direction is denoted by $\mathbb{D}_z$.

Stokes structure in the level $\boldsymbol{m}$ For any multi-sector S in $\mathcal{X} - W$, let $\overline{S}$ denote the closure of S in $\widetilde{\mathcal{X}}(W)$, and let Z denote $\overline{S} \cap \pi^{-1}(W)$. The irregular decomposition (24) on $\widehat{W}$ induces the decomposition on $\widehat{Z}$:

$$(E, \mathbb{D})_{|\widehat{Z}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (\widehat{E}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}})_{|\widehat{Z}} \tag{25}$$

We put $\mathcal{F}_{\mathfrak{a}}^Z := \bigoplus_{\mathfrak{b} \leq_S \mathfrak{a}} \widehat{E}_{\mathfrak{b}|\widehat{Z}}$, and then we obtain the filtration $\mathcal{F}^Z$ of $E_{|\widehat{Z}}$ indexed by $\bigl(\mathcal{I}, \leq_S\bigr)$. We can show the following proposition. (See Subsections 7.2.1 and 8.1.1 of [19].)

Proposition 5.5 *For any point $P \in \mathfrak{Z}$, there exists $S \in \mathcal{MS}(P, \mathcal{X} - W, \mathcal{I})$ such that the following holds:*

- *There exists the unique $\mathbb{D}$-flat filtration $\mathcal{F}^S$ of $E_{|\overline{S}}$ indexed by $\bigl(\mathcal{I}, \leq_S\bigr)$ such that $\mathcal{F}_{|\widehat{Z}}^S = \mathcal{F}^Z$. Moreover, if a $\mathbb{D}_z$-flat filtration $\mathcal{F}'^S$ of $E_{|\overline{S}}$ indexed by $(\mathcal{I}, \leq_S)$ satisfies $\mathcal{F}'^S_{|\widehat{Z}} = \mathcal{F}^Z$, then $\mathcal{F}'^S = \mathcal{F}^S$.*
- *There exists a $\mathbb{D}_z$-flat splitting of $\mathcal{F}^S$ on $\overline{S}$. Note that if we take such a splitting, the restriction to $\widehat{Z}$ is the same as* (25).

We call $\mathcal{F}^S$ the Stokes filtration of $(E,\mathbb{D})$ in the level $\boldsymbol{m}$. ∎

Notation 5.6 *For any $P \in \mathfrak{Z}$, let $\mathcal{MS}^*(P,\mathcal{X}-W,\mathcal{I})$ denote the set of $S \in \mathcal{MS}(P,\mathcal{X}-W,\mathcal{I})$ as in Proposition 5.5. Let $\mathcal{MS}^*(\mathcal{X}-W,\mathcal{I})$ denote the union of $\mathcal{MS}^*(P,\mathcal{X}-W,\mathcal{I})$ for $P \in \mathfrak{Z}$.* ∎

The following lemma is clear.

Lemma 5.7 *Let $S,S' \in \mathcal{MS}(P,\mathcal{X}-W,\mathcal{I})$. Assume (i) $S' \subset S$, (ii) $S \in \mathcal{MS}^*(P,\mathcal{X}-W,\mathcal{I})$. Then, $S' \in \mathcal{MS}^*(P,\mathcal{X}-W,\mathcal{I})$. The filtration $\mathcal{F}^{S'}$ is the restriction of $\mathcal{F}^S$.* ∎

Compatibility of the Stokes filtrations Let $S,S' \in \mathcal{MS}^*(\mathcal{X}-W,\mathcal{I})$ such that $S' \subset S$. The natural map $\big(\mathcal{I},\leq_S\big) \longrightarrow \big(\mathcal{I},\leq_{S'}\big)$ is order-preserving. We can show the following lemma easily by using Proposition 5.5. (See Subsections 7.2.2 and 8.1.2 of [19].)

Lemma 5.8 *The filtrations $\mathcal{F}^S$ and $\mathcal{F}^{S'}$ are compatible with respect to $\big(\mathcal{I},\leq_S\big) \longrightarrow \big(\mathcal{I},\leq_{S'}\big)$ in the following sense:*

- $\mathcal{F}^{S'}_{\mathfrak{a}}(E_{|\overline{S'}}) = \mathcal{F}^{S'}_{<\mathfrak{a}}(E_{|\overline{S'}}) + \mathcal{F}^{S}_{\mathfrak{a}}(E_{|\overline{S}})_{|\overline{S'}}$.
- *The induced morphisms* $\mathrm{Gr}^{\mathcal{F}^S}_{\mathfrak{a}}(E_{|S})_{|S'} \longrightarrow \mathrm{Gr}^{\mathcal{F}^{S'}}_{\mathfrak{a}}(E_{|S'})$ *are isomorphisms.*

In particular, we have $\mathcal{F}^S(E_{|\overline{S}})_{|\overline{S'}} = \mathcal{F}^{S'}(E_{|\overline{S'}})$, if $\big(\mathcal{I},\leq_S\big) \longrightarrow \big(\mathcal{I},\leq_{S'}\big)$ is isomorphic. ∎

Splitting with nice property We have the induced morphisms $\mathrm{Res}_j(\mathbb{D}) : E_{|\mathcal{D}_{\zeta,j}} \longrightarrow \boldsymbol{z}^{\boldsymbol{m}(1)} \cdot E_{|\mathcal{D}_{\zeta,j}}$ for $j=1,\ldots,\ell$. Since $\mathcal{F}^S$ is $\mathbb{D}$-flat, $\mathrm{Res}_j(\mathbb{D})$ preserves $\mathcal{F}^S_{|\mathcal{D}_{\zeta,j}}$. If we fix the coordinate, we have the induced family of flat λ-connections of $E_{|\mathcal{D}_{\zeta,j}}$ which is denoted by ${}^j\mathbb{D}$. It also preserves the filtration $\mathcal{F}^S_{|\mathcal{D}_{\zeta,j}}$. Let ${}^jF$ $(j=1,\ldots,\ell)$ be filtrations of $E_{|\mathcal{D}_{\zeta,j}}$, which are preserved by the endomorphism $\mathrm{Res}_j(\mathbb{D})$ and the flat connection ${}^j\mathbb{D}$ of $E_{|\mathcal{D}_{\zeta,j}}$. We can show the following (Subsections 7.2.3 and 8.1.3 of [19]).

Proposition 5.9 *Let $P \in \mathfrak{Z}$. There exist $S \in \mathcal{MS}^*(P,\mathcal{X}-W,\mathcal{I})$ and a $\mathbb{D}_z$-flat splitting of the filtration $\mathcal{F}^S$, whose restriction to $\overline{S} \cap \mathcal{D}_{\zeta,j}$ is compatible with* $\mathrm{Res}_j(\mathbb{D})$ *and the filtrations ${}^jF$ for $j=1,\ldots,\ell$.* ∎

Under some more assumption, we can take a $\mathbb{D}$-flat splitting. (See Subsection 7.2.3 of [19].)

Proposition 5.10 *Assume that $\mathcal{K}$ is a point or a compact region in $\boldsymbol{C}^*_\lambda$. Assume that the eigenvalues α,β of* $\mathrm{Res}_j(\mathbb{D}^f)_{|D_j\times\{\lambda\}}$ *satisfy $\alpha-\beta \notin (\mathbb{Z}-\{0\})$ for any $j=1,\ldots,\ell$ and for any $\lambda \in \mathcal{K}$. Then, we have a $\mathbb{D}$-flat splitting of $\mathcal{F}^S$, whose restriction to $\mathcal{D}_{\zeta,j}$ is compatible with ${}^jF$ for each $j=1,\ldots,\ell$.* ∎

Some functoriality of Stokes filtrations We explain functoriality of Stokes filtrations. See Subsections 7.2.4 and 8.1.4 of [19] for more details.

In general, when we are given vector spaces $U \subset V$, let $U^\perp$ denote the subspace of the dual $V^\vee$ given by $U^\perp = \big\{f \in V^\vee \,\big|\, f(U)=0\big\}$. It is naturally generalized for vector bundles. Let $(E,\mathbb{D},\mathcal{I})$ be an unramifiedly good lattice of a family of meromorphic λ-flat bundles in the level $(\boldsymbol{m},i(0))$ on $(\mathcal{X},\mathcal{D}_z)$. Let $S \in \mathcal{MS}^*(\mathcal{X}-W,\mathcal{I})$. We have the following for any $\mathfrak{a} \in \mathcal{I}^\vee := \big\{-\mathfrak{b} \,\big|\, \mathfrak{b} \in \mathcal{I}\big\}$:

$$\mathcal{F}^S_{\mathfrak{a}}(E^\vee_{|\overline{S}}) = \left(\sum_{\substack{\mathfrak{c}\in\mathcal{I}\\ \mathfrak{c}\not\geq_S -\mathfrak{a}}} \mathcal{F}^S_{\mathfrak{c}}(E_{|\overline{S}})\right)^{\perp}$$

Let $(E_p,\mathbb{D}_p,\mathcal{I}_p)$ $(p=1,2)$ be good lattices of families of meromorphic λ-flat bundles in the level $(\boldsymbol{m},i(0))$. We assume that $\mathcal{I}_1\otimes\mathcal{I}_2 := \big\{\mathfrak{a}_1+\mathfrak{a}_2 \,\big|\, \mathfrak{a}_p \in \mathcal{I}_p\big\}$ is a good set of irregular values in the level $(\boldsymbol{m},i(0))$. We put $(\widetilde{E},\widetilde{\mathbb{D}}) := (E_1,\mathbb{D}_1)\otimes(E_2,\mathbb{D}_2)$. Let $S \in \bigcap_{p=1,2}\mathcal{MS}^*\big(\mathcal{X}-W,\mathcal{I}_p\big)$. We have the following for each $\mathfrak{a} \in \mathcal{I}_1\otimes\mathcal{I}_2$:

$$\mathcal{F}^S_{\mathfrak{a}}(\widetilde{E}_{|\overline{S}}) = \sum_{\mathfrak{a}_1+\mathfrak{a}_2\leq_S\mathfrak{a}} \mathcal{F}^S_{\mathfrak{a}_1}(E_{1|\overline{S}})\otimes\mathcal{F}^S_{\mathfrak{a}_2}(E_{2|\overline{S}}).$$

Assume that $\mathcal{I}_1 \oplus \mathcal{I}_2 := \mathcal{I}_1 \cup \mathcal{I}_2$ is a good set of irregular values in the level $(\boldsymbol{m}, i(0))$. Let $S \in \bigcap_{p=1,2} \mathcal{MS}^*\big(\mathcal{X} - W, \mathcal{I}_p\big)$. We have the following for each $\mathfrak{a} \in \mathcal{I}_1 \oplus \mathcal{I}_2$:

$$\mathcal{F}^S_{\mathfrak{a}}\big((E_1 \oplus E_2)_{|\overline{S}}\big) = \mathcal{F}^S_{\mathfrak{a}}(E_{1|\overline{S}}) \oplus \mathcal{F}^S_{\mathfrak{a}}(E_{2|\overline{S}}).$$

Let $F : (E_1, \mathbb{D}_1) \longrightarrow (E_2, \mathbb{D}_2)$ be a flat morphism. For simplicity, we assume that $\mathcal{I}_1 \cup \mathcal{I}_2$ is a good set of irregular values in the level $(\boldsymbol{m}, i(0))$.

Lemma 5.11 *Let $S \in \bigcap_{p=1,2} \mathcal{MS}^*(\mathcal{X} - W, \mathcal{I}_p)$. The restriction $F_{|\overline{S}}$ preserves the Stokes filtrations. As a result, we obtain the following.*

- *If the restriction of F to $\mathcal{X} - \mathcal{D}$ is isomorphic, we have $\mathcal{I}_1 = \mathcal{I}_2$ and $\mathcal{F}^S_{\mathfrak{a}}(E_{1|S\setminus\mathcal{D}}) = \mathcal{F}^S_{\mathfrak{a}}(E_{2|S\setminus\mathcal{D}})$.*
- *In particular, the Stokes filtration $\mathcal{F}^S$ depends only on the family of meromorphic λ-flat bundles $\big(E(*\mathcal{D}), \mathbb{D}\big)$ in the sense that it is independent of the choice of an unramifiedly good lattice $E \subset E(*\mathcal{D})$ in the level $(\boldsymbol{m}, i(0))$.* ∎

The associated graded bundle in the level $\boldsymbol{m}$ For sectors $S \in \mathcal{MS}^*(\mathcal{X} - W, \mathcal{I})$ and each $\mathfrak{a} \in \mathcal{I}$, we obtain the bundle $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_{|\overline{S}})$ on $\overline{S}$ associated to the Stokes filtration $\mathcal{F}^S$ in the level $\boldsymbol{m}$. By varying S and gluing $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_{|\overline{S}})$, we obtain the bundle $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_{|\widetilde{\mathcal{V}}(W)})$ on $\widetilde{\mathcal{V}}(W)$ with the induced family of flat λ-connections $\mathbb{D}_{\mathfrak{a}}$, where $\mathcal{V}$ denotes some neighbourhood of $O_z \times \mathcal{Y}$, and $\widetilde{\mathcal{V}}(W)$ denotes the real blow up of $\mathcal{V}$ along $W \cap \mathcal{V}$. It is shown that we have the descent of $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_{|\widetilde{\mathcal{V}}(W)})$ to $\mathcal{V}$, i.e., there exists a locally free sheaf $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)$ on $\mathcal{V}$ with a family of meromorphic flat λ-connections $\mathbb{D}_{\mathfrak{a}}$, such that

$$\pi^{-1}(\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E), \mathbb{D}_{\mathfrak{a}}) \simeq (\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_{|\widetilde{\mathcal{V}}(W)}), \mathbb{D}_{\mathfrak{a}}), \qquad (\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E), \mathbb{D}_{\mathfrak{a}})_{|\widehat{W}\cap\mathcal{V}} \simeq \big(\widehat{E}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}}\big)_{|\widehat{W}\cap\mathcal{V}}$$

(See Subsection 7.3 and Subsection 8.1.5 of [19].) If we set $\mathbb{D}'_{\mathfrak{a}} := \mathbb{D}_{\mathfrak{a}} - d_X\mathfrak{a}$, we have

$$\mathbb{D}'_{\mathfrak{a}} E_{\mathfrak{a}} \subset \boldsymbol{z}^{\boldsymbol{m}(1)} \cdot E_{\mathfrak{a}} \otimes p_\lambda^* \Omega^1_X(\log D).$$

We give some statements for functoriality. See Subsections 7.3.2 and 8.1.6 of [19] for more details.

By taking Gr of the Stokes filtrations of $(E^\vee, \mathbb{D}^\vee, \mathcal{I}^\vee)$, we obtain the associated graded bundle $\mathrm{Gr}^{\boldsymbol{m}}(E^\vee) = \bigoplus_{\mathfrak{a}\in\mathcal{I}^\vee} \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E^\vee)$. We have the natural flat isomorphism

$$\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E^\vee) \simeq \mathrm{Gr}^{\boldsymbol{m}}_{-\mathfrak{a}}(E)^\vee. \tag{26}$$

Actually, by construction, we have such an isomorphism on the real blow up, which induces (26).

Let $(E_p, \nabla_p, \mathcal{I}_p)$ $(p = 1, 2)$ be unramifiedly good lattices of families of meromorphic λ-flat bundles. Assume $\mathcal{I}_1 \otimes \mathcal{I}_2$ is a good set of irregular values in the level $\boldsymbol{m}$. Let $(\widetilde{E}, \widetilde{\mathbb{D}}) := (E_p, \mathbb{D}_1) \otimes (E_p, \mathbb{D}_2)$. We have the following natural isomorphism for each $\mathfrak{a} \in \mathcal{I}_1 \otimes \mathcal{I}_2$:

$$\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(\widetilde{E}) \simeq \bigoplus_{\substack{(\mathfrak{a}_1,\mathfrak{a}_2)\in\mathcal{I}_1\times\mathcal{I}_2\\ \mathfrak{a}_1+\mathfrak{a}_2=\mathfrak{a}}} \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}_1}(E_1) \otimes \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}_2}(E_2) \tag{27}$$

Assume $\mathcal{I}_1 \oplus \mathcal{I}_2$ is a good set of irregular values in the level $(\boldsymbol{m}, i(0))$. For each $\mathfrak{a} \in \mathcal{I}_1 \oplus \mathcal{I}_2$, we obviously have

$$\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1 \oplus E_2) \simeq \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1) \oplus \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2).$$

Lemma 5.12 *Let $F : (E_1, \mathbb{D}_1) \longrightarrow (E_2, \mathbb{D}_2)$ be a flat morphism. Assume $\mathcal{I}_1 \oplus \mathcal{I}_2$ is a good set of irregular values in the level $(\boldsymbol{m}, i(0))$. We have the naturally induced morphism $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(F) : \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1) \longrightarrow \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2)$. If the restriction $E_{1|\mathcal{X}-\mathcal{D}} \longrightarrow E_{2|\mathcal{X}-\mathcal{D}}$ is an isomorphism, the induced morphism*

$$\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1) \otimes \mathcal{O}(*\mathcal{D}) \longrightarrow \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2) \otimes \mathcal{O}(*\mathcal{D})$$

is an isomorphism.

Hence, the associated meromorphic flat bundles $\big(\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E) \otimes \mathcal{O}(*\mathcal{D}), \mathbb{D}_{\mathfrak{a}}\big)$ are well defined for the meromorphic flat bundle $\big(E(*\mathcal{D}), \mathbb{D}\big)$. ∎

A characterization of sections of E Let $\boldsymbol{w}_{\mathfrak{a}}$ be a frame of $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)$. Let $S \in \mathcal{MS}^*(\mathcal{X}-W,\mathcal{I})$, and let $E_{|\overline{S}} = \bigoplus E_{\mathfrak{a},S}$ be a $\mathbb{D}_z$-flat splitting of the Stokes filtration $\mathcal{F}^S$. By the natural isomorphism $E_{\mathfrak{a},S} \simeq \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)_{|\overline{S}}$, we take a lift $\boldsymbol{w}_{\mathfrak{a},S}$ of $\boldsymbol{w}_{\mathfrak{a}}$. Thus, we obtain a frame $\boldsymbol{w}_S = \bigl(\boldsymbol{w}_{\mathfrak{a},S}\bigr)$ of $E_{|\overline{S}}$. The following proposition is clear, which implies a characterization of sections of E by growth order with respect to the frames $\boldsymbol{w}_S$ ($S \in \mathcal{MS}^*(\mathcal{X}-W,\mathcal{I})$).

Proposition 5.13 *Let $\boldsymbol{v}$ be a frame of E, and let G_S be determined by $\boldsymbol{v} = \boldsymbol{w}_S \cdot G_S$. Then, G_S and G_S^{-1} are bounded on S.* ▮

Complement on the induced flat connection along the λ-direction Assume that we are given a connection along the λ-direction $\nabla_\lambda : E \longrightarrow E \otimes \Omega^1_{\mathcal{K}}(*W)$ such that $\mathbb{D}^f + \nabla_\lambda$ is a meromorphic flat connection of E.

Lemma 5.14 *The Stokes filtrations are flat with respect to ∇_λ, and we have the induced meromorphic flat connection ∇_λ along the λ-direction on $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)$.*

Proof Take N such that $\lambda^N \nabla_\lambda(\partial_\lambda) E \subset E \otimes \mathcal{O}_{\mathcal{X}}(*\mathcal{D})$. Let $\boldsymbol{w}_S = (\boldsymbol{w}_{\mathfrak{a},S})$ be a frame of $E_{|\overline{S}}$ as above. Let $A = (A_{\mathfrak{a},\mathfrak{b}})$ be the matrix-valued holomorphic function on S determined by $\lambda^N \nabla(\partial_\lambda)\boldsymbol{w}_S = \boldsymbol{w}_S \cdot A$. By using Proposition 5.13, we can show that $A_{\mathfrak{a},\mathfrak{b}}$ are of polynomial order.

Let $B_{\mathfrak{a}}$ be the matrix-valued meromorphic one-forms determined by $\mathbb{D}_{\mathfrak{a},z}\boldsymbol{w}_{\mathfrak{a}} = \boldsymbol{w}_{\mathfrak{a}} \cdot \bigl(d_z\mathfrak{a} + B_{\mathfrak{a}}\bigr)$. Note that $\boldsymbol{z}^{-\boldsymbol{m}(1)} B_{\mathfrak{a}}$ is logarithmic. By the commutativity $[\mathbb{D}^f, \nabla_\lambda] = 0$, we obtain the following relation for $\mathfrak{a} \neq \mathfrak{b}$:

$$\lambda \cdot d_z A_{\mathfrak{a},\mathfrak{b}} + \bigl(d_z(\mathfrak{a}-\mathfrak{b})\bigr) \cdot A_{\mathfrak{a},\mathfrak{b}} + \bigl(A_{\mathfrak{a},\mathfrak{b}} B_{\mathfrak{b}} - B_{\mathfrak{a}} A_{\mathfrak{a},\mathfrak{b}}\bigr) = 0 \tag{28}$$

By applying the results in Subsection 4.3 of [19] to (28), we obtain $A_{\mathfrak{a},\mathfrak{b}} = 0$ unless $\mathfrak{a} \leq_S \mathfrak{b}$, which implies the first claim. Since $A_{\mathfrak{a},\mathfrak{a}}$ is of polynomial order, the induced connection along the λ-direction is meromorphic. ▮

Prolongment of morphisms Let $(E_p, \mathbb{D}_p, \mathcal{I}_p)$ ($p = 1,2$) be good lattices in the level $(\boldsymbol{m}, i(0))$. Assume that $\mathcal{I}_1 \cup \mathcal{I}_2$ is a good set of irregular values in the level $(\boldsymbol{m}, i(0))$. Assume that we are given a flat morphism $F : (E_1, \mathbb{D}_1)_{|\mathcal{X}-\mathcal{D}_z} \longrightarrow (E_2, \mathbb{D}_2)_{|\mathcal{X}-\mathcal{D}_z}$ with the following property:

- For each small sector $S \in \mathcal{MS}(\mathcal{X}-\mathcal{D}_z, \mathcal{I}_1 \cup \mathcal{I}_2)$, the Stokes filtrations are preserved by $F_{|S}$.
- The induced maps $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(F) : \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1)_{|\mathcal{X}-\mathcal{D}_z} \longrightarrow \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2)_{|\mathcal{X}-\mathcal{D}_z}$ are extended to $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1) \longrightarrow \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2)$ for any $\mathfrak{a} \in \mathcal{I}_1 \cup \mathcal{I}_2$.

Lemma 5.15 *F is extended to a morphism $E_1 \longrightarrow E_2$.*

Proof Let $\boldsymbol{w}_{p,S} = (\boldsymbol{w}_{p,\mathfrak{a},S})$ be frames of $E_{p|\overline{S}}$ as above. Let $A = (A_{\mathfrak{a},\mathfrak{b}})$ be determined by $F(\boldsymbol{w}_{1,S}) = \boldsymbol{w}_{2,S} \cdot A$. By the assumption, $A_{\mathfrak{a},\mathfrak{b}} = 0$ unless $\mathfrak{a} \leq_S \mathfrak{b}$, and $A_{\mathfrak{a},\mathfrak{a}}$ is bounded. By applying an argument in the proof of Lemma 5.14 to $A_{\mathfrak{a},\mathfrak{b}}$ for $\mathfrak{a} <_S \mathfrak{b}$, and by shrinking X, we obtain $A_{\mathfrak{a},\mathfrak{b}} = O\bigl(\exp\bigl(-\epsilon|\lambda^{-1} \cdot \boldsymbol{z}^{\boldsymbol{m}}|\bigr)\bigr)$ on $S \cap (\mathcal{X}-\mathcal{D}_z)$. Then, the claim follows from Proposition 5.13. ▮

5.1.3 Pseudo-good lattice in the level $\boldsymbol{m}$

Let Y be a complex manifold. Let $X := \Delta_z^k \times Y$, $D_{z,i} := \{z_i = 0\}$ and $D := \bigcup_{i=1}^k D_{z,i}$. Let E be a locally free $\mathcal{O}_X$-module. For simplicity, we consider a meromorphic flat connection $\nabla : E \longrightarrow E \otimes \Omega^1_X(*D)$ instead of a family of meromorphic flat λ-connections. Let $\boldsymbol{m} \in \mathbb{Z}^k_{<0}$ and $i(0) \in [1,k]$. We put $\boldsymbol{m}(1) := \boldsymbol{m} + \boldsymbol{\delta}_{i(0)}$.

Definition 5.16 *We say that (E,∇) is an unramifiedly pseudo-good lattice in the level $(\boldsymbol{m}, i(0))$, if there exists an unramifiedly good lattice $E' \supset E$ of $\bigl(E(*D),\nabla\bigr)$ with the irregular decomposition $(E',\mathbb{D})_{|\widehat{D}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}}(\widehat{E}'_{\mathfrak{a}}, \widehat{\nabla}_{\mathfrak{a}})$ in the level $(\boldsymbol{m}, i(0))$, such that*

$$E_{|\widehat{D}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \bigl(\widehat{E}'_{\mathfrak{a}} \cap E_{|\widehat{D}}\bigr) \tag{29}$$

The decomposition (29) is called the irregular decomposition of $(E,\mathbb{D})$ in the level $(\boldsymbol{m}, i(0))$. ▮

It is easy to observe that $\widehat{E}_{\mathfrak{a}} := \widehat{E}'_{\mathfrak{a}} \cap E_{|\widehat{\mathcal{D}}_z}$ in (29) is independent of the choice of a good lattice $E' \supset E$ in the level $\boldsymbol{m}$. We have straightforward generalizations of the results in Subsection 5.1.2. We naturally identify X with $\{1\} \times X \subset \boldsymbol{C}_\lambda \times X$ when we consider the order $\leq_S$ for multi-sectors $S \subset X - D$.

Construction of Gr We take an unramifiedly good lattice $E' \supset E$ in the level $(\boldsymbol{m}, i(0))$. By shrinking X around $O_z \times Y$, we have the vector bundle $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E')$ on X with a meromorphic flat connection $\nabla_{\mathfrak{a}}$ for each $\mathfrak{a} \in \mathcal{I}$. Recall that we have the natural isomorphism $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E')_{|\widehat{D}} \simeq \widehat{E}'_{\mathfrak{a}}$. Hence, we have the sub-lattice of $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E')$ corresponding to $\widehat{E}_{\mathfrak{a}} \subset \widehat{E}'_{\mathfrak{a}}$, which is denoted by $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)$. It is equipped with a meromorphic flat connection $\nabla_{\mathfrak{a}}$. By construction, we have the isomorphism

$$(\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E), \nabla_{\mathfrak{a}})_{|\widehat{D}} \simeq (\widehat{E}_{\mathfrak{a}}, \widehat{\nabla}_{\mathfrak{a}}). \tag{30}$$

Lemma 5.17 *Let (E_i, ∇_i) $(i = 1, 2)$ be pseudo-good lattices in the level $(\boldsymbol{m}, i(0))$. Let $F : (E_1, \nabla_1) \longrightarrow (E_2, \nabla_2)$ be a flat morphism. Assume $\mathcal{I}_1 \oplus \mathcal{I}_2$ is a good set of irregular values in the level $(\boldsymbol{m}, i(0))$. We have the naturally induced morphism $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(F) : \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1) \longrightarrow \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2)$.*

Proof We can take good lattices (E'_i, ∇_i) in the level $(\boldsymbol{m}, i(0))$ such that $E_i \subset E'_i$ and $F(E'_1) \subset E'_2$. By Lemma 5.12, we have the induced morphism $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(F) : \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E'_1) \longrightarrow \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E'_2)$. By considering the completion, it is easy to observe that $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_1) \longrightarrow \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E_2)$ is induced. ∎

Flat splitting and Stokes filtration Let $\pi : \widetilde{X}(D) \longrightarrow X$ be the real blow up. Let $S \in \mathcal{MS}^*(X - D, \mathcal{I})$. Let $\overline{S}$ denote the closure of S in $\widetilde{X}(D)$, and let Z denote $\overline{S} \cap \pi^{-1}(D)$. We have the Stokes filtration $\mathcal{F}^S$ of $E'_{|\overline{S}}$, and we can take a flat splitting $E'_{|\overline{S}} = \bigoplus E'_{\mathfrak{a},S}$ such that $E'_{\mathfrak{a},S|\widehat{Z}} = \pi^{-1}(\widehat{E}'_{\mathfrak{a}})$. Because $E_{|X-D} = E'_{|X-D}$, it induces the flat decomposition of $E_{|S}$.

Lemma 5.18 *It is extended to the decomposition $E_{|\overline{S}} = \bigoplus E_{\mathfrak{a},S}$ such that $E_{\mathfrak{a},S|\widehat{Z}} = \pi^{-1}(\widehat{E}_{\mathfrak{a}})$.*

Proof Let $\boldsymbol{w}_{\mathfrak{a}}$ and $\boldsymbol{w}'_{\mathfrak{a}}$ be frames of $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)$ and $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E')$. Let $G_{\mathfrak{a}}$ be determined by $\boldsymbol{w}_{\mathfrak{a}} = \boldsymbol{w}'_{\mathfrak{a}} \cdot G_{\mathfrak{a}}$. They induce the frames $\widehat{\boldsymbol{w}}_{\mathfrak{a}}$ and $\widehat{\boldsymbol{w}}'_{\mathfrak{a}}$ of $\widehat{E}_{\mathfrak{a}}$ and $\widehat{E}'_{\mathfrak{a}}$, respectively.

By the isomorphism $E'_{\mathfrak{a},S} \simeq \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E')_{|\overline{S}}$, we obtain the frames $\boldsymbol{w}'_{\mathfrak{a},S}$ of $E'_{\mathfrak{a},S}$. Then, $\boldsymbol{w}_{\mathfrak{a},S} := \boldsymbol{w}'_{\mathfrak{a},S} \cdot G_{\mathfrak{a}}$ gives a tuple of sections of $E'_{\mathfrak{a},S}$, and we can observe that $\boldsymbol{w}_{\mathfrak{a}|\widehat{Z}} = \pi^{-1}(\widehat{\boldsymbol{w}}_{\mathfrak{a}})$. Let $E_{\mathfrak{a},S}$ be generated by $\boldsymbol{w}_{\mathfrak{a},S}$, and then we obtain the desired decomposition $E = \bigoplus E_{\mathfrak{a},S}$. ∎

Let $\boldsymbol{w}_S = (\boldsymbol{w}_{\mathfrak{a},S})$ be as above. Let $\boldsymbol{v}$ be a frame of E on X. Let G_S be determined by $\boldsymbol{v} = \boldsymbol{w}_S \cdot G_S$. Both $\boldsymbol{v}_{|\widehat{Z}}$ and $\boldsymbol{w}_{S|\widehat{Z}}$ give the frame of $E_{|\widehat{Z}}$, we obtain the following.

Proposition 5.19 *G_S and G_S^{-1} are bounded on S.* ∎

Proposition 5.20 *The flat subbundle $\mathcal{F}^S_{\mathfrak{a}}(E_{|\overline{S}}) := \bigoplus_{\mathfrak{b} \leq_S \mathfrak{a}} \overline{E}_{\mathfrak{b},S}$ is independent of the choice of a flat decomposition $E_{|\overline{S}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \overline{E}_{\mathfrak{a},S}$ such that $\overline{E}_{\mathfrak{a},S|\widehat{Z}} = \pi^{-1}\widehat{E}_{\mathfrak{a}}$.*

Proof Let $E_{|\overline{S}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \overline{E}_{\mathfrak{a},S}$ be another flat decomposition such that $\overline{E}_{\mathfrak{a},S|\widehat{Z}} = \pi^{-1}\widehat{E}_{\mathfrak{a}}$. We take a frame $\overline{\boldsymbol{w}}_{\mathfrak{a},S}$ of $\overline{E}_{\mathfrak{a},S}$ such that $\overline{\boldsymbol{w}}_{\mathfrak{a},S|\widehat{Z}} = \widehat{\boldsymbol{w}}_{\mathfrak{a}}$. We set $\overline{\boldsymbol{w}}'_{\mathfrak{a}} := \overline{\boldsymbol{w}}_{\mathfrak{a}} \cdot G_{\mathfrak{a}}^{-1}$. Then, $\overline{\boldsymbol{w}}'_{\mathfrak{a}|\widehat{Z}} = \pi^{-1}\widehat{\boldsymbol{w}}'_{\mathfrak{a}}$. Let $\overline{E}'_{\mathfrak{a}}$ be generated by $\overline{\boldsymbol{w}}'_{\mathfrak{a}}$. Then, we obtain a flat decomposition $E'_{|\overline{S}} = \bigoplus \overline{E}'_{\mathfrak{a}}$, which has to be a splitting of the Stokes filtration $\mathcal{F}^S(E'_{|\overline{S}})$. Because $\overline{E}'_{\mathfrak{a}|S} = \overline{E}_{\mathfrak{a}|S}$, we obtain the well definedness of the filtration. ∎

Thus, we obtain the filtration $\mathcal{F}^S$ of $E_{|\overline{S}}$, which is called the Stokes filtration.

Lemma 5.21 *We have the natural isomorphism $\mathrm{Gr}^{\mathcal{F}^S}_{\mathfrak{a}}(E_{|\overline{S}}) \simeq \mathrm{Gr}_{\mathfrak{a}}(E)_{|\overline{S}}$.*

Proof We use the notation in the proof of Lemma 5.18. By the comparison of $\boldsymbol{w}_{\mathfrak{a}}$ and $\boldsymbol{w}_{\mathfrak{a},S}$, we obtain $E_{\mathfrak{a},S} \simeq \mathrm{Gr}_{\mathfrak{a}}(E)_{|\overline{S}}$. By the construction of the Stokes filtration, we have the natural isomorphism $\mathrm{Gr}^{\mathcal{F}^S}_{\mathfrak{a}}(E_{|\overline{S}}) \simeq E_{\mathfrak{a},S}$. Then, the claim of Lemma 5.21 is clear. ∎

5.1.4 A comparison

Let Y be a complex manifold. Let $X := \Delta_z^k \times Y$, $D_{z,i} := \{z_i = 0\}$ and $D := \bigcup_{i=1}^k D_{z,i}$. Let $\widetilde{\mathcal{K}}$ be a compact region in $\boldsymbol{C}_{z_{k+1}}$. We set $\widetilde{\mathcal{X}} := \widetilde{\mathcal{K}} \times X$. We use the symbol $\widetilde{\mathcal{D}}$ in a similar meaning. We set $\widetilde{W} := \widetilde{\mathcal{D}} \cup (\{0\} \times X)$.

Let $\mathcal{I} \subset M(X,D)$ be a good set of irregular values in the level $(\boldsymbol{m}, i(0))$. We set $\widetilde{\boldsymbol{m}} := (\boldsymbol{m}, -1) \in \mathbb{Z}_{<0}^{k+1}$. We put $\widetilde{\mathfrak{a}} := z_{k+1}^{-1}\mathfrak{a}$ for $\mathfrak{a} \in \mathcal{I}$, and we set

$$\widetilde{\mathcal{I}} := \bigl\{\widetilde{\mathfrak{a}} \bigm| \mathfrak{a} \in \mathcal{I}\bigr\} \subset M(\widetilde{\mathcal{X}}, \widetilde{W})/H(\widetilde{\mathcal{X}}).$$

Then, it is a good set of irregular values in the level $(\widetilde{\boldsymbol{m}}, i(0))$.

Let $\widetilde{E}$ be a holomorphic vector bundle on $\widetilde{\mathcal{X}}$ with a meromorphic flat connection $\nabla : \widetilde{E} \longrightarrow \widetilde{E} \otimes \Omega^1_{\widetilde{\mathcal{X}}}(*\widetilde{W})$ such that $(\widetilde{E}, \nabla)$ is an unramifiedly good lattice in the level $(\widetilde{\boldsymbol{m}}, i(0))$ on $(\widetilde{\mathcal{X}}, \widetilde{W})$ with the irregular decomposition:

$$(\widetilde{E}, \nabla)_{|\widehat{\widetilde{W}}} = \bigoplus_{\widetilde{\mathfrak{a}} \in \widetilde{\mathcal{I}}} (\widehat{\widetilde{E}}_{\widetilde{\mathfrak{a}}}, \widehat{\nabla}_{\widetilde{\mathfrak{a}}}) \tag{31}$$

Applying a general theory in Subsection 5.1.3, we obtain a holomorphic vector bundle $\mathrm{Gr}^{\widetilde{\boldsymbol{m}}}_{\widetilde{\mathfrak{a}}}(\widetilde{E})$ on $\widetilde{\mathcal{X}}$ with the induced meromorphic flat connection $\nabla_{\widetilde{\mathfrak{a}}}$ for each $\widetilde{\mathfrak{a}} \in \widetilde{\mathcal{I}}$.

By setting $\lambda = z_{k+1}$, we obtain the isomorphism $\boldsymbol{C}_{z_{k+1}} \simeq \boldsymbol{C}_\lambda$. Let $\mathcal{K} \subset \boldsymbol{C}_\lambda$ be the image of $\widetilde{\mathcal{K}}$. We put $\mathcal{X} := \mathcal{K} \times X$ and we use the symbol $\mathcal{D}$ in a similar meaning. We set $W := \mathcal{D} \cup \bigl(\{0\} \times X\bigr)$. We have the natural isomorphism $\iota : (\mathcal{X}, \mathcal{D}) \longrightarrow (\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})$. The pull back of $\widetilde{E}$ is denoted by E. Let $\mathbb{D}^f$ denote the restriction of $\iota^*\nabla$ to the X-direction. We set $\mathbb{D} := \lambda \cdot \mathbb{D}^f$. Note the following:

- $\mathbb{D}(E) \subset E \otimes p_\lambda^* \Omega^1_X(*D)$, i.e., $\mathbb{D}$ gives a family of meromorphic λ-connections of E.
- $(E, \mathbb{D})$ is a good lattice in the level $(\boldsymbol{m}, i(0))$ on $(\mathcal{X}, W)$, and (31) naturally induces the irregular decomposition of $(E, \mathbb{D})_{|\widehat{W}}$.

By applying a general theory explained in Subsection 5.1.2, for each $\mathfrak{a} \in \mathcal{I}$, we obtain $\mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E, \mathbb{D})$.

Let $\widetilde{S}$ be a small sector in $\widetilde{\mathcal{X}} - \widetilde{W}$. We have the Stokes filtration $\mathcal{F}^{\widetilde{S}}$ of $\widetilde{E}_{|\overline{\widetilde{S}}}$ in the level $\widetilde{\boldsymbol{m}}$ indexed by $(\widetilde{\mathcal{I}}, \leq_{\widetilde{S}})$ (Proposition 5.20). For $S := \iota^{-1}(\widetilde{S})$, we have the Stokes filtration $\mathcal{F}^S$ of $E_{|\overline{S}}$ in the level $\boldsymbol{m}$ indexed by $(\mathcal{I}, \leq_S)$. We remark the following.

Lemma 5.22 *Under the natural identification $\widetilde{\mathcal{I}} = \mathcal{I}$, the orders $\leq_{\widetilde{S}}$ and $\leq_S$ are the same. Under the natural isomorphism $E \simeq \iota^*\widetilde{E}$, the filtrations $\mathcal{F}^S$ and $\mathcal{F}^{\widetilde{S}}$ are the same.*

Proof For the order $\leq_{\widetilde{S}}$, we use the identification $\widetilde{\mathcal{X}} = \{1\} \times \widetilde{\mathcal{X}} \subset \boldsymbol{C}_\lambda \times \widetilde{\mathcal{X}}$. Then, the first claim is clear. Note that both $\iota^*\widetilde{\mathcal{F}}^{\widetilde{S}}$ and $\mathcal{F}^S$ satisfy the condition in Proposition 5.5. Hence, they are the same. ∎

Corollary 5.23 *We have the natural isomorphism $\iota^* \mathrm{Gr}^{\widetilde{\boldsymbol{m}}}_{\widetilde{\mathfrak{a}}}(\widetilde{E}) \simeq \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)$, and $\mathbb{D}_{\mathfrak{a}}$ is induced by $\iota^*\nabla_{\widetilde{\mathfrak{a}}}$ via the above procedure.*

Proof By Lemma 5.22, we obtain the isomorphism $j : \iota^* \mathrm{Gr}^{\widetilde{\boldsymbol{m}}}_{\widetilde{\mathfrak{a}}}(\widetilde{E})_{|\mathcal{X}-W} \simeq \mathrm{Gr}^{\boldsymbol{m}}_{\mathfrak{a}}(E)_{|\mathcal{X}-W}$, on which $\mathbb{D}_{\mathfrak{a}}$ is induced by $\nabla_{\mathfrak{a}}$ via the above procedure. Since j is extended on $\widetilde{\mathcal{X}}(W)$, it is extended on $\mathcal{X}$. ∎

5.1.5 Stokes filtration of the associated flat bundle on the real blow up

We use the setting in Subsection 5.1.3. Let $\mathcal{I} \subset M(X,D)$ be a good set of irregular values in the level $(\boldsymbol{m}, i(0))$. Let E be a holomorphic vector bundle on X with a meromorphic flat connection $\nabla : E \longrightarrow E \otimes \Omega^1_X(*D)$ such that (E, ∇) is a pseudo-good lattice in the level $(\boldsymbol{m}, i(0))$. (In other words, we consider a family of meromorphic λ-flat bundles on $\{1\} \times (X, D)$.) Let $\pi : \widetilde{X}(D) \longrightarrow X$ be a real blow up of X along D. The flat bundle $E_{|X-D}$ is naturally extended to the flat bundle $\mathfrak{V}$ on $\widetilde{X}(D)$.

We set $\mathfrak{Z} := \pi^{-1}(O_z \times Y)$. For each $P \in \mathfrak{Z}$, we take a small sector $S \in \mathcal{MS}(P, X-D, \mathcal{I})$ on which we have the Stokes filtration $\mathcal{F}^S$ of $E_{|S}$. The filtration is naturally extended to the flat filtration of $\mathfrak{V}_{|\overline{S}}$. By restricting it to the fiber $\mathfrak{V}_{|P}$, we obtain the filtration $\mathcal{F}^P$ indexed by $\bigl(\mathcal{I}, \leq_P\bigr)$. It is easy to observe that $\mathcal{F}^P$ is well defined.

If $Q \in \pi^{-1}(\mathfrak{Z})$ is sufficiently close to P, the map $\bigl(\mathcal{I}, \leq_P\bigr) \longrightarrow \bigl(\mathcal{I}, \leq_Q\bigr)$ preserves the orders, and the filtrations $\mathcal{F}^P$ and $\mathcal{F}^Q$ are compatible under the identification $\mathfrak{V}_{|P} \simeq \mathfrak{V}_{|Q}$ given by the parallel transport in $\overline{S}_P$. In particular, we have $\mathcal{F}^P = \mathcal{F}^Q$ if $\leq_P = \leq_Q$.

We have the functoriality of the filtrations $\mathcal{F}^P$ for dual, tensor product and direct sum as in the case of $\mathcal{F}^S$.

Lemma 5.24 *Let $F : (E_1, \nabla_1) \longrightarrow (E_2, \nabla_2)$ be a flat morphism. For simplicity, we assume that $\mathcal{I}_1 \cup \mathcal{I}_2$ is a good set of irregular values in the level $(\boldsymbol{m}, i(0))$. The induced morphism $F_{|P} : \mathfrak{V}_{1|P} \longrightarrow \mathfrak{V}_{2|P}$ preserves the Stokes filtrations $\mathcal{F}^P$.* ∎

Remark 5.25 *We considered two vector bundles on $\widetilde{X}(D)$. One is $\pi^{-1}(E)$ and the other is $\mathfrak{V}$. We should emphasize that they are different in general. The bundle $\mathfrak{V}$ depends only on the flat bundle $(E, \nabla)_{|X-D}$, and $\pi^{-1}(E)$ depends on the prolongment (E, ∇).*

Let us see the simplest example $E = \mathcal{O} \cdot e$ and $\nabla(e) = e \cdot d(z^{-1})$. A trivialization of $\pi^{-1}(E)$ is given by $\pi^{-1}(e)$. A trivialization of $\mathfrak{V}$ is induced by $\exp(-z^{-1}) \cdot e$. ∎

5.2 Unramifiedly good lattices of a family of meromorphic λ-flat bundles

5.2.1 Preliminary

Good set of irregular values We use the partial order $\leq_{\mathbb{Z}^n}$ of $\mathbb{Z}^n$ given by $\boldsymbol{a} \leq_{\mathbb{Z}^n} \boldsymbol{b} \Longleftrightarrow a_i \leq b_i$, $(\forall i)$. We say $\boldsymbol{a} <_{\mathbb{Z}^n} \boldsymbol{b}$ in the case $a_i < b_i$ for any i, and we say $\boldsymbol{a} \lneq_{\mathbb{Z}^n} \boldsymbol{b}$ in the case $\boldsymbol{a} \leq_{\mathbb{Z}^n} \boldsymbol{b}$ and $\boldsymbol{a} \neq \boldsymbol{b}$. Let $\boldsymbol{\delta}_j$ denote the element $(\overbrace{0, \ldots, 0}^{j-1}, 1, 0, \ldots, 0)$, and let $\boldsymbol{0}$ denote the zero in $\mathbb{Z}^n$. We also use $\boldsymbol{0}_n$ when we distinguish the dependence on n.

Let Y be a complex manifold. Let $X := \Delta^\ell \times Y$. Let $D_i := \{z_i = 0\} \times Y$ and $D := \bigcup_{i=1}^{\ell} D_i$ be the hypersurfaces of X. We also put $D_{\underline{\ell}} = \bigcap_{i=1}^{\ell} D_i$, which is naturally identified with Y.

For any $f \in M(X, D)$, we have the Laurent expansion:

$$f = \sum_{\boldsymbol{m} \in \mathbb{Z}^\ell} f_{\boldsymbol{m}}(y) \cdot \boldsymbol{z}^{\boldsymbol{m}}.$$

Here $f_{\boldsymbol{m}}$ are holomorphic functions on $D_{\underline{\ell}}$. We often use the following identification implicitly:

$$M(X, D) \big/ \boldsymbol{z}^{\boldsymbol{n}} \cdot H(X) \simeq \Bigl\{ f \in M(X, D) \Bigm| f_{\boldsymbol{m}} = 0,\ \forall \boldsymbol{m} \geq \boldsymbol{n} \Bigr\} \tag{32}$$

For any $f \in M(X, D)$, let $\mathrm{ord}(f)$ denote the minimum of the set $\bigl\{\boldsymbol{m} \in \mathbb{Z}^\ell \bigm| f_{\boldsymbol{m}} \neq 0\bigr\} \cup \{\boldsymbol{0}\}$ with respect to $\leq_{\mathbb{Z}^\ell}$, if it exists. It is always contained in $\mathbb{Z}^\ell_{\leq 0}$, if it exists.

For any $\mathfrak{a} \in M(X, D)/H(X)$, we take any lift $\widetilde{\mathfrak{a}}$ to $M(X, D)$, and we set $\mathrm{ord}(\mathfrak{a}) := \mathrm{ord}(\widetilde{\mathfrak{a}})$, if the right hand side exists. If $\mathrm{ord}(\mathfrak{a})$ exists in $\mathbb{Z}^\ell - \{\boldsymbol{0}\}$, $\widetilde{\mathfrak{a}}_{\mathrm{ord}(\mathfrak{a})}$ is independent of the choice of a lift $\widetilde{\mathfrak{a}}$, which is denoted by $\mathfrak{a}_{\mathrm{ord}(\mathfrak{a})}$.

Definition 5.26 *A finite subset $\mathcal{I} \subset M(X, D)\big/H(X)$ is called a good set of irregular values on (X, D), if the following conditions are satisfied:*

- $\mathrm{ord}(\mathfrak{a})$ *exists for each $\mathfrak{a} \in \mathcal{I}$, and $\mathfrak{a}_{\mathrm{ord}(\mathfrak{a})}$ is nowhere vanishing on $D_{\underline{\ell}}$ for $\mathfrak{a} \neq 0$.*
- *For any two distinct $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$, $\mathrm{ord}(\mathfrak{a} - \mathfrak{b})$ exists in $\mathbb{Z}^\ell_{\leq 0} - \{\boldsymbol{0}\}$, and $(\mathfrak{a} - \mathfrak{b})_{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}$ is nowhere vanishing on $D_{\underline{\ell}}$.*
- *The set $\mathcal{T}(\mathcal{I}) := \bigl\{\mathrm{ord}(\mathfrak{a} - \mathfrak{b}) \bigm| \mathfrak{a}, \mathfrak{b} \in \mathcal{I}\bigr\}$ is totally ordered with respect to the partial order on $\mathbb{Z}^\ell$.* ∎

The condition in Definition 5.26 does not depend on the choice of a holomorphic coordinate such that $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$.

We will use the following lemma implicitly.

Lemma 5.27 *The set $\{\mathrm{ord}(\mathfrak{a}) \,|\, \mathfrak{a} \in \mathcal{I}\}$ is totally ordered. In particular, the minimum*

$$\boldsymbol{m}(0) := \min\{\mathrm{ord}(\mathfrak{a}) \,|\, \mathfrak{a} \in \mathcal{I}\}$$

exists. Moreover, $\boldsymbol{m}(0) \leq_{\mathbb{Z}^\ell} \boldsymbol{m}$ for any $\boldsymbol{m} \in \mathcal{T}(\mathcal{I})$.

Proof Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$. Assume $\mathrm{ord}(\mathfrak{a}) \not\leq \mathrm{ord}(\mathfrak{b})$ and $\mathrm{ord}(\mathfrak{a}) \not\geq \mathrm{ord}(\mathfrak{b})$. Then, $\mathrm{ord}(\mathfrak{a}-\mathfrak{b})$ does not exist, which contradicts the second condition. Hence, we obtain the first claim of the lemma. For any $\boldsymbol{m} \in \mathcal{T}(\mathcal{I})$, there exists $\mathfrak{a} \in \mathcal{I}$ such that $\mathfrak{a}_{\boldsymbol{m}} \neq 0$. Hence, $\boldsymbol{m}(0) \leq_{\mathbb{Z}^\ell} \boldsymbol{m}$. ∎

Remark 5.28 *It is often convenient to use a coordinate such that $\mathcal{T}(\mathcal{I}) \cup \{\boldsymbol{m}(0)\} \subset \coprod_{i=0}^{\ell} \mathbb{Z}_{<0}^{i} \times \boldsymbol{0}_{\ell-i}$.* ∎

Auxiliary sequence Let $\mathcal{I}$ be a good set of irregular values on (X, D). Since the set $\mathcal{T}(\mathcal{I})$ is totally ordered with respect to the partial order $\leq_{\mathbb{Z}^\ell}$, we can take a sequence

$$\mathcal{M} := \big(\boldsymbol{m}(0), \boldsymbol{m}(1), \boldsymbol{m}(2), \ldots, \boldsymbol{m}(L), \boldsymbol{m}(L+1)\big) \subset \mathbb{Z}_{\leq 0}^{\ell}$$

with the following property:

- $\mathcal{T}(\mathcal{I}) \subset \mathcal{M}$ and $\boldsymbol{m}(L+1) = \boldsymbol{0}_\ell$.
- We have $1 \leq \mathfrak{h}(i) \leq \ell$ such that $\boldsymbol{m}(i+1) = \boldsymbol{m}(i) + \boldsymbol{\delta}_{\mathfrak{h}(i)}$ for each $i \leq L$.

Such a sequence is called an auxiliary sequence for $\mathcal{I}$. It is not uniquely determined for $\mathcal{I}$. It is convenient for an inductive argument.

Truncation Let $\mathcal{I}$ be a good set of irregular values. We take an auxiliary sequence for $\mathcal{I}$, and let $\overline{\eta}_{\boldsymbol{m}(0)} : \mathcal{I} \longrightarrow M(X, D)/H(X)$ be given as follows:

$$\overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}) := \sum_{\boldsymbol{n} \not\geq \boldsymbol{m}(1)} \mathfrak{a}_{\boldsymbol{n}} \cdot \boldsymbol{z}^{\boldsymbol{n}}$$

Then, the image is a good set of irregular values in the level $(\boldsymbol{m}(0), i(0))$. More generally, $\overline{\eta}_{\boldsymbol{m}(j)}$ is defined as follows:

$$\overline{\eta}_{\boldsymbol{m}(j)}(\mathfrak{a}) := \sum_{\boldsymbol{n} \not\geq \boldsymbol{m}(j)} \mathfrak{a}_{\boldsymbol{n}} \cdot \boldsymbol{z}^{\boldsymbol{n}}$$

We have $\overline{\eta}_{\boldsymbol{m}(L)}(\mathfrak{a}) = \mathfrak{a}$. We set $\zeta_{\boldsymbol{m}(0)}(\mathfrak{a}) := \overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a})$ and $\zeta_{\boldsymbol{m}(j)}(\mathfrak{a}) := \overline{\eta}_{\boldsymbol{m}(j)}(\mathfrak{a}) - \overline{\eta}_{\boldsymbol{m}(j-1)}(\mathfrak{a})$ for $j = 1, \ldots, L$. Then, we have the decomposition $\overline{\eta}_{\boldsymbol{m}(i)}(\mathfrak{a}) = \sum_{j \leq i} \zeta_{\boldsymbol{m}(j)}(\mathfrak{a})$.

Let $\mathcal{I}(\boldsymbol{m}(i))$ denote the image of $\overline{\eta}_{\boldsymbol{m}(i)} : \mathcal{I} \longrightarrow M(X, D)/H(X)$.

Lemma 5.29 *If we shrink X appropriately, $\mathcal{I}(\boldsymbol{m}(0))$ is a good set of irregular values in the level $\big(\boldsymbol{m}(0), \mathfrak{h}(0)\big)$.*

Proof If $\overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}-\mathfrak{b}) \neq 0$ for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$, we have $\mathrm{ord}(\mathfrak{a}-\mathfrak{b}) = \boldsymbol{m}(0)$ and $\big(\boldsymbol{z}^{-\boldsymbol{m}(0)}\overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}-\mathfrak{b})\big)_{|D_\ell}$ is nowhere vanishing. Hence, $\big(\boldsymbol{z}^{-\boldsymbol{m}(0)}\overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}-\mathfrak{b})\big)$ is nowhere vanishing on X after X is shrinked appropriately. Similarly, we may have $\big(\boldsymbol{z}^{-\boldsymbol{m}(0)}\overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a})\big)$ is nowhere vanishing on X after X is shrinked appropriately. ∎

We can use the following lemma for inductive arguments.

Lemma 5.30 *For any $\mathfrak{b} \in \mathcal{I}(\boldsymbol{m}(0))$, we fix any element $\mathfrak{a}^{(0)} \in \overline{\eta}_{\boldsymbol{m}(0)}^{-1}(\mathfrak{b})$. Then, the set*

$$\{\mathfrak{a} - \mathfrak{a}^{(0)} \,|\, \overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}) = \mathfrak{b}\}$$

is also a good set of irregular values. ∎

Example We give some examples.

$$\mathfrak{a}^{(1)} := z_1^{-1} \cdot z_2^{-1}, \quad \mathfrak{a}^{(2)} := z_1^{-1}, \quad \mathfrak{a}^{(3)} := 0.$$

An auxiliary sequence is unique in this case, and given as follows:

$$\boldsymbol{m}(0) = (-1,-1),\ \mathfrak{h}(0) = 2, \quad \boldsymbol{m}(1) = (-1,0),\ \mathfrak{h}(1) = 1, \quad \boldsymbol{m}(2) = (0,0) \tag{33}$$

The truncations are given as follows:

$$\overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}^{(1)}) = \mathfrak{a}^{(1)}, \quad \overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}^{(2)}) = 0, \quad \overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{a}^{(3)}) = 0$$

$$\overline{\eta}_{\boldsymbol{m}(1)}(\mathfrak{a}^{(1)}) = \mathfrak{a}^{(1)}, \quad \overline{\eta}_{\boldsymbol{m}(1)}(\mathfrak{a}^{(2)}) = \mathfrak{a}^{(2)}, \quad \overline{\eta}_{\boldsymbol{m}(1)}(\mathfrak{a}^{(3)}) = \mathfrak{a}^{(3)}$$

The image of $\mathcal{I}$ via $\overline{\eta}_{\boldsymbol{m}(0)}$ is $\{\mathfrak{a}^{(1)}, 0\}$.

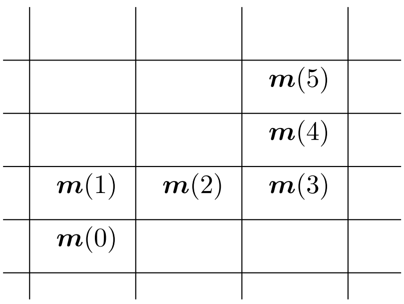
Let us consider the following set:

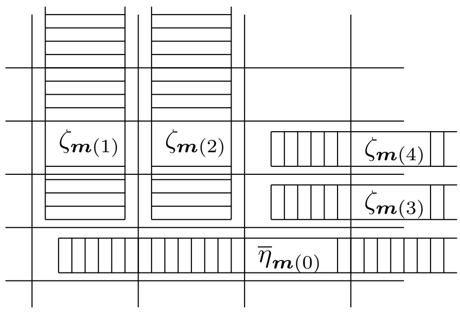
$$\mathfrak{b}^{(1)} = z_1^{-1} \cdot z_2^{-1} + a \cdot z_2^{-1} + b \cdot z_1^{-1}, \quad \mathfrak{b}^{(2)} = z_1^{-1}$$

An auxiliary sequence is given by (33). The truncation is given as follows:

$$\overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{b}^{(1)}) = z_1^{-1} \cdot z_2^{-1} + a \cdot z_2^{-1}, \quad \overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{b}^{(2)}) = 0$$

We have the following picture in our mind for truncation.





$L = 4$, $\boldsymbol{m}(0) = (-2,-3)$, $\boldsymbol{m}(1) = (-2,-2)$, $\boldsymbol{m}(2) = (-1,-2)$,
$\boldsymbol{m}(3) = (0,-2)$, $\boldsymbol{m}(4) = (0,-1)$, $\boldsymbol{m}(5) = (0,0)$.

5.2.2 Unramifiedly good lattices of a family of meromorphic λ-flat bundles

Let X be a complex manifold, and let D be a normal crossing divisor of X. Let $\mathcal{K}$ be a point or a compact region in $\boldsymbol{C}_\lambda$. Let $\mathcal{X}$ and $\mathcal{D}$ denote $\mathcal{K} \times X$ and $\mathcal{K} \times D$, respectively. For $\lambda \in \mathcal{K}$, we set $\mathcal{X}^\lambda := \{\lambda\} \times X$ and $\mathcal{D}^\lambda := \{\lambda\} \times D$. Let $(\mathcal{E}, \mathbb{D})$ be a family of meromorphic λ-flat bundles on $(\mathcal{X}, \mathcal{D})$, i.e., $\mathcal{E}$ is an $\mathcal{O}_\mathcal{X}(*\mathcal{D})$-coherent sheaf with a holomorphic family of flat λ-connections $\mathbb{D} : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{\mathcal{X}/\mathcal{K}}$. The restriction to $(\mathcal{X}^\lambda, \mathcal{D}^\lambda)$ is denoted by $(\mathcal{E}^\lambda, \mathbb{D}^\lambda)$.

Remark 5.31 *If $\mathcal{K}$ is a point, "family" can be omitted.* ∎

Let E be an $\mathcal{O}_\mathcal{X}$-locally free lattice of $(\mathcal{E}, \mathbb{D})$. Let P be any point of $\mathcal{D}$. We can take a holomorphic coordinate $(\mathcal{U}, \lambda, z_1, \ldots, z_n)$ around P such that $\mathcal{D}_\mathcal{U} := \mathcal{D} \cap \mathcal{U} = \bigcup_{i=1}^{\ell} \mathcal{D}_{\mathcal{U},i}$, where $\mathcal{D}_{\mathcal{U},i} := \{z_i = 0\}$. We put $\mathcal{D}_{\mathcal{U},I} := \bigcap_{i\in I} \mathcal{D}_{\mathcal{U},i}$ and $\mathcal{D}_\mathcal{U}(I) := \bigcup_{i\in I} \mathcal{D}_{\mathcal{U},i}$. For any subset $I \subset \underline{\ell}$, we put $I^c := \underline{\ell} - I$. The completion of $\mathcal{X}$ along $\mathcal{D}_{\mathcal{U},I}$ (resp. $\mathcal{D}_\mathcal{U}(I)$) is denoted by $\widehat{\mathcal{D}}_{\mathcal{U},I}$ (resp. $\widehat{\mathcal{D}}_\mathcal{U}(I)$).

Definition 5.32 *We say that E is unramifiedly good at P, if the following holds:*

- *We are given a good set of irregular values $S \subset M(\mathcal{U},\mathcal{D}_{\mathcal{U}})/H(\mathcal{U})$.*
- *For any $\emptyset \neq I \subset \underline{\ell}$, we have the decomposition:*

$$(E,\mathbb{D})_{|\widehat{\mathcal{D}}_{\mathcal{U},I}} = \bigoplus_{\mathfrak{a}\in S(I)} \bigl({}^{I}\widehat{E}_{\mathfrak{a}}, {}^{I}\widehat{\mathbb{D}}_{\mathfrak{a}}\bigr) \tag{34}$$

  Here $S(I)$ denotes the image of S via the map $M(\mathcal{U},\mathcal{D}_{\mathcal{U}})/H(\mathcal{U}) \longrightarrow M(\mathcal{U},\mathcal{D}_{\mathcal{U}})/M\bigl(\mathcal{U},\mathcal{D}_{\mathcal{U}}(I^c)\bigr)$.

- *$(\mathbb{D}_{\mathfrak{a}}-d\mathfrak{a})\bigl({}^{I}\widehat{E}_{\mathfrak{a}}\bigr)$ is contained in ${}^{I}\widehat{E}_{\mathfrak{a}}\otimes\Bigl(\Omega^1_{\mathcal{X}/\mathcal{K}}\bigl(\log\mathcal{D}_{\mathcal{U}}(I)\bigr)+\Omega^1_{\mathcal{X}/\mathcal{K}}\bigl(*\mathcal{D}_{\mathcal{U}}(I^c)\bigr)\Bigr)$, where $\mathfrak{a}$ is lifted to $M(\mathcal{U},\mathcal{D}_{\mathcal{U}})$. This condition is independent of the choice of a lift.*

The property is independent of the choice of the coordinate $(\mathcal{U},\lambda,z_1,\ldots,z_n)$.

We say that $(E,\mathbb{D})$ is unramifiedly good, if it is unramifiedly good at any point. ∎

See Subsection 5.7 of [19] for another but equivalent formulation, which seems easier to state.

The decomposition (34) is called the irregular decomposition of $E_{|\widehat{\mathcal{D}}_{\mathcal{U},I}}$. The set S is uniquely determined if ${}^{\underline{\ell}}E_{\mathfrak{a}} \neq 0$ for each $\mathfrak{a}\in S$. So, it is denoted by $\mathrm{Irr}(\mathbb{D},P)$. The restriction of E to $\{\lambda\}\times X$ is denoted by E^{λ}.

If E is an unramifiedly good lattice of $(\mathcal{E},\mathbb{D})$, we have the well defined endomorphism $\mathrm{Res}_i(\mathbb{D})$ of $E_{|\mathcal{D}_i}$ for each irreducible component D_i of D. It is called the residue of $\mathbb{D}$ at D_i with respect to the lattice E. If $\mathcal{K}\neq\{0\}$, the eigenvalues of $\mathrm{Res}_i(\mathbb{D})$ are constant on $\mathcal{D}_i^{\lambda}$ for each $\lambda\in\mathcal{K}$. (See Subsection 5.1.3 of [19], for example.)

Remark 5.33 *We have the notion of good lattice which is locally a descent of an unramifiedly good lattice. See* [19]. *See also Definition* 5.42 *below.* ∎

Irregular decompositions in the level $\boldsymbol{m}(j)$ In the following, let $X:=\Delta^n$, $D_i:=\{z_i=0\}$ and $D:=\bigcup_{i=1}^{\ell}D_i$. We set $\mathcal{D}(\leq p):=\bigcup_{i\leq p}\mathcal{D}_i$. Let $(E,\mathbb{D})$ be an unramifiedly good lattice of a family of meromorphic λ-flat bundles on $(\mathcal{X},\mathcal{D})$ with the good set $\mathrm{Irr}(\mathbb{D})=\mathrm{Irr}(\mathbb{D},O)$. We assume that the coordinate is as in Remark 5.28 for $\mathrm{Irr}(\mathbb{D})$. Let $\mathrm{Irr}(\mathbb{D},p)$ and $\mathrm{Irr}'(\mathbb{D},p)$ denote the image of $\mathrm{Irr}(\mathbb{D})$ by the natural maps

$$\pi_p : M(\mathcal{X},\mathcal{D})/H(\mathcal{D}) \longrightarrow M(\mathcal{X},\mathcal{D})/M(\mathcal{X},\mathcal{D}(\leq p-1)),\quad \pi'_p : M(\mathcal{X},\mathcal{D})/H(\mathcal{D}) \longrightarrow M(\mathcal{X},\mathcal{D})/M(\mathcal{X},\mathcal{D}(\neq p)).$$

Note that the naturally induced map $\mathrm{Irr}(\mathbb{D},p)\longrightarrow\mathrm{Irr}'(\mathbb{D},p)$ is bijective, via which we identify them.

Take an auxiliary sequence $\boldsymbol{m}(0),\ldots,\boldsymbol{m}(L)$ for the good set $\mathrm{Irr}(\mathbb{D})$. Let $\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(0))$ denote the image of $\mathrm{Irr}(\mathbb{D})$ via $\overline{\eta}_{\boldsymbol{m}(0)}$. Let $k(j)$ denote the number determined by $\boldsymbol{m}(j)\in\mathbb{Z}^{k(j)}_{<0}\times\boldsymbol{0}_{\ell-k(j)}$. For $p\leq k(j)$, we have the map $\mathrm{Irr}(\mathbb{D},p)\longrightarrow M(\mathcal{X},\mathcal{D})/M(\mathcal{X},\mathcal{D}(\leq p-1))$ induced by $\overline{\eta}_{\boldsymbol{m}(j)}$ which is denoted by $\overline{\eta}_{\boldsymbol{m}(j),p}$.

By using a lemma in Subsection 5.1.2 of [19] and the uniqueness of the decompositions, we obtain the following decomposition on the completion $\widehat{\mathcal{D}}(\leq k(j))$ along $\mathcal{D}(\leq k(j))$:

$$(E,\mathbb{D})_{|\widehat{\mathcal{D}}(\leq k(j))} = \bigoplus_{\mathfrak{b}\in\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(j))}\bigl(\widehat{E}^{\boldsymbol{m}(j)}_{\mathfrak{b}},\mathbb{D}_{\mathfrak{b}}\bigr),\quad\text{where}\quad \widehat{E}^{\boldsymbol{m}(j)}_{\mathfrak{b}|\widehat{\mathcal{D}}_p} = \bigoplus_{\substack{\mathfrak{c}\in\mathrm{Irr}(\mathbb{D},p)\\ \overline{\eta}_{\boldsymbol{m}(j),p}(\mathfrak{c})=\pi_p(\mathfrak{b})}} {}^{p}\widehat{E}_{\mathfrak{c}},\ \ \bigl(p\leq k(j)\bigr) \tag{35}$$

The decomposition (35) is called the irregular decomposition in the level $\boldsymbol{m}(j)$.

Remark 5.34 *We do not have the irregular decomposition in the level $\boldsymbol{m}(j)$ on $\widehat{D}$ in general, which Sabbah remarked in* [20] *for the surface case.* ∎

The associated graded bundles with the family of meromorphic flat λ-connections Assume $\mathcal{K}\neq\{0\}$. We set $W:=\mathcal{X}^0\cup\mathcal{D}(\leq k(0))$. It is easy to observe that $(E,\mathbb{D})$ is an unramifiedly good lattice in the level $\bigl(\boldsymbol{m}(0),i(0)\bigr)$ with the decomposition (35) for $j=0$. The set of the irregular values in the level $\bigl(\boldsymbol{m}(0),i(0)\bigr)$ is $\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(0))$.

As stated in Subsection 5.1.2, we obtain the holomorphic bundle $\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(E)$ with a family of meromorphic flat λ-connections $\mathbb{D}^{\boldsymbol{m}(0)}_{\mathfrak{a}}$ on $(\mathcal{V},\mathcal{V}\cap\mathcal{D})$ for each $\mathfrak{a}\in\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(0))$, where $\mathcal{V}$ denotes a neighbourhood of $\bigcap_{1\leq i\leq k(0)}\mathcal{D}_i$.

Let $\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(E,\mathbb{D}) := \big(\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(E), \mathbb{D}^{\boldsymbol{m}(0)}_{\mathfrak{a}}\big)$. We obtain the following isomorphisms for any $\mathfrak{a} \in \overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(0))$ from (30):

$$\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(E,\mathbb{D})_{|\widehat{W}} \simeq (\widehat{E}^{\boldsymbol{m}(0)}_{\mathfrak{a}}, \mathbb{D}_{\mathfrak{a}})$$

In particular, $\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(E,\mathbb{D})$ are unramifiedly good lattices whose set of irregular values is $\mathrm{Irr}(\mathbb{D}^{\boldsymbol{m}(0)}_{\mathfrak{a}}) = \overline{\eta}^{-1}_{\boldsymbol{m}(0)}(\mathfrak{a})$.

Let $\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(j))$ denote the image of $\overline{\eta}_{\boldsymbol{m}(j)} : \mathrm{Irr}(\mathbb{D}) \longrightarrow M(X,D)/H(X)$ for any j. Let us consider the case in which $\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(j-1))$ consists of a unique element. We take any element $\mathfrak{a}^{(1)} \in \mathrm{Irr}(\mathbb{D})$. Let $\mathcal{L}(\pm\mathfrak{a}^{(1)})$ be a line bundle $\mathcal{O}_{\mathcal{X}} \cdot e$ with a family of meromorphic flat λ-connections $\mathbb{D}e = e \cdot (\pm d\mathfrak{a}^{(1)})$. Then, $(E',\mathbb{D}') := (E,\mathbb{D}) \otimes \mathcal{L}(-\mathfrak{a}^{(1)})$ is an unramifiedly good lattice with the good set

$$\mathrm{Irr}(\mathbb{D}') = \big\{\mathfrak{a} - \mathfrak{a}^{(1)} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\mathbb{D})\big\}.$$

The sequence $\boldsymbol{m}(j), \boldsymbol{m}(j+1), \ldots, \boldsymbol{m}(L)$ gives an auxiliary sequence for $\mathrm{Irr}(\mathbb{D}')$. By applying the above procedure to $(E',\mathbb{D}')$ and shrinking X, we obtain $\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{c}}(E',\mathbb{D}')$ for each $\mathfrak{c} \in \overline{\mathrm{Irr}}(\mathbb{D}',\boldsymbol{m}(j))$. For any $\mathfrak{b} \in \overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(j))$, we define

$$\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{b}}(E,\mathbb{D}) := \mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{b}-\overline{\eta}_{\boldsymbol{m}(j)}(\mathfrak{a}^{(1)})}(E',\mathbb{D}') \otimes \mathcal{L}(\mathfrak{a}^{(1)})$$

It is independent of the choice of $\mathfrak{a}^{(1)}$ up to canonical isomorphisms. (We may avoid tensor products.) It is easy to observe that $\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{b}}(E,\mathbb{D})$ are also unramifiedly good lattices with the good sets of irregular values $\mathrm{Irr}(\mathbb{D}^{\boldsymbol{m}(j)}_{\mathfrak{b}}) = \overline{\eta}^{-1}_{\boldsymbol{m}(j)}(\mathfrak{b})$. By construction, $\overline{\mathrm{Irr}}\big(\mathbb{D}^{\boldsymbol{m}(j)}_{\mathfrak{b}}, \boldsymbol{m}(j)\big)$ consists of the unique element $\mathfrak{b}$.

Let us consider the general case. Let $\overline{\eta}_{\boldsymbol{m}(j-1),\boldsymbol{m}(j)} : \overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(j)) \longrightarrow \overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(j-1))$ be the induced map. For any $\mathfrak{a} \in \overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(j))$, we inductively define

$$\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{a}}(E,\mathbb{D}) := \mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{a}}\,\mathrm{Gr}^{\boldsymbol{m}(j-1)}_{\overline{\eta}_{\boldsymbol{m}(j-1),\boldsymbol{m}(j)}(\mathfrak{a})}(E,\mathbb{D})$$

For each $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D})$, we set $\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(E,\mathbb{D}) := \mathrm{Gr}^{\boldsymbol{m}(L)}_{\mathfrak{a}}(E,\mathbb{D})$, which is called the full reduction. By construction, $\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(E,\mathbb{D}) \otimes \mathcal{L}(-\mathfrak{a})$ is logarithmic.

We have the functoriality as in Subsection 5.1.2.

Deformation Assume $0 \notin \mathcal{K}$. We would like to regard $(E,\mathbb{D})$ as a prolongment of $(E,\mathbb{D})_{|\mathcal{X}-\mathcal{D}(\leq k(0))}$. For a given holomorphic function $T(\lambda)$ with $\mathrm{Re}\big(T(\lambda)\big) > 0$, we have the other prolongment $(E^{(T)},\mathbb{D}^{(T)})$ of $(E,\mathbb{D})_{|\mathcal{X}-\mathcal{D}(\leq k(0))}$, which is also an unramifiedly good lattice with the set of irregular values

$$\mathrm{Irr}\big(E^{(T)},\mathbb{D}^{(T)}\big) := \big\{T\cdot\mathfrak{a} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\mathbb{D})\big\}.$$

We refer to Subsections 7.5, 7.8–7.9 of [19] for the construction. We mention some properties (Subsection 7.8 of [19]):

(D1): $E^{(T_1\cdot T_2)} \simeq \big(E^{(T_1)}\big)^{(T_2)}$, if $\mathrm{Re}(T_i) > 0$ and $\mathrm{Re}(T_1 \cdot T_2) > 0$.

(D2): $(E^{(T)},\mathbb{D}^{(T)})_{|\widehat{\mathcal{D}}_{I_0}} \simeq \bigoplus_{\mathfrak{a}\in\mathcal{I}}\big({}^{I_0}\widehat{E}_{\mathfrak{a}}, {}^{I_0}\widehat{\mathbb{D}}_{\mathfrak{a}} + (T-1)d\mathfrak{a}\big)$, where $I_0 := \{1,\ldots,k(0)\}$. In other brief words, the deformation does not change the regular part.

We give some statements for functoriality. See Subsection 7.8.1 of [19] for more details. Let $(E_p,\mathbb{D}_p)$ $(p = 1,2)$ be unramifiedly good. We have the following natural isomorphisms:

$$(E_1 \oplus E_2)^{(T)} \simeq E_1^{(T)} \oplus E_2^{(T)}, \quad (E_1 \otimes E_2)^{(T)} \simeq E_1^{(T)} \otimes E_2^{(T)}, \quad \big(E^\vee\big)^{(T)} \simeq (E^{(T)})^\vee$$

Here, we have assumed that $(E_1,\mathbb{D}_1) \oplus (E_2,\mathbb{D}_2)$ and $(E_1,\mathbb{D}_1) \otimes (E_2,\mathbb{D}_2)$ are unramifiedly good. Moreover, let $F : (E_1,\mathbb{D}_1) \longrightarrow (E_2,\mathbb{D}_2)$ be a flat morphism. Assume $\mathcal{I}_1 \cup \mathcal{I}_2$ is a good set of irregular values in the level $(\boldsymbol{m}, i(0))$. Then, we have the naturally induced morphism $(E_1^{(T)},\mathbb{D}_1^{(T)}) \longrightarrow (E_2^{(T)},\mathbb{D}_2^{(T)})$.

5.3 Smooth divisor case

Let $X := \Delta^n$ and $D := \{z_1 = 0\}$. Let $\mathcal{K} \subset \boldsymbol{C}_\lambda$. Let $(E, \mathbb{D})$ be an unramifiedly good lattice of a family of meromorphic λ-bundles on $(\mathcal{X}, \mathcal{D})$ with a good set of irregular values $\mathrm{Irr}(\mathbb{D}) = \mathrm{Irr}(\mathbb{D}, O)$. We have the formal decomposition $(E, \mathbb{D})_{|\widehat{\mathcal{D}}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathbb{D})} (\widehat{E}_\mathfrak{a}, \widehat{\mathbb{D}}_\mathfrak{a})$, where $\widehat{\mathbb{D}}_\mathfrak{a} - d\mathfrak{a} \cdot \mathrm{id}_{\widehat{E}_\mathfrak{a}}$ are logarithmic. We set $W := \mathcal{D} \cup \mathcal{X}^0$ in the case $0 \in \mathcal{K}$, and $W := \mathcal{D}$ otherwise. We obtain the decomposition on $\widehat{W}$:

$$(E, \mathbb{D})_{|\widehat{W}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathbb{D})} (\widehat{E}_\mathfrak{a}, \widehat{\mathbb{D}}_\mathfrak{a}) \tag{36}$$

Full Stokes filtration In this case, it is also easy and convenient to consider full Stokes filtration. We explain it in the following. Let $\pi : \widetilde{\mathcal{X}}(W) \longrightarrow \mathcal{X}$ denote the real blow up of $\mathcal{X}$ along W. We put $\mathfrak{Z} := \pi^{-1}(\mathcal{D})$.

For any multi-sector S in $\mathcal{X} - W$, the order $\leq_S$ on $\mathrm{Irr}(\mathbb{D})$ is defined as follows:

- $\mathfrak{a} \leq_S \mathfrak{b}$ if and only if $-\mathrm{Re}\big(\lambda^{-1}\mathfrak{a}(\lambda, \boldsymbol{z})\big) \leq_S -\mathrm{Re}\big(\lambda^{-1}\mathfrak{b}(\lambda, \boldsymbol{z})\big)$ for any $\boldsymbol{z} \in S$ such that $|z_1|$ is sufficiently small.

Let $\overline{S}$ denote the closure of S in $\widetilde{\mathcal{X}}(W)$, and let Z denote $\overline{S} \cap \pi^{-1}(W)$. The irregular decomposition (36) on $\widehat{W}$ induces the decomposition on $\widehat{Z}$:

$$(E, \mathbb{D})_{|\widehat{Z}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathbb{D})} (\widehat{E}_\mathfrak{a}, \widehat{\mathbb{D}}_\mathfrak{a})_{|\widehat{Z}} \tag{37}$$

We put $\mathcal{F}^Z_\mathfrak{a} := \bigoplus_{\mathfrak{b} \leq_S \mathfrak{a}} \widehat{E}_{\mathfrak{b}|\widehat{Z}}$, and then we obtain the filtration $\mathcal{F}^Z$ indexed by $\big(\mathrm{Irr}(\mathbb{D}), \leq_S\big)$. By using Proposition 5.5 and Lemma 5.8 successively (or by using more classical results), we obtain the following.

Proposition 5.35 *For any point $P \in \mathfrak{Z}$, there exists $S \in \mathcal{MS}(P, \mathcal{X} - W)$ such that the following holds:*

- *There exists the unique $\mathbb{D}$-flat filtration $\widetilde{\mathcal{F}}^S$ of $E_{|\overline{S}}$ on $\overline{S}$ indexed by $\big(\mathrm{Irr}(\mathbb{D}), \leq_S\big)$ such that $\widetilde{\mathcal{F}}^S_{|\widehat{Z}} = \mathcal{F}^Z$.*
- *There exists a $\mathbb{D}$-flat splitting of $\widetilde{\mathcal{F}}^S$ on $\overline{S}$.*

We call $\widetilde{\mathcal{F}}^S$ the full Stokes filtration of $(E, \mathbb{D})$.

For $S' \subset S$, the filtrations $\widetilde{\mathcal{F}}^{S'}$ and $\widetilde{\mathcal{F}}^S$ satisfy the compatibility condition as in Lemma 5.8. ∎

The following lemma is clear from the definition of full Stokes filtrations.

Lemma 5.36 *Let $S, S' \in \mathcal{MS}(P, \mathcal{X} - W)$. Assume (i) $S' \subset S$, (ii) $E_{|\overline{S}}$ has the full Stokes filtration $\widetilde{\mathcal{F}}^S$ as above. Then, the restriction of $\widetilde{\mathcal{F}}^S$ to $\overline{S}'$ is the full Stokes filtration of $E_{|\overline{S}'}$.* ∎

We have functoriality of full Stokes filtrations as in the case of Stokes filtrations in the level $(\boldsymbol{m}, i(0))$.

The associated graded bundle For any sectors S and each $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D})$, we obtain the bundle $\mathrm{Gr}^{\mathrm{full}}_\mathfrak{a}(E_{|S})$ on S associated to the full Stokes filtration $\widetilde{\mathcal{F}}^S$. By varying S and gluing $\mathrm{Gr}^{\mathrm{full}}_\mathfrak{a}(E_{|\overline{S}})$, we obtain the bundle $\mathrm{Gr}^{\mathrm{full}}_\mathfrak{a}(E_{|\widetilde{\mathcal{V}}(W)})$ on $\widetilde{\mathcal{V}}(W)$ with the induced family of flat λ-connections $\mathbb{D}_\mathfrak{a}$, where $\mathcal{V}$ denotes some neighbourhood of $\mathcal{D}$, and $\widetilde{\mathcal{V}}(W)$ denote the real blow up of $\mathcal{V}$ along $W \cap \mathcal{V}$. As in Subsection 5.1.2, we can show that $\mathrm{Gr}^{\mathrm{full}}_\mathfrak{a}(E_{|\widetilde{\mathcal{V}}(W)})$ has the descent to $\mathcal{V}$, i.e., there exists a locally free sheaf $\mathrm{Gr}^{\mathrm{full}}_\mathfrak{a}(E)$ on $\mathcal{V}$ with a family of meromorphic flat λ-connections $\mathbb{D}_\mathfrak{a}$, such that

$$\pi^{-1}\big(\mathrm{Gr}^{\mathrm{full}}_\mathfrak{a}(E), \mathbb{D}_\mathfrak{a}\big) \simeq \big(\mathrm{Gr}^{\mathrm{full}}_\mathfrak{a}(E_{|\widetilde{\mathcal{V}}(W)}), \mathbb{D}_\mathfrak{a}\big), \qquad (\mathrm{Gr}^{\mathrm{full}}_\mathfrak{a}(E), \mathbb{D}_\mathfrak{a})_{|\widehat{W} \cap \mathcal{V}} \simeq \big(\widehat{E}_\mathfrak{a}, \mathbb{D}_\mathfrak{a}\big)_{|\widehat{W} \cap \mathcal{V}}.$$

By construction, $\mathbb{D}_\mathfrak{a} - d\mathfrak{a}$ is logarithmic for each $\mathfrak{a} \in \mathrm{Irr}(\mathbb{D})$.

As in the case of Gr with respect to Stokes filtrations in the level $(\boldsymbol{m}, i(0))$, we have the following isomorphisms:

$$\mathrm{Gr}^{\mathrm{full}}_\mathfrak{a}(E^\vee) \simeq \mathrm{Gr}^{\mathrm{full}}_{-\mathfrak{a}}(E)^\vee,$$

$$\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(E_1\otimes E_2)\simeq\bigoplus_{\substack{\mathfrak{a}_i\in\mathrm{Irr}(\mathbb{D}_i)\\ \mathfrak{a}_1+\mathfrak{a}_2=\mathfrak{a}}}\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}_1}(E_1)\otimes\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}_2}(E_2),$$

$$\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(E_1\oplus E_2)\simeq\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(E_1)\oplus\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(E_2).$$

Here, we have assumed that $(E_1,\mathbb{D}_1)\otimes(E_2,\mathbb{D}_2)$ and $(E_1,\mathbb{D}_1)\oplus(E_2,\mathbb{D}_2)$ are unramifiedly good lattices.

Lemma 5.37 *Let $(E_p,\mathbb{D}_p)$ $(p=1,2)$ be unramifiedly good lattices on $(\mathcal{X},\mathcal{D})$. Assume $\mathcal{I}_1\cup\mathcal{I}_2$ is a good set of irregular values. Let $F:(E_1,\mathbb{D}_1)\longrightarrow(E_2,\mathbb{D}_2)$ be a flat morphism. We have the naturally induced morphism $\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(F):\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(E_1)\longrightarrow\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(E_2)$.* ∎

A characterization of sections of E Let $\boldsymbol{w}_{\mathfrak{a}}$ be a frame of $\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(E)$. Let S be a small multi-sector, and let $E_{|\overline{S}}=\bigoplus E_{\mathfrak{a},S}$ be a $\mathbb{D}$-flat splitting of the full Stokes filtration $\widetilde{\mathcal{F}}^S$. By the natural isomorphism $E_{\mathfrak{a},S}\simeq\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(E)_{|\overline{S}}$, we take a lift $\boldsymbol{w}_{\mathfrak{a},S}$ of $\boldsymbol{w}_{\mathfrak{a}}$. Thus, we obtain the frame $\boldsymbol{w}_S=\bigl(\boldsymbol{w}_{\mathfrak{a},S}\bigr)$ of $E_{|\overline{S}}$. The following proposition implies a characterization of sections of E by growth order with respect to the frames $\boldsymbol{w}_S$ for small multi-sectors S.

Proposition 5.38 *Let $\boldsymbol{v}$ be a frame of E, and let G_S be determined by $\boldsymbol{v}=\boldsymbol{w}_S\cdot G_S$. Then, G_S and G_S^{-1} are bounded on S.* ∎

Deformation When $|\arg(T)|$ is sufficiently small, we have a more direct local construction of the deformation $(E,\mathbb{D})^{(T)}$. We explain it in the smooth divisor case.

We take a covering $\mathcal{X}-\mathcal{D}=\bigcup_{i=1}^N S^{(i)}$ by sectors $S^{(i)}$ on which we have the full Stokes filtrations. Assume that $\bigl|\arg(T)\bigr|$ is sufficiently small such that the following holds:

- $\mathfrak{a}\leq_{S^{(i)}}\mathfrak{b}\Longleftrightarrow T\mathfrak{a}\leq_{S^{(i)}}T\mathfrak{b}$ for any $\mathfrak{a},\mathfrak{b}\in\mathrm{Irr}(\mathbb{D})$ and for any $S^{(i)}$.

We take frames $\boldsymbol{w}_{\mathfrak{a}}$ of $\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(E)$. For each $S=S^{(i)}$, we take a $\mathbb{D}$-flat splitting $E_{|S}=\bigoplus E_{\mathfrak{a},S}$ of the full Stokes filtration. Let $\boldsymbol{w}_S=(\boldsymbol{w}_{\mathfrak{a},S})$ be as above. We put $\boldsymbol{w}^{(T)}_{\mathfrak{a},S}:=\boldsymbol{w}_{\mathfrak{a},S}\cdot\exp\bigl((T-1)\cdot\lambda^{-1}\cdot\mathfrak{a}\bigr)$ and $\boldsymbol{w}^{(T)}_S:=\bigl(\boldsymbol{w}^{(T)}_{\mathfrak{a},S}\bigr)$. Let f be a holomorphic section of $E_{|\mathcal{X}-\mathcal{D}}$. We have the corresponding decomposition $f=\sum f_{\mathfrak{a},S}$ on each S. We have the expression $f_{\mathfrak{a},S}=\sum f^{(T)}_{\mathfrak{a},S,j}\cdot w^{(T)}_{\mathfrak{a},S,j}$. We put $\boldsymbol{f}_{\mathfrak{a},S}:=\bigl(f^{(T)}_{\mathfrak{a},S,j}\bigr)$.

Lemma 5.39 *f gives a section of $E^{(T)}$ if and only if $\boldsymbol{f}^{(T)}_{\mathfrak{a},S^{(i)}}$ is bounded for each $S^{(i)}$ and $\boldsymbol{w}_{S^{(i)}}$. (See Subsection* 7.9.1 *of* [19].*)* ∎

Prolongation of a flat morphism Let $(E_p,\mathbb{D}_p)$ $(p=1,2)$ be unramifiedly good lattices on $(\mathcal{X},\mathcal{D})$. Assume $\mathrm{Irr}(\mathbb{D}_1)\cup\mathrm{Irr}(\mathbb{D}_2)$ is a good set of irregular values. Let $F:(E_1,\mathbb{D}_1)_{|\mathcal{X}-\mathcal{D}}\longrightarrow(E_2,\mathbb{D}_2)_{|\mathcal{X}-\mathcal{D}}$ be a flat morphism.

Lemma 5.40 *If F preserves the full Stokes filtrations $\widetilde{\mathcal{F}}^S$ for each small sector S, F is extended to the meromorphic morphism $F:E_1(*\mathcal{D})\longrightarrow E_2(*\mathcal{D})$.*

Proof We have only to consider the case $0\notin\mathcal{K}$ according to the Hartogs theorem. Then, the claim follows from a result in Subsection 7.7.6 of [19]. As another argument, let $\boldsymbol{w}^{(i)}_S$ be frames of $E_{i|\overline{S}}$ as in Proposition 5.38. We can directly show that $F_{|S}$ is of polynomial order with respect to the frames $\boldsymbol{w}^{(i)}_S$. ∎

Complement on a connection along the λ-direction let $X:=\Delta^n$, $D_i:=\{z_i=0\}$ and $D:=\bigcup_{i=1}^{\ell}D_i$. Let $\mathcal{K}\subset\boldsymbol{C}^*_\lambda$ be a compact region. Let $(E,\mathbb{D})$ be an unramifiedly good lattice of a family of meromorphic λ-flat bundles on $(\mathcal{X},\mathcal{D})$ with a good set $\mathrm{Irr}(\mathbb{D})$. Assume that E is equipped with a meromorphic connection along the λ-direction $\nabla_\lambda:E\longrightarrow E\otimes\Omega^1_{\mathcal{K}}(*\mathcal{D})$, such that $\mathbb{D}^f+\nabla_\lambda$ is flat.

Lemma 5.41 *∇_λ naturally induces a meromorphic connection of $E^{(T)}$ along the λ-direction.*

Proof It is easy to observe that we have only to consider the case in which D is smooth and $|\arg(T)|$ is sufficiently small. Take N such that $\lambda^N\nabla_\lambda(\partial_\lambda)E \subset E\otimes\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$. For $S=S^{(i)}$, let $\boldsymbol{w}_S=(\boldsymbol{w}_{\mathfrak{a},S})$ be a frame of $E_{|\overline{S}}$ as above. Let $A_S=(A_{S,\mathfrak{a},\mathfrak{a}'})$ be the matrix-valued holomorphic function on S determined by $\lambda^N\nabla_\lambda(\partial_\lambda)\boldsymbol{w}_S=\boldsymbol{w}_S\cdot A_S$. Let $B_{\mathfrak{a}}$ be the matrix-valued holomorphic function on $\mathcal{X}$ determined by $\mathbb{D}_{\mathfrak{a}}(z_1\partial_1)\boldsymbol{w}_{\mathfrak{a}}=\boldsymbol{w}_{\mathfrak{a}}\cdot\big(z_1\partial_1\mathfrak{a}+B_{\mathfrak{a}}\big)$. Because $[\mathbb{D}^f,\nabla_\lambda]=0$, we have the following relation in the case $\mathfrak{a}\neq\mathfrak{b}$:

$$\lambda\cdot z_1\partial_1A_{S,\mathfrak{a},\mathfrak{b}}+\big(z_1\partial_1(\mathfrak{a}-\mathfrak{b})\big)\cdot A_{S,\mathfrak{a},\mathfrak{b}}+\big(A_{S,\mathfrak{a},\mathfrak{b}}B_{\mathfrak{b}}-B_{\mathfrak{a}}A_{S,\mathfrak{a},\mathfrak{b}}\big)=0$$

Hence, we have $A_{S,\mathfrak{a},\mathfrak{b}}=0$ unless $\mathfrak{a}\leq_S\mathfrak{b}$, and we obtain the estimate

$$A_{S,\mathfrak{a},\mathfrak{b}}\cdot\exp\big(\lambda^{-1}(\mathfrak{a}-\mathfrak{b})\big)=O\Big(\exp\big(C|\lambda^{-1}|\cdot\log|z_1^{-1}|\big)\Big)$$

for some $C>0$ in the case $\mathfrak{a}<_S\mathfrak{b}$.

Let $A_S^{(T)}$ be the matrix-valued holomorphic function on S determined by $\lambda^N\nabla(\partial_\lambda)\boldsymbol{w}_S^{(T)}=\boldsymbol{w}_S^{(T)}\cdot A_S^{(T)}$. We have $A^{(T)}_{S,\mathfrak{a},\mathfrak{b}}=0$ unless $\mathfrak{a}\leq_S\mathfrak{b}$. In the case $\mathfrak{a}<_S\mathfrak{b}$, we have

$$A^{(T)}_{S,\mathfrak{a},\mathfrak{b}}\cdot\exp\big(\lambda^{-1}\cdot T\cdot(\mathfrak{a}-\mathfrak{b})\big)=A_{S,\mathfrak{a},\mathfrak{b}}\cdot\exp\big(\lambda^{-1}\cdot(\mathfrak{a}-\mathfrak{b})\big)=O\Big(\exp\big(C|\lambda^{-1}|\cdot\log|z_1^{-1}|\big)\Big).$$

Therefore, we obtain $A^{(T)}_{S,\mathfrak{a},\mathfrak{b}}=O\big(\exp(-\epsilon|z_1^{-1}|)\big)$ for some $\epsilon>0$. By a direct calculation, we obtain $A^{(T)}_{S,\mathfrak{a},\mathfrak{a}}=A_{S,\mathfrak{a},\mathfrak{a}}+\lambda^N\cdot\partial_\lambda\big(\lambda^{-1}\cdot(1-T)\cdot\mathfrak{a}\big)$, which is of polynomial order. Hence, the claim of the lemma follows from Lemma 5.39. ∎

5.4 Family of good filtered λ-flat bundles

Pull back of filtered bundle via a ramified covering The notion of filtered bundle is introduced in [25] (1 dimension), and studied in [18] (arbitrary dimension). Let X be a complex manifold, and let D be a simple normal crossing hypersurface with the irreducible decomposition $D=\bigcup_{i\in I}D_i$. A filtered bundle on (X,D) is defined to be a sequence of locally free sheaves $\boldsymbol{E}_*=\big({}_{\boldsymbol{a}}E\,\big|\,\boldsymbol{a}\in\boldsymbol{R}^I\big)$ such that (i) ${}_{\boldsymbol{a}}E\subset{}_{\boldsymbol{b}}E$ for $\boldsymbol{a}\leq\boldsymbol{b}$ and ${}_{\boldsymbol{a}}E$ is the intersection of ${}_{\boldsymbol{b}}E$ for $\boldsymbol{b}>\boldsymbol{a}$, (ii) ${}_{\boldsymbol{a}}E_{|X-D}={}_{\boldsymbol{b}}E_{|X-D}$, (iii) ${}_{\boldsymbol{a}}E\otimes\mathcal{O}(\sum n_i\cdot D_i)={}_{\boldsymbol{a}-\boldsymbol{n}}E$, where $\boldsymbol{n}=(n_i)\in\mathbb{Z}^I$, (iv) it satisfies some compatibility condition at the intersection of the divisors. The compatibility condition is given in Definition 4.37 of [18]. Although it is not difficult, it is slightly complicated to state. Later, Iyer and Simpson [11] introduced the notion of locally abelian condition, which is equivalent to our compatibility condition. Hertling and Sevenheck (Chapter 4 of [9]) showed that it is equivalent to another simple condition. We refer to the above papers for more details.

Let us recall the pull back of a filtered bundle via a ramified covering. See [11] for more systematic treatment. See also Subsection 2.9.1 of [19]. Let $X:=\Delta_z^n$, $D:=\bigcup_{i=1}^{\ell}\{z_i=0\}$, $\widetilde{X}:=\Delta_w^n$ and $\widetilde{D}:=\bigcup_{j=1}^{\ell}\{w_j=0\}$. Let $\varphi_e:\widetilde{X}\longrightarrow X$ be a ramified covering $\varphi_e(w_1,\ldots,w_n)=(w_1^e,\ldots,w_\ell^e,w_{\ell+1},\ldots,w_n)$. For $\boldsymbol{b}\in\boldsymbol{R}^\ell$, we put $\mathcal{S}(\boldsymbol{b}):=\big\{(\boldsymbol{a},\boldsymbol{n})\in\boldsymbol{R}^\ell\times\mathbb{Z}^\ell_{\geq 0}\,\big|\,e\cdot\boldsymbol{a}+\boldsymbol{n}\leq\boldsymbol{b}\big\}$. For a given filtered bundle $\boldsymbol{E}_*$ on (X,D), we set

$${}_{\boldsymbol{b}}\widetilde{E}=\sum_{(\boldsymbol{a},\boldsymbol{n})\in\mathcal{S}(\boldsymbol{b})}\boldsymbol{w}^{-\boldsymbol{n}}\cdot\varphi_e^*\big({}_{\boldsymbol{a}}E\big).$$

Then, it is easy to show that $\widetilde{\boldsymbol{E}}_*$ is also a filtered bundle. Let $\mathrm{Gal}(\widetilde{X}/X)$ denote the Galois group of the ramified covering. We can reconstruct $\boldsymbol{E}_*$ from $\widetilde{\boldsymbol{E}}_*$ with the natural $\mathrm{Gal}(\widetilde{X}/X)$-action, and hence $\boldsymbol{E}_*$ is called the descent of $\widetilde{\boldsymbol{E}}_*$. Since the construction is independent of the choice of coordinates, it can be globalized.

Family of good filtered λ-flat bundles We use the notation in Subsection 5.2. A family of filtered λ-flat bundles on $(\mathcal{X},\mathcal{D})$ is defined to be a filtered bundle $\boldsymbol{E}_*$ on $(\mathcal{X},\mathcal{D})$ with a family of meromorphic flat λ-connections $\mathbb{D}$ of $\boldsymbol{E}=\bigcup{}_{\boldsymbol{a}}E$.

Definition 5.42 *Let $(\boldsymbol{E}_*,\mathbb{D})$ be a family of filtered λ-flat bundles on $(\mathcal{X},\mathcal{D})$.*

- *We say that $(\boldsymbol{E}_*, \mathbb{D})$ is unramifiedly good, if ${}_{\boldsymbol{c}}E$ are unramifiedly good lattices for any $\boldsymbol{c} \in \boldsymbol{R}^\ell$.*
- *Let $P \in \mathcal{D}$. We say that $(\boldsymbol{E}_*, \mathbb{D})$ is good at P, if there exist a ramified covering $\varphi_e : (\widetilde{\mathcal{U}}, \widetilde{\mathcal{D}}_{\mathcal{U}}) \longrightarrow (\mathcal{U}, \mathcal{D}_{\mathcal{U}})$ such that $\big(\widetilde{\boldsymbol{E}}_*, \varphi_e^*\mathbb{D}\big)$ on $(\widetilde{\mathcal{U}}, \widetilde{\mathcal{D}}_{\mathcal{U}})$ is unramifiedly good. Here, $\mathcal{U}$ is a coordinate neighbourhood of P, φ_e is a ramified covering, and $\widetilde{\boldsymbol{E}}_*$ is induced by φ and $\boldsymbol{E}_*$ as above.*
- *We say that $(\boldsymbol{E}_*, \mathbb{D})$ is good, if it is good at any point $P \in \mathcal{D}$.* ∎

Induced filtrations Let $(\boldsymbol{E}_*, \mathbb{D})$ be good family of filtered λ-flat bundles. Let ${}^iF$ denote the induced filtration of ${}_{\boldsymbol{c}}E_{|\mathcal{D}_i}$. We set ${}^i\operatorname{Gr}^F_a\big({}_{\boldsymbol{c}}E\big) := {}^iF_a \big/ {}^iF_{<a}$. It can be shown that (i) we have the well defined residue endomorphism $\operatorname{Gr}^F_a \operatorname{Res}_i(\mathbb{D})$ of ${}^i\operatorname{Gr}^F_a\big({}_{\boldsymbol{c}}E\big)$ on $\mathcal{D}_i$ for each $i \in \underline{\ell}$, (ii) it preserves the induced filtrations ${}^jF$ of ${}^i\operatorname{Gr}^F_a\big({}_{\boldsymbol{c}}E\big)_{|\mathcal{D}_i\cap\mathcal{D}_j}$. (See Subsection 6.1.3 of [19]. The residues are well defined as endomorphisms of ${}_{\boldsymbol{c}}E_{|\mathcal{D}_i}$ in the non-ramified case, and as endomorphisms of ${}^i\operatorname{Gr}^F_a\big({}_{\boldsymbol{c}}E\big)$ even in the ramified case.) In the following, $\operatorname{Gr}^F_a \operatorname{Res}_i(\mathbb{D})$ are often denoted by $\operatorname{Res}_i(\mathbb{D})$ for simplicity of the description.

Let I be a subset of $\underline{\ell}$. We set $\mathcal{D}_I := \bigcap_{i\in I} \mathcal{D}_i$. For $\boldsymbol{a} \in \boldsymbol{R}^I$, we put

$$
{}^IF_{\boldsymbol{a}}\big({}_{\boldsymbol{c}}E_{|\mathcal{D}_I}\big) := \bigcap_{i\in I} {}^iF_{a_i}\big({}_{\boldsymbol{c}}E_{|\mathcal{D}_I}\big), \qquad {}^I\operatorname{Gr}^F_{\boldsymbol{a}}\big({}_{\boldsymbol{c}}E\big) := \frac{{}^IF_{\boldsymbol{a}}\big({}_{\boldsymbol{c}}E_{|\mathcal{D}_I}\big)}{\sum_{\boldsymbol{b}\lneq\boldsymbol{a}} {}^IF_{\boldsymbol{b}}\big({}_{\boldsymbol{c}}E_{|\mathcal{D}_I}\big)}.
$$

We often consider the following sets:

$$
\mathcal{P}ar\big({}_{\boldsymbol{c}}E, I\big) := \big\{\boldsymbol{a} \in \boldsymbol{R}^I \,\big|\, {}^I\operatorname{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E) \neq 0\big\}, \quad \mathcal{P}ar\big(\boldsymbol{E}_*, I\big) := \bigcup_{\boldsymbol{c}\in\boldsymbol{R}^\ell} \mathcal{P}ar\big({}_{\boldsymbol{c}}E, I\big)
$$

We have the induced endomorphisms $\operatorname{Res}_i(\mathbb{D})$ $(i \in I)$ of ${}^I\operatorname{Gr}^F_{\boldsymbol{a}}({}_{\boldsymbol{c}}E)$, which are mutually commutative.

KMS structure for fixed λ Let us consider the case in which $\mathcal{K}$ is a point $\{\lambda\}$. In this case, we prefer the symbol $\mathbb{D}^\lambda$ to $\mathbb{D}$. If $\lambda \neq 0$, the eigenvalues of $\operatorname{Res}_i(\mathbb{D}^\lambda)$ are constant. Hence, we have the generalized eigen decomposition ${}^I\operatorname{Gr}^F_{\boldsymbol{a}}\big({}_{\boldsymbol{c}}E\big) = \bigoplus_{\boldsymbol{\alpha}} {}^I\operatorname{Gr}^{F,\mathbb{E}}_{(\boldsymbol{a},\boldsymbol{\alpha})}\big({}_{\boldsymbol{c}}E\big)$, where the eigenvalues of $\operatorname{Gr}^F \operatorname{Res}_i(\mathbb{D}^\lambda)$ on ${}^I\operatorname{Gr}^{F,\mathbb{E}}_{(\boldsymbol{a},\boldsymbol{\alpha})}\big({}_{\boldsymbol{c}}E\big)$ are the i-th components of $\boldsymbol{\alpha}$. We put

$$
\mathcal{KMS}({}_{\boldsymbol{c}}E, \mathbb{D}^\lambda, I) := \big\{(\boldsymbol{a}, \boldsymbol{\alpha}) \,\big|\, {}^I\operatorname{Gr}^{F,\mathbb{E}}_{(\boldsymbol{a},\boldsymbol{\alpha})}({}_{\boldsymbol{c}}E) \neq 0\big\}, \quad \mathcal{KMS}(\boldsymbol{E}_*, \mathbb{D}^\lambda, I) := \bigcup_{\boldsymbol{c}\in\boldsymbol{R}^S} \mathcal{KMS}({}_{\boldsymbol{c}}E, \mathbb{D}^\lambda, I)
$$

$$
\mathcal{S}p({}_{\boldsymbol{c}}E, \mathbb{D}^\lambda, I) := \big\{\boldsymbol{\alpha} \in \boldsymbol{C}^I \,\big|\, \exists \boldsymbol{a} \in \boldsymbol{R}^I,\ (\boldsymbol{a}, \boldsymbol{\alpha}) \in \mathcal{KMS}\big({}_{\boldsymbol{c}}E, \mathbb{D}^\lambda, I\big)\big\}, \quad \mathcal{S}p(\boldsymbol{E}_*, \mathbb{D}^\lambda, I) := \bigcup_{\boldsymbol{c}\in\boldsymbol{R}^S} \mathcal{S}p({}_{\boldsymbol{c}}E, \mathbb{D}^\lambda, I)
$$

Each element of $\mathcal{KMS}(\boldsymbol{E}_*, \mathbb{D}^\lambda, I)$ is called a KMS-spectrum of $(\boldsymbol{E}_*, \mathbb{D}^\lambda)$ at D_I.

Even in the case $\lambda = 0$, a similar definition makes sense if the eigenvalues of $\operatorname{Res}_i(\mathbb{D}^\lambda)$ are constant. It is satisfied when we consider wild harmonic bundles.

KMS structure around λ_0 Assume that $\mathcal{K}$ is a neighbourhood of $\lambda_0 \in \boldsymbol{C}$, and we regard that $(\boldsymbol{E}_*, \mathbb{D})$ is given around $\{\lambda_0\} \times X$. In this case, we prefer the symbols ${}^iF^{(\lambda_0)}$ to ${}^iF$. Let $\mathfrak{p}(\lambda) : \boldsymbol{R} \times \boldsymbol{C} \longrightarrow \boldsymbol{R}$ and $\mathfrak{e}(\lambda) : \boldsymbol{R} \times \boldsymbol{C} \longrightarrow \boldsymbol{C}$ be given as follows:

$$
\mathfrak{p}\big(\lambda, (a, \alpha)\big) = a + 2\operatorname{Re}(\lambda \cdot \overline{\alpha}), \quad \mathfrak{e}\big(\lambda, (a, \alpha)\big) = \alpha - a \cdot \lambda - \overline{\alpha} \cdot \lambda^2
$$

The induced map $\boldsymbol{R} \times \boldsymbol{C} \longrightarrow \boldsymbol{R} \times \boldsymbol{C}$ is denoted by $\mathfrak{k}(\lambda)$.

Definition 5.43 *We say that $(\boldsymbol{E}_*, \mathbb{D})$ has the KMS-structure at λ_0 indexed by $T(i) \subset \boldsymbol{R} \times \boldsymbol{C}$ $(i \in S)$, if the following holds:*

- *$\mathcal{P}ar(\boldsymbol{E}_*, i)$ is the image of $T(i)$ via the map $\mathfrak{p}(\lambda_0)$.*

- *For each $a \in \mathcal{P}ar(\boldsymbol{E}_*, i)$, we put $\mathcal{K}(a,i) := \bigl\{u \in T(i) \,\big|\, \mathfrak{p}(\lambda_0, u) = a\bigr\}$. Then, the restrictions of $\mathrm{Res}_i(\mathbb{D})$ to ${}^i\mathrm{Gr}^{F^{(\lambda_0)}}_a\bigl({}_{\boldsymbol{c}}E\bigr)_{|\mathcal{D}_i^\lambda}$ have the unique eigenvalue $\mathfrak{e}(\lambda, u)$ for any $u \in \mathcal{K}(a,i)$.* ∎

Assume $(\boldsymbol{E}_*, \mathbb{D})$ has the KMS-structure at λ_0. We have the decomposition

$$
{}^i\mathrm{Gr}^{F^{(\lambda_0)}}_a\bigl({}_{\boldsymbol{c}}E\bigr) = \bigoplus_{u \in \mathcal{K}(a,i)} {}^i\mathcal{G}^{(\lambda_0)}_u\bigl({}_{\boldsymbol{c}}E\bigr), \tag{38}
$$

such that (i) it is preserved by $\mathrm{Res}_i(\mathbb{D})$, (ii) the restriction of $\mathrm{Res}_i(\mathbb{D}) - \mathfrak{e}(\lambda, u)$ to ${}^i\mathcal{G}^{(\lambda_0)}_u\bigl({}_{\boldsymbol{c}}E\bigr)$ is nilpotent. More generally, we have the decomposition on $\mathcal{D}_I$

$$
{}^I\mathrm{Gr}^{F^{(\lambda_0)}}_{\boldsymbol{a}}\bigl({}_{\boldsymbol{c}}E\bigr) = \bigoplus_{\boldsymbol{u} \in \prod \mathcal{K}(a_i,i)} {}^I\mathcal{G}^{(\lambda_0)}_{\boldsymbol{u}}\bigl({}_{\boldsymbol{c}}E\bigr), \tag{39}
$$

such that (i) it is preserved by $\mathrm{Res}_i(\mathbb{D})$ $(i \in I)$, (ii) the restrictions of $\mathrm{Res}_i(\mathbb{D}) - \mathfrak{e}(\lambda, u_i)$ $(i \in I)$ are nilpotent, where u_i denotes the i-th component of $\boldsymbol{u}$. Note ${}^I\mathcal{G}^{(\lambda_0)}_{\boldsymbol{u}}\bigl({}_{\boldsymbol{c}}E\bigr)$ can be 0.

The following lemma is standard in our works. (See Subsection 6.2.5 of [19].)

Lemma 5.44 *Let $(\boldsymbol{E}_{1*}, \mathbb{D}_1)$ and $(\boldsymbol{E}_{2*}, \mathbb{D}_2)$ be good filtered λ-flat bundles on $(\mathcal{X}, \mathcal{D})$ which have the KMS-structures at λ_0. An isomorphism $\varphi : (\boldsymbol{E}_1, \mathbb{D}_1) \simeq (\boldsymbol{E}_2, \mathbb{D}_2)$ of families of meromorphic λ-flat bundles induces the isomorphism $\varphi : (\boldsymbol{E}_{1*}, \mathbb{D}_1) \simeq (\boldsymbol{E}_{2*}, \mathbb{D}_2)$ of families of filtered λ-flat bundles.* ∎

We say that $(\boldsymbol{E}, \mathbb{D})$ has the KMS-structure at λ_0, if there exists a good filtered λ-flat bundle $(\boldsymbol{E}_*, \mathbb{D})$ which has the KMS-structure at λ_0, such that $\boldsymbol{E} = \bigcup_{\boldsymbol{a}} E$. It makes sense by the above lemma.

Pick $\boldsymbol{c} \in \boldsymbol{R}^S$ such that $c_i \notin \mathcal{P}ar\bigl(\boldsymbol{E}_*, i\bigr)$ for each $i \in S$. Assume that $\mathcal{K}$ is a sufficiently small neighbourhood of λ_0. Take $\lambda_1 \in \mathcal{K}$, and let $U(\lambda_1) \subset \mathcal{K}$ be a neighbourhood of λ_1. We set $\mathcal{X}^{(\lambda_1)} := U(\lambda_1) \times X$. We use the symbols $\mathcal{D}_i^{(\lambda_1)}$ and $\mathcal{D}^{(\lambda_1)}$ in similar meanings. Let $\pi_{i,a}$ denote the projection ${}^iF^{(\lambda_0)}_a\bigl({}_{\boldsymbol{c}}E_{|\mathcal{D}_i}\bigr) \longrightarrow {}^i\mathrm{Gr}^{F^{(\lambda_0)}}_a\bigl({}_{\boldsymbol{c}}E\bigr)$ for any $a \in \mathcal{P}ar({}_{\boldsymbol{c}}E, i)$. Let $b \in]c_i - 1, c_i]$. If $\mathfrak{p}(\lambda_1, v) = b$ for some $v \in \mathcal{K}(a,i)$, we put on $\mathcal{D}_i^{(\lambda_1)}$

$$
{}^iF^{(\lambda_1)}_b := \bigoplus_{\substack{u \in \mathcal{K}(a,i) \\ \mathfrak{p}(\lambda_1, u) \leq b}} \pi_{i,a}^{-1}\bigl({}^i\mathcal{G}^{(\lambda_0)}_u({}_{\boldsymbol{c}}E)\bigr).
$$

Otherwise, let $b_0 := \max\bigl\{\mathfrak{p}(\lambda_1, v) < b \,\big|\, v \in \mathcal{K}(a,i)\bigr\}$, and we set ${}^iF^{(\lambda_1)}_b := {}^iF^{(\lambda_1)}_{b_0}$. Thus, we obtain the filtration ${}^iF^{(\lambda_1)}$ of ${}_{\boldsymbol{c}}E_{|\mathcal{D}_i^{(\lambda_1)}}$. It induces the family of the filtered λ-flat bundles $(\boldsymbol{E}^{(\lambda_1)}_*, \mathbb{D})$ on $(\mathcal{X}^{(\lambda_1)}, \mathcal{D}^{(\lambda_1)})$. By construction, $\mathrm{Res}_i(\mathbb{D}) - \mathfrak{e}(\lambda, u)$ are nilpotent on ${}^i\mathrm{Gr}^{F^{(\lambda_1)}}_{\mathfrak{p}(\lambda_1,u)}\bigl({}_{\boldsymbol{c}}E\bigr)$. Namely, $(\boldsymbol{E}^{(\lambda_1)}_*, \mathbb{D})$ has the KMS-structure at λ_1 indexed by $T(i)$. Hence, if $(\boldsymbol{E}, \mathbb{D})$ has the KMS-structure at λ_0, it has the KMS-structure at any λ sufficiently close to λ_0, and the index set is independent of λ. For each $\lambda \in \mathcal{K}$, we put $\boldsymbol{E}^\lambda_* := (\boldsymbol{E}^{(\lambda)}_*)_{|\mathcal{X}^\lambda}$, which is the good filtered λ-flat bundle. The set $\mathcal{KMS}(\boldsymbol{E}^\lambda_*, i)$ is the image of $T(i)$ via the map $\mathfrak{k}(\lambda)$. Note $\mathcal{KMS}(\boldsymbol{E}^0_*, i) = T(i)$ if $0 \in \mathcal{K}$. We often identify them.

Deformation Let $T(\lambda)$ be a holomorphic function with $\mathrm{Re}\bigl(T(\lambda)\bigr) > 0$. We obtain the deformation $(\boldsymbol{E}^{(T)}_*, \mathbb{D})$. If $(\boldsymbol{E}_*, \mathbb{D})$ is unramified, the set of irregular values is given by

$$
\mathrm{Irr}\bigl(\mathbb{D}, E^{(T)}\bigr) := \bigl\{T \cdot \mathfrak{a} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\mathbb{D})\bigr\}.
$$

Since the regular part of the completion is unchanged, the set of KMS-spectra is unchanged.

6 Wild harmonic bundle

6.1 Definition of wild harmonic bundle

Local condition for Higgs fields Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on $X - D$, where X is a complex manifold, and D is a normal crossing divisor of X. We would like to explain some conditions for the Higgs field θ. First, let us consider the case $X=\Delta^n=\bigl\{\boldsymbol{z}=(z_1,\ldots,z_n)\,\big|\,|z_i|<1\bigr\}$, $D_i=\{z_i=0\}$ and $D=\bigcup_{i=1}^{\ell}D_i$. We have the expression:

$$\theta=\sum_{j=1}^{\ell}F_j\cdot\frac{dz_j}{z_j}+\sum_{j=\ell+1}^{n}G_j\cdot dz_j$$

We have the characteristic polynomials $\det\bigl(T-F_j(\boldsymbol{z})\bigr)=\sum A_{j,k}(\boldsymbol{z})\cdot T^k$ and $\det\bigl(T-G_j(\boldsymbol{z})\bigr)=\sum B_{j,k}(\boldsymbol{z})\cdot T^k$. The coefficients $A_{j,k}$ and $B_{j,k}$ are holomorphic on $X-D$.

- We say that θ is tame, if the following conditions are satisfied:

  (T1): $A_{j,k}$ and $B_{j,k}$ are holomorphic on X for any k.

  (T2): The restriction of $A_{j,k}$ to D_j are constant for any $j=1,\ldots,\ell$ and any k. In other words, roots of $\sum A_{j,k}(\boldsymbol{z})\cdot T^k$ are independent of $\boldsymbol{z}\in D_j$.

- We say that θ is unramifiedly good, if there exists a good set of irregular values $\mathrm{Irr}(\theta)\subset M(X,D)\big/H(X)$ and a decomposition $(E,\theta)=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)}(E_{\mathfrak{a}},\theta_{\mathfrak{a}})$, such that $\theta_{\mathfrak{a}}-d\mathfrak{a}\cdot\pi_{\mathfrak{a}}$ are tame, where $\pi_{\mathfrak{a}}$ denotes the projection onto $E_{\mathfrak{a}}$ with respect to the decomposition.

- We say that θ is good, if $\varphi_e^*(\theta)$ is unramifiedly good for some $e\in\mathbb{Z}_{>0}$, where φ_e is the covering given by $\varphi_e(z_1,\ldots,z_n)=(z_1^e,\ldots,z_\ell^e,z_{\ell+1},\ldots,z_n)$. ▮

Global condition for Higgs fields Let us consider the case in which X is a general complex manifold. Let D be a normal crossing hypersurface of X, and let (E,θ) be a Higgs bundle on $X-D$.

- We say that θ is (unramifiedly) good at $P\in D$, if it is (unramifiedly) good on some holomorphic coordinate neighbourhood of P.

- We say that θ is (unramifiedly) good, if it is (unramifiedly) good at any point $P\in D$. ▮

Let Z be a closed analytic subset of X, and let (E,θ) be a Higgs bundle on $X-Z$. The Higgs field θ is called wild, if there exists a regular birational map $\varphi:X'\longrightarrow X$ such that (i) $\varphi^{-1}(D)$ is normal crossing, (ii) $\varphi^{-1}\theta$ is good.

Remark 6.1 *Even if Z is a normal crossing divisor, wild θ is not necessarily good.* ▮

Conditions for harmonic bundles Let X be a complex manifold. Let D be a normal crossing hypersurface of X, and let $(E,\overline{\partial}_E,\theta,h)$ be a harmonic bundle on $X-D$.

- It is called tame, if θ is tame.

- It is called (unramifiedly) good wild harmonic bundle, if θ is (unramifiedly) good. ▮

Let Z be a closed analytic subset of X. A harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ on $X-Z$ is called wild, if θ is wild.

Remark We give some remarks on the condition **(T2)** for tameness.

1. If θ comes from a harmonic bundle $(E,\overline{\partial}_E,\theta,h)$, **(T2)** is implied by **(T1)**. (See Lemma 8.2 of [18].)

2. Let $(E,\overline{\partial}_E,\theta,h)$ be a harmonic bundle with a good set of irregular values $\mathrm{Irr}(\theta)$ and a decomposition $(E,\overline{\partial}_E,\theta)=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)}(E_{\mathfrak{a}},\overline{\partial}_{E_{\mathfrak{a}}},\theta_{\mathfrak{a}})$ such that $\widetilde{\theta}_{\mathfrak{a}}:=\theta_{\mathfrak{a}}-d\mathfrak{a}\cdot\pi_{\mathfrak{a}}$ satisfy the condition **(T1)**. The author does not know whether **(T2)** for $\widetilde{\theta}_{\mathfrak{a}}$ is automatically satisfied or not. But, if moreover $(E,\overline{\partial}_E,\theta,h)$ underlies a variation of polarized pure integrable structures, **(T2)** is satisfied. Actually, the roots of the polynomials are 0. (See Lemma 7.10 below.)

6.2 Simpson's main estimate

The first fundamental result is an estimate of Higgs field, so called Simpson's main estimate. For later use, we recall it in the case that D is smooth. Let $X := \Delta^n$ and $D := \{z_1 = 0\}$. Let $(E, \overline{\partial}_E, \theta, h)$ be an unramifiedly good wild harmonic bundle on $X - D$. (See Subsections 11.2 and 11.3 of [19] for more details.) We will be interested in the behaviour around O. Hence, by shrinking X, we assume that there exists a holomorphic decomposition $(E,\theta) = \bigoplus_{(\mathfrak{a},\alpha)\in \mathrm{Irr}(\theta)\times \boldsymbol{C}}(E_{\mathfrak{a},\alpha}, \theta_{\mathfrak{a},\alpha})$ satisfying the following conditions:

- For each $(\mathfrak{a},\alpha)$, let $\pi_{\mathfrak{a},\alpha}$ denote the projection onto $E_{\mathfrak{a},\alpha}$ with respect to the decomposition. We have the expression
$$\theta_{\mathfrak{a},\alpha} - \bigl(\alpha\cdot dz_1/z_1 + d\mathfrak{a}\bigr)\cdot \pi_{\mathfrak{a},\alpha} = F_1\cdot \frac{dz_1}{z_1} + \sum_{j=2}^{n} G_j\cdot dz_j.$$
Then, the coefficients of $\det(T-F_1)$ and $\det(T-G_j)$ are holomorphic on X, and $\det(T-F_1)_{|D} = T^{\mathrm{rank}\, E_{\mathfrak{a},\alpha}}$.

We also set $E_{\mathfrak{a}} := \bigoplus_{\alpha\in\boldsymbol{C}} E_{\mathfrak{a},\alpha}$, and let $\pi_{\mathfrak{a}}$ denote the projection onto $E_{\mathfrak{a}}$ with respect to the decomposition $E = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} E_{\mathfrak{a}}$.

Truncation For any $\mathfrak{a}\in \mathrm{Irr}(\theta)$, we have the expression $\mathfrak{a} = \sum_{j\le -1}\mathfrak{a}_j\cdot z_1^j$. We put $\eta_p(\mathfrak{a}) := \sum_{j\le p}\mathfrak{a}_j\cdot z_1^j$ and $\mathrm{Irr}(\theta,p) := \bigl\{\eta_p(\mathfrak{a})\,\big|\,\mathfrak{a}\in\mathrm{Irr}(\theta)\bigr\}$. For each $\mathfrak{b}\in\mathrm{Irr}(\theta,p)$, let $E^{(p)}_{\mathfrak{b}}$ denote the direct sum of $E_{\mathfrak{a}}$ ($\mathfrak{a}\in\mathrm{Irr}(\theta), \eta_p(\mathfrak{a}) = \mathfrak{b}$), and let $\pi^{(p)}_{\mathfrak{b}}$ denote the projection onto $E^{(p)}_{\mathfrak{b}}$ with respect to the decomposition $E = \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\theta,p)} E^{(p)}_{\mathfrak{b}}$. We have $\mathrm{Irr}(\theta,-1) = \mathrm{Irr}(\theta)$ and $E_{\mathfrak{a}} = E^{(-1)}_{\mathfrak{a}}$. We have the induced maps $\eta_{q,p} : \mathrm{Irr}(\theta,p)\longrightarrow \mathrm{Irr}(\theta,q)$ for $q\le p$.

Asymptotic orthogonality We take total orders $\le'$ on $\mathrm{Irr}(\theta,p)$ ($p\le -1$) which are preserved by $\eta_{q,p}$. For each $\mathfrak{b}\in\mathrm{Irr}(\theta,p)$, we set $F^{(p)}_{\mathfrak{b}}(E) := \bigoplus_{\mathfrak{a}\le'\mathfrak{b}} E^{(p)}_{\mathfrak{a}}$. Let $E^{(p)\prime}_{\mathfrak{b}}$ be the orthogonal complement of $F^{(p)}_{<\mathfrak{b}}(E)$ in $F^{(p)}_{\mathfrak{b}}(E)$. We obtain an orthogonal decomposition $E = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta,p)} E^{(p)\prime}_{\mathfrak{a}}$. Let $\pi^{(p)\prime}_{\mathfrak{a}}$ denote the orthogonal projection onto $E^{(p)\prime}_{\mathfrak{a}}$.

We take a total order $\le'$ on $\boldsymbol{C}$. Then, we obtain the lexicographic order on $\mathrm{Irr}(\theta)\times\boldsymbol{C}$. We obtain the orthogonal decomposition $E = \bigoplus E'_{\mathfrak{a},\alpha}$ by the procedure as above, and let $\pi'_{\mathfrak{a},\alpha}$ denote the orthogonal projection onto $E'_{\mathfrak{a},\alpha}$.

Proposition 6.2 *We have the following estimates with respect to h.*

- $\pi^{(p)}_{\mathfrak{a}} - \pi^{(p)\prime}_{\mathfrak{a}} = O\Bigl(\exp\bigl(-\epsilon|z_1^p|\bigr)\Bigr)$ *for some $\epsilon > 0$. In particular, the decomposition $E = \bigoplus E^{(p)}_{\mathfrak{b}}$ is $O\bigl(\exp(-\epsilon|z_1^p|)\bigr)$-asymptotically orthogonal in the sense that there exists $A>0$ such that*
$$\bigl|h(u,v)\bigr| \le A\cdot |u|_h\cdot |v|_h\cdot \exp\bigl(-\epsilon|z_1(Q)|^p\bigr)$$
for any $Q\in X-D$, $u\in E_{\mathfrak{a}|Q}$ and $v\in E_{\mathfrak{b}|Q}$ ($\mathfrak{a}\neq\mathfrak{b}$).
- $\pi_{\mathfrak{a},\alpha} - \pi'_{\mathfrak{a},\alpha} = O\bigl(|z_1|^{\epsilon}\bigr)$ *for some $\epsilon>0$. In particular, the decomposition $E = \bigoplus E_{\mathfrak{a},\alpha}$ is $O\bigl(|z_1|^{\epsilon}\bigr)$-asymptotically orthogonal.* ∎

Estimate of Higgs field We set $\widetilde{\theta} := \theta - \bigoplus_{\mathfrak{a},\alpha}\bigl(d\mathfrak{a} + \alpha\cdot dz_1/z_1\bigr)\pi_{\mathfrak{a},\alpha}$. Let $g_{\mathbf{p}}$ denote the Poincaré metric of $X-D$. The estimates in Subsection 11.2 of [19] implies the following.

Proposition 6.3 *$\widetilde{\theta}$ is bounded with respect to h and $g_{\mathbf{p}}$.* ∎

Estimate of curvatures As mentioned in Subsection 2.1.7, we obtain a holomorphic vector bundle $\mathcal{E}^{\lambda} = (E, \overline{\partial}_E + \lambda\theta^{\dagger})$ on $X-D$. The curvature of the unitary connection associated to $(\mathcal{E}^{\lambda}, h)$ equals to $-(1+|\lambda|^2)\cdot\bigl[\theta,\theta^{\dagger}\bigr]$.

Proposition 6.4 $[\theta,\theta^{\dagger}]$ *is bounded with respect to h and $g_{\mathbf{p}}$. In particular, $(\mathcal{E}^{\lambda},h)$ is acceptable, i.e., the curvature of $(\mathcal{E}^{\lambda},h)$ is bounded with respect to h and $g_{\mathbf{p}}$.* ∎

6.3 Prolongation of unramifiedly good wild harmonic bundles

6.3.1 Prolongment $\mathcal{P}\mathcal{E}^{\lambda}$

Let $(E, \overline{\partial}_E, \theta, h)$ be a good wild harmonic bundle on $X - D$, where X is a complex manifold and D is a normal crossing divisor. As mentioned in Subsection 2.1.7, we obtain a holomorphic vector bundle $\mathcal{E}^{\lambda} = (E, \overline{\partial}_E + \lambda\theta^{\dagger})$ on $X - D$ for each complex number λ. It is important to prolong it to a good filtered λ-flat bundle on (X, D). For simplicity, we explain it assuming the following. (The general case can be easily reduced to this case.)

- $X = \Delta^n$ and $D = \bigcup_{i=1}^{\ell}\{z_i = 0\}$.
- $(E, \overline{\partial}_E, \theta, h)$ is unramifiedly good wild, and the underlying Higgs bundle has the following decomposition

$$(E, \theta) = \bigoplus_{\substack{\mathfrak{a} \in \mathrm{Irr}(\theta) \\ \boldsymbol{\alpha} \in \boldsymbol{C}^{\ell}}} (E_{\mathfrak{a},\boldsymbol{\alpha}}, \theta_{\mathfrak{a},\boldsymbol{\alpha}}), \tag{40}$$

  such that (i) $\widetilde{\theta}_{\mathfrak{a}} = \theta_{\mathfrak{a}} - \big(d\mathfrak{a} + \sum_{j=1}^{\ell} \alpha_j \cdot dz_j/z_j\big) \cdot \pi_{\mathfrak{a},\boldsymbol{\alpha}}$ are tame, where $\pi_{\mathfrak{a},\boldsymbol{\alpha}}$ denote the projections onto $E_{\mathfrak{a},\boldsymbol{\alpha}}$, (ii) $\det(T - F_j)_{|D_j} = T^{\mathrm{rank}\, E_{\mathfrak{a},\boldsymbol{\alpha}}}$ for the expression $\widetilde{\theta}_{\mathfrak{a}} = \sum_{j=1}^{\ell} F_j \cdot dz_j/z_j + \sum_{j=\ell+1}^{n} G_j \cdot dz_j$.

For any open subset $U \subset X$ and $\boldsymbol{a} \in \boldsymbol{R}^{\ell}$, we set

$$\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}(U) := \Big\{ f \in \mathcal{E}^{\lambda}(U \setminus D) \,\Big|\, |f|_h = O\Big(\prod_{i=1}^{\ell} |z_i|^{-a_i-\epsilon}\Big)\ \forall \epsilon > 0 \Big\}$$

Thus, we obtain an increasing sequence of $\mathcal{O}_X$-modules $\mathcal{P}_*\mathcal{E}^{\lambda} := \big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda} \,\big|\, \boldsymbol{a} \in \boldsymbol{R}^{\ell}\big)$. We obtain an $\mathcal{O}_X(*D)$-module $\mathcal{P}\mathcal{E}^{\lambda} := \bigcup_{\boldsymbol{a}} \mathcal{P}_{\boldsymbol{a}}\mathcal{E}^{\lambda}$.

Proposition 6.5

- *(Subsection* 11.4 *of* [19]*)* $(\mathcal{P}_*\mathcal{E}^{\lambda}, \mathbb{D}^{\lambda})$ *is an unramifiedly good filtered λ-flat bundle. The set of irregular values is given by*
$$\mathrm{Irr}(\mathbb{D}^{\lambda}, \mathcal{P}\mathcal{E}^{\lambda}) = \big\{(1 + |\lambda|^2) \cdot \mathfrak{a} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\theta)\big\}.$$
- *(Subsection* 12.2 *of* [19]*)* $\mathfrak{k}(\lambda)$ *induces the bijection* $\mathcal{KMS}(\mathcal{E}^0, i) \longrightarrow \mathcal{KMS}(\mathcal{E}^{\lambda}, i)$ *for each* i*. We also have* $\dim{}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{a,\alpha}(\mathcal{P}\mathcal{E}^0) = \dim{}^i\,\mathrm{Gr}^{F,\mathbb{E}}_{\mathfrak{k}(\lambda,(a,\alpha))}\big(\mathcal{P}\mathcal{E}^{\lambda}\big)$. ∎

Take an auxiliary sequence for $\mathrm{Irr}(\theta)$. Let $\mathrm{Irr}(\theta, \boldsymbol{m}(0))$ denote the image of $\mathrm{Irr}(\theta)$ via $\overline{\eta}_{\boldsymbol{m}(0)}$. If $\lambda \neq 0$, for each small sector S in $\{\lambda\} \times (X - D)$, we have the Stokes filtration $\mathcal{F}^S$ in the level $\boldsymbol{m}(0)$, indexed by the ordered set $\big\{(1 + |\lambda|^2) \cdot \mathfrak{a} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\theta, \boldsymbol{m}(0))\big\}$ with $\leq_S$. We have the following characterization of the filtration by the growth order of the norms of flat sections with respect to h. (See Subsection 11.4.1 of [19] for more details.)

Proposition 6.6 *Assume $\lambda \neq 0$. Let f be a flat section of $\mathcal{E}^{\lambda}_{|S}$. We have $f \in \mathcal{F}^S_{(1+|\lambda|^2)\mathfrak{b}}$ for $\mathfrak{b} \in \mathrm{Irr}(\theta, \boldsymbol{m}(0))$, if and only if*

$$\Big| f \cdot \exp\big((\lambda^{-1} + \overline{\lambda}) \cdot \mathfrak{b}\big) \Big|_h = O\Big(\exp\big(C \cdot |\boldsymbol{z}^{\boldsymbol{m}(1)}|\big) \cdot \prod_{k(1) < j \leq \ell} |z_j|^{-N}\Big)$$

holds for some $C > 0$ and $N > 0$, where $k(1)$ is determined by $\boldsymbol{m}(1) \in \mathbb{Z}^{k(1)}_{<0} \times \boldsymbol{0}_{\ell-k(1)}$. ∎

6.3.2 Prolongment $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}$

It is important to consider families for λ. In the tame case, the family $\bigcup_{\lambda} \mathcal{P}\mathcal{E}^{\lambda}$ gives a regular family of meromorphic λ-flat bundles. More precisely, if we consider the sheaf of holomorphic sections of $\mathcal{E}$ of polynomial growth, then (i) it is a locally free $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$-module, (ii) the specialization at each $\{\lambda\} \times X$ is naturally isomorphic to $\mathcal{P}\mathcal{E}^{\lambda}$. (We need some more consideration to take nice lattices.)

However, it does not in the non-tame case, as suggested by the fact that the sets

$$\mathrm{Irr}(\mathcal{PE}^\lambda, \mathbb{D}^\lambda) = \bigl\{(1+|\lambda|^2)\mathfrak{a} \bigm| \mathfrak{a} \in \mathrm{Irr}(\theta)\bigr\}$$

depend on λ in a non-holomorphic way. We consider an auxiliary family of meromorphic λ-flat bundles $\mathcal{P}^{(\lambda_0)}\mathcal{E}$. We explain it under the above setting.

Let $\pi_{\mathfrak{a},\boldsymbol{\alpha}}$ denote the projection onto $E_{\mathfrak{a},\boldsymbol{\alpha}}$ in (40). We set

$$g(\lambda) := \prod_{\mathfrak{a},\boldsymbol{\alpha}} \exp\Bigl(\lambda\cdot\bigl(\overline{\mathfrak{a}} + \sum \overline{\alpha}_j \cdot \log|z_j|^2\bigr)\Bigr)\cdot \pi_{\mathfrak{a},\boldsymbol{\alpha}}$$

Let $U(\lambda_0)$ denote a small neighbourhood of $\lambda_0 \in \boldsymbol{C}$. We set $\mathcal{X}^{(\lambda_0)} := U(\lambda_0)\times X$, and $\mathcal{D}^{(\lambda_0)} := U(\lambda_0)\times D$. We also set $\mathcal{X}^\lambda := \{\lambda\}\times X$ and $\mathcal{D}^\lambda := \{\lambda\}\times D$. Let p_λ be the projection of $\mathcal{X}^{(\lambda_0)} - \mathcal{D}^{(\lambda_0)}$ onto $X - D$. We consider the hermitian metric

$$\mathcal{P}^{(\lambda_0)}h := g(\lambda-\lambda_0)^* h$$

of $p_\lambda^{-1}E$ on $\mathcal{X}^{(\lambda_0)} - \mathcal{D}^{(\lambda_0)}$. Let $\boldsymbol{a} \in \boldsymbol{R}^\ell$. For any open subset V of $\mathcal{X}^{(\lambda_0)}$, we define

$$\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}(V) := \Bigl\{ f \in \mathcal{E}(V^*) \Bigm| |f|_{\mathcal{P}^{(\lambda_0)}h} = O\Bigl(\prod_{j=1}^{\ell}|z_j|^{-a_j-\epsilon}\Bigr),\ \forall\epsilon>0\Bigr\}$$

where $V^* := V \setminus \mathcal{D}^{(\lambda_0)}$. Thus, we obtain an increasing sequence $\mathcal{P}^{(\lambda_0)}_*\mathcal{E} = \bigl(\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E} \bigm| \boldsymbol{a}\in\boldsymbol{R}^\ell\bigr)$ of $\mathcal{O}_{\mathcal{X}^{(\lambda_0)}}$-modules. We put $\mathcal{P}^{(\lambda_0)}\mathcal{E} := \bigcup_{\boldsymbol{a}\in\boldsymbol{R}^\ell}\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}$. The restrictions to $\mathcal{X}^\lambda$ are denoted by $\mathcal{P}^{(\lambda_0)}_*\mathcal{E}^\lambda$ and $\mathcal{P}^{(\lambda_0)}\mathcal{E}^\lambda$.

Proposition 6.7

- *(Subsections* 13.1 *and* 13.2 *of* [19]*)* $(\mathcal{P}^{(\lambda_0)}_*\mathcal{E}, \mathbb{D})$ *is an unramifiedly good family of filtered λ-flat bundles. The set of irregular values is given by*

$$\mathrm{Irr}(\mathcal{P}^{(\lambda_0)}\mathcal{E}, \mathbb{D}) = \bigl\{(1+\lambda\overline{\lambda}_0)\cdot\mathfrak{a} \bigm| \mathfrak{a}\in\mathrm{Irr}(\theta)\bigr\}.$$

- *(Subsection* 13.2.1 *of* [19]*) Recall that we have the deformation mentioned in Subsections* 5.2.2 *and* 5.4*, for which* $(\mathcal{P}^{(\lambda_0)}\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ *is isomorphic to* $(\mathcal{PE}^\lambda, \mathbb{D}^\lambda)^{T(\lambda)}$ *with* $T(\lambda) = (1+|\lambda|^2)^{-1}\cdot(1+\lambda\overline{\lambda}_0)$.
- *(Subsection* 13.2.3 *of* [19]*) Let* $U(\lambda_1)\subset U(\lambda_0)$ *be small, and we set* $\mathcal{X}^{(\lambda_1)} := U(\lambda_1)\times X$*. Then,* $(\mathcal{P}^{(\lambda_1)}\mathcal{E}, \mathbb{D})$ *on* $\mathcal{X}^{(\lambda_1)}$ *is isomorphic to the deformation* $(\mathcal{P}^{(\lambda_0)}\mathcal{E}, \mathbb{D})^{(T(\lambda_0,\lambda_1))}_{|\mathcal{X}^{(\lambda_1)}}$ *with* $T(\lambda_0,\lambda_1) = (1+\lambda\overline{\lambda}_0)^{-1}(1+\lambda\overline{\lambda}_1)$. ∎

We should remark that $\mathcal{P}^{(\lambda_0)}h \neq h$ even in the tame case, and hence $\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}$ are different from ${}_{\boldsymbol{a}}\mathcal{E}$ in [18] in the tame case. We can avoid to use $\mathcal{P}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}$ by considering KMS structure in the tame case.

By the property (D2) of the deformation (Subsection 5.2.2) and the correspondence between $\mathcal{KMS}(\mathcal{PE}^\lambda, i)$ and $\mathcal{KMS}(\mathcal{PE}^0, i)$, we can show the following.

Lemma 6.8 *It has the KMS-structure at λ_0 with the index sets $\mathcal{KMS}(\mathcal{PE}^0, i)$ $(i=1,\ldots,\ell)$.* ∎

6.3.3 Prolongment $\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}$ and $\mathcal{QE}$

Applying the deformation procedure to $(\mathcal{P}^{(\lambda_0)}_*\mathcal{E}, \mathbb{D})$ with $T = (1+\lambda\overline{\lambda}_0)^{-1}$, we obtain a family of good filtered λ-flat bundles $\bigl(\mathcal{Q}^{(\lambda_0)}_*\mathcal{E}, \mathbb{D}\bigr)$ on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$. Then, $\mathcal{Q}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}$ is an unramifiedly good lattice of $\mathcal{Q}^{(\lambda_0)}\mathcal{E}$ with the good set of irregular values $\mathrm{Irr}(\mathcal{Q}^{(\lambda_0)}\mathcal{E}, \mathbb{D}) = \mathrm{Irr}(\theta)$, i.e.,

$$(\mathcal{Q}^{(\lambda_0)}_{\boldsymbol{a}}\mathcal{E}, \mathbb{D})_{|\widehat{\mathcal{D}}} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \bigl(\mathcal{Q}^{(\lambda_0)}_{\boldsymbol{a}}\widehat{\mathcal{E}}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}}\bigr),$$

such that $\widehat{\mathbb{D}}_{\mathfrak{a}} - d\mathfrak{a}\cdot\mathrm{id}$ has logarithmic singularity for each $\mathfrak{a}$. By using the property (D1) of the deformation explained in Subsection 5.2.2, we obtain the following. (See Subsection 15.1.1 of [19] for more details.)

Lemma 6.9 *$(\mathcal{Q}^{(\lambda_0)}\mathcal{E},\mathbb{D})_{|\mathcal{X}^\lambda}$ is naturally isomorphic to $\big(\mathcal{P}\mathcal{E}^\lambda,\mathbb{D}^\lambda\big)^{T_1(\lambda)}$ with $T_1(\lambda)=(1+|\lambda|^2)^{-1}>0$.* ∎

By the property (D1) of the deformation, we have $\mathcal{Q}^{(\lambda_0)}\mathcal{E}_{|\mathcal{X}^{(\lambda_1)}}=\mathcal{Q}^{(\lambda_1)}\mathcal{E}$. Hence, we obtain the global family of meromorphic λ-flat bundles $(\mathcal{Q}\mathcal{E},\mathbb{D})$ on $\boldsymbol{C}_\lambda\times(X,D)$. By using the property (D2) of the deformation and Lemma 6.8, we can show the following.

Lemma 6.10 *For each λ_0, $(\mathcal{Q}\mathcal{E},\mathbb{D})$ has the KMS-structure at λ_0 indexed by $\mathcal{KMS}\big(\mathcal{P}\mathcal{E}^0,i\big)$ $(i=1,\ldots,\ell)$.* ∎

Let S be a small sector in $\{\lambda\}\times(X-D)$. By Lemma 6.9, the Stokes filtrations of $\mathcal{Q}\mathcal{E}^\lambda$ and $\mathcal{P}\mathcal{E}^\lambda$ in the level $\boldsymbol{m}(0)$ are related as follows:

$$\mathcal{F}^S_{\mathfrak{a}}\big(\mathcal{Q}\mathcal{E}^\lambda_{|\overline{S}}\big)=\mathcal{F}^S_{(1+|\lambda|^2)\mathfrak{a}}\big(\mathcal{P}\mathcal{E}^\lambda_{|\overline{S}}\big),\quad \mathfrak{a}\in\overline{\mathrm{Irr}}\big(\theta,\boldsymbol{m}(0)\big)$$

Hence, we have the characterization of the Stokes filtrations of $\mathcal{Q}$ in the level $\boldsymbol{m}(0)$, by growth order of the norms of flat sections with respect to h. (See Subsection 15.1.1 of [19] for more details.)

Proposition 6.11 *Let f be a flat section of $\mathcal{E}^\lambda_{|S}$. We have $f\in\mathcal{F}^S_{\mathfrak{b}}(\mathcal{Q}\mathcal{E}^\lambda_{|S})$ for $\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(0))$, if and only if*

$$\Big|f\cdot\exp\big((\lambda^{-1}+\overline{\lambda})\cdot\mathfrak{b}\big)\Big|_h=O\Big(\exp\big(C\cdot|\boldsymbol{z}^{\boldsymbol{m}(1)}|\big)\cdot\prod_{k(1)<j\le\ell}|z_j|^{-N}\Big)$$

holds for some $C>0$ and $N>0$, where $k(1)$ is determined by $\boldsymbol{m}(1)\in\mathbb{Z}^{k(1)}_{<0}\times\boldsymbol{0}_{\ell-k(1)}$. ∎

By taking Gr with respect to the Stokes filtration $\mathcal{F}^S$ in the level $\boldsymbol{m}(0)$ explained in Subsection 5.2.2, we obtain an unramifiedly good lattice $\big(\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(\mathcal{Q}\mathcal{E}),\mathbb{D}_{\mathfrak{a}}\big)$.

In the case that D is smooth, we have the following characterization of the full Stokes filtration $\widetilde{\mathcal{F}}^S$ (Subsection 15.1.1 of [19]).

Proposition 6.12 *Let f be a flat section of $\mathcal{E}^\lambda_{|S}$. We have $f\in\widetilde{\mathcal{F}}^S_{\mathfrak{b}}(\mathcal{Q}\mathcal{E}^\lambda_{|S})$ for $\mathfrak{b}\in\overline{\mathrm{Irr}}(\theta)$, if and only if*

$$\Big|f\cdot\exp\big((\lambda^{-1}+\overline{\lambda})\cdot\mathfrak{b}\big)\Big|_h=O\big(|z_1|^{-N}\big)$$

holds for some $N>0$. ∎

Remark 6.13 *We have a characterization of full Stokes filtrations or more general Stokes filtrations in the level $\boldsymbol{m}(i)$, even in the general normal crossing case.* ∎

6.4 Reduction from wild to tame

Let X, D and $(E,\overline{\partial}_E,\theta,h)$ be as in Subsection 6.3. By making the same procedure to $(E,\partial_E,\theta^\dagger,h)$ on $X^\dagger-D^\dagger$, we obtain the family of meromorphic μ-flat bundles $\big(\mathcal{Q}\mathcal{E}^\dagger,\mathbb{D}^\dagger\big)$ on $\boldsymbol{C}_\mu\times(X^\dagger,D^\dagger)$.

Lemma 6.14 *The correspondence $(a,\alpha)\longleftrightarrow(-a,\overline{\alpha})$ induces a bijection $\mathcal{KMS}(\mathcal{P}\mathcal{E}^0,i)\simeq\mathcal{KMS}(\mathcal{P}\mathcal{E}^{\dagger\,0},i)$. We also have the bijection $\mathrm{Irr}(\theta)\simeq\mathrm{Irr}(\theta^\dagger)$ given by $\mathfrak{a}\longleftrightarrow\overline{\mathfrak{a}}$.*

Proof The claim for $\mathrm{Irr}(\theta)$ and $\mathrm{Irr}(\theta^\dagger)$ is clear. See Corollary 11.12 of [18] for the correspondence between $\mathcal{KMS}(\mathcal{P}\mathcal{E}^0,i)$ and $\mathcal{KMS}(\mathcal{P}\mathcal{E}^{\dagger\,0},i)$. ∎

One step reduction I Since both the Stokes filtrations of $(\mathcal{QE}^{\lambda},\mathbb{D}^{\lambda})$ and $\big(\mathcal{QE}^{\dagger\,\mu},\mathbb{D}^{\dagger\mu}\big)$ are characterized by growth order of the norms of flat sections with respect to h, we have the induced isomorphisms of the associated graded family of flat bundles for $\mathfrak{a}\in\mathrm{Irr}(\theta,\boldsymbol{m}(0))$:

$$\big(\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}\,\mathcal{QE},\mathbb{D}^{f}_{\mathfrak{a}}\big)_{|\boldsymbol{C}^{*}_{\lambda}\times(X-D)}\simeq\big(\mathrm{Gr}^{\boldsymbol{m}(0)}_{\overline{\mathfrak{a}}}\,\mathcal{QE}^{\dagger},\mathbb{D}^{\dagger\,f}_{\overline{\mathfrak{a}}}\big)_{|\boldsymbol{C}^{*}_{\mu}\times(X-D)}$$

Hence, they give a variation of $\mathbb{P}^1$-holomorphic vector bundles denoted by $\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle})$ on $\mathbb{P}^1\times(X-D)$.

We can show that the pairing $\mathcal{S}:(\mathcal{E},\mathbb{D})\otimes\sigma^{*}(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle})\longrightarrow\mathcal{O}_{\mathcal{X}-\mathcal{D}}$ is extended to

$$\mathcal{QE}\otimes\sigma^{*}\mathcal{QE}^{\dagger}\longrightarrow\mathcal{O}_{\boldsymbol{C}_{\lambda}\times X}\big(*(\boldsymbol{C}_{\lambda}\times D)\big).$$

(See Subsection 15.1.3 of [19].) By functoriality of Gr with respect to Stokes structures, we obtain

$$\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(\mathcal{QE},\mathbb{D})\otimes\sigma^{*}\,\mathrm{Gr}^{\boldsymbol{m}(0)}_{\overline{\mathfrak{a}}}(\mathcal{QE}^{\dagger},\mathbb{D}^{\dagger})\longrightarrow\mathcal{O}_{\boldsymbol{C}_{\lambda}\times X}\big(*(\boldsymbol{C}_{\lambda}\times D)\big).$$

Similarly, we obtain $\mathrm{Gr}^{\boldsymbol{m}(0)}_{\overline{\mathfrak{a}}}(\mathcal{QE}^{\dagger},\mathbb{D}^{\dagger})\otimes\sigma^{*}\,\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(\mathcal{QE},\mathbb{D})\longrightarrow\mathcal{O}_{\boldsymbol{C}_{\mu}\times X^{\dagger}}\big(*(\boldsymbol{C}_{\mu}\times D^{\dagger})\big)$. They give a morphism of variations of $\mathbb{P}^1$-holomorphic vector bundles:

$$\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(\mathcal{S}):\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle})\otimes\sigma^{*}\,\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle})\longrightarrow\mathbb{T}(0)$$

One of the main result in the study of wild harmonic bundles is the following. (See Subsection 15.2 of [19] for more details.)

Proposition 6.15 *If we shrink X appropriately, the following holds:*

- $\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}\big(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle},\mathcal{S}\big)$ *is a variation of pure polarized twistor structures.*
- *Let $(E_{\mathfrak{a}},\overline{\partial}_{\mathfrak{a}},h_{\mathfrak{a}},\theta_{\mathfrak{a}})$ denote the underlying harmonic bundle for $\mathfrak{a}\in\mathrm{Irr}(\theta,\boldsymbol{m}(0))$. By construction, the Higgs bundle $(E_{\mathfrak{a}},\theta_{\mathfrak{a}})$ is naturally isomorphic to*

$$\bigoplus_{\substack{\mathfrak{b}\in\mathrm{Irr}(\theta)\\ \overline{\eta}_{\boldsymbol{m}(0)}(\mathfrak{b})=\mathfrak{a}}}\bigoplus_{\boldsymbol{\alpha}}(E_{\mathfrak{b},\boldsymbol{\alpha}},\theta_{\mathfrak{b},\boldsymbol{\alpha}})$$

  (Recall the decomposition (40)). In particular, the harmonic bundle is unramifiedly good wild. The set of irregular values is $\overline{\eta}^{-1}_{\boldsymbol{m}(0)}(\mathfrak{a})$.
- *Let $(\mathcal{QE}_{\mathfrak{a}},\mathbb{D}_{\mathfrak{a}})$ be the family of meromorphic λ-flat bundles on $\boldsymbol{C}_{\lambda}\times(X,D)$ associated to $(E_{\mathfrak{a}},\overline{\partial}_{\mathfrak{a}},h_{\mathfrak{a}},\theta_{\mathfrak{a}})$. Then, we have the natural isomorphism $(\mathcal{QE}_{\mathfrak{a}},\mathbb{D}_{\mathfrak{a}})\simeq\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(\mathcal{QE},\mathbb{D})$.*
- *Similarly, let $(\mathcal{QE}^{\dagger}_{\overline{\mathfrak{a}}},\mathbb{D}^{\dagger}_{\overline{\mathfrak{a}}})$ denote the associated family of meromorphic μ-flat bundles on $\boldsymbol{C}_{\mu}\times(X,D)$. Then, we have the natural isomorphism $(\mathcal{QE}^{\dagger}_{\overline{\mathfrak{a}}},\mathbb{D}^{\dagger}_{\overline{\mathfrak{a}}})\simeq\mathrm{Gr}^{\boldsymbol{m}(0)}_{\overline{\mathfrak{a}}}\big(\mathcal{QE}^{\dagger},\mathbb{D}^{\dagger}\big)$.* ∎

One step reduction II Let $\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(j))$ denote the image of $\mathrm{Irr}(\theta)$ via $\overline{\eta}_{\boldsymbol{m}(j)}$. For each $\mathfrak{a}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(j))$, we obtain a variation of $\mathbb{P}^1$-holomorphic bundles with a pairing $\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{a}}\big(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle},\mathcal{S}\big)$, which is naturally isomorphic to $\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{a}}\,\mathrm{Gr}^{\boldsymbol{m}(j-1)}_{\overline{\eta}_{\boldsymbol{m}(j-1)}(\mathfrak{a})}(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle},\mathcal{S})$. We explain how to apply Proposition 6.15 in this situation.

Let us consider the case in which $\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(j-1))$ consists one element. We take any $\mathfrak{a}\in\mathrm{Irr}(\theta)$. Let $L(-\mathfrak{a})$ be the variation of polarized pure twistor structures as in Subsection 2.2.1. The underlying harmonic bundle is also denoted by $L(-\mathfrak{a})$. We set $(E',\overline{\partial}_{E'},\theta',h'):=(E,\overline{\partial}_{E},\theta,h)\otimes L(-\mathfrak{a})$. Note $\mathrm{Irr}(\theta'):=\big\{\mathfrak{a}'-\mathfrak{a}\,\big|\,\mathfrak{a}'\in\mathrm{Irr}(\theta)\big\}$, and hence $\boldsymbol{m}(j),\boldsymbol{m}(j+1),\ldots,\boldsymbol{m}(L)$ give an auxiliary sequence for $\mathrm{Irr}(\theta')$. We have the natural isomorphisms of the associated variation of polarized pure twistor structures:

$$(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle},\mathcal{S})\simeq(\mathcal{E}'^{\triangle},\mathbb{D}'^{\triangle},\mathcal{S}')\otimes L(\mathfrak{a})$$

For each $\mathfrak{b}\in\mathrm{Irr}(\theta,\boldsymbol{m}(j))$, we have the natural isomorphism:

$$\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{b}}(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle},\mathcal{S})\simeq\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{b}-\overline{\eta}_{\boldsymbol{m}(j)}(\mathfrak{a})}(\mathcal{E}'^{\triangle},\mathbb{D}'^{\triangle},\mathcal{S}')\otimes L(\mathfrak{a})$$

Hence, by shrinking X appropriately, we obtain that $\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{b}}(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle},\mathcal{S})$ is also a variation of pure twistor structures, due to Proposition 6.15.

Full reduction Let us consider the general case. By using the above result inductively, we obtain that $\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{a}}\big(\mathcal{E}^{\triangle},\mathbb{D}^{\triangle},\mathcal{S}\big)$ are variations of polarized pure twistor structures for any $\mathfrak{a}\in\mathrm{Irr}(\theta,\boldsymbol{m}(j))$. The underlying Higgs field is

$$\bigoplus_{\substack{\mathfrak{b}\in\mathrm{Irr}(\theta)\\ \overline{\eta}_{\boldsymbol{m}(j)}(\mathfrak{b})=\mathfrak{a}}}\bigoplus_{\boldsymbol{\alpha}}(E_{\mathfrak{b},\boldsymbol{\alpha}},\theta_{\mathfrak{b},\boldsymbol{\alpha}})$$

For any $\mathfrak{a}\in\mathrm{Irr}(\theta)$, we set $\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(\mathcal{E},\mathbb{D}^{\triangle},\mathcal{S}):=\mathrm{Gr}^{\boldsymbol{m}(L)}_{\mathfrak{a}}(\mathcal{E},\mathbb{D}^{\triangle},\mathcal{S})$, which are called the full reductions. Let $(E_{\mathfrak{a}},\overline{\partial}_{\mathfrak{a}},h_{\mathfrak{a}})$ be the underlying harmonic bundles. Then, $(E_{\mathfrak{a}},\overline{\partial}_{\mathfrak{a}},h_{\mathfrak{a}})\otimes L(-\mathfrak{a})$ are tame. This procedure is the reduction from wild harmonic bundles to tame harmonic bundles.

6.5 Reduction from tame to twistor nilpotent orbit

Let $X:=\Delta^n$, $D_i=\{z_i=0\}$ and $D:=\bigcup_{i=1}^{\ell}D_i$. Let $(E,\overline{\partial}_E,\theta,h)$ be a tame harmonic bundle on $X-D$. The family of λ-flat bundles $(\mathcal{E},\mathbb{D})$ is prolonged to a family of meromorphic λ-flat bundle $\big(\mathcal{Q}\mathcal{E},\mathbb{D}\big)$, which has the KMS-structure at λ_0 indexed by $\mathcal{KMS}(\mathcal{P}\mathcal{E}^0,i)$ $(i=1,\dots,\ell)$ for each $\lambda_0\in\boldsymbol{C}_\lambda$. For later use, we recall how to obtain the limiting mixed twistor structure. For simplicity, we assume $\mathcal{KMS}\big(\mathcal{E}^0,i\big)\subset\boldsymbol{R}\times\{0\}$. See Section 11 of [18] for the general case. See also an account due to Hertling and Sevenheck in [9] for this case.

In a neighbourhood $U(\lambda_0)$ of λ_0, we set

$$\mathcal{G}^{(\lambda_0)}_{(\boldsymbol{a},0)}(E):={}^{\underline{\ell}}\mathcal{G}^{(\lambda_0)}_{(\boldsymbol{a},0)}(\mathcal{Q}^{(\lambda_0)}\mathcal{E})_{|U(\lambda_0)\times\{O\}}$$

for $\boldsymbol{a}\in\mathcal{P}ar(\mathcal{P}\mathcal{E}^0,\underline{\ell})$. (See (39) for the right hand side. In this simpler case, we have only to take Gr with respect to parabolic filtrations.) By varying $\lambda_0\in\boldsymbol{C}_\lambda$ and gluing them, we obtain the vector bundle $\mathcal{G}_{(\boldsymbol{a},0)}(E)$ on $\boldsymbol{C}_\lambda$. It is endowed with the nilpotent maps $\mathcal{N}_i$ $(i=1,\dots,\ell)$, which are the nilpotent part of the residues $\mathrm{Res}_i(\mathbb{D})$. By applying the same procedure to $(E,\partial_E,\theta^\dagger,h)$ on $X^\dagger-D^\dagger$, we obtain the vector bundle $\mathcal{G}^\dagger_{(-\boldsymbol{a},0)}(E)$ on $\boldsymbol{C}_\mu$ with nilpotent endomorphisms $\mathcal{N}^\dagger_i$ induced by residues $\mathrm{Res}_i(\mathbb{D}^\dagger)$. We would like to glue $\mathcal{G}_{(\boldsymbol{a},0)}(E)$ and $\mathcal{G}^\dagger_{(-\boldsymbol{a},0)}(E)$, to obtain a vector bundle $S^{\mathrm{can}}_{(\boldsymbol{a},0)}(E)$ on $\mathbb{P}^1$.

We have the $\mathbb{D}$-flat decomposition $\mathcal{Q}_0\mathcal{E}_{|\boldsymbol{C}^*_\lambda\times X}=\bigoplus_{\boldsymbol{a}\in\mathcal{P}ar(\mathcal{P}_0\mathcal{E}^0,\underline{\ell})}\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{E}$ with the following property:

- Let M_i be the family of the monodromy endomorphisms along the path $(z_1,\dots,e^{2\pi\sqrt{-1}\theta}z_i,\dots,z_n)$ $(0\le\theta\le1)$ with respect to $\mathbb{D}^f$. Then, the restriction of M_i to $\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{E}$ has the unique eigenvalue $\exp\big(2\pi\sqrt{-1}a_i\big)$.
- $\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{E}_{|\boldsymbol{C}^*_\lambda\times O}\simeq\mathcal{G}_{(\boldsymbol{a},0)}(E)$.

For $\lambda\neq0$, let $H(\mathcal{E}^\lambda)$ be the space of multivalued flat sections of $(\mathcal{E}^\lambda,\mathbb{D}^\lambda)$. We have the holomorphic vector bundle $\mathcal{H}(E)$ on $\boldsymbol{C}^*_\lambda$ whose fiber over λ is $H(\mathcal{E}^\lambda)$. We have the decomposition

$$\mathcal{H}(E)=\bigoplus_{\boldsymbol{a}\in\mathcal{P}ar(\mathcal{P}_0\mathcal{E}^0,\underline{\ell})}\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{H}(E)$$

such that (i) it is preserved by the monodromy M_i, (ii) the restriction of M_i to $\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{H}(E)$ has the unique eigenvalue $\exp\big(2\pi\sqrt{-1}a_i\big)$.

Let $U\subset\boldsymbol{C}^*_\lambda$, and let s be a section of $\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{H}(E)$ on U. We regard s as a multi-valued flat section of $\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{E}$. It is expressed as a finite sum:

$$s=\sum f_{\boldsymbol{m}}\cdot\prod_{i=1}^{\ell}\exp\big(a_i\log z_i\big)\cdot\big(\log z_i\big)^{m_i}$$

Here, $f_{\boldsymbol{m}}$ are holomorphic sections of $\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{E}_{|U\times X}$. We set $\Phi^{\mathrm{can}}_{(\boldsymbol{a},0)}(s)=f_{0|U\times O}$, and thus we obtain an isomorphism

$$\Phi^{\mathrm{can}}_{(\boldsymbol{a},0)}:\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{H}(E)\longrightarrow\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{E}_{|\boldsymbol{C}^*_\lambda\times O}=\mathcal{G}_{(\boldsymbol{a},0)}(E).$$

Let $\boldsymbol{\delta}=(1,\dots,1)\in\boldsymbol{R}^\ell$. We have the $\mathbb{D}^\dagger$-flat decomposition $\mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger_{|\boldsymbol{C}^*_\lambda\times X}=\bigoplus_{\boldsymbol{a}\in\mathcal{P}ar(\mathcal{P}_0\mathcal{E}^0,\underline{\ell})}\mathcal{G}_{(-\boldsymbol{a},0)}\mathcal{E}^\dagger$ with the following property:

- The restriction of M_i^{-1} to $\mathcal{G}_{(-\boldsymbol{a},0)}\mathcal{E}^\dagger$ has the unique eigenvalue $\exp\bigl(-2\pi\sqrt{-1}a_i\bigr)$. (Because the base space is the complex conjugate $X^\dagger - D^\dagger$, the direction of the loop is reversed.)

- $\mathcal{G}_{(-\boldsymbol{a},0)}\mathcal{E}^\dagger_{|\boldsymbol{C}^*_\mu\times O} \simeq \mathcal{G}^\dagger_{(-\boldsymbol{a},0)}(E)$.

Similarly, let $\mathcal{H}^\dagger(E)$ be the holomorphic vector bundle on $\boldsymbol{C}^*_\mu$ whose fiber over μ is the space of the multi-valued flat sections of $(\mathcal{E}^{\dagger\,\mu}, \mathbb{D}^{\dagger\,\mu})$. We have the decomposition

$$\mathcal{H}^\dagger(E) = \bigoplus_{\boldsymbol{a}\in\mathcal{P}ar(\mathcal{P}_0\mathcal{E}^0,\underline{\ell})} \mathcal{G}_{(-\boldsymbol{a},0)}\mathcal{H}^\dagger(E)$$

such that the restriction of M_i^{-1} to $\mathcal{G}_{(-\boldsymbol{a},0)}\mathcal{H}^\dagger(E)$ has the unique eigenvalue $\exp\bigl(-2\pi\sqrt{-1}a_i\bigr)$. For a section s of $\mathcal{G}_{(-\boldsymbol{a},0)}\mathcal{H}^\dagger(E)_{|U}$, we have an expression

$$s = \sum f^\dagger_{\boldsymbol{m}} \cdot \prod_{i=1}^{\ell} \exp\bigl(-a_i \log \overline{z}_i\bigr)\cdot \bigl(\log \overline{z}_i\bigr)^{m_i},$$

where $f^\dagger_{\boldsymbol{m}}$ are sections of $\mathcal{Q}\mathcal{E}^\dagger_{|U\times X}$. We set $\Phi^{\mathrm{can}\,\dagger}_{(\boldsymbol{a},0)}(s) = f^\dagger_{0|U\times O}$, and thus we obtain an isomorphism

$$\Phi^{\mathrm{can}\,\dagger}_{(\boldsymbol{a},0)} : \mathcal{G}_{(-\boldsymbol{a},0)}\mathcal{H}^\dagger(E) \longrightarrow \mathcal{G}_{(-\boldsymbol{a},0)}\mathcal{E}^\dagger_{|\boldsymbol{C}^*_\mu\times O} = \mathcal{G}^\dagger_{(-\boldsymbol{a},0)}(E).$$

By construction, we have the natural isomorphism $\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{H}(E) \simeq \mathcal{G}_{(-\boldsymbol{a},0)}\mathcal{H}^\dagger(E)$ under the identification of $\boldsymbol{C}^*_\lambda = \boldsymbol{C}^*_\mu$ via $\mu = \lambda^{-1}$. Thus, we obtain the vector bundle $S^{\mathrm{can}}_{(\boldsymbol{a},0)}(E)$ by gluing $\mathcal{G}_{(\boldsymbol{a},0)}(E)$ and $\mathcal{G}^\dagger_{(-\boldsymbol{a},0)}(E)$. Under the gluing, we have the relation

$$\lambda^{-1}\mathcal{N}_i = -\mu^{-1}\mathcal{N}_i^\dagger.$$

Thus, $\mathcal{N}_i\cdot t_0^{(-1)}$ and $\mathcal{N}_\infty \cdot t_\infty^{(-1)}$ give the morphism $\mathcal{N}_i^\triangle : S^{\mathrm{can}}_{(\boldsymbol{a},0)}(E) \longrightarrow S^{\mathrm{can}}_{(\boldsymbol{a},0)}(E)\otimes\mathbb{T}(-1)$. The tuple of them is denoted by $\boldsymbol{N}^\triangle$.

The morphism $\mathcal{S}_0 : \mathcal{E}\otimes\sigma^*\mathcal{E}^\dagger \longrightarrow \mathcal{O}_{\mathcal{X}-\mathcal{D}}$ is extended to $\mathcal{Q}_0\mathcal{E}\otimes\sigma^*\mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger \longrightarrow \mathcal{O}_{\mathcal{X}}$. Similarly, we have $\mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger\otimes\sigma^*\mathcal{Q}_0\mathcal{E} \longrightarrow \mathcal{O}_{\mathcal{X}^\dagger}$. They induce

$$\mathcal{G}_{(\boldsymbol{a},0)}(E)\otimes\sigma^*\mathcal{G}^\dagger_{(-\boldsymbol{a},0)}(E) \longrightarrow \mathcal{O}_{\boldsymbol{C}_\lambda},$$

$$\mathcal{G}^\dagger_{(-\boldsymbol{a},0)}(E)\otimes\sigma^*\mathcal{G}_{(\boldsymbol{a},0)}(E) \longrightarrow \mathcal{O}_{\boldsymbol{C}_\mu},$$

$$\mathcal{G}_{(-\boldsymbol{a},0)}\mathcal{H}^\dagger(E)\otimes\sigma^*\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{H}(E) \longrightarrow \mathcal{O}_{\boldsymbol{C}^*_\lambda}$$

They are preserved by the above isomorphisms. Hence, we obtain $\mathcal{S}_{(\boldsymbol{a},0)} : S^{\mathrm{can}}_{(\boldsymbol{a},0)}(E)\otimes\sigma^*S^{\mathrm{can}}_{(\boldsymbol{a},0)}(E) \longrightarrow \mathbb{T}(0)$. Theorem 12.22 of [18] implies the following.

Proposition 6.16 *$(S^{\mathrm{can}}_{(\boldsymbol{a},0)}(E), \boldsymbol{N}^\triangle, \mathcal{S})$ is a polarized mixed twistor structure of weight 0 in ℓ-variables.* ∎

By Theorem 4.1, a polarized mixed twistor structure induces a nilpotent orbit. This is the reduction from tame harmonic bundles to nilpotent orbits.

Remark 6.17 *Although the notation is changed, the construction explained in this subsection is the same as that in* [18]*. In the tame case, $\mathcal{Q}\mathcal{E}$ is equal to the sheaf of holomorphic sections whose norms with respect to h are of polynomial growth order. We also remark the uniqueness in Lemma* 5.44*.* ∎

Family version The construction can be done in family on $D_{\underline{\ell}} := \bigcap_{i=1}^{\ell} D_i$. As in the construction of $\mathcal{G}_{(\boldsymbol{a},0)}(E)$, we obtain the vector bundle ${}^{\underline{\ell}}\mathcal{G}_{(\boldsymbol{a},0)}(\mathcal{QE})$ on $\mathcal{D}_{\underline{\ell}} := \boldsymbol{C}_\lambda \times D_{\underline{\ell}}$, as the gluing of ${}^{\underline{\ell}}\mathcal{G}^{(\lambda_0)}_{(\boldsymbol{a},0)}(\mathcal{Q}^{(\lambda_0)}\mathcal{E})$. They are equipped with the nilpotent maps $\mathcal{N}_i$ $(i = 1,\dots,\ell)$. By applying the nearby cycle functors for $\mathcal{R}$-modules along z_i $(i = 1,\dots,\ell)$, or by a direct consideration as in Subsection 8.8.3 of [18], we obtain the induced family of flat λ-connections $\mathbb{D}_{\boldsymbol{a},0}$ of ${}^{\underline{\ell}}\mathcal{G}_{(\boldsymbol{a},0)}(\mathcal{QE})$ for which $\mathcal{N}_i$ are flat. Similarly, we obtain a family of μ-flat bundles $\bigl({}^{\underline{\ell}}\mathcal{G}_{(-\boldsymbol{a},0)}(\mathcal{QE}^\dagger), \mathbb{D}^\dagger_{(-\boldsymbol{a},0)}\bigr)$ on $\boldsymbol{C}_\mu \times D^\dagger_{\underline{\ell}}$ with flat nilpotent maps $\mathcal{N}_i^\dagger$.

Let $q : X - D \longrightarrow D_{\underline{\ell}}$ be the projection. We naturally obtain a holomorphic vector bundle $\widetilde{\mathcal{H}}(E)$ on $\boldsymbol{C}_\lambda^* \times D_{\underline{\ell}}$, whose fiber over (λ, P) is the space of multi-valued flat sections of $(\mathcal{E}^\lambda, \mathbb{D}^\lambda)_{|q^{-1}(P)}$. It has the generalized eigen decomposition $\widetilde{\mathcal{H}}(E) = \bigoplus {}^{\underline{\ell}}\mathcal{G}_{(\boldsymbol{a},0)}\widetilde{\mathcal{H}}(E)$ with respect to the monodromy endomorphisms around D_i. $(i = 1,\dots,\ell)$. It is naturally equipped with the family of flat connections $\mathbb{D}^f_{\boldsymbol{a},0}$.

By using the family of flat bundles $\bigl(\mathcal{G}_{(\boldsymbol{a},0)}\mathcal{E}, \mathbb{D}^f\bigr)$, we obtain the flat isomorphisms

$$\Phi^{\mathrm{can}}_{(\boldsymbol{a},0)} : {}^{\underline{\ell}}\mathcal{G}_{(\boldsymbol{a},0)}\widetilde{\mathcal{H}}(E) \longrightarrow {}^{\underline{\ell}}\mathcal{G}_{(\boldsymbol{a},0)}(\mathcal{QE})_{|\boldsymbol{C}_\lambda^* \times D_{\underline{\ell}}}.$$

Similarly, we obtain the flat isomorphisms $\Phi^{\mathrm{can}}_{(\boldsymbol{a},0)} : {}^{\underline{\ell}}\mathcal{G}_{(\boldsymbol{a},0)}\widetilde{\mathcal{H}}(E) \longrightarrow {}^{\underline{\ell}}\mathcal{G}_{(-\boldsymbol{a},0)}(\mathcal{QE}^\dagger)_{|\boldsymbol{C}_\mu^* \times D^\dagger_{\underline{\ell}}}$. As the gluing, we obtain a variation of $\mathbb{P}^1$-holomorphic vector bundles $\bigl({}^{\underline{\ell}}\mathcal{E}^\triangle_{\boldsymbol{a},0}, \mathbb{D}^\triangle_{\boldsymbol{a},0}\bigr)$ with a tuple $\boldsymbol{N}^\triangle$ of flat nilpotent morphisms

$$\mathcal{N}_i^\triangle : {}^{\underline{\ell}}\mathcal{E}^\triangle_{\boldsymbol{a},0} \longrightarrow {}^{\underline{\ell}}\mathcal{E}^\triangle_{\boldsymbol{a},0} \otimes \mathbb{T}(-1), \quad (i = 1,\dots,\ell)$$

We also have the induced flat symmetric pairing $\mathcal{S} : {}^{\underline{\ell}}\mathcal{E}^\triangle_{\boldsymbol{a},0} \otimes \sigma^* {}^{\underline{\ell}}\mathcal{E}^\triangle_{\boldsymbol{a},0} \longrightarrow \mathbb{T}(0)$. By Proposition 6.16,

$$\bigl({}^{\underline{\ell}}\mathcal{E}^\triangle_{\boldsymbol{a},0}, \boldsymbol{N}^\triangle, \mathbb{D}^\triangle_{\boldsymbol{a},0}, \mathcal{S}_{\boldsymbol{a},0}\bigr)$$

is a variation of polarized mixed twistor structures of weight 0 in ℓ-variables. (See Subsection 2.4.1.)

7 Prolongation and reductions in the integrable case

7.1 Preliminary Estimate

7.1.1 Statements

Let $X := \Delta^n$ and $D := \{z_1 = 0\}$. Let $(E, \overline{\partial}_E, \theta, h)$ be an unramifiedly good wild harmonic bundle on $X - D$. For simplicity, we assume that there exists a holomorphic decomposition

$$(E, \theta) = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta)} (E_{\mathfrak{a}}, \theta_{\mathfrak{a}}) \tag{41}$$

such that $\theta_{\mathfrak{a}} - d\mathfrak{a} \cdot \pi_{\mathfrak{a}}$ are tame, where $\pi_{\mathfrak{a}}$ denotes the projection onto $E_{\mathfrak{a}}$ with respect to the decomposition (41).

Remark 7.1 *Since $(E, \overline{\partial}_E, \theta, h)$ is assumed to be unramifiedly good, such a decomposition exists on a neighbourhood of each point of D. Because we are interested in the behaviour around O, we may assume such a decomposition exists globally by replacing X with a small neighbourhood of O.* ∎

Let $\mathcal{U}$ be a holomorphic section of $\mathrm{End}(E)$ on $X - D$ such that $[\theta, \mathcal{U}] = 0$. Let $\mathcal{Q}$ be a C^∞-section of $\mathrm{End}(E)$ on $X - D$ such that $\mathcal{Q} = \mathcal{Q}^\dagger$. We assume the following equations:

$$\partial_E \mathcal{U} - [\theta, \mathcal{Q}] + \theta = 0 \tag{42}$$

$$\partial_E \mathcal{Q} + [\theta, \mathcal{U}^\dagger] = 0 \tag{43}$$

We set $\widetilde{\mathcal{U}} := \mathcal{U} + \sum_{\mathfrak{a} \in \mathrm{Irr}(\theta)} \mathfrak{a} \cdot \pi_{\mathfrak{a}}$. We will prove the following proposition in Subsections 7.1.2–7.1.6.

Proposition 7.2 *$\widetilde{\mathcal{U}} = O(1)$ and $\mathcal{Q} = O\Bigl(\bigl(-\log|z_1|\bigr)^M\Bigr)$ for some $M > 0$ with respect to h.*

Remark 7.3 *Eventually, we obtain that $\mathcal{Q}$ is bounded. (See Corollary 7.16 and Corollary 7.22.) See Corollary 7.8 for the boundedness of $\widetilde{\mathcal{U}}$ in the case that D is normal crossing.* ∎

We set $g_{\mathrm{irr}}(\lambda)=\exp\Bigl(\bigoplus\lambda\overline{\mathfrak{a}}\cdot\pi_{\mathfrak{a}}\Bigr)$. Let $\lambda_0\in\boldsymbol{C}$, and let $U(\lambda_0)$ be a small neighbourhood of λ_0 in $\boldsymbol{C}$. Let p_λ be the projection of $U(\lambda_0)\times(X-D)$ onto $X-D$. We consider the hermitian metric

$$\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h:=g_{\mathrm{irr}}(\lambda-\lambda_0)^{*}h \tag{44}$$

of $p_\lambda^{-1}E$ on $U(\lambda_0)\times(X-D)$. We regard $\mathcal{U}$ and $\mathcal{Q}$ as C^∞-sections of $\mathrm{End}\bigl(p_\lambda^{-1}E\bigr)$. We will prove the following lemma in Subsection 7.1.7.

Proposition 7.4 *Assume $U(\lambda_0)$ is sufficiently small. Then, $\widetilde{\mathcal{U}}=O(1)$ and $\mathcal{Q}=O\bigl((-\log|z_1|)^M\bigr)$ with respect to $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h$.*

7.1.2 Preliminary

We take orthogonal decompositions $E=\bigoplus E'_{\mathfrak{a},\alpha}=\bigoplus E'_{\mathfrak{a}}$ as in Subsection 6.2. For any $f\in\mathrm{End}(E)$, we have the decompositions:

$$f=\sum f'_{\mathfrak{a},\mathfrak{b}},\quad f'_{\mathfrak{a},\mathfrak{b}}\in\mathrm{Hom}\bigl(E'_{\mathfrak{b}},E'_{\mathfrak{a}}\bigr)$$

$$f=\sum f'_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)},\quad f'_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)}\in\mathrm{Hom}\bigl(E'_{\mathfrak{b},\beta},E'_{\mathfrak{a},\alpha}\bigr).$$

We have similar decompositions for sections of $\mathrm{End}(E)\otimes\Omega^{p,q}$. The following lemma is easy to show by using Proposition 6.2.

Lemma 7.5 *Let f be a C^∞-section of $\mathrm{End}(E)$ such that f commutes with θ.*

- *If $\mathfrak{a}\neq\mathfrak{b}$, we have $|f'_{\mathfrak{a},\mathfrak{b}}|_h=O\Bigl(\exp\bigl(-\epsilon|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}\bigr)\Bigr)\cdot|f|_h$ for some $\epsilon>0$.*
- *If $\alpha\neq\beta$, $\bigl|f'_{(\mathfrak{a},\alpha),(\mathfrak{a},\beta)}\bigr|_h=O\bigl(|z_1|^{\epsilon}\bigr)\cdot|f|_h$ for some $\epsilon>0$.* ∎

7.1.3 Step 1

Let θ_1 denote the dz_1-component of θ.

Lemma 7.6 *We have the following estimate with respect to h:*

$$\bigl[\theta_1^{\dagger},\mathcal{U}\bigr]=O\Bigl(\frac{d\bar{z}_1}{|z_1|\cdot\bigl(-\log|z_1|\bigr)}\Bigr)\cdot\bigl|\mathcal{U}\bigr|_h$$

Proof In the following, ϵ_i denote some positive constants. We have the decomposition:

$$\bigl[\theta_1^{\dagger},\mathcal{U}\bigr]=\sum_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}\bigl(\theta^{\dagger\prime}_{1,\mathfrak{a},\mathfrak{b}}\circ\mathcal{U}'_{\mathfrak{b},\mathfrak{c}}-\mathcal{U}'_{\mathfrak{a},\mathfrak{b}}\circ\theta^{\dagger\prime}_{1,\mathfrak{b},\mathfrak{c}}\bigr)$$

By the estimates in Subsection 11.2 of [19] (see Subsection 6.2), we have the following estimates for $\mathfrak{a}\neq\mathfrak{b}$:

$$\theta^{\dagger\prime}_{1,\mathfrak{a},\mathfrak{b}}=O\Bigl(\exp\bigl(-\epsilon_1|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}\bigr)\cdot d\bar{z}_1\Bigr)$$

Because $\mathcal{U}$ and θ are commutative, we have the following estimate for $\mathfrak{a}\neq\mathfrak{b}$ due to Lemma 7.5:

$$\mathcal{U}'_{\mathfrak{a},\mathfrak{b}}=O\Bigl(\exp\bigl(-\epsilon_2|z_1|^{-1}\bigr)\Bigr)\cdot\bigl|\mathcal{U}\bigr|_h$$

Hence, we have the following estimate with respect to h:

$$\bigl[\theta_1^{\dagger},\mathcal{U}\bigr]=\sum_{\mathfrak{a}}\bigl[\theta^{\dagger\prime}_{1,\mathfrak{a},\mathfrak{a}},\mathcal{U}'_{\mathfrak{a},\mathfrak{a}}\bigr]+O\Bigl(\exp\bigl(-\epsilon_3|z_1|^{-1}\bigr)\cdot d\bar{z}_1\Bigr)\cdot\bigl|\mathcal{U}\bigr|_h$$

Similarly, we have the following estimates for $\alpha \neq \beta$, by Theorem 11.12 of [19] and Lemma 7.5:

$$\theta^{\dagger\prime}_{1,(\mathfrak{a},\alpha),(\mathfrak{a},\beta)} = O\Big(|z_1|^{\epsilon_4}\Big) \cdot \frac{d\overline{z}_1}{\overline{z}_1}, \qquad \mathcal{U}'_{(\mathfrak{a},\alpha),(\mathfrak{a},\beta)} = O\big(|z_1|^{\epsilon_4}\big)$$

By Proposition 6.3, $\theta^{\dagger\prime}_{(\mathfrak{a},\alpha),(\mathfrak{a},\alpha)} - \big(d\mathfrak{a} + \alpha \cdot dz_1/z_1\big) \cdot \pi'_{\mathfrak{a},\alpha}$ is bounded with respect to h and Poincaré metric of $X - D$. Hence, we obtain

$$[\theta_1^{\dagger}, \mathcal{U}] = \sum_{\mathfrak{a},\alpha} \big[\theta^{\dagger\prime}_{1,(\mathfrak{a},\alpha),(\mathfrak{a},\alpha)}, \mathcal{U}'_{(\mathfrak{a},\alpha),(\mathfrak{a},\alpha)}\big] + O\Big(|z_1|^{\epsilon_5}\Big) \cdot \frac{d\overline{z}_1}{\overline{z}_1} \cdot \big|\mathcal{U}\big|_h = O\Big(\frac{d\overline{z}_1}{|z_1| \cdot \big(-\log|z_1|\big)}\Big) \cdot \big|\mathcal{U}\big|_h.$$

Thus, we obtain Lemma 7.6. ∎

7.1.4 Step 2

Let $\overline{\partial}_1$ denote the $d\overline{z}_1$-components of $\overline{\partial}_E$ and $\overline{\partial}$. Similarly, let ∂_1 denote the dz_1-component of ∂_E and ∂. The following holds:

$$\overline{\partial}_1 \big|\mathcal{U}\big|_h^2 = \big(\mathcal{U}, \partial_1 \mathcal{U}\big)_h = \big(\mathcal{U}, [\theta_1, \mathcal{Q}] - \theta_1\big)_h = -\operatorname{tr}\Big(\mathcal{U} \cdot \big[\theta_1^{\dagger}, \mathcal{Q}\big]\Big) - \operatorname{tr}\Big(\mathcal{U} \cdot \theta_1^{\dagger}\Big) = -\operatorname{tr}\Big(\big[\mathcal{U}, \theta_1^{\dagger}\big] \cdot \mathcal{Q}\Big) - \operatorname{tr}\Big(\mathcal{U} \cdot \theta_1^{\dagger}\Big)$$

Hence, we obtain

$$\overline{\partial}_1 \big|\mathcal{U}\big|_h^2 = O\Big(\frac{d\overline{z}_1}{|z_1| \cdot (-\log|z_1|)}\Big) \cdot \big|\mathcal{U}\big|_h \cdot \big|\mathcal{Q}\big|_h + O\Big(\frac{d\overline{z}_1}{|z_1|^N}\Big) \cdot \big|\mathcal{U}\big|_h.$$

We also have

$$\overline{\partial}_1 |\mathcal{Q}|_h^2 = -\big(\mathcal{Q}, [\theta_1, \mathcal{U}^{\dagger}]\big)_h + \big([\theta_1^{\dagger}, \mathcal{U}], \mathcal{Q}\big) = O\Big(\frac{d\overline{z}_1}{|z_1| \cdot \big(-\log|z_1|\big)}\Big) \cdot \big|\mathcal{U}\big|_h \cdot \big|\mathcal{Q}\big|_h.$$

Therefore, we obtain

$$\overline{\partial}_1 \Big(\big|\mathcal{U}\big|_h^2 + \big|\mathcal{Q}\big|_h^2\Big) = O\Big(\frac{d\overline{z}_1}{|z_1|\big(-\log|z_1|\big)}\Big) \cdot \big|\mathcal{U}\big|_h \cdot \big|\mathcal{Q}\big|_h + O\Big(\frac{d\overline{z}_1}{|z_1|^N}\Big) \cdot \big|\mathcal{U}\big|_h. \tag{45}$$

We set $r := |z_1|$ and $F := \big(\big|\mathcal{U}\big|_h^2 + \big|\mathcal{Q}\big|_h^2 + 1\big)^{1/2}$. We use the polar coordinate $\big(r, \arg(z_1), z_2, \ldots, z_n\big)$. We consider the estimate on a simply connected region $Z(\vartheta_0, \vartheta_1) := \{\vartheta_0 < \arg(z_1) < \vartheta_1\}$ for some fixed $\vartheta_0 < \vartheta_1$. We obtain the following estimate from (45):

$$\frac{\partial}{\partial r} F^2 = G_1 \cdot F^2 + G_2 \cdot F, \qquad G_1 = O\Big(\frac{1}{r \cdot (-\log r)}\Big), \quad G_2 = O\big(\frac{1}{r^N}\big)$$

We take a solution $H \neq 0$ of the differential equation:

$$\frac{\partial}{\partial r} H = -G_1 \cdot H$$

There exist $C_1 > 0$ and $M_0 > 0$ such that $C_1^{-1} \cdot (-\log r)^{-M_0} \leq |H| \leq C_1 \cdot (-\log r)^{M_0}$. Since $Z(\vartheta_0, \vartheta_1)$ is simply connected, we can take $H^{1/2}$. Then, we have

$$\frac{\partial}{\partial r}\big(H \cdot F^2\big) = G_2 \cdot H \cdot F = (G_2 \cdot H^{1/2}) \cdot (H^{1/2} \cdot F).$$

Because $G_2 \cdot H^{1/2} = O\Big(r^{-M_1}\Big)$, we obtain $H \cdot F^2 = O(r^{-M_2})$, and hence $F = O\big(r^{-M_3}\big)$. Thus, we obtain the following estimate on $Z(\vartheta_0, \vartheta_1)$ for some $M_4 > 0$:

$$\big|\mathcal{U}\big|_h = O(r^{-M_4}), \quad \big|\mathcal{Q}\big|_h = O\big(r^{-M_4}\big) \tag{46}$$

By varying θ_0 and θ_1, we obtain the estimate (46) on $X - D$. In particular, we obtain the following estimate on $X - D$ for $\mathfrak{a} \neq \mathfrak{b}$:

$$\mathcal{U}'_{\mathfrak{a},\mathfrak{b}} = O\Big(\exp\big(-\epsilon |z_1|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})}\big)\Big)$$

7.1.5 Step 3

We have $\big[\theta_1,\mathcal{U}^{\dagger}\big]=\big[\theta_1,\widetilde{\mathcal{U}}^{\dagger}\big]+O\Big(\exp\big(-\epsilon|z_1|^{-1}\big)\cdot dz_1\Big)$ with respect to h. By an argument in the proof of Lemma 7.6, we obtain the following estimate with respect to h:

$$\big[\theta_1,\mathcal{U}^{\dagger}\big]=O\Big(\frac{dz_1}{|z_1|\cdot(-\log|z_1|)}\Big)\cdot\big|\widetilde{\mathcal{U}}\big|_h+O\Big(\exp\big(-\epsilon|z_1|^{-1}\big)\cdot dz_1\Big) \tag{47}$$

According to an estimate in Subsection 11.5.2 of [19], we have

$$\partial_1\mathcal{U}=\partial_1\widetilde{\mathcal{U}}-\sum_{\mathfrak{a}\in\mathrm{Irr}(\theta)}\partial_1\mathfrak{a}\cdot\pi_{\mathfrak{a}}+O\Big(\exp\big(-\epsilon|z_1|^{-1}\big)\cdot dz_1\Big).$$

We set $\widetilde{\theta}:=\theta-\sum_{\mathfrak{a}\in\mathrm{Irr}(\theta)}d\mathfrak{a}\cdot\pi_{\mathfrak{a}}$. We obtain the following estimates with respect to h:

$$\partial_1\widetilde{\mathcal{U}}-\big[\theta_1,\mathcal{Q}\big]+\widetilde{\theta}_1=O\Big(\exp\big(-\epsilon|z_1|^{-1}\big)\Big) \tag{48}$$

$$\partial_1\mathcal{Q}+\big[\theta_1,\widetilde{\mathcal{U}}^{\dagger}\big]=O\Big(\exp\big(-\epsilon|z_1|^{-1}\big)\Big) \tag{49}$$

We set $\widetilde{F}:=\big(\big|\widetilde{\mathcal{U}}\big|_h^2+\big|\mathcal{Q}\big|_h^2+1\big)^{1/2}$. As in Step 2, we consider the estimates on $Z(\vartheta_0,\vartheta_1)$. By using an argument in Subsection 7.1.4, we obtain

$$\frac{\partial}{\partial r}\widetilde{F}^2=\widetilde{G}_1\cdot\widetilde{F}^2+\widetilde{G}_2\cdot\widetilde{F},\qquad \widetilde{G}_1=O\Big(\frac{1}{r\cdot(-\log r)}\Big),\quad \widetilde{G}_2=O\Big(\frac{1}{r}\Big).$$

We take a solution $\widetilde{H}_1\neq 0$ of the differential equation:

$$\frac{\partial}{\partial r}\widetilde{H}_1=-\widetilde{G}_1\cdot\widetilde{H}_1$$

Note $\log\big|\widetilde{H}_1\big|=O\Big(\log\big(-\log r\big)\Big)$. By choosing $\widetilde{H}_1^{1/2}$, we obtain

$$\frac{\partial}{\partial r}\big(\widetilde{H}_1\cdot\widetilde{F}^2\big)=\big(\widetilde{G}_2\cdot\widetilde{H}_1^{1/2}\big)\cdot\big(\widetilde{H}_1^{1/2}\cdot\widetilde{F}\big).$$

Because $\widetilde{G}_2\cdot\widetilde{H}_1^{1/2}=O\Big(r^{-1}\cdot\big(-\log r\big)^{M_5}\Big)$ for some $M_5>0$, we obtain $\widetilde{H}_1\cdot\widetilde{F}^2=O\Big(\big(-\log r\big)^{M_6}\Big)$ for some $M_6>0$, and thus $\widetilde{F}=O\Big(\big(-\log r\big)^{M_7}\Big)$ for some $M_7>0$. Therefore, we obtain the following estimates with respect to h:

$$\widetilde{\mathcal{U}}=O\Big(\big(-\log r\big)^{M_7}\Big),\quad \mathcal{Q}=O\Big(\big(-\log r\big)^{M_7}\Big) \tag{50}$$

7.1.6 Step 4

By (50), $\widetilde{\mathcal{U}}$ is a holomorphic section of $\mathcal{P}_0\operatorname{End}(E)$. Because $[\theta,\widetilde{\mathcal{U}}]=0$, we obtain the boundedness of $\big|\widetilde{\mathcal{U}}\big|_h$ by an estimate in Subsection 11.7 of [19]. Thus, the proof of Proposition 7.2 is finished. ∎

Remark 7.7 *From* (47) *and* (49)*, we also have the following estimate:*

$$\partial_1\mathcal{Q}=O\Big(\frac{dz_1}{|z_1|\cdot(-\log|z_1|)}\Big)$$

Hence, we actually obtain $\mathcal{Q}=O\Big(\log\big(-\log|z_1|\big)\Big)$*. However, we will obtain the boundedness later.* ∎

7.1.7 Proof of Proposition 7.4

For an endomorphism f of E, we have the following:

$$\big|f\big|_{\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h} = \big|g_{\mathrm{irr}}(\lambda-\lambda_0)\circ f\circ g_{\mathrm{irr}}(\lambda-\lambda_0)^{-1}\big|_h \tag{51}$$

Hence, the claim for $\widetilde{\mathcal{U}}$ is clear from $[\widetilde{\mathcal{U}}, g_{\mathrm{irr}}(\lambda-\lambda_0)]=0$. We have the decomposition $\mathcal{P}_0\mathcal{E}^0=\bigoplus\mathcal{P}_0\mathcal{E}^0_{\mathfrak{a}}$ extending $E=\bigoplus E_{\mathfrak{a}}$. Let $\boldsymbol{v}=(\boldsymbol{v}_{\mathfrak{a}})$ be a holomorphic frame of $\mathcal{P}_0\mathcal{E}^0$ compatible with the decomposition. Let C be the matrix-valued function determined by $\partial_1\boldsymbol{v}=\boldsymbol{v}\cdot C\cdot dz_1$. We have the decomposition $C=(C_{\mathfrak{a},\mathfrak{b}})$ corresponding to the decomposition $\boldsymbol{v}=(\boldsymbol{v}_{\mathfrak{a}})$. According to an estimate in Subsection 11.5.2 of [19], there exists $\epsilon_1>0$ such that the following holds for $\mathfrak{a}\neq\mathfrak{b}$:

$$C_{\mathfrak{a},\mathfrak{b}}=O\Big(\exp\big(-\epsilon_1|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}\big)\Big)$$

Let A be the matrix-valued function determined by $\mathcal{U}\boldsymbol{v}=\boldsymbol{v}\cdot A$. Note A is block-diagonal, i.e., $A=\bigoplus A_{\mathfrak{a},\mathfrak{a}}$. We have $(\partial_1\mathcal{U})\boldsymbol{v}=\boldsymbol{v}\cdot\big(\partial_1A+[C,A]\cdot dz_1\big)$. We set $B\cdot dz_1:=\partial_1A+[C,A]\cdot dz_1=\big(B_{\mathfrak{a},\mathfrak{b}}\cdot dz_1\big)$. Then, there exists $\epsilon_2>0$ such that the following holds for $\mathfrak{a}\neq\mathfrak{b}$:

$$B_{\mathfrak{a},\mathfrak{b}}=C_{\mathfrak{a},\mathfrak{b}}\cdot A_{\mathfrak{b},\mathfrak{b}}-A_{\mathfrak{a},\mathfrak{a}}\cdot C_{\mathfrak{a},\mathfrak{b}}=O\Big(\exp\big(-\epsilon_2|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}\big)\Big) \tag{52}$$

For any section f of $\mathrm{End}(E)\otimes\Omega^{1,0}$, we have the decomposition

$$f=\sum f_{\mathfrak{a},\mathfrak{b}},\quad f_{\mathfrak{a},\mathfrak{b}}\in\mathrm{Hom}(E_{\mathfrak{b}},E_{\mathfrak{a}})\otimes\Omega^{1,0}.$$

From the relation $\partial_1\mathcal{U}-[\theta_1,\mathcal{Q}]+\theta_1=0$, we obtain the following:

$$(\partial_1\mathcal{U})_{\mathfrak{a},\mathfrak{b}}-\partial_1(\mathfrak{a}-\mathfrak{b})\cdot\mathcal{U}_{\mathfrak{a},\mathfrak{b}}-(\theta_{1,\mathfrak{a}}-\partial_1\mathfrak{a})\cdot\mathcal{U}_{\mathfrak{a},\mathfrak{b}}+\mathcal{U}_{\mathfrak{a},\mathfrak{b}}\cdot(\theta_{1,\mathfrak{b}}-\partial_1\mathfrak{b})=0$$

Note the following (see Proposition 6.2 and Proposition 6.3):

$$\partial(\mathfrak{a}-\mathfrak{b})/\partial z_1\sim|z_1^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})-1}|\cdot dz_1,\quad \big|\theta_{1,\mathfrak{a}}-\partial_1\mathfrak{a}\big|_h=O\big(dz_1/z_1\big),\quad \big|\theta_{1,\mathfrak{b}}-\partial_1\mathfrak{b}\big|_h=O\big(dz_1/z_1\big) \tag{53}$$

The estimate (52) implies the following:

$$\big|(\partial_1\mathcal{U})_{\mathfrak{a},\mathfrak{b}}\big|=O\Big(\exp\big(-\epsilon_2|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}\big)\Big) \tag{54}$$

Due to (53) and (54), there exists $\epsilon_3>0$ such that the following holds for $\mathfrak{a}\neq\mathfrak{b}$:

$$\big|\mathcal{Q}_{\mathfrak{a},\mathfrak{b}}\big|_h=O\Big(\exp\big(-\epsilon_3|z_1|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}\big)\Big)$$

By using (51), we obtain the desired estimate for $\mathcal{Q}$ with respect to $\mathcal{P}^{(\lambda_0)}h$, if $U(\lambda_0)$ is sufficiently small. ∎

7.1.8 Complement for the normal crossing case

Let $X:=\Delta^n$ and $D:=\bigcup_{i=1}^{\ell}\{z_i=0\}$. Let $(E,\overline{\partial}_E,\theta,h)$ be an unramifiedly good wild harmonic bundle on $X-D$. Let $\mathcal{U}$ be a holomorphic section of $\mathrm{End}(E)$ on $X-D$ such that $[\theta,\mathcal{U}]=0$. Let $\mathcal{Q}$ be a C^∞-section of $\mathrm{End}(E)$ on $X-D$ such that $\mathcal{Q}^\dagger=\mathcal{Q}$. Assume that they satisfy the equations (42) and (43). We also assume that there exists a holomorphic decomposition $(E,\theta)=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)}(E_{\mathfrak{a}},\theta_{\mathfrak{a}})$ such that $\theta_{\mathfrak{a}}-d\mathfrak{a}\cdot\pi_{\mathfrak{a}}$ are tame, where $\pi_{\mathfrak{a}}$ denotes the projection onto $E_{\mathfrak{a}}$ with respect to the above decomposition. We set $\widetilde{\mathcal{U}}:=\mathcal{U}+\sum_{\mathfrak{a}\in\mathrm{Irr}(\theta)}\mathfrak{a}\cdot\pi_{\mathfrak{a}}$.

Corollary 7.8 *$\widetilde{\mathcal{U}}$ is bounded with respect to h.*

Proof It follows from Proposition 7.2 above and the estimate in Subsection 11.7 of [19]. ∎

7.2 Prolongation of variation of integrable twistor structures

7.2.1 Statements

Let X be a complex manifold, and let D be a simple normal crossing divisor of X. Let $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$ be a variation of pure polarized integrable twistor structures of weight 0 on $\mathbb{P}^1 \times (X - D)$. We have the underlying harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X - D$.

Definition 7.9

- *We say that $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$ is tame (wild, good wild, unramifiedly good wild), if $(E, \overline{\partial}_E, \theta, h)$ is tame (wild, good wild, unramifiedly good wild).*
- *If we are given a real structure κ of $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$, we say that the variation of polarized pure twistor-TERP structures $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S}, \kappa, 0)$ is tame (wild, good wild, unramifiedly good wild), if $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$ is tame (wild, good wild, unramifiedly good wild).*

Note that "wild" does not imply "good wild" as remarked in Remark 6.1. ∎

Assume that $(E, \overline{\partial}_E, \theta, h)$ is good wild. We will show the following proposition later. (The tame case was shown in [9].)

Lemma 7.10 *The sets of $\mathcal{KMS}\big(\mathcal{P}\mathcal{E}^0, i\big)$ are contained in $\boldsymbol{R} \times \{0\}$.*

We use the notation in Subsection 2.1.7. As explained in Subsection 6.3, $(\mathcal{E}, \mathbb{D})$ is prolonged to the family of meromorphic λ-flat bundles $(\mathcal{Q}\mathcal{E}, \mathbb{D})$ on $\boldsymbol{C}_{\lambda} \times (X, D)$, and $(\mathcal{E}^{\dagger}, \mathbb{D}^{\dagger})$ is prolonged to the family of meromorphic μ-flat bundles $(\mathcal{Q}\mathcal{E}^{\dagger}, \mathbb{D}^{\dagger})$ on $\boldsymbol{C}_{\mu} \times (X^{\dagger}, D^{\dagger})$.

Proposition 7.11

- *$\widetilde{\mathbb{D}}^f$ (resp. $\widetilde{\mathbb{D}}^{\dagger\, f}$) gives a meromorphic flat connection of $\mathcal{Q}\mathcal{E}$ (resp. $\mathcal{Q}\mathcal{E}^{\dagger}$).*
- *If a real structure κ of $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$ is given, $\kappa_0 : \gamma^*\mathcal{E}^{\dagger} \simeq \mathcal{E}$ is extended to the isomorphism $\gamma^*\mathcal{Q}\mathcal{E}^{\dagger} \simeq \mathcal{Q}\mathcal{E}$. Similarly, $\kappa_{\infty} : \gamma^*\mathcal{E} \simeq \mathcal{E}^{\dagger}$ is extended to $\gamma^*\mathcal{Q}\mathcal{E} \simeq \mathcal{Q}\mathcal{E}^{\dagger}$.*

For the proof of Lemma 7.10 and Proposition 7.11, we may and will assume (i) D is smooth, i.e., $\ell = 1$, (ii) $(E, \overline{\partial}_E, \theta, h)$ is unramified.

7.2.2 Meromorphic connection of $\mathcal{P}^{(\lambda_0)}\mathcal{E}$

Let $\lambda_0 \in \boldsymbol{C}_{\lambda}$, and let $U(\lambda_0)$ be a small neighbourhood of λ_0 in $\boldsymbol{C}_{\lambda}$. We set $\mathcal{X}^{(\lambda_0)} := U(\lambda_0) \times X$ and $\mathcal{D}^{(\lambda_0)} := U(\lambda_0) \times D$. Recall that we have a family of meromorphic λ-flat bundles $(\mathcal{P}^{(\lambda_0)}\mathcal{E}, \mathbb{D})$ on $(\mathcal{X}^{(\lambda_0)}, \mathcal{D}^{(\lambda_0)})$, as explained in Subsection 6.3. Note that $\mathcal{P}^{(\lambda_0)}\mathcal{E}$ is identified with the sheaf of holomorphic sections of $\mathcal{E}$ of polynomial order with respect to $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}} h$, because $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}} h$ and $\mathcal{P}^{(\lambda_0)} h$ are mutually bounded up to polynomial orders. (See (44) for $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}} h$. They are different in general.)

Proposition 7.12 $\widetilde{\mathbb{D}}^f$ *gives a meromorphic flat connection of $\mathcal{P}^{(\lambda_0)}\mathcal{E}$.*

Proof We have only to show $\lambda^2 \nabla_{\lambda}\big(\partial_{\lambda}\big)\mathcal{P}^{(\lambda_0)}\mathcal{E} \subset \mathcal{P}^{(\lambda_0)}\mathcal{E}$. As mentioned in Subsection 2.1.7, we have the induced holomorphic section $\mathcal{U}$ of $\mathrm{End}(E)$ on $X - D$ such that $[\theta, \mathcal{U}] = 0$, and the C^{∞}-section $\mathcal{Q}$ of $\mathrm{End}(E)$ such that $\mathcal{Q}^{\dagger} = \mathcal{Q}$, determined by

$$\nabla_{\lambda} = d_{\lambda} + \big(\lambda^{-1}\mathcal{U} - \mathcal{Q} - \lambda \cdot \mathcal{U}^{\dagger}\big)\frac{d\lambda}{\lambda},$$

where d_{λ} denote the naturally induced flat connection of $p_{\lambda}^{-1}E$ along the λ-direction. They satisfy the equations (42) and (43).

Let $\boldsymbol{v} = (\boldsymbol{v}_{\mathfrak{a}})$ be a holomorphic frame of $\mathcal{P}_0\mathcal{E}^0$ compatible with the decomposition $\mathcal{P}_0\mathcal{E}^0 = \bigoplus_{\mathfrak{a}} \mathcal{P}_0\mathcal{E}^0_{\mathfrak{a}}$. Corresponding to the decomposition $\boldsymbol{v} = (\boldsymbol{v}_{\mathfrak{a}})$, the identity matrix is decomposed into $\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} I_{\mathfrak{a}}$. We regard $\boldsymbol{v}$ as a C^∞-frame of $\mathcal{E}_{|\mathcal{X}^{(\lambda_0)}-\mathcal{D}^{(\lambda_0)}}$, and we set

$$\widetilde{\boldsymbol{v}} = g_{\mathrm{irr}}(\lambda-\lambda_0)^{-1}\boldsymbol{v} = \boldsymbol{v}\cdot\Bigl(\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \exp\bigl(-(\lambda-\lambda_0)\cdot\overline{\mathfrak{a}}\bigr)\cdot I_{\mathfrak{a}}\Bigr)$$

Let $H\bigl(\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h,\widetilde{\boldsymbol{v}}\bigr)$ denote the Hermitian matrix-valued function whose (i,j)-entries are given by $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h(\widetilde{v}_i,\widetilde{v}_j)$. Then, it is clear that $H\bigl(\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h,\widetilde{\boldsymbol{v}}\bigr)$ and its inverse are of polynomial order. We also have the following relation:

$$d_\lambda\widetilde{\boldsymbol{v}} = \widetilde{\boldsymbol{v}}\cdot A, \qquad A := -\bigoplus \overline{\mathfrak{a}}\cdot d\lambda\cdot I_{\mathfrak{a}}$$

Let $\boldsymbol{w}$ be a holomorphic frame of $\mathcal{P}^{(\lambda_0)}_a\mathcal{E}$. Let $H\bigl(\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h,\boldsymbol{w}\bigr)$ denote the Hermitian matrix-valued function whose (i,j)-entries are given by $\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h(w_i,w_j)$. Then, $H\bigl(\mathcal{P}^{(\lambda_0)}_{\mathrm{irr}}h,\boldsymbol{w}\bigr)$ and its inverse are of polynomial order. (See Subsection 13.1.2 of [19], for example.) Let G be the matrix-valued function determined by $\boldsymbol{w} = \widetilde{\boldsymbol{v}}\cdot G$. Then, G and G^{-1} are of polynomial order. We have

$$d_\lambda\boldsymbol{w} = \widetilde{\boldsymbol{v}}\cdot\Bigl(A\cdot G + d_\lambda G\Bigr) = \boldsymbol{w}\cdot\Bigl(G^{-1}\cdot A\cdot G + G^{-1}d_\lambda G\Bigr).$$

Since $\widetilde{\boldsymbol{v}}$ and $\boldsymbol{w}$ are λ-holomorphic, G is λ-holomorphic. Hence, $d_\lambda G$ and $G^{-1}\cdot A\cdot G + G^{-1}d_\lambda G$ are of polynomial order

Let B be determined by $\lambda^2\nabla_\lambda(\partial_\lambda)\boldsymbol{w} = \boldsymbol{w}\cdot B$. Then, B is of polynomial order, and hence meromorphic. Thus, the proof of Proposition 7.12 is finished. ∎

We have the irregular decomposition:

$$(\mathcal{P}^{(\lambda_0)}_a\mathcal{E},\mathbb{D})_{|\widehat{\mathcal{D}}^{(\lambda_0)}} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} (\mathcal{P}^{(\lambda_0)}_a\widehat{\mathcal{E}}_{\mathfrak{a}},\widehat{\mathbb{D}}_{\mathfrak{a}}) \tag{55}$$

Lemma 7.13

- $\lambda^2\nabla_\lambda(\partial_\lambda)$ *preserves the decomposition* (55).
- *Assume* $\lambda_0\neq 0$. *Then,* (55) *is the irregular decomposition for* $\bigl(\mathcal{P}^{(\lambda_0)}\mathcal{E},\widetilde{\mathbb{D}}^f\bigr)$, *and* $\mathcal{P}^{(\lambda_0)}_a\mathcal{E}$ *is an unramifiedly good lattice of* $\mathcal{P}^{(\lambda_0)}\mathcal{E}$.

Proof Since it can be shown by a standard argument, we give only an outline. Let $\widehat{\boldsymbol{v}} = (\widehat{\boldsymbol{v}}_{\mathfrak{a}})$ be a frame of $\mathcal{P}^{(\lambda_0)}_a\mathcal{E}_{|\widehat{\mathcal{D}}}$ compatible with the decomposition (55). Let $A = \sum A_{\mathfrak{b},\mathfrak{a}}$ be determined by $\lambda^2\nabla_\lambda(\partial_\lambda)\widehat{\boldsymbol{v}} = \widehat{\boldsymbol{v}}\cdot A$. For $\mathfrak{a}\neq\mathfrak{b}$, let $F_{\mathfrak{b},\mathfrak{a}}:\mathcal{P}^{(\lambda_0)}_a\widehat{\mathcal{E}}_{\mathfrak{a}}\longrightarrow\mathcal{P}^{(\lambda_0)}_a\widehat{\mathcal{E}}_{\mathfrak{b}}$ be given by $F_{\mathfrak{b},\mathfrak{a}}\widehat{\boldsymbol{v}}_{\mathfrak{a}} = \widehat{\boldsymbol{v}}_{\mathfrak{b}}\cdot A_{\mathfrak{b},\mathfrak{a}}$. Because $\bigl[\lambda^2\nabla_\lambda(\partial_\lambda),\mathbb{D}^f\bigr]=0$, we obtain that $F_{\mathfrak{b},\mathfrak{a}}$ is flat. However, meromorphic flat section has to be 0 in the case $\mathfrak{b}\neq\mathfrak{a}$. Thus, we obtain the first claim.

Let us show the second claim. Let $B_{\mathfrak{a}}$ be determined by

$$\mathbb{D}^f(z_1\partial_1)\widehat{\boldsymbol{v}}_{\mathfrak{a}} = \widehat{\boldsymbol{v}}_{\mathfrak{a}}\cdot\bigl((\lambda^{-1}+\overline{\lambda}_0)\cdot z_1\partial_1\mathfrak{a} + B_{\mathfrak{a}}\bigr)$$

Then, $B_{\mathfrak{a}}$ is regular. For $\mathfrak{a}=0$, the following holds:

$$\lambda^2\partial_\lambda B_0 + A_{0,0}\cdot B_0 - B_0\cdot A_{0,0} - z_1\partial_1 A_{0,0} = 0$$

We have the expansions $B_0 = \sum_{m\geq 0} B_{0;m}\cdot z_1^m$ and $A_{0,0} = \sum_{m\geq N} A_{0,0;m}\cdot z_1^m$. We assume $N<0$ and $A_{0,0;N}\neq 0$. We obtain the relation $\bigl[B_{0;0},A_{0,0;N}\bigr] - NA_{0,0;N} = 0$ on $\mathcal{D}^{(\lambda_0)}$. Note that the eigenvalues of $B_{0;0}$ are of the form $\lambda^{-1}\mathfrak{e}(\lambda,u)$, where $u\in\mathcal{KMS}(\mathcal{PE}^0)$ and $a-1<\mathfrak{p}(\lambda_0,u)\leq a$. It implies that the difference of two distinct eigenvalues of $B_{0,0}$ cannot be N. Therefore, we obtain $A_{0,0;N}=0$, which contradicts with our assumption. Hence, we obtain $N\geq 0$.

By considering a twist by a meromorphic flat line bundle given by $\nabla e = e \cdot d\big((\lambda^{-1}+\overline{\lambda}_0)\mathfrak{a}\big)$, we obtain that $\widetilde{\mathbb{D}}^f = \mathbb{D}^f + \nabla_\lambda$ on $\mathcal{P}^{(\lambda_0)}\mathcal{E}$ is of the form

$$\widetilde{\mathbb{D}}^f = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\theta)} \Big(d\big((\lambda^{-1}+\overline{\lambda}_0)\cdot\mathfrak{a}\big) + \widetilde{\mathbb{D}}^f_{\mathcal{P}^{(\lambda_0)}\widehat{\mathcal{E}},\mathfrak{a}} \Big),$$

where $\widetilde{\mathbb{D}}^f_{\mathcal{P}^{(\lambda_0)}\widehat{\mathcal{E}},\mathfrak{a}}$ are logarithmic with respect to $\mathcal{P}^{(\lambda_0)}\widehat{\mathcal{E}}_{\mathfrak{a}}$. Thus, the proof of Lemma 7.13 is finished. ∎

7.2.3 Proof

By Lemma 7.13, the eigenvalues of $\mathrm{Res}(\widetilde{\mathbb{D}}^f)$ on $\mathcal{P}^{(\lambda_0)}_b\mathcal{E}_{|\mathcal{D}^{(\lambda_0)}}$ are constant. On the other hand, the eigenvalues of $\mathrm{Res}(\mathbb{D}^f) = \mathrm{Res}(\widetilde{\mathbb{D}}^f)$ on $\mathcal{P}^{(\lambda_0)}_b\mathcal{E}_{|\mathcal{D}^{(\lambda_0)}}$ have to be of the form $\lambda^{-1}\alpha - a - \lambda\overline{\alpha}$ for $(a,\alpha)\in\mathcal{KMS}(\mathcal{PE}^0)$ by Lemma 6.8. Hence, we obtain $\alpha = 0$ for any $(a,\alpha)\in\mathcal{KMS}(\mathcal{PE}^0)$, i.e., $\mathcal{KMS}(\mathcal{PE}^0)\subset\boldsymbol{R}\times\{0\}$. Thus, Lemma 7.10 is proved.

Let us show Proposition 7.11. The first claim follows from Lemma 5.41, Proposition 7.12 and the definition of $\mathcal{QE}$ in Subsection 6.3.3. To show the second claim, we remark that κ is flat and preserves the pluri-harmonic metrics for $(\mathcal{E}^\triangle,\mathbb{D}^\triangle,\mathcal{S})$ and $\gamma^*(\mathcal{E}^\triangle,\mathbb{D}^\triangle,\mathcal{S})$. We also remark that we have only to consider the case in which D is smooth. We have $\mathrm{Irr}(\mathbb{D}^\lambda,\mathcal{QE}^\lambda)=\mathrm{Irr}(\theta)$ and $\mathrm{Irr}\big(\mathbb{D}^{\dagger\,\overline{\lambda}},\mathcal{QE}^{\dagger\,\overline{\lambda}}\big)=\mathrm{Irr}(\theta^\dagger)=\big\{\overline{\mathfrak{a}}\,\big|\,\mathfrak{a}\in\mathrm{Irr}(\theta)\big\}$. Hence, we have the natural identification $\mathrm{Irr}(\mathbb{D}^\lambda,\mathcal{QE}^\lambda)=\mathrm{Irr}(\gamma^*\mathbb{D}^{\dagger\overline{\lambda}},\gamma^*\mathcal{QE}^{\dagger\,\overline{\lambda}})$. Since the full Stokes filtrations are characterized by growth order of the norms of flat sections with respect to the pluri-harmonic metrics (Proposition 6.12), the full Stokes filtrations are preserved by κ. Thus, the second claim of Proposition 7.11 follows from Lemma 5.40. ∎

Remark 7.14 *Because $\mathcal{KMS}(\mathcal{PE}^0)\subset\boldsymbol{R}\times\{0\}$, it turns out that any $\lambda\neq 0$ is generic.* ∎

7.3 Reduction from wild to tame

7.3.1 Construction of the reductions

Let $X := \Delta^n$ and $D := \bigcup_{i=1}^{\ell}\{z_i=0\}$. Let $(\mathcal{E}^\triangle,\widetilde{\mathbb{D}}^\triangle,\mathcal{S})$ be an unramifiedly good wild variation of pure polarized integrable twistor structures of weight 0 on $\mathbb{P}^1\times(X-D)$. We have the underlying harmonic bundle $(E,\overline{\partial}_E,\theta,h)$. We take an auxiliary sequence $\mathcal{M}=\big(\boldsymbol{m}(0),\boldsymbol{m}(1),\ldots,\boldsymbol{m}(L)\big)$ for $\mathrm{Irr}(\theta)$ as in Subsection 3.1.2 of [19].

For each $\mathfrak{a}\in\overline{\mathrm{Irr}}\big(\theta,\boldsymbol{m}(0)\big)$, we obtain the variation of pure polarized twistor structures $\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}\big(\mathcal{E}^\triangle,\mathbb{D}^\triangle,\mathcal{S}\big)$ by taking Gr with respect to Stokes filtrations in the level $\boldsymbol{m}(0)$, as explained in Subsection 6.4. By Proposition 7.11 and Lemma 5.14, it is enriched to integrable $\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(\mathcal{E}^\triangle,\widetilde{\mathbb{D}}^\triangle,\mathcal{S})$. If a real structure κ of $(\mathcal{E}^\triangle,\widetilde{\mathbb{D}}^\triangle,\mathcal{S})$ is given, κ_0 and κ_∞ preserve the Stokes filtration in the level $\boldsymbol{m}(0)$, which follows from Proposition 7.11 and Lemma 5.11. Hence, we also have the induced real structure $\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(\kappa)$ of $\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}\big(\mathcal{E}^\triangle,\widetilde{\mathbb{D}}^\triangle,\mathcal{S}\big)$, and we obtain a pure polarized variation of twistor-TERP structures $\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}\big(\mathcal{E}^\triangle,\widetilde{\mathbb{D}}^\triangle,\mathcal{S},\kappa,0\big)$ for each $\mathfrak{a}\in\overline{\mathrm{Irr}}(\mathbb{D},\boldsymbol{m}(0))$.

Applying the above procedure inductively, $\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{a}}(\mathcal{E}^\triangle,\mathbb{D}^\triangle,\mathcal{S})$ are enriched to integrable $\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{a}}(\mathcal{E}^\triangle,\widetilde{\mathbb{D}}^\triangle,\mathcal{S})$ for any $\mathfrak{a}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(j))$. (See the argument in Subsection 6.4.) If a real structure κ is provided, the reductions are also equipped with induced real structures, and we obtain variation of twistor-TERP structures $\mathrm{Gr}^{\boldsymbol{m}(j)}_{\mathfrak{a}}(\mathcal{E}^\triangle,\widetilde{\mathbb{D}}^\triangle,\mathcal{S},\kappa,0)$. In the case $\boldsymbol{m}(L)$, we use the symbols $\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(\mathcal{E}^\triangle,\widetilde{\mathbb{D}}^\triangle,\mathcal{S})$ and $\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(\mathcal{E}^\triangle,\widetilde{\mathbb{D}}^\triangle,\mathcal{S},\kappa,0)$. They are called the full reductions.

For any $\mathfrak{a}\in\mathrm{Irr}(\theta)$, we have the harmonic bundles $L(-\mathfrak{a})$ as in Subsection 6.4. The associated variation of polarized pure twistor structures is also denoted by the same symbol $L(-\mathfrak{a})$. As explained in Subsection 2.2.1, it is naturally enriched to a variation of pure twistor-TERP structures of weight 0. The underlying harmonic bundle of $\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}(\mathcal{E}^\triangle,\widetilde{\mathbb{D}}^\triangle,\mathcal{S})\otimes L(-\mathfrak{a})$ is tame for each $\mathfrak{a}\in\mathrm{Irr}(\theta)$. This procedure is the reduction "from wild to tame" in the integrable case. We have a similar reduction in the twistor-TERP case.

7.3.2 Approximating map and estimate of the new supersymmetric index

Let $(\mathcal{E}^\triangle,\widetilde{\mathbb{D}}^\triangle,\mathcal{S})$ and $(E,\overline{\partial}_E,\theta,h)$ be as above. Let $\overline{\partial}_{\mathbb{P}^1,\mathcal{E}^\triangle}$ denote the λ-holomorphic structure of $\mathcal{E}^\triangle$.

One step reduction By the one step reduction in Subsection 7.3.1, we have obtained the unramifiedly good wild variation of polarized pure integrable twistor structures:

$$(\mathcal{E}_0^{\triangle},\widetilde{\mathbb{D}}_0^{\triangle},\mathcal{S}_0):=\bigoplus_{\mathfrak{a}\in\overline{\mathrm{Irr}}(\theta,\boldsymbol{m}(0))}\mathrm{Gr}^{\boldsymbol{m}(0)}_{\mathfrak{a}}(\mathcal{E}^{\triangle},\widetilde{\mathbb{D}}^{\triangle},\mathcal{S})$$

Let $(E_0,\overline{\partial}_{E_0},\theta_0,h_0)$ be the underlying harmonic bundle. Let $\overline{\partial}_{\mathbb{P}^1,\mathcal{E}_0^{\triangle}}$ denote the λ-holomorphic structure of $\mathcal{E}_0^{\triangle}$. We fix a hermitian metric $g_{\mathbb{P}^1}$ of $\Omega^{0,1}_{\mathbb{P}^1}\oplus\Omega^{1,0}_{\mathbb{P}^1}\bigl(2\{0,\infty\}\bigr)$. We will prove the following proposition in Subsection 7.3.3.

Proposition 7.15 *There exists a C^{∞}-map $\Phi:\mathcal{E}_0^{\triangle}\longrightarrow\mathcal{E}^{\triangle}$ such that the following holds for some $\epsilon>0$ with respect to h_0 and $g_{\mathbb{P}^1}$:*

$$\Phi^{*}\mathcal{S}-\mathcal{S}_0=O\Bigl(\exp\bigl(-\epsilon|\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\Bigr),\quad \overline{\partial}_{\mathbb{P}^1,\mathcal{E}_0^{\triangle}}\bigl(\Phi^{*}\mathcal{S}-\mathcal{S}_0\bigr)=O\Bigl(\exp\bigl(-\epsilon|\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\Bigr),\tag{56}$$

$$\Phi^{*}\nabla_{\lambda}-\nabla_{\lambda,0}=O\Bigl(\exp\bigl(-\epsilon|\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\Bigr)\tag{57}$$

In fact, the order of the estimates can be improved as $O\bigl(\exp(-\epsilon(|\lambda|+|\lambda^{-1}|)|\boldsymbol{z}^{\boldsymbol{m}(0)}|)\bigr)$. We give a consequence. Let $\mathcal{Q}_0$ denote the new supersymmetric index of $(\mathcal{E}_0^{\triangle},\widetilde{\mathbb{D}}_0^{\triangle},\mathcal{S}_0)$.

Corollary 7.16 *We have the following estimate for some $\epsilon>0$ with respect to h_0:*

$$\bigl|\Phi^{*}h-h_0\bigr|_{h_0}=O\Bigl(\exp\bigl(-\epsilon|\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\Bigr),\quad \bigl|\Phi^{*}\mathcal{Q}-\mathcal{Q}_0\bigr|_{h_0}=O\Bigl(\exp\bigl(-\epsilon|\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\Bigr)$$

Proof It follows from Lemma 2.20. ∎

Full reduction By taking the full reduction in Subsection 7.3.1, we have obtained the unramifiedly good wild variation of polarized pure integrable twistor structures:

$$\bigl(\mathcal{E}_1^{\triangle},\widetilde{\mathbb{D}}_1^{\triangle},\mathcal{S}_1\bigr):=\bigoplus_{\mathfrak{a}\in\overline{\mathrm{Irr}}(\theta)}\mathrm{Gr}^{\mathrm{full}}_{\mathfrak{a}}\bigl(\mathcal{E}^{\triangle},\widetilde{\mathbb{D}}^{\triangle},\mathcal{S}\bigr)$$

Let $(E_1,\overline{\partial}_{E_1},\theta_1,h_1)$ be the underlying harmonic bundle, and let $\mathcal{Q}_1$ denote the supersymmetric index for $(\mathcal{E}_1^{\triangle},\mathbb{D}_1^{\triangle})$. By applying Proposition 7.15 and Corollary 7.16 inductively (see Subsection 6.4 for an inductive use), we obtain a C^{∞}-map $\Phi_1:\mathcal{E}_1^{\triangle}\longrightarrow\mathcal{E}^{\triangle}$ such that the following holds for some $\epsilon>0$ with respect to h_1:

$$\bigl|\Phi_1^{*}h-h_1\bigr|_{h_1}=O\Bigl(\exp(-\epsilon|\boldsymbol{z}^{\boldsymbol{m}(L)}|)\Bigr),\quad \bigl|\Phi_1^{*}\mathcal{Q}-\mathcal{Q}_1\bigr|_{h_1}=O\Bigl(\exp(-\epsilon|\boldsymbol{z}^{\boldsymbol{m}(L)}|)\Bigr)$$

Note that the new supersymmetric index is unchanged after taking the tensor product with $L(-\mathfrak{a})$. (See Subsection 2.2.1.) Hence, the study of asymptotic behaviour of new supersymmetric index is reduced to the study in the tame case, up to decay with exponential orders.

7.3.3 Construction of an approximating map

We assume that the coordinate is as in Remark 5.28 for the good set $\mathrm{Irr}(\theta)$. Let k be determined by $\boldsymbol{m}(0)\in\mathbb{Z}^k_{<0}\times\boldsymbol{0}_{\ell-k}$. Let $\lambda_0\in\boldsymbol{C}_{\lambda}$. Let $U(\lambda_0)$ denote a small neighbourhood of λ_0. We set $\mathcal{X}^{(\lambda_0)}:=U(\lambda_0)\times X$ and $\mathcal{D}^{(\lambda_0)}(\leq k):=U(\lambda_0)\times D(\leq k)$. We also use the symbol $\mathcal{D}_i^{(\lambda_0)}$ in a similar meaning. We set $W:=\mathcal{D}^{(\lambda_0)}(\leq k)$ if $\lambda_0\neq 0$, and $W:=\mathcal{D}^{(\lambda_0)}(\leq k)\cup(\{0\}\times X)$. Let $\sigma:\boldsymbol{C}_{\lambda}\longrightarrow\boldsymbol{C}_{\mu}$ be given by $\sigma(\lambda)=-\overline{\lambda}$, which induces the anti-holomorphic map $\boldsymbol{C}_{\lambda}\times X\longrightarrow\boldsymbol{C}_{\mu}\times X^{\dagger}$. We set $\mathcal{X}^{\dagger\,(-\overline{\lambda}_0)}:=\sigma\bigl(\mathcal{X}^{(\lambda_0)}\bigr)$.

From $(E,\overline{\partial}_E,\theta,h)$, we obtain the vector bundle $\mathcal{P}_0^{(\lambda_0)}\mathcal{E}$ on $\mathcal{X}^{(\lambda_0)}$ with a meromorphic flat connection $\widetilde{\mathbb{D}}^f:=\mathbb{D}^f+\nabla_{\lambda}$. Similarly we obtain $\mathcal{P}_0^{(\lambda_0)}\mathcal{E}_0$ with $\widetilde{\mathbb{D}}_0^f=\mathbb{D}_0^f+\nabla_{\lambda,0}$ from $(E_0,\overline{\partial}_{E_0},\theta_0,h_0)$.

We also obtain the vector bundle $\mathcal{P}_0^{(\mu_0)}\mathcal{E}^{\dagger}$ with the meromorphic flat connection $\widetilde{\mathbb{D}}^{\dagger\,f}=\mathbb{D}^{\dagger\,f}+\nabla_{\mu}$ on $\mathcal{X}^{(\mu_0)}$ from $(E,\overline{\partial}_E,\theta,h)$, and the vector bundle $\mathcal{P}_0^{(\mu_0)}\mathcal{E}_0^{\dagger}$ with the meromorphic flat connection $\widetilde{\mathbb{D}}_0^{\dagger\,f}=\mathbb{D}_0^{\dagger\,f}+\nabla_{\mu,0}$ from $(E_0,\overline{\partial}_{E_0},\theta_0,h_0)$.

Let $\mathbb{D}_{\leq k}$ denote the restriction of $\mathbb{D}$ to the $(z_1,\ldots,z_k)$-direction.

Preliminary Let S be a small multi-sector of $\mathcal{X}^{(\lambda_0)} - W$. By Proposition 5.9, we take a $\mathbb{D}_{\leq k}$-flat splitting

$$\mathcal{P}_0^{(\lambda_0)}\mathcal{E}_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\theta)} \mathcal{P}_0^{(\lambda_0)}\mathcal{E}_{\mathfrak{a},S}$$

of the Stokes filtration in the level $\boldsymbol{m}(0)$, such that the restrictions to $\mathcal{D}_j^{(\lambda_0)} \cap S$ ($j = k+1,\ldots,\ell$) are compatible with $\mathrm{Res}_j(\mathbb{D})$ and the filtrations ${}^jF^{(\lambda_0)}$. If $\lambda_0 \neq 0$, we may assume that it is $\mathbb{D}^f$-flat by Proposition 5.10. (Note that the $\mathbb{D}^f$-flatness implies the compatibility with the residues and the parabolic filtrations.) By construction of $\mathrm{Gr}^{\boldsymbol{m}(0)}$, it induces the isomorphism $\bigl(\mathcal{P}_0^{(\lambda_0)}\mathcal{E}_0, \mathbb{D}_{0,\leq k}\bigr)_{|\overline{S}} \simeq \bigl(\mathcal{P}_0^{(\lambda_0)}\mathcal{E}, \mathbb{D}_{\leq k}\bigr)_{|\overline{S}}$. Let Φ_S^p ($p = 0,\ldots,m$) be such isomorphisms. Let a_p ($p = 0,\ldots,m$) be non-negative C^∞-functions on S such that (i) $\sum a_p = 1$, (ii) $\partial_i a_p$ and $\partial_\lambda a_p$ are $O\bigl(|\lambda|^{-C}\cdot\prod_{i=1}^k |z_i|^{-C}\bigr)$ for some $C > 0$. We set $\Phi_S := \sum a_p\cdot\Phi_S^p$. We also set $G := (\Phi_S^0)^{-1}\circ\Phi_S$ and $G^p := (\Phi_S^0)^{-1}\circ\Phi_S^p$.

Lemma 7.17 *We have the following estimates with respect to h_0 for some $\epsilon > 0$:*

$$G^p - \mathrm{id} = O\Bigl(\exp\bigl(-\epsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\Bigr) \tag{58}$$

$$(\Phi_S^0)^{-1}\circ\bigl(\lambda^2\nabla_\lambda(\partial_\lambda)\bigr)\circ\Phi_S^0 - \lambda^2\nabla_{\lambda,0}(\partial_\lambda) = O\bigl(\exp(-\epsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|)\bigr) \tag{59}$$

Proof Let $\mathcal{G}$ be the left hand side of (58) or (59). It is flat with respect to $\mathbb{D}_{0\leq k}$, and strictly decreases the Stokes filtration in the level $\boldsymbol{m}(0)$. Moreover, $\mathcal{G}_{|\mathcal{D}_i^{(\lambda_0)}\cap S}$ preserves the filtrations ${}^iF^{(\lambda_0)}$ and the residues $\mathrm{Res}_j(\mathbb{D})$ for $j = k+1,\ldots,\ell$. Then, we obtain the desired estimate by using the estimate in Subsection 13.3 of [19]. (It is easy to show it directly.) ∎

Hence, we have $\bigl|G - \mathrm{id}\bigr|_{h_0} = O\bigl(\exp\bigl(-\epsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\bigr)$. We set $\Phi_S^*\nabla_\lambda(\partial_\lambda) := \Phi_S^{-1}\circ\bigl(\nabla_\lambda(\partial_\lambda)\bigr)\circ\Phi_S$. We use the symbol $(\Phi_S^0)^*\nabla_\lambda(\partial_\lambda)$ in a similar meaning. By the previous lemma, we have the following estimate for some $\epsilon > 0$ with respect to h_0:

$$(\Phi_S^0)^*\nabla_\lambda(\partial_\lambda) - \nabla_{\lambda,0}(\partial_\lambda) = O\bigl(\exp\bigl(-\epsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\bigr)$$

Lemma 7.18 *The following estimate holds for some $\epsilon > 0$ with respect to h_0:*

$$\Phi_S^*\nabla_\lambda(\partial_\lambda) - \nabla_{\lambda,0}(\partial_\lambda) = O\Bigl(\exp\bigl(-\epsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\Bigr)$$

Proof We have the following equalities:

$$\begin{aligned}
\Phi_S^*\nabla_\lambda(\partial_\lambda) - \nabla_{\lambda,0}(\partial_\lambda) &= \bigl(\Phi_S^{-1}\circ\Phi_S^0\bigr)\circ(\Phi_S^0)^*\nabla_\lambda(\partial_\lambda)\circ\bigl((\Phi_S^0)^{-1}\circ\Phi_S\bigr) - \nabla_{\lambda,0}(\partial_\lambda)\\
&= G^{-1}\circ\bigl((\Phi_S^0)^*\nabla_\lambda(\partial_\lambda) - \nabla_{\lambda,0}(\partial_\lambda)\bigr)\circ G + G^{-1}\circ\nabla_{\lambda,0}(\partial_\lambda)\circ G - \nabla_{\lambda,0}(\partial_\lambda)\\
&= G^{-1}\circ\bigl((\Phi_S^0)^*\nabla_\lambda(\partial_\lambda) - \nabla_{\lambda,0}(\partial_\lambda)\bigr)\circ G + G^{-1}\cdot\bigl(\nabla_{\lambda,0}(\partial_\lambda)G\bigr)
\end{aligned} \tag{60}$$

We have the following:

$$\nabla_{\lambda,0}(\partial_\lambda)G = \sum\frac{\partial a_p}{\partial\lambda}\cdot G^p = \sum\frac{\partial a_p}{\partial\lambda}\cdot(G^p - \mathrm{id}) = O\Bigl(\exp\bigl(-\epsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\bigr)\Bigr)$$

Thus, we obtain Lemma 7.18. ∎

Assume we are also given morphisms on sectors $\sigma(S)$ of $\mathcal{X}^{\dagger(-\overline{\lambda}_0)} - W^\dagger$

$$\Phi^{\dagger\, q}_{\sigma(S)} : \bigl(\mathcal{P}^{(-\overline{\lambda}_0)}\mathcal{E}_0^\dagger, \mathbb{D}_0^\dagger\bigr)_{|\sigma(\overline{S})} \longrightarrow \bigl(\mathcal{P}^{(-\overline{\lambda}_0)}\mathcal{E}^\dagger, \mathbb{D}^\dagger\bigr)_{|\sigma(\overline{S})}, \quad (q = 0,\ldots,m'),$$

induced by $\mathbb{D}^\dagger_{\leq k}$-flat of the Stokes filtration in the level $\boldsymbol{m}(0)$ such that the restriction to $\sigma(S)\cap\mathcal{D}_j^{\dagger(-\overline{\lambda}_0)}$ ($j = k+1,\ldots,\ell$) are compatible with the residue $\mathrm{Res}_j(\mathbb{D}^\dagger)$ and the filtration ${}^jF^{(-\overline{\lambda}_0)}$. If $\lambda_0 \neq 0$, we may assume that the splittings are $\mathbb{D}^\dagger$-flat. Let b_q ($q = 0,\ldots,m'$) be non-negative C^∞-functions on $\sigma(S)$ satisfying similar conditions for a_p. We set $\Phi^\dagger_{\sigma(S)} := \sum b_q\cdot\Phi^{\dagger\, q}_{\sigma(S)}$.

Lemma 7.19 *We set* $H := \mathcal{S} \circ \big(\Phi_S \otimes \sigma^* \Phi^{\dagger}_{\sigma(S)}\big) - \mathcal{S}_0$. *Then, we have the following estimate with respect to* h_0 *for some* $\epsilon > 0$*:*

$$H = O\Big(\exp\big(-\epsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big), \quad \overline{\partial}_{\mathcal{E}_0^{\triangle},\mathbb{P}^1} H = O\Big(\exp\big(-\epsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big)$$

Proof We set $H_{p,q} := \mathcal{S} \circ \big(\Phi_S^p \otimes \sigma^* \Phi^{\dagger\, q}_{\sigma(S)}\big) - \mathcal{S}_0$. According to an estimate in Subsection 15.3.2 of [19], we have

$$H_{p,q} = O\Big(\exp\big(-\epsilon|\lambda^{-1}\boldsymbol{z}^{\boldsymbol{m}(0)}|\big)\Big)$$

with respect to h_0 for some $\epsilon > 0$. We also have $\overline{\partial}_{\mathcal{E}_0^{\triangle},\mathbb{P}^1} H_{p,q} = 0$. Then, the claim of Lemma 7.19 follows. ∎

Construction We take a compact region $\mathcal{K}$ of $\boldsymbol{C}_\lambda$ such that the union of the interior parts of $\mathcal{K}$ and $\sigma(\mathcal{K})$ cover $\mathbb{P}^1$. We take a covering of

$$\big(\mathcal{K} \times X\big) - \Big(\big(\mathcal{K} \times D(\leq k)\big) \cup \big(\{0\} \times X\big)\Big)$$

by multi-sectors S_i $(i = 1, \ldots, N)$ such that S_i are sufficiently small as in **Preliminary** above. Then, we have $\mathbb{P}^1 = \bigcup S_i \cup \bigcup \sigma(S_i)$. We take a partition $\big(\chi_{S_i}, \chi_{\sigma(S_i)} \,\big|\, i = 1, \ldots, N\big)$ of unity on $\mathbb{P}^1$ subordinated to the covering. We assume that $\partial_j \chi_{S_i}$ and $\partial_\lambda \chi_{S_i}$ are $O\big(|\lambda|^{-C} \cdot \prod_{i=1}^k |z_i|^{-C}\big)$ for some $C > 0$. We assume similar conditions for $\partial_j \chi_{\sigma(S_i)}$ and $\partial_\mu \chi_{\sigma(S_i)}$.

For each $S_i \subset \mathcal{X}^{(\lambda_0)} - W$, we take isomorphisms:

$$\Phi_{S_i} : \big(\mathcal{P}_0^{(\lambda_0)}\mathcal{E}_0, \mathbb{D}_0\big)_{|\overline{S}_i} \simeq \big(\mathcal{P}_0^{(\lambda_0)}\mathcal{E}, \mathbb{D}\big)_{|\overline{S}_i}, \quad \Phi^{\dagger}_{\sigma(S_i)} : \big(\mathcal{P}_0^{(-\overline{\lambda}_0)}\mathcal{E}_0^{\dagger}, \mathbb{D}_0^{\dagger}\big)_{|\sigma(\overline{S}_i)} \longrightarrow \big(\mathcal{P}_0^{(-\overline{\lambda}_0)}\mathcal{E}^{\dagger}, \mathbb{D}^{\dagger}\big)_{|\sigma(\overline{S}_i)}$$

induced by $\mathbb{D}_{\leq k}$-flat or $\mathbb{D}^{\dagger}_{\leq k}$-flat splittings of Stokes filtrations as above. If $\lambda_0 \neq 0$, we assume that $\mathbb{D}^f$-flatness and $\mathbb{D}^{\dagger\, f}$-flatness. We set

$$\Phi := \sum_{i=1}^N \chi_{S_i} \cdot \Phi_{S_i} + \sum_{i=1}^N \chi_{\sigma(S_i)} \cdot \Phi_{\sigma(S_i)}.$$

It is easy to check that Φ satisfies the desired estimates (56) and (57), by using Lemma 7.18 and Lemma 7.19. Note that a $\mathbb{D}$-flat splitting of the Stokes filtration of $\mathcal{P}^{(\lambda_0)}\mathcal{E}_{|\overline{S}}$ in the level $\boldsymbol{m}(0)$ naturally gives a $\mathbb{D}^{\dagger}$-flat splitting of the Stokes filtration of $\mathcal{P}^{(\lambda_0^{-1})}\mathcal{E}^{\dagger}_{|\overline{S}'}$ in the level $\boldsymbol{m}(0)$, where S' is the multi-sector of $\mathcal{X}^{\dagger(\lambda_0^{-1})} \setminus W^{\dagger}$, which follows from the characterization of the Stokes filtrations by the growth order of the norms of flat sections. Thus, we obtain Proposition 7.15. ∎

7.4 Reduction from tame to twistor nilpotent orbit

7.4.1 Reduction

Let $X := \Delta^n$, $D_i := \{z_i = 0\}$, $D := \bigcup_{i=1}^{\ell} D_i$ and $D_{\underline{\ell}} = \bigcap_{i=1}^{\ell} D_i$. Let $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$ be a tame variation of pure polarized integrable twistor structures of weight 0 on $\mathbb{P}^1 \times (X - D)$. We have the underlying harmonic bundle $(E, \overline{\partial}_E, \theta, h)$. As explained in Subsection 6.5, we have the limiting polarized mixed twistor structure $\big(S^{\mathrm{can}}_{\boldsymbol{a},0}(E), \boldsymbol{N}, \mathcal{S}_{\boldsymbol{a},0}\big)$ associated to $(E, \overline{\partial}_E, \theta, h)$. We also have the variation of polarized mixed twistor structures $\big({}^{\underline{\ell}}\mathcal{E}^{\triangle}_{\boldsymbol{a},0}, \boldsymbol{N}^{\triangle}, \mathbb{D}^{\triangle}_{\boldsymbol{a},0}, \mathcal{S}_{\boldsymbol{a},0}\big)$ of weight 0 in ℓ-variables. Hertling and Sevenheck observed the following (see [9]).

Proposition 7.20 $\big(S^{\mathrm{can}}_{\boldsymbol{a},0}(E), \boldsymbol{N}, \mathcal{S}_{\boldsymbol{a},0}\big)$ *is naturally enriched to a polarized mixed integrable twistor structure* $\big(S^{\mathrm{can}}_{\boldsymbol{a},0}(E), \nabla, \boldsymbol{N}, \mathcal{S}_{\boldsymbol{a},0}\big)$. *Similarly,* $\big({}^{\underline{\ell}}\mathcal{E}^{\triangle}_{\boldsymbol{a},0}, \boldsymbol{N}^{\triangle}, \mathbb{D}^{\triangle}_{\boldsymbol{a},0}, \mathcal{S}_{\boldsymbol{a},0}\big)$ *is naturally enriched to a variation of polarized mixed integrable twistor structures* $\big({}^{\underline{\ell}}\mathcal{E}^{\triangle}_{\boldsymbol{a},0}, \boldsymbol{N}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}_{\boldsymbol{a},0}, \mathcal{S}_{\boldsymbol{a},0}\big)$.

If $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S})$ *has a real structure* κ, *they are also equipped with induced real structures.* ∎

7.4.2 Approximating maps

For $0<R<1$, we set $X^*(R):=\big\{(z_1,\ldots,z_n)\,\big|\,0<|z_i|<R,\ i=1,\ldots,n\big\}$ and $D_{\underline{\ell}}:=\big\{(z_{\ell+1},\ldots,z_n)\,\big|\,|z_i|<R\big\}$. By the natural projection $X^*(R)\longrightarrow D_{\underline{\ell}}(R)$, we regard $X^*(R)$ as $D_{\underline{\ell}}(R)\times\big\{(z_1,\ldots,z_\ell)\,\big|\,0<|z_i|<R\big\}$. Due to Theorem 4.1, we have the integrable twistor nilpotent orbit $\mathrm{TNIL}(\mathcal{E}^{\triangle}_{\boldsymbol{a},0},\widetilde{\mathbb{D}}^{\triangle}_{\boldsymbol{a},0},\boldsymbol{N},\mathcal{S}_{\boldsymbol{a},0})$ on $X^*(R)$ for some R. Thus, we obtain a tame variation of pure polarized integrable twistor structures:

$$\big(\mathcal{E}_0^{\triangle},\widetilde{\mathbb{D}}_0^{\triangle},\mathcal{S}_0\big):=\bigoplus_{\boldsymbol{a}\in\mathcal{P}ar(\mathcal{P}_0\mathcal{E}^0,\underline{\ell})}\mathrm{TNIL}(\mathcal{E}^{\triangle}_{\boldsymbol{a},0},\widetilde{\mathbb{D}}^{\triangle}_{\boldsymbol{a},0},\boldsymbol{N},\mathcal{S}_{\boldsymbol{a},0})\otimes L(\boldsymbol{a})$$

(See Subsection 2.2.2 for $L(\boldsymbol{a})$.) We have the underlying tame harmonic bundle

$$\big(E_0,\overline{\partial}_{E_0},\theta_0,h_0\big)=\bigoplus\big(E_{\boldsymbol{a}},\overline{\partial}_{\boldsymbol{a}},\theta_{\boldsymbol{a}},h_{\boldsymbol{a}}\big).$$

We would like to explain that we can approximate the original $(\mathcal{E}^{\triangle},\widetilde{\mathbb{D}}^{\triangle},\mathcal{S})$ with $(\mathcal{E}_0^{\triangle},\widetilde{\mathbb{D}}_0^{\triangle},\mathcal{S}_0)$.

Let $\overline{\partial}_{\mathbb{P}^1,\mathcal{E}_0^{\triangle}}$ denote the λ-holomorphic structure of $\mathcal{E}_0^{\triangle}$. We fix a hermitian metric $g_{\mathbb{P}^1}$ of $\Omega^{0,1}_{\mathbb{P}^1}\oplus\Omega^{1,0}_{\mathbb{P}^1}\big(2\{0,\infty\}\big)$. For a permutation σ of $\{1,\ldots,\ell\}$ and for $C>0$, we set

$$Z(\sigma,C):=\big\{(z_1,\ldots,z_n)\in X^*(R)\,\big|\,|z_{\sigma(i-1)}|^C<|z_{\sigma(i)}|,\ i=1,\ldots,\ell-1\big\}$$

If we take a sufficiently large $C>0$, we have $X^*(R)=\bigcup_{\sigma}Z(\sigma,C)$. For any $\epsilon>0$, we set $\Lambda_0(\epsilon):=\sum_{i=1}^{\ell}|z_i|^{\epsilon}$. We will prove the following proposition in Subsection 7.4.3.

Proposition 7.21 *There exists a C^∞-map $\Phi_\sigma:\mathcal{E}_0^{\triangle}\longrightarrow\mathcal{E}^{\triangle}$ such that the following estimate holds for some $\epsilon>0$ with respect to h_0 and $g_{\mathbb{P}^1}$ on $\mathbb{P}^1\times Z(\sigma,C)$:*

$$\Phi_\sigma^*\mathcal{S}-\mathcal{S}_0=O\big(\Lambda_0(\epsilon)\big),\quad\overline{\partial}_{\mathbb{P}^1,\mathcal{E}_0^{\triangle}}\big(\Phi_\sigma^*\mathcal{S}-\mathcal{S}_0\big)=O\big(\Lambda_0(\epsilon)\big),\quad\Phi_\sigma^*\nabla_\lambda-\nabla_{\lambda,0}=O\big(\Lambda_0(\epsilon)\big)\tag{61}$$

Before going into the proof, we give a consequence. Let $\mathcal{Q}_0$ and $\mathcal{Q}$ denote the new supersymmetric indices of $(\mathcal{E}_0^{\triangle},\widetilde{\mathbb{D}}_0^{\triangle})$ and $(\mathcal{E}^{\triangle},\widetilde{\mathbb{D}}^{\triangle})$. By using Lemma 2.20, we obtain the following estimates on $Z(\sigma,C)$ for some $\epsilon>0$ with respect to h_0:

$$\big|\Phi_\sigma^*h-h_0\big|_{h_0}=O\big(\Lambda_0(\epsilon)\big),\quad\big|\Phi_\sigma^*\mathcal{Q}-\mathcal{Q}_0\big|_{h_0}=O\big(\Lambda_0(\epsilon)\big)\tag{62}$$

Corollary 7.22 *The eigenvalues of $\mathcal{Q}$ and $\mathcal{Q}_0$ are the same up to $O\big(\Lambda_0(\epsilon)\big)$ for some $\epsilon>0$.*

Proof By using (62), we obtain the estimate on $Z(\sigma,C)$. Because $X^*(R)=\bigcup Z(\sigma,C)$, the claim of the corollary follows. ∎

We also give a more rough but global estimate, for which the proof is much simpler. For $M>0$ and $\epsilon>0$, we set

$$\Lambda(M,\epsilon):=\prod_{i=1}^{\ell}(-\log|z_i|)^M\sum_{i=1}^{\ell}|z_i|^{\epsilon}.$$

Proposition 7.23 *There exists a C^∞-map $\Phi:\mathcal{E}_0^{\triangle}\longrightarrow\mathcal{E}^{\triangle}$ such that the following holds for some $\epsilon>0$ and $M>0$ with respect to h_0 and $g_{\mathbb{P}^1}$:*

$$\Phi^*\mathcal{S}-\mathcal{S}_0=O\big(\Lambda(M,\epsilon)\big),\quad\overline{\partial}_{\mathbb{P}^1,\mathcal{E}_0^{\triangle}}\big(\Phi^*\mathcal{S}-\mathcal{S}_0\big)=O\big(\Lambda(M,\epsilon)\big),\quad\Phi^*\nabla_\lambda-\nabla_{\lambda,0}=O\big(\Lambda(M,\epsilon)\big)\tag{63}$$

Note that Φ^*h and h_0 are mutually bounded up to log order, which follows from the weak norm estimate for acceptable bundles. (See Lemma 7.35 below.) Hence, we obtain the following estimate for some $M'>0$ and $\epsilon'>0$ by using Lemma 2.19:

$$\big|\Phi^*\mathcal{Q}-\mathcal{Q}_0\big|_{h_0}=O\big(\Lambda(M',\epsilon')\big)$$

In the one dimensional case, the estimates in the two propositions are not so different. We also remark that Φ_σ in Proposition 7.21 also satisfies the estimates (63).

7.4.3 Proof of Proposition 7.21

For the proof of Proposition 7.21, we have only to consider the case that σ is the identity. We use the symbol $Z(C)$ instead of $Z(\mathrm{id}, C)$. Instead of considering $X^*(R)$, we will shrink X around the origin.

Decomposition For any subset $I \subset \underline{\ell}$, let $m(I)$ be determined by the condition $m(I) := \min\bigl\{m \in I \,|\, m+1 \notin I\bigr\}$, in other words, $\{1, \ldots, m(I)\} \subset I$ but $m(I)+1 \notin I$. Let $q_I : \mathcal{P}ar\bigl(\mathcal{P}_0\mathcal{E}^0, \underline{\ell}\bigr) \longrightarrow \mathcal{P}ar(\mathcal{P}_0\mathcal{E}^0, I)$ and $r_{m(I)} : \mathbb{Z}^\ell \longrightarrow \mathbb{Z}^{m(I)}$ be the natural projections. Let $\lambda_0 \in \boldsymbol{C}_\lambda$. Let $\mathcal{K}$ denote a small neighbourhood of λ_0 in $\boldsymbol{C}_\lambda$. We set $\mathcal{X} := \mathcal{K} \times X$. We use the symbols $\mathcal{D}_i$, $\mathcal{D}_I$, $\mathcal{D}$, etc., in similar meanings.

We have the induced filtrations ${}^iF$ $(i \in I)$ of $\mathcal{Q}_0\mathcal{E}_{|\mathcal{D}_I}$. For any $i \in I$, we have the residue endomorphisms $\mathrm{Res}_i(\mathbb{D})$ on ${}^I\mathrm{Gr}_{\boldsymbol{b}}(\mathcal{Q}_0\mathcal{E}_{|\mathcal{D}_I})$, which have the unique eigenvalues $-b_i \cdot \lambda$. Hence, the nilpotent part $\mathcal{N}_i$ is well defined. For $i \leq m(I)$, we set $\mathcal{N}(\underline{i}) := \sum_{j \leq i} \mathcal{N}_j$. Recall that the conjugacy classes of $\mathcal{N}(\underline{i})_{|(\lambda,P)}$ are independent of $(\lambda, P) \in \mathcal{D}_I$ (Lemma 12.47 of [18]). By considering the weight filtration of $\mathcal{N}(\underline{i})$, we obtain the filtration $W(\underline{i})$ of ${}^I\mathrm{Gr}_{\boldsymbol{b}}(\mathcal{Q}_0\mathcal{E}_{|\mathcal{D}_I})$ indexed by $\mathbb{Z}$ in the category of vector bundles on $\mathcal{D}_I$.

Lemma 7.24 *We have a decomposition*

$$\mathcal{Q}_0\mathcal{E}_{|\mathcal{X}} = \bigoplus_{\substack{\boldsymbol{a} \in \mathcal{P}ar(\mathcal{P}_0\mathcal{E}^0, \underline{\ell}) \\ \boldsymbol{k} \in \mathbb{Z}^\ell}} U_{\boldsymbol{a},\boldsymbol{k}} \tag{64}$$

with the following property:

- *For any subset $I \subset \underline{\ell}$, $\boldsymbol{b} \in \mathcal{P}ar(\mathcal{P}_0\mathcal{E}^0, I)$ and $\boldsymbol{h} \in \mathbb{Z}^{m(I)}$, we put*

$${}^IU_{\boldsymbol{b},\boldsymbol{h}} = \bigoplus_{\substack{\boldsymbol{a} \in q_I^{-1}(\boldsymbol{b}) \\ \boldsymbol{k} \in r_{m(I)}^{-1}(\boldsymbol{h})}} U_{\boldsymbol{a},\boldsymbol{k}} \quad \text{and} \quad {}^IU_{\boldsymbol{b}} = \bigoplus_{\boldsymbol{h} \in \mathbb{Z}^{m(I)}} {}^IU_{\boldsymbol{b},\boldsymbol{h}}$$

  Then, the following holds for any $\boldsymbol{c} \in \boldsymbol{R}^I$:

$$\bigoplus_{\boldsymbol{b} \leq \boldsymbol{c}} {}^IU_{\boldsymbol{b}|\mathcal{D}_I} = \bigcap_{i \in I} {}^iF_{c_i}\bigl(\mathcal{Q}_0\mathcal{E}_{|\mathcal{D}_I}\bigr) \tag{65}$$

  Moreover, the following holds for any $\boldsymbol{n} \in \mathbb{Z}^{m(I)}$ under the identification ${}^IU_{\boldsymbol{b}|\mathcal{D}_I} \simeq {}^I\mathrm{Gr}_{\boldsymbol{b}}(\mathcal{Q}_0\mathcal{E})$ induced by (65):

$$\bigoplus_{\boldsymbol{h} \leq \boldsymbol{n}} {}^IU_{\boldsymbol{b},\boldsymbol{h}|\mathcal{D}_I} = \bigcap_{1 \leq i \leq m(I)} W_{n_i}(\underline{i})\bigl({}^I\mathrm{Gr}_{\boldsymbol{b}}(\mathcal{Q}_0\mathcal{E}_{|\mathcal{D}_I})\bigr)$$

Proof Although this is essentially Corollary 4.47 of [18], we recall an outline for later use. The theorems and the definitions referred in this proof are given in [18]. By Theorem 12.43, the tuple $\bigl({}^iF, \mathcal{N}(\underline{j}) \,\big|\, i \in \underline{\ell}, j \in \underline{\ell}\bigr)$ is sequentially compatible in the sense of Definition 4.43. Hence, $\bigl({}^iF, W(\underline{j}) \,\big|\, i \in \underline{\ell}, j \in \underline{\ell}\bigr)$ is compatible in the sense of Definition 4.39, as remarked in Lemma 4.44. By Proposition 4.41, there exists a splitting of $\bigl({}^iF, W(\underline{j}) \,\big|\, i \in \underline{\ell}, j \in \underline{\ell}\bigr)$ in the sense of Definition 4.40. By applying Lemma 2.16, we can take a frame compatible with splittings. It is easy to take a decomposition as in the claim of Lemma 7.24, by using such a compatible frame. ∎

Let $(\mathcal{Q}\mathcal{E}_{\boldsymbol{a}}, \mathbb{D})$ be the prolongment of $(E_{\boldsymbol{a}}, \overline{\partial}_{\boldsymbol{a}}, \theta_{\boldsymbol{a}}, h_{\boldsymbol{a}})$. Similarly, we have a decomposition

$$\mathcal{Q}_0\mathcal{E}_{\boldsymbol{a}|\mathcal{X}} = \bigoplus_{\boldsymbol{k} \in \mathbb{Z}^\ell} U_{0,\boldsymbol{a},\boldsymbol{k}} \tag{66}$$

satisfying a similar condition. By our construction of $(\mathcal{E}_0^{\triangle}, \widetilde{\mathbb{D}}_0^{\triangle}, \mathcal{S}_0)$, we are given the isomorphism for each $\boldsymbol{a} \in \mathcal{P}ar(\mathcal{P}_0\mathcal{E}^0, \underline{\ell})$:

$$\nu_{\boldsymbol{a}} : {}^{\underline{\ell}}\mathrm{Gr}_{\boldsymbol{a}}(\mathcal{Q}_0\mathcal{E}) \simeq \mathcal{Q}_0\mathcal{E}_{\boldsymbol{a}|\mathcal{D}_{\underline{\ell}}}.$$

Lemma 7.25 *We may assume that $\nu_{\boldsymbol{a}}$ preserves the decompositions $\bigoplus_{\boldsymbol{k}} U_{0,\boldsymbol{a},\boldsymbol{k}|\mathcal{D}_{\underline{\ell}}}$ and $\bigoplus_{\boldsymbol{k}} U_{\boldsymbol{a},\boldsymbol{k}|\mathcal{D}_{\underline{\ell}}}$.*

Proof In Proposition 4.41 of [18], the construction of a splitting is given in a descending inductive way, and we can take any splitting of ${}^{\underline{\ell}}\mathrm{Gr}_{\boldsymbol{a}}(\mathcal{Q}\mathcal{E})$ of the filtrations $W(\underline{j})$ $(j=1,\ldots,\ell)$ in the beginning. Thus, we obtain Lemma 7.25. ∎

Let $\nu_{\boldsymbol{a},\boldsymbol{k}}$ denote the induced map $U_{0,\boldsymbol{a},\boldsymbol{k}|\mathcal{D}_{\underline{\ell}}} \simeq U_{\boldsymbol{a},\boldsymbol{k}|\mathcal{D}_{\underline{\ell}}}$.

Norm estimate We recall the norm estimate for tame harmonic bundles. We take a C^∞-frame $h'_{\boldsymbol{a},\boldsymbol{k}}$ of $U_{\boldsymbol{a},\boldsymbol{k}}$ in (64). We set

$$h^{(1)}_{\boldsymbol{a},\boldsymbol{k}} := h'_{\boldsymbol{a},\boldsymbol{k}} \cdot \prod_{j=1}^{\ell} |z_j|^{-2a_j}\bigl(-\log|z_j|\bigr)^{k_j-k_{j-1}} = h'_{\boldsymbol{a},\boldsymbol{k}} \cdot \prod_{j=1}^{\ell} |z_j|^{-2a_j} \prod_{j=1}^{\ell-1}\Bigl(\frac{-\log|z_j|}{-\log|z_{j+1}|}\Bigr)^{k_j} \cdot \bigl(-\log|z_\ell|\bigr)^{k_\ell}$$

(We formally set $k_0 := 0$.) We obtain a C^∞-hermitian metric $h^{(1)} = \bigoplus h^{(1)}_{\boldsymbol{a},\boldsymbol{k}}$ of $\mathcal{Q}\mathcal{E}_{|\mathcal{X}-\mathcal{D}}$. Theorem 13.25 of [18] implies the following lemma.

Lemma 7.26 *h and $h^{(1)}$ are mutually bounded on $\mathcal{K}\times Z(C)$.* ∎

An estimate

Lemma 7.27 *Let f be a holomorphic endomorphism of $\mathcal{Q}_0\mathcal{E}_0$ satisfying the following conditions:*

- *It preserves the filtrations ${}^{i}F$ $(i=1,\ldots,\ell)$.*
- *For each $\boldsymbol{b}\in\boldsymbol{R}^I$, the induced endomorphism ${}^{I}\mathrm{Gr}^F_{\boldsymbol{b}}(f)$ of $\bigoplus_{q_I(\boldsymbol{a})=\boldsymbol{b}} \mathcal{Q}_0\mathcal{E}_{\boldsymbol{a}|\mathcal{D}_I}$ preserves the weight filtrations $W(\underline{j})$ $(j=1,\ldots,m(I))$.*
- *For each $\boldsymbol{a}\in\boldsymbol{R}^\ell$, the induced endomorphism ${}^{\underline{\ell}}\mathrm{Gr}^F_{\boldsymbol{a}}(f)$ of $\mathcal{Q}_0\mathcal{E}_{\boldsymbol{a}|\mathcal{D}_{\underline{\ell}}}$ is 0.*

Then, we have $|f|_{h_0} = O\bigl(\Lambda_0(\epsilon)\bigr)$ for some $\epsilon>0$ on $\mathcal{K}\times Z(C)$.

Proof We take decompositions (66). Applying Lemma 7.26 to $(E_{\boldsymbol{a}},\overline{\partial}_{\boldsymbol{a}},\theta_{\boldsymbol{a}},h_{\boldsymbol{a}})$ with the decomposition (66), we take a C^∞-hermitian metric $h^{(1)}_{0,\boldsymbol{a}} = \bigoplus h^{(1)}_{0,\boldsymbol{a},\boldsymbol{k}}$ of $\mathcal{Q}_0\mathcal{E}_{\boldsymbol{a}|\mathcal{X}-\mathcal{D}}$ and $h^{(1)}_0 := \bigoplus h^{(1)}_{0,\boldsymbol{a}}$ of $\mathcal{Q}_0\mathcal{E}_{0|\mathcal{X}-\mathcal{D}}$ as above. We have the decomposition:

$$f = \sum f_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')}, \qquad f_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')} \in \mathrm{Hom}\bigl(U_{0,\boldsymbol{a}',\boldsymbol{k}'},\,U_{0,\boldsymbol{a},\boldsymbol{k}}\bigr)$$

We have only to show

$$\bigl|f_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')}\bigr|_{h^{(1)}_0} = O\bigl(\Lambda_0(\epsilon)\bigr) \tag{67}$$

for any $(\boldsymbol{a},\boldsymbol{k})$ and $(\boldsymbol{a}',\boldsymbol{k}')$ on $\mathcal{K}\times Z(C)$. Note that the induced metrics on $\mathrm{Hom}\bigl(U_{0,\boldsymbol{a}',\boldsymbol{k}'},\,U_{0,\boldsymbol{a},\boldsymbol{k}}\bigr)_{|\mathcal{X}-\mathcal{D}}$ are of the form

$$g_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')} \cdot \prod_{j=1}^{\ell} |z_j|^{2(-a_j+a'_j)} \cdot \prod_{j=1}^{\ell-1}\Bigl(\frac{-\log|z_j|}{-\log|z_{j+1}|}\Bigr)^{k_j-k'_j} \cdot \bigl(-\log|z_\ell|\bigr)^{k_\ell-k'_\ell}, \tag{68}$$

where $g_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')}$ are C^∞-metrics of $\mathrm{Hom}\bigl(U_{0,\boldsymbol{a}',\boldsymbol{k}'},\,U_{0,\boldsymbol{a},\boldsymbol{k}}\bigr)$ on $\mathcal{X}$.

(I) Let us consider the case $\boldsymbol{a}\neq\boldsymbol{a}'$. We define

$$I_+ := \bigl\{i \,\big|\, a_i > a'_i\bigr\}, \quad I_- := \bigl\{i \,\big|\, a_i < a'_i\bigr\}, \quad I_0 := \bigl\{i \,\big|\, a_i = a'_i\bigr\}.$$

Let m be the number determined by $\{1,\ldots,m\}\subset I_0$ and $m+1\notin I_0$. Since the parabolic filtrations are preserved, we have $f_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')|\mathcal{D}_i} = 0$ for any $i\in I_+$. Hence, there exists a holomorphic section $f'_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')}$ of $\mathrm{Hom}\bigl(U_{0,\boldsymbol{a}',\boldsymbol{k}'},\,U_{0,\boldsymbol{a},\boldsymbol{k}}\bigr)$ such that

$$f_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')} = f'_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')} \cdot \prod_{i\in I_+} z_i \tag{69}$$

We have the following inequality for some $\epsilon > 0$:

$$\prod_{i\in I_+} |z_i|^{1-a_i+a_i'} \cdot \prod_{i\in I_-} |z_i|^{-a_i+a_i'} \le \prod_{i\in I_+\cup I_-} |z_i|^{\epsilon} \le \big|z_{m+1}\big|^{\epsilon}. \tag{70}$$

Let us consider the set $S = \big\{p \le m \,\big|\, k_p > k_p'\big\}$. If S is not empty, let p be the minimum. Note that $k_t \le k_t'$ for any $t < p$ and $k_p > k_p'$ by our choice. Since the weight filtrations $W(\underline{j})$ $(j = 1, \dots, p)$ are preserved on ${}^{\underline{p}}\operatorname{Gr}^F$, we have $f'_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')|\mathcal{D}_{\underline{p}}} = 0$. Hence, there exist holomorphic sections $f''_{t,(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')}$ $(t = 1, \dots, p)$ of $\operatorname{Hom}\big(U_{0,\boldsymbol{a}',\boldsymbol{k}'},\, U_{0,\boldsymbol{a},\boldsymbol{k}}\big)$ such that

$$f'_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')} = \sum_{t=1}^{p} z_t \cdot f''_{t,(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')}. \tag{71}$$

We remark the following for any $t \le p$:

$$|z_t| \cdot \prod_{j=1}^{\ell-1} \left(\frac{-\log|z_j|}{-\log|z_{j+1}|}\right)^{k_j-k_j'} \big(-\log|z_\ell|\big)^{k_\ell-k_\ell'} \le |z_t| \cdot \prod_{j=1}^{t-1} C^{k_j-k_j'} \prod_{j=t}^{\ell-1} \left(\frac{-\log|z_j|}{-\log|z_{j+1}|}\right)^{k_j-k_j'} \big(-\log|z_\ell|\big)^{k_\ell-k_\ell'}$$
$$= O\big(|z_t|^{1/2}\big) \tag{72}$$

By using (68), (69), (70), (71) and (72), we obtain $\big|f_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a}',\boldsymbol{k}')}\big|_{h_0^{(1)}} = \sum_{t=1}^{p} O\big(|z_t|^{1/2}\big) = O\big(\Lambda_0(1/2)\big)$.

If S is empty, we have $k_j \le k_j'$ for $j = 1, \dots, m$. Hence, we have the following:

$$|z_{m+1}|^{\epsilon} \cdot \prod_{j=1}^{\ell-1} \left(\frac{-\log|z_j|}{-\log|z_{j+1}|}\right)^{k_j-k_j'} \big(-\log|z_\ell|\big)^{k_\ell-k_\ell'}$$
$$\le |z_{m+1}|^{\epsilon} \cdot \prod_{j=1}^{m} C^{k_j-k_j'} \prod_{j=m+1}^{\ell-1} \left(\frac{-\log|z_j|}{-\log|z_{j+1}|}\right)^{k_j-k_j'} \big(-\log|z_\ell|\big)^{k_\ell-k_\ell'} = O\big(|z_{m+1}|^{\epsilon/2}\big) \tag{73}$$

By using (68), (69) (70) and (73), we obtain (67).

(II) Let us consider the case $\boldsymbol{a} = \boldsymbol{a}'$. Because $\operatorname{Gr}^F_{\boldsymbol{a}} \Phi_{\mathcal{K}} = \operatorname{Gr}^F_{\boldsymbol{a}} \Phi'_{\mathcal{K}} = \nu_{\boldsymbol{a}}$, there exist holomorphic sections $f_{i,\boldsymbol{a},(\boldsymbol{k},\boldsymbol{k}')}$ of $\operatorname{Hom}\big(U_{0,\boldsymbol{a},\boldsymbol{k}'},\, U_{0,\boldsymbol{a},\boldsymbol{k}}\big)$ such that

$$f_{(\boldsymbol{a},\boldsymbol{k}),(\boldsymbol{a},\boldsymbol{k}')} = \sum_{i=1}^{\ell} z_i \cdot f_{i,\boldsymbol{a},(\boldsymbol{k},\boldsymbol{k}')} \tag{74}$$

Let us consider the set $S = \big\{p \,\big|\, k_p > k_p'\big\}$. If S is not empty, let p be the minimum. Note that $k_t \le k_t'$ for any $t < p$ and $k_p > k_p'$ by our choice. Since the weight filtrations $W(\underline{j})$ $(j = 1, \dots, p)$ are preserved on ${}^{\underline{p}}\operatorname{Gr}^F$, we have $f_{i,\boldsymbol{a},(\boldsymbol{k},\boldsymbol{k}')|\mathcal{D}_{\underline{p}}} = 0$. Hence, there exist holomorphic sections $f'_{t,i,\boldsymbol{a},(\boldsymbol{k},\boldsymbol{k}')}$ $(t = 1, \dots, p)$ of $\operatorname{Hom}\big(U_{0,\boldsymbol{a},\boldsymbol{k}'},\, U_{0,\boldsymbol{a},\boldsymbol{k}}\big)$ such that

$$f_{i,\boldsymbol{a},(\boldsymbol{k},\boldsymbol{k}')} = \sum_{t=1}^{p} z_t \cdot f'_{t,i,\boldsymbol{a},(\boldsymbol{k},\boldsymbol{k}')}. \tag{75}$$

By using (72) and (75), we obtain $\big|f_{i,\boldsymbol{a},(\boldsymbol{k},\boldsymbol{k}')}\big|_{h_0^{(1)}} = O\big(\Lambda_0(1/2)\big)$.

If S is empty, we have $k_j \le k_j'$ for $j = 1, \dots, \ell$. Hence, we have the following:

$$\prod_{j=1}^{\ell-1} \left(\frac{-\log|z_j|}{-\log|z_{j+1}|}\right)^{k_j-k_j'} \big(-\log|z_\ell|\big)^{k_\ell-k_\ell'} = O(1). \tag{76}$$

Hence, we obtain $|f_{i,\boldsymbol{a},(\boldsymbol{k},\boldsymbol{k}')}|_{h_0^{(1)}} = O(1)$. By using (74), we obtain (67). Thus, we obtain Lemma 7.27. ∎

Local isomorphism with a nice property

Lemma 7.28 *There exists a holomorphic isomorphism* $\Phi_{\mathcal{K}} : \mathcal{Q}_0\mathcal{E}_{0|\mathcal{X}} \longrightarrow \mathcal{Q}_0\mathcal{E}_{|\mathcal{X}}$ *with the following property:*

- *It preserves the filtrations* ${}^iF$ $(i = 1, \ldots, \ell)$.
- *For each* $\boldsymbol{b} \in \boldsymbol{R}^I$, *the induced map* $\bigoplus_{q_I(\boldsymbol{a})=\boldsymbol{b}} \mathcal{Q}_0\mathcal{E}_{\boldsymbol{a}|\mathcal{D}_I} \longrightarrow {}^I\mathrm{Gr}^F_{\boldsymbol{b}}(\mathcal{Q}_0\mathcal{E}_{|\mathcal{D}_I})$ *preserves the weight filtrations* $W(\underline{j})$ $(j = 1, \ldots, m(I))$.
- *For each* $\boldsymbol{a} \in \boldsymbol{R}^\ell$, *the induced map* $\mathcal{Q}_0\mathcal{E}_{\boldsymbol{a}|\mathcal{D}_{\underline{\ell}}} \longrightarrow {}^{\underline{\ell}}\mathrm{Gr}^F_{\boldsymbol{b}}(\mathcal{Q}_0\mathcal{E}_{|\mathcal{D}_{\underline{\ell}}})$ *is equal to* $\nu_{\boldsymbol{a}}$.

Proof We take decompositions (64) and (66) as in Lemma 7.25. We take an isomorphism $\widetilde{\nu}_{\boldsymbol{a},\boldsymbol{k}} : U_{0,\boldsymbol{a},\boldsymbol{k}} \simeq U_{\boldsymbol{a},\boldsymbol{k}}$ such that $\widetilde{\nu}_{\boldsymbol{a},\boldsymbol{k}|\mathcal{D}_{\underline{\ell}}} = \nu_{\boldsymbol{a},\boldsymbol{k}}$. We set $\Phi_{\mathcal{K}} := \sum \widetilde{\nu}_{\boldsymbol{a},\boldsymbol{k}}$. It is easy to check that $\Phi_{\mathcal{K}}$ has the desired property. Thus, we obtain Lemma 7.28. ∎

By the norm estimate (Lemma 7.26), $\Phi_{\mathcal{K}}$ and $\Phi_{\mathcal{K}}^{-1}$ are bounded on $\mathcal{K} \times Z(C)$.

Lemma 7.29 *We have the following estimate for some* $\epsilon > 0$ *with respect to* h_0 *on* $\mathcal{K} \times Z(C)$:

$$\Phi_{\mathcal{K}}^*\nabla_\lambda - \nabla_{\lambda,0} = O\big(\Lambda_0(\epsilon)\big) \tag{77}$$

Proof Let F denote the left hand side of (77). It is easy to observe that F satisfies the conditions in Lemma 7.27. Hence, Lemma 7.29 follows from Lemma 7.27. ∎

Let $\Phi_{\mathcal{K}}$ and $\Phi'_{\mathcal{K}}$ be morphisms as in Lemma 7.28. We set $G := \Phi_{\mathcal{K}}^{-1} \circ \Phi'_{\mathcal{K}}$.

Lemma 7.30 *We have the following estimates for some* $\epsilon > 0$ *on* $\mathcal{K} \times Z(C)$:

$$\big|G - \mathrm{id}\big|_{h_0} = O\big(\Lambda_0(\epsilon)\big), \quad \big|\nabla_{\lambda,0}(\lambda^2\partial_\lambda)G\big|_{h_0} = O\big(\Lambda_0(\epsilon)\big)$$

Proof We have only to apply Lemma 7.27 to $G - \mathrm{id}$ and $\nabla_{\lambda,0}(\lambda^2\partial_\lambda)G$. ∎

Let $\sigma : \boldsymbol{C}_\lambda \longrightarrow \boldsymbol{C}_\mu$ given by $\sigma(\lambda) = -\overline{\lambda}$. The induced map $\boldsymbol{C}_\lambda \times X \longrightarrow \boldsymbol{C}_\mu \times X^\dagger$ is also denoted by σ.

Lemma 7.31 *We can take a holomorphic isomorphism* $\Phi^\dagger_{\sigma(\mathcal{K})} : \mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger_{0|\sigma(\mathcal{X})} \longrightarrow \mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger_{|\sigma(\mathcal{X})}$ *satisfying the conditions (i) it preserves the filtrations* ${}^iF$ $(i = 1, \ldots, \ell)$, *(ii) the induced morphism on* ${}^I\mathrm{Gr}^F_{-\boldsymbol{a}}$ *preserves the weight filtrations* $W(\underline{j})$ $(j = 1, \ldots, m(I))$, *(iii) the induced morphism on* ${}^{\underline{\ell}}\mathrm{Gr}^F_{-\boldsymbol{a}}$ *is equal to the given one.*

Proof It can be shown by the argument in the proof of Lemma 7.28. More directly, we have the isomorphisms $\mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger_{0|\sigma(\mathcal{X})} \simeq \sigma^*\big(\mathcal{Q}_0\mathcal{E}_{0|\mathcal{X}}\big)^\vee$ and $\mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger_{|\sigma(\mathcal{X})} \simeq \sigma^*\big(\mathcal{Q}_0\mathcal{E}_{|\mathcal{X}}\big)^\vee$, and $\sigma^*(\Phi_{\mathcal{K}})^\vee$ satisfies the conditions. ∎

Lemma 7.32 *Let* $\Phi_{\mathcal{K}}$ *and* $\Phi^\dagger_{\sigma(\mathcal{K})}$ *satisfy the above conditions. We set*

$$H := \mathcal{S}_0 - \mathcal{S}\big(\Phi_{\mathcal{K}} \otimes \sigma^*\Phi^\dagger_{\sigma(\mathcal{K})}\big) : \mathcal{Q}_0\mathcal{E}_{0|\mathcal{X}} \otimes \sigma^*\big(\mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger_{0|\sigma(\mathcal{X})}\big) \longrightarrow \mathcal{O}_{\mathcal{X}}$$

Then, $H = O\big(\Lambda_0(\epsilon)\big)$ *with respect to* h_0 *for some* $\epsilon > 0$ *on* $\mathcal{K} \times Z(C)$.

Proof If $\Phi^\dagger_{\sigma(\mathcal{K})}$ is given by $\sigma^*\Phi_{\mathcal{K}}^\vee$, we have $H = 0$. Hence, we have only to show that the property is independent of the choice of $\Phi^\dagger_{\sigma(\mathcal{K})}$.

Let $\Phi^\dagger_{i,\sigma(\mathcal{K})}$ $(i = 1, 2)$ be as in Lemma 7.31. Note that h and h_0 are mutually bounded through $\Phi^\dagger_{1,\sigma(\mathcal{K})}$ on $\sigma(\mathcal{K}) \times Z(C)$. By using Lemma 7.32, we obtain $\Phi^\dagger_{1,\sigma(\mathcal{K})} - \Phi^\dagger_{2,\sigma(\mathcal{K})} = O\big(\Lambda_0(\epsilon)\big)$ for some $\epsilon > 0$ with respect to h and h_0. Then, we obtain $\mathcal{S} \circ \Big(\Phi_{\mathcal{K}} \otimes \sigma^*\big(\Phi^\dagger_{1,\sigma(\mathcal{K})} - \Phi^\dagger_{2,\sigma(\mathcal{K})}\big)\Big) = O\big(\Lambda_0(\epsilon)\big)$ with respect to h_0. Thus, the proof of Lemma 7.32 is finished. ∎

Local C^∞-isomorphisms Let $\Phi^p_{\mathcal{K}}$ $(p=0,\ldots,m)$ be as in Lemma 7.28, and let a_p $(p=0,\ldots,m)$ be non-negative C^∞-functions on $\mathcal{K}$ such that $\sum a_p=1$. We set $\Phi_{\mathcal{K}}:=\sum_{p=0}^m a_p\cdot\Phi^p_{\mathcal{K}}$. We also set $G:=(\Phi^0_{\mathcal{K}})^{-1}\circ\Phi_{\mathcal{K}}$ and $G^p:=(\Phi^0_{\mathcal{K}})^{-1}\circ\Phi^p_{\mathcal{K}}$. By Lemma 7.30, $|G^p-\mathrm{id}|_{h_0}=O\bigl(\Lambda_0(\epsilon)\bigr)$, and hence $|G-\mathrm{id}|_{h_0}=O\bigl(\Lambda_0(\epsilon)\bigr)$ for some $\epsilon>0$ on $\mathcal{K}\times Z(C)$.

Lemma 7.33 *The following estimate holds for some $\epsilon>0$ with respect to h_0 on $\mathcal{K}\times Z(C)$:*

$$\Phi_{\mathcal{K}}^{-1}\circ\nabla_\lambda(\lambda^2\partial_\lambda)\circ\Phi_{\mathcal{K}}-\nabla_{\lambda,0}(\lambda^2\partial_\lambda)=O\bigl(\Lambda_0(\epsilon)\bigr)$$

Proof We have the following equalities:

$$\begin{aligned}\Phi_{\mathcal{K}}^{-1}\circ\nabla_{\lambda,0}(\partial_\lambda)\circ\Phi_{\mathcal{K}}-\nabla_{\lambda,0}(\partial_\lambda)&=\bigl(\Phi_{\mathcal{K}}^{-1}\circ\Phi^0_{\mathcal{K}}\bigr)\circ(\Phi^0_{\mathcal{K}})^{-1}\circ\nabla_\lambda(\partial_\lambda)\circ\bigl(\Phi^0_{\mathcal{K}}\bigr)\circ\bigl((\Phi^0_{\mathcal{K}})^{-1}\circ\Phi_{\mathcal{K}}\bigr)-\nabla_{\lambda,0}(\partial_\lambda)\\&=G^{-1}\circ\bigl((\Phi^0_{\mathcal{K}})^*\nabla_\lambda(\partial_\lambda)-\nabla_{\lambda,0}(\partial_\lambda)\bigr)\circ G+G^{-1}\cdot\nabla_{\lambda,0}(\partial_\lambda)G \qquad (78)\end{aligned}$$

By Lemma 7.29, we have $(\Phi^0_{\mathcal{K}})^*\nabla_\lambda(\lambda^2\partial_\lambda)-\nabla_{\lambda,0}(\lambda^2\partial_\lambda)=O\bigl(\Lambda_0(\epsilon)\bigr)$. We also have

$$\nabla_{\lambda,0}(\lambda^2\partial_\lambda)G=\sum\lambda^2\frac{\partial a_p}{\partial\lambda}\cdot(G^p-\mathrm{id})=O\bigl(\Lambda_0(\epsilon)\bigr)$$

Thus, we obtain Lemma 7.33. ∎

Let $\Phi^{\dagger\,q}_{\sigma(\mathcal{K})}$ $(q=0,1,\ldots,m')$ be as in Lemma 7.31, and let b_q be non-negative C^∞-functions on $\sigma(\mathcal{K})$ such that $\sum b_q=1$. We set $\Phi^\dagger_{\sigma(\mathcal{K})}:=\sum b_q\cdot\Phi^{\dagger\,q}_{\sigma(S)}$.

Lemma 7.34 *We set $H:=\mathcal{S}\bigl(\Phi_{\mathcal{K}}\otimes\sigma^*(\Phi^\dagger_{\sigma(\mathcal{K})})\bigr)-\mathcal{S}_0$. Then, we have the following estimate on $\mathcal{K}\times Z(C)$ with respect to h_0 for some $\epsilon>0$:*

$$H=O\bigl(\Lambda_0(\epsilon)\bigr),\quad \overline{\partial}_{\mathcal{E}_0^{\triangle},\mathbb{P}^1}H=O\bigl(\Lambda_0(\epsilon)\bigr)$$

Proof It follows from Lemma 7.32. ∎

Construction of an approximating map We take $0<R_1<R_2<1$. We set $\mathcal{K}_1:=\bigl\{\lambda\,\big|\,|\lambda|\leq R_2\bigr\}$ and $\mathcal{K}_2:=\bigl\{\lambda\,\big|\,R_1\leq|\lambda|\leq R_1^{-1}\bigr\}$. We take a partition of unity $\bigl(\chi_{\mathcal{K}_1},\chi_{\mathcal{K}_2},\chi_{\sigma(\mathcal{K}_1)}\bigr)$ on $\mathbb{P}^1$ which subordinates to $\{\mathcal{K}_1,\mathcal{K}_2,\sigma(\mathcal{K}_1)\}$.

We take a holomorphic isomorphism $\Phi_{\mathcal{K}_1}:\mathcal{Q}_0\mathcal{E}_{0|\mathcal{K}_1\times X}\longrightarrow\mathcal{Q}_0\mathcal{E}_{|\mathcal{K}_1\times X}$ as in Lemma 7.28. Similarly, we take a holomorphic isomorphism $\Phi^\dagger_{\sigma(\mathcal{K}_1)}:\mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger_{0|\sigma(\mathcal{K})\times X^\dagger}\longrightarrow\mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger_{|\sigma(\mathcal{K})\times X^\dagger}$ as in Lemma 7.31.

We can take a flat isomorphism $\Phi_{\mathcal{K}_2}:\bigl(\mathcal{E}_0,\widetilde{\mathbb{D}}^f_0\bigr)_{|\mathcal{K}_2\times(X-D)}\longrightarrow\bigl(\mathcal{E},\widetilde{\mathbb{D}}^f\bigr)_{|\mathcal{K}_2\times(X-D)}$. We may assume that $\Phi_{\mathcal{K}_2}$ is extended to the isomorphisms $\mathcal{Q}_0\mathcal{E}_{0|\mathcal{K}_2\times X}\simeq\mathcal{Q}_0\mathcal{E}_{|\mathcal{K}_2\times X}$ and $\mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger_{0|\mathcal{K}_2\times X^\dagger}\simeq\mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^\dagger_{|\mathcal{K}_2\times X^\dagger}$ equipped with the property in Lemmas 7.28 and 7.31. We set

$$\Phi:=\chi_{\mathcal{K}_1}\cdot\Phi_{\mathcal{K}_1}+\chi_{\mathcal{K}_2}\cdot\Phi_{\mathcal{K}_2}+\chi_{\sigma(\mathcal{K}_1)}\cdot\Phi^\dagger_{\sigma(\mathcal{K}_1)}.$$

By using Lemmas 7.33 and 7.34, we can check that Φ satisfies the estimates in (61). Thus, the proof of Proposition 7.21 is finished. ∎

7.4.4 Proof of Proposition 7.23

Decomposition We have a decomposition

$$\mathcal{Q}_0\mathcal{E}_{\mathcal{X}}=\bigoplus_{\boldsymbol{a}\in\mathcal{P}ar(\mathcal{P}_0\mathcal{E}^0,\underline{\ell})}U_{\boldsymbol{a}} \qquad (79)$$

with the following property:

- For any subset $I\subset\underline{\ell}$ and $\boldsymbol{b}\in\mathcal{P}ar(\mathcal{P}_0\mathcal{E}^0,I)$, we put ${}^IU_{\boldsymbol{b}}=\bigoplus_{\boldsymbol{a}\in q_I^{-1}(\boldsymbol{b})}U_{\boldsymbol{a}}$. Then, the following holds for any $\boldsymbol{c}\in\boldsymbol{R}^I$:

$$\bigoplus_{\boldsymbol{b}\leq\boldsymbol{c}}{}^IU_{\boldsymbol{b}|\mathcal{D}_I}=\bigcap_{i\in I}{}^iF_{c_i}\bigl(\mathcal{Q}_0\mathcal{E}_{|\mathcal{D}_I}\bigr) \qquad (80)$$

Weak norm estimate We take a C^{∞}-frame $h'_{\boldsymbol{a}}$ of $U_{\boldsymbol{a}}$ in (79). We set $h^{(2)}_{\boldsymbol{a}} := h'_{\boldsymbol{a}} \cdot \prod_{j=1}^{\ell} |z_j|^{-2a_j}$. We obtain a C^{∞}-hermitian metric $h^{(2)} = \bigoplus h^{(2)}_{\boldsymbol{a}}$ of $\mathcal{QE}_{|\mathcal{X}-\mathcal{D}}$. Proposition 8.70 of [18] implies the following lemma.

Lemma 7.35 *h and $h^{(2)}$ are mutually bounded up to log order, namely,*

$$h^{(2)} \cdot C^{-1} \cdot \Big(\sum_{i=1}^{\ell} -\log|z_i|\Big)^{-N} \leq h \leq h^{(2)} \cdot C \cdot \Big(\sum_{i=1}^{\ell} -\log|z_i|\Big)^{N}$$

holds for some $C > 0$ and $N > 0$. ∎

An estimate

Lemma 7.36 *Let f be a holomorphic endomorphism of $\mathcal{Q}_0\mathcal{E}_0$ satisfying the following conditions:*

- *It preserves the filtrations ${}^{i}F$ $(i = 1, \ldots, \ell)$.*
- *For each $\boldsymbol{a} \in \boldsymbol{R}^{\ell}$, the induced endomorphism ${}^{\underline{\ell}}\mathrm{Gr}^{F}_{\boldsymbol{a}}(f)$ of $\mathcal{QE}_{\boldsymbol{a}|\mathcal{D}_{\underline{\ell}}}$ is 0.*

Then, we have $|f|_{h_0} = O\big(\Lambda(M, \epsilon)\big)$ for some $M > 0$ and $\epsilon > 0$.

Proof We take a decomposition of $\mathcal{Q}_0\mathcal{E}_0$ like (79). Applying the weak norm estimate to $(E_{\boldsymbol{a}}, \overline{\partial}_{\boldsymbol{a}}, \theta_{\boldsymbol{a}}, h_{\boldsymbol{a}})$ with the decomposition (66), we take a C^{∞}-hermitian metric $h^{(2)}_{\boldsymbol{a}}$ of $\mathcal{QE}_{\boldsymbol{a}|\mathcal{X}-\mathcal{D}}$, and $h^{(2)}_0 = \bigoplus h^{(2)}_{\boldsymbol{a}}$ of $\mathcal{QE}_{0|\mathcal{X}-\mathcal{D}}$. We have the decomposition:

$$f = \sum f_{\boldsymbol{a},\boldsymbol{a}'}, \qquad f_{\boldsymbol{a},\boldsymbol{a}'} \in \mathrm{Hom}\big(U_{0,\boldsymbol{a}'}, U_{0,\boldsymbol{a}}\big)$$

We have only to show $\big|f_{\boldsymbol{a},\boldsymbol{a}'}\big|_{h^{(2)}_0} = O\big(\Lambda(M, \epsilon)\big)$ for any $\boldsymbol{a}$ and $\boldsymbol{a}'$. Assume $\boldsymbol{a} \neq \boldsymbol{a}'$. We define

$$I_+ := \big\{i \,\big|\, a_i > a'_i\big\}, \quad I_- := \big\{i \,\big|\, a_i < a'_i\big\}, \quad I_0 := \big\{i \,\big|\, a_i = a'_i\big\}.$$

Since the parabolic filtrations are preserved, we have $f_{\boldsymbol{a},\boldsymbol{a}',|\mathcal{D}_i} = 0$ for any $i \in I_+$. Hence, there exists a holomorphic section $f'_{\boldsymbol{a},\boldsymbol{a}'}$ such that $f_{\boldsymbol{a},\boldsymbol{a}'} = f'_{\boldsymbol{a},\boldsymbol{a}'} \cdot \prod_{i \in I_+} z_i$. We have the inequality as in (70). Then, we obtain the desired estimate for $f_{\boldsymbol{a},\boldsymbol{a}'}$ in the case $\boldsymbol{a} \neq \boldsymbol{a}'$.

If $\boldsymbol{a} = \boldsymbol{a}'$, $f_{\boldsymbol{a},\boldsymbol{a}|\mathcal{D}_{\underline{\ell}}} = 0$. Hence, there are holomorphic sections $f_{t,\boldsymbol{a}}$ of $\mathrm{Hom}\big(\mathcal{Q}_0\mathcal{E}_{\boldsymbol{a}}, \mathcal{Q}_0\mathcal{E}_{\boldsymbol{a}}\big)$ such that $f_{\boldsymbol{a},\boldsymbol{a}} = \sum z_t \cdot f_{t,\boldsymbol{a}}$. Because $|f_{t,\boldsymbol{a}}|_{h_0} = O\Big(\big(\sum_{i=1}^{\ell} -\log|z_i|\big)^{N}\Big)$, we obtain the desired estimate. ∎

Local isomorphism with a nice property We can show the following lemma by the argument in the proof of Lemma 7.28.

Lemma 7.37 *There exists a holomorphic isomorphism $\Phi_{\mathcal{K}} : \mathcal{QE}_{0|\mathcal{X}} \longrightarrow \mathcal{QE}_{|\mathcal{X}}$ such that (i) it preserves the filtrations ${}^{i}F$ $(i = 1, \ldots, \ell)$, (ii) for each $\boldsymbol{a} \in \boldsymbol{R}^{\ell}$, the induced map $\mathcal{QE}_{\boldsymbol{a}|\mathcal{D}_{\underline{\ell}}} \longrightarrow {}^{\underline{\ell}}\mathrm{Gr}^{F}_{\boldsymbol{b}}(\mathcal{QE}_{|\mathcal{D}_{\underline{\ell}}})$ is equal to $\nu_{\boldsymbol{a}}$.*

Similarly, we can take a holomorphic isomorphism $\Phi^{\dagger}_{\sigma(\mathcal{K})} : \mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^{\dagger}_{0|\sigma(\mathcal{X})} \longrightarrow \mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^{\dagger}_{|\sigma(\mathcal{X})}$ satisfying the conditions (i) it preserves the filtrations ${}^{i}F$ $(i = 1, \ldots, \ell)$, (iii) the induced morphism on ${}^{\underline{\ell}}\mathrm{Gr}^{F}_{-\boldsymbol{a}}$ is equal to the given one. ∎

By the weak norm estimate, $\Phi_{\mathcal{K}}$ and $\Phi_{\mathcal{K}}^{-1}$ are bounded up to log order. We can show the following lemma by using Lemma 7.36.

Lemma 7.38 *We have $\Phi^{*}_{\mathcal{K}}\nabla_{\lambda} - \nabla_{\lambda,0} = O\big(\Lambda(M, \epsilon)\big)$ for some $\epsilon > 0$ and $M > 0$ with respect to h_0.* ∎

Let $\Phi_{\mathcal{K}}$ and $\Phi'_{\mathcal{K}}$ be morphisms as in Lemma 7.37. We set $G := \Phi_{\mathcal{K}}^{-1} \circ \Phi'_{\mathcal{K}}$.

Lemma 7.39 *We have the following estimates for some $\epsilon > 0$ and $M > 0$:*

$$\big|G - \mathrm{id}\big|_{h_0} = O\big(\Lambda(M, \epsilon)\big), \quad \big|\nabla_{\lambda,0}(\lambda^2\partial_{\lambda})G\big|_{h_0} = O\big(\Lambda(M, \epsilon)\big)$$

Proof It follows from Lemma 7.36. ∎

Lemma 7.40 *Let $\Phi_{\mathcal{K}}$ and $\Phi^{\dagger}_{\sigma(\mathcal{K})}$ satisfy the above conditions. We set*

$$H := \mathcal{S}_0 - \mathcal{S}\big(\Phi_{\mathcal{K}} \otimes \sigma^*\Phi^{\dagger}_{\sigma(\mathcal{K})}\big) : \mathcal{Q}_0\mathcal{E}_{0|\mathcal{X}} \otimes \sigma^*\big(\mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^{\dagger}_{0|\sigma(\mathcal{X})}\big) \longrightarrow \mathcal{O}_{\mathcal{X}}$$

Then, $H = O\big(\Lambda(M,\epsilon)\big)$ with respect to h_0 for some $\epsilon > 0$ and $M > 0$.

Proof It can be shown by the argument in the proof of Lemma 7.32. ∎

Local C^{∞}-isomorphisms Let $\Phi^p_{\mathcal{K}}$ $(p = 0,\ldots,m)$ be as in Lemma 7.37, and let a_p $(p = 0,\ldots,m)$ be non-negative C^{∞}-functions on $\mathcal{K}$ such that $\sum a_p = 1$. We set $\Phi_{\mathcal{K}} := \sum_{p=0}^m a_p \cdot \Phi^p_{\mathcal{K}}$. We also set $G := (\Phi^0_{\mathcal{K}})^{-1} \circ \Phi_{\mathcal{K}}$ and $G^p := (\Phi^0_{\mathcal{K}})^{-1} \circ \Phi^p_{\mathcal{K}}$. By Lemma 7.39, $|G^p - \mathrm{id}\,|_{h_0} = O\big(\Lambda_0(\epsilon)\big)$, and hence $|G - \mathrm{id}\,|_{h_0} = O\big(\Lambda_0(\epsilon)\big)$ for some $\epsilon > 0$ and $M > 0$.

We can show the following estimate by using an argument in the proof of Lemma 7.33 with Lemma 7.38:

$$\Phi_{\mathcal{K}}^{-1} \circ \nabla_{\lambda}(\lambda^2\partial_{\lambda}) \circ \Phi_{\mathcal{K}} - \nabla_{\lambda,0}(\lambda^2\partial_{\lambda}) = O\big(\Lambda(M,\epsilon)\big) \tag{81}$$

Let $\Phi^{\dagger\,q}_{\sigma(\mathcal{K})}$ $(q = 0,1,\ldots,m')$ be as in Lemma 7.31, and let b_q be non-negative C^{∞}-functions on $\sigma(\mathcal{K})$ such that $\sum b_q = 1$. We set $\Phi^{\dagger}_{\sigma(\mathcal{K})} := \sum b_q \cdot \Phi^{\dagger\,q}_{\sigma(S)}$. We set $H := \mathcal{S}\big(\Phi_{\mathcal{K}} \otimes \sigma^*(\Phi^{\dagger}_{\sigma(\mathcal{K})})\big) - \mathcal{S}_0$. Then, we can show the following estimate with respect to h_0 for some $\epsilon > 0$ and $M > 0$, by using Lemma 7.40:

$$H = O\big(\Lambda(M,\epsilon)\big), \quad \overline{\partial}_{\mathcal{E}_0^{\triangle},\mathbb{P}^1} H = O\big(\Lambda(M,\epsilon)\big) \tag{82}$$

Construction We take $0 < R_1 < R_2 < 1$. We set $\mathcal{K}_1 := \big\{\lambda \,\big|\, |\lambda| \le R_2\big\}$ and $\mathcal{K}_2 := \big\{\lambda \,\big|\, R_1 \le |\lambda| \le R_1^{-1}\big\}$. We take a partition of unity $\big(\chi_{\mathcal{K}_1}, \chi_{\mathcal{K}_2}, \chi_{\sigma(\mathcal{K}_1)}\big)$ on $\mathbb{P}^1$ which subordinates to $\{\mathcal{K}_1, \mathcal{K}_2, \sigma(\mathcal{K}_1)\}$.

We take a holomorphic isomorphism $\Phi_{\mathcal{K}_1} : \mathcal{Q}\mathcal{E}_{0|\mathcal{K}\times X} \longrightarrow \mathcal{Q}\mathcal{E}_{|\mathcal{K}\times X}$ as in Lemma 7.37. Similarly, we take a holomorphic isomorphism $\Phi^{\dagger}_{\sigma(\mathcal{K}_1)} : \mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^{\dagger}_{0|\sigma(\mathcal{K})\times X^{\dagger}} \longrightarrow \mathcal{Q}_{<\boldsymbol{\delta}}\mathcal{E}^{\dagger}_{|\sigma(\mathcal{K})\times X^{\dagger}}$ as in Lemma 7.37.

We can take a flat isomorphism $\Phi_{\mathcal{K}_2} : \big(\mathcal{E}_0, \widetilde{\mathbb{D}}^f_0\big)_{|\mathcal{K}_2\times(X-D)} \longrightarrow \big(\mathcal{E}, \widetilde{\mathbb{D}}^f\big)_{|\mathcal{K}_2\times(X-D)}$. We set

$$\Phi := \chi_{\mathcal{K}_1} \cdot \Phi_{\mathcal{K}_1} + \chi_{\mathcal{K}_2} \cdot \Phi_{\mathcal{K}_2} + \chi_{\sigma(\mathcal{K}_1)} \cdot \Phi^{\dagger}_{\sigma(\mathcal{K}_1)}.$$

By using (81) and (82), we can check that Φ satisfies the estimates in (63). Thus, the proof of Proposition 7.23 is finished. ∎

8 An application to HS-orbit

8.1 Preliminary

8.1.1 Compatibility of real structure and Stokes structure

Let X be a complex manifold. We set $\mathcal{X} := \boldsymbol{C}_{\lambda} \times X$ and $\mathcal{X}^0 := \{0\} \times X$. Let $(H, H'_{\boldsymbol{R}}, \nabla)$ be a TER-structure on $\mathcal{X}$. We say that H is unramifiedly pseudo-good if the following holds:

- We are given a good set of irregular values $\mathrm{Irr}(\nabla) \subset M(\mathcal{X},\mathcal{X}^0)/H(\mathcal{X})$ in the level -1. Namely, (i) any elements $\mathfrak{a}$ of $\mathrm{Irr}(\nabla)$ are of the form $\mathfrak{a} = \lambda^{-1}\mathfrak{a}'$ for some holomorphic functions $\mathfrak{a}'$ on X, (ii) $\mathfrak{a}' - \mathfrak{b}'$ are nowhere vanishing for distinct $\lambda^{-1}\mathfrak{a}', \lambda^{-1}\mathfrak{b}' \in \mathrm{Irr}(\nabla)$.
- H has the formal decomposition
$$(H,\nabla)_{|\widehat{\mathcal{X}}^0} = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\nabla)} (\widehat{H}_{\mathfrak{a}}, \widehat{\nabla}_{\mathfrak{a}}),$$
such that $\widehat{\nabla}_{\mathfrak{a}} - d\mathfrak{a}$ is regular. Note that they are not assumed to be logarithmic.

(See also Subsection 5.1.3.) If X is a point, it means that H requires no ramification in the sense of [8].

By a classical theory (see also Subsection 5.1.3), we have the Stokes filtration $\mathcal{F}^S$ indexed by $\big(\mathrm{Irr}(\nabla), \leq_S\big)$ for each small sector S of $\mathcal{X} - \mathcal{X}^0$. We say that the real structure and the Stokes structure are compatible, if the Stokes filtrations on any small sectors S come from a flat filtration of $H'_{\boldsymbol{R}|S}$. (See [14].)

By taking Gr of (H, ∇) with respect to the Stokes filtrations, we obtain a TE-structure $\mathrm{Gr}_{\mathfrak{a}}(H, \nabla)$ for $\mathfrak{a} \in \mathrm{Irr}(\nabla)$. As observed in [8], if the real structure and the Stokes structure are compatible, $\mathrm{Gr}_{\mathfrak{a}}(H, \nabla)$ is enriched to a TER-structure denoted by $\mathrm{Gr}_{\mathfrak{a}}(H, H'_{\boldsymbol{R}}, \nabla)$. If $(H, H'_{\boldsymbol{R}}, \nabla)$ is enriched to a TERP-structure $(H, H'_{\boldsymbol{R}}, \nabla, P, w)$, $\mathrm{Gr}_{\mathfrak{a}}(H, H'_{\boldsymbol{R}}, \nabla)$ is also naturally enriched to a TERP-structure denoted by $\mathrm{Gr}_{\mathfrak{a}}(H, H'_{\boldsymbol{R}}, \nabla, P, w)$.

Another formulation In [8], a compatibility of real structure and Stokes structure is formulated in a slightly different way. Let us check that it is equivalent to the above. For simplicity, we consider the case in which X is a point.

Let H be a vector bundle on $\boldsymbol{C}_\lambda$ with a meromorphic flat connection $\nabla : H \longrightarrow H \otimes \Omega^1_{\boldsymbol{C}_\lambda}(*0)$ such that H requires no ramification with the good set of irregular values $\mathrm{Irr}(\nabla) \subset \lambda^{-1} \cdot \boldsymbol{C}$. Take $\theta_0 \in \boldsymbol{R}$ such that $\mathrm{Re}(\mathfrak{a} - \mathfrak{b})(r \cdot e^{\sqrt{-1}\theta_0}) \neq 0$ for any distinct $\mathfrak{a}, \mathfrak{b} \in \mathrm{Irr}(\nabla)$. Take a sufficiently small $\epsilon > 0$, and let us consider the sector

$$\mathcal{S} := \big\{ r \cdot e^{\sqrt{-1}\theta} \,\big|\, \theta_0 - \epsilon \leq \theta \leq \theta_0 + \pi + \epsilon \big\}$$

Let $\overline{\mathcal{S}}$ denote the closure of $\mathcal{S}$ in the real blow up $\widetilde{\boldsymbol{C}}_\lambda(0) \longrightarrow \boldsymbol{C}_\lambda$ along 0. Let $\mathcal{Z} := \overline{\mathcal{S}} \cap \pi^{-1}(0)$. As a version of Hukuhara-Turrittin theorem, it is well known that we have a *unique* flat decomposition

$$(H, \nabla)_{|\overline{\mathcal{S}}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla)} \big(H_{\mathfrak{a},\mathcal{S}}, \nabla_{\mathfrak{a},\mathcal{S}}\big) \tag{83}$$

such that the restriction of (83) to $\widehat{\mathcal{Z}}$ is the same as the pull back of the irregular decomposition of $H_{|\widehat{0}}$.

Assume that the flat bundle $(H, \nabla)_{|\boldsymbol{C}^*_\lambda}$ is equipped with a real structure, i.e., a $\boldsymbol{C}$-anti-linear flat involution $\kappa : H \longrightarrow H$. In other words, (H, ∇, κ) is a TER-structure. In Section 8 of [8], the real structure and the Stokes structure are defined to be compatible, if $\kappa(H_{\mathfrak{a},\mathcal{S}}) = H_{\mathfrak{a},\mathcal{S}}$ for any $\mathfrak{a} \in \mathrm{Irr}(\nabla)$ and any $\mathcal{S}$ as above.

If a small sector S is contained in $\mathcal{S}$, the restriction of (83) to S gives a splitting of $\mathcal{F}^S$. Hence, if $H_{\mathfrak{a},\mathcal{S}}$ are preserved by κ for any $\mathfrak{a}$, the filtration $\mathcal{F}^S$ is also preserved by κ. Let S_1 and S_2 be small sectors containing the rays $\{r \cdot e^{\sqrt{-1}\theta_0} \,|\, r > 0\}$ and $\{-r \cdot e^{\sqrt{-1}\theta_0} | \, r > 0\}$, respectively. Then, $\mathfrak{a} \leq_{S_1} \mathfrak{b}$ if and only if $\mathfrak{a} \geq_{S_2} \mathfrak{b}$. By the parallel transform on $\mathcal{S}$, the flat bundle $H_{|\mathcal{S}}$ is trivialized, and we can observe that $H_{\mathfrak{a},\mathcal{S}} = \mathcal{F}^{S_1}_{\mathfrak{a}} \cap \mathcal{F}^{S_2}_{\mathfrak{a}}$. Hence, if $\mathcal{F}^{S_i}_{\mathfrak{a}}$ $(i = 1, 2)$ are preserved by κ, $H_{\mathfrak{a},\mathcal{S}}$ is also preserved by κ. The equivalence of two notions of compatibilities follows from these considerations.

8.1.2 Two Stokes filtrations of integrable twistor structures

Let $(V, \widetilde{\mathbb{D}}^{\triangle})$ be a variation of integrable twistor structures over $\mathbb{P}^1 \times X$. It is obtained as the gluing of TE-structure $(V_0, \widetilde{\mathbb{D}}^f_0)$ on $\mathcal{X} := \boldsymbol{C}_\lambda \times X$ and $\widetilde{T}E$-structure $(V_\infty, \widetilde{\mathbb{D}}^{\dagger f}_\infty)$ on $\mathcal{X}^\dagger := \boldsymbol{C}_\mu \times X^\dagger$. We set $\mathcal{X}^0 := \{0\} \times X \subset \mathcal{X}$ and $\mathcal{X}^{\dagger\, 0} := \{0\} \times X^\dagger \subset \boldsymbol{C}_\mu \times X^\dagger$.

Definition 8.1 *We say that $(V, \widetilde{\mathbb{D}}^{\triangle})$ is unramifiedly pseudo-good, if both $(V_0, \widetilde{\mathbb{D}}^f_0)$ and $(V_\infty, \widetilde{\mathbb{D}}^{\dagger f}_\infty)$ is unramifiedly pseudo-good. In that case, let $\mathrm{Irr}(\widetilde{\mathbb{D}}^f_0)$ and $\mathrm{Irr}(\widetilde{\mathbb{D}}^{\dagger f}_\infty)$ denote the sets of irregular values of $\widetilde{\mathbb{D}}^f_0$ and $\widetilde{\mathbb{D}}^{\dagger f}_\infty$, respectively.*

If X is a point, it is also said that $(V, \widetilde{\mathbb{D}}^{\triangle})$ requires no ramification. ∎

Definition 8.2 *Assume $(V, \widetilde{\mathbb{D}}^{\triangle})$ is unramifiedly pseudo-good.*

- *We say that the sets of the irregular values of $(V, \widetilde{\mathbb{D}}^{\triangle})$ are compatible, if $\mathrm{Irr}\big(\widetilde{\mathbb{D}}^f_0\big)$ and $\mathrm{Irr}\big(\widetilde{\mathbb{D}}^f_\infty\big)$ bijectively correspond by $\mathfrak{a} \longleftrightarrow \overline{\gamma^* \mathfrak{a}}$.*
- *We say that $(V, \widetilde{\mathbb{D}}^{\triangle})$ has compatible Stokes structures, if the following holds:*
  - *The sets of irregular values of $(V, \widetilde{\mathbb{D}}^{\triangle})$ are compatible.*

– *For a small sector S of $\mathcal{X}-\mathcal{X}^0$, we have the Stokes filtration $\mathcal{F}^S$ of $(V_0,\widetilde{\mathbb{D}}_0^f)$. We also have the Stokes filtration $\mathcal{F}^{\gamma(S)}$ of $(V_\infty,\widetilde{\mathbb{D}}_\infty^f)$, where we regard $\gamma(S)$ as a small sector of $\mathcal{X}^\dagger-\mathcal{X}^{\dagger\,0}$. Then, $\mathcal{F}^S$ and $\mathcal{F}^{\gamma(S)}$ are the same under the parallel transform along any rays connecting S and $\gamma(S)$.* ∎

Remark 8.3 *In the above definition, a ray means a line $\bigl\{(t\cdot e^{\sqrt{-1}\varphi},P)\,\big|\,0<t<\infty\bigr\}$ in $\boldsymbol{C}_\lambda^*\times\{P\}\subset\boldsymbol{C}_\lambda^*\times X$. We say that it connects S and $\gamma(S)$, if (i) $(t\cdot e^{\sqrt{-1}\varphi},P)$ is contained in S for any sufficiently small t, (ii) $(t\cdot e^{\sqrt{-1}\varphi},P)$ is contained in $\gamma(S)$ for any sufficiently large t.* ∎

Lemma 8.4 *If $(V,\widetilde{\mathbb{D}}^\triangle)$ is equipped with either a real structure κ or a perfect pairing $\mathcal{S}$ of weight w, then the irregular values of $\widetilde{\mathbb{D}}_0^f$ and $\widetilde{\mathbb{D}}_\infty^f$ are compatible.*

Proof We have $\mathrm{Irr}\bigl(\gamma^*\widetilde{\mathbb{D}}_\infty^f\bigr)=\bigl\{\overline{\gamma^*\mathfrak{a}}\,\big|\,\mathfrak{a}\in\mathrm{Irr}(\widetilde{\mathbb{D}}_\infty^f)\bigr\}$. If $(V,\widetilde{\mathbb{D}}^\triangle)$ is equipped with a real structure, $\gamma^*(V_\infty,\widetilde{\mathbb{D}}_\infty^f)\simeq(V_0,\widetilde{\mathbb{D}}_0^f)$. Hence, the irregular values of $\widetilde{\mathbb{D}}_0^f$ and $\widetilde{\mathbb{D}}_\infty^f$ are compatible.

We have $\mathrm{Irr}\bigl(\sigma^*\widetilde{\mathbb{D}}_\infty^f\bigr)=\bigl\{\overline{\sigma^*\mathfrak{a}}\,\big|\,\mathfrak{a}\in\mathrm{Irr}(\widetilde{\mathbb{D}}_\infty^f)\bigr\}$. Note that $\mathfrak{a}\in\mathrm{Irr}(\widetilde{\mathbb{D}}_\infty^f)$ are of the form $\mu^{-1}\mathfrak{a}'$, where $\mathfrak{a}'$ are holomorphic functions on $X^\dagger$. Hence, $\overline{\sigma^*\mathfrak{a}}=-\overline{\gamma^*\mathfrak{a}}$. If $(V,\widetilde{\mathbb{D}}_f^\triangle)$ is equipped with a perfect pairing, $(V_0,\widetilde{\mathbb{D}}_0^f)$ is isomorphic to the dual of $\sigma^*(V_\infty,\widetilde{\mathbb{D}}_\infty^f)$. Therefore, the irregular values of $\widetilde{\mathbb{D}}_0^f$ and $\widetilde{\mathbb{D}}_\infty^f$ are compatible. ∎

If $(V,\widetilde{\mathbb{D}}^\triangle)$ is unramifiedly pseudo-good, we obtain TE-structure $\mathrm{Gr}_\mathfrak{a}(V_0,\widetilde{\mathbb{D}}_0^f)$ on $\mathcal{X}$ for $\mathfrak{a}\in\mathrm{Irr}(\widetilde{\mathbb{D}}_0^f)$, and $\widetilde{T}E$-structure $\mathrm{Gr}_\mathfrak{b}(V_\infty,\widetilde{\mathbb{D}}_\infty^f)$ on $\mathcal{X}^\dagger$ for $\mathfrak{b}\in\mathrm{Irr}(\widetilde{\mathbb{D}}_\infty^f)$, by taking Gr with respect to the Stokes filtrations. If $(V,\widetilde{\mathbb{D}}^\triangle)$ has compatible Stokes structures, we have the natural isomorphism

$$\mathrm{Gr}_\mathfrak{a}(V_0,\widetilde{\mathbb{D}}_0^f)_{|\mathcal{X}-\mathcal{X}^0}\simeq\mathrm{Gr}_{\overline{\gamma^*\mathfrak{a}}}\bigl(V_\infty,\widetilde{\mathbb{D}}_\infty^f\bigr)_{|\mathcal{X}^\dagger-\mathcal{X}^{\dagger\,0}}.$$

Hence, we obtain a variation of integrable twistor structures $\mathrm{Gr}_\mathfrak{a}(V,\widetilde{\mathbb{D}}^\triangle)$ for each $\mathfrak{a}\in\mathrm{Irr}(\widetilde{\mathbb{D}}_0^f)$ as the gluing of them. We have the following functoriality (Lemma 5.17).

Lemma 8.5 *Let $(V^{(a)},\widetilde{\mathbb{D}}^{(a)\triangle})$ be unramifiedly pseudo-good. Assume (i) $(V^{(a)},\widetilde{\mathbb{D}}^{(a)\triangle})$ $(a=1,2)$ have compatible Stokes filtrations, (ii) the union $\mathcal{I}:=\mathrm{Irr}(\widetilde{\mathbb{D}}_0^{(1)\,f})\cup\mathrm{Irr}(\widetilde{\mathbb{D}}_0^{(2)\,f})$ is good. Then, a morphism $(V^{(1)},\widetilde{\mathbb{D}}^{(1)\,\triangle})\longrightarrow(V^{(2)},\widetilde{\mathbb{D}}^{(2)\,\triangle})$ induces $\mathrm{Gr}_\mathfrak{a}\bigl(V^{(1)},\widetilde{\mathbb{D}}^{(1)\,\triangle}\bigr)\longrightarrow\mathrm{Gr}_\mathfrak{a}\bigl(V^{(2)},\widetilde{\mathbb{D}}^{(2)\triangle}\bigr)$ for each $\mathfrak{a}\in\mathcal{I}$.* ∎

We have the natural isomorphisms

$$\gamma^*\,\mathrm{Gr}_\mathfrak{a}(V,\widetilde{\mathbb{D}}^\triangle)\simeq\mathrm{Gr}_\mathfrak{a}\bigl(\gamma^*(V,\widetilde{\mathbb{D}}^\triangle)\bigr),\quad\sigma^*\,\mathrm{Gr}_\mathfrak{a}(V,\widetilde{\mathbb{D}}^\triangle)\simeq\mathrm{Gr}_{-\mathfrak{a}}\bigl(\sigma^*(V,\widetilde{\mathbb{D}}^\triangle)\bigr).$$

The following lemma follows from functoriality.

Lemma 8.6 *Assume $(V,\widetilde{\mathbb{D}}^\triangle)$ has compatible Stokes structures. If $(V,\widetilde{\mathbb{D}}^\triangle)$ is equipped with a real structure, (resp. a perfect pairing of weight w), each $\mathrm{Gr}_\mathfrak{a}(V,\widetilde{\mathbb{D}}^\triangle)$ is also equipped with an induced real structure (resp. an induced perfect pairing of weight w).* ∎

Lemma 8.7 *Let $(H,H'_{\boldsymbol{R}},\nabla,P',-w)$ be a variation of TERP-structures, and let $(V,\widetilde{\mathbb{D}}^\triangle,\mathcal{S},\kappa,-w)$ be the corresponding variation of twistor-TERP structures. (See Subsection 2.1.8 for the correspondence.) Assume that $(H,H'_{\boldsymbol{R}},\nabla,P',-w)$ is unramifiedly pseudo-good, or equivalently, $(V,\widetilde{\mathbb{D}}^\triangle,\mathcal{S},\kappa,-w)$ is unramifiedly pseudo-good.*

- *The real structure and the Stokes structures of (H,∇) are compatible, if and only if $(V,\widetilde{\mathbb{D}}^\triangle)$ has compatible Stokes structures.*
- *If the real structures and the Stokes structures are compatible, $\mathrm{Gr}_\mathfrak{a}(V,\widetilde{\mathbb{D}}^\triangle,\mathcal{S},\kappa,-w)$ is the variation of twistor-TERP structures corresponding to $\mathrm{Gr}_\mathfrak{a}(H,H'_{\boldsymbol{R}},\nabla,P',-w)$.*

Proof Note that the Stokes filtrations of $\gamma^*(H,\nabla)$ on $\gamma^*(S)$ is given by the composite of the conjugate with respect to $H'_{\boldsymbol{R}}$ and the parallel transport along the rays connecting S and $\gamma(S)$, with the change of the index sets from $\mathrm{Irr}(\nabla)$ to $\bigl\{\overline{\gamma^*\mathfrak{a}}\,\big|\,\mathfrak{a}\in\mathrm{Irr}(\nabla)\bigr\}$. Then, the first claim follows.

Let us consider the second claim. We have only to consider the case $w = 0$. We may assume that $(H, H'_{\boldsymbol{R}}, \nabla, P')$ is obtained from $(V, \nabla, \mathcal{S}, \kappa)$ by the procedure explained in Subsection 2.1.8. By construction, we have $\mathrm{Gr}_{\mathfrak{a}}(H, \nabla) = \mathrm{Gr}_{\mathfrak{a}}(V_0, \nabla_0)$. For comparison of induced real structures and pairings, we have only to consider the case in which X is a point.

Let us compare the induced real structures. The flat real structure of H' is obtained as the composite:

$$\overline{H}_{|\lambda} \xrightarrow{\text{parallel transform}} \overline{H}_{|\overline{\lambda}^{-1}} \xrightarrow{\kappa_{|\lambda}} H_{|\lambda}$$

Hence, we have the following factorization of the real structure on $\mathrm{Gr}_{\mathfrak{a}}(H)_{|\lambda}$ obtained as Gr of the Stokes filtration:

$$\overline{\mathrm{Gr}_{\mathfrak{a}}(H)}_{|\lambda} \xrightarrow{\text{parallel transform}} \overline{\mathrm{Gr}_{\mathfrak{a}}(H)}_{|\overline{\lambda}^{-1}} \xrightarrow{\mathrm{Gr}_{\mathfrak{a}}(\kappa)_{|\lambda}} \mathrm{Gr}_{\mathfrak{a}}(H)_{|\lambda}$$

It is the same as the real structure induced by $\mathrm{Gr}_{\mathfrak{a}}(\kappa)$ on $\mathrm{Gr}_{\mathfrak{a}}(V, \nabla)$.

Let $P : H \otimes j^*H \longrightarrow \mathcal{O}_{\boldsymbol{C}_\lambda}$ be the pairing induced by κ and $\mathcal{S}$ as in (6), whose restriction to H' is P'. Let S be a small sector in $\boldsymbol{C}_\lambda^*$. We have the following factorization of $P_{|S}$:

$$\mathcal{F}^S_{\mathfrak{a}}(H) \otimes j^*\mathcal{F}^{j(S)}_{\mathfrak{b}}(H) = \mathcal{F}^S_{\mathfrak{a}}(V_0) \otimes \sigma^*\gamma^*\mathcal{F}^{j(S)}_{\mathfrak{b}}(V_0) \xrightarrow{1\otimes\sigma^*\kappa} \mathcal{F}^S_{\mathfrak{a}}(V_0) \otimes \sigma^*\mathcal{F}^{\sigma(S)}_{\overline{\gamma^*(\mathfrak{b})}}(V_\infty) \xrightarrow{\mathcal{S}} \mathcal{O}_S$$

The restriction to $\mathcal{F}^S_{\mathfrak{a}}(H) \otimes j^*\mathcal{F}^{j(S)}_{\mathfrak{b}}(H)$ is 0 unless $\mathfrak{a} - \mathfrak{b} \geq_S 0$. The induced pairing $P_{\mathfrak{a}}$ for $\mathrm{Gr}_{\mathfrak{a}}(V_0)$ is factorized as follows:

$$\mathrm{Gr}_{\mathfrak{a}}(V_0)_{|S} \otimes j^*\,\mathrm{Gr}_{\mathfrak{a}}(V_0)_{|j(S)} \xrightarrow{1\otimes\sigma^*\,\mathrm{Gr}_{\mathfrak{a}}\kappa} \mathrm{Gr}_{\mathfrak{a}}(V_0)_{|S} \otimes \sigma^*\,\mathrm{Gr}_{\overline{\gamma^*(\mathfrak{a})}}(V_\infty)_{|\sigma(S)} \xrightarrow{\mathrm{Gr}_{\mathfrak{a}}\mathcal{S}} \mathcal{O}_S$$

Hence, it is the same as the pairing induced by $\mathrm{Gr}_{\mathfrak{a}}(V, \nabla, \mathcal{S}, \kappa)$. Thus, the proof of Lemma 8.7 is finished. ∎

8.1.3 Preliminary for pull back

We set $X := \boldsymbol{C}_z$, $D = \{0\}$, $\mathcal{X} := \boldsymbol{C}_\lambda \times X$, $\mathcal{D} := \boldsymbol{C}_\lambda \times D$ and $W := \mathcal{D} \cup (\{0\} \times X)$. Let $\pi : \widetilde{\mathcal{X}}(W) \longrightarrow \mathcal{X}$ be a real blow up of $\mathcal{X}$ along W. Let $\pi_1 : \widetilde{\boldsymbol{C}}_\lambda(0) \longrightarrow \boldsymbol{C}_\lambda$ be the real blow up of $\boldsymbol{C}_\lambda$ along $\{0\}$. Let $\phi_0 : \mathcal{X} \longrightarrow \boldsymbol{C}_\lambda$ be given by $\phi_0(\lambda, z) = \lambda \cdot z$. It induces the map $\widetilde{\phi}_0 : \widetilde{\mathcal{X}}(W) \longrightarrow \widetilde{\boldsymbol{C}}_\lambda(0)$.

Let H be a vector bundle on $\boldsymbol{C}_\lambda$ with a meromorphic flat connection $\nabla : H \longrightarrow H \otimes \Omega^1_{\boldsymbol{C}_\lambda}(*0)$ such that (H, ∇) requires no ramification with the good set of irregular values $\mathcal{I} \subset \boldsymbol{C}\cdot\lambda^{-1}$. Let $\mathfrak{V}$ denote the flat bundle on $\widetilde{\boldsymbol{C}}_\lambda(0)$ associated to $H_{|\boldsymbol{C}_\lambda^*}$. For each $Q \in \pi_1^{-1}(0)$, we have the Stokes filtration $\mathcal{F}^Q$ of $\mathfrak{V}_{|Q}$ for the meromorphic prolongment H. (See Subsection 5.1.5) We can naturally regard $\widetilde{\phi}_0^*\mathfrak{V}$ as the flat bundle on $\widetilde{\mathcal{X}}(W)$ associated to $(\phi_0^*H)_{|\mathcal{X}-W}$.

Lemma 8.8 *The following holds:*

- *$\phi_0^*(H, \nabla)$ is unramifiedly pseudo-good in the level $\boldsymbol{m} = (-1, -1)$. (See Subsection 5.1.3.) The set of irregular values is given by $\phi_0^*\mathcal{I} := \{\phi_0^*\mathfrak{a} \,|\, \mathfrak{a} \in \mathcal{I}\}$.*
- *For each $P \in \pi^{-1}(W)$, the Stokes filtration $\mathcal{F}^P$ of $\widetilde{\phi}_0^*(\mathfrak{V})_{|P}$ for ϕ_0^*H is the pull back of the Stokes filtration of $\mathfrak{V}_{|\widetilde{\phi}_0(P)}$.*
- *We have the natural isomorphism $\phi_0^*\,\mathrm{Gr}_{\mathfrak{a}}(H) \simeq \mathrm{Gr}_{\phi_0^*\mathfrak{a}}(\phi_0^*H)$.*

Proof We have the decomposition $(H, \nabla)_{|\widehat{0}} = \bigoplus_{\mathfrak{a}\in\mathcal{I}}(H_{\mathfrak{a}}, \widehat{\nabla}_{\mathfrak{a}})$, where $\widehat{\nabla}_{\mathfrak{a}} - d\mathfrak{a}$ are regular. It induces the decomposition of $\phi_0^*(H, \nabla)_{|\widehat{W}}$. Hence, the first claim is clear.

We set $Q := \widetilde{\phi}_0(P)$. Note that the orders $\leq_Q$ and $\leq_P$ are the same under the identification $\mathcal{I} \simeq \phi_0^*\mathcal{I}$. Let $H_1 \supset H$ be an unramifiedly good lattice. Then, $\phi_0^*H_1$ is an unramifiedly good lattice. We take a small sector $S_Q \in \mathcal{MS}(Q, \boldsymbol{C}_\lambda^*, \mathcal{I})$ such that there exists the Stokes filtration $\mathcal{F}^{S_Q}$ of $H_{1|\overline{S}_Q}$. We take a small multi-sector $S_P \in \mathcal{MS}(P, \mathcal{X} - W, \phi_0^*\mathcal{I})$ such that $\phi_0(S_P) \subset S_Q$. Then, we obtain the filtration $\widetilde{\phi}_0^*\mathcal{F}^{S_Q}$ of $\phi_0^*(H_1)_{|\overline{S}_P}$ indexed by $(\phi_0^*\mathcal{I}, \leq_P)$. It gives the Stokes filtration of $\phi_0^*(H_1)_{|\overline{S}_P}$, which follows from the characterization in Proposition

5.5. Since the filtration of $\widetilde{\phi}_0^*(\mathfrak{V})_{|P}$ induced by $\widetilde{\phi}_0^*\mathcal{F}^{S_Q}$ is the same as the pull back of $\mathcal{F}^Q$ on $\mathfrak{V}_{|Q}$, we obtain the second claim. Note that we also obtain that the Stokes filtration of $\phi_0^*(H)_{|\overline{S}_P}$ is given by the pull back of the Stokes filtration of $H_{|\overline{S}_Q}$.

Let S_P be a small multi-sector as above. By the above compatibility of the Stokes filtrations and Lemma 5.21, we obtain the natural isomorphisms

$$\phi_0^*\big(\mathrm{Gr}_{\mathfrak{a}}(H)\big)_{|\overline{S}_P} \simeq \mathrm{Gr}_{\phi_0^*\mathfrak{a}}(\phi_0^* H)_{|\overline{S}_P}. \tag{84}$$

By varying S_P and gluing them, we obtain $\phi_0^*\big(\mathrm{Gr}_{\mathfrak{a}}(H)\big)_{|\widetilde{\mathcal{U}}(W)} \simeq \mathrm{Gr}_{\phi_0^*\mathfrak{a}}(\phi_0^* H)_{|\widetilde{\mathcal{U}}(W)}$, where $\mathcal{U}$ is a neighbourhood of W, and $\widetilde{\mathcal{U}}(W)$ denote the real blow up of $\mathcal{U}$ along W. By using the flatness, it is extended to $\phi_0^*\big(\mathrm{Gr}_{\mathfrak{a}}(H)\big)_{|\widetilde{\mathcal{X}}(W)} \simeq \mathrm{Gr}_{\phi_0^*\mathfrak{a}}(\phi_0^* H)_{|\widetilde{\mathcal{X}}(W)}$. Hence, we obtain an isomorphism on $\mathcal{X}$. ∎

8.1.4 Rescaling and HS-orbit

We recall a rescaling construction in [7] and [8]. See also [22]. We set $X := \boldsymbol{C}_z$, $D = \big\{0\big\}$ and $X^* := X - D$. For $R > 0$, we set $X(R) := \big\{z \in X \,\big|\, |z| < R\big\}$ and $X^*(R) := X(R) \cap X^*$. We set $\mathcal{X} := \boldsymbol{C}_\lambda \times X$. We use the symbols $\mathcal{X}^*$, $\mathcal{D}$, $\mathcal{X}(R)$ and $\mathcal{X}^*(R)$ in similar meanings. Let $\phi_0 : \mathcal{X} \longrightarrow \boldsymbol{C}_\lambda$ be given by $\phi_0(\lambda, z) = \lambda \cdot z$. The restriction to $\mathcal{X}^*$ is denoted by ψ_0.

TERP-structure We consider only TERP-structures of weight 0. Hence, we omit to specify weights. Let $(H, H'_{\boldsymbol{R}}, \nabla, P)$ be a TERP-structure. Hertling and Sevenheck studied the variation of TERP-structures $\psi_0^*(H, H'_{\boldsymbol{R}}, \nabla, P)$ on X^*. If there exists an $R > 0$ such that $\psi_0^*(H, H'_{\boldsymbol{R}}, \nabla, P)_{|X^*(R)}$ is pure and polarized, the variation is called an HS-orbit (Hertling-Sevenheck orbit), and we say in this paper that $(H, H'_{\boldsymbol{R}}, \nabla, P)$ induces an HS-orbit.

Remark 8.9 *An HS-orbit is called a "nilpotent orbit" in* [8]. *We use "HS-orbit" for distinction from twistor nilpotent orbit. It matches their terminology "Sabbah-orbit".* ∎

Lemma 8.10 *We assume (i) (H, ∇) requires no ramification, (ii) the Stokes structure and the real structure of $(H, H'_{\boldsymbol{R}}, \nabla)$ are compatible. Then, the following holds:*

- *$\psi_0^*(H, \nabla)$ is unramifiedly pseudo-good. The set of irregular values is given by $\big\{\psi_0^*\mathfrak{a} \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\nabla)\big\}$.*
- *The real structure and the Stokes structure of $\psi_0^*(H, \nabla)$ are compatible.*
- *We have the natural isomorphism $\psi_0^* \mathrm{Gr}_{\mathfrak{a}}\big(H, H'_{\boldsymbol{R}}, \nabla, P\big) \simeq \mathrm{Gr}_{\psi_0^*\mathfrak{a}}\, \psi_0^*\big(H, H'_{\boldsymbol{R}}, \nabla, P\big)$.*

Proof The first two claims follow from Lemma 8.8. To show the third claim, we have only to compare the induced flat pairings. It can be done directly, or by considering the restriction to $\boldsymbol{C}_\lambda \times \{1\}$. ∎

Integrable twistor structure We set $\mathcal{X}^\dagger := \boldsymbol{C}_\mu \times X^\dagger$, $\mathcal{D}^\dagger := \boldsymbol{C}_\mu \times D^\dagger$, $\mathcal{X}^{*\,\dagger} := \mathcal{X}^\dagger - \mathcal{D}^\dagger$ and $W^\dagger := \mathcal{D}^\dagger \cup \big(\{0\} \times X^\dagger\big)$. Let $\phi_\infty : \mathcal{X}^\dagger \longrightarrow \boldsymbol{C}_\mu$ be given by $\phi_\infty(\mu, z) = \mu \cdot \overline{z}$. The restriction to $\mathcal{X}^{*\,\dagger}$ is denoted by ψ_∞.

Let (V, ∇) be an integrable twistor structure on $\mathbb{P}^1$ which requires no ramification. It is obtained as the gluing of (V_0, ∇_0) and $(V_\infty, \nabla_\infty)$. The gluing is denoted by $g : V_{0|\boldsymbol{C}_\lambda^*} \simeq V_{\infty|\boldsymbol{C}_\mu^*}$, which is flat with respect to ∇.

We set $\mathrm{HS}(V)_0 := \psi_0^*(V_0)$ and $\mathrm{HS}(V)_\infty := \psi_\infty^*(V_\infty)$. They are naturally equipped with TE-structure $\mathrm{HS}(\nabla)_0$ and $\widetilde{T}E$-structure $\mathrm{HS}(\nabla)_\infty$. Note that $\mathrm{HS}(V, \nabla)_0$ and $\mathrm{HS}(V, \nabla)_\infty$ are unramifiedly pseudo good. Let us construct a flat isomorphism Φ between $\mathrm{HS}(V, \nabla)_{0|\boldsymbol{C}_\lambda^* \times X^*}$ and $\mathrm{HS}(V, \nabla)_{\infty|\boldsymbol{C}_\mu^* \times X^{\dagger *}}$. The fibers $\mathrm{HS}(V)_{0|(\lambda,z)}$ and $\mathrm{HS}(V)_{\infty|(\mu,z)}$ are naturally identified with $V_{0|\lambda\cdot z}$ and $V_{\infty|\mu\cdot\overline{z}}$, respectively. If $\lambda = \mu^{-1}$, we have $(\lambda \cdot z)^{-1} = \mu \cdot \overline{z} \cdot |z|^{-2}$. Hence, we have an isomorphism $\Phi_{(\lambda,z)} : H(V)_{0|(\lambda,z)} \simeq H(V)_{\infty|(\lambda^{-1},z)}$ induced by the gluing g with the parallel transform along the segments connecting $\lambda^{-1} \cdot \overline{z}$ and $\lambda^{-1} \cdot \overline{z} \cdot |z|^{-2}$. Thus, we obtain the isomorphism Φ as desired.

Let $\mathrm{HS}(V, \nabla)$ denote the variation of integrable twistor structures obtained as the gluing of $\mathrm{HS}(V, \nabla)_0$ and $\mathrm{HS}(V, \nabla)_\infty$. The following lemma is clear from the construction and the functoriality (Lemma 5.17).

Lemma 8.11

- *Let* $F : (V^{(1)}, \nabla^{(1)}) \longrightarrow (V^{(2)}, \nabla^{(2)})$ *be a morphism of integrable pure twistor structures. Then, we have the induced morphisms* $\mathrm{HS}(F) : \mathrm{HS}(V^{(1)}, \nabla^{(1)}) \longrightarrow \mathrm{HS}(V^{(2)}, \nabla^{(2)})$.

- *Let* f *be* γ *or* σ*. Then,* $\mathrm{HS} \circ f^*(V, \nabla)$ *is naturally isomorphic to* $f^* \mathrm{HS}(V, \nabla)$*.* ∎

By the above lemma, a real structure κ of (V, ∇) induces a real structure $\mathrm{HS}(\kappa)$ of $\mathrm{HS}(V, \nabla)$. Since we have the natural isomorphism $\mathrm{HS}\big(\mathbb{T}(0)\big) \simeq \mathbb{T}(0)_{X^*}$, a paring $\mathcal{S}$ of (V, ∇) with weight 0 induces a pairing $\mathrm{HS}(\mathcal{S})$ of $\mathrm{HS}(V, \nabla)$ with weight 0. Hence, an integrable twistor structure with a pairing $(V, \nabla, \mathcal{S})$ induces $\mathrm{HS}(V, \nabla, \mathcal{S})$ on $\mathbb{P}^1 \times X^*$, and if $(V, \nabla, \mathcal{S})$ is equipped with a real structure, $\mathrm{HS}(V, \nabla, \mathcal{S})$ is also equipped with a naturally induced real structure.

Lemma 8.12 *Assume that* (V, ∇) *has compatible Stokes structures. Then,* $\mathrm{HS}(V, \nabla)$ *also has compatible Stokes structures, and we have the natural isomorphism*

$$\mathrm{HS}\,\mathrm{Gr}_{\mathfrak{a}}(V, \nabla) \simeq \mathrm{Gr}_{\psi_0^* \mathfrak{a}} \mathrm{HS}(V, \nabla) \tag{85}$$

If (V, ∇) *is equipped with a pairing of weight* 0 *(resp. a real structure),* (85) *preserves the induced pairings (resp. real structures).*

Proof It follows from Lemma 8.8. ∎

Lemma 8.13 *Let* $(H, H'_{\boldsymbol{R}}, \nabla, P')$ *be a TERP-structure, and let* $(V, \nabla, \mathcal{S}, \kappa)$ *be the corresponding twistor-TERP structure. Then,* $\mathrm{HS}(V, \nabla, \mathcal{S}, \kappa)$ *is the variation of twistor-TERP structure corresponding to* $\psi_0^*(H, H'_{\boldsymbol{R}}, \nabla, P')$*.*

Proof By construction, we have the natural isomorphism $\mathrm{HS}(V, \nabla)_0 \simeq (H, \nabla)$. We have only to compare the induced real structures and pairings on them. Since they are flat, we have only to compare them on the fiber over $z = 1$. Then, the claim is clear. ∎

If there exists an $R > 0$ such that $\mathrm{HS}(V, \nabla, \mathcal{S})_{|\mathbb{P}^1 \times X^*(R)}$ is pure and polarized, it is called a twistor HS-orbit, and we say that $(V, \nabla, \mathcal{S})$ induces a twistor HS-orbit.

8.2 Reduction of wild HS-orbit

8.2.1 Statement

We use the notation in Subsection 8.1.4. Let (V, ∇) be an integrable twistor structure with a perfect pairing $\mathcal{S}$ of weight 0, which requires no ramification. Assume that $(V, \nabla, \mathcal{S})$ induces a twistor HS-orbit on $\mathbb{P}^1 \times X^*(R)$ for some $R > 0$. We obtain the underlying unramifiedly good wild harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X^*(R)$ of $\mathrm{HS}(V, \nabla, \mathcal{S})_{|\mathbb{P}^1 \times X^*(R)}$, which is unramifiedly good. Let $\mathcal{I}$ denote the set of irregular values of (V, ∇) at 0. It is easy to see

$$\mathrm{Irr}(\theta) = \big\{ \mathfrak{a}(z) \,\big|\, \mathfrak{a}(\lambda) \in \mathcal{I} \big\} \simeq \mathcal{I}.$$

We will not distinguish them in the following.

Let $(\mathcal{E}^{\triangle}, \mathbb{D}^{\triangle}, \mathcal{S}_E)$ denote the variation of polarized pure twistor structure associated to $(E, \overline{\partial}_E, \theta, h)$. It is enriched to integrable one $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S}_E)$. Although it is naturally isomorphic to $\mathrm{HS}(V, \nabla, \mathcal{S})$, it is non-trivial that the natural meromorphic extensions $\mathcal{Q}\mathcal{E}_0$ and $\phi_0^*(V_0) \otimes \mathcal{O}_{\mathcal{D}}(*\mathcal{D})$ are isomorphic. Hence, we use the symbol $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S}_E)$ for distinction. By applying the construction in Subsection 7.3.1 to $(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S}_E)$, we obtain a wild variation of pure polarized integrable twistor structures $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S}_E)$ for each $\mathfrak{a} \in \mathcal{I}$. We will prove the following theorem in Subsection 8.2.2.

Theorem 8.14

- (V, ∇) *has compatible Stokes structures.*

- $\mathrm{HS}\,\mathrm{Gr}_{\mathfrak{a}}(V, \nabla, \mathcal{S})$ *is naturally isomorphic to* $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{E}^{\triangle}, \widetilde{\mathbb{D}}^{\triangle}, \mathcal{S}_E)$ *for each* $\mathfrak{a} \in \mathcal{I}$*. In particular,* $\mathrm{Gr}_{\mathfrak{a}}(V, \nabla, \mathcal{S})$ *induces a twistor HS-orbit.*

Before going into the proof, we give a consequence.

Corollary 8.15 *Let $(H, H'_{\boldsymbol{R}}, \nabla, P, 0)$ be a TERP structure which requires no ramification. If $(H, H'_{\boldsymbol{R}}, \nabla, P, 0)$ induces an HS-orbit, it is a mixed-TERP structure in the sense of Definition* 9.1 *of* [8].

Proof Thanks to Theorem 9.3 of [8], $(H, H'_{\boldsymbol{R}}, \nabla, P, 0)$ is a mixed-TERP structure, if and only if (i) the real structure and the Stokes structure of $(H, H'_{\boldsymbol{R}}, \nabla)$ are compatible, (ii) $\mathrm{Gr}_{\mathfrak{a}}(H, H'_{\boldsymbol{R}}, \nabla, P, 0)$ induces an HS-orbit for each $\mathfrak{a} \in \mathrm{Irr}(\nabla)$. Hence, this corollary follows from Theorem 8.14, Lemma 8.7 and Lemma 8.13. ∎

The claim of the corollary was established by Hertling and Sevenheck [8] in the case that (H, ∇) has regular singularity. They also showed the converse of the claim in general.

Remark 8.16 *In their study of the case that (H, ∇) has regular singularity, Hertling and Sevenheck closely investigated the limiting object. In particular, they showed that the limiting TERP-structure is generated by elementary sections, for which the eigenvalues of the new supersymmetric index can be described in terms of the Hodge filtrations of the corresponding mixed Hodge structure.*

Even in the irregular case, the limiting object can be obtained from the reduced regular one. Hence, the limit of the eigenvalues of the new supersymmetric index of $\phi_0^*(H, H'_{\boldsymbol{R}}, \nabla, P)$ can be described in terms of their mixed Hodge structures. ∎

8.2.2 Proof of Theorem 8.14

We have the natural identifications $\mathrm{HS}(V, \nabla)_0 \simeq \big(\mathcal{E}, \widetilde{\mathbb{D}}^f\big)$ and $\mathrm{HS}(V, \nabla)_\infty \simeq \big(\mathcal{E}^\dagger, \widetilde{\mathbb{D}}^{\dagger f}\big)$. We have the following locally free $\mathcal{O}_{\mathcal{X}}(*\mathcal{D})$-modules

$$\overline{\mathrm{HS}}(V)_0 := \phi_0^*(V_0) \otimes \mathcal{O}_{\mathcal{X}}(*\mathcal{D}).$$

We also have the following locally free $\mathcal{O}_{\mathcal{X}^\dagger}(*\mathcal{D}^\dagger)$-modules

$$\overline{\mathrm{HS}}(V)_\infty := \phi_\infty^*(V_\infty) \otimes \mathcal{O}_{\mathcal{X}^\dagger}(*\mathcal{D}^\dagger).$$

Comparison of $\mathcal{QE}$ and $\overline{\mathrm{HS}}(V)_0$ We would like to show that $\mathcal{QE}$ and $\overline{\mathrm{HS}}(V)_0$ are naturally isomorphic. We set $W := \mathcal{D} \cup \big(\{0\} \times X\big)$. Let $\pi : \widetilde{\mathcal{X}}(W) \longrightarrow \mathcal{X}$ be the real blow up of $\mathcal{X}$ along W. Let $\mathfrak{V}$ be the flat bundle on $\widetilde{\mathcal{X}}(W)$ associated to $(\mathcal{E}, \widetilde{\mathbb{D}}^f)_{|\mathcal{X}-W}$. We set $\phi_0^*\mathcal{I} := \big\{\phi_0^*\mathfrak{a} \,\big|\, \mathfrak{a} \in \mathcal{I}\big\}$.

As remarked in Lemma 8.8, $\phi_0^* H$ is a pseudo-good lattice of $\overline{\mathrm{HS}}(V)_0 \otimes \mathcal{O}_{\mathcal{X}}(*W)$ in the level $\boldsymbol{m} = (-1, -1)$.

Lemma 8.17 *$\mathcal{Q}_0\mathcal{E}$ is a good lattice of $\mathcal{QE} \otimes \mathcal{O}_{\mathcal{X}}(*W)$ in the level $\boldsymbol{m} = (-1, -1)$ around $(\lambda, z) = (0, 0)$.*

Proof We have the decomposition $(\mathcal{Q}_0\mathcal{E}, \mathbb{D})_{|\widehat{W}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (\mathcal{Q}_0\widehat{\mathcal{E}}_{\mathfrak{a}}, \widehat{\mathbb{D}}_{\mathfrak{a}})$ such that $\widehat{\mathbb{D}}_{\mathfrak{a}} - d_X\mathfrak{a}$ is logarithmic. We have the corresponding decomposition $\widetilde{\mathbb{D}}^f = \bigoplus \widetilde{\mathbb{D}}^f_{\mathfrak{a}}$. Let us show that

$$\Big(\widetilde{\mathbb{D}}^f_{\mathfrak{a}} - d\big(\mathfrak{a}(z)/\lambda\big)\Big)\mathcal{Q}\widehat{\mathcal{E}}_{\mathfrak{a}} \subset \lambda^{-1} \cdot \mathcal{Q}\widehat{\mathcal{E}}_{\mathfrak{a}} \otimes \Omega^{1,0}_{\mathcal{X}}(\log W)$$

Since $\mathbb{D}_{\mathfrak{a}} - d_X\mathfrak{a}$ is logarithmic, it is satisfied for the derivatives along the X-direction. Since we have already known that the restriction of $\mathcal{Q}_0\mathcal{E}$ to $\boldsymbol{C}^*_\lambda \times X$ is unramifiedly good by Lemma 7.13, it is also satisfied for the derivatives along the λ-direction. ∎

Let $P \in \pi^{-1}(0, 0)$. We have the Stokes filtration $\mathcal{F}^P_1$ of $\mathfrak{V}_{|P}$ corresponding to the meromorphic prolongment $\mathcal{QE} \otimes \mathcal{O}_{\mathcal{X}}(*W)$, and the Stokes filtration $\mathcal{F}^P_2$ of $\mathfrak{V}_{|P}$ corresponding to the meromorphic prolongment $\overline{\mathrm{HS}}(V)_0 \otimes \mathcal{O}_{\mathcal{X}}(*W)$. (See Subsection 5.1.5 for such filtrations in the pseudo-good case.)

Lemma 8.18 $\mathcal{F}^P_1 = \mathcal{F}^P_2$.

Proof Let $S_P \in \mathcal{MS}(P, \mathcal{X} - W, \phi_0^*\mathcal{I})$ be a small sector such that there exist the Stokes filtrations $\mathcal{F}^{S_P}_1$ of $\mathcal{QE}_{|\overline{S}_P}$ and $\mathcal{F}^{S_P}_2$ of $\phi_0^*(H)_{|\overline{S}_P}$. We can take $Q \in \overline{S}_P \cap \pi^{-1}(W \setminus \mathcal{D})$ such that the orders $\leq_Q$ and $\leq_P$ on $\mathcal{I}$ are the same. We have the filtrations $\mathcal{F}^Q_i$ $(i = 1, 2)$ of $\mathfrak{V}_{|Q}$ corresponding to the meromorphic prolongments $\mathcal{QE}(*W)$ and $\overline{\mathrm{HS}}(V)_0(*W)$. Because $\mathrm{HS}(V, \nabla)_0 \simeq \big(\mathcal{E}, \widetilde{\mathbb{D}}^f\big)$, we have $\mathcal{F}^Q_1 = \mathcal{F}^Q_2$.

Let us show that $\mathcal{F}_i^P$ is obtained as the parallel transport of $\mathcal{F}_i^Q$, which implies $\mathcal{F}_1^P = \mathcal{F}_2^P$. We take $S_Q \in \mathcal{MS}(Q, \mathcal{X} - W, \phi_0^*\mathcal{I})$ such that there exist the Stokes filtrations $\mathcal{F}_1^{S_Q}$ of $\mathcal{QE}_{|\overline{S}_Q}$ and $\mathcal{F}_2^{S_Q}$ of $\phi_0^*(H)_{|\overline{S}_Q}$. By using the characterization in Proposition 5.20, we obtain $\big(\mathcal{F}_i^{S_P}\big)_{|\overline{S}_Q} = \mathcal{F}_i^{S_Q}$. Hence, we can conclude that $\mathcal{F}_i^P$ are obtained as the parallel transport of $\mathcal{F}_i^Q$. ∎

Lemma 8.19 *The isomorphism $\mathcal{E} \simeq \mathrm{HS}(V)_0$ on $\mathcal{X} - \mathcal{D}$ is extended to the isomorphism $\mathcal{QE} \simeq \overline{\mathrm{HS}}(V)_0$ on $\mathcal{X}$.*

Proof Let $P \in \pi^{-1}(0,0)$. We take a small multi-sector $S_P \in \mathcal{MS}(P, \mathcal{X} - W, \mathcal{I})$ such that we have the Stokes filtrations $\mathcal{F}^{S_P}$ for $\mathcal{Q}_0\mathcal{E}_{|\overline{S}_P}$ and $\phi_0^*(H)_{|\overline{S}_P}$. By Lemma 8.18, the restrictions of them to S_P are the same. We take a flat splitting $\mathcal{E}_{|S_P} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \mathcal{E}_{\mathfrak{a},S}$, which is extended to the decompositions:

$$\mathcal{QE}_{|\overline{S}_P} = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \mathcal{QE}_{\mathfrak{a},S}, \quad \phi_0^*(H)_{|\overline{S}_P} = \bigoplus \phi_0^*(H)_{\mathfrak{a},S}$$

Let $\widetilde{\mathcal{L}}(-\mathfrak{a})$ be a line bundle $\mathcal{O}_{\mathcal{X}}(*W) \cdot e$ with $\nabla e = e \cdot \big(-d(\lambda^{-1}\mathfrak{a})\big)$. We remark that $\mathrm{Gr}_{\mathfrak{a}}(\phi_0^*H) \otimes \widetilde{\mathcal{L}}(-\mathfrak{a})$ and $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{QE}) \otimes \widetilde{\mathcal{L}}(-\mathfrak{a})$ have regular singularity along W. Hence, the isomorphism on $\mathcal{X} - W$ is naturally extended to the isomorphism $\mathrm{Gr}_{\mathfrak{a}}(\phi_0^*H) \otimes \widetilde{\mathcal{L}}(-\mathfrak{a}) \simeq \mathrm{Gr}_{\mathfrak{a}}(\mathcal{QE}) \otimes \widetilde{\mathcal{L}}(-\mathfrak{a})$. Since the restrictions of $\mathrm{Gr}_{\mathfrak{a}}(\phi_0^*H) \otimes \mathcal{O}(*\mathcal{D})$ and $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{QE})$ to $\mathcal{X} - \mathcal{D}$ are naturally isomorphic, we obtain the isomorphism $\mathrm{Gr}_{\mathfrak{a}}(\phi_0^*H) \otimes \mathcal{O}(*\mathcal{D}) \simeq \mathrm{Gr}_{\mathfrak{a}}(\mathcal{QE})$.

Let $\boldsymbol{w}_{\mathfrak{a}}$ and $\boldsymbol{v}_{\mathfrak{a}}$ be frames of $\mathrm{Gr}_{\mathfrak{a}}(\phi_0^*H) \otimes \mathcal{O}(*\mathcal{D})$ and $\mathrm{Gr}_{\mathfrak{a}}(\mathcal{QE})$, respectively. We have the relation $\boldsymbol{w}_{\mathfrak{a}} = \boldsymbol{v}_{\mathfrak{a}} \cdot A_{\mathfrak{a}}$, where $A_{\mathfrak{a}}$ are meromorphic along $\mathcal{D}$. We take lifts $\boldsymbol{w}_{\mathfrak{a},S}$ and $\boldsymbol{v}_{\mathfrak{a},S}$ to $\mathcal{E}_{\mathfrak{a},S}$ by using the above splittings for any small sectors in $\mathcal{X} - W$. We have the relation $\boldsymbol{w}_{\mathfrak{a},S} = \boldsymbol{v}_{\mathfrak{a},S} \cdot A_{\mathfrak{a}}$. Then, the isomorphism $\mathcal{E} \simeq \mathrm{HS}(V)_0$ is extended to $\mathcal{QE}_{|\mathcal{U}} \simeq \overline{\mathrm{HS}}(V)_{0|\mathcal{U}}$ on some small neighbourhood $\mathcal{U}$ of $(0,0)$, which follows from Proposition 5.19. (We may apply Lemma 5.15. But, since $\phi_0^*V_0$ may not be a good lattice, we replace V_0 with an unramifiedly good lattice, or we use a variant of Lemma 5.15 for a pseudo-good lattice.) Then, it is easy to observe that the isomorphism is extended to $\mathcal{QE} \simeq \overline{\mathrm{HS}}(V)_0$ by using Hartogs theorem. (Sabbah also independently obtained an argument to extend such isomorphisms in this kind of situation.) ∎

Similarly, $\mathcal{E}^\dagger \simeq \mathrm{HS}(V)_\infty$ on $\mathcal{X}^\dagger - \mathcal{D}^\dagger$ is extended to the isomorphism $\mathcal{QE}^\dagger \simeq \overline{\mathrm{HS}}(V)_\infty$ on $\mathcal{X}^\dagger$.

Proof of the first claim Let $\widetilde{\mathcal{X}}^\triangle$ denote the real blow up of $\mathbb{P}^1 \times X$ along $(\mathbb{P}^1 \times D) \cup (\{0\} \times X) \cup (\{\infty\} \times X)$. Let $\pi^\dagger : \widetilde{\mathcal{X}}^\dagger(W^\dagger) \longrightarrow \mathcal{X}^\dagger$ denote the real blow up of $\boldsymbol{C}_\mu \times X^\dagger$ along $W^\dagger = \mathcal{D}^\dagger \cup (\{0\} \times X^\dagger)$. We have

$$\widetilde{\mathcal{X}}^\triangle = \widetilde{\mathcal{X}}(W) \cup \widetilde{\mathcal{X}}^\dagger(W^\dagger).$$

Let $\mathfrak{V}^\triangle$ denote the flat bundle on $\widetilde{\mathcal{X}}^\triangle$ associated to the flat bundle $(\mathcal{E}, \widetilde{\mathbb{D}}^f)_{|\boldsymbol{C}_\lambda^* \times (X-D)}$.

We have the C^∞-map $\mathcal{X} - W \longrightarrow (\boldsymbol{R}_{\geq 0} \times S^1)^2$ given by

$$(\lambda, z) \longmapsto \Big(\big(|\lambda|,\, \lambda/|\lambda|\big),\ \big(|z|,\, z/|z|\big)\Big)$$

It induces the natural identification $\widetilde{\mathcal{X}}(W) \simeq (\boldsymbol{R}_{\geq 0} \times S^1)^2$. We set

$$P_0 = \Big(\big(0, \exp(\sqrt{-1}\varphi)\big), (1,1)\Big) \in \pi^{-1}\big((0,1)\big) \subset \widetilde{\mathcal{X}}(W).$$

Similarly, we identify $\widetilde{\mathcal{X}}^\dagger(W^\dagger)$ with $(\boldsymbol{R}_{\geq 0} \times S^1)^2$ via the map induced by

$$(\mu, z) \longmapsto \Big(\big(|\mu|,\, \mu/|\mu|\big),\ \big(|z|,\, z/|z|\big)\Big).$$

We set $Q_0 := \Big(\big(0, \exp(-\sqrt{-1}\varphi)\big), (1,1)\Big) \in (\pi^\dagger)^{-1}\big((0,1)\big) \subset \widetilde{\mathcal{X}}^\dagger(W^\dagger)$. Note that we can identify (V, ∇) with $\mathrm{HS}(V, \nabla)_{|\mathbb{P}^1 \times \{1\}}$. Hence, we have only to compare the Stokes filtrations $\mathcal{F}^{P_0}(\mathfrak{V}^\triangle_{|P_0})$ and $\mathcal{F}^{Q_0}(\mathfrak{V}^\triangle_{|Q_0})$ under the parallel transport along the ray $\Big(\big(s, \exp(\sqrt{-1}\varphi)\big), (1,1)\Big)$ $(s \in \boldsymbol{R}_{\geq 0} \cup \{+\infty\})$ connecting P_0 and Q_0. (Note that the signature of the arguments are reversed by the coordinate change $\lambda^{-1} = \mu$.)

Let us consider the map $G : [0,1] \times [0,1] \longrightarrow \widetilde{\mathcal{X}}(W)$ given by

$$G(s,t) = \Big(\big(s, \exp(\sqrt{-1}\varphi)\big), (t,1)\Big).$$

Note $G(0,1) = P_0$. We set $P_1 := G(1,0)$ and $P_2 := G(1,1)$. The image of $\Gamma_0 := \big([0,1] \times \{0\}\big) \cup \big(\{0\} \times [0,1]\big)$ is contained in $\pi^{-1}(W)$. The orders $\leq_P$ are independent of $P \in G(\Gamma_0)$. Hence, the Stokes filtrations are unchanged along $G(\Gamma_0)$.

Similarly, let us consider the map $G^\dagger : [0,1] \times [0,1] \longrightarrow \widetilde{\mathcal{X}}^\dagger(W^\dagger)$ given by

$$G^\dagger(s,t) = \Big(\big(s, \exp(-\sqrt{-1}\varphi)\big), (t,1)\Big).$$

Note $G^\dagger(0,1) = Q_0$. We set $Q_1 := G^\dagger(1,0)$ and $Q_2 := G^\dagger(1,1)$. The image of $\Gamma_\infty := \big([0,1] \times \{0\}\big) \cup \big(\{0\} \times [0,1]\big)$ is contained in $(\pi^\dagger)^{-1}(W^\dagger)$. The orders $\leq_Q$ are independent of the choice of $Q \in G^\dagger(\Gamma_\infty)$. Hence, the Stokes filtrations are unchanged along $G^\dagger(\Gamma_\infty)$.

Under the identification $\mathcal{X} - W = \mathcal{X}^\dagger - W^\dagger$, we have $P_2 = Q_2$, and the union of the paths $G([0,1] \times \{1\})$ and $G^\dagger([0,1] \times \{1\})$ is the ray connecting P_0 and Q_0. Hence, for the comparison of $\mathcal{F}^{P_0}$ and $\mathcal{F}^{Q_0}$, we have only to show that $\mathcal{F}^{P_1}(\mathfrak{V}_{0|P_1})$ of $\mathfrak{V}_{|P_1}$ and $\mathcal{F}^{Q_1}(\mathfrak{V}_{\infty|Q_1})$ of $\mathfrak{V}_{|Q_1}$ are the same. It follows from the characterization of the Stokes filtrations of $(\mathcal{QE}, \mathbb{D})$ and $(\mathcal{QE}^\dagger, \mathbb{D}^\dagger)$ by growth order of the norms of flat sections with respect to the metric h. (See Subsection 6.3.) Thus, we obtain the first claim of Theorem 8.14.

Proof of the second claim By using Corollary 5.23 and Lemma 8.12, we obtain the isomorphisms on $\mathbb{P}^1 \times X^*(R)$ for some $R > 0$:

$$\mathrm{Gr}_{\mathfrak{a}}(\mathcal{E}^\triangle, \widetilde{\mathbb{D}}^\triangle, \mathcal{S}_E) \simeq \mathrm{Gr}_{\psi_0^*\mathfrak{a}} \mathrm{HS}(V, \nabla, \mathcal{S}) \simeq \mathrm{HS}\,\mathrm{Gr}_{\mathfrak{a}}(V, \nabla, \mathcal{S})$$

Thus, the second claim is also proved. ∎

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