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By

Michael Semenov-Tian-Shansky

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Michael Semenov-Tian-Shansky

Université de Bourgogne UFR Sciences & Techniques 9 avenue Alain Savary BP 47970 - 21078 Dijon Cedex FRANCE

Current address: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 JAPAN

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Abstract. These lectures cover the theory of classical r-matrices satisfying the classical Yang–Baxter equations and their basic applications in the theory of Integrable Systems as well as in geometry of Poisson Lie groups. Special attention is given to the factorization problems associated with classical r-matrices, the theory of dressing transformations, the geometric theory of difference zero curvature equations. We also discuss the relations between the Virasoro algebra and the Poisson Lie groups.

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Introduction

The present notes are based on the lectures which the author was giving in 2006 and in 2008 at the Université de Bourgogne, Dijon, France during two special semesters on Integrable Systems. They were completed during his visit to the Kyoto Research Institute for Mathematical Science. The author expresses his sincere gratitude to RIMS for hospitality and excellent working conditions. His work was partly supported by the ANR program "GIMP" ANR-05-BLAN-0029-01 and the RFFI grant 05-01-00922.

The notion of classical r-matrix introduced in late 1970's by Sklyanin [**S1**], as a part of a vast research program launched by L. D. Faddeev, which culminated in the discovery of the Quantum Inverse Scattering Method and of Quantum Groups [**F**]. It has proved to be an extremely useful tool which allows to understand geometric and algebraic aspects of the Theory of Integrable Systems from a single point of view and also lead to the discovery of a new kind of Poisson structures, the quadratic Poisson brackets on Poisson Lie groups. The r-matrix approach allows to take into account in a natural way several important features which are typical for various concrete examples of Integrable Systems:

(i) Nonlinear integrable equations arise as compatibility conditions for an auxiliary system of linear equations.
(ii) They are Hamiltonian with respect to a natural Poisson bracket.
(iii) The integrals of motion are spectral invariants of the auxiliary linear problem. They are in involution with respect to the Poisson bracket associated withour system.
(iv) The solution of these equations is reduced to some version of the Riemann–Hilbert problem.

Somewhat earlier, a geometric approach to integrable equations of the Lax type has been proposed in the important works of Kostant [**K**] and Adler [**A**]; some key ingredients were already implicit in the earlier work of Zakharov and Shabat on dressing transformations. A link of these earlier ideas with the notion of the classical r-matrix was established in [**STS1**]. The r-matrix naturally unites all properties listed above and allows to deduce them from a single geometric theorem. Importantly, the notion of r-matrix relates the Hamiltonian structure of integrable equations with the Riemann problem (factorization problem) which provides the main technical tool of their explicit solution.

Nonlinear integrable equations may be divided into several classes depending on the type of the associated linear problem:

(a) Finite dimensional systems.
(b) Infinite-dimensional systems with one or two spatial variables.
(c) Discrete systems on one or two-dimensional lattices.

In the first case the auxiliary linear problem is the eigenvalue problem for finite-dimensional matrices (possibly depending on an additional parameter). In case (b) the auxiliary linear operator is differential. Finally, in case (c) it is a difference operator.

The formulation and the proof of the main theorem are particularly simple for systems of the first two types. Integrable systems associated with difference operators require a separate study, since they are related to a rather complicated type of nonlinear Poisson structures. This case is particularly interesting, because it may be regarded as a semiclassical approximation in the Quantum Group theory. The associated notions of Poisson Lie groups and Poisson Hopf algebras was introduced by V.Drinfeld [**D1**], following the seminal work of Sklyanin and of the Leningrad school in general on the Quantum Inverse Scattering Method.

In the context of difference systems on the lattice the role of classical r-matrices is in fact two-fold: On the one hand, it defines the factorization problem needed to solve nonlinear difference equations, and on the other hand, it determines a Poisson–Hopf structure which plays a major role in the study of lattice systems. These two aspects lead to two inequivalent abstract definitions. The first one, due to Drinfeld, is the notion of *Lie bialgebra*, associated with the Hopf properties. The second one, due to the present author, is associated with factorization aspect of classical r-matrices and is suited for the study of integrable systems. In these lectures we prefer to coin for this notion a new term "Lie dialgebra" suggested to the author by N.Reshetikhin. The mismatch between the two notions may be sometimes misleading. In particular, while there are sharp classification results for Lie bialgebras, much less is known for the case of Lie dialgebras. The common ground is the important case of *factorizable Lie bialgebras* [**RS1**] which satisfy both sets of definitions.

The method of classical r-matrix provides a way for the systematic construction of examples of integrable systems. Its starting point is the choice of an underlying Lie algebra. There are several important classes of Lie algebras which give rise to different kinds of interesting examples:

1. Finite-dimensional semisimple Lie algebras. The associated examples include open Toda lattices and other integrable systems which are integrable with the help of *elementary functions* (rational functions of time t or of $\exp t$.
2. Loop algebras, or affine Lie algebras. They lead to systems which are integrable in abelian functions of t. Examples which may be obtained in this way include periodic Toda lattices and the majority

of classical examples from the XIX crntury textbooks on Analytical Mechanics.

3. Double loop algebras and their central extensions. This class of Lie algebras leads to integrable PDE's with two independent variables which admit the so called zero curvature representation, such as the nonlinear Schroedinger equation, the sine-Gordon equation, etc.
4. Lie algebras of formal pseudo-differential operators. They may be used to get the Korteweg – de Vries equation and its higher analogs, although it is more practical to derive these equations from from the double loop algebras.
5. Lie algebras of vector fields on the line or on the circle and the associated loop algebras. These algebras (or, more precisely, their central extension, the Virasoro algebra) are again connected with the Korteweg – de Vries type equations.

This list may be further extended, for instance, the Lie algebra of functions on the plane (x, p) with the canonical Poisson bracket as the commutator leads to the so called dispersionless equations.

Remarkably, the choice of a Lie algebra determines both the class of the associated phase spaces (i.e., the kinematics of the dynamical systems related to this algebra) and also the functional space of its solutions. In all cases it is very important to examine central extensions of the Lie algebras in question (whenever they exist) and also their non-trivial automorphisms. Usually this leads to new interesting examples. Non-trivial central extensions exist in cases 2, 3, 4, 5; in all cases they lead to important new constructions. In particular, central extensions of loop algebras are related to zero curvature equations. External automorphisms are used to construct twisted loop algebras which have a number of important applications; in the case of difference zero curvature equations on a lattice they play the same rôle as do central extensions in continuous case.

In the present lectures we shall be unable to describe the full range of applications. We shall mainly focus on questions related to the theory of Poisson Lie groups. In particular, we describe the general theory of zero curvature equations, both in discrete and in continuous case. An important technical tool which we will frequently use is the Hamiltonian, or, more generally, Poisson reduction. Although this subject is well known, I have included its brief description in the form which is particularly well suited for the applications to Poisson Lie groups and lattice systems.

LECTURE 1

Preliminaries: Poisson Brackets, Poisson and Symplectic Manifolds, Symplectic Leaves, Reduction

1.1. Poisson Manifolds

Recall that a Poisson bracket on a smooth manifold $\mathcal{M}$ is a Lie bracket on the space $C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$ which satisfies the Leibniz rule

$$\{\varphi_1,\, \varphi_2\varphi_3\} = \{\varphi_1,\, \varphi_2\}\varphi_3 + \{\varphi_1,\, \varphi_3\}\varphi_2.$$

In local coordinates $\{x_i\}$ the Poisson bracket is given by

$$\{\varphi,\, \psi\}(\mathbf{x}) = \sum_{i,j} \pi_{ij}(\mathbf{x}) \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j},$$

where π_{ij} is a skew symmetric tensor, the *Poisson tensor*. The Jacobi identity for the Poisson bracket is equivalent to the system of first order differential equations for π,

$$\sum_l \left(\pi^{lj} \frac{\partial \pi^{ik}}{\partial x^l} + \pi^{lk} \frac{\partial \pi^{ji}}{\partial x^l} + \pi^{li} \frac{\partial \pi^{kj}}{\partial x^l} \right) = 0. \tag{1.1}$$

A manifold $\mathcal{M}$ equipped with a Poisson bracket is called a Poisson manifold. Each function φ on $\mathcal{M}$ determines a Hamiltonian vector field $\xi_\varphi \in \operatorname{Vect} \mathcal{M}$ defined by the formula $\xi_\varphi \cdot \psi = \{\varphi,\, \psi\}$, or, in coordinate form,

$$\xi_\varphi = \sum_{ij} \pi^{ij}(\mathbf{x}) \frac{\partial \varphi}{\partial x^i} \frac{\partial}{\partial x^j}. \tag{1.2}$$

The function φ is called the Hamiltonian of ξ_φ.

A mapping $f\colon \mathcal{M} \to \mathcal{N}$ of Poisson manifolds is called a Poisson mapping if it preserves the Poisson brackets, i. e.,

$$f^*\{\varphi,\, \psi\}_{\mathcal{N}} = \{f^*\varphi,\, f^*\psi\}_{\mathcal{M}}$$

for any $\varphi,\, \psi \in C^\infty(\mathcal{N})$. In this case f maps the flow of the Hamiltonian $f^*\varphi = \varphi \circ f$ on $\mathcal{M}$ onto the flow of φ on $\mathcal{N}$.

A submanifold $\mathcal{M}$ of a Poisson manifold $\mathcal{N}$ is a Poisson submanifold if there is a Poisson structure on $\mathcal{M}$ such that the embedding $\mathcal{M} \hookrightarrow \mathcal{N}$ is a Poisson mapping (such a structure is unique if it exists). It is easy to check that $\mathcal{M}$ is a Poisson submanifold if and only if all Hamiltonian vector fields in $\mathcal{N}$ are tangent to $\mathcal{M}$.

A manifold is called symplectic if it carries a non-degenerate closed 2-form. Every symplectic manifolds has a natural Poisson structure. Let Ω be a non-degenerate closed 2-form on a manifold $\mathcal{S}$; in local coordinates we have

$$\Omega = \sum_{i<j} \Omega_{ij}(x) dx^i \wedge dx^j,$$

where the matrix $\Omega_{ij}(x)$ in non-degenerate. In local coordinates, the condition $d\Omega = 0$ amounts to the system of linear differential equations for the coefficients $\Omega_{ij}(x)$:

$$\frac{\partial \Omega_{jk}}{\partial x^i} + \frac{\partial \Omega_{ki}}{\partial x^j} + \frac{\partial \Omega_{ij}}{\partial x^k} = 0 \quad \text{for all} \quad i < j < k. \tag{1.3}$$

The Poisson bracket on $\mathcal{S}$ is defined by the formula

$$\{\varphi, \psi\} = \sum_{i,j} \Omega_{ij}(x)^{-1} \frac{\partial \varphi}{\partial x^i} \frac{\partial \psi}{\partial x^j}. \tag{1.4}$$

It is easy to see that the system (1.3) is equivalent to the quadratic relation (1.1) for the inverse matrix; thus the bracket (1.4) automatically satisfies the Jacobi identity. Conversely, if the Poisson tensor $\pi^{ij}(x)$ on a manifold $\mathcal{M}$ is non-degenerate, the differential form $\Omega = \sum \pi_{ij}^{-1}\, dx^i \wedge dx^j$ is closed and defines of $\mathcal{M}$ the structure of a symplectic manifold. The non-degeneracy of the Poisson tensor implies that Hamiltonian vector fields on a symplectic manifold span the entire tangent space at each point, and hence connected symplectic manifolds do not contain any nontrivial Poisson submanifolds.

There is an obvious and a very profound difference between the functorial properties of the Poisson and symplectic manifolds. The point is that the symplectic form transforms by the pullback, while the Poisson tensor transforms by the pushforward. For this reason, although the categories of Poisson and symplectic manifolds have many common objects, the sets of their natural morphisms are different. This is one more reason to bring into play Poisson manifolds, even if one starts with the Hamiltonian mechanics on symplectic manifolds.

The main theorem of the theory of Poisson manifolds which goes back to Lie [1893] asserts that an arbitrary Poisson manifold admits a stratification whose strata are symplectic manifolds (they may also be characterized as *the minimal Poisson submanifolds*). The strata of this stratification are called *symplectic leaves*. The stratification into symplectic leaves has a very simple geometrical meaning. Any function $H \in C^\infty(M)$ gives rise to a Hamiltonian vector field on M which acts on smooth functions $\varphi \in C^\infty(M)$ via $\xi_H \varphi = \{H, \varphi\}$; for any $x \in M$ the tangent vectors $\xi_H(x)$ span a linear subspace $\mathcal{H}_x$ in the tangent space $T_x M$; this is precisely the tangent space to the symplectic leaf passing through x.

The integrability condition which guarantees the existence of a submanifold with the tangent distribution $\{\mathcal{H}_x\}$ immediately follows from the Jacobi identity and the Frobenius theorem. The subtle part of the proof consists in checking that the possible jumps of the rank of the Poisson tensor π do not lead to singularities of symplectic leaves. It is easy to see that the Lie derivative of π along any Hamiltonian vector field is zero, and hence its rank is constant along the leaves (although it may change in transversal directions.

By construction, Hamiltonian vector fields are tangent to symplectic leaves, and hence the Hamiltonian flows preserve each symplectic leaf separately.

The existence of symplectic leaves is closely related to the existence of nontrivial Casimir functions of a given Poisson structure. A function φ is called a Casimir function (on a Poisson manifold $\mathcal{M}$) if $\{\varphi, \psi\} = 0$ for all $\psi \in C^\infty(\mathcal{M})$. (In other words, Casimir functions are those functions which give rise to trivial equations of motion.) A function φ is a Casimir function if and only if it is constant on each symplectic leaf in $\mathcal{M}$. The common level surfaces of Casimir functions define a stratification on $\mathcal{M}$; in general, it is more coarse than the stratification into symplectic leaves. Still, in many applications the knowledge of Casimir functions provides a rather sharp (or even a precise) description of symplectic leaves.

1.2. Lie–Poisson Brackets

The simplest example of a Poisson manifold which is not symplectic is the dual space of a Lie algebra $\mathfrak{g}$ equipped with the Lie–Poisson bracket (or, the Kirillov bracket) defined by the condition that for linear functions on $\mathfrak{g}$ it coincides with the Lie bracket on $\mathfrak{g}$:

$$\{X, Y\}(L) = \langle L, [X, Y]\rangle, \quad X, Y \in \mathfrak{g}, \quad L \in \mathfrak{g}^*. \tag{1.5}$$

By the Leibniz identity, the Lie–Poisson bracket immediately extends from linear functions to polynomials and then to arbitrary smooth functions. This argument shows that the Jacobi identity for the Lie–Poisson bracket automatically follows from the Jacobi identity for the underlying Lie algebra. Explicitly, the Lie–Poisson bracket of two functions $\varphi, \psi \in C^\infty(\mathfrak{g}^*)$ is given by

$$\{\varphi, \psi\}(L) = \langle L, [d\varphi(L), d\psi(L)]\rangle. \tag{1.6}$$

(Note that $d\varphi(L),\ d\psi(L) \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}$, and hence the Lie bracket $[d\varphi(L), d\psi(L)]$ is well defined!)

Choose a basis $\{e_i\}$ in $\mathfrak{g}$. The tensor $\pi_{ij}(L)$ for the Lie–Poisson bracket has the form

$$\pi_{ij}(L) = \sum_k c_{ij}^k L_k,$$

where $L_k = \langle L, e_k \rangle$ and c_{ij}^k are the structure constants of $\mathfrak{g}$. Thus in local coordinates formula (1.6) takes the form

$$\{\varphi, \psi\}(L) = \sum_{i<j} c_{ij}^k L_k \frac{\partial \varphi(L)}{\partial L_i} \frac{\partial \psi(L)}{\partial L_j}.$$

Recall the definition of the *adjoint representation* of a Lie group G acting in its Lie algebra $\mathfrak{g}$:

$$\operatorname{Ad} g \cdot X \quad = \quad \left(\tfrac{d}{dt}\right)_{t=0} g \cdot \exp tX \cdot g^{-1}, \ X \in \mathfrak{g}.$$

The dual representation defined by

$$\langle \operatorname{Ad}^* g \cdot L, X \rangle = \left\langle L, \operatorname{Ad} g^{-1} \cdot X \right\rangle, \ X \in \mathfrak{g}, \ L \in \mathfrak{g}^*,$$

which is acting in the space $\mathfrak{g}^*$ is called the *coadjoint representation.* Put

$$\operatorname{ad} X \cdot Y = \left(\tfrac{d}{dt}\right)_{t=0} \operatorname{Ad} \exp tX \cdot Y, \ \operatorname{ad}^* X \cdot L = \left(\tfrac{d}{dt}\right)_{t=0} \operatorname{Ad}^* \exp tX \cdot L.$$

We have, obviously, $\operatorname{ad} X \cdot Y = [X, Y]$, $\operatorname{ad}^* X = -(\operatorname{ad} X)^*$.

The following fundamental theorem was basically proved by Lie; it has been rediscovered by Kirillov and Kostant in the early 1960's.

THEOREM 1.1. *(i) Symplectic leaves of the Lie–Poisson bracket coincide with the orbits of coadjoint representation.*

(ii) The Casimir functions of the Lie–Poisson bracket are the Ad^**-invariant functions on* $\mathfrak{g}^*$*.*

We shall prove this theorem in Section 1.4 below with the help of Hamiltonian reduction (this is basically the proof found by Lie himself). It is very easy to check a slightly weaker assertion:

PROPOSITION 1.2. *The Hamiltonian equation of motion on* $\mathfrak{g}^*$ *generated by an arbitrary function* $\varphi \in C^\infty(\mathfrak{g}^*)$ *with respect to the Lie–Poisson bracket has the form*

$$\frac{dL}{dt} = -\operatorname{ad}^* d\varphi(L) \cdot L, \ L \in \mathfrak{g}^*; \tag{1.7}$$

in other words, the velocity vector for any Hamiltonian system on $\mathfrak{g}^*$ *computed at some point* $L \in \mathfrak{g}^*$ *is automatically tangent to the coadjoint orbit which passes through* L*.*

Indeed, let us fix a vector $X \in \mathfrak{g}$. The Lie derivative of the linear function $X(L) = \langle L, X \rangle$ along the Hamiltonian vector field ξ_φ is given by

$$\frac{dX}{dt} = \{\varphi, X\}(L) = \langle L, [d\varphi(L), X] \rangle = -\langle \operatorname{ad}^* d\varphi(L) \cdot L, X \rangle.$$

On the other hand, $\frac{dX(L)}{dt} = \langle \dot{L}, X \rangle$; since X is arbitrary, this implies that $\dot{L} = -\operatorname{ad}^* d\varphi(L) \cdot L$.

Coadjoint orbits play a very important role in the study of integrable systems: in many cases their phase spaces are just coadjoint orbits of some suitable Lie group. However, in the case of lattice integrable systems the use of coadjoint orbits and of linear Poisson brackets is not sufficient. As we shall see, the natural Poisson structures associated with lattice systems are quadratic. Remarkably, the symplectic leaves of these more general Poisson structures are again orbits of a natural group action. This is the so called *dressing action*, described in lecture 5.

Classification of coadjoint orbits for various particular groups is a good exercise, sometimes, a fairly nontrivial one: for instance, the complete classification of coadjoint orbits of the unipotent group of all upper triangular $n \times n$-matrices for any n is still unknown! Let us give a few simple examples.

EXAMPLE 1.3. Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ be the full matrix algebra; its dual space $\mathfrak{g}^*$ may be identified with $\mathfrak{g}$ by means of the invariant inner product[1]

$$\langle X, Y \rangle = \operatorname{tr} XY. \tag{1.8}$$

Hence, the adjoint and the coadjoint representations of the full linear group $G = GL(n, \mathbb{C})$ are identical; we have $\operatorname{Ad}^* g \cdot L = gLg^{-1}$. Orbits of coadjoint representation consist of conjugate matrices; their classification is provided by the Jordan theorem. The Casimir functions are spectral invariants of matrices; according to a classical theorem of the theory of invariants, the ring of the Casimir functions is a free algebra generated by the coefficients of the characteristic polynomial. In this case, the level surface of Casimirs consists of a finite number of coadjoint orbits; if the spectrum is simple, the level surface contains a single orbit.

A similar classification for the algebra $\mathfrak{gl}(n, \mathbb{R})$ is more subtle: one has to take into account the type of the matrix which may be reducible to its Jordan form over $\mathbb{R}$, or only over $\mathbb{C}$.

EXERCISE 1.4. Give a complete classification of coadjoint orbits for $G = GL(2, \mathbb{C})$ and for $G = GL(2, \mathbb{R})$.

EXAMPLE 1.5. Let $\mathfrak{b}_+ \subset \mathfrak{g}$ be the solvable algebra of all upper triangular matrices The coupling (1.8) allows to identify its dual with the space $\mathfrak{b}_-$ of lower triangular matrices. In this realization, the coadjoint action of the group B_+ of invertible upper triangular matrices is given by

$$\operatorname{Ad}^* b \cdot F = P_- \left(b \cdot F \cdot b^{-1} \right), \; b \in B_+, \; F \in \mathfrak{b}_-,$$

where the projection operator $P_- : \mathfrak{g} \to \mathfrak{b}_-$ replaces with zeros all matrix coefficients above the principle diagonal.

In this example the coadjoint and the adjoint representations are *inequivalent.*

EXERCISE 1.6. Describe all orbits of the group of upper triangular 2×2-matrices.

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[1] In these lectures the term *inner product* usually means a non-degenerate bilinear form (which is $\mathbb{C}$-bilinear for complex Lie algebras); for real algebras we do not impose, unless it is stated explicitly, any positivity conditions.

1.3. Poisson and Hamilton Reduction

Suppose that a group G acts on a symplectic manifold $\mathcal{M}$. The space of its orbits $\mathcal{M}/G$ is almost never a symplectic manifold; indeed, the projection map $\pi\colon \mathcal{M} \to \mathcal{M}/G$ transforms differential forms via the pullback and hence there is no immediate way to replant the symplectic form on $\mathcal{M}$ to the quotient space. On the other hand, the Poisson tensor transforms by push-forward, and hence under rather mild conditions on the group action the quotient space inherits a Poisson structure. This structure is in general degenerate, even though the initial manifold is symplectic.

Let $\mathcal{M}$ be a Poisson manifold. The action $G \times \mathcal{M} \to \mathcal{M}$ of a Lie group G on $\mathcal{M}$ is said to be *admissible*, if the subspace $C^\infty(\mathcal{M})^G \subset C^\infty(\mathcal{M})$ of G-invariant functions on $\mathcal{M}$ is a Lie subalgebra with respect to the Poisson bracket; in other words, we assume that the Poisson bracket of G-invariant functions is again G-invariant.

PROPOSITION 1.7. *Let us assume that the action $G \times \mathcal{M} \to \mathcal{M}$ is admissible and the space $\mathcal{M}/G$ of its orbits is a smooth manifold. Then $\mathcal{M}/G$ carries a natural Poisson structure such that the canonical projection map $\pi\colon \mathcal{M} \to \mathcal{M}/G$ is a Poisson map.*

PROOF. We can identify the space $C^\infty(\mathcal{M}/G)$ with the space $C^\infty(\mathcal{M})^G$ of G-invariant functions on $\mathcal{M}$. By assumption, $C^\infty(\mathcal{M})^G$ is a Lie subalgebra in $C^\infty(\mathcal{M})$. $\square$

The Poisson manifold $\mathcal{M}/G$ is called the *reduced manifold* obtained by reduction of $\mathcal{M}$ over the action of G.

The traditional class of admissible group actions are Hamiltonian, or, slightly more generally, symplectic group actions.

DEFINITION 1.8. An action of G on a symplectic manifold $\mathcal{M}$ is called *symplectic*, if it preserves the symplectic form.

PROPOSITION 1.9. *Symplectic actions are admissible.*

Indeed, invariance of the symplectic form means that the Poisson bracket is also G-invariant. Hence for any $\varphi, \psi \in C^\infty(\mathcal{M})^G$ and for any $g \in G$ we have

$$g^*\{\varphi, \psi\} = \{g^*\varphi, g^*\psi\} = \{\varphi, \psi\}.$$

The converse is *not true*: admissible group action do not necessarily preserve Poisson brackets. Nontrivial examples of admissible actions will be given in Lectures 5 and 6 in connection with the theory of Poisson Lie groups.

The difficult part of reduction consists in the description of symplectic leaves in the reduced space $\mathcal{M}/G$. We shall now do it for the so called *Hamiltonian* actions.

DEFINITION 1.10. Let us suppose there is an action $G\times\mathcal{M} \to \mathcal{M}$ of a Lie group G on a symplectic manifold $\mathcal{M}$, and let $\mathfrak{g} \to \operatorname{Vect}\mathcal{M}$ be the associated homomorphism of its Lie algebra into the Lie algebra of vector fields on $\mathcal{M}$. The action of G on $\mathcal{M}$ is *Hamiltonian*, if there is a homomorphism $H\colon \mathfrak{g} \to C^\infty(\mathcal{M})$ into the Lie algebra of functions on $\mathcal{M}$ such that the following diagram is commutative:

$$\begin{tikzcd} & C^\infty(\mathcal{M}) \arrow[dr, "\xi"] & \\ \mathfrak{g} \arrow[ur, "H"] \arrow[rr] & & \operatorname{Vect}\mathcal{M} \end{tikzcd}$$

In other words, an action of G on $\mathcal{M}$ is *Hamiltonian,* if the action of any of its one-parametric subgroups $\exp tX$, $X \in \mathfrak{g}$, is given by a Hamiltonian H_X[2], and, moreover, the map $X \mapsto H_X$ is linear and

$$H_{[X,Y]} = \{H_X,\, H_Y\}. \tag{1.9}$$

Clearly, Hamiltonian actions preserve the symplectic form and hence are admissible.

REMARK 1.11. Symplectic actions, i. e. actions which preserve symplectic form, in general are not Hamiltonian. One possible obstruction arises if $\mathcal{M}$ is not simply connected (more precisely, if its first cohomology group $H^1(\mathcal{M}, \mathbb{R})$ is nontrivial). Indeed, let us assume that a vector field $X \in \operatorname{Vect}\mathcal{M}$ is Hamiltonian with Hamiltonian h. The definition of the Poisson bracket immediately implies that the the contraction of X with the symplectic form is an *exact* 1-form,

$$i_X\omega \stackrel{\text{def}}{=} \omega(X, \cdot) = -dh.$$

On the other hand, let us assume barely that $X \in \operatorname{Vect}\mathcal{M}$ preserves ω, i. e. the Lie derivative $\mathcal{L}_X\omega$ is zero. The Cartan formula $\mathcal{L}_X\omega = di_X\omega + i_X d\omega$, implies that the 1-form $i_X\omega$ is closed; however, it need not be exact, as shown by the simple example of the vector field $X = \frac{\partial}{\partial\varphi}$ on the cylinder. In this case one speaks of vector fields which are locally (and not globally) Hamiltonian. Second, even if all vector fields generated by the action of G are globally Hamiltonian, relation (1.9) may be valid only up to a constant,

$$H_{[X,Y]} - \{H_X,\, H_Y\} = c(X,Y) = const, \tag{1.10}$$

since constant functions lie in the center of the Poisson bracket. It is easy to see that $c(X,Y)$ regarded as a function of its two arguments is a *2-cocycle* on $\mathfrak{g}$. In other words, $c(X,Y)$ is a skew bilinear form on $\mathfrak{g}$ and

$$c([X,Y],Z) + c([Y,Z],X) + c([Z,X],Y) = 0.$$

[2]It is worth to make a comment on the choice of signs in our definitions. We define the vector field $\widehat{X} \in \operatorname{Vect}\mathcal{M}$ generated by $X \in \mathfrak{g}$ by setting

$$\widehat{X}\varphi(x) = \left(\tfrac{d}{dt}\right)_{t=0}\varphi(e^{-tX}\cdot x);$$

with this choice, the commutator of vector fields on $\mathcal{M}$ generated by $X, Y \in \mathfrak{g}$ is in agreement with the Lie bracket in $\mathfrak{g}$.

Let $\widehat{\mathfrak{g}}$ be the central extension of $\mathfrak{g}$ which corresponds to c. It is easy to see that if we replace $\mathfrak{g}$ with $\widehat{\mathfrak{g}}$, the obstruction will be killed , so the initial action of G becomes a Hamiltonian action of its central extension. We shall encounter non-trivial examples of this type in Lectures 7 and 8.

Since H_X linearly depends on $X \in \mathfrak{g}$, a Hamiltonian action gives rise to a mapping $\mu\colon \mathcal{M} \to \mathfrak{g}^*$, called *moment mapping*, which is defined by

$$\mu(m)(X) = H_X(m), \quad m \in M.$$

PROPOSITION 1.12. *(i) Let us equip $\mathfrak{g}^*$ with the Lie–Poisson bracket. The moment mapping is a G-equivariant Poisson map into $\mathfrak{g}^*$. In other words, we have a commutative diagram in which all vertical arrows are Poisson mappings:*

$$\begin{array}{ccc} G \times \mathcal{M} & \longrightarrow & \mathcal{M} \\ {\scriptstyle id\times\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times \mathfrak{g}^* & \xrightarrow{\mathrm{Ad}^*} & \mathfrak{g}^* \end{array}$$

(It is convenient to assume that G is also a Poisson manifold equipped with the zero Poisson bracket.) (ii) Conversely any Poisson mapping $\mu\colon \mathcal{M} \to \mathfrak{g}^*$ gives rise to a Hamiltonian action of $\mathfrak{g}$ on $\mathcal{M}$ for which μ is the moment mapping.

DEFINITION 1.13. Hamiltonian G-spaces are symplectic manifolds equipped with a Hamiltonian action of G and an equivariant moment map; a morphism $F : \mathcal{M} \to \mathcal{M}'$ of Hamiltonian G-spaces with moment maps μ, μ' is a G-equivariant map such that $\mu = \mu' \circ F$.

The mapping $\mu\colon \mathcal{M} \to \mathfrak{g}^*$ and the canonical projection $\pi\colon \mathcal{M} \to \mathcal{M}/G$ to the space of G-orbits in $\mathcal{M}$ gives an example of a *dual pair* of Poisson mappings . Dual pairs were studied already by Lie himself; they represent a key tool in Poisson geometry. The general definition is as follows:

Let $\mathcal{M}$ be a symplectic manifold and $F_1\colon \mathcal{M} \to \mathcal{U}_1$, $F_2\colon \mathcal{M} \to \mathcal{U}_2$ two surjective Poisson mappings onto Poisson manifolds $\mathcal{U}_1$ and $\mathcal{U}_2$. Let $F_1^* C^\infty(\mathcal{U}_1)$ and $F_2^* C^\infty(\mathcal{U}_2) \subset C^\infty(\mathcal{M})$ be the subalgebras of smooth functions on $\mathcal{M}$, which are the pullbacks of functions on $\mathcal{U}_1$ and on $\mathcal{U}_2$, respectively. Since the mappings F_1, F_2 are Poisson, these subalgebras are closed with respect to the Poisson bracket on $\mathcal{M}$.

DEFINITION 1.14. Two Poisson mappings F_1, F_2 form a *dual pair*, if the Lie subalgebra $F_1^* C^\infty(\mathcal{U}_1)$ is the centralizer in $C^\infty(\mathcal{M})$ of $F_2^* C^\infty(\mathcal{U}_2)$ with respect to the Poisson bracket on $\mathcal{M}$ (in this case $F_2^* C^\infty(\mathcal{U}_2)$ is also the centralizer of $F_1^* C^\infty(\mathcal{U}_1)$).

The importance of the notion of dual pairs is explained by the following theorem.

THEOREM 1.15. *Let*

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\ \mathcal{U}_1 & & \mathcal{U}_2 \end{tikzcd}$$

be a dual pair. Then the connected components of the sets $F_1(F_2^{-1}(v))$, $v \in \mathcal{U}_2$ *and* $F_2(F_1^{-1}(u))$, $u \in \mathcal{U}_1$ *are symplectic leaves in* $\mathcal{U}_1$ *and* $\mathcal{U}_2$*, respectively.*

In the special case of Hamiltonian group actions we have the following result:

PROPOSITION 1.16. *Let* $G \times \mathcal{M}$ *be a Hamiltonian group action,* $\pi\colon \mathcal{M} \to \mathcal{M}/G$ *the canonical projection to the space of* G*-orbits and* $\mu\colon \mathcal{M} \to \mathfrak{g}^*$ *the associated moment map. Then*

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "\pi"'] \arrow[dr, "\mu"] & \\ \mathcal{M}/G & & \mathfrak{g}^* \end{tikzcd}$$

is a dual pair.

Indeed, a function φ on $\mathcal{M}$ is G-invariant if and only if $\{H_X, \varphi\} = 0$ for all $X \in \mathfrak{g}$; this implies that for any $\Phi \in C^\infty(\mathfrak{g}^*)$ $\{\varphi, \Phi \circ \mu\} = 0$; conversely, if $\{\varphi, \Phi \circ \mu\} = 0$ for all $\Phi \in C^\infty(\mathfrak{g}^*)$, the function φ is G-invariant.

COROLLARY 1.17. *Symplectic leaves in* $\mathcal{M}/G$ *are connected components of the sets* $\pi(\mu^{-1}(f))$, $f \in \mathfrak{g}^*$.

We shall say that the symplectic manifold $\overline{\mathcal{M}}_f = \pi(\mu^{-1}(f))$ is obtained by *reduction* of $\mathcal{M}$ over the point $f \in \mathfrak{g}^*$.

A more traditional definition of reduction (which avoids the explicit use of the Poisson structure in the quotient space $\mathcal{M}/G$) is given as follows. For $f \in \mathfrak{g}^*$ let $\mathcal{M}_f = \mu^{-1}(f)$ be the corresponding *level surface* of the moment map; let us assume that f is a regular value of μ, and hence $\mathcal{M}_f \subset \mathcal{M}$ is a smooth submanifold. Let $G^f \subset G$ be the identity component of the isotropy subgroup of f. Since μ is G-eqivariant, the action of G^f leaves $\mathcal{M}_f \subset \mathcal{M}$ invariant.

PROPOSITION 1.18. *The restriction of the symplectic form* ω *to* $\mathcal{M}_f$ *is degenerate; its kernel* $\operatorname{Ker} \omega|_{\mathcal{M}_f}(x)$ *at the point* $x \in \mathcal{M}_f$ *coincides with the tangent space to the* G^f*-orbit of* x.

Notice that since $\omega|_{\mathcal{M}_f}$ is closed, its kernels form an integrable distribution of the tangent bundle $T\mathcal{M}_f$. Indeed, by the classical Frobenius theorem, a subbundle of the tangent bundle is integrable if and only if the Lie bracket of any two vector fields which are sections of this subbundle lies in the same subbundle. In order to check integrability of the subbundle

$\operatorname{Ker}\omega|_{\mathcal{M}_f} \subset T\mathcal{M}_f$, let us use Cartan's formula for the exterior differential of the 2-form:

$$
\begin{aligned}
0 = d\omega(X,Y,Z) &= X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y) - \\
&\quad - \omega([Y,Z],X) - \omega([Z,X],Y) - \omega([X,Y],Z).
\end{aligned}
\tag{1.11}
$$

Assume that X, Y, Z are tangent to $\mathcal{M}_f$ and, moreover, $X, Y \in \operatorname{Ker}\omega|_{\mathcal{M}_f}$; in this case all terms in the r.h.s. of (1.11) except the last one are identically zero, and hence also $\omega([X,Y],Z) = 0$ for any $Z \in \operatorname{Vect}\mathcal{M}_f$, i.e. $[X,Y] \in \operatorname{Ker}\omega|_{\mathcal{M}_f}$. Integral submanifolds of this subbundle form a *null-foliation* of the symplectic form restricted to $\mathcal{M}_f$. The leaves of the foliation are precisely the G^f-orbits in $\mathcal{M}_f$.

PROPOSITION 1.19. *The quotient space of $\mathcal{M}_f/G^f$ over the action of G^f is a symplectic manifold.*

This quotient space is called *reduced symplectic manifold.* Clearly, $\mathcal{M}_f/G^f$ coincides with $\pi(\mu^{-1}(f))$, and hence this construction is equivalent to the former one.

REMARK 1.20. A convenient way to describe the reduced manifold is to choose a *cross-section* of the subbundle $\operatorname{Ker}\omega|_{\mathcal{M}_f}$, i.e. a submanifold $\Sigma \subset \mathcal{M}_f$ which intersects all G^f-orbits in $\mathcal{M}_f$ transversally and in a single point each. Clearly such a section Σ provides a model for the reduced manifold. (In physics the choice of a section is called *gauge fixing.*) In some cases, the subbundle $\operatorname{Ker}\omega|_{\mathcal{M}_f}$ does not admit global cross-sections, due to topological obstructions; one can still choose a cross-section in the complement to the cycle of positive codimension which defines the obstruction and hence construct a model of an open cell of the reduced space.

The computation of the quotient Poisson bracket on Σ is *local* (in other words, the reduced Poisson bracket at some point $x \in \Sigma$ depends on the behaviour of the moment map and of the cross-section in its infinitesimal neighborhood); this computation may be performed without assuming that Σ is a global cross-section.

EXERCISE 1.21. Let $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ be the complex linear space equipped with the standard Poisson bracket

$$\{\varphi, \psi\} = \sum_i \left(\frac{\partial\varphi}{\partial p_i}\frac{\partial\psi}{\partial q_i} - \frac{\partial\psi}{\partial p_i}\frac{\partial\varphi}{\partial q_i} \right),\ z = p + \sqrt{-1}q \in \mathbb{C}^n;$$

let $\mathbb{T} = S^1$ be the multiplicative group of complex numbers of absolute value 1 and $\mathbb{T} \times \mathbb{C}^n \to \mathbb{C}^n$ its standard action on $\mathbb{C}^n$.

(i) Check that action is Hamiltonian anf the associated moment map $\mu\colon \mathbb{C}^n \longrightarrow \mathfrak{t} \simeq \mathbb{R}$ is given by $\mu(z) = \frac{1}{2}|z|^2$.

(ii) Show that the reduced space $\mu^{-1}(1)/\mathbb{T}$ is isomorphic to the projective space $\mathbb{C}P_{n-1}$.

(iii) Describe obstruction to the construction of a global cross-section

$$\Sigma\colon \mathbb{C}P_{n-1} \longrightarrow S^{2n-1}$$

and construct such a cross-section over the affine part of $\mathbb{C}P_{n-1}$.

(iv) Describe explicitly the dual pair of Poisson mappings associated with the action $\mathbb{T} \times \mathbb{C}^n \to \mathbb{C}^n$.

A G-invariant Hamiltonian φ on $\mathcal{M}$ gives rise to a *reduced Hamiltonian* $\overline{\varphi}$ on the quotient space $\mathcal{M}/G$; by definition, $\overline{\varphi}$ is a function on $\mathcal{M}/G$ such that $\varphi = \overline{\varphi} \circ \pi$; if $\mathcal{M}/G$ is a smooth manifold, this function exists and is unique). The projection π maps the integral curves of φ onto the integral curves of the reduced Hamiltonian Let F_t be the dynamical flow on $\mathcal{M}$ defined by the Hamiltonian φ and $\overline{F}_t$ the flow of the reduced Hamiltonian on $\mathcal{M}/G$. The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F_t} & \mathcal{M} \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ \mathcal{M}/G & \xrightarrow{\overline{F}_t} & \mathcal{M}/G \end{array} .$$

In other words, the flow F_t *factorizes* over $\mathcal{M}/G$.

Remark 1.22. If we assume in Theorem 1.15 that the manifold $\mathcal{M}$ carries only a Poisson structure, the connected components of the sets $F_1(F_2^{-1}(v))$, $v \in \mathcal{U}_2$, and $F_2(F_1^{-1}(u))$, $u \in \mathcal{U}_1$, need not be symplectic leaves in the target spaces; however, they are still Poisson submanifolds. Thus a dual pair provides, in this case as well, a useful (though more coarse) stratification of the target spaces. Restricting the mappings F_1, F_2 to the symplectic leaves in $\mathcal{M}$, we get a family of "genuine" dual pairs which yield a refinement of this stratification.

Traditionally, the reduction technique (which goes back to classical work of Euler, Lagrange and Jacobi) is used to simplify the equations of motion in the presence of nontrivial first integrals by excluding some of the degrees of freedom. As noted by Kazhdan, Kostant and Sternberg [**KKS**], it is sometimes possible to use reduction in the opposite direction, i. e. to produce complicated dynamical systems from simple ones. As a matter of fact, we shall see below that all integrable systems admitting a natural Lax representation arise from very simple standard systems with a high number of degrees of freedom. (Due to the "complexity conservation", Lax equations with complicated dynamics, e. g., equations which are integrated in abelian functions, stem from *infinite dimensional* systems.) Actually in practical study of Integrable Systems reduction is frequently used in both directions, producing "simpler" systems from "complicated" ones and vice versa. Our first example is Hamiltonian mechanics on the cotangent bundle of a Lie group which yields a description of symplectic leaves for the Lie–Poisson brackets.

1.4. Cotangent Bundle of a Lie Group.

Let $\mathcal{M}$ be a smooth manifold and $T^*\mathcal{M}$ its cotangent bundle. We start by recalling the standard definition of the symplectic structure on $T^*\mathcal{M}$. Fix a point $q \in \mathcal{M}$ and $p \in T_q^*\mathcal{M}$. The tangent space $T_{(q,p)}(T^*\mathcal{M})$ may be naturally identified with $T_q\mathcal{M} \oplus T_q^*\mathcal{M}$; let v' be the projection of $v \in T_{(q,p)}(T^*\mathcal{M})$ onto $T_q\mathcal{M}$ in this splitting. The canonical 1-form on $T^*\mathcal{M}$ is defined by

$$\theta(v) = \langle p, v' \rangle \tag{1.12}$$

It is easy to see that (1.12) defines a global 1-form on $T^*\mathcal{M}$; in local coordinates q^i, p_i on $T^*\mathcal{M}$ it is expressed by the standard formula $\theta = \sum_i p_i dq^i$. The canonical symplectic form on $T^*\mathcal{M}$ is its derivative, $\Omega = d\theta$.

Let now G be a Lie group and T^*G its cotangent bundle. G acts freely on itself by left and right translations, and hence both its tangent and cotangent bundles are trivial. Let $\lambda_x \colon G \to G,\ x \in G$ be the left translation map, $\lambda_x \colon g \mapsto xg$ and λ'_x its differential at the unit element, $\lambda'_x : T_eG \simeq \mathfrak{g} \to T_xG$. We may fix the trivialization of the cotangent bundle T^*G with the help of the dual map ${\lambda'_x}^* \colon T_x^*G \to T_e^*G \simeq \mathfrak{g}^*$. Another distinguished trivialization of T^*G uses the differentials of *right* translations $\rho_x,\ x \in G$.

Proposition 1.23. *In left trivialization, the natural action of G on $T^*G \simeq G \times \mathfrak{g}^*$ is given by*

$$\begin{aligned} &\lambda_x \colon (g, L) \longmapsto (xg, L), \\ &\rho_x \colon (g, L) \longmapsto (gx^{-1}, \mathrm{Ad}^*\, x \cdot L). \end{aligned} \tag{1.13}$$

Proof. The first formula is obvious. To prove the second, observe that in left trivialization the action of G on itself by right translations induces a nontrivial action in the fiber $\mathfrak{g}^*$; it is easy to check that this is precisely the coadjoint action.

The canonical symplectic form on T^*G is invariant with respect to both left and right translations (this follows. e. g., from the explicit formula given below); hence both actions λ, ρ are symplectic. As a matter of fact, the following stronger assertion holds true:

Proposition 1.24. *(i) The actions $\lambda, \rho \colon G \times T^*G \to T^*G$ are Hamiltonian. (ii) In left trivialization, the associated moment maps are given by*

$$\begin{aligned} &\mu_l \colon (x, L) \longmapsto -\mathrm{Ad}^* x \cdot L, \\ &\mu_r \colon (x, L) \longmapsto L, \end{aligned} \tag{1.14}$$

(iii) The quotient Poisson manifold T^*G/G over the action of G by left translations is isomorphic to $\mathfrak{g}^*$ equipped with the Lie–Poisson bracket; the quotient manifold $G\backslash T^*G$ over the action of G by right translations is anti-isomorphic to $\mathfrak{g}^*$ (In other words, the Lie–Poisson bracket changes sign.) (iv) The mappings μ_l, μ_r form a dual pair; in particular, μ_l is constant on the orbits of ρ, and μ_r is constant on the orbits of λ.

REMARK 1.25. The extra minus which arises in the switch from left to right is due to the fact that right multiplication reverses the order of factors and hence the Lie algebras of left- and right-invariant vector fields on G are *anti-isomorphic.* We included one more sign flip into the definition of the Hamiltonian group action in order to get a simple formula (1.9).

Formula (1.14) is a special case of the following more general one. Assume that G is acting on a smooth manifold $\mathcal{M}$; this action canonically lifts to an action of G on its cotangent bundle $T^*\mathcal{M}$ by the formula

$$g\colon (q,p) \longmapsto \left(g\cdot q, (g'(q)^*)^{-1}\cdot p\right), \; q\in\mathcal{M}, \; p\in T_q^*\mathcal{M}. \tag{1.15}$$

For $X\in\mathfrak{g}$ we denote by $\widehat{X}$ the vector field on $T^*\mathcal{M}$ generated by the 1-parameter transformation group $\exp tX$.

PROPOSITION 1.26. *The vector field $\widehat{X}$ is Hamiltonian with the Hamiltonian $h_X = \langle\theta,\widehat{X}\rangle$; in other words, in canonical coordinates on $T^*\mathcal{M}$ we have $h_X(q,p) = \langle p, \pi_*\widehat{X}\rangle$, where $\pi\colon T^*\mathcal{M}\to\mathcal{M}$ is the canonical projection.*

REMARK 1.27. The Hamiltonian h_X, which is linear in canonical momenta, is effectively the so called *Noether integral of motion* associated with the 1-parameter symmetry group $\{\exp tX; t\in\mathbb{R}\}$ of the configuration space.

To prove Proposition 1.24, it is useful do obtain an explicit formula for the canonical Poisson bracket on T^*G. (In the next lecture this formula will be also used in the proof of Factorization Theorem 2.4.) It is more convenient to expand the symplectic form with respect to a basis of left- or right-invariant differential forms, rather than using local coordinates on the base and in the fiber as usual. It is for this reason that we have preferred the invariant definition of the canonical one-form on T^*G to the more simple one which uses local coordinates. Choose a basis $\{e_i\}$ in the Lie algebra $\mathfrak{g}$, and let $\{e^i\}$ be the dual basis in $\mathfrak{g}^*$. Let us denote by $\widehat{X}$ the left-invariant field on G, generated by $X\in\mathfrak{g}$,

$$\widehat{X}\varphi(x) = \left(\tfrac{d}{dt}\right)_{t=0}\varphi(x\cdot e^{tX}).$$

Let ω^i be the left-invariant 1-form on G which corresponds to $e^i\in\mathfrak{g}^*$. The differential form

$$\omega = \sum_i e_i\otimes\omega^i$$

with values in the Lie algebra $\mathfrak{g}$ is called the *Maurer–Cartan form*. Clearly, the Maurer–Cartan form is left-invariant and does not depend on the choice of basis

EXERCISE 1.28. Suppose that G is a matrix group; in this case the Maurer–Cartan form is matrix-valued and is given by $\omega = g^{-1}dg$.

In a similar way, one can define the right-invariant Maurer–Cartan form ω^r.

EXERCISE 1.29. Prove that for matrix groups $\omega^r = dg\, g^{-1}$.

The Maurer–Cartan form satisfies the differential equation

$$d\omega + \tfrac{1}{2}[\omega, \omega] = 0, \tag{1.16}$$

called the Maurer–Cartan equation[3].

EXERCISE 1.30. (i) Check (1.16) for the matrix groups using formula $\omega = g^{-1}dg$ from exercise 1.28. (ii) Give a general proof using the Cartan formula for the differential of a 1-form on a smooth manifold $\mathcal{M}$:

$$d\alpha(X,Y) = X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X,Y]),\ X,\, Y \in \operatorname{Vect}\mathcal{M}.$$

(iii) Compute the differential of the right-invariant Maurer–Cartan form.

PROPOSITION 1.31. *(i) In the left trivialization of the cotangent bundle the canonical 1-form on T^*G is given by*

$$\theta(g, L) = \langle L, \omega\rangle. \tag{1.17}$$

(ii) The symplectic form on T^*G is given by

$$\Omega(g, L) = \langle dL, \omega\rangle - \tfrac{1}{2}\langle L, [\omega,\, \omega]\rangle. \tag{1.18}$$

To prove (1.17) notice that the coupling of the Maurer–Cartan form with a left-invariant vector field on G yields the corresponding element of the Lie algebra, $\langle\theta, \widehat{X}\rangle = X$; since $T_{(g,L)}(T^*G) \simeq \mathfrak{g} \oplus \mathfrak{g}^*$, this implies that the action of the 1-form (1.17) on the tangent vectors to T^*G agrees with the definition of the canonical 1-form. Formula (1.18) immediately follows from the Maurer–Cartan equation.

Identify the tangent space to T^*G with $\mathfrak{g} \oplus \mathfrak{g}^*$ with the help of left translations. Then the skew-symmetric bilinear form on $\mathfrak{g} \oplus \mathfrak{g}^*$ which corresponds to the 2-form (1.18) is given by the 2×2 block matrix

$$\Omega = \begin{pmatrix} A(L) & I \\ -I & 0 \end{pmatrix}, \quad \text{where } A_{ij}(L) = \tfrac{1}{2}\sum c_{ij}^k L_k.$$

It is easy to check that the inverse matrix is equal to

$$H = \begin{pmatrix} 0 & -I \\ I & A(L) \end{pmatrix},$$

which immediately yields the formula for the canonical Poisson bracket on T^*G:

$$\{\varphi,\, \psi\}(g, L) = \sum_i \left(\partial^i\varphi\, \widehat{e_i}\psi - \partial^i\psi\, \widehat{e_i}\varphi\right) + \tfrac{1}{2}\sum_{i,\,j,\,k} c_{ij}^k\, L_k \partial^i\varphi\, \partial^j\psi, \tag{1.19}$$

where we have set $L = \sum L_i e^i$, $\partial^i = \partial/\partial L_i$. Note that the second term in (1.19) is precisely the Lie–Poisson bracket of φ, ψ regarded as functions of the second argument $L \in \mathfrak{g}^*$.

[3] The commutator of $\mathfrak{g}$-valued differential forms in this formula is defined componentwise, i. e.,

$$[e_i \otimes \omega^i,\ e_j \otimes \omega^j] = [e_i,\, e_j] \otimes \omega^i \wedge \omega^j = \sum_k c_{ij}^k e_k \otimes \omega^i \wedge \omega^j,$$

where c_{ij}^k are the structure constants of $\mathfrak{g}$.

EXERCISE 1.32. (i) Compute the canonical 1-form and the symplectic form on T^*G using the *right* trivialization of T^*G. (ii) Prove Proposition 1.24 using formulae (1.19), (1.17).

Getting back to Proposition 1.24, we can deduce its immediate corollary:

PROPOSITION 1.33. *Symplectic leaves in* $\mathfrak{g}^*$ *coincide with the orbits of the coadjoint representation.*

Indeed, formula (1.14) immediately implies that

$$\mu_l\left(\mu_r(f)\right) = \{-\mathrm{Ad}^* g \cdot f;\ g \in G\} = \mathcal{O}_{-f}.$$

In the next lecture we shall apply Hamiltonian reduction of T^*G to the study of Lax equations.

LECTURE 2

The r-Matrix Method and the Main Theorem

2.1. Introduction

The modern theory of Integrable Systems began in 1967 with the observation, due to Peter Lax, that it is sometimes possible to write a nonlinear equation in the Lax form

$$\frac{dL}{dt} = [L,\, M]. \tag{2.1}$$

Here L and M are linear operators which depend on dynamical variables; it is usually assumed that L contains all information on the initial data and M is a function of L. Spectral invariants of L are integrals of motion for this evolution equation. Since the right hand side contains only the commutator, one can try to rewrite (2.1) in a representation independent way and assume that L, M are elements of a Lie algebra. A few years later Faddeev and Zakharov noticed, on the example of the KdV equation, that Lax equations are Hamiltonian and the spectral invariants of the Lax operator form a complete set of integrals in involution. The Hamiltonian aspect of Lax equations discovered in [**ZF**] is truly fundamental, but the way to combine the Hamiltonian structure with the commutator representation is quite non-trivial. It is tempting to compare (2.1) with the Hamiltonian equations of motion associated with the Lie–Poisson brackets. The idea looks seducing as confirmed by the following easy assertion which specializes Proposition 1.2:

PROPOSITION 2.1. *Assume that the general linear algebra $\mathfrak{g} = \mathfrak{gl}(n)$ is identified with its dual space with the help of the invariant inner product (1.8) and is equipped with the Lie–Poisson bracket. The Hamiltonian equation of motion on $\mathfrak{g}$ with a Hamiltonian φ, $\varphi \in C^\infty(\mathfrak{g})$, has the form*

$$\frac{dL}{dt} = -\left[d\varphi\left(L\right), L\right]; \tag{2.2}$$

all Hamiltonian flows on $\mathfrak{g}$ preserve the spectra of matrices.

Equation (2.2) looks like a genuine Lax equation. However, a more careful inspection leads to a deception. Indeed, spectral invariants of matrices are *Casimir functions* of our Lie–Poisson bracket, and their conservation is a trivial fact which has nothing to do with integrability of equation (2.2).

There are no chances whatsoever that this equation with arbitrary Hamiltonian φ will be completely integrable; on the other hand, the spectral invariants themselves, which one would like to use as the natural Hamiltonians for our future examples of integrable systems, give rise to *trivial* equations of motion. Indeed, we have:

PROPOSITION 2.2. *For any Lie algebra $\mathfrak{g}$, a function $\varphi \in C^\infty(\mathfrak{g}^*)$ is a Casimir function of the Lie–Poisson bracket on $\mathfrak{g}^*$ if and only if*

$$ad^*\, d\varphi\,(L) \cdot L = 0$$

for all $L \in \mathfrak{g}^*$. If the spaces $\mathfrak{g}$ and $\mathfrak{g}^*$ are identified, this relation takes the form $[d\varphi\,(L)\,, L] = 0$.

In spite of this initial drawback, the idea of using the Lie–Poisson brackets and coadjoint orbits may be saved. However, instead of the initial Lie–Poisson bracket whose Casimir functions are spectral invariants of matrices, one has to find some *other* one. To implement this idea we shall use the notion of *classical r-matrix.* With its help, we shall define another structure of the Lie algebra in the same linear space $\mathfrak{g}$ (and, by duality, another Lie–Poisson bracket in the dual space $\mathfrak{g}^*$). The interplay of two different structures of a Lie algebra (or two Lie–Poisson brackets) on the same space proves to be a key property of Lax equations.

2.2. Lie Dialgebras and Involutivity Theorem

DEFINITION 2.3. Let $\mathfrak{g}$ be a Lie algebra, $r \in \operatorname{End}\mathfrak{g}$ a linear operator. We shall say that r is a classical r-matrix (or simply r-matrix, for short) if the bracket

$$[X,\, Y]_r = \tfrac{1}{2}\,([rX,\, Y] + [X,\, rY]) \tag{2.3}$$

is a Lie bracket, i.e. if it satisfies the Jacobi identity (the skew symmetry of (2.1) is obvious for any r). We denote the Lie algebra with the bracket (2.1) by $\mathfrak{g}_r$. In the dual space $\mathfrak{g}^* \simeq \mathfrak{g}_r^*$ there are therefore two Poisson brackets: the Lie–Poisson brackets of $\mathfrak{g}$ and $\mathfrak{g}_r$. A pair $(\mathfrak{g},\, \mathfrak{g}_r)$ is called a *Lie dialgebra.*

The Lie–Poisson bracket of $\mathfrak{g}_r$ will be referred to as the *r-bracket*, for short.

REMARK 2.4. In earlier work, the term “double Lie algebra” was used; we preferred to coin a new term “Lie dialgebra” because the word “double” in these lectures is too heavily overburdened. This term also stresses the important and often neglected difference between Lie dialgebras and another important object, Lie bialgebras, which will play a key role in the study of non-linear Poisson structures (see Lecture 4). Both objects are related to classical r-matrices, but the relevant definitions in these two cases are *different.* This difference is sometimes the source of confusion; in particular, it affects very heavily the classification problem.

We can immediately define one important class of Lie dialgebras. Assume that there is a vector space decomposition of $\mathfrak{g}$ into a direct sum of two Lie subalgebras, $\mathfrak{g} = \mathfrak{g}_+ \dot{+} \mathfrak{g}_-$. Let $P_\pm$ be the projection operator onto $\mathfrak{g}_\pm$ parallel to the complementary subalgebra and put

$$r = P_+ - P_-. \tag{2.4}$$

Formula (2.3) immediately implies that in this case the r-bracket is given by

$$[X, Y]_r = [X_+, Y_+] - [X_-, Y_-], \quad X_\pm = P_\pm X, \quad Y_\pm = P_\pm Y, \tag{2.5}$$

i. e., it is the difference of Lie brackets in $\mathfrak{g}_+$ and $\mathfrak{g}_-$. The Jacobi identity for $[X, Y]_r$ is obvious from (2.5). More examples and the general theory of r-matrices will be discussed in Section 2.5.

The definition of Lie dialgebras is motivated by the following basic theorem. Let $I(\mathfrak{g})$ be the ring of Casimir functions on $\mathfrak{g}^*$ (i.e. the ring of its coadjoint invariants).

Theorem 2.5. *(i) Functions from $I(\mathfrak{g})$ are in involution with respect to the r-bracket on $\mathfrak{g}^*$.*

(ii) The equations of motion induced by a function $\varphi \in I(\mathfrak{g})$ with respect to the r-bracket have the form

$$\frac{dL}{dt} = -\mathrm{ad}^*{}_{\mathfrak{g}} M \cdot L, \quad M = \tfrac{1}{2}\, r(d\varphi(L)). \tag{2.6}$$

If $\mathfrak{g}$ admits a non-degenerate invariant bilinear form, so that $\mathfrak{g}^* = \mathfrak{g}$ and $\mathrm{ad}^* = \mathrm{ad}$*, equations* (2.6) *take the Lax form*

$$\frac{dL}{dt} = [L, M]. \tag{2.7}$$

Proof. The r-bracket of two functions φ, ψ on $\mathfrak{g}^*$ is given by

$$\{\varphi \cdot \psi\}_r(L) = \tfrac{1}{2}\langle [rd\varphi(L),\, d\psi(L)],\, L\rangle + \tfrac{1}{2}\langle [d\varphi(L),\, rd\psi(L)] L\rangle.$$

The invariance of φ is equivalent to the relation $\langle [d\varphi(L),\, X],\, L\rangle = 0$ for all $X \in \mathfrak{g}$. This proves (i). For any $\psi \in C^\infty(\mathfrak{g}^*)$ the equation of motion $\dot{\psi} = \{\varphi,\, \psi\}_r$ with an invariant Hamiltonian φ takes the form

$$\dot{\psi}(L) = \tfrac{1}{2}\langle [rd\varphi(L),\, d\psi(L)],\, L\rangle = -\tfrac{1}{2}\langle d\psi(L), \mathrm{ad}^* rd\varphi(L) \cdot L\rangle;$$

Taking as ψ a set of linear coordinate functions in $\mathfrak{g}^*$, we conclude that $\dot{L} = -\frac{1}{2}\,\mathrm{ad}^* rd\varphi(L) \cdot L$. $\square$

Remark 2.6. The definition of M in equation (2.6) is not canonical: we may always add to it an arbitrary element from the centralizer of L; as we know, this centralizer contains in particular the differentials of all invariant Hamiltonians. It is convenient to choose $M_\pm(L) = \frac{1}{2}\,(r \pm 1)(d\varphi(L))$; equation (2.6) then takes the form

$$\frac{dL}{dt} = -\,\mathrm{ad}^*\, M_\pm \cdot L. \tag{2.8}$$

The freedom in the choice of the sign is closely related to the formulation of the factorization problem associated with the Lax equation which we consider in Section 2.3 below.

Theorem 2.5 has a transparent geometrical meaning: it shows that the trajectories of the dynamical systems with Hamiltonians $\varphi \in I(\mathfrak{g})$ lie in the intersection of two families of orbits in $\mathfrak{g}^*$, the coadjoint orbits of $\mathfrak{g}$ and $\mathfrak{g}_r$. Indeed, the coadjoint orbits of $\mathfrak{g}_r$ are preserved by all Hamiltonian flows in $\mathfrak{g}^*$. On the other hand, formula (2.6) shows that the flow is always tangent to $\mathfrak{g}$-orbits in $\mathfrak{g}^*$. In many cases the intersections of orbits are precisely the "Liouville tori" for our dynamical systems.

REMARK 2.7. For general Lie algebras, the adjoint and the coadjoint representations are inequivalent, and hence the right hand side of the generalized Lax equations (2.6) is not written in simple commutator form. There are several important classes of Lie algebras where this is indeed the case: these are central extensions of the current algebras and the algebra of vector fields on the circle which we discuss in Lectures 7 and 8 below. One can show that the commutator form can be restored in these cases as well by adding extra elements to our Lie algebra. In applications to finite dimensional integrable systems, the basic Lie algebras are equipped with a non-degenerate invariant inner product and hence their adjoint and coadjoint representations coincide.

2.3. Factorization Theorem

The scheme outlined so far incorporates only two of the three main features of the inverse scattering method: the Poisson brackets and the Lax form of the equations of motion. As it happens, the most important feature of this method, the reduction of the equations of motion to the Riemann problem, is already implicit in our scheme (under some additional conditions on the choice of the classical r-matrix, in particular for the r-matrices of the form (2.4)[1]). An abstract version of the Riemann problem is provided by the factorization problem in Lie groups.

We shall state the factorization theorem (which is a global version of Theorem 2.5) for the simplest r-matrices of the form (2.4). Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$, and let $G_\pm$ be its subgroups corresponding to $\mathfrak{g}_\pm$.

THEOREM 2.8. *Let $\varphi \in I(\mathfrak{g})$, $X = d\varphi(L)$. Let $g_\pm(t)$ be the smooth curves in $G_\pm$ which solve the factorization problem*

$$\exp tX = g_+(t) g_-(t)^{-1}, \quad g_\pm(0) = e. \tag{2.9}$$

Then the integral curve $L(t)$ of equation (2.8) with $L(0) = L$ is given by

$$L(t) = \mathrm{Ad}^*_G\, g_+(t)^{-1} \cdot L = \mathrm{Ad}^*_G\, g_-(t)^{-1} \cdot L. \tag{2.10}$$

[1]The natural general condition imposed on r is the so called *modified classical Yang-Baxter equation* discussed in section 2.5.

PROOF. Let us notice first of all that the two formulae in (2.10) are equivalent, since $g_+(t)g_-(t)^{-1} = \exp tX$ lies in the centralizer of L. Differentiating (2.10) with respect to t we get

$$\frac{dL}{dt} = -\operatorname{ad}^*\left(g_+^{-1}\dot{g}_+\right)\cdot L = -\operatorname{ad}^*\left(g_-^{-1}\dot{g}_-\right)\cdot L.$$

It remains to check that $g_\pm^{-1}\dot{g}_\pm = M_\pm$. Recall that

$$M_\pm(t) = \pm P_\pm X(t).$$

The $\operatorname{Ad}\mathfrak{g}$-invariance of φ implies that

$$X(t) = \operatorname{Ad}_G g_\pm(t)^{-1}\cdot X. \tag{2.11}$$

Writing (2.9) in the form $g_+(t)^{-1}\exp tX = g_-(t)^{-1}$ and differentiating with respect to t we get

$$\operatorname{Ad} g_-(t)^{-1}\cdot X = g_+(t)^{-1}\dot{g}_+(t) - g_-(t)^{-1}\dot{g}_-(t).$$

Since

$$g_\pm(t)^{-1}\dot{g}_\pm(t) \in \mathfrak{g}_\pm,$$

this implies $g_\pm(t)^{-1}\dot{g}_\pm(t) = \pm P_\pm X(t)$. □

2.4. Factorization Theorem and Hamiltonian Reduction

A more geometric proof of Theorem 2.8 is based on Hamiltonian reduction. As already noted, the reduction technique in the study of Lax equations works in a non-traditional way: we begin with a simple multi-dimensional system with admits an explicit description of the dynamical flows and possesses a rich symmetry group ("free dynamics") and apply to it the reduction procedure yielding an apparently more complicated system described by Lax equations. The realization of this idea breaks up into several steps.

1. Choose the "big" phase space which will produce the phase space of the Lax system as a reduced symplectic manifold.
2. Choose a symmetry group and the "free" dynamics which possesses the suitable invariance.
3. Describe the reduced system.

Although the proof of Factorization Theorem which is based on this approach is much longer than the elementary proof given above, it is more transparent and explains the origin of the result.

The natural choice of the "big" phase space is the cotangent bundle T^*G already considered in Section 1.4. To implement our program we will start with the description of the free dynamics on T^*G. Fix a trivialization of $T^*G \simeq G\times\mathfrak{g}^*$ by means of left translations.

LEMMA 2.9. *A Casimir function $\varphi\in I(\mathfrak{g})$ may be canonically lifted to T^*G by the formula*

$$h_\varphi = \varphi\circ\mu_l, \quad \textit{or, equivalently,} \quad h_\varphi = \varphi\circ\mu_r,$$

where μ_l, μ_r *are the moment maps from* (1.14). *The resulting functions* h_φ, $\varphi \in I(\mathfrak{g})$, *may be characterized as* bi-invariant *functions on* T^*G.

LEMMA 2.10. *In the left trivialization of* T^*G *the Hamiltonian flow on* T^*G *generated by a bi-invariant Hamiltonian* h_φ *is given by*

$$F_t\colon (g, L) \longmapsto (g \cdot \exp t\, d\varphi(L),\ L)\,,\ g \in G,\ L \in \mathfrak{g}^*. \tag{2.12}$$

In other words, its integral curves project onto the left translates of one-parameter subgroups in G*; the choice of* φ *determines the "dispersion law", i. e., the dependence of the (constant) velocity vector* $d\varphi(L)$ *on the initial canonical momentum* L.

SKETCH OF A PROOF. Consider the dual pair

$$\begin{tikzcd} & T^*G \arrow[dl, "\mu_l"'] \arrow[dr, "\mu_r"] & \\ \mathfrak{g}^* & & \mathfrak{g}^* \end{tikzcd}$$

from Proposition 1.24. The Hamiltonian h_φ is constant on the fibers of the projection maps μ_l, μ_r; for both maps, the reduced Hamiltonians coincide with φ, i. e., they are the Casimir functions of the quotient Poisson structure. Hence μ_l, μ_r project integral curves of h_φ into points; this immediately implies that the velocity vector is constant.

EXERCISE 2.11. Using formula (1.19), check that the velocity vector is equal to $d\varphi(L)$.

Since the Hamiltonian h_φ is invariant with resect to the action of G by left and right translations, the flow F_t admits reduction with respect to *any* subgroup $U \subset G \times G$. There exists therefore a tremendous freedom in the choice of such a subgroup, different subgroups giving rise to various interesting reduced systems. The particular choice which leads to the Lax system (2.6) corresponds to our choice of the r-matrix; let us set, namely, $U = G_+ \times G_-$. (This is precisely the Lie group associated with the Lie algebra $\mathfrak{g}_r$!) Due to (1.13), the action of $G_+ \times G_- \simeq G_r$ on $T^*G \simeq G \times \mathfrak{g}^*$ is given by

$$(h_+, h_-)\colon (g, L) \longmapsto \left(h_+ g h_-^{-1},\ \mathrm{Ad}_G^* \cdot L\right). \tag{2.13}$$

To perform reduction over G_r, we produce a mapping s which is constant on its orbits in T^*G; its image then yields a *model* of the quotient space. For $g \in G$ denote by $g_\pm$ the solutions of the factorization problem

$$g = g_+ \cdot g_-^{-1}, \quad g_+ \in G_+,\ g_- \in G_-. \tag{2.14}$$

THEOREM 2.12. *(i) the Hamiltonian* h_φ *is invariant with respect to the action* (2.13).

(ii) The mapping

$$s\colon T^*G \longrightarrow \mathfrak{g}^*\colon (g, L) \longmapsto \mathrm{Ad}_G^*\, g_-^{-1} \cdot L$$

is constant on G_r-orbits in T^*G. If G is globally diffeomorphic to $G_+ \times G_-$, i. e., the factorization problem (2.14) *always admits a unique solution, s is a global cross-section of the action of G_r.*

(iii) The quotient Poisson structure on $T^*G/G_r \simeq \mathfrak{g}^*$ coincides with the Lie–Poisson bracket of $\mathfrak{g}_r$.

(iv) The quotient flow $\bar{F}_t$ on $\mathfrak{g}_r^*$ obtained by reduction of the "free" flow (2.12) *is given by*

$$\bar{F}_t\colon L \longmapsto \mathrm{Ad}^*{}_G g_\pm(t)^{-1} L, \tag{2.15}$$

where $g_+(t), g_-(t)$ are the solutions of the factorization problem $\exp t\nabla\varphi(L) = g_+(t)\, g_-(t)^{-1}$, and satisfy the Lax equation (2.8); the reduced Hamiltonian on $\mathfrak{g}_r^*$ coincides with φ.

Only the third assertion requires a special proof. Consider the mapping

$$\sigma\colon G_r \to G\colon h = (h_+, h_-) \mapsto h_+ h_-^{-1}.$$

By assumption, σ is a diffeomorphism; hence its derivative is a non-degenerate linear mapping and

$$T\sigma\colon T^*G_r \longrightarrow T^*G\colon (h,\xi) \longmapsto \big(\sigma(h), (\sigma(h)')^{-1}\cdot\xi\big) \tag{2.16}$$

is a symplectic isomorphism of the cotangent bundles. This isomorphism transforms the action of G_r on T^*G defined by (2.13), into the standard action of G_r by left translations on its own cotangent bundle (check!). Now (iii) follows from Proposition 1.24 applied to G_r.

In the general case, of course, G need not be diffeomorphic to G_r; still, the space $\mathfrak{g}_r^*$ provides a model for a "big cell" in the quotient phase space T^*G/G_r; under some mild additional assumptions one can show that the action of G_r on T^*G given by (2.13) is *proper*, and hence the quotient space T^*G/G_r is well defined and the reduced dynamical flow on this space is complete[2]. This allows to construct a canonical completion of the dynamical flow associated with the Lax equation. Of course, the flow on $\mathfrak{g}_r^*$ need not be complete: its integral curves "escape to infinity" precisely at the moments when the integral curves of the non-reduced system (alias, the one-parameter subgroups in G) escape from the "big cell" in G, in which the factorization problem is solvable. In typical examples one can check that the one-parameter subgroups intersect transversally the "complementary" cells of positive codimension in G and return back to the big cell; the associated solution displays a polar singularity in time variable.

Remark 2.13. The change of variables (2.16) in the proof of theorem 2.12 may be replaced by an easy computation. For any two functions $\varphi, \psi \in C^\infty(\mathfrak{g}^*)$ set $H_\varphi = s^*\varphi$, $H_\psi = s^*\psi$. Let $\widetilde{L} = \mathrm{Ad}^*\, x_-^{-1}\cdot L$. For any $\xi \in \mathfrak{g}$ we have

$$\left(\tfrac{d}{dt}\right)_{t=0} s^*\varphi(xe^{t\xi}, L) = \left\langle \left[\xi_-,\, d\varphi(\widetilde{L})\right], \widetilde{L}\right\rangle = \left\langle \xi, P_-^*(\mathrm{ad}^* d\varphi(\widetilde{L})\cdot\widetilde{L})\right\rangle.$$

[2]Note that the flow (2.12) is always complete, since its integral curves are one-parameter subgroups!

Using this formula (and a similar formula for the right derivatives of $s^*\psi$), it is easy to compute the canonical Poisson bracket $\{s^*\varphi, s^*\psi\}$. We have

$$\begin{aligned}\{s^*\varphi, s^*\psi\}(x, L) = \left\langle P_-^*(\mathrm{ad}^*\, d\psi(\widetilde{L})),\, d\varphi(\widetilde{L})\right\rangle -\\ - \left\langle P_-^*(\mathrm{ad}^*\, d\varphi(\widetilde{L})),\, d\psi(\widetilde{L})\right\rangle + \left\langle \left[d\varphi(\widetilde{L}),\, d\psi(\widetilde{L})\right],\, \widetilde{L}\right\rangle =\\ = \left\langle \left[d\varphi(\widetilde{L}),\, d\psi(\widetilde{L})\right]_r,\, \widetilde{L}\right\rangle.\end{aligned}$$

In Lecture 6 we shall make a similar computation in the study of Lax equations on Poisson Lie groups. (In that case, a more geometrical proof based on the symplectic change of variables given above does not apply.)

REMARK 2.14. The geometric construction which underlies theorem 2.12 may be generalized in several important ways. For instance, instead of the group $G_r = G_+ \times G_-$ associated with the special r-matrix (2.4) we can consider the action on G of an *arbitrary subgroup* $H \subset G \times G$, *which is transversal to the diagonal* $G^\delta = \{(x, x) \in G \times G; x \in G\}$. (In Section 2.6 we shall see that this generalization precisely corresponds to the case of general r-matrices satisfying the modified Yang-Baxter equation.) In this case the orbit of the unit element is open in G, and the restriction of the action

$$H \times G \to G\colon (h_1, h_2)\colon x \mapsto h_1 x h_2^{-1} \tag{2.17}$$

to this orbit is locally free. Extend the action (2.17) to T^*G; it is easy to see that the Poisson reduced space T^*G/H contains $\mathfrak{h}^*$ as an open cell. Actually, the entire construction is based just on the assumption that the action $H \times G \to G$ is *free* (or locally free) and admits an open orbit. Finally, one can also drop the assumption that the subgroup $H \subset G \times G$ is transversal to the diagonal. One can consider, in particular, the action of G on itself by conjugations. With the suitable choice of a symplectic leaf in the reduced space T^*G/G one arrives in this way to the *Calogero–Moser systems,* cf. [**KKS**]. In this case, however, the reduced space does no longer have the structure of a coadjoint orbit and in fact is not even a homogeneous space of some group.

In Lecture 6 we shall generalize the above proof of Factorization Theorem to the case of Poisson–Lie groups and difference Lax equations. In order to do it we shall have to replace the phase space of the “free system”, i. e., the cotangent bundle T^*G, with a more complicated one, the “symplectic double” or the “twisted double” of G. In this more general case the integral curves of the free system are again one-parameter subgroups, and the main steps of the proof remain the same. Another important generalization of Factorization Theorem is the theory of *dressing transformations* which allow, in certain cases, to produce new solutions of integrable equations from simple ones (see Section 7.6).

REMARK 2.15. Theorem 2.12 shows that Lax equations obtained by reduction of bi-invariant Hamiltonian equations on T^*G are in some sense *explicitly solvable*. However, the question whether these equations are *completely integrable in the Liouville sense* requires a separate study. Recall that according to the classical Liouville–Arnold theorem [**A**] a Hamiltonian system on a $2n$-dimensional symplectic manifold is completely integrable if it admits exactly n independent first integrals of motion; in that case, the generic level surface of these integrals is a Lagrangian submanifold. The integrals of motion of the Lax equations stem from the Casimir functions on $\mathfrak{g}^*$. If G is a finite-dimensional semisimple group, the ring $I(\mathfrak{g})$ of its coadjoint invariants is finitely generated; according to the classical Chevalley theorem, the number of independent generators of this ring is equal to $l = \operatorname{rank} G$. On the other hand, generic coadjoint orbits of G_r usually have high dimension of order l^2. Thus for typical orbits the number of independent integrals provided by Theorem 2.5 is insufficient. For generic Hamiltonian systems this would mean that their trajectories densely span a subset of *small codimension* in the phase space. In practice, however, the situation is exactly opposite: the trajectories of the Hamiltonian flows generated by coadjoint invariants span a subset of *small dimension*. This means that the behaviour of our exactly solvable Lax equations is much more regular than for typical Hamiltonian systems. However, this regular, or resonance behaviour also prevents us from constructing immediately a complete set of integrals of motion in involution. In other words, generically the exactly solvable Lax equations associated with finite-dimensional semisimple Lie algebras fall within the class of the so called *degenerate integrable systems*. A version of the Liouville–Arnold theorem for such systems has been obtained by Nekhoroshev [**N**]. Degenerate integrability is discussed in detail in a series of recent papers by Reshetikhin [**R1, R2**]. To complete the system of integrals one can use various sets of "semi-invariants" (examples of such completion may be found in [**DLNT**]). This difficulty does not arise only for Lax systems supported on low-dimensional coadjoint orbits of the r-bracket (for instance, for the open Toda lattices). The situation changes drastically for Lax equations with spectral parameter. In this case the ring of invariants of the underlying Lie algebra (affine Lie algebra, or loop algebra) has *infinitely many* independent generators; accordingly, Lax equations supported on generic finite-dimensional orbits of the r-bracket are automatically completely integrable in the Liouville sense (at least, under some mild technical assumptions).

2.5. Classical Yang-Baxter Identity and the General Theory of Classical r-Matrices

Let us now discuss the conditions imposed on an r-matrix by the Jacobi identity for the r-bracket. Recall that the r-bracket associated with $r \in \operatorname{End} \mathfrak{g}$

is given by

$$[X,\, Y]_r = \tfrac{1}{2}([rX,\, Y] + [X,\, rY]).$$

Put

$$B_r(X,Y) = [rX,\, rY] - r([rX,\, Y] + [X,\, rY]). \tag{2.18}$$

PROPOSITION 2.16. *The r-bracket* (2.3) *satisfies the Jacobi identity if and only if for any* $X, Y, Z \in \mathfrak{g}$ *we have*

$$[B_r(X,Y),\, Z] + [B_r(Y,Z),\, X] + [B_r(Z,X),\, Y] = 0. \tag{2.19}$$

The necessary and sufficient condition (2.19) is usually replaced by sufficient conditions which are bilinear rather than trilinear. The simplest sufficient condition is the so called classical Yang-Baxter identity (CYBE)

$$B_r(X,Y) = 0. \tag{2.20}$$

Another important sufficient condition is the modified classical Yang-Baxter identity (mCYBE)

$$B_r(X,Y) = -c^2[X,\, Y]. \tag{2.21}$$

If $\mathfrak{g}$ is a real Lie algebra, we may always assume by rescaling that $c = \pm 1$. Note that the r-matrices (2.4) satisfy mCYBE with $c = 1$. The case $c = 1$, or the *split case*, is more important; although general r-matrices which satisfy the the modified classical Yang-Baxter identity (2.21) do not in general have the simple form (2.4), one can still associate with them a factorization problem. By contrast, the ordinary classical Yang-Baxter identity (2.20) represents a degenerate case and does not lead to a factorization problem. In the *non-split* case $c = -1$ the factorization problem still makes sense, but the factors lie in the complex hull of the group G.

Let us describe the factorization problem associated with an arbitrary split r-matrix satisfying the modified classical Yang–Baxter equation. Suppose that $r \in \operatorname{End} \mathfrak{g}$ satisfies mCYBE with $c = 1$; put

$$r_\pm = \tfrac{1}{2}(r \pm id). \tag{2.22}$$

PROPOSITION 2.17. *We have*

$$[r_\pm X,\, r_\pm Y] = r_\pm[X,\, Y], \quad X, Y \in \mathfrak{g}, \tag{2.23}$$

i.e. $r_\pm \colon \mathfrak{g}_r \to \mathfrak{g}$ *is a Lie algebra homomorphism.*

Put $\mathfrak{g}_\pm = \operatorname{Im} r_\pm$, $\mathfrak{k}_\pm = \operatorname{Ker} r_\mp$.

PROPOSITION 2.18. *(i)* $\mathfrak{g}_\pm \subset \mathfrak{g}$ *is a Lie subalgebra; (ii)* $\mathfrak{k}_\pm \subset \mathfrak{g}_\pm$ *is an ideal.*

Define the mapping $\theta_r \colon \mathfrak{g}_+/\mathfrak{k}_+ \longrightarrow \mathfrak{g}_-/\mathfrak{k}_-$ by setting

$$\theta_r \colon (r + id)X \mapsto (r - id)X. \tag{2.24}$$

Note that θ_r is well defined, since $X \in \mathfrak{k}_\pm$ implies $(r \mp id)X = 0$.

PROPOSITION 2.19. *(i) θ_r is a Lie algebra isomorphism. (ii) Consider the combined mapping $i_r = r_+ \oplus r_-$:*

$$i_r\colon \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}\colon \quad X \mapsto (r_+X,\, r_-X).$$

This is a Lie algebra embedding and its image consists of those pairs $(X, Y) \in \mathfrak{g}_+ \oplus \mathfrak{g}_-$ for which $\theta_r \bar{X} = \theta_r \bar{Y}$ (here $\bar{X}, \bar{Y}$ is the residue class of X, Y in $\mathfrak{g}_+/\mathfrak{k}_+, \mathfrak{g}_-/\mathfrak{k}_-$, respectively).

(iii) Each $X \in \mathfrak{g}$ has a unique decomposition

$$X = X_+ - X_-, \quad \textit{where} \quad (X_+,\, X_-) \in \operatorname{Im} i_r.$$

The operator θ_r is called the *Cayley transform* of r.

The proof of Proposition 2.19 immediately follows from the standard homomorphism theorem; to check the last assertion one has to note that $r_+ - r_- = id$ is the identity mapping.

Now let G, G_r be local Lie groups which correspond to $\mathfrak{g}, \mathfrak{g}_r$. The homomorphisms $r_\pm$ give rise to Lie group homomorphisms which we denote by the same letters. We put $G_\pm = r_\pm(G_r)$, $K_\pm = \operatorname{Ker} r_\mp$ and extend the Cayley transform θ_r to a Lie group isomorphism $\theta_r : G_+/K_+ \to G_-/K_-$.

THEOREM 2.20. *(i) The mapping*

$$i_r\colon G_r \to G \times G\colon h \longmapsto (r_+h, r_-h) \tag{2.25}$$

is a Lie group embedding; its image consists of those pairs $(x, y) \in G_+ \times G_-$ for which $\theta_r \bar{x} = \bar{y}$.

(ii) Consider the mapping $m\colon G \times G \to G\colon (x, y) \mapsto xy^{-1}$; the combined map

$$f\colon\; G \xrightarrow{i_r} G \times G \xrightarrow{m} G$$

is a local homeomorphism, and hence an arbitrary element $x \in G$ which is sufficiently close to unity admits a unique representation

$$x = x_+x_-^{-1}, \quad \textit{where} \quad (x_+,\, x_-) \in \operatorname{Im}(r_+ \times r_-). \tag{2.26}$$

The proof of Theorem 2.8 given in Section 2.4 extends to the present setting with only minor changes[3]. Just notice that the mapping (2.25) defines an embedding of G_r into $G \times G$; we may define an action $G_r \times G \to G$ by $h\colon g \mapsto h_+gh_-^{-1}$, $h_\pm = r_\pm(h)$ and extend it to T^*G.

2.6. Lie Dialgebras and their Doubles.

As already noted, linear operators $r \in \operatorname{End}\mathfrak{g}$ which satisfy the modified classical Yang–Baxter identity do not in general have the special form (2.4), i. e., are not represented as the difference of two complementary projection operators[4]. However, it is possible to reduce the problem to the special case (2.4) by "squaring" the initial Lie algebra. Let $r \in \operatorname{End}\mathfrak{g}$ be a solution of the

[3]The same is true, of course, for the computational proof from Section 2.3.

[4]We shall return to the description of "general" classical r-matrices in Lecture 4 (Section 4.5.1), where we shall explain a connection of this problem with the extension theory of linear operators.

modified Yang–Baxter equation Set $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ (direct sum of Lie algebras). Recall that according to Proposition 2.19 the Lie algebra $\mathfrak{g}_r$ is canonically embedded into $\mathfrak{d}$. Let $\mathfrak{g}^\delta \subset \mathfrak{d}$ be the diagonal subalgebra.

PROPOSITION 2.21. *(i) $\mathfrak{d}$ admits a direct sum decomposition*

$$\mathfrak{d} = \mathfrak{g}^\delta \dot{+} \mathfrak{g}_r. \tag{2.27}$$

(ii) Conversely, any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{g}$ which is transversal to the diagonal gives rise to a solution of the modified classical Yang–Baxter equation on $\mathfrak{g}$.

PROOF. It is easy to derive explicit formulae for the projection operators onto $\mathfrak{g}^\delta$, $\mathfrak{g}_r \subset \mathfrak{d}$ in the decomposition (2.27). We have

$$P_{\mathfrak{g}^\delta}(X,Y) = (Y_+ - X_-,\, Y_+ - X_-),$$
$$P_{\mathfrak{g}_r}(X,Y) = (X_+ - Y_+,\, X_- - Y_-),$$

where $X_\pm = r_\pm X$, $Y_\pm = r_\pm Y$. Let $p_\pm \colon \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ be the canonical projection operators associating to $(X,Y) \in \mathfrak{d}$ its first and second components. Set $r_\pm = p_\pm|_{\mathfrak{h}}$; then $r_\pm \colon \mathfrak{h} \to \mathfrak{g}$ are Lie algebra homomorphisms and $r_+ - r_-$ defines a linear space isomorphism between $\mathfrak{h}$ and $\mathfrak{g}$. Thus $(\mathfrak{g}, \mathfrak{h})$ is a Lie dialgebra and $r = \frac{1}{2}(r_+ + r_-)$ satisfies (2.21). $\square$

Put

$$r_{\mathfrak{d}} = P_{\mathfrak{g}_r} - P_{\mathfrak{g}^\delta}.$$

Clearly, $r_{\mathfrak{d}}$ satisfies (2.21) and equips $\mathfrak{d}$ with the structure of a double Lie algebra. The pair $(\mathfrak{d}, \mathfrak{d}_{r_{\mathfrak{d}}})$ is called *the double of* $(\mathfrak{g}, \mathfrak{g}_r)$.

Let $D = G \times G$, $G^\delta \subset D$ be the diagonal subgroup; we may identify G_r with the embedded subgroup $i_r(G_r) \subset D$.

LEMMA 2.22. *An arbitrary element $(x,y) \in D$ sufficiently close to unity admits a unique factorization*

$$(x,y) = (h_+, h_-)(g,g), \quad (h_+, h_-) \in G_r, \quad g \in G;$$

we have $h_\pm = (xy^{-1})_\pm$, $g = x(xy^{-1})_+^{-1} = y(xy^{-1})_-^{-1}$.

Let us describe the relation between Lax equations in $\mathfrak{g}^*$ and in $\mathfrak{d}^*$. Let

$$\mathfrak{d}^* = (\mathfrak{g}^\delta)^* \dot{+} \mathfrak{g}_r^*, \quad (\mathfrak{g}^\delta)^* = \mathfrak{g}_r^\perp,\ \mathfrak{g}_r^* = (\mathfrak{g}^\delta)^\perp,$$

be the biorthogonal decomposition of the dual space.

PROPOSITION 2.23. *Let $\varphi \in I(\mathfrak{d})$ be a Casimir function of $\mathfrak{d}$. The generalized Lax equation associated with φ with respect to the Lie–Poisson bracket of $\mathfrak{d}_{r_{\mathfrak{d}}}$ leaves the subspace $\mathfrak{g}_r^* \subset \mathfrak{d}^*$ invariant; the induced vector field on this subspace is Hamiltonian with respect to the Lie–Poisson bracket of $\mathfrak{g}_r$ with the Hamiltonian $\varphi\big|_{\mathfrak{g}_r^*}$.*

PROOF. Identify the dual space $\mathfrak{d}^*$ with $\mathfrak{g}^* \oplus \mathfrak{g}^*$ with the help of the coupling

$$\langle(\xi,\eta),\,(X,Y)\rangle = \langle \xi, X\rangle - \langle \eta, Y\rangle.$$

It is easy to see that in this realization we have

$$(\mathfrak{g}^\delta)^\perp = \{(\xi,\xi);\, \xi \in \mathfrak{g}^*\}\,,\ \mathfrak{g}_r^\perp = \left\{(r_-^*\xi,\, r_+^*\xi);\, \xi \in \mathfrak{g}^*\right\}.$$

Let $(\xi,\xi) \in \mathfrak{g}_r^* \simeq (\mathfrak{g}^\delta)^\perp$. It is easy to check that if $d\varphi(\xi,\xi) = (X,Y)$, we have $d\varphi\,|_{\mathfrak{g}_r^*}(\xi) = X - Y$ and, moreover, $[X,Y] = 0$, $\operatorname{ad}^* X \cdot \xi = \operatorname{ad}^* Y \cdot \xi = 0$. Hence, $\varphi\big|_{\mathfrak{g}_r^*}$ is a Casimir function of $\mathfrak{g}$. To check that the Lax equations generated by φ and $\varphi\big|_{\mathfrak{g}_r^*}$ coincide on $\mathfrak{g}_r^* \subset \mathfrak{d}^*$, we may compare their integral curves, using the relation between the factorization problems in D and G, respectively. The integral curve in $\mathfrak{d}^*$ which emanates from $(\xi,\xi) \in \mathfrak{d}^*$, is given by

$$(\xi(t),\xi(t)) = \operatorname{Ad}^*{}_D(h_+(t),h_-(t))^{-1}\cdot(\xi,\xi) = (\operatorname{Ad}^*{}_G h_+^{-1}(t)\xi, \operatorname{Ad}^*{}_G h_-^{-1}(t)\xi),$$

where

$$(h_+(t),h_-(t))(g(t),g(t)) = \exp t d\varphi(\xi,\xi)$$

is the solution of the factorization problem in D. Lemma 2.22 implies that

$$h_\pm = (\exp t(X-Y))_\pm = (\exp\, t\varphi\big|_{\mathfrak{g}_r^*}(\xi))_\pm.$$

Let us note one more useful formula which easily follows from the relation between the decompositions of $\mathfrak{g}$ and $\mathfrak{d}$.

EXERCISE 2.24. The coadjoint representation of G_r is given by the formula

$$\operatorname{Ad}^*_{G_r}(h_+,h_-)\cdot\xi = r_+^*(\operatorname{Ad}^*_G h_+\cdot\xi) - r_-^*(\operatorname{Ad}^*_G h_-\cdot\xi). \tag{2.28}$$

Using (2.28) one can easily get one more formula for the solutions of Lax equations.

EXERCISE 2.25. Let $\sigma\colon G_r \longrightarrow G$ be the local diffeomorphism defined by the factorization problem (2.26). The solution of the Lax equation (2.8) is given by

$$L(t) = \operatorname{Ad}^*{}_{G_r}\sigma^{-1}(\exp t d\varphi(L_0))\cdot L_0. \tag{2.29}$$

Geometrically, formulae (2.10), (2.29) correspond to the possibility to reach the point $L(t)$ lying in the inersection of the coadjoint orbits of G and G_r acting on the initial point by the coadjoint representation operators of any of these groups.

REMARK 2.26. The construction of the double of a Lie dialgebra is a version (though not a special case!) of a similar construction proposed by Drinfeld [**D1, D2**] for Lie bialgebras which we shall discuss in detail in Lecture 5.

LECTURE 3

Classical r-matrices Related to Affine Lie Algebras

Finite-dimensional semi-simple Lie algebras admit several important decompositions which give rise to the structure of a Lie dialgebra. For instance, we can consider the Gauss decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \dot{+}\, \mathfrak{h} \dot{+}\, \mathfrak{n}_-,$$

where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}_\pm$ are the opposite nilpotent subalgebras spanned by root space vectors associated respectively with positive and negative roots. Let $P_\pm$, $P_\mathfrak{h}$ be the projection operators onto $\mathfrak{n}_\pm$, $\mathfrak{h}$ in this decomposition. We can put

$$r_{\text{Gauss}} = P_+ + P_\mathfrak{h} - P_-, \quad r_0 = P_+ - P_-.$$

Both r-matrices satisfy the modified Yang–Baxter identity and define on $\mathfrak{g}$ the structure of a a Lie dialgebra. Another possibility is to consider the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} \dot{+}\, \mathfrak{a} \dot{+}\, \mathfrak{n},$$

where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$, $\mathfrak{a}$ its split Cartan subalgebra, and $\mathfrak{n}$ is spanned by the root space vectors associated with the positive roots of $(\mathfrak{g}, \mathfrak{a})$. We can set

$$\mathfrak{g}_+ = \mathfrak{a} \dot{+}\, \mathfrak{n}, \quad \mathfrak{g}_- = \mathfrak{k}, \quad r_{\text{Iw}} = P_+ - P_-.$$

All these decompositions give rise to interesting examples of integrable systems supported on coadjoint orbits of the corresponding Lie algebras $\mathfrak{g}_r$. However, the most interesting examples of Lax equations are those where the Lax matrices depend on a spectral parameter. They are connected with the so-called affine Lie algebras (loop algebras).

3.1. Loop Algebras and their Standard Decompositions

Let $\mathfrak{g}$ be a Lie algebra. Its loop algebra $\mathcal{L}(\mathfrak{g})$ is the Lie algebra of Laurent polynomials in the variable λ with coefficients in $\mathfrak{g}$:

$$\mathcal{L}(\mathfrak{g}) = \mathfrak{g}[\lambda, \lambda^{-1}] = \left\{ X(\lambda) = \sum_i x_i \lambda^i,\, x_i \in \mathfrak{g} \right\}.$$

The commutator in $\mathcal{L}(\mathfrak{g})$ is given by

$$[x\lambda^i, y\lambda^j] = [x, y]\lambda^{i+j},$$

or

$$[X, Y](\lambda) = [X(\lambda), Y(\lambda)].$$

This means that the algebra $\mathcal{L}(\mathfrak{g})$ has a natural grading by powers of λ:

$$\mathcal{L}(\mathfrak{g}) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}\,\lambda^i. \tag{3.1}$$

If (,) is a non-degenerate invariant bilinear form on $\mathfrak{g}$, we may define a non-degnerate invariant form on $\mathcal{L}(\mathfrak{g})$ by

$$\langle X, Y\rangle = \mathrm{Res}_{\lambda=0}\lambda^{-1}(X(\lambda), Y(\lambda)). \tag{3.2}$$

The algebraic dual of $\mathcal{L}(\mathfrak{g}, \sigma)$ consists of formal Laurent series with coefficients in $\mathfrak{g}^*$. In this section we shall consider the "polynomial" dual space

$$\mathcal{L}^*(\mathfrak{g}) = \bigoplus_{i} \mathfrak{g}^*\,\lambda^i,$$

which consists of Laurent polynomials. Slightly more subtle questions related to the completion of loop algebras will be discussed in Section 3.3. The non-degenerate invariant bilinear form (3.2) allows to identify $\mathcal{L}^*(\mathfrak{g})$ with $\mathcal{L}(\mathfrak{g})$, so that the coadjoint representation of $\mathcal{L}(\mathfrak{g})$ is identified with the adjoint representation. In the sequel we shall assume that $\mathfrak{g}$ admits a non-degenerate invariant bilinear form, so that $\mathcal{L}^*(\mathfrak{g})$ and $\mathcal{L}(\mathfrak{g})$ are identified.

Loop algebras $\mathcal{L}(\mathfrak{g}, \sigma)$, where $\mathfrak{g}$ is semi-simple, are also called *affine Lie algebras.*

Following the general scheme of Section 2.2, in order to construct Lax equations associated with $\mathcal{L}(\mathfrak{g}, \sigma)$ we must indicate:

(1) a decomposition of $\mathcal{L}(\mathfrak{g})$ into two subalgebras;
(2) the invariants of $\mathcal{L}(\mathfrak{g}, \sigma)$;
(3) the orbits or suitable Poisson subspaces of the r-bracket.

This may easily be done. A decomposition into two subalgebras is defined by the grading (3.1). Put

$$\mathcal{L}(\mathfrak{g})_+ = \bigoplus_{i\geqslant 0} \mathfrak{g}\,\lambda^i, \quad \mathcal{L}(\mathfrak{g})_- = \bigoplus_{i<0} \mathfrak{g}\,\lambda^i. \tag{3.3}$$

The pairing (3.2) allows us to identify the duals $\mathcal{L}(\mathfrak{g})^*_\pm$ with the subspaces

$$\mathcal{L}(\mathfrak{g})^*_+ = \bigoplus_{i\leqslant 0} \mathfrak{g}\,\lambda^i, \quad \mathcal{L}(\mathfrak{g})^*_- = \bigoplus_{i>0} \mathfrak{g}\,\lambda^i.$$

Since $\mathcal{L}(\mathfrak{g})_r$ is isomorphic to the direct sum of complementary subalgebras $\mathcal{L}(\mathfrak{g})_+$ and $\mathcal{L}(\mathfrak{g})_-$, its coadjoint orbits are direct products of coadjoint orbits of $\mathcal{L}(\mathfrak{g})_+$ and $\mathcal{L}(\mathfrak{g})_-$ lying in the complementary subspaces $\mathcal{L}(\mathfrak{g})^*_+ = \mathcal{L}(\mathfrak{g})^\perp_-$ and $\mathcal{L}(\mathfrak{g})^*_- = \mathcal{L}(\mathfrak{g})^\perp_+$ of the dual space $\mathcal{L}(\mathfrak{g})^*$. More generally, Poisson subspaces in $\mathcal{L}(\mathfrak{g})^*_r$ are direct products of Poisson subspaces in $\mathcal{L}(\mathfrak{g})^*_+$ and in $\mathcal{L}(\mathfrak{g})^*_-$. In order to give a coordinate expression for the r-bracket in $\mathcal{L}(\mathfrak{g})^*$

defined by the decomposition (3.3), let us choose a basis $\{e^a\}$ in $\mathfrak{g}$ and let C_c^{ab} be the structure constants of $\mathfrak{g}$:

$$[e^a, e^b] = \sum C_c^{ab} e^c.$$

Let

$$L(\lambda) = \sum_i u_i \lambda^i, \ u_i \in \mathfrak{g}^*. \tag{3.4}$$

We define linear functionals u_i^a on $\mathcal{L}(\mathfrak{g})^*$ by $u_i^a[L] = \langle u_i, e^a \rangle$. Then

$$\left\{ u_i^a, u_j^b \right\} = \varepsilon_{ij} C_c^{ab} u_{i+j}^c, \tag{3.5}$$

where

$$\varepsilon_{ij} = \begin{cases} 1 & \text{for } i, j \leq 0, \\ -1 & \text{for } i, j > 0, \\ 0 & \text{for } i \leq 0, \ j > 0. \end{cases} \tag{3.6}$$

The invariants of $\mathcal{L}(\mathfrak{g})$ are easily described.

LEMMA 3.1. *Let φ be an invariant polynomial on $\mathfrak{g}^*$. For any $n, m \in \mathbb{Z}$ and $L \in \mathcal{L}(\mathfrak{g})^*$ set*

$$\varphi_{mn}[L] = \mathrm{Res}_{\lambda=0} \lambda^{-n} \varphi(\lambda^m X(\lambda)).$$

Then $\varphi_{mn} \in I(\mathcal{L}(\mathfrak{g}, \sigma))$.

Formula 2.28 immediately implies that for any $m \geq -1$, $n \geq 0$ the subspace

$$\bigoplus_{-m}^{n} \mathfrak{g} \lambda^i \subset \mathcal{L}(\mathfrak{g})^*$$

is a Poisson subspace of the r-bracket. In other words, the subspaces

$$\bigoplus_{-m}^{-1} \mathfrak{g} \lambda^i \subset \mathcal{L}(\mathfrak{g})_+^*, \qquad \bigoplus_{0}^{n} \mathfrak{g} \lambda^i \subset \mathcal{L}(\mathfrak{g})_-^*$$

are invariant under the coadjoint action of the subalgebras $\mathcal{L}(\mathfrak{g})_+$ and $\mathcal{L}(\mathfrak{g})_-$, respectively.

Thus, in spite of the fact that the original algebra $\mathcal{L}(\mathfrak{g}, \sigma)$ is infinite-dimensional, the orbits of the r-bracket in the polynomial dual of $\mathcal{L}(\mathfrak{g}, \sigma)$ are finite-dimensional. Since the ring of invariants of $\mathcal{L}(\mathfrak{g}, \sigma))$ has infinitely many independent generators, it is natural to expect that when restricted to an orbit of the r-bracket, these invariants become a complete Poisson-commuting family. One can prove that this is indeed true for almost all orbits using the technique of algebraic geometry.

REMARK 3.2. In applications it is sometimes useful to consider Lax operators which possess an additional discrete symmetry. This symmetry is accounted for by the notion of *twisted loop algebras.* Let $\sigma \in \operatorname{Aut}\mathfrak{g}$ be an automorphism of $\mathfrak{g}$ of finite order n. Put

$$\mathcal{L}(\mathfrak{g}, \sigma) = \{X \in \mathcal{L}(\mathfrak{g})\colon X(\varepsilon\lambda) = \sigma X(\lambda)\},$$

where $\varepsilon = \exp 2\pi i/n$. (If $n > 2$, it is assumed that $\mathfrak{g}$ is complex.) All constructions described above easily extend to the case of twisted loop algebras. The classification of all automorphisms of finite order of semisimple Lie algebras has its important counterpart in the classification of gradings of the associated loop algebras [**Kac**]. Different gradings define on loop algebras different structures of a Lie dialgebra. It is not difficult to classify all graded r-matrices, for which $\operatorname{Ker} r_\pm$ and $\operatorname{Im} r_\pm$ are graded subspaces in $\mathcal{L}(\mathfrak{g})$. However, general classification of different structures of a Lie dialgebra on affine Lie algebras is still unknown. The situation here is much less rigid than in the better known case of Lie bialgebras which will discuss in Lecture 4.

3.2. Factorization Theorem for Loop Algebras

The following theorem is the specialization of Theorem 2.8 for the case of Lax equations on loop algebras.

THEOREM 3.3. *(i) Invariant functionals $\varphi_{n,m}$ give rise to equations of motion on $\mathcal{L}(\mathfrak{g}) \subset \mathcal{L}(\mathfrak{g})^*_r$ which are Hamiltonian with respect to the Lie–Poisson bracket of $\mathcal{L}(\mathfrak{g})_r$ and admit a Lax representation of the following form:*

$$\frac{dL}{dt} = [L, M_\pm], \quad M_\pm = \pm P_\pm(\operatorname{grad}\varphi_{n,m}[L]). \tag{3.7}$$

(ii) The integral curve of (3.7) *which starts at L_0 is given by*

$$L(t,\lambda) = g_\pm(t,\lambda)^{-1} L_0(\lambda) g_\pm(t,\lambda), \tag{3.8}$$

where $g_+(t,\cdot)$, $g_-(t,\cdot)$ are matrix valued functions which are holomorphic (along with their inverses) in $\mathbb{C}$ and in $\mathbb{C}P_1 \setminus \{0\}$, respectively, such that

$$\exp t \operatorname{grad}\varphi_{n,m}[L](\lambda) = g_+(t,\lambda) g_-(t,\lambda)^{-1}. \tag{3.9}$$

Factorization problem (3.9) is called *matrix Riemann problem.* Notice that $\operatorname{grad}\varphi_{n,m}[L]$ is a Laurent polynomial and hence is regular in the punctured Riemann sphere. This implies the following important geometrical interpretation of the Riemann problem.

Let us consider the covering of the Riemann sphere with open domains

$$U_+ = \mathbb{C}P_1 \setminus \{\infty\}, \quad U_- = \mathbb{C}P_1 \setminus \{0\}, \quad U_+ \cap U_- = \mathbb{C}^*.$$

Let us suppose that $\mathfrak{g}$ is a matrix Lie algebra, i.e., $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$. Let $G \subset GL(n)$ be the corresponding Lie group. The group $\mathcal{L}(G)$ consisting of holomorphic mappings $g\colon \mathbb{C}^* \to G$ is called the *loop group* of G. A function $g \in \mathcal{L}(G)$ may be regarded as the transition function of an n-dimensional holomorphic vector bundle over $\mathbb{C}P_1$. It is well known that not all holomorphic bundles over $\mathbb{C}P_1$ are holomorphically trivial. Namely, according to the classical Birkhoff–Grothendieck theorem, an n-dimensional holomorphic vector bundle over $\mathbb{C}P_1$ is isomorphic to the direct sum of linear bundles; their degrees $d_1, \dots, d_n$ form a complete set of holomorphic invariants of the given bundle. (Thus the moduli space of holomorphic vector bundle over $\mathbb{C}P_1$ is discrete.) In the language of transition functions this means that an arbitrary function $g \in \mathcal{L}(GL(n))$ admits a factorization of the following form:

$$g = g_+(t, \lambda) d(\lambda) g_-(t, \lambda)^{-1},$$

where $g_\pm$ are regular (along with their inverses) in $U_\pm$ and

$$d(\lambda) = \operatorname{diag}(\lambda^{d_1}, \dots, \lambda^{d_n})$$

Hence the special factorization (3.9) is possible if and only if all "partial indices" $d_1, \dots, d_n$ associated with the transition function $\exp t \operatorname{grad} \varphi_{m,n}$ are zero. One can show that this is true at least for sufficiently small $t \in \mathbb{C}$.

PROPOSITION 3.4. *(Gohberg and Feldman* **[GF]**.*) Let $g \in \mathcal{L}(GL(n))$ and let $\sum_{-\infty}^{\infty} c_i \lambda^i$ be the Fourier expansion of the function $g(\lambda) - \mathbf{1}$. If $\sum_{-\infty}^{\infty} ||c_i|| < 1$, factorization problem $g = g_+(t, \lambda) g_-(t, \lambda)^{-1}$ is solvable, i.e., all partial indices associated with g are zero.*

Clearly, the solution of (3.9) is unique if it exists; indeed, if $g_+ g_-^{-1} = h_+ h_-^{-1}$, the function $g_+^{-1} h_+ = g_-^{-1} h_-$ is regular in the entire Riemann sphere $\mathbb{C}P^1$ and hence is a constant. The normalization condition implies that $g_+ = h_+$.

Proposition 3.4 easily implies the solvability (for small t) of the factorization problem for other Lie groups.

PROPOSITION 3.5. *Let $G \subset GL(n, \mathbb{C})$ be a matrix group, $M \in \mathcal{L}(\mathfrak{g}, \sigma)$. Let $g_\pm(t)$ be the solution of the factorization problem* (3.9) *in $\mathcal{L}(GL(n, \mathbb{C}))$. Then $g_\pm(t) \in \mathcal{L}(G, \sigma)_\pm$.*

The exceptional values $t \in \mathbb{C}$ for which the problem (3.9) is not solvable form a discrete subset in $\mathbb{C}$; for these values t the integral curve $L(t)$ has a pole; in other words, the integral curve of the Lax equation "escapes to infinity".

3.3. Rational r-matrix and Multi-pole Lax Equations

Loop algebras decompositions discussed in the previous lecture give rise to Lax matrices which are rational functions on the Riemann sphere with poles only at $\lambda = 0, \infty$. After an appropriate completion, loop algebras admit different decompositions into the sum of two subalgebras which yield Lax equations with arbitrary poles. This construction can be generalized to include Lax equations with spectral parameter on an elliptic curve **[ReyS2]**.

Let $\mathfrak{g}$ be a complex Lie algebra. Fix a finite set $D \subset \mathbb{C}P_1$ (we assume that $\infty \in D$). Let $R_D(\mathfrak{g})$ be the algebra of rational functions with values in $\mathfrak{g}$ that are regular outside D. For $\nu \in D$ let λ_ν be the local parameter at ν, i.e. $\lambda_\nu = \lambda - \nu$ for $\nu \neq \infty$, $\lambda_\infty = 1/\lambda$. By definition, the local algebra (the localization of $R_D(\mathfrak{g})$ at $\nu \in D$) is the algebra of formal Laurent series in local parameter λ_ν with coefficients in $\mathfrak{g}$,

$$\mathfrak{L}(\mathfrak{g})_\nu = \mathfrak{g} \otimes \mathbb{C}((\lambda_\nu)).$$

Clearly, $\mathfrak{L}(\mathfrak{g})_\nu$ is a completion of the polynomial loop algebra $\mathcal{L}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. Let

$$\mathfrak{L}(\mathfrak{g})_\nu^+ = \mathfrak{g} \otimes \mathbb{C}[[\lambda_\nu]] \tag{3.10}$$

be its subalgebra consisting of formal Taylor series. For $\nu = \infty$ we put

$$\mathfrak{L}(\mathfrak{g})_\infty^+ = \mathfrak{g} \otimes \lambda^{-1}\mathbb{C}[[\lambda_\infty]]; \tag{3.11}$$

(in other words, $\mathfrak{L}(\mathfrak{g})_\infty^+$ consists of formal power series in $\lambda_\infty = 1/\lambda$ without constant term). Put

$$\mathfrak{L}(\mathfrak{g})_D = \bigoplus_{\nu \in D} \mathfrak{L}(\mathfrak{g})_\nu, \quad \mathfrak{L}(\mathfrak{g})_D^+ = \bigoplus_{\nu \in D} \mathfrak{L}(\mathfrak{g})_\nu^+ \tag{3.12}$$

(direct sum of Lie algebras). There is a natural embedding

$$R_D(\mathfrak{g}) \longrightarrow \mathfrak{L}(\mathfrak{g})_D,$$

which assigns to a rational function $L \in R_D(\mathfrak{g})$ the set of its Laurent expansions at the points $\nu \in D$. In the sequel we shall frequently identify $R_D(\mathfrak{g})$ with its canonical image in $\mathfrak{L}(\mathfrak{g})_D$.

PROPOSITION 3.6. *We have*

$$\mathfrak{L}(\mathfrak{g})_D = R_D(\mathfrak{g}) \dot{+} \mathfrak{L}(\mathfrak{g})_D^+. \tag{3.13}$$

PROOF. Our assertion is basically a reformulation of the following classical statement from the elementary theory of rational functions: any rational function of the Riemann sphere admits a unique decomposition into elementary fractions and conversely, there is a unique rational function on the sphere with prescribed singularities and with the given principal parts at its poles. [1]

Indeed, fix an element $X = \{X_\nu\}_{\nu \in D} \in \mathfrak{L}(\mathfrak{g})_D$; for $\nu \in D$ let us denote by X_ν^- the principal part of the Laurent series X_ν, i. e., a polynomial in λ_ν^{-1} without constant term such that $X_\nu - X_\nu^- \in \mathfrak{L}(\mathfrak{g})_\nu^+$. (For $\nu = \infty$ the constant term is included into the principal part.) There exists a unique rational function $X^0 \in R_D(\mathfrak{g})$ whose principal parts at $\nu \in D$ coincide with X_ν^-. (Notice the rôle of the normalization condition at $\nu = \infty$.) Clearly, $X - X^0 \in \mathfrak{L}(\mathfrak{g})_D^+$. $\square$

Let P_0 be the projection operator onto $R_D(\mathfrak{g})$ parallel to $\mathfrak{L}(\mathfrak{g})_D^+$. It is given by the following formula

$$(P_0 X)(\lambda) = \sum_{\nu \in D} \mathrm{Res}_\nu(r(\lambda, \mu) X_\nu(\mu_\nu) d\mu), \tag{3.14}$$

[1] The construction of a meromorphic function from the given set of its principal parts is the so called *Mittag-Leffler problem*.

where

$$r(\lambda,\mu) = \frac{I}{\mu-\lambda} \tag{3.15}$$

and I is the identity operator in $\mathfrak{g}$. The kernel (3.15) (which is essentially the Cauchy kernel) is called the *rational r-matrix.*

Let (,) be a non-degenerate invariant bilinear form on $\mathfrak{g}$. We put

$$\langle X, Y\rangle = \sum_{\nu\in D} \operatorname{Res}_\nu (X_\nu, Y_\nu)\, d\lambda. \tag{3.16}$$

PROPOSITION 3.7. *(i) The bilinear form* (3.16) *is invariant and non-degenerate on $\mathfrak{L}(\mathfrak{g})_D$. (ii) The subalgebras $R_D(\mathfrak{g})$, $\mathfrak{L}(\mathfrak{g})_D^+$ are isotropic.*

PROOF. The isotropy of $R_D(\mathfrak{g})$ is a reformulation of the classical assertion: the sum of residues of a rational function is equal to zero. It is obvious that for $\nu \neq \infty$ the residues of Taylor series in local parameter are zero. For $\nu = \infty$ $\mathfrak{L}(\mathfrak{g})_\infty^+$ is a Taylor series in $1/\lambda$ without constant term; the product of two such series begins with the term of order λ^{-2} and hence its residue at infinity is also zero.

COROLLARY 3.8. $(\mathfrak{L}(\mathfrak{g})_D^+)^* \simeq R_D(\mathfrak{g})$.

We shall apply Theorem 2.5 to the decomposition (3.13). The coadjoint orbits of $(\text{Ł}(\mathfrak{g})_D)_r$ are direct products of the coadjoint orbits of the complementary subalgebras $\mathfrak{L}(\mathfrak{g})_D^+$, which lie in $R_D(\mathfrak{g})$, and the coadjoint orbits of $R_D(\mathfrak{g})$, which lie in $\mathfrak{L}(\mathfrak{g})_D^+$. Since we are interested in Lax equations with rational Lax matrices, we shall consider only orbits of $\mathfrak{L}(\mathfrak{g})_D^+$ in $R_D(\mathfrak{g})$. In other words, we take the trivial zero orbit of the complementary subalgebra $R_D(\mathfrak{g})$. In order to understand the Lie–Poisson structure in the space $R_D(\mathfrak{g})$ observe first that there is another model of the dual space $\mathfrak{L}(\mathfrak{g})_D^{+*}$ associated with the decomposition

$$\mathfrak{L}(\mathfrak{g})_D = \mathfrak{L}(\mathfrak{g})_D^+ \dotplus \mathfrak{L}(\mathfrak{g})_D^-, \tag{3.17}$$

where

$$\mathfrak{L}(\mathfrak{g})_D^- = \bigoplus_{\nu\in D} \mathfrak{L}(\mathfrak{g})_\nu^-,$$

$$\mathfrak{L}(\mathfrak{g})_\nu^- = \mathfrak{g} \oplus \lambda_\nu^{-1}\mathbb{C}[\lambda_\nu^{-1}] \quad (\nu\neq\infty), \quad \mathfrak{L}(\mathfrak{g})_\infty^- = \mathfrak{g}\oplus\mathbb{C}[\lambda].$$

Clearly, decomposition (3.17) allows to identify the dual space $(\mathfrak{L}(\mathfrak{g})_D^+)^*$ with $\mathfrak{L}(\mathfrak{g})_D^-$. An element

$$X = \left\{\sum_i X_{i\nu}\lambda_\nu^{-i}\right\}_{\nu\in D} \in \mathfrak{L}(\mathfrak{g})_D^-$$

may be regarded as the set of principal parts of a rational function (with values in $\mathfrak{g}$) at the points $\nu \in D$. The two realizations of the dual are related by the mapping $\mathfrak{L}(\mathfrak{g})_D^- \to R_D(\mathfrak{g}))$ which assigns to X the function $f_X = \sum_\nu \sum_i X_{i\nu}\lambda_\nu^{-i}$. The realization of $(\mathfrak{L}(\mathfrak{g})_D^+)^*$ as $\mathfrak{L}(\mathfrak{g})_D^-$ makes quite transparent the Lie–Poisson structure in the space of rational functions in terms of the decomposition of a function into partial fractions (or principal parts). Namely, the Poisson submanifolds in $R_D(\mathfrak{g})$ are direct products over $\nu \in D$ of the Poisson submanifolds of local algebras $\mathfrak{L}(\mathfrak{g})^+_\nu$. In particular, coadjoint orbits in $R_D(\mathfrak{g})$ are direct products of coadjoint orbits of local algebras in in $\mathfrak{L}(\mathfrak{g})^-_\nu$.

From the discussion in Section 3.1 we know that $\left\{\sum_{1\leqslant i\leqslant N} X_i\lambda^{-i}\right\}$ is a Poisson subspace in $(\mathfrak{L}(\mathfrak{g})^+_\nu)^*$. Let $\mathfrak{D} = \sum_{\nu\in D} n_\nu \cdot \nu$ be a divisor supported in D; set

$$R(\mathfrak{g};\, \mathfrak{D}) = \left\{ \sum_{\nu} \sum_{i\leqslant n_\nu} X_{\nu i}\lambda^{-i} \right\}.$$

In other words, $R(\mathfrak{g};\, \mathfrak{D})$ consists of rational functions which are regular outside D and such that the order of their pole at $\nu \in D$ does not exceed n_ν. Then $R(\mathfrak{g};\, \mathfrak{D})$ is a Poisson subspace in $R_D(\mathfrak{g})$.

REMARK 3.9. The use of formal power series is well suited for the study of coadjoint orbits in $R_D(\mathfrak{g})$; they are also sufficient for the definition of the pronilpotent Lie groups which correspond to the Lie algebra $\mathfrak{L}(\mathfrak{g})^+_D$ and its subalgebras. However, in order to study the global properties of Lax equations supported on $R_D(\mathfrak{g})$, we also need a Lie group which corresponds to the "big" algebra $\mathfrak{L}(\mathfrak{g})_D$; in order to define such a group, we need to modify the topology in the underlying Lie algebra. A natural choice consists in the use of the *Wiener loop algebras* $(\mathfrak{L}(\mathfrak{g})_D)_{\mathcal{W}}$ in which formal Laurent series are replaced by the absolutely convergent Fourier series. The main advantage of the Wiener algebra is that by the Wiener–Levy theorem the exponential of an absolutely convergent series is again absolutely convergent, which allows to define a Lie group with the Lie algebra $(\mathfrak{L}(\mathfrak{g})_D)_{\mathcal{W}}$. For our goals there is no need of a truly serious theory of the corresponding Wiener Lie groups; it is sufficient to describe their coadjoint invariants and their gradients and to be able to define one-parameter subgroups $\exp t \operatorname{grad}\varphi$ generated by these gradients.

Let us notice, first of all, that the ring of invariants of $\mathfrak{L}(\mathfrak{g})_D$ is generated by the functionals of the following form:

$$\varphi_{f,\mu}\colon X = \{X_\nu\}_{\nu\in D} \longmapsto \operatorname{Res}_\mu(f(\lambda_\mu)\varphi(X_\mu(\lambda_\mu))d\lambda), \tag{3.18}$$

where $\mu \in D$, $\varphi \in I(\mathfrak{g})$ and $f_\mu(\lambda_\mu) \in \mathbb{C}((\lambda_\mu))$ is a formal Laurent series. Since we are interested only in restrictions of these functionals to the finite dimensional Poisson subspaces described above, we may assume without loss of generality that f is a *Laurent polynomial.* (Indeed, one can drop all terms of f except for a finite number without affecting the value of our functional on the given finite dimensional subspace in $R(\mathfrak{g};\, \mathfrak{D})$.) Let $M_\varphi = \{M_\nu\}_{\nu\in D}$ be the gradient of $\varphi[X]$; as usual, we have

$$\operatorname{ad} M_\nu[X] \cdot X_\nu = 0 \quad \text{for all} \quad \nu \in D.$$

If X_ν is the Laurent expansion of a rational function $X \in R_D(\mathfrak{g})$, all series $M_\nu[X]$ are locally convergent(we assume that all series $f_\mu(\lambda_\mu)$ in the definition of basic invariants are Laurent polynomials).

PROPOSITION 3.10. *(i) The Hamiltonian equation of motion on $R_D(\mathfrak{g})$, generated by an invariant Hamiltonian φ, has the Lax form*

$$\frac{dL}{dt} = [L,\, M_0], \quad L \in R_D(\mathfrak{g}), \quad M_0 = P_0(M_\varphi[L]). \tag{3.19}$$

(ii) For sufficiently small $t \in \mathbb{C}$ there exists a holomorphic function $g_0(\cdot, t)$ regular in $\mathbb{C}P_1 \setminus D$ with values in the Lie group G such that for all $\nu \in D$ the functions $\exp tM_\nu[L] \cdot g_0(\lambda, t)^{-1}$ *are regular in the vicinity of ν. The integral curve of the Lax equation* (3.19)*, starting from L is given by*

$$L(t) = \mathrm{Ad} g_0(\lambda, t) \cdot L. \tag{3.20}$$

The factorization problem described in Proposition 3.10 is the so-called *matrix Cousin problem.* Its geometric interpretation is the same as for the matrix Riemann problem. Namely, let $U_\nu \subset \mathbb{C}P_1$ be a small disc around ν. Consider the covering of the Riemann sphere by the open domains U_ν ($\nu \in D$), $U_0 = \mathbb{C}P_1 \setminus D$; we can assume that all discs U_ν are so small that $U_\mu \cap U_\nu = \emptyset$ for $\mu \neq \nu$, $U_0 \cap U_\nu = U_\nu \setminus \{\nu\}$ and that all series $\exp tM_\nu(\lambda_\nu)$ are absolutely convergent in $U_\nu \setminus \{\nu\}$. The functions $\exp tM_\nu$ form the system of transition functions of a holomorphic vector bundle over $\mathbb{C}P_1$ with respect to the given covering; the Cousin problem consists in its holomorphic trivialization.

REMARK 3.11. The choice of the set D in the definition of our basic Lie algebra $\mathfrak{L}(\mathfrak{g})_D$ is, of course, arbitrary. It would be more natural to deal, instead of $\mathfrak{L}(\mathfrak{g})_D$, with the algebra of *adèles*

$$\mathfrak{L}(\mathfrak{g})_A = \coprod_{\nu \in \mathbb{C}P_1} \mathfrak{g} \otimes \mathbb{C}((\lambda_\nu)).$$

(By definition, the elements of $\mathfrak{L}(\mathfrak{g})_A$ are the sets $X = \{X_\nu\}_{\nu \in \mathbb{C}P_1}$ such that all X_ν, except for a finite number of them, belong to $\mathfrak{L}(\mathfrak{g})_\nu^+$.) The canonical embedding of the algebra $\mathfrak{g}(\lambda)$ of all rational functions with values in $\mathfrak{g}$ into the algebra of adeles $\mathfrak{L}(\mathfrak{g})_A$ assigns to a rational function the set of *all* its Laurent expansions at all points of the Riemann sphere. All constructions and arguments described above extend to the adèlic setting without any major change.

REMARK 3.12. There is another approach to Lax equations with rational Lax matrices which avoids localization. Let us fix two disjoint sets of poles $D, D' \subset \mathbb{C}P_1$, $\infty \in D$, and let $R_{D\cup D'}(\mathfrak{g})$ be the algebra of rational functions with values in $\mathfrak{g}$ which are regular outside of $D \cup D'$. Then

$$R_{D\cup D'}(\mathfrak{g}) = R_D(\mathfrak{g}) \dot{+} {}^0R_{D'}(\mathfrak{g}), \tag{3.21}$$

where the functions from ${}^0R_{D'}(\mathfrak{g})$ are required to be zero at $X(\infty) = 0$. Define the inner product on $R_{D\cup D'}(\mathfrak{g})$ by

$$\langle X, Y\rangle = \sum_{\nu\in D} \mathrm{Res}_\nu (X_\nu, Y_\nu) d\lambda. \tag{3.22}$$

This pairing sets the subspaces $R_D(\mathfrak{g})$, ${}^0R_{D'}(\mathfrak{g})$ into duality, and we may equip $R_D(\mathfrak{g})$ with the Lie–Poisson bracket associated with ${}^0R_{D'}(\mathfrak{g})$. It is easy to see that this construction reduces to the preceding one. Namely, consider the embedding

$$i_D\colon R_{D\cup D'}(\mathfrak{g}) \hookrightarrow \mathfrak{L}(\mathfrak{g})_D,$$

which assigns to each function $X \in R_{D\cup D'}(\mathfrak{g})$ the set of all its Laurent expansions at the points $\nu \in D$. Clearly, i_D maps ${}^0R_{D'}(\mathfrak{g})$ into the subalgebra $\mathfrak{L}(\mathfrak{g})^+_D$, and the bilinear form (3.22) is compatible with the form (3.16) on $\mathfrak{L}(\mathfrak{g})_D$. Hence the subspaces $R(\mathfrak{g};\mathfrak{D}) \subset R_D(\mathfrak{g})$, associated with divisors $\mathfrak{D} = \sum_{\nu\in D} n_\nu \cdot \nu$ are Poisson subspaces in $R_D(\mathfrak{g}) \subset {}^0R_{D'}(\mathfrak{g})^*$ and the Poisson structure in these subspaces coincides with the one induced by $\mathfrak{L}(\mathfrak{g})^+_D$. It is also easy to establish a correspondence between Lax equations on $R_D(\mathfrak{g})$ that are constructed with the use of decompositions (3.13) and (3.21). The choice of D' in (3.21) is arbitrary and does not affect the Poisson structure on the subspaces $R(\mathfrak{g};\mathfrak{D}) \subset R_D(\mathfrak{g})$ or the supply of invariant Hamiltonians.

LECTURE 4

Poisson Lie Groups

4.1. Introduction

The Poisson structures which we studied so far are the Lie–Poisson brackets associated with various Lie algebras. An important characteristic of these Poisson structures is connected with the additive structure of the underlying linear space $\mathfrak{g}^*$. We start with the following general definition.

DEFINITION 4.1. Let $\mathcal{M}$, $\mathcal{N}$ be two Poisson manifolds. Their product is the manifold $\mathcal{M} \times \mathcal{N}$ equipped with the Poisson bracket

$$\{\varphi, \psi\}_{\mathcal{M}\times\mathcal{N}}(x, y) = \big\{\varphi(\,\cdot\,, y), \psi(\,\cdot\,, y)\big\}_{\mathcal{M}}(x) + \\ + \big\{\varphi(x, \,\cdot\,), \psi(x, \,\cdot\,)\big\}_{\mathcal{N}}(y). \tag{4.1}$$

In other words, (4.1) is the unique Poisson structure on $\mathcal{M} \times \mathcal{N}$ such that:

(i) Natural projections $p_{\mathcal{M}} \colon \mathcal{M} \times \mathcal{N} \to \mathcal{M}$, $p_{\mathcal{N}} \colon \mathcal{M} \times \mathcal{N} \to \mathcal{N}$ are Poisson mappings;

(ii) $\{p_{\mathcal{M}}^*\varphi, p_{\mathcal{N}}^*\psi\} = 0$ for any $\varphi \in C^\infty(\mathcal{M})$, $\psi \in C^\infty(\mathcal{N})$.

PROPOSITION 4.2. *(i) Let $\mathfrak{g}^*$ be the dual of a Lie algebra equipped with the Lie–Poisson bracket. Then*

$$\mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^* : (L, L') \mapsto L + L' \tag{4.2}$$

is a Poisson mapping.

(ii) Conversely, if V is a linear space equipped with a Poisson structure such that the addition $V \times V \to V$ is a Poisson mapping, then the dual of V is a Lie algebra and the Poisson structure on V coincides with its Lie–Poisson bracket.

The definition of Poisson Lie groups generalizes the linearity property (4.2) to the non-abelian setting. We shall see in Lecture 6 that the corresponding Poisson structures are well adapted for the study of difference Lax equations. In this case the Lax matrix should be treated as an element of a Lie group rather than that of a Lie algebra or of its dual.

4.2. Main Definitions. Poisson Lie Groups and Lie Bialgebras

DEFINITION 4.3. A Poisson Lie group is a Lie group G equipped with a Poisson bracket such that multiplication in G defines a Poisson mapping $G \times G \to G$.

Poisson groups form a category where the morphisms are Lie group homomorphisms which are also Poisson mappings.

EXAMPLE 4.4. (i) Let $\mathfrak{g}^*$ be the dual of a Lie algebra $\mathfrak{g}$ equipped with the Lie–Poisson bracket. The additive group of the space $\mathfrak{g}^*$ is a Poisson Lie group. This example was our starting point in the previous section.

(ii) Any Lie group G equipped with the zero Poisson is a Poisson Lie group.

We shall see later that examples (i), (ii) are dual to each other. The duality theory for general Poisson Lie groups is a key part of their geometry which generalizes the theory of coadjoint orbits.

4.2.1. The Hopf Algebra Language. There is a dual way to formulate the multiplicativity axiom using the language of Hopf algebras. Recall that a Hopf algebra over k (the ground field k is either $\mathbb{R}$ or $\mathbb{C}$) is a k-algebra A equipped with homomorphisms $\Delta : A \to A \otimes A$ (the coproduct) and $\varepsilon : A \to k$ (the counit) such that the following diagrams are commutative:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \otimes A \arrow[d, "\Delta\otimes\mathrm{id}"] \\ A \otimes A \arrow[r, "\mathrm{id}\otimes\Delta"'] & A \otimes A \otimes A \end{tikzcd} \tag{4.3}$$

$$\begin{tikzcd} & A \arrow[dr, "\Delta"] & \\ A \otimes k \arrow[ur, "\simeq"] & & A \otimes A \arrow[ll, "\mathrm{id}\otimes\varepsilon"] \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dr, "\Delta"] & \\ k \otimes A \arrow[ur, "\simeq"] & & A \otimes A \arrow[ll, "\varepsilon\otimes\mathrm{id}"] \end{tikzcd} \tag{4.4}$$

Let $P \in \mathrm{End}(A \otimes A)$ be the permutation map defined by $P(x \otimes y) = y \otimes x$. A Hopf algebra is *cocommutative* if $P \circ \Delta = \Delta$. If A is a topological algebra, we replace the algebraic tensor product $A \otimes A$ in the definition by its appropriate completion. The commutative algebra $C^\infty(G)$ has a natural structure of topological Hopf algebra with the coproduct $\Delta : C^\infty(G) \to C^\infty(G \times G)$ and the counit $\varepsilon : C^\infty(G) \to \mathbb{R}$ defined by

$$\Delta\varphi(x, y) = \varphi(xy), \quad x, y \in G, \quad \varepsilon(\varphi) = \varphi(e).$$

The axioms (4.3), (4.4) immediately follow from the associativity of multiplication in G and the standard properties of the unit element e. If the group G is commutative, the coproduct in $C^\infty(G)$ is cocommutative.

Suppose that A is commutative and carries a Poisson structure. We extend it to $A \otimes A$ by setting

$$\left\{u \otimes v,\, u' \otimes v'\right\} = \left\{u, u'\right\} \otimes vv' + uu' \otimes \left\{v, v'\right\}. \tag{4.5}$$

DEFINITION 4.5. A commutative Hopf algebra equipped with a Poisson bracket is called a Poisson Hopf algebra if

$$\Delta\left\{u,\, v\right\} = \left\{\Delta u,\, \Delta v\right\} \quad \text{for any} \quad u, v \in A \tag{4.6}$$

It is easy to see that (4.5) is a reformulation of (4.1) and (4.6) is equivalent to the multiplicativity axiom for Poisson brackets on a Lie group G. Hence the algebra of functions on a Poisson Lie group has the structure of a Poisson Hopf algebra. The language of Hopf algebras plays a key rôle in the quantization of Poisson Lie groups developed by Drinfeld [**D2**].

4.2.2. Poisson Tensors and Tangent Lie Bialgebras. Let us now formulate the multiplicativity property of a Poisson bracket more explicitly. Let λ_x, ρ_x be the left and right translation operators on $C^\infty(G)$ by an element $x \in G$,

$$\lambda_x \varphi(y) = \varphi(xy),\ \rho_x \varphi(y) = \varphi(yx).$$

Multiplication in G induces a Poisson mapping $G \times G \to G$ if

$$\{\varphi, \psi\}(xy) = \{\lambda_x \varphi, \lambda_x \psi\}(y) + \{\rho_y \varphi, \rho_y \psi\}(x). \tag{4.7}$$

More explicitly, for any two functions $\varphi, \psi \in C^\infty(G)$ let us set $\Phi(x, y) = \varphi(xy), \Psi(x, y) = \psi(xy),\ \Phi, \Psi \in C^\infty(G \times G)$. When the bracket $\{\Phi, \Psi\}$ is computed, Φ, Ψ are regarded as functions of *two* variables, i. e., we compute the derivatives of Φ and Ψ with respect to x for fixed y and with respect to y for fixed x and add up both terms; on the other hand, we can compute the bracket $\{\varphi, \psi\}$ for functions of *one* variable $z \in G$ and then put $z = xy$. Multiplicativity means that both results are identical.[1]

Recall that any Poisson bracket is bilinear in derivatives of functions. It is convenient to write down Poisson brackets on a Lie group in the right- or left-invariant frame. Define the left and right differentials of a function $\varphi \in C^\infty(G)$ by the formulae

$$\begin{gathered}\langle D\varphi(x), X\rangle = \left(\tfrac{d}{dt}\right)_{t=0} \varphi(e^{tX}x),\ \langle D'\varphi(x), X\rangle = \left(\tfrac{d}{dt}\right)_{t=0} \varphi(xe^{tX}),\\ X \in \mathfrak{g},\ D\varphi(x), D'\varphi(x) \in \mathfrak{g}^*.\end{gathered} \tag{4.8}$$

Let us define the Poisson operators $\eta, \eta' \colon G \to \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g})$ which correspond to our bracket by setting

$$\{\varphi, \psi\}(x) = \langle \eta(x) D\varphi(x), D\psi\rangle = \langle \eta'(x) D'\varphi(x), D'\psi\rangle. \tag{4.9}$$

PROPOSITION 4.6. *Suppose G is a Poisson Lie group; then the functions η, η' satisfy the functional equations*

$$\begin{aligned}\eta(xy) &= \mathrm{Ad}\, x \circ \eta(y) \circ \mathrm{Ad}^* x^{-1} + \eta(x),\\ \eta'(xy) &= \mathrm{Ad}\, y^{-1} \circ \eta'(x) \circ \mathrm{Ad}^* y + \eta'(y).\end{aligned} \tag{4.10}$$

PROOF. Obviously,

$$D(\lambda_x \varphi)(y) = \mathrm{Ad}^* x^{-1} \circ D\varphi(xy),\ D'(\lambda_x \varphi)(y) = D'\varphi(xy). \tag{4.11}$$

Clearly, (4.11) together with (4.7) and (4.9) imply (4.10). □

Functional equations (4.10) mean that η, η' are 1-cocycles on G with values in $\mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g})$.

[1] The reader will notice that (4.7) is still another way to rewrite the Poisson–Hopf axiom (4.6).

It is convenient to identify the space $\operatorname{Hom}(\mathfrak{g}^*, \mathfrak{g})$ with $\mathfrak{g} \otimes \mathfrak{g}$;[2] Under this isomorphism the functional equations for cocycles η, η' become

$$\begin{aligned} \eta(xy) &= (\operatorname{Ad} x \otimes \operatorname{Ad} x) \cdot \eta(y) + \eta(x), \\ \eta'(xy) &= (\operatorname{Ad} y^{-1} \otimes \operatorname{Ad} y^{-1}) \cdot \eta'(x) + \eta'(y). \end{aligned} \tag{4.12}$$

The cocycles η, η' give rise to a cocycle on $\mathfrak{g}$. Indeed, let us put

$$\delta(X) = \left(\tfrac{d}{dt}\right)_{t=0} \eta(e^{tX});$$

then (4.12) immediately implies that

$$\delta([X, Y]) = [X \otimes I + I \otimes X, \delta(Y)] - [Y \otimes I + I \otimes Y, \delta(X)],$$

i. e., δ is a 1-cocycle on $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$ (with the natural structure of a $\mathfrak{g}$-module).

Functional equations (4.10) imply, in particular, that $\eta(e) = \eta'(e) = 0$, hence the Poisson structure on G is always degenerate at the unit element e. Linearizing the Poisson bracket at the point e gives a Lie–Poisson structure on $\mathfrak{g}$, i.e. a Lie algebra structure on $\mathfrak{g}^*$. To be more precise, let $\xi, \xi' \in \mathfrak{g}^*$ and choose φ, $\varphi' \in C^\infty(G)$ such that $d_e\varphi = \xi$, $d_e\varphi' = \xi'$. Put

$$[\xi,\, \xi']_* = d_e\{\varphi,\, \varphi'\}. \tag{4.13}$$

PROPOSITION 4.7. *Formula* (4.13) *defines the structure of a Lie algebra on* $\mathfrak{g}^*$.

PROOF. Formulae (4.9), (4.13) imply

$$[\xi_1,\, \xi_2]_* = \langle d\eta(e)\xi_1,\, \xi_2\rangle. \tag{4.14}$$

Hence the definition (4.13) is unambiguous. The Jacobi identity for (4.13) is obvious. $\square$

DEFINITION 4.8. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^*$ its dual. Suppose there is a Lie algebra structure on $\mathfrak{g}^*$, i.e. a mapping

$$[\,,\,]_* \colon \mathfrak{g}^* \otimes \mathfrak{g}^* \longrightarrow \mathfrak{g}^*,$$

satisfying the Jacobi identity. Lie brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$ are said to be consistent if the dual mapping

$$\delta \colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$$

is a 1-cocycle on $\mathfrak{g}$ (with respect to the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$). A pair $(\mathfrak{g},\, \mathfrak{g}^*)$ with consistent Lie brackets is called a *Lie bialgebra.*

[2]Strictly speaking, the canonical isomorphism $\operatorname{Hom}(\mathfrak{g}^*, \mathfrak{g}) \simeq \mathfrak{g} \otimes \mathfrak{g}$ holds only for finite dimensional Lie algebras; traditionally. η, η' (and the closely related classical r-matrices, see formula (4.15) below) are regarded as elements of $\mathfrak{g} \otimes \mathfrak{g}$ for infinite dimensional Lie algebras as well (e.g., for loop algebras); of course such cocycles are given by singular kernels which belong not to the algebraic tensor product, but rather to its appropriate completion. (See examples in Section 4.5.)

Thus if G is a Poisson Lie group, the pair $(\mathfrak{g},\, \mathfrak{g}^*)$ is a Lie bialgebra. We shall refer to it as the *tangent Lie bialgebra* of G.

It is easy to check straightforwardly that the definition of Lie bialgebras is actually symmetric: if $\delta\colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}$, the mapping $\delta^*\colon \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$, which is dual to the Lie bracket

$$[\,,\,]\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g},$$

is a 1-cocycle on $\mathfrak{g}^*$. (We shall give a proof of this assertion in Section 5.1 below in connection with the theory of *Drinfeld's double*) Hence if $(\mathfrak{g},\, \mathfrak{g}^*)$ is a Lie bialgebra, $(\mathfrak{g}^*,\, \mathfrak{g})$ is also a Lie bialgebra . This duality allows to associate a *dual Poisson group* with the given one.

Definition 4.9. A morphism of Lie bialgebras $p\colon (\mathfrak{g},\, \mathfrak{g}^*) \rightsquigarrow (\mathfrak{h},\, \mathfrak{h}^*)$ is a Lie algebra homomorphism $p\colon \mathfrak{g} \to \mathfrak{h}$ such that the dual map $p^*\colon \mathfrak{h}^* \to \mathfrak{g}^*$ is also a Lie algebra homomorphism.

Proposition 4.10 ("The homomorphism theorem"). *Let* $p\colon\ (\mathfrak{g},\, \mathfrak{g}^*) \rightsquigarrow (\mathfrak{h},\, \mathfrak{h}^*)$ *be a morphism of Lie bialgebras. Then* $\operatorname{Im} p \subset \mathfrak{h}$ *is a Lie subalgebra and* $\operatorname{Ker} p^* \subset \mathfrak{h}^*$ *is an ideal; moreover,* $(\operatorname{Im} p,\, \mathfrak{h}^*/\operatorname{Ker} p^*)$ *is a Lie bialgebra.*

Theorem 4.11. *Let* $(\mathfrak{g},\, \mathfrak{g}^*)$ *be a Lie bialgebra; let* G *be the connected simply connected Lie group which corresponds to* $\mathfrak{g}$. *There is a unique multiplicative Poisson bracket on* G *which makes it a Poisson Lie group with tangent Lie bialgebra* $(\mathfrak{g},\, \mathfrak{g}^*)$.

The difficult part of the proof is the check of the Jacobi identity. We shall postpone it until Section 4.7.

The correspondence between Poisson Lie groups and Lie bialgebras is functorial.

Proposition 4.12. *Let* $G \to H$ *be a morphism of Poisson Lie groups; then the associated tangent map is a morphism of Lie bialgebras. Conversely, any morphism* p *of Lie bialgebras gives ries to a morphism of simply connected Poisson Lie groups such that* p *is its tangent map at the identity.*

The category of Poisson groups is rather wide; some special types of Poisson groups traditionally bear somewhat argotic names. We shall say that G is a *coboundary Poisson group* if the cocycle η which determines the Poisson structure on G is a coboundary. In a similar way, a Lie bialgebra $(\mathfrak{g},\, \mathfrak{g}^*)$ is called a cobounary bialgebra if the cobracket δ on $\mathfrak{g}$ is a coboundary. The tangent Lie bialgebra of a coboundary Poisson group is a coboundary bialgebra and vice versa. Clearly, we have in this case:

$$\begin{aligned} \eta(g) &= \frac{1}{2}(r - \operatorname{Ad} g \otimes \operatorname{Ad} g \cdot r), \\ \delta(X) &= \frac{1}{2}[X \otimes I + I \otimes X,\, r], \end{aligned} \tag{4.15}$$

where $r \in \mathfrak{g} \wedge \mathfrak{g}$ is a fixed element (*classical* r*-matrix*). Notice that if G is semisimple, the first cohomology group $H^1(G, V)$ with values in any G-module V is trivial, and hence any structure of a Poisson group on G is of coboundary type; on the other hand, the dual group of a coboundary Poisson group is in general no longer a coboundary group.

REMARK 4.13. Let $P \in \operatorname{End}(\mathfrak{g} \otimes \mathfrak{g})$ be the permutation operator defined by $P(x \otimes y) = y \otimes x$. In formula (4.15), r need not be a skew symmetric tensor; it is sufficient to admit that its symmetric part $t = r + P(r)$ is $\operatorname{ad}\mathfrak{g}$-invariant, i.e. $[X \otimes I + I \otimes X, t] = 0$ for all $X \in \mathfrak{g}$, and hence also $t = \operatorname{Ad} g \otimes \operatorname{Ad} g \cdot t$ for all $g \in G$. In this case the symmetric part of r drops out from η and δ. We shall see, however, that the presence of t *does* affect the Yang–Baxter identity which should be satisfied by r.

Taking into account the possible non-skew-symmetry of r, the dual Lie bracket of a coboundary Lie bialgebra takes the form

$$[\xi, \xi']_* = \frac{1}{2}(\operatorname{ad}^* r\, \xi \cdot \xi' + \operatorname{ad}^* r^* \xi' \cdot \xi), \tag{4.16}$$

where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is identified with a linear operator acting from $\mathfrak{g}^*$ into $\mathfrak{g}$ (clearly, under this identification $P(r)$ goes into the adjoint operator r^*).

REMARK 4.14. It is useful to compare the definition of Lie bialgebras with that of Lie dialgebras which we discussed in Section 2.2. These definitions are *different* and are related to different notions of classical r-matrix. Indeed, in the case of Lie dialgebras we are dealing with two structures of a Lie algebra on the *same linear space* and the associated classical r-matrix is a linear operator on $\mathfrak{g}$; by contrast, in the case of Lie bialgebras the brackets $[\ ,\ ]$ and $[\ ,\ ]_*$ are defined on *dual linear spaces* $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively, and $r \in \operatorname{Hom}(\mathfrak{g}^*, \mathfrak{g})$. The motivations of these two definitions are very different as well: as we saw in Section 2.2, the definition of Lie dialgebras is motivated by the involutivity theorem (Theorem 2.5), while Lie bialgebras arise in the study of multiplicative Poisson brackets. Our next assertion describes the situation where the two notions match.

PROPOSITION 4.15. *Let $(\mathfrak{g}, \mathfrak{g}_r)$ be a Lie dialgebra. Assume that $\mathfrak{g}$ is equipped with an invariant inner product which allows to identify $\mathfrak{g}$ and $\mathfrak{g}^*$ and that $r \in \operatorname{End}\mathfrak{g}$ is skew symmetric. The Lie algebras $(\mathfrak{g}, \mathfrak{g}_r)$ which are set into duality by means of this inner product form a Lie bialgebra.*

Indeed, the invariance of inner product on $\mathfrak{g}$ means that the operators $\operatorname{ad} X$ and $\operatorname{ad}^* X$ coincide. Under this identification, formulae (2.3) and (4.16) coincide as well[3].

WARNING 1. As we have just seen, the symmetric part of r drops out from (4.16) (on the other hand, it is severely restricted by the condition of $\mathfrak{g}$-invariance). By contrast, formula (2.3) does not imply any particular restrictions on the symmetric part of r and it does not drop out at all from the r-bracket. Hence there exists a wide and important class of Lie dialgebras

[3]The somewhat queer normalization in (4.15) is chosen precisely for this purpose.

which do *not* have the structure of Lie bialgebras[4]. In Chapters 1 and 2 we have encountered numerous examples of classical r-matrices which are not skew with respect to the natural invariant inner product. Such r-do not fit the framework of Poisson Lie groups, although they are quite interesting; in some cases it is possible to associate with them nonlinear Poisson brackets on Lie groups (which are of course not multiplicative!).[5] In common practice, one usually starts with formula (4.16), which means that "genuine" non-skew-symmetric r-matrices are excluded right away. It is mainly for this reason that their classification remains unknown (the very question of this classification is partly hidden by the confusion in the definitions). This classification is certainly much less rigid than in the case of skew r-matrices (where it is 'almost' complete).

4.3. Yang–Baxter Identity and Tensor Formalism

Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a coboundary Lie bialgebra. The Jacobi identity for $\mathfrak{g}^*$ imposes strong restrictions on the choice of r. Let us state them first in the operator form, by analogy with Proposition 2.16.

Given $r \in \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g})$, let us define a linear operator $B_r \in \mathrm{Hom}(\mathfrak{g}^* \wedge \mathfrak{g}^*, \mathfrak{g})$ by

$$B_r(\xi \wedge \eta) = \tfrac{1}{2}[r\xi, r\eta] - r[\xi, \eta]_*. \tag{4.17}$$

PROPOSITION 4.16. *The bracket* (4.16) *satisfies the Jacobi identity if and only if*

$$\mathrm{ad}^* B_r(\xi_1 \wedge \xi_2) \cdot \xi_3 + \mathrm{ad}^* B_r(\xi_2 \wedge \xi_3) \cdot \xi_1 + \mathrm{ad}^* B_r(\xi_3 \wedge \xi_1) \cdot \xi_2 = 0$$

for all $\xi_1, \xi_2, \xi_3 \in \mathfrak{g}^*$.

A more traditional form of this assertion uses the "tensor" language. Before we pass to this formulation we must introduce a couple of standard (but rather lengthy) definitions.

We shall label different factors in tensor products by frozen indices 1, 2, 3 indicated the order of factors. For simplicity we may assume that $\mathfrak{g}$ is embedded into an associative algebra $\mathcal{N}$ with unit. Define the embeddings

$$i_{12},\, i_{23},\, i_{13} \colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathcal{N} \otimes \mathcal{N} \otimes \mathcal{N}, \tag{4.18}$$

setting $i_{12}(X \otimes Y) = X \otimes Y \otimes I,$ and similarly in other cases. For $a \in \mathfrak{g} \otimes \mathfrak{g}$ we shall write $i_{12}(a) = a_{12}$, and so on. Let us set also $a_{21} = P_{12}(a_{12})$, where P_{12} is the permutation operator in the tensor product. The commutator $[a_{12}, b_{13}]$ is computed in the the associative algebra $\mathcal{N} \otimes \mathcal{N} \otimes \mathcal{N}$. (As usual, we define the commutator at first for decomposable elements $a, b \in \mathfrak{g} \otimes \mathfrak{g}$ and then extend the definition to arbitrary a, b by universality of the tensor product.)

[4]Here is a simple "shocking example": the pair $(\mathfrak{g}, \mathfrak{g})$ with $r = id$ is a Lie dialgebra, but not a Lie bialgebra!

[5]Cf. for instance [**LP**].

EXERCISE 4.17. The commutator $[a_{12}, b_{13}]$ lies in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \subset \mathcal{N}\otimes\mathcal{N}\otimes\mathcal{N}$ and depends only on the Lie bracket in $\mathfrak{g}$ and not on the embedding $\mathfrak{g} \subset \mathcal{N}$.

EXERCISE 4.18. Under the isomorphism $\mathrm{Hom}(\mathfrak{g}^*\otimes\mathfrak{g}^*, \mathfrak{g}) \simeq \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$, B_r goes to $\frac{1}{2}[[r,r]]$, where

$$[[r,r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]. \tag{4.19}$$

REMARK 4.19. Formula (4.19) is a special case of the so called *Schouten bracket* defined on $\bigwedge^* \mathfrak{g}$. We shall return to the general definition of the Schouten bracket in Section 4.7 below.

Our next statement is the tensor version of Proposition 4.16.

PROPOSITION 4.20. *The bracket* (4.16) *satisfies the Jacobi identity if and only if* $[[r,r]] \in \mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g}$ $\mathrm{ad}\,\mathfrak{g}$ *is* $\mathfrak{g}$*-invariant, i. e.,*

$$[X_1 + X_2 + X_3, [[r,r]]] = 0 \quad \textit{for all} \quad X \in \mathfrak{g}.$$

DEFINITION 4.21. An element $r \in \mathfrak{g}\otimes\mathfrak{g}$ satisfies the classical Yang–Baxter identity if $[[r,r]] = 0$.

As we know, factorization problems associated with Lax equations are related to the *modified* Yang–Baxter identity. Its tensor counterpart is defined as follows.

Let us assume that $\mathfrak{g}$ admits an invariant inner product. Let $\{e_i\}$ be an arbitrary linear basis in $\mathfrak{g}$, and $\{e^i\}$ the dual basis in $\mathfrak{g}^*$; let

$$I = \sum_i e_i \otimes e^i \in \mathfrak{g}\otimes\mathfrak{g}^*$$

be the canonical element. Its image $t \in \mathfrak{g}\otimes\mathfrak{g}$ under the isomorphism $\mathfrak{g}\otimes\mathfrak{g}^* \simeq \mathfrak{g}\otimes\mathfrak{g}$ induced by the inner product in $\mathfrak{g}$ is called the Casimir tensor, or Casimir element. If the adjoint representation of $\mathfrak{g}$ is irreducible, the Casimir element is the unique (up to a scalar factor) invariant symmetric tensor in $\mathfrak{g}\otimes\mathfrak{g}$.

LEMMA 4.22. *Under the assumptions made above, the subspace of* $\mathrm{ad}\,\mathfrak{g}$*-invariants in* $\mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g}$ *is one-dimensional; it is generated by*

$$t_{123} = [t_{12}, t_{13}] + [t_{12}, t_{23}] + [t_{13}, t_{23}].$$

DEFINITION 4.23. An element $r \in \mathfrak{g}\wedge\mathfrak{g}$ satisfies the modified classical Yang–Baxter identity if, if

$$[[r,r]] = -t_{123}. \tag{4.20}$$

It is easy to see that under the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ induced by the inner product this formula goes to the modified Yang–Baxter identity (2.21).

EXERCISE 4.24. The identity $[[r,r]] = -t_{123}$ implies that $[[r\pm t, r\pm t]] = 0$.

This assertion is practically identical to formulae (2.23), but the context now is different: as we saw, the symmetric part of r drops out from the formula (4.16) for the dual bracket. (Clearly, the symmetric part of $r \pm t$ is

$\pm t$, which is of course ad$\mathfrak{g}$-invariant!), Hence for coboundary Lie bialgebras one can get back from the modified classical Yang–Baxter equation to the simpler equation with zero right hand side by replacing the skew symmetric r-matrix with $r \pm t$.

WARNING 2. This trick does not work for operator r-matrices associated with Lie dialgebras! Once again, we see that two parallel definitions of classical r-matrices are not completely equivalent.

4.4. Factorizable Lie Bialgebras

There are two important types of Lie bialgebras which are frequently discussed in the literature. If the dual bracket is defined my means of a classical r-matrix (i. e., $(\mathfrak{g}, \mathfrak{g}^*)$ is a coboundary bialgebra) and moreover r is skew and satisfies the classical Yang–Baxter identity, $(\mathfrak{g}, \mathfrak{g}^*)$ is called *triangular Lie bialgebra* ; if the skew symmetry condition is dropped, the Lie balgebra is called *quasitriangular*. Both terms, which belong to the professional jargon, stem from the term 'triangles equation' which is sometimes used as the synonyme of the 'Yang–Baxter equation'. (In its turn, this latter term is due to the role of the *quantum* Yang–Baxter equation in the so called *factorized scattering theory* and refers to the convenient diagrammatic representation of this equation [**ZZ**].) According to this definition, it may happen that a quasitriangular Lie bialgebra is in fact triangular (if the symmetric part of the r-matrix happens to be zero), although in general this need not be the case.

A correct way out of this slightly confusing situation consists in adopting the following important working definition [**RS1**].

DEFINITION 4.25. A Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is called *factorizable*, if

(i) Lie algebra $\mathfrak{g}$ is equipped with a (fixed) invariant inner product which allows to identify $\mathfrak{g}$ with its dual space.

(ii) The Lie bracket on $\mathfrak{g}^* \simeq \mathfrak{g}$ is given by

$$[X, Y]_* = \tfrac{1}{2}([rX, Y] + [X, rY]),$$

where $r \in \operatorname{End}\mathfrak{g}$ is skew and satisfies the modified classical Yang–Baxter identity

$$[rX, rY] - r([rX, Y] + [X, rY]) = -[X, Y].$$

Equivalent condition:

(ii)$'$ The cobracket on $\mathfrak{g}$ is given by

$$\delta(X) = [X \otimes I + I \otimes X,\ r_\pm],$$

where $r_\pm \in \mathfrak{g} \otimes \mathfrak{g}$ and, moreover, $r_- = -P(r_+)$, $r_\pm$ satisfy the Yang–Baxter identity $[[r_\pm, r_\pm]] = 0$ and, finally, $r_+ - r_- = t$ is the Casimir element which represents the given inner product on $\mathfrak{g}$.

Clearly $r_\pm = \frac{1}{2}(r \pm t)$ are the tensor kernels of the operators $r_\pm = \frac{1}{2}(r \pm id)$ (up to the isomorphism $\mathfrak{g} \otimes \mathfrak{g} \simeq \operatorname{End}\mathfrak{g}$ which maps t to the identity operator).

A Poisson Lie group is called *factorizable*, if its tangent Lie bialgebra is factorizable. The name 'factorizable' is of course due to the fact that we can apply to such groups Theorem 2.20. Poisson brackets on factorizable Poisson groups are the so called *Sklyanin brackets*; they will be studied in detail in Section 4.6.

One may fear that the definition of factorizable Lie bialgebras and of factorizable Poisson groups is too restrictive, since it leaves aside all Lie bialgebras and the associated Poisson groups except for a seemingly narrow special class. We shall see, however, that any Lie bialgebra may be canonically embedded into a factorizable one (namely, in its own double); in a similar way, any Poisson group may be realized as a Poisson subgroup of factorizable Poisson group.

4.5. Examples of Lie Bialgebras

1°. Let $\mathfrak{g} = \mathfrak{gl}(n)$. Consider the Gauss decomposition of $\mathfrak{g}$ into the sum of Lie subalgebras of lower triangular, diagonal and upper triangular matrices, $\mathfrak{g} = \mathfrak{n}_- \dot{+} \mathfrak{h} \dot{+} \mathfrak{n}_+$; let $P_\pm, P_0$ be the projection operators onto $\mathfrak{n}_\pm, \mathfrak{h}$, respectively, associated with this decomposition. Define the inner product on $\mathfrak{g}$ by $(X, Y) = \operatorname{tr} XY$. The Gauss decomposition defines on $\mathfrak{g}$ the structure of a factorizable Lie bialgebra (this is the so called *standard* Lie bialgebra structure on $\mathfrak{g} = \mathfrak{gl}(n)$). This structure is defined by the classical r-matrix $r_0 = P_+ - P_-$. Let $\{e_{ij}\}$be the basis in $\mathfrak{g}$ which consists of elementary matrices. In tensor form, r_0 is given by

$$r_0 = \sum_{i<j} e_{ij} \wedge e_{ji};$$

We have, moreover,

$$(r_0)_+ = \sum_{i<j} e_{ij} \otimes e_{ji} + \tfrac{1}{2} \sum_i e_{ii} \otimes e_{ii},$$
$$(r_0)_- = -\sum_{i>j} e_{ij} \otimes e_{ji} - \tfrac{1}{2} \sum_i e_{ii} \otimes e_{ii}.$$

In Section 4.5.1 we shall describe a generalization of this example (as well as of the next one); it is connected with the possibility to add to r_0 a nontrivial Cartan summand, i.e., an extra term with values in $\mathfrak{h} \otimes \mathfrak{h}$.

2°. In a more general way, let $\mathfrak{g}$ be a split real semisimple Lie algebra,$\mathfrak{h}$ its Cartan subalgebra, Δ_+ a system of its positive roots. Fix an invariant inner product on $\mathfrak{g}$, and let $\{e_\alpha; \alpha \in \pm\Delta_+\}$ be the root space vectors normalized in such a way that $(e_\alpha, e_{-\alpha}) = 1$. Let

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_+} \mathbb{R} \cdot e_{\pm\alpha}.$$

We have $\mathfrak{g} = \mathfrak{n}_- \dot{+} \mathfrak{h} \dot{+} \mathfrak{n}_+$ (*Bruhat decomposition*). Let $P_\pm$ be the projection operators onto $\mathfrak{n}_\pm$ associated with this decomposition. *The standard Lie* *bialgebra structure* on $\mathfrak{g}$ is defined by the r-matrix $r_0 = P_+ - P_-$. In tensor form,

$$r_0 = \sum_{\alpha\in\Delta_+} e_\alpha \wedge e_{-\alpha}. \tag{4.21}$$

Let $\mathfrak{b}_\pm = \mathfrak{h} \dot{+} \mathfrak{n}_\pm$. The subalgebras $\mathfrak{b}_\pm$ (Borel subalgebras in $\mathfrak{g}$) are set in duality by the inner product; it is easy to see that $(\mathfrak{b}_+, \mathfrak{b}_-)$ is a Lie subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$.

3° We keep to the notation of Example 1°. Let $\mathfrak{G} = \mathcal{L}(\mathfrak{gl}(n))$ be the associated loop algebra with the inner product

$$(X, Y) = \operatorname{Res}\operatorname{tr} X(\lambda)Y(\lambda)\, d\lambda/\lambda.$$

The standard structure of a factorizable Lie bialgebra on $\mathfrak{G}$ is associated with the decomposition

$$\mathfrak{G} = \mathfrak{N}_- \dot{+} \mathfrak{h} \dot{+} \mathfrak{N}_+, \text{ where } \mathfrak{N}_+ = \mathfrak{n}_+ \dot{+} \bigoplus_{n\geqslant 1} \mathfrak{g}\otimes\lambda^n,\ \mathfrak{N}_- = \mathfrak{n}_- \dot{+} \bigoplus_{n\geqslant 1} \mathfrak{g}\otimes\lambda^{-n}.$$

Let $\mathcal{P}_\pm$ be the projection operators onto $\mathfrak{N}_\pm$ in the decomposition $\boldsymbol{r} = \mathcal{P}_+ - \mathcal{P}_-$. Their tensor kernels are given

$$\mathcal{P}_+ = P_+ + \sum_{n=1}^{\infty}\sum_{i,j} e_{ij}\lambda^n \otimes e_{ji}\mu^{-n} = P_+ + \sum_{n=1}^{\infty} t\left(\frac{\lambda}{\mu}\right)^n,$$
$$\mathcal{P}_- = P_- + \sum_{n=1}^{\infty}\sum_{i,j} e_{ij}\lambda^{-n} \otimes e_{ji}\mu^{n} = P_- + \sum_{n=1}^{\infty} t\left(\frac{\lambda}{\mu}\right)^{-n}.$$

Computing the sum of these geometric progressions, we conclude that

$$\boldsymbol{r}(\lambda,\mu) = r_0 + \frac{\mu+\lambda}{\mu-\lambda} t, \tag{4.22}$$

where t is the tensor Casimir,

$$t = \sum_{i,j} e_{ij}\otimes e_{ji}.$$

This is the so called *trigonometric* r-matrix. In trigonometric parametrization, $\lambda = e^{iu}$, $\mu = e^{iv}$, $\boldsymbol{r}(\lambda,\mu)$ depends only on the difference $u - v$.

4°. Let us choose another inner product on $\mathfrak{G}$, setting:

$$\langle X, Y\rangle = \operatorname{Res}\operatorname{tr} X(\lambda)Y(\lambda)\, d\lambda; \tag{4.23}$$

Set

$$\mathfrak{G} = \mathfrak{G}_- \dot{+} \mathfrak{G}_+, \text{ where } \mathfrak{G}_+ = \bigoplus_{i\geqslant 0} \mathfrak{g}\otimes\lambda^i, \quad \mathfrak{G}_- = \bigoplus_{i<0} \mathfrak{g}\otimes\lambda^i.$$

Let $\widehat{\mathcal{P}}_\pm$ be the associated projection operators and $r_{\text{rat}} = \widehat{\mathcal{P}}_+ - \widehat{\mathcal{P}}_-$. The tensor kernels of the projection operators associated with this decomposition are now given be the series

$$\widehat{\mathcal{P}}_+ = \sum_{n=0}^{\infty} \sum_{i,j} e_{ij}\lambda^n \otimes e_{ji}\mu^{-n-1},$$

$$\widehat{\mathcal{P}}_- = \sum_{n=1}^{\infty} \sum_{i,j} e_{ij}\lambda^{-n} \otimes e_{ji}\mu^{n-1},$$

Summing up the progressions, we get:

$$\boldsymbol{r}_{\mathrm{rat}}(\lambda,\mu) = \frac{t}{\mu-\lambda}. \tag{4.24}$$

This is the *rational r-matrix*. From the operator point of view, $\widehat{\mathcal{P}}_+, \widehat{\mathcal{P}}_-, \boldsymbol{r}_{\mathrm{rat}}$ are distribution kernels of singular integral operators; more accurately, they must be written as

$$\widehat{\mathcal{P}}_\pm(\lambda,\mu) = \frac{t}{\mu-\lambda\mp i0}, \ \boldsymbol{r}_{\mathrm{rat}} = \text{v.p.}\frac{t}{\mu-\lambda}.$$

4. Let

$$\mathfrak{L}(\mathfrak{g})_D = R_D(\mathfrak{g}) \dotplus \mathfrak{L}(\mathfrak{g})_D^+$$

be the algebra of formal Laurent series from Section 3.3 equipped with the inner product (3.16). The Mittag–Leffler decomposition (3.13) equips it with the structure of a factorizable Lie bialgebra. Lie subalgebras $R_D(\mathfrak{g}), \mathfrak{L}(\mathfrak{g})_D^+ \subset \mathfrak{L}(\mathfrak{g})_D$ are dual to each other with respect to the inner product; hence, $(R_D(\mathfrak{g}), \mathfrak{L}(\mathfrak{g})_D^+)$ — is a sub-bialgebra of $\mathfrak{L}(\mathfrak{g})_D$. (This sub-bialgebra is no longer factorizable!) We shall see below that $\mathfrak{L}(\mathfrak{g})_D$ is precisely the double of $(R_D(\mathfrak{g}), \mathfrak{L}(\mathfrak{g})_D^+)$. The tensor kernel of the r-matrix associated with this bialgebra is again given by (4.24) (basically, it is just the Cauchy kernel).

REMARK 4.26. As we see, classical r-matrices for affine Lie algebras are given by singular kernels which are close to the Cauchy kernel. For such kernels the difference between the ordinary and the modified Yang–Baxter equation is rather subtle: the right hand side of the modified Yang–Baxter equation is a singular kernel containing delta-functions.

EXERCISE 4.27. Let $r(\lambda,\mu) = \text{v.p.}\frac{t}{\mu-\lambda}$. Show that

$$\begin{aligned}[r_{12}(\lambda,\mu),\, r_{13}(\lambda,\nu)] + [r_{12}(\lambda,\mu),\, r_{23}(\mu,\nu)] + \\ + [r_{13}(\lambda,\nu),\, r_{23}(\mu,\nu)] = t_{123}\,\delta(\lambda-\mu)\delta(\mu-\nu).\end{aligned} \tag{4.25}$$

Formula (4.25) is closely related to the *Poincaré–Bertrand formula* in the theory of singular integrals:

$$\begin{aligned}\frac{1}{(\pi i)^2}\int_C d\lambda \int_C \frac{\varphi(\lambda,\mu)}{(\mu-\lambda)(\lambda-\lambda_0)}\,d\mu = \\ = \varphi(\lambda_0,\lambda_0) + \frac{1}{(\pi i)^2}\int_C d\mu \int_C \frac{\varphi(\lambda,\mu)}{(\mu-\lambda)(\lambda-\lambda_0)}\,d\lambda.\end{aligned} \tag{4.26}$$

EXERCISE 4.28. Deduce (4.25) from (4.26).

Of course, the operator point of view on the r-matrix (r-matrix as the difference of projection operators) makes the modified Yang–Baxter equation obvious without almost any calculations. If, on the other hand, we regard the r-matrix just as an ordinary meromorphic function of two variables (as people frequently do), the delta-function terms become 'invisible'.

4.5.1. Classification of r-matrices and Extension of Linear Operators. After discussing the examples of classical r-matrices and of the associated Lie bialgebras let us make some remarks on the structure of "general" solutions of the modified classical Yang–Baxter equation. These solutions are not exhausted by the simplest r-matrices of the form (2.4); however, there exists an important class of r-matrices which are "close to standard ones". (Moreover, under some additional assumptions such r-matrices already exhaust *all* solutions.) Let $r \in \operatorname{End}\mathfrak{g}$ be a solution of the modified classical Yang–Baxter equation. As before, we set $\mathfrak{g}_\pm = \operatorname{Im} r_\pm$, $\mathfrak{k}_\pm = \operatorname{Ker} r_\mp$; as we know, $\mathfrak{k}_\pm \subset \mathfrak{g}_\pm$; clearly, $\mathfrak{k}_+ \cap \mathfrak{k}_- = 0$ and $r|_{\mathfrak{k}_\pm} = \pm I$. Hence our r-matrix reduces to the standard form on the linear subspace $\mathfrak{k}_+ \oplus \mathfrak{k}_- \subset \mathfrak{g}$. Let us denote by r_0 *the partially defined linear operator* in $\mathfrak{g}$ with domain $D_0 = \mathfrak{k}_+ \oplus \mathfrak{k}_-$ which is given by the formula

$$r_0(X_+ - X_-) = X_+ + X_-, \quad X_\pm \in \mathfrak{k}_\pm.$$

Clearly, the initial operator r is an extension of r_0 (in the sense of the standard theory of operator extensions, see e.g. [**GL**]). This interpretation suggests the following natural question: describe all linear operators $r \in \operatorname{End}\mathfrak{g}$ which satisfy the modified classical Yang–Baxter equation, such that $\mathfrak{g}_\pm = \operatorname{Im} r_\pm$, $\mathfrak{k}_\pm = \operatorname{Ker} r_\mp$ are given Lie subalgeras of $\mathfrak{g}$. This question may be split into the two following problems:

(a) Describe all extensions of r_0 for which $\operatorname{Ker} r_\pm = \mathfrak{g}_\pm$, $\operatorname{Im} r_\pm = \mathfrak{k}_\mp$.
(b) Single out those operators from this list which satisfy the modified classical Yang–Baxter equation.

The first question is the standard problem of the theory of operator extensions.

Fix a splitting $\mathfrak{g}_\pm = \mathfrak{k}_\pm \dot{+} \mathfrak{m}_\pm$. We shall say that a linear operator $\theta \in \operatorname{Hom}(\mathfrak{m}_+, \mathfrak{m}_-)$ is regular, if the subspace $(I - \theta)\mathfrak{m}_+$ is complementary to $\mathfrak{k}_+ \oplus \mathfrak{k}_-$. (Note that $\dim \mathfrak{m}_+ = \operatorname{codim}(\mathfrak{k}_+ \oplus \mathfrak{k}_-)$.)

Proposition 4.29. *(i) Linear extensions r^θ of r_0 such that* $\operatorname{Im} r^\theta_+ = \mathfrak{g}_+$, $\operatorname{Im} r^\theta_- \subseteq \mathfrak{g}_-$ *are determined by the* von Neumann formulae

$$\begin{aligned} X &= X_+ - X_- + (I - \theta)X_0, \\ r^\theta X &= X_+ + X_- + (I + \theta)X_0, \quad X_\pm \in \mathfrak{k}_\pm, \quad X_0 \in \mathfrak{m}_+, \end{aligned} \tag{4.27}$$

where $\theta \in \operatorname{Hom}(\mathfrak{m}_+, \mathfrak{m}_-)$ is a regular linear operator.

(ii) Suppose that θ is a linear isomorphism of $\mathfrak{m}_+$ onto $\mathfrak{m}_-$. Then $\operatorname{Im} r^\theta_- = \mathfrak{g}_-, \operatorname{Ker} r^\theta_+ = \mathfrak{k}_+$.

(iii) Let $P^0_\pm$ *be the projection operators onto* $\mathfrak{m}_\pm$ *in the decomposition* $\mathfrak{g}_\pm = \mathfrak{k}_\pm \dot{+} \mathfrak{m}_\pm$*; let* $P_\pm$ *be the complementary projection operators. Operator* r^θ *satisfies the modified classical Yang–Baxter equation if and only if*

$$[\theta(P^0_+X),\, \theta(P^0_+Y)] = P^-_0\,(\theta(P^0_+[X,Y])) \tag{4.28}$$

for all $X, Y \in \mathfrak{g}$.

Assertion (i) is a standard result of the operator extensions theory. Formula (4.28) is a reformulation of the main property of the Cayley transform (see Proposition 2.19, (i)); finally, assertion (iii) is tantamount to saying that an r-matrix is uniquely restored from its Cayley transform and satisfies (2.21) if and only if its Cayley transform is a homomorphism of the quotient Lie algebras $\mathfrak{g}_+/\mathfrak{k}_+ \to \mathfrak{g}_-/\mathfrak{k}_-$.

A thorough study of the conditions imposed on the Cayley transform of r allows to construct a series of nontrivial examples and, moreover, to get a complete classification of the solutions of the modified classical Yang–Baxter equation (this classification applies under some additional assumptions). The simplest case of such classification theorem is the descriptions of all skew symmetric classical r-matrices on finite-dimensional semisimple Lie algebras **[BD]** The point of view based on the theory of operator extensions makes much more transparent the formulae for 'generic' r-matrices which arise in this way.

Example 4.30. We start with an example of a family of classical r-matrices satisfying the modified classical Yang–Baxter equation which is based on the Gauss–Bruhat decomposition. The simplest r-matrices from this family we already considered in Examples 1° and 2° of the previous section. We keep to the notation we introduced there.

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra,

$$\mathfrak{g} = \mathfrak{n}_- \dot{+} \mathfrak{h} \dot{+} \mathfrak{n}_+$$

its Bruhat decomposition. Let $P_\pm, P_0$ be the projection operators onto $\mathfrak{n}_\pm, \mathfrak{h}$ in this decomposition. We shall describe a family of r-matrices related to the Bruhat decomposition for which $\mathrm{Im}(r \pm I) = \mathfrak{b}_\pm$, $\mathrm{Ker}(r \mp I) = \mathfrak{n}_\pm$; all such r-matrices are extensions of a single partially defined operator r_0 with the domain $\mathfrak{n}_+ \dot{+} \mathfrak{n}_-$,

$$r_0 X = \begin{cases} X & \text{for } X \in \mathfrak{n}_+, \\ -X & \text{for } X \in \mathfrak{n}_-. \end{cases}$$

Clearly, in this case the quotient algebras $\mathfrak{b}_+/\mathfrak{n}_+$, $\mathfrak{b}_-/\mathfrak{n}_-$ and the splitting spaces $\mathfrak{m}_+, \mathfrak{m}_-$ may be identified with the Cartan subalgebra $\mathfrak{h}$; since $\mathfrak{h}$ is abelian, *any* extension of r_0 will automatically satisfy the modified classical Yang–Baxter equation (condition (4.28) is void). Thus we get a family of linear operators

$$r_\theta = P_+ - P_- + \frac{I+\theta}{I-\theta} P_0, \quad \theta \in \operatorname{End}\mathfrak{h}, \quad \det(I-\theta) \neq 0. \tag{4.29}$$

It is easy to see that r_θ is skew if and only if $\theta \in \operatorname{End}\mathfrak{h}$ is orthogonal.

If $\mathfrak{g} = \mathfrak{sl}(n)$ and $\mathfrak{h}$ is the diagonal subalgebra, the Bruhat decomposition coincides with the Gauss decomposition into upper and lower triangular matrices. One can obtain more general solutions of the modified Yang–Baxter equation by passing to *block-triangular matrices* (their counterpart for general semisimple Lie algebras are *parabolic subalgebras*). In this case, the Yang–Baxter equation leads already to strong restrictions on the choice of admissible pairs of subalgebras which may serve as the image of $r \pm I$. We explain the situation on a simple example.

EXAMPLE 4.31. In the notation introduced above, let $\mathfrak{p}_+ \supset \mathfrak{b}_+$ be a parabolic subalgebra of $\mathfrak{g}$ which contains the standard Borel subalgebra. All such parabolic subalgebras are parameterized by the subsets of the set $P \subset \Delta_+$ of simple roots of $(\mathfrak{g}, \mathfrak{h})$. More precisely, if $F \subset P$ is such a subset, put

$$\mathfrak{h}_F = \{H \in \mathfrak{h};\, \alpha(H) = 0 \text{ for all } \alpha \in F\}.$$

Let $\mathfrak{m}_F$ be the semisimple Lie algebra whose system of simple roots is F; $\mathfrak{m}_F$ is canonically embedded into $\mathfrak{g}$, and its Cartan subalgebra ${}^*\mathfrak{h}_F \subset \mathfrak{h}$ is the orthogonal complement of $\mathfrak{h}_F \subset \mathfrak{h}$. Let $\Delta_F^+ \subset \Delta_+$ be the set of positive roots of $(\mathfrak{g}, , \mathfrak{h})$ which are identically zero on $\mathfrak{h}_F$, ${}'\Delta_F^+ = \Delta_+ \setminus \Delta_F^+$. Let ${}^*\Delta_F$ -be the set of positive roots which lie in the linear envelope of F. Put

$$\mathfrak{n}_F^+ = \bigoplus_{\alpha \in' \Delta_F^+} \mathfrak{g}_\alpha.$$

By definition, the parabolic subalgebra $\mathfrak{p}_F^+$ associated with F is

$$\mathfrak{p}_F^+ = \mathfrak{m}_F \dot{+} \mathfrak{h}_f \dot{+} \mathfrak{n}_F.$$

Let $\mathfrak{p}_F^-$ be the opposite parabolic subalgebra, $\mathfrak{p}_F^- = \mathfrak{m}_F \dot{+} \oplus_{\alpha \in' \Delta_F^+} \mathfrak{g}_{-\alpha}$. We have $\mathfrak{p}_F^\pm / \mathfrak{n}_F^\pm \simeq \mathfrak{m}_F \dot{+} \mathfrak{h}_F$; the Lie algebra $\mathfrak{l}_F = \mathfrak{m}_F \dot{+} \mathfrak{h}_F$ is reductive, and its center coincides with $\mathfrak{h}_F$.

PROPOSITION 4.32. *There are no solutions of the modified Yang–Baxter equation for which* $\operatorname{Im} r_\pm = \mathfrak{p}_F^\pm$, $\operatorname{Ker} r_\pm = \mathfrak{n}_F^\pm$.

Indeed, in this case $\operatorname{Im} r_\pm / \operatorname{Ker} r_\pm \simeq \mathfrak{l}_F$ and the Cayley transform of r is an automorphism of the reductive Lie algebra $\theta\colon \mathfrak{l}_F \longrightarrow \mathfrak{l}_F$. Any such automorphism induces an automorphism of its semisimple $\mathfrak{m}_F \subset \mathfrak{l}_F$; but according to the standard fundamental theorem, every automorphism of a semisimple Lie algebra has at least one fixed vector. This is incompatible with the non-degeneracy condition $\det(I - \theta) \neq 0$.

One can save the idea to make use of parabolic subalgebras for the construction of r-matrices by the following trick (it is precisely on this way that the examples of r-matrices in the classification theorem of Belavin and Drinfeld [**BD**] are constructed): instead of a single parabolic subalgebra and its associated subset $F \subset P$ one has to consider two *different* parabolics with the associated subsets $F_1, F_2 \subset P$ and an isometric map $\tau\colon F_1 \xrightarrow{\text{onto}} F_2$. Any such map gives rise to an isomorphism of Lie algebras $\theta\colon \mathfrak{m}_{F_1} \to \mathfrak{m}_{F_2}$. We have yet to keep an eye on the crucial condition: our isomorphism has to be induced by the Cayley transform of an r-matrix and hence should have no fixed vectors. The following condition on $F_1, F_2 \subset P$ and τ which assures this property:

> The mapping $\tau\colon F_1 \to F_2$, defined on $F_1 \subset P$ should admit only a *finite number of iterations.*

More precisely, let $\alpha \in F_1$; if $\tau\alpha \in F_2 \cap F_1$, we can define an element $\tau^2\alpha$, and so on. We shall say that the triple $\tau\colon F_1 \to F_2$ *is admissible* if the chain $\alpha, \tau\alpha, \dots,$ breaks up after a finite number of steps for any $\alpha \in F_1$ (i. e. there is a $k \in \mathbb{N}$ such that $\tau\alpha, \dots, \tau^{k-1} \in F_1 \cap F_2$, but $\tau^k \notin F_1$). If the triple $\tau\colon F_1 \to F_2$ is admissible, the Belavin–Drinfeld construction yields an operator $\theta : \mathfrak{p}^+_{F_1}/\mathfrak{n}^+_{F_1} \to \mathfrak{p}^-_{F_2}/\mathfrak{n}^-_{F_2}$ without fixed vectors and hence a solution of the modified Yang–Baxter equation on $\mathfrak{g}$.

The remaining freedom in this construction is associated with the extension of θ to the center of $\mathfrak{l}_{F_1}$, which remains arbitrary. If this extension is an isometry, the corresponding r-matrix is skew; hence any admissible triple $\tau\colon F_1 \to F_2$ gives rise to a family of classical r-matrices.

Let us give an explicit formula for the Belavin–Drinfeld r-matrix. Choose the root vectors $e_\alpha \in \mathfrak{g}_\alpha$ in such a way that $\langle e_\alpha, e_{-\alpha}\rangle = 1$. Let $\mathfrak{m}_{F_1}, \mathfrak{m}_{F_2}$ be the Lie subalgebras in $\mathfrak{g}$ generated by $e_{\pm\alpha}$, where $\alpha \in F_1$ (respectively, $\alpha \in F_2$).The mapping $\tau\colon F_1 \to F_2$ extends to an isomorphism $\theta\colon \mathfrak{m}_{F_1} \to \mathfrak{m}_{F_2}$ such that $\theta(e_\alpha) = e_{\tau(\alpha)}, \alpha \in F_1$. The mapping θ is defined in particular on the Cartan subalgebra of $\mathfrak{m}_{F_1}$; clearly, it extends here by linearity the mapping τ (originally defined on the finite set of simple roots). Let us introduce a partial ordering in the set of the roots of $\mathfrak{g}$, setting $\alpha < \beta$, if $\beta = \theta^m\alpha$ for some $m \in \mathbb{N}$ (hence two roots are comparable if and only if both of them belong to the linear envelope of one of the sets $F_1, \tau(F_1), \dots, \tau^m(F_1)$).Then

$$r = r_c + \sum_{\alpha\in\Delta_+} e_\alpha \wedge e_{-\alpha} + \sum_{\alpha,\beta\in\Delta_+,\alpha<\beta} e_\alpha \wedge e_{-\beta}, \tag{4.30}$$

where r_c is the "Cartan component" of r associated with the center of the Lévi subalgebra $\mathfrak{l}_{F_1}$.

This construction immediately extends to affine Lie algebras. In Section 4.5, Example 3° we have of the standard "trigonometric" r-matrix on the loop algebra $\mathcal{L}(\mathfrak{g})$. "General" trigonometric r-matrices are again extensions of a partially defined operator $r_0 = P_+ - P_-$ with a *finite* deficiency index. Hence the Cayley transform of r is again an isomorphism of finite-dimensional reductive Lie algebras. Moreover, both these subalgebras are realized as embedded subalgebras of the initial affine Lie algebra and the isomorphism should not have any fixed vectors. This latter condition binds very severely the restriction of the Cayley transform to the semisimple component of the reductive algebra; its restriction to its center remains largely arbitrary, which yields continuous parameters in the family of solutions.The discrete set of parameters remains basically the same: it is an admissible triple $\tau\colon F_1 \to F_2$, where F_1, F_2 are two subsets of simple roots of the affine Lie algebra.

REMARK 4.33. The classification of Belavin and Drinfeld crucially uses the skew symmetry condition. (This condition can be somewhat weakened but cannot be dropped altogether.) The r-matrices associated with Lie dialgebras, as opposed to Lie bialgebras, are not necessarily skew. This key difference between the two cases makes the classification of Lie dialgebras substantially less rigid. Even in the finite-dimensional case, a completely general classification theorem for Lie dialgebras remains unknown.

4.6. Sklyanin Brackets

Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a factorizable Lie bialgebra, G a Lie group with Lie algebra $\mathfrak{g}$. There is a natural Poisson bracket on G which gives it the structure of a Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. The explicit formula for this bracket, called *Sklyanin bracket*, is given below.

For any function $\varphi \in C^\infty(G)$ we denote by $\nabla\varphi$, $\nabla'\varphi \in \mathfrak{g}$ its left and right gradients. By definition,

$$\begin{aligned} \langle \nabla\varphi(x),\, \xi\rangle &= \left(\frac{d}{dt}\right)_{t=0} \varphi\big(e^{t\xi}x\big), \\ \langle \nabla'\varphi(x),\, \xi\rangle &= \left(\frac{d}{dt}\right)_{t=0} \varphi\big(x\, e^{t\xi}x\big) \end{aligned} \tag{4.31}$$

for all $\xi \in \mathfrak{g}$. The Sklyanin bracket of φ, $\psi \in C^\infty(G)$ is given by

$$\{\varphi,\, \psi\} = \tfrac{1}{2}\langle r(\nabla'\varphi),\, \nabla'\psi\rangle - \tfrac{1}{2}\langle r(\nabla\varphi),\, \nabla\psi\rangle. \tag{4.32}$$

THEOREM 4.34. *The bracket* (4.32) *is multiplicative and satisfies the Jacobi identity. The tangent Lie bialgebra of G coincides with* $(\mathfrak{g}, \mathfrak{g}^*)$.

Expressing the Sklyanin bracket by means of left differentials, we have

$$\{\varphi, \psi\}\,(x) = \langle \eta_r(x)\cdot(\nabla\varphi),\, \nabla\psi\rangle,$$

where

$$\eta_r\,(x) = \frac{1}{2}\left(r - Ad\, x^{-1} \circ r \circ Ad\, x\right).$$

Clearly, η_r is a trivial 1-cocycle on G; hence the multiplicativity (4.32) immediately follows from Proposition 4.6. A comparison with (4.15) immediately shows that G has the correct tangent Lie bialgebra. The proof of the Jacobi identity for (4.32) will be postponed until Section 4.7, when the necessary technical tools (*Schouten bracket*) will become available.

The Sklyanin bracket is frequently written in "tensor form", in which the linear operator $r \in \operatorname{End}\mathfrak{g}$ is replaced with its tensor kernel $r \in \mathfrak{g}\otimes\mathfrak{g}$ (by an abuse of notation, we constantly denote both objects by the same letter). This formula applies, however, not to the whole ring $C^\infty(G)$, but rather to a set of its generators. This tensor form becomes particularly natural if we regard G as an algebraic group whose affine ring is generated by the matrix coefficients of its fixed matrix representation.

Let $G = GL\,(n)$ be the full linear group. Consider the "tautological" functions ϕ_{ij} on G which assign to $L \in G$ matrix coefficients $\phi_{ij}\,(L) = L_{ij}$; Clearly, the coordinate ring $\mathbb{C}\,[\phi_{ij}]$ is a dense subset in $C^{\infty}\,(G)$ in an appropriate topology (by the Stone–Weierstrass theorem, say), and hence the Poisson bracket on G is completely determined by its values on the 'generators' ϕ_{ij}. Let us fix an invariant inner product on $\mathfrak{g} = \mathfrak{gl}\,(n)$ and a classical r-matrix $r \in \operatorname{End}\mathfrak{gl}(n) \simeq \mathfrak{gl}(n) \otimes \mathfrak{gl}(n) \simeq \operatorname{Mat}\left(n^2\right)$. (Examples of r-matrices which make $\mathfrak{g} = \mathfrak{gl}\,(n)$ a factorizable Lie bialgebra were discussed in the previous section.)

PROPOSITION 4.35. *The Sklyanin bracket (4.32) of the matrix coefficients is given by*

$$\{\phi_{ij},\, \phi_{km}\}\,(L) = [r,\, L \otimes L]_{ikjm}\,. \tag{4.33}$$

The commutator in the r.h s. is computed in $\operatorname{Mat}\bigl(n^2\bigr)$. By an abuse of notation, people frequently do not distinguish the functions ϕ_{ij} and their values and write this formula (with suppressed matrix indices!) simply as[6]

$$\left\{L \overset{\otimes}{,} L\right\} = [r,\, L \otimes L]\,,$$

or equivalently, using our convention on frozen tensor indices and setting $L_1 = L \otimes I$, $L_2 = I \otimes L$:

$$\{L_1\,, L_2\} = [r_{12},\, L_1 L_2] \tag{4.34}$$

Note that the r.h.s. in (4.33) is a *quadratic* expression in the matrix coefficients (in contrast with the Lie–Poisson bracket which is *linear*).

In applications, it is more interesting to suppose that G is a loop group, or, more generally, a matrix group over the ring of rational functions. Its affine ring is generated by the "evaluation functionals"

$$\Phi_{ik,\lambda}\colon L \longmapsto L_{ik}(\lambda), \quad L \in G.$$

As before, we shall not distinguish these functionals from their values; suppressing matrix indices, we get the following formula for the Sklyanin bracket on the loop group:

$$\left\{L(\lambda) \overset{\otimes}{,} L(\mu)\right\} = [r(\lambda,\mu), L(\lambda) \otimes L(\mu)]. \tag{4.35}$$

The tensor kernel $r(\lambda,\, \mu)$ in the r.h s is a distribution in the L.Schwartz sense. (In the previous section we have explicitly computed it in two typical cases.)

As every multiplicative bracket, the Sklyanin bracket is degenerate at the unit element; we have already seen in Proposition 4.7 that its linearization at

[6]One can give a precise meaning to this formula if we agree to consider Poisson brackets of functions with values in Lie groups or even in arbitrary manifolds. One has simply to suppose that L in the l.h.s stands for the identity mapping of G to itself (or else for a matrix representation of an abstract group G).

the unit element defines the the structure of a Lie algebra in $\mathfrak{g}^*$. By duality, we get a linear Poisson bracket in the tangent space $\mathfrak{g}$; this is precisely the r-bracket studied in previous lectures.[7]. Thus the Sklyanin bracket may be regarded as a *nonlinear analog of the r-bracket*; its role in the theory of integrable systems supported on Poisson submanifolds of G (in particular, for difference Lax systems) will be completely similar.

In order to compare (4.35) with the linear r-bracket defined by means of the same r-matrix, let us express the bracket (2.3) in tensor form; as above, we assume that the basic Lie algebra is a loop algebra $\mathcal{L}(\mathfrak{g})$ with fixed inner product, so that we may identify $\mathcal{L}(\mathfrak{g}) \simeq \mathcal{L}(\mathfrak{g})^*$; for the moment we do not suppose, however, that $r \in \mathrm{End}(\mathcal{L}(\mathfrak{g}))$ is skew. We shall consider $L(\lambda)$ as the "evaluation map" on $\mathcal{L}(\mathfrak{g})$ which assigns to a matrix-valued function $L \in \mathcal{L}(\mathfrak{g})$ its value at λ.

Exercise 4.36. (i) Prove that

$$\left\{L(\lambda) \overset{\otimes}{,} L(\mu)\right\}_r = [r(\lambda,\mu),\, L(\lambda) \otimes I] - [r^*(\lambda,\mu), I \otimes L(\mu)]; \tag{4.36}$$

We denoted by $r^*(\lambda,\mu) = P(r(\mu,\lambda))$ the kernel of the adjoint operator. If r is skew, so that $r^*(\lambda,\mu) = -r(\lambda,\mu)$, formula (4.36) takes the form

$$\left\{L(\lambda) \overset{\otimes}{,} L(\mu)\right\}_r = [r(\lambda,\mu),\, L(\lambda) \otimes I + I \otimes L(\mu)]. \tag{4.37}$$

(ii) Deduce (4.37) by formal linearization from (4.35). (iii) Compare the result with formula (7.11).

4.7. Schouten bracket and the Gelfand–Dorfman Theorem

The most difficult part of theorem 4.34 is the check of the Jacobi identity. Let us consider the most general skew symmetric bracket which is a derivation with respect to both its arguments. Such bracket may be written as

$$\{h_1,\, h_2\} = \langle H(dh_1),\, dh_2\rangle. \tag{4.38}$$

It is easy to see that the obstruction for the Jacobi identity associated with (4.38) is a trilinear form in the differentials of functions. A convenient formalism to compute this obstruction was proposed by Gelfand and Dorfman **[GD]**. The Gelfand–Dorfman construction allows to define a Poisson bracket in a fairly general setting with the help of some elementary homological algebra. We start with the description of this general construction.

Let $\mathcal{G}$ be a Lie algebra, and M a $\mathcal{G}$-module; let $\Omega = \bigwedge^* \mathcal{G}^* \otimes M = \oplus_{p \geq 0} \bigwedge^p \mathcal{G}^* \otimes M$ be the complex of exterior form on $\mathcal{G}$ with values in M.

[7] In the case of Lie dialgebras we studied before the r-bracket is defined on the *dual* space; an "extra" dualization is connected with the change of the definition of the r-matrix. Since for factorizable Lie algebras we identify $\mathfrak{g}$ with its dual, this "extra" dualization is totally harmless.

The Chevalley differential in $d\colon \Omega_p \to \Omega_{p+1}$ is defined in a standard way,

$$d\alpha(X_0 \dots X_p) = \sum_i (-1)^i X\alpha(X_0 \dots \widehat{X}_i \dots X_p) - \\ - \sum_{i+j} (-1)^{i+j} \alpha\big([X_i, X_j], X_0 \dots \widehat{X}_i \dots \widehat{X}_j \dots X_p\big).$$

Let us associate with each $X \in \mathcal{G}$ the interior derivative $i_X\colon \Omega_p \to \Omega_{p-1}$:

$$i_X\alpha(X_1 \dots X_{p-1}) = \alpha(X, X_1 \dots X_{p-1}). \tag{4.39}$$

For $p = 1$, formula (4.39) defines a coupling

$$\mathcal{G} \times \Omega_1 \longrightarrow M\colon \langle X, \alpha\rangle = i_X\alpha = \alpha(X).$$

The Lie derivative is defined by the Cartan formula

$$\mathcal{L}_X = d\, i_X + i_X d; \tag{4.40}$$

it satisfies $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$, i.e., gives a representation of $\mathcal{G}$ in Ω.

Suppose now that $H, K\colon \Omega_1 \to \mathcal{G}$ are linear operators. Their *Schouten bracket* is a trilinear map from $\Omega_1 \times \Omega_1 \times \Omega_1$ into M defined by[8]

$$[H, K](\alpha_1, \alpha_2, \alpha_3) = \langle H\mathcal{L}_{K\alpha_1}\alpha_2, \alpha_3\rangle + \langle K\mathcal{L}_{H\alpha_1}\alpha_2, \alpha_3\rangle + c.p. \tag{4.41}$$

Let us now define the Poisson bracket associated with the Poisson operator $H\colon \Omega_1 \to \mathcal{G}$, by the formula

$$\{\varphi, \psi\} = \langle H\, d\varphi, d\psi\rangle = i_{Hd\varphi} d\psi, \quad \varphi, \psi \in M. \tag{4.42}$$

REMARK 4.37. In this definition we do not suppose that the "space of functions" Ω_0 has the structure of an algebra, so that the Leibniz rule for (4.42) makes sense. As a matter of fact, quite often it is convenient to define the Poisson bracket on some set of "distinguished observables". Such a space need not be an algebra, but one can always consider the algebra they generate and apply the Leibniz rule in order to extend the Poisson bracket to this larger setting.

THEOREM (Gelfand and Dorfman [**GD**]). *The bracket* (4.42) *is a Lie bracket on* Ω_0 *if and only if* H *is skew and its Schouten bracket with itself is zero. Poisson brackets with Poisson operators* H, K *are compatible if and only if* $[H, K] = 0$.

PROOF. We have

$$\begin{aligned} 0 = \tfrac{1}{2}[H, H](d\varphi_1, d\varphi_2, d\varphi_3) &= \\ = \langle H\mathcal{L}_{H\, d\varphi_1} d\varphi_2, d\varphi_3\rangle + c.p. &= \langle H\, d\, i_{H\, d\varphi_1} d\varphi_2, d\varphi_3\rangle + c.p. = \\ &= \langle Hd\{\varphi_1, \varphi_2\}, d\varphi_3\rangle + c.p. = \{\{\varphi_1, \varphi_2\}, d\varphi_3\} + c.p. \end{aligned} \tag{4.43}$$

[8]Here and below we denote by $c.p$ the sum of terms obtained from the ones listed by a cyclic permutation of the arguments.

4.8. Jacobi Identity and Yang–Baxter Equation

Let G be a Lie group. Let $r, r' \in \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g})$ be skew symmetric linear maps.

For $\varphi \in C^\infty(G)$ we denote by $D\varphi,\, D'\varphi \in \mathfrak{g}^*$ its left nd right diffentials which are defined, in analogy with (4.31), by

$$\begin{aligned}\langle D\varphi(x),\, \xi\rangle &= \big(\tfrac{d}{dt}\big)_{t=0}\, \varphi\big(e^{t\xi}x\big),\\ \langle D'\varphi(x),\, \xi\rangle &= \big(\tfrac{d}{dt}\big)_{t=0}\, \varphi\big(x\, e^{t\xi}x\big)\end{aligned} \tag{4.44}$$

for any $\xi \in \mathfrak{g}$. Let us define on G the left and right brackets (in general, neither of them satisfies the Jacobi identity):

$$\{\varphi,\, \psi\}_r = \langle r(D\varphi),\, D\psi\rangle, \quad \{\varphi,\, \psi\}_{r'} = \langle r'(D'\varphi),\, D'\psi\rangle, . \tag{4.45}$$

The complex associated with these brackets is the usual de Rham complex of G. Let $\mathcal{G} = \mathrm{Vect}\, G$ be the Lie algebra of vector fields on G. For $x \in G$ we denote by $\rho_x,\, \lambda_x \colon \mathfrak{g} \to T_xG$ the diffentials of the right and left translations by x. Let $\rho_x^*,\, \lambda_x^* \colon T_xG \to \mathfrak{g}$ be the dual maps. The Poisson operators associated with the brackets (4.45), are given by

$$H_r(x) = \lambda_x \circ r \circ \lambda_x^*, \quad H'_{r'}(x) = \rho_x \circ r' \circ \rho_x^*. \tag{4.46}$$

Lemma 4.38. *the Schouten brackets of $H_r,\, H'_{r'}$ are given by*

$$\begin{aligned}[][H_r,\, H_r](d\varphi_1,\, d\varphi_2,\, d\varphi_3) &= \langle D\varphi_1,\, \big[r(D\varphi_2),\, r(D\varphi_3)\big]\rangle + c.p.,\\ [H'_{r'},\, H'_{r'}](d\varphi_1,\, d\varphi_2,\, d\varphi_3) &= -\langle D'\varphi_1,\, \big[r'(D'\varphi_2),\, r'(D'\varphi_3)\big]\rangle - c.p.,\\ [H_r,\, H'_{r'}] &= 0.\end{aligned} \tag{4.47}$$

Proof. The equality $[H_r,\, H'_{r'}] = 0$ is immediate, since left and right translations commute with each other. For ny $\alpha \in \Omega_1$, $X, Y \in \mathrm{Vect}\, G$ we have

$$X \cdot \langle \alpha, Y\rangle = \langle \mathcal{L}_X\alpha, Y\rangle + \langle \alpha, [X, Y]\rangle$$

(where we denoted by $X \cdot f$ the action of X on f); hence the definition of the Schouten bracket (4.41) and the skew symmetry of H_r yield

$$[H_r,\, H_r](\alpha,\, \beta,\, \gamma) = H_r\alpha \cdot \beta(H_r\,\gamma) - \beta\big([H_r\alpha, H_r\gamma]\big) + c.p. \tag{4.48}$$

Choose a basis $\{\omega^i\}$ in the space of right-invariant 1-forms on G (which we identify with $\mathfrak{g}^*$), and let

$$\alpha = \sum \alpha_i\omega^i, \quad \beta = \sum \beta_i\omega^i, \quad \gamma = \sum \gamma_i\omega^i, \quad \alpha_i,\, \beta_i,\, \gamma_i \in C^\infty(G).$$

Then $H_r\alpha = \sum \alpha_i r\omega^i$, etc. The Lie bracket $H_r\alpha,\, H_r\beta$ is equal to

$$[H_r\alpha,\, H_r\gamma] = \sum_{i,j} \alpha_i\gamma_j[r\omega^i,\, r\omega^j] + \sum_i (H_r\alpha \cdot \gamma_i - H_r\gamma \cdot \alpha_i)r\omega^i.$$

In a similar way,

$$H_r\alpha \cdot \beta(H_r\gamma) = \sum_{i,j} H_r\alpha \cdot (\beta_i\gamma_j)(r\omega^j,\, \omega^i).$$

Substitute these expressions into (4.48); after cyclic permutation all terms containing derivatives of α_i, β_i, γ_i cancel out, and we get

$$\tfrac{1}{2}[H_r,\, H_r](\alpha,\, \beta,\, \gamma) = \sum_{i,j,k} \alpha_i\beta_j\gamma_k \langle \omega^i,\, [r\omega^j,\, r\omega^k]\rangle + c.p.$$

The second formula in (4.47) is proved in a similar way. The important sign difference is due to the fact that the Lie algebras of left- and right-invariant vector fields on G are *anti-isomorphic.*

Let us now assume that $\mathfrak{g}$ is equipped with an invariant inner product which we use to identify $\mathfrak{g}$ and $\mathfrak{g}^*$; we assume, moreover, that $r, r' \in \operatorname{End}\mathfrak{g}$ satisfy the modified Yang–Baxter identity. For any $\xi \in \mathfrak{g}^* \simeq \mathfrak{g}$ let us denote by $\omega_\xi, \omega'_\xi \in \Omega_1(G)$ the corresponding right- and left-invariant 1-forms on G.

LEMMA 4.39. *We have*

$$[H_r,\ H_r]\,(\omega_{\xi_1}, \omega_{\xi_2}, \omega_{\xi_3}) = -\langle \xi_1, [\xi_2, \xi_3]\rangle,$$
$$[H'_{r'},\ H'_{r'}]\,(\omega'_{\xi_1}, \omega'_{\xi_2}, \omega'_{\xi_3}) = \langle \xi_1, [\xi_2, \xi_3]\rangle,$$

and hence $[H_r \pm H'_{r'},\, H_r \pm H'_{r'}] = 0.$

PROOF. By (4.47) we have

$$[H_r,\, H_r]\,(\omega_{\xi_1}, \omega_{\xi_2}, \omega_{\xi_3}) = \langle \xi_1, [r\xi_2, r\xi_3]\rangle + \langle \xi_2, [r\xi_3, r\xi_1]\rangle + \langle \xi_3, [r\xi_1, r\xi_2]\rangle.$$

Now, (2.21) implies that

$$\langle \xi_1, [r\xi_2, r\xi_3]\rangle = -\langle \xi_1, [\xi_2, \xi_3]\rangle - \langle \xi_1, [r\xi_2, \xi_3]\rangle - \langle \xi_1, [\xi_2, r\xi_3]\rangle;$$

use the invariance of the inner product and the skew symmetry of r to see all terms containing r cancel.

Lemma 4.39 implies that the brackets (4.45) do not satisfy the Jacobi identity separately. However, for the Sklyanin bracket the obstructions cancel each other.

In a more general way, we can consider on G the Poisson bracket

$$\{\varphi,\, \psi\}_{r,r'} = \tfrac{1}{2}\langle r'(\nabla'\varphi\rangle,\, \nabla'\psi\big) + \tfrac{1}{2}\langle r(\nabla\varphi),\, \nabla\psi\rangle. \tag{4.49}$$

this bracket also satisfies the Jacobi identity. Note in particular that both r and $-r$ satisfy the modified Yang–Baxter identity simultaneously. Hence the bracket

$$\{\varphi,\, \psi\}_{r,r} = \tfrac{1}{2}\langle r(\nabla\varphi),\, \nabla\psi\rangle + \tfrac{1}{2}\langle r(\nabla\varphi),\, \nabla\psi\rangle \tag{4.50}$$

satisfies the Jacobi identity. The brackets (4.49), (4.50) play an important role in the study of the Poisson geometry of G and in the geometric theory of Lax equations. Poisson structures of the form (4.49) are sometimes called *affine Poisson structures*[9].

[9] An abstract definition of affine Poisson structures may be found in [**Wein2**]. Brackets (4.49) (introduced in [**STS2**]) give the main example of such structures. When G is semisimple, all affine Poisson structures on G have the form (4.49) for some appropriate r, r'.

REMARK 4.40. By a simple modification of the argument in Lemma 4.38, one can prove the following assertion.

THEOREM. *A multiplicative Poisson bracket on a Lie group satisfies the Jacobi identity if and only it is true for its linearization at the unit element.*

4.9. Properties of Affine Poisson Structures on G

The brackets (4.49), (4.50) defined above are not multiplicative. However, their behavior under multiplicatiion in G admits a nice characteristic. Let us denote by $G_{(r,r')}$ a copy of G equipped with the bracket (4.49); in particular, $G_{(r,-r)}$ is a Poisson Lie group equipped with the Sklyanin bracket.

PROPOSITION 4.41. *Multiplication in G defines Poisson mappings*

$$G_{(r,-r)} \times G_{(r,r')} \longrightarrow G_{(r,r')}, \tag{4.51}$$

$$G_{(r,r')} \times G_{(r',-r')} \longrightarrow G_{(r,r')}. \tag{4.52}$$

To put it in another way, the Poisson bracket (4.49) is left-$G_{(r,-r)}$-covariant and right-$G_{(r',-r')}$-covariant.

PROOF. Let λ_x, ρ_x be the left and right translation operators by $x \in G$ acting in $C^\infty(G)$. The covariance of (4.49) with respect to left translations is expressed by the identity

$$\{x,\, y\}_{r,r'}(x,\, y) = \{\lambda_x\varphi,\, \lambda_x\psi\}_{r,r'}(y) + \{\rho_y\varphi,\, \rho_y\psi\}_{r,-r}(x), \tag{4.53}$$

which should be valid for any φ, $\psi \in C^\infty(G)$ and x, $y \in G$. Let $\eta'_{r,r'}$ be the Poisson operator associated with (4.49) in the left-invariant frame,

$$\eta'_{r,r'}(x) = \tfrac{1}{2}(r' + \mathrm{Ad}\, x^{-1} \circ r \circ \mathrm{Ad} x). \tag{4.54}$$

We have

$$\eta'_{r,r'}(xy) = \eta'_{r,r'}(y) + \mathrm{Ad}\, y^{-1} \circ \eta'_{r,-r} \circ \mathrm{Ad}\, y. \tag{4.55}$$

The definition of gradients (4.31) implies that

$$\nabla'\lambda_x\varphi(y) = \nabla'\varphi(xy), \quad \nabla'\rho_y\varphi(x) = \mathrm{Ad}\, y \cdot \nabla'\varphi(x). \tag{4.56}$$

It is now clear from (4.56) that formulae (4.53) and (4.55) are equivalent. The right covariance of (4.49) is proved in a similar way. (It is now convenient to express the Poisson operator associated with (4.49) in the right-invariant frame.)

Multiplication maps in (4.51), (4.52) are examples of *Poisson group actions*. Note that in the abstract group theory a group G may be regarded as its own *principal homogeneous space* equipped with the free action of G by left and right translations. Moreover, a principal homogeneous space (for a given G) is unique up to a G-equivariant isomorphism. In the theory of Poisson groups, for a given G there exist plenty of different principal homogeneous spaces, since the covariance condition fixes the Poisson structure in a not too rigid way. As a result, the category of Poisson G-spaces (in particular, the category of homogeneous G-spaces) is also much richer: for instance, left homogeneous G-spaces may be obtained from the principal homogeneous spaces $G_{(r,r')}$ by means of reduction over the right action of an appropriate subgroup of G. A classification theorem for the Poisson homogeneous spaces was proved by Drinfeld [**D3**]. We shall return to the reduction theory for Poisson groups in Section 5.4 below.

LECTURE 5

Duality Theory for Poisson Lie Groups

5.1. The Drinfeld Double

As already noted, Lie bialgebras possess a remarkable symmetry: if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, the same is true for $(\mathfrak{g}^*, \mathfrak{g})$. Hence *the dual Lie group* G^*, which corresponds to the Lie algebra $\mathfrak{g}^*$, also carries a multiplicative Poisson structure. For factorizable Lie groups this Poisson structure may be pushed forward to G by means of the factorization mapping, and hence we get *two* Poisson brackets on G which play an important role in the geometric theory of Lax equations which we shall discuss in the next lecture. The simplest way to describe this duality is based on the following important observation, due to Drinfeld: both G and G^* are Poisson subgroups of a bigger Poisson group D, their common *double.*

Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra; the linear space $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ is equipped with the natural inner product

$$\left\langle\left\langle (X, F), (X', F') \right\rangle\right\rangle = \left\langle F, X' \right\rangle + \left\langle F', X \right\rangle. \tag{5.1}$$

THEOREM 5.1. *There is a unique structure of a Lie algebra in $\mathfrak{d}$ such that: (i) $\mathfrak{g}, \mathfrak{g}^* \subset \mathfrak{d}$ are its Lie subalgebras. (ii) The inner product (5.1) is* $\operatorname{ad}\mathfrak{d}$*-invariant.*

Conversely, if $\mathfrak{d}$ is a Lie algebra with an invariant inner product and $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{d}$ are its isotropic Lie subalgebras such that $\mathfrak{d} = \mathfrak{a} \dot{+} \mathfrak{b}$ as a linear space, then $(\mathfrak{a}, \mathfrak{b})$ and $(\mathfrak{b}, \mathfrak{a})$ are mutually dual Lie bialgebras.

Triples $(\mathfrak{d}, \mathfrak{a}, \mathfrak{b})$ of the form described above are called *Manin triples.*

PROOF. Both $\mathfrak{g}$ and $\mathfrak{g}^*$ carry the natural structure of $\mathfrak{g}$-modules (the adjoint action of $\mathfrak{g}$ in itself and the coadjoint action in $\mathfrak{g}^*$) and of $\mathfrak{g}^*$-modules (the coadjoint action of $\mathfrak{g}^*$ in itself and the adjoint action in $\mathfrak{g}$). Let us define the Lie bracket in $\mathfrak{d} = \mathfrak{g} \dot{+} \mathfrak{g}^*$ by

$$[(X,\xi),\, (Y,\eta)] = ([X,Y] + \operatorname{ad}^*_{\mathfrak{g}^*}\xi \cdot Y - \operatorname{ad}^*_{\mathfrak{g}^*}\eta \cdot X,\, [\xi,\eta]_* + \operatorname{ad}^*_{\mathfrak{g}} X \cdot \eta - \operatorname{ad}^*_{\mathfrak{g}} Y \cdot \xi).$$

It is easy so see that this is the only definition which is compatible with the invariance of the inner prooduct (5.1). The Jacobi identity for this bracket is equivalent to the Jacobi identities for $\mathfrak{g}, \mathfrak{g}^*$ completed by the matching condition of the two brackets (i. e., the cocycle equation). One can say that $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ is a *bi-cross-product* of $\mathfrak{g}$ and $\mathfrak{g}^*$.

COROLLARY 5.2. *Let* $P_{\mathfrak{g}}, P_{\mathfrak{g}^*}$ *be the projection maps onto* $\mathfrak{g}, \mathfrak{g}^*$ *in the decomposition* $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$. *Put* $r_{\mathfrak{d}} = P_{\mathfrak{g}} - P_{\mathfrak{g}^*}$*; then* $r_{\mathfrak{d}}$ *defines on* $\mathfrak{d}$ *the structure of a factorizable Lie bialgebra.*

The pair $(\mathfrak{d}, \mathfrak{d}^*)$ is called the *double* of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. The group $D(G)$ with the Lie algebra $\mathfrak{d}$ is called the *double*, or, more precisely, the *Drinfeld double* of G.

PROPOSITION 5.3. *Let us equip* $D(G)$ *with the Sklyanin bracket associated with* $r_{\mathfrak{d}}$. *Then* $G^{\delta}, G^* \subset D(G)$ *are Poisson subgroups in* D*, i. e., they are both Poisson submanifolds in* D *and the induced Poisson brackets are multiplicative. The bracket induced on* G *coincides with the initial one; the bracket induced on* G^* *is mapped to the standard multiplicative bracket on* G^* *with the tangent Lie bialgebra* $(\mathfrak{g}^*, \mathfrak{g})$ *by means of inversion* $x \mapsto x^{-1}$.[1]

5.2. Examples

1°. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be the trivial Lie bialgebra, i. e., the Lie bracket on $\mathfrak{g}^*$ is identically zero. Then $\mathfrak{d} = \mathfrak{g} \ltimes \mathfrak{g}^*$, and $D = G \ltimes \mathfrak{g}^*$ is the semi-direct product of G and of the additive group of $\mathfrak{g}^*$. The Sklyanin bracket on D is the Lie–Poisson bracket on $\mathfrak{g}^*$, extended by zero onto G. Note that as a manifold D coincides with the cotangent bundle T^*G, but the Poisson bracket on D is highly degenerate. We shall return to this example in Section 5.12, where we are going to define another Poisson structure on D which is the natural analog of the canonical Poisson bracket on T^*G (and reduces to this bracket for trivial Lie bialgebras).

2°. Let $\mathfrak{g}$ be a complex semisimple Lie algebra considred as a Lie algebra over $\mathbb{R}$. Let us equip $\mathfrak{g}$ with an inner product

$$(X, Y) = \operatorname{Im} B(X, Y),$$

where B is the complex Killing form on $\mathfrak{g}$. Let $\mathfrak{k}$ be the compact real form of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{b}$ the Iwasawa decomposition of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{k}, \mathfrak{b})$ is a Manin triple. We conclude that every compact semisimple Lie algebra carries a canonical structure of a Lie bialgebra; its double is the corresponding complex Lie algebra. The double of a compact semisimple Lie group K is the associated complex group G regarded as a real group. The particular advantage of this example is that the factorization problem in G coincides with the Iwasawa decomposition and hence is always globally solvable.

REMARK 5.4. The associated r-matrix on $\mathfrak{k}$ is an example of a *non-split classical r-matrix* which satisfies the modified Yang-Baxter equation

$$[rX, rY] - r([rX, Y] + [X, rY]) = [X, Y].$$

[1]The sign difference is due the the minus sign in $r_{\mathfrak{d}} = P_{\mathfrak{g}} - P_{\mathfrak{g}^*}$.

3°. We return to the setting of Section 3.3. Let $\mathfrak{L}(\mathfrak{g})_D = \bigoplus_{\nu\in D} \mathfrak{g} \otimes \mathbb{C}((\lambda_\nu))$ be the algebra of formal Laurent series associated with a finite set D of points on the Riemann sphere, $\infty \in D$. Then

$$\mathfrak{L}(\mathfrak{g})_D = R_D(\mathfrak{g}) \dotplus \mathfrak{L}(\mathfrak{g})^+_D$$

and $\mathfrak{L}(\mathfrak{g})_D,\ R_D(\mathfrak{g})\ \mathfrak{L}(\mathfrak{g})^+_D$ is a Manin triple. This assertion immediately follows from Proposition 3.7. More generally, the algebra of adèles

$$\mathfrak{L}(\mathfrak{g})_A = \coprod_{\nu\in\mathbb{C}P_1} \mathfrak{g} \otimes \mathbb{C}((\lambda_\nu)).$$

is the double of the Lie bialgebra of rational functions $R(\mathfrak{g})$.

4°. Our next example is based on an entirely different choice of the basic Lie algebra. Let $\mathfrak{A}$ be the Lie algebra of *formal pseudodifferential operators* on the line (or on the circle). By definition, elements of $\mathfrak{A}$ are formal Laurent series of the form

$$X(x,\xi) = \sum_{-N}^{\infty} X_n(x)\xi^{-n}, \quad X_n \in C^\infty(\mathbb{R}). \tag{5.2}$$

There is a unique way to define on $\mathfrak{A}$ an associative product which is compatible with the Leibniz rule

$$\xi \circ X - X \circ \xi = \partial_x X, \quad X \in C^\infty(\mathbb{R}).$$

Explicitly, the product of two pseudodifferential operators is given by

$$a \circ b(x,\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_\xi^n a(x,\xi)\, \partial_x^n b(x,\xi), \tag{5.3}$$

where the product in the r.h.s. is the usual product of formal series. The structure of a Lie algebra on $\mathfrak{A}$ is defined by the standard commutator $[X, Y] = X \circ Y - Y \circ X$. It is easy to see that the subspaces $\mathfrak{A}_+, \mathfrak{A}_- \subset \mathfrak{A}$ which consist of polynomials in ξ and of formal series in ξ^{-1} without constant term, respectively, are Lie subalgebras in $\mathfrak{A}$. Put

$$\operatorname{Tr} X = \int \operatorname{Res} X(x,\xi)\, dx, \tag{5.4}$$

where, as usual, $\operatorname{Res} X(x,\xi) = X_{-1}(x)$.

EXERCISE 5.5. Prove that $\operatorname{Tr} X \circ Y = \operatorname{Tr} Y \circ X$ for any $X, Y \in \mathfrak{A}$.

Hence formula

$$\langle X, Y\rangle = \operatorname{Tr} X \circ Y \tag{5.5}$$

defines on $\mathfrak{A}$ a nondegenrate invariant inner product[2].

[2]The only condition which should be imposed on the coefficients $X_n \in C^\infty(\mathbb{R})$ should provide the possibility to drop out total derivatives in the integration by parts. This condition is satisfied e.g. if all coefficients are rapidly decreasing, or if they are periodic.

PROPOSITION 5.6. $(\mathfrak{A}, \mathfrak{A}_+, \mathfrak{A}_-)$ *is a Manin triple. Hence the inner product* (5.5) *sets the subalgebras* $\mathfrak{A}_+$ *and* $\mathfrak{A}_-$ *into duality; both carry the structure of a Lie bialgebra and* $\mathfrak{A}$ *is their common double.*

There is just one obvious Lie group which can be associated with this Manin triple: this is the *group* $\mathfrak{I}$ *of formal integral operators* which consists of formal series of the form $1 + X$, $X \in \mathfrak{A}_-$. However, one can notice that invertibility is not really needed to define the Sklyanin bracket. Its definition perfectly makes sense for functions defined on an associative algebra. The corresponding bracket on $\mathfrak{A}$ is called *the second Gelfand–Dickey bracket*, or Adler–Gelfand–Dickey bracket (the *first* bracket is just the Lie–Poisson bracket associated with $\mathfrak{A}$). Explicitly, the Poisson bracket of two functionals φ, ψ on $\mathfrak{A}$ is given by

$$\begin{aligned}(5.6)\qquad \{\varphi, \psi\}(L) = \tfrac{1}{2}\operatorname{Tr}((\operatorname{grad}\varphi \circ L)_+ \circ \operatorname{grad}\psi \circ L) - \\ - \tfrac{1}{2}\operatorname{Tr}((L \circ \operatorname{grad}\varphi)_+ \circ L \circ \operatorname{grad}\psi),\end{aligned}$$

where $\operatorname{grad}\varphi, \operatorname{grad}\psi \in \mathfrak{A}_-$ are the variational derivatives of φ, ψ, and X_+ is the natural projection of $X \in \mathfrak{A}$ onto $\mathfrak{A}_+$, with all termes of non-positive degree wiped off. The subspaces $\mathfrak{A}_\pm \subset \mathfrak{A}$ are Poisson subspaces with respect to this bracket. In particular, we get a quadratic Poisson bracket on the space of ordinary differential operators.

5.3. Double of a Factorizable Lie Bialgebra

If the initial Lie bialgebra is itself factorizable, its double admits a simple explicit description. Let us consider the Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ (the direct sum of two copies of $\mathfrak{g}$) equipped with the inner product

$$(5.7)\qquad \langle\langle (X, Y), (X', Y') \rangle\rangle = \langle X, X' \rangle - \langle Y, Y' \rangle,$$

where $\langle\ ,\ \rangle$ is the invariant inner product in $\mathfrak{g}$.

PROPOSITION 5.7. *The double of a factorizable Lie bialgebra is canonically isomorphic (as a Lie algebra) to* $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$.

PROOF. Recall (see Section 2.5) that there are two homomorphisms $r_\pm \colon \mathfrak{g}^* \to \mathfrak{g}$ associated with a given r-matrix satisfying the modified Yang–Baxter identity; together, they define an embedding $\mathfrak{g}^* \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$ (cf. Proposition 2.19). Let $\mathfrak{g}^\delta \subset \mathfrak{g} \oplus \mathfrak{g}$ be the diagonal subalgebra, $\mathfrak{g}^\delta = \{(X, X);\ X \in \mathfrak{g}\}$.

LEMMA 5.8. *(i) We have:* $\mathfrak{d} = \mathfrak{g}^\delta \dot{+} \mathfrak{g}^*$. *(ii) The suabalgebras* $\mathfrak{g}^*$, $\mathfrak{g}^\delta \subset \mathfrak{g} \oplus \mathfrak{g}$ *are isotropic with respect to the inner product (5.7)*[3].

[3]In Section 2.6 we have already considered a version of this construction for double Lie algebras. The only difference is that now $\mathfrak{g}$ are $\mathfrak{d}$ are equipped with an invariant inner product and the subalgebras $\mathfrak{g}^\delta$, $\mathfrak{g}^* \subset \mathfrak{d}$ must be isotropic.

Clearly, the isotropy condition is equivalent to the skew symmetry of $r_{\mathfrak{d}} = P_{\mathfrak{g}} - P_{\mathfrak{g}^*}$. It is useful to write down $r_{\mathfrak{d}}$ in the block form associated with the decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$. We have

$$r_{\mathfrak{d}} = \begin{pmatrix} r & -2r_+ \\ 2r_- & -r \end{pmatrix}. \tag{5.8}$$

The double of a factorizable Lie group G is $D\,(G) = G \times G$. Let G^δ, $G^* \subset G \times G$ be the subgroups which correspond to the Lie subalgebras $\mathfrak{g}^\delta, \mathfrak{g}^*$. Clearly, $G^\delta \subset D\,(G)$ is the diagonal subgroup. The decomposition $\mathfrak{d} = \mathfrak{g}^\delta \dot{+} \mathfrak{g}^*$ gives rise to a factorization problem in $D = G \times G$. As we have already seen in 2.6, this factorization problem may be reduced to the one in G. Let us assume for simplicity that the factorization problem (2.26) in G associated with the original r-matrix is globally solvable. In this case we have:

PROPOSITION 5.9. *$D(G) = G^* \cdot G^\delta$; in other words, factorization problem in $D(G)$,*

$$(x, y) = (\eta_+, \eta_-) \cdot (\xi, \xi)\,, \quad (\eta_+, \eta_-) \in G^* \subset D,\, \xi \in G, \tag{5.9}$$

is also globally solvable.

Indeed, we have

$$\eta_\pm = (xy^{-1})_\pm, \quad \xi = (xy^{-1})_+^{-1} x = (xy^{-1})_-^{-1} y.$$

REMARK 5.10. As we have just seen, the double of a factorizable Lie bialgebra is itself factorizable. The converse is not true: a factorizable Lie bialgebra need not be the double of some other one. However, the obstruction is not too serious: it is related to the overlap of the subalgebras $\mathfrak{g}_\pm \subset \mathfrak{g}$ and may be removed with the help of the Cayley transform of r. Let us illustrate the issue on a simple example.

Let $\mathfrak{g}$ be a semisimple Lie algebra equipped with the standard structure of a Lie bialgebra which we described in Example 2°, Section 4.2 (formula (4.21)). As we have seen the Borel subalgebras $\mathfrak{b}_\pm \subset \mathfrak{g}$ are sub-bialgebras in $\mathfrak{g}$. Let us identify $\mathfrak{h}$ with the quotient Lie algebra $\mathfrak{b}_\pm/\mathfrak{n}_\pm$, and let $p\colon \mathfrak{b}_\pm \to \mathfrak{h}$be the canonical projection map. Set $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$; let us equip $\mathfrak{d}$ with the inner product

$$\langle\langle (X, H),\, (X', H') \rangle\rangle = \langle X, X' \rangle - \langle H, H' \rangle.$$

Let us define the embeddings of the opposite Borel subalgebras $\mathfrak{b}_\pm$ into $\mathfrak{d}$ by

$$i_\pm\colon \mathfrak{b}_\pm \hookrightarrow \mathfrak{g} \oplus \mathfrak{h} \colon X_\pm \longmapsto (X_\pm, \pm p(X_\pm)).$$

Then $(\mathfrak{d},\, \mathfrak{b}_+,\, \mathfrak{b}_-)$ is a Manin triple and the natural embedding $\mathfrak{g} \hookrightarrow \mathfrak{d}$ is a morphism of Lie bialgebras

The inversion $X \mapsto -X$ on $\mathfrak{h}$ is of course just the Cayley transform of the standard r-matrix on $\mathfrak{g}$.

5.4. Poisson Reduction.

We shall continue the study of the Poisson structure on Poisson groups. In order to give an explicit description of symplectic leaves in G, G^* it is useful to realize both groups not as Poisson subgroups in $D(G)$, but rather as Poisson quotient spaces. Let us start with the general reduction theory for Poisson group actions. For the reader's convenience we recall two definitions which were already discussed earlier.

DEFINITION 5.11. (i) An action $G \times \mathcal{M} \longrightarrow \mathcal{M}$ of a Lie group G on a Poisson manifold $\mathcal{M}$ is called *admissible* if the subspace $C^\infty(\mathcal{M})^G$ of G-invariant functions is a Lie subalgebra in $C^\infty(\mathcal{M})$ with respect to the Poisson bracket.

(ii) An action of a Poisson group G on a Poisson manifold $\mathcal{M}$ is *Poisson* if $G \times \mathcal{M} \to \mathcal{M}$ is a Poisson mapping.

Let G be a connected Poisson group, let $(\mathfrak{g}, \mathfrak{g}^*)$ be its tangent Lie bialgebra. Let $G \times \mathcal{M} \to \mathcal{M}$ be an action of G on a Poisson manifold $\mathcal{M}$. For $\varphi \in C^\infty(\mathcal{M})$ we define a covector $\xi_\varphi \in \mathfrak{g}^*$ by

$$\langle \xi_\varphi \,, X \rangle = \left(\tfrac{d}{dt}\right)_{t=0} \varphi(e^{tX} \cdot x), \ X \in \mathfrak{g}.$$

Denote by $\widehat{X}$ the vector field on $\mathcal{M}$ generated by $X \in \mathfrak{g}$.

PROPOSITION 5.12. *An action $G \times \mathcal{M} \to \mathcal{M}$ is Poisson if and only if*

$$\widehat{X}\{\varphi, \psi\}_{\mathcal{M}} - \{\widehat{X}\varphi, \psi\}_{\mathcal{M}} - \{\varphi, \widehat{X}\psi\}_{\mathcal{M}} = \langle [\xi_\varphi, \xi_\psi]_* \,, X \rangle \tag{5.10}$$

for all $X \in \mathfrak{g}$.

PROOF. For $\varphi \in C^\infty(\mathcal{M})$ we denote by $\lambda_g\varphi$ the shifted function, $\lambda_g\varphi(x) = \varphi(g \cdot x)$. Let us denote by $\widehat{\varphi}_x$, $x \in \mathcal{M}$ the pullback of φ to G, $\widehat{\varphi}_x(g) = \varphi(g \cdot x)$. The action $\lambda\colon G \times \mathcal{M} \to \mathcal{M}$ is Poisson if and only if for any φ, $\psi \in C^\infty(\mathcal{M})$ we have

$$\Big\{\lambda^*\varphi, \lambda^*\psi\Big\}_{G\times\mathcal{M}}(g, x) = \Big\{\lambda_g\varphi, \lambda_g\psi\Big\}_{\mathcal{M}}(x) + \Big\{\widehat{\varphi}_x, \widehat{\psi}_x\Big\}_G(g). \tag{5.11}$$

Setting $g = \exp sX$ and taking the derivative with respect to s for $s = 0$, we get (5.12). Conversely, (5.12) implies that (5.11) holds for any one-parameter subgroup in G and hence, since G is connected, for the entire group G.

THEOREM 5.13. *Let us suppose that $G \times \mathcal{M} \to \mathcal{M}$ is a Poisson groups action and $H \subset G$ is a closed connected subgroup. Assume that the annihilator of its Lie algebra $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is a Lie subalgebra in $\mathfrak{g}^*$. Then the restriction of the action of G to H is admissible.*

PROOF. Suppose that φ, $\psi \in C^\infty(\mathcal{M})^H$. Then $\widehat{X}\varphi = \widehat{X}\psi = 0$ for all $X \in \mathfrak{h}$, and the covectors ξ_φ, ξ_ψ belong to $\mathfrak{h}^\perp$. Formula (5.10) immediately implies that

$$\widehat{X}\{\varphi, \psi\} = \langle [\xi_\varphi, \xi_\psi]_* \,, X \rangle = 0,$$

and hence $\{\varphi, \psi\} \in C^\infty(\mathcal{M})^H$.

COROLLARY 5.14. *Suppose that the space* $H\backslash\mathcal{M}$ *of* H*-orbits in* $\mathcal{M}$ *is a smooth manifold. There exists a unique Poisson structure on* $H\backslash\mathcal{M}$ *such that the canonical projection*$\pi\colon \mathcal{M} \to H\backslash\mathcal{M}$ *is a Poisson map.*

The passage from $\mathcal{M}$ to $H\backslash\mathcal{M}$ is called *Poisson reduction.*

REMARK 5.15. A Lie subgroup $H \subset G$ is a Poisson subgroup (i.e., a subgroup in G which is also a Poisson submanifold) if and only if the annihilator of its Lie algebra $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is an *ideal* in $\mathfrak{g}^*$. In this case H is itself a Poisson Lie group, its tangent Lie bialgebra is $(\mathfrak{h},\, \mathfrak{g}^*/\mathfrak{h}^\perp)$ and the restriction of the action of G to H is again a Poisson action. We see that the admissibility condition on $H \subset G$, which assures the existence of the quotient Poisson structure on $H\backslash\mathcal{M}$, is less restrictive.

5.5. Symplectic Double and Dressing Transformations

Let G be a Poisson group, and D its double. The Sklyanin bracket on D associated with the r-matrix $r_{\mathfrak{d}}$, is degenerate; however, there is another Poisson structure on D which is nondegenerate, except perhaps for a subset of positive codimension, and hence makes it a symplectic manifold (at least away from this exceptional subset). The rôle of this *symplectic double* in the theory of Poisson Lie groups is similar to that of the ordinary cotangent bundle T^*G in the conventional case.

The Poisson bracket on D which endows it with an almost nondegenerate Poisson structure is defined by

$$\{\varphi,\, \psi\}_+ = \tfrac{1}{2}\big\langle r_d(\nabla'\varphi),\, \nabla'\psi\big\rangle + \tfrac{1}{2}\big\langle r_d(\nabla\varphi),\, \nabla\psi\big\rangle. \tag{5.12}$$

Due to Theorem 4.34, the bracket (5.12) satisfies the Jacobi identity.

The study of the bracket (5.12) becomes more simple if the factorization problem in D is globally solvable, i.e. if $D \simeq G \times G^*$ as a manifold. We shall start with the study of this case.

PROPOSITION 5.16. *Assume that* $D \simeq G \times G^*$*; then the bracket (5.12) is nondegenerate.*

EXERCISE 5.17. If $(\mathfrak{g}, \mathfrak{g}^*)$ is a trivial Lie bialgebra, the bracket (5.12) coincides with the canonical Poisson bracket on $T^*G \simeq G \ltimes \mathfrak{g}^*$.

Let us equip D with the Sklyanin bracket (4.32) associated with the r-matrix $r_{\mathfrak{d}}$.

PROPOSITION 5.18. *Left and right multiplications*

$$D_{(r_{\mathfrak{d}},-r_{\mathfrak{d}})} \times D_{(r_{\mathfrak{d}},\,r_{\mathfrak{d}})} \to D_{(r_{\mathfrak{d}},r_{\mathfrak{d}})},$$
$$D_{(r_{\mathfrak{d}},r_{\mathfrak{d}})} \times D_{(r_{\mathfrak{d}},\,-r_{\mathfrak{d}})} \to D_{(r_{\mathfrak{d}},r_{\mathfrak{d}})}$$

are Poisson mappings.

This assertion is a special case of Proposition 4.41. We shall write $D_{(r_{\mathfrak{d}},\,r_{\mathfrak{d}})} = D_+$, $D_{(r_{\mathfrak{d}},-r_{\mathfrak{d}})} = D_-$ for brevity. As we know, G and G^* are

Poisson subgroups in D_-; hence theorem 5.13 allows to construct the Poisson quotients of D_+ with respect to the left and right action of G, G^* on D_+.

PROPOSITION 5.19. (i) *Canonical projections*

$$\begin{array}{ccccc} & & D & & \\ & \swarrow^{\pi_*} & & \searrow^{\pi'_*} & \\ D/G^* & & & & G^*\backslash D \end{array}$$

form a dual pair. (ii) *In the dual way, projections*

$$\begin{array}{ccccc} & & D & & \\ & \swarrow^{\pi} & & \searrow^{\pi'} & \\ D/G & & & & G\backslash D \end{array}$$

also form a dual pair.

PROOF. Let us prove (i); the second assertion is completely similar. Suppose that $\varphi \in C^\infty(D)$ is left-G^*-invariant and $\psi \in C^\infty(D)$ is right-G^*-invariant. Then $\nabla\varphi,\ \nabla'\psi \in (\mathfrak{g}^*)^\perp = \mathfrak{g}^*$. Hence

$$2\{\varphi,\, \psi\} = \langle r_{\mathfrak{d}}\nabla\varphi,\, \nabla\psi\rangle - \langle\nabla'\varphi,\, r_{\mathfrak{d}}\nabla'\psi\rangle = -\langle\nabla\varphi,\, \nabla\psi\rangle + \langle\nabla'\varphi,\, \nabla'\psi\rangle = 0.$$

By assumption, $D = G \cdot G^*$, i. e., the factorization problem in D is globally solvable. Thus we may identify the quotient spaces D/G^*, $G^*\backslash D$ (respectively, D/G, $G\backslash D$) with G (respectively, with G^*).

PROPOSITION 5.20. *The quotient Poisson brackets on $D/G^* \simeq G$, $D/G \simeq G^*$ coincide with the initial multiplicative brackets on G and G^*, respectively.*

The action of G^* on D by left translations gives rise to a natural action $G^* \times D/G^* \longrightarrow D/G^*$. We get a commutative diagram in which all arrows are Poisson maps:

$$\begin{array}{ccc} G^* \times D_+ & \xrightarrow{m} & D_+ \\ \Big\downarrow{\scriptstyle id\times\pi_*} & & \Big\downarrow{\scriptstyle \pi_*} \\ G^* \times D/G^* & \longrightarrow & D/G^*. \\ \Big\| & & \Big\| \\ G^* \times G & \xrightarrow{\mathrm{Dr}} & G \end{array} \tag{5.13}$$

In the dual way, we also get the commutative diagram

$$\begin{array}{ccc} G \times D_+ & \xrightarrow{m} & D_+ \\ \Big\downarrow{\scriptstyle id\times\pi} & & \Big\downarrow{\scriptstyle \pi} \\ G \times D/G & \longrightarrow & D/G. \\ \Big\| & & \Big\| \\ G \times G^* & \xrightarrow{\mathrm{Dr}} & G^* \end{array} \tag{5.14}$$

The mappings $\mathrm{Dr} : G^* \times G \to G,\ G \times G^* \to G^*$ in the bottom rows of these diagrams define an action of G and G^* on their duals; this is the so called *dressing action*[4]. As is clear from the diagrams (5.13), (5.14), dressing is a Poisson action; hence dressing transformations provide a nontrivial example of a *Poisson group action*[5]. Comparing the definition of dressing transformations with Theorem 1.15 and Proposition 5.19, we get the following result:

PROPOSITION 5.21. *Symplectic leaves in G and in G^* are orbits of dressing transformations.*

The definition of dressing transformations implies the following explicit formula:

PROPOSITION 5.22. *For $g \in G$, $g_* \in G^*$ let $h(g, g_*)$, $h_*(g, g_*)$ be the solutions of the factorization problem in D: $g_* \cdot g = h \cdot h_*$, $h \in G$, $h_* \in G^*$. Then $\mathrm{Dr}(g_*) \cdot g = h(g, g_*)$.*

The role of dressing transformations is quite similar to that of the coadjoint representation in the conventional case. This analogy is further confirmed by the following simple assertion.

PROPOSITION 5.23. *Dressing action $\mathrm{Dr} : G^* \times G \to G$ leaves the unit element in G invariant; linearization of this action in the tangent space $T_eG \simeq \mathfrak{g}$ coincides with the coadjoint action of G^*. In the dual way, dressing action $\mathrm{Dr} : G \times G^* \to G^*$ leaves invariant the unit element in G^*; its linearization in $T_eG^* \simeq \mathfrak{g}^*$ coincides with the coadjoint action of G.*

Let us describe the properties of (5.12) in general case. The subset $G \cdot G^*$ is open in D; this implies that D is the union of disjoint double coset classes,

$$D = \bigcup_i G\, d_i G^*,$$

where $d_0 = e$ and the set $\{d_i\} \simeq G\backslash D/G^*$ is finite or countable. Put $C_i = G\, d_i G^*, C^j = G^* d_j^{-1} G, C_i^j = C_i \cap C^j$.

[4]The term "dressing transformations" was introduced in **[STS2]** by analogy with the Zakharov–Shabat dressing in the soliton theory. We shall describe dressing transformations for zero curvature equations and their Poisson properties in Lecture 7.

[5]Note that in general a Poisson group action need not preserve symplectic leaves (example: the action of G on itself by left translations). Dressing action not only preserves symplectic leaves, but is in fact transitive on each leaf, at least locally.

PROPOSITION 5.24. [**AM**] *Symplectic leaves in D_+ coincide with the submanifolds $C_i^j \subset D$.*

In general case it is still possible to use projection maps π, π' and π_*, π'_* to describe symplectic leaves in the quotient manifolds D/G, D/G^*, as in Proposition 5.19; however before applying Theorem 1.15, one has to restrict these maps to symplectic leaves $C_i^j \subset D_+$. Another difficulty, which arises when the factorization problem is not globally solvable, is the incompleteness of dressing transformations. Indeed, we identified the group G with an open cell in D/G^* (this is true if $G \cap G^* = \{e\}$). In general, dressing transformations do not preserve this cell.

5.6. Factorizable Groups and their Duals.

For applications to Lax equations, we are mainly interested in the case when the group G and its tangent Lie bialgebra are factorizable. Recall that in this case the double of G is canonically identified with $G \times G$. Equip $D = G \times G$ with the Poisson structure (5.12). According to Proposition 5.20, the quotient Poisson bracket D_+/G^* coincides with the Sklyanin bracket. In order to check it explicitly let us notice first of all that since $D = G^\delta \cdot G^*$, we may use the diagonal subgroup $G^\delta \subset D$ as a model for the quotient space D_+/G^*. Let $\varphi, \psi \in C^\infty(G)$; we can extend them to the vicinity of the diagonal subgroup $G^\delta \subset D$ as right-G^*-invariant functions:

$$\Phi((x,x)\cdot(h_+,h_-)) = \varphi(x), \quad \Psi((x,x)\cdot(h_+,h_-)) = \psi(x).$$

We have

$$\{\varphi,\psi\}_* = \left.\{\Phi,\Psi\}\right|_{G^\delta},$$

where the bracket in the r.h.s of this equality is computed in D_+ and the resulting functions are then restricted to the diagonal subgroup.

To compute this bracket it is enough to know the gradients of Φ, Ψ on $G^\delta \subset D$.

LEMMA 5.25. *(i) The right gradient of Φ on $G^\delta \subset D$ is given by*

$$\left.\nabla'\Phi\right|_{G^\delta} = (X_+, X_-) \in \mathfrak{d}, \text{ where } X = \nabla\varphi,\ X_\pm = r_\pm X.$$

(ii) The left gradient Φ is given by

$$\nabla\Phi(x,x) = (\operatorname{Ad} x \cdot X_+,\ \operatorname{Ad} x \cdot X_-).$$

(Note that $\nabla'\Phi \in \mathfrak{g}^*$; this is of course compatible with the right invariance of Φ, which implies that $\nabla'\Phi \in (\mathfrak{g}^*)^\perp = \mathfrak{g}^*$.)

Since $\mathfrak{g}^* \subset \mathfrak{d}$ is isotropic, the contribution of the right gradients to the bracket $\left.\{\Phi,\Psi\}\right|_{G^\delta}$ vanishes, and hence

$$\{\varphi,\psi\}_* = \tfrac{1}{2}\left.\langle\langle r_{\mathfrak{d}}(\nabla\Phi),\ \nabla\Psi\rangle\rangle\right|_{G^\delta}.$$

The r.h.s. of this formula may be computed directly using lemma 5.25 and formula (5.8) for $r_{\mathfrak{d}}$. After several remarkable cancellations (based on the identity $r_+ - r_- = I$), one gets for the quotient bracket the usual formula $\{\varphi, \psi\}_* = \frac{1}{2}(\langle rX, Y\rangle - \langle rX', Y'\rangle)$.

PROPOSITION 5.26. *Let us identify D/G^* with G with the help of decomposition $D = G^\delta \cdot G^*$. Dressing action $G^* \times G \longrightarrow G$ defined by the commutative diagram (5.13) is given by*

$$\mathrm{Dr}\,(h_+, h_-) \cdot x = h_+ x \left(x^{-1} h_+^{-1} h_- x\right)_+ = h_- x \left(x^{-1} h_+^{-1} h_- x\right)_-, \tag{5.15}$$

where the factors $\left(x^{-1} h_+^{-1} h_- x\right)_\pm$ are defined by the factorization problem

$$x^{-1} h_+^{-1} h_- x = \left(x^{-1} h_+^{-1} h_- x\right)_+ \cdot \left(x^{-1} h_+^{-1} h_- x\right)_-^{-1}$$

in G associated with the initial r-matrix.

By duality, reduction of D_+ with respect to the action of the diagonal subgroup $G^\delta \subset D$ leads to the dual Poisson group G^*; however, the quotient space D_+/G^δ has got yet another important realization.

PROPOSITION 5.27. *The quotient spaces $D/G^\delta, G^\delta \backslash D$ may be canonically identified with G; the projection maps π, π', which form a dual pair, are given by*

$$\pi \colon (x,\, y) \to xy^{-1}, \quad \pi' \colon (xy) \to y^{-1}x. \tag{5.16}$$

The quotient Poisson bracket on $G \simeq D/G^\delta$ is given by

$$\begin{aligned}\{\varphi, \psi\}_* = \tfrac{1}{2} \langle r\nabla\varphi, \nabla\psi\rangle + \tfrac{1}{2} \left\langle r\nabla'\varphi, \nabla'\psi\right\rangle \\ - \left\langle r_+\nabla\varphi, \nabla'\psi\right\rangle - \left\langle r_-\nabla'\varphi, \nabla\psi\right\rangle.\end{aligned} \tag{5.17}$$

PROOF. For $\varphi,\, \psi \in C^\infty(G)$ we set $h_\varphi = \varphi \cdot \pi,\ h_\psi = \psi \cdot \pi$. We have $\nabla' h_\varphi \in \mathfrak{g}^\delta \subset \mathfrak{d},\ \nabla h_\varphi = (\nabla\varphi,\, \nabla'\varphi) \in \mathfrak{d}$. Similar formulae hold for gradients of h_ψ. Since ${}^\delta\mathfrak{g} \subset \mathfrak{d}$ is isotropic, the first term in formula (5.12), which gives the bracket $\{h_\varphi,\, h_\psi\}$, drops out. Now formula (5.17) directly follows from the 'block realization' of $r_\mathfrak{d}$ described in formula(5.8).

EXERCISE 5.28. Check that the linearization of (5.17) at the unit element reproduces the the structure of the initial Lie algebra in the cotangent space $T_e^*G = \mathfrak{g}^* \simeq \mathfrak{g}$.

Formula (5.17) looks complicated, but as a matter of fact, the bracket (5.17) is very remarkable.

PROPOSITION 5.29. *Let us equip G with the quotient Poisson structure. Its symplectic leaves coincide with conjugacy classes in G.*

PROOF. According to Theorem 1.15, symplectic leaves in G are obtained by the "blow-up" of points in the double fibering

$$\begin{tikzcd} & D \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ D/G & & G\backslash D \end{tikzcd}$$

From (5.16) we immediately get $\pi\big(\pi'^{-1}(g)\big) = \{h^{-1}g\,h;\ h \in G\}$. As is easy to see, conjugations $x \longmapsto hxh^{-1}$ are precisely dressing transformations in this realization of the quotient space.

COROLLARY 5.30. *The Poisson bracket* (5.17) *is covariant with respect to the action of G on itself by conjugations. Its Casimir functions are central functions on G.*

Thus the Poisson bracket (5.17) provides the still missing element for the geometric theory of Lax equations on G: as in Theorem 2.5, we have got now *two* Poisson bracket defined on the same manifold (the Sklyanin bracket and the dual bracket (5.17), and the Casimir functions of the dual bracket give rise to Lax equations with respect to the Sklyanin bracket; the associated Hamiltonian flows preserve intersections of two systems of symplectic leaves in G.

As we already mentioned, another model for the quotient space $D/^{\delta}G$ is provided by the group G^*. The two models are connected by the canonical mapping $G_r \longrightarrow G\colon (h_+, h_-) \longmapsto h_+h_-^{-1}$. For completeness we give explicit formulae for the dual bracket (5.17) in tensor form, by analogy with (4.34). We use the tensor notation introduced in Section 4.2 (formula (4.18)) and write $x_1 = x \otimes I$, $x_2 = I \otimes x$; we have[6]:

$$\{x_1\,,\,x_2\}_* = \tfrac{1}{2}rx_1x_2 + \tfrac{1}{2}x_1x_2r - x_2r_+x_1 - x_1r_-x_2. \tag{5.18}$$

Alternatively, we may define the Poisson bracket on the affine ring of G^* generated by matrix coefficients of the pair of matrices $(h_+, h_-) \in G^* \subset G \times G$:

$$\big\{h_\pm \overset{\otimes}{,} h_\pm\big\}_* = \tfrac{1}{2}[r,\, h_\pm \otimes h_\pm], \quad \big\{h_+ \overset{\otimes}{,} h_-\big\}_* = [r_+,\, h_+ \otimes h_-]. \tag{5.19}$$

EXERCISE 5.31. (i) Verify that formula (5.19) implies (5.18) for $x = h_+h_-^{-1}$. (ii) Write down the corresponding relations for loop groups.

5.7. Nonabelian Moment and Dressing Transformations

Let $\mathcal{M}$ be a Poisson manifold, H a Poisson Lie group, H^* the dual Poisson Lie group. Let π be the Poisson tensor of $\mathcal{M}$, and $\Theta_* \in \mathfrak{h}^* \otimes \Omega_1(H^*)$ the left-invariant Maurer–Cartan form on H^*. We shall consider π as a mapping $\pi\colon \Omega_1(\mathcal{M}) \to \mathrm{Vect}(\mathcal{M})$. Let $\delta^*\colon \mathfrak{h}^* \to \mathfrak{h}^* \wedge \mathfrak{h}^*$ be the cobracket on $\mathfrak{h}^*$ which is dual to the commutator $\mathfrak{h} \wedge \mathfrak{h} \to \mathfrak{h}$. Let $H \times \mathcal{M} \to \mathcal{M}$ be an action of H and $\mathfrak{h} \to \mathrm{Vect}\,\mathcal{M}\colon X \mapsto \widehat{X}$ the associated infinitesimal action, i. e., a homomorphism of the Lie algebra of H into the Lie algebra of vector fields on $\mathcal{M}$.

[6]The Poisson algebra 5.18 arises in Quantum Group theory as a semiclassical limit of the so called *reflection equation;* for this reason it is sometimes called *classical reflection equation*. Both terms point to the important paper of Sklyanin **[S3]** on the boundary conditions for integrable systems. The bracket itself has appeared in **[STS2]** in connection with the duality theory for Poisson groups.

DEFINITION 5.32. We shall say that the action $H \times \mathcal{M} \to \mathcal{M}$ admits a nonabelian moment map if there is a smooth mapping $\mu\colon \mathcal{M} \to H^*$ such that

$$\widehat{X}\varphi = \langle X,\, \mu^{-1}\{\mu,\, \varphi\}\rangle, \quad \varphi \in C^\infty(\mathcal{M}). \tag{5.20}$$

Note that for any $x \in \mathcal{M}$ $\{\mu,\, \cdot\,\}(x)$ is an element of $T_{\mu(x)}H^* \otimes T_x\mathcal{M}$; left translation by $\mu^{-1}(x)$ takes this element to $\mathfrak{h}^* \otimes T_x\mathcal{M}$; the natural coupling with $X \in \mathfrak{h}$ yields a tangent vector $\widehat{X}(x) \in T_x\mathcal{M}$. In more formal way, we may rewrite formula (5.20) as

$$\widehat{X} = \langle X \otimes id,\, \pi(\mu^*\Theta_*)\rangle, \quad X \in \mathfrak{h}. \tag{5.21}$$

The mapping μ is said to be *H-equivariant* if

$$[\pi(\mu^*\Theta_*),\, \pi(\mu^*\Theta_*)]_{\mathrm{Vect}(\mathcal{M})} = (\delta^* \otimes id)\pi(\mu^*\Theta_*). \tag{5.22}$$

PROPOSITION 5.33. *Suppose that $\mu\colon \mathcal{M} \to H^*$ is H-equivariant; then formula* (5.20) *(or the equivalent formula* (5.21) *) define an infinitesimal Poisson action of H on $\mathcal{M}$.*

Sketch of a proof. First of all, we have to check that the r.h.s. of (5.20) defines a homomorphism of Lie algebras, i. e.,

$$[\widehat{X},\, \widehat{Y}] = \langle [X, Y] \otimes id,\, \pi(\mu^*\Theta_*)\rangle.$$

This formula immediately follows from (5.22). Second, we have to check relation (5.10) which gives a criterion for the infinitesimal action to be Poisson. Note that for any $\varphi \in C^\infty(\mathcal{M})$ the covector $\pi(\mu^*\Theta_*) \cdot \varphi(x) \in \mathfrak{h}^*$ coincides with the covector $\xi_\varphi(x)$ from (5.10). The definition of the vector field $\widehat{X}$ immediately implies that

$$\begin{aligned}\{\widehat{X}\varphi,\, \psi\} + \{\varphi,\, \widehat{X}\psi\} - \widehat{X}\{\varphi, \psi\} &= \\ &= \{\mu^{-1},\, \psi\}\{\mu,\, \varphi\} + \{\varphi,\, \mu^{-1}\}\{\mu,\, \psi\} = -[\xi_\varphi, \xi_\psi]_*.\end{aligned}$$

In the last line we have effectively used the Maurer-Cartan equation for H^*.

The equivariance condition is satisfied if the mapping $\mu\colon \mathcal{M} \to H^*$ is Poisson.

EXERCISE 5.34. Check this assertion using the definition (5.21)the Maurer-Cartan equation Θ_*.

Hint: For 1-forms, the cobracket δ^* coincides with the Chevalley differential.

The simplest example of an equivariant nonabelian moment map is connected with the dressing action of H on the dual group H^*.

THEOREM. *The nonabelian moment map associated with the infinitesimal dressing action is the identity map $id : H^* \to H^*$.*

This characteristic of infinitesimal dressing action may be used for their definition; global dressing transformations are then constructed by integration of the corresponding vector fields (which in general are incomplete). This approach is used in [**Wein1**]. In this section we do not aim at full generality and will check this assertion for the standard dressing action of the dual group G^* of a factorizable Poisson group G. Recall that in this case the dressing action $G^* \times G \to G$ is given by

$$\begin{aligned}\mathrm{Dr}_{(h_+,h_-)} \cdot x &= \left(xh_+h_-^{-1}x^{-1}\right)_+^{-1} x\, h_+ = \\ &= \left(xh_+h_-^{-1}x^{-1}\right)_-^{-1} x\, h_-, \quad x \in G,\ (h_+,h_-) \in G^*.\end{aligned} \tag{5.23}$$

The corresponding infinitesimal action is given by

$$\mathrm{dr}_{(X_+,X_-)} \cdot g = (g\, X\, g^{-1})_\pm g - g\, X_\pm, \quad X = X_+ - X_-. \tag{5.24}$$

The r.h.s. of this formula defines a tangent vector to G at the point g.

Exercise 5.35. Using the formula $\{g_1, g_2\} = [r,\, g_1 g_2]$ for the Poisson bracket on G check that

$$\mathrm{dr}_{(X_+,X_-)} \cdot g = \mathrm{tr}_2 \left(X_2 g_2^{-1} \{g_1, g_2\}\right). \tag{5.25}$$

(The symbol tr_2 stands for the "inner product in the second space"defined by

$$\mathrm{tr}_2(a \otimes b) \circ (I \otimes X) = a \cdot \langle b,\, X\rangle = a\, \mathrm{tr}\, bX.$$

In a slightly more general way, let $A \in \mathrm{End}\,\mathfrak{g}$ be a linear operator with kernel $A_{12} \in \mathfrak{g} \otimes \mathfrak{g}$; then easy to see that $AX = \mathrm{tr}_2 A_{12} X_2$ for all $X \in \mathfrak{g}$.[7])

[7] Of course, for loop algebras "the inner product with respect to the second space" includes also integration (or residue) with respect to loop variable.

LECTURE 6

Lax Equations on Lie groups

6.1. Introduction

So far, the only motivation of the Poisson Lie groups definition was to provide a non-linear generalization of the Poisson–Lie brackets, with the multiplicativity axiom replacing the simple linearity condition. We shall now examine the role of this new kind of Poisson structures in the theory of Integrable Systems. The natural context in which these structures do appear is the case of lattice systems which admit a *difference zero curvature representation*. By definition, zero curvature equations are compatibility conditions for a system of linear differential equations of the first order

$$\begin{aligned} \partial_x \psi &= L\psi, \\ \partial_t \psi &= M\psi. \end{aligned} \tag{6.1}$$

We suppose that L, M are functions with values in a Lie algebra $\mathfrak{g}$ and ψ is a fundamental solution of this system with values in the corresponding Lie group. The compatibility condition for system (6.1) is

$$\partial_t L - \partial_x M + [L,\, M] = 0. \tag{6.2}$$

Equivalently, this means that the $\mathfrak{g}$-valued 1-form $Ldx + Mdt$ on $\mathbb{R}^2$ has zero curvature. In a similar way, difference zero curvature equations arise as the compatibility conditions for linear differential-difference systems of the form

$$\begin{aligned} \psi_{m+1} &= L_m \psi_m, \\ \frac{d\psi_m}{dt} &= M_m \psi_m, \quad m \in \mathbb{Z}. \end{aligned} \tag{6.3}$$

In this case we assume that the "lattice connection" matrices L_m, $m \in \mathbb{Z}$, take values in a Lie group G, while M_m are still elements of the corresponding Lie algebra. If G is a matrix group, the compatibility condition may be written in the form

$$\frac{dL_m}{dt} = M_{m+1}L_m - L_m M_m, \quad m \in \mathbb{Z}. \tag{6.4}$$

In the general case, we have to use left or right translation in order to make the velocity vector $\dot{L}_m$ an element of the Lie algebra, and the compatibility condition takes the form

$$\dot{L}_m L_m^{-1} = M_{m+1} - \operatorname{Ad} L_m \cdot M_m, \quad m \in \mathbb{Z},$$

In the sequel we shall keep to the more simple matrix notation; the generalization to the abstract case is always straightforward.

The use of difference operators associated with a one-dimensional lattice is particularly well suited for the study of "multiparticle" integrable problems. Let us assume the "potentials" L_m in (6.3) are periodic, $L_{m+N} = L_m$; the period N may be interpreted as the number of copies of an "elementary" system. The auxiliary linear problem allows to construct families of Hamiltonians which remain integrable for any N. The phase spaces for such systems are direct products of a large number of elementary "one-particle" phase spaces. Typical dynamical systems which arise in this way are classical analogues of lattice systems in Quantum Statistical Mechanics. In degenerate case when the lattice shrinks to a single point the lattice zero curvature equation reduces to an ordinary Lax equation for a G-valued function $L(t)$. This case deserves special attention; we shall state for it an analogue of Theorem 2.8.

6.2. Difference Equations with Periodic Coefficients and the Main Motivations

Consider the auxiliary linear system (6.3). Let us suppose that $L_{m+N} = L_m$. This system is covariant with respect to gauge transformations

$$\psi_m \mapsto g_m\psi_m,\ L_m \mapsto g_{m+1}L_m g_m^{-1},\ M_m \mapsto g_m M_m g_m^{-1} - \partial_t g_m \cdot g_m^{-1}, \tag{6.5}$$

where the transformation matrices $g_m \in G$ are periodic, $g_{m+N} = g_m$. The infinitesimal gauge transformations are given by

$$\psi_m \mapsto X_m\psi_m,\ L_m \mapsto X_{m+1}L_m - L_m X_m,\ M_m \mapsto X_m M_m - M_m X_m - \partial_t X_m, \tag{6.6}$$

where $X_m \in \mathfrak{g},\ X_{m+N} = X_m$. Note that the right hand side of zero curvature may be regarded as the result of an *infinitesimal* gauge transformation. Let us consider the *monodromy map* associated with the difference system (6.3) which assigns to the set of Lax matrices $\{L_m\}$ their ordered product

$$T(L) = \prod_k^{\curvearrowleft} L_k. \tag{6.7}$$

THEOREM 6.1 (Floquet). *Two difference operators with periodic coefficients are gauge equivalent if and only if their monodromies are conjugate in G.*

PROOF. Monodromy matrices transform by conjugation under gauge transformations with periodic coefficients. Conversely, let us suppose that two monodromy matrices $M, \tilde{M}$ are conjugate; by performing, if necessary, a constant gauge transformation we may assume that they are equal. Let $\psi, \tilde{\psi}$ be the corresponding fundamental solutions. Put $g_m = \psi_m\tilde{\psi}_m^{-1}$; then g_m is periodic, $g_{m+N} = g_m$, and defines the gauge transformation which transforms L_m into $\tilde{L}_m$.

COROLLARY 6.2. *Let $\phi \in C^\infty(G)$ be a central function on G. Then $H_\phi[L] = \phi(T(L))$ is a spectral invariant of the auxiliary linear problem; conversely, all spectral invariants are constructed in this way.*

Following the usual logic of the Classical Inverse Scattering Method, spectral invariants of the auxiliary linear problem provide the set of first integrals for lattice zero curvature equations. Functions h_ϕ are first integrals, if the monodromy matrix itself satisfies a Lax equation:

$$\frac{dT_L}{dt} = [T_L, A_L]\,. \tag{6.8}$$

Let $F_t\colon G^N \longrightarrow G^N$ be the dynamical flow associated with equations (6.4) and $\bar{F}_t\colon G \longrightarrow G$ the corresponding flow for (6.8). Then the following diagram is commutative:

$$\begin{tikzcd} G^N \arrow[r, "F_t"] \arrow[d, "T_L"] & G^N \arrow[d, "T_L"] \\ G \arrow[r, "\bar{F}_t"] & G \end{tikzcd} \tag{6.9}$$

We want to equip our phase space (i. e., the direct product of several copies of G) with a Poisson structure such that all maps in this diagram are Poisson mappings. Moreover, we want to preserve the simple geometric picture connected with Theorem 2.5; this means that we have to construct *two* Poisson structures in each space in diagram (6.9) in such a way that

(i) *Spectral invariants of the monodromy are Casimir functions for the* first *structure.*
(ii) *They are in involution with respect to the* second one *and give rixe to lattice zero curvature equations (respectively, to ordinary Lax equations for the monodromy).*
(iii) *The flows $F_t, \bar{F}_t$ preserve intersections of the symplectic leaves for the two structures.*
(iv) *Vertical arrows in diagram (6.9) are Poisson mappings with respect to both brackets.*
(v) *Finally, the solution of the equations of motion (both for local Lax matrices and for the monodromy) is reduced to a* factorization problem.

It is natural to split the problem into two parts:

1. *Construct Poisson brackets on $\mathbf{G} = G^N$ and on G such that spectral invariants of the monodromy are their Casimir functions.*
2. *Construct Poisson brackets on $\mathbf{G} = G^N$ and on G such that spectral invariants of the monodromy give rise to lattice zero curvature equations (6.4)and the monodromy map is a Poisson mapping.*

The second one of these two questions is better known than the first one; it is precisely this question which has led to the theory of *Poisson groups.* The key step consists in the following simplifying assumption:

Dinamical variables associated with different copies of G commute with each other with respect to the Poisson bracket on G^N.

By induction, $T\colon G^N \longrightarrow G$ is a Poisson mapping if the multiplication $m\colon G \times G \to G$ is itself a Poisson mapping. This is precisely the multiplicativity axiom which was our starting point in Lecture 4. By analogy with the linear case, we expect that the appropriate class of Poisson structures to be used for lattice zero curvature equations is provided by factorizable Lie bialgebas. Rather surprisingly, we shall be able to answer the first question as well using the same classical r-matrix. Of course, the corresponding Poisson structure on $\mathbf{G} = G^N$ is no longer multiplicative. We describe it in Section 6.5 below.

6.3. Lax Equations on Lie Groups

Let us now state the analog of the Factorization Theorem 2.8 which applies to factorizable Poisson groups equipped with the Sklyanin bracket.

EXERCISE 6.3. The Hamiltonian equation of motion on G associated with a Hamiltonian $H \in C^\infty(G)$ with respect to the Sklyanin bracket has the form

$$\frac{dL}{dt} = LA - BL, \text{ where } L \in G,\ A = r(\nabla H(L)),\ B = r(\nabla' H(L)). \tag{6.10}$$

REMARK 6.4. In formula (6.10) the velocity vector dL/dt belongs to the tangent space $T_L G$; in a slightly more accurate way, we may write (6.10) as an equation in the Lie algebra:

$$L^{-1} dL/dt = A - \operatorname{Ad} L^{-1} \cdot B.$$

Assume that the tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ of G is factorizable. Let $I(G) \in C^\infty(G)$ be the algebra of central functions on G (a function $\varphi \in C^\infty(G)$ is central if $\varphi(xy) = \varphi(yx)$ for all $x, y \in G$).

THEOREM 6.5. *(i) Central functions are in involution with respect to the Sklyanin bracket (4.32). (ii) The Hamiltonian equation of motion on G with Hamiltonian $h \in I(G)$ may be written in Lax form*

$$\frac{dL}{dt} = LM_\pm - M_\pm L, \quad \textit{where} \quad M_\pm = r_\pm(\nabla h(L)). \tag{6.11}$$

(iii) The integral curve $L(t)$ of equation (6.11), which starts at $L(0) = L_0$ is given by

$$L(t) = g_\pm(t)^{-1} L_0\, g_\pm(t), \tag{6.12}$$

where $g_+(t)$, $g_-(t)$ are the solutions of the factorization problem in G associated with r:

$$g_+(t)\, g_-(t)^{-1} = \exp t\nabla h(L_0), \tag{6.13}$$

As in the case of Theorem 2.8, there is a simple direct way to check Theorem 6.5: in order to verify that (6.12) is an integral curve of (6.11), one has simply to compute the velocity vector. On the other hand, there again exists a geometric proof which explains the origin of the result. This time, however, it is rather complicated. It is given in the next two section. Following the plan described in Section 2.4, we must, first of all, exhibit a "big phase space" and a free dynamics on it; reduction over an appropriate symmetry should then yield Lax equations (6.11). As we have seen in the previous section, the natural substitute of the cotangent bundle for Poisson group is the symplectic double D_+; it is quite natural to use it as the "big phase space" (however, as we shall see, difference Lax equations require a more general space, the "twisted double'").

6.4. Free Dynamics

Let $\varphi \in I(G)$ be a central function on G. Let us lift it to $D = G \times G$ by means of the projection map $\pi\colon D \to G\colon (x,\, y) \mapsto xy^{-1}$.We equip D with the Poisson structure (5.12).

PROPOSITION 6.6. *The trajectories of the Hamiltonian $h_\varphi = \pi^*\varphi$ on D have the form*

$$\big(e^{tX}x_0,\, e^{tX}y_0\big), \quad X = \nabla\varphi(x_0 y_0^{-1}). \tag{6.14}$$

PROOF. The projections of the trajectories of h_φ to the quotient spaces D/G^δ , $G^\delta\backslash D$ reduce to points, since the quotient Hamiltonian φ is a Casimir function of the quotient Poisson structure. Since h_φ is invariant with respect to left and right translations from G^δ, we have $\nabla h_\varphi,\, \nabla h'_\varphi \in \mathfrak{g}^\delta \subset \mathfrak{d}$. Hence $\nabla h_\varphi,\, \nabla h'_\varphi$ are eigenvectors of $r_{\mathfrak{d}}$; The $\operatorname{Ad} D$-invariance of the inner product on $\mathfrak{d}$ implies that for any function $\psi \in C^\infty(D)$ we have

$$\{h_\varphi,\, \psi\} = \langle \nabla h_\varphi,\, \nabla h_\psi\rangle.$$

It is easy to see that $\nabla h_\varphi = (X,\, X)$, where $X = \nabla\varphi\big(\pi(x,\, y)\big)$, and X is time-independent. Thus the derivative of ψ along the Hamiltonian vector field generated by h_φ is equal to

$$\dot\psi = \{h_\varphi,\, \psi\} = \left(\tfrac{d}{dt}\right)_{t=0} \psi(e^{tX}\cdot(x_0, y_0)).$$

Now (6.14) follows immediately.

Consider the action of G^* on D given by

$$G^* \times D \longrightarrow D\colon\ (h_+, h_-) : (x,\, y) \longmapsto (h_+ x h_-^{-1},\, h_+ y h_-^{-1}). \tag{6.15}$$

THEOREM 6.7. *(i) The Hamiltonian h_φ is invariant with respect to to the action* (6.15).

(ii) The mapping

$$s\colon D\,(G)_+ \longrightarrow G\colon\ (x, y) \longmapsto y_+^{-1} x y_-$$

is constant on G^*-orbits in $D\,(G)_+$ and allows to identify the quotient space $D\,(G)_+\,/G^*$ with the subgroup $G = \{(x, e)\,; x \in G\} \subset D\,(G)$. [1]

[1] As usual, for $g \in G$ we denote by g_+, g_- the solutions of the factorization problem $g = g_+ g_-^{-1}$, $(g_+, g_-) \in G^* \subset D$.

(iii) G^* is an admissible subgroup of $D(G) \times D(G)$ which acts on $D(G)_+$ by left and right translations.

(iv) The quotient Poisson structure on $D(G)_+/G^* \simeq G$ coincides with the Sklyanin bracket[2]*.*

(v) The quotient flow $\bar{F}_t$ on G obtained by reduction of the "free" flow (6.14)*, is given by $\bar{F}_t\colon L \mapsto g_\pm(t)^{-1} L g_\pm(t)$, where $g_+(t), g_-(t)$ are solutions of the factorization problem $\exp t\nabla\varphi(L) = g_+(t)\, g_-(t)^{-1}$, and satisfy Lax equation (6.11); the quotient Hamiltonian on G coincides with φ.*

PROOF. Assertions (i), (ii) are obvious. Since φ is a central function on G, we have $s^*\varphi(x,y) = \varphi(y_+^{-1}xy_-) = \varphi(xy^{-1}) = h_\varphi(x,y)$. Let $c(t) = (e^{tX}L_0, e^{tX})$ be the integral curve of the free system which starts from $(L_0, e) \in G \times \{e\}$. Clearly,

$$s(c(t)) = g_+(t)^{-1}e^{tX}L_0\, g_-(t) = g_-(t)^{-1}L_0\, g_-(t),$$

where $g_+(t)g_-(t)^{-1} = e^{tX}$. Since the velocity vector is given by $X = \nabla\varphi(L_0)$, this formula reproduces the formula for the solution of the Lax equation. The difficult part of the proof consists in the check of the admissibility condition for the action (6.15) and in the computation of the quotient Poisson structure. We postpone the proof till next section, where a more general result will be obtained (Theorem 6.8 below).

6.5. Twisted Double and General Reduction Theorem

Theorem 6.7 cannot be applied to the study of difference Lax equations. Indeed, the Hamiltonians discussed in this theorem are central functions on G; on the other hand, natural Hamiltonians for lattice systems are invariant with respect to lattice gauge transformations. In a more general way, one can consider generalized Lax equations whose Hamiltonians are invariants of "twisted conjugations" associated with an appropriate automorphism of G.

Let $\tau \in \operatorname{Aut} G$ be an automorphism of G; we shall write it exponentially, $g \mapsto {}^\tau g$. Suppose that the corresponding automorphism of $\mathfrak{g}$ (which we denote by the same letter) preserves the inner product in $\mathfrak{g}$ and commutes with r.

Define the action $G \times G \to G$ by the formula

$$g\colon h \mapsto {}^\tau g\, h\, g^{-1}. \tag{6.16}$$

Transformations (6.16) will be called *twisted conjugations.* Let ${}^\tau I(G) \subset C^\infty(G)$ be the space of functions on G which are invariant with respect to twisted conjugations.

THEOREM 6.8. *(i) Functions $\varphi \in {}^\tau I(G)$ are in involution with respect to the Sklyanin bracket on G.*

[2] This assertion requires a small precision which we postpone till the proof of Proposition6.19 below.

(ii) Equation of motion defined by $\varphi \in {}^{\tau}I(G)$ *with respect to the Sklyanin bracket* (4.32) *has the generalized Lax form*

$$\frac{dL}{dt} = LM_{\pm} - {}^{\tau}M_{\pm}L, \quad M_{\pm} = r_{\pm}\big(\nabla\varphi(L)\big). \tag{6.17}$$

(iii) Let $g_{+}(t)$, $g_{-}(t)$ *be the solution of the factorization problem in* G *with the l.h.s.*

$$g(t) = \exp t\nabla\varphi(L_0). \tag{6.18}$$

The integral curve of (6.17) *which starts from* $L_0 \in G$ *is given by*

$$L(t) = {}^{\tau}g_{\pm}(t)^{-1}L_0\, g_{\pm}(t). \tag{6.19}$$

Let us prove first of all the involutivity of $\varphi \in {}^{\tau}I(G)$. We have $\varphi({}^{\tau}g \cdot x) = \varphi(x \cdot g)$, and hence $\nabla\varphi = {}^{\tau}\nabla'\varphi$. Thus for φ, $\psi \in {}^{\tau}I(G)$ we get

$$\begin{aligned} 2\{\varphi, \psi\} &= \langle r(\nabla\varphi), \nabla\psi\rangle - \big\langle r(\nabla'\varphi), \nabla'\psi\big\rangle = \\ &= \big\langle r({}^{\tau}\nabla'\varphi), {}^{\tau}\nabla'\psi\big\rangle - \big\langle r(\nabla'\varphi), \nabla'\psi\big\rangle = \\ &= \big\langle {}^{\tau}r(\nabla'\varphi), {}^{\tau}\nabla'\psi\big\rangle - \big\langle r(\nabla'\varphi), \nabla'\psi\big\rangle = 0. \end{aligned} \tag{6.20}$$

The same relation immediately implies (6.17).

"The big phase space" associated with the twisted Lax equations (6.19) is the so called *twisted double*, i. e. the double $D(G) = G \times G$ equipped with the twisted Poisson structure. Its definition is given as follows.

Extend the automorphism $\tau \in \operatorname{Aut}\mathfrak{g}$ to $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ by setting

$$\hat{\tau}(X, Y) = ({}^{\tau}X, Y). \tag{6.21}$$

Put ${}^{\tau}\mathfrak{g} = \{({}^{\tau}X, X);\ X \in \mathfrak{g}\}$, ${}^{\tau}\mathfrak{g}^* = \{({}^{\tau}X_{+}, X_{-});\ (X_{+}, X_{-}) \in \mathfrak{g}^*\} \subset \mathfrak{d}$. Clearly, the subalgebras ${}^{\tau}\mathfrak{g}$, ${}^{\tau}\mathfrak{g}^*$ are isotropic with respect to the inner product (5.7); thus $(\mathfrak{d}, {}^{\tau}\mathfrak{g}, {}^{\tau}\mathfrak{g}^*)$ is a Manin triple. We can associate with this triple a classical r-matrix

$$^{\tau}r_{\mathfrak{d}} = \hat{\tau} \circ r_{\mathfrak{d}} \circ \hat{\tau}^{-1}. \tag{6.22}$$

Define the Poisson bracket on D by the formula

$$\{\varphi, \psi\}_{\tau} = \{\varphi, \psi\}_{{}^{\tau}r_d, r_d} = \tfrac{1}{2}\big\langle {}^{\tau}r_d(\nabla\varphi), \nabla\psi\big\rangle + \tfrac{1}{2}\big\langle r_d(\nabla'\varphi), \nabla'\psi\big\rangle. \tag{6.23}$$

By Theorem 4.34, this bracket satisfies the Jacobi identity.

Definition 6.9. The twisted double associated with an automorphism τ is the group D equipped with the Poisson structure (6.23).

The twisted double is a principal homogeneous space for two different Poisson groups.

Proposition 6.10. *Multiplication* $D \times D \to D$ *defines Poisson mappings*

$$\begin{aligned} D_{({}^{\tau}r_d, -{}^{\tau}r_d)} \times D_{({}^{\tau}r_d, r_d)} &\longrightarrow D_{({}^{\tau}r_d, r_d)}, \\ D_{({}^{\tau}r_d, r_d)} \times D_{(r_d, -r_d)} &\longrightarrow D_{({}^{\tau}r_d, r_d)}. \end{aligned}$$

This assertion is a specialization of Proposition 4.41.

Let $G^{\delta} \subset D$ be the diagonal subgroup and ${}^{\tau}G = \{({}^{\tau}x, x);\ x \in G\}$.

COROLLARY 6.11. *(i) The action of* ${}^{\tau}G$ *on* $D_{({}^{\tau}r_d, r_d)}$ *by left translations is Poisson. (ii) The right action of* G^{δ} *on* $D_{({}^{\tau}r_d, r_d)}$ *by right translations in Poisson.*

Indeed, ${}^{\tau}G \subset D_{({}^{\tau}r_d, -{}^{\tau}r_d)}$ and $G^{\delta} \subset D_{(r_d, -r_d)}$ are clearly Poisson subgroups.

PROPOSITION 6.12. *Canonical projections*

$$\begin{tikzcd} & D \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ D/G^{\delta} & & G^{\tau}\backslash D \end{tikzcd} \tag{6.24}$$

form a dual pair.

The group G is a natural model of the quotient spaces ${}^{\tau}G \setminus D$, $D/{}^{\delta}G$; the projection maps π, π' are given by

$$\pi' \colon (x, y) \mapsto {}^{\tau}y^{-1}x, \quad \pi \colon (x, y) \mapsto xy^{-1}. \tag{6.25}$$

PROPOSITION 6.13. *The quotient Poisson structure on* G *is covariant with respect to twisted conjugations; orbits of twisted conjugations in* G *are Poisson submanifolds.*

In contrast to Proposition 5.29, we do not claim in general that orbits of twisted conjugations are symplectic leaves of the quotient Poisson structure; the point is that the twisted bracket (6.23) may be degenerate. Recall that in this case the blow-up of points in the double fibering associated with the dual pair π, π' yields Poisson submanifolds in the quotient spaces (cf. remark 1.22). We have

$$\pi^{-1}(x) = \{(xy, y);\, y \in G\}\,,\ \pi'\big(\pi^{-1}(x)\big) = \left\{{}^{\tau}y\, xy^{-1};\, y \in G\right\}.$$

COROLLARY 6.14. *Invariants of twisted conjugations* (6.16) *are Casimir functions of the quotient Poisson structure on* G*.*

An explicit formula for the quotient Poisson structure on G is proved in the same way as in Proposition 5.27.

PROPOSITION 6.15. *The quotient Poisson bracket on* G *is given by*

$$\begin{aligned} \{\varphi, \psi\}_{red} = \tfrac{1}{2}\langle r(\nabla\varphi), \nabla\psi\rangle + \tfrac{1}{2}\left\langle r(\nabla'\varphi), \nabla'\psi\right\rangle \\ - \left\langle r_+({}^{\tau}\nabla\varphi), \nabla'\psi\right\rangle - \left\langle r_-(\nabla'\varphi), {}^{\tau}\nabla\varphi\right\rangle. \end{aligned} \tag{6.26}$$

In tensor notation this be bracket may be written as

$$\{L_1, L_2\} = \tfrac{1}{2}r\, L_1L_2 + \tfrac{1}{2}L_1L_2\, r - L_1\,(\tau\otimes id)r_+\, L_2 - L_2\,(id\otimes\tau)r_-\, L_1.$$

We shall discuss the meaning and the origin of (6.26) in Section 6.8.

6.6. Proof of Reduction Theorem.

Let us discuss first of all the "free dynamics" defined by a Hamiltonian $\varphi \in {}^\tau I(G)$ on the twisted double. Let $\pi\colon D \to G$ be the projection map from the dual pair (6.24) and $h_\varphi = \varphi \circ \pi$.

PROPOSITION 6.16. *The trajectories of the Hamiltonian h_φ on $D_{{}^\tau r_d, r_d}$ have the form*

$$\left(e^{t\,{}^\tau X} x_0,\, e^{tX} y_0\right), \quad X = \nabla\varphi\bigl(\pi(x_0,\, y_0)\bigr). \tag{6.27}$$

The proof is similar to that in Proposition 6.6. The definition of h_φ implies that

$$\nabla h_\varphi \in {}^\tau\mathfrak{g}, \quad \nabla' h_\varphi \in \mathfrak{g}^\delta,$$

and hence for any $\psi \in C^\infty(D)$ we get

$$\{h_\varphi,\, \psi\} = \langle \nabla h_\varphi,\, \nabla\psi\rangle.$$

Moreover, $\nabla h_\varphi(x,\, y) = ({}^\tau X,\, X)$, where $X = \nabla\varphi\bigl(\pi(x,\, y)\bigr)$ and the vector X does not depend on t.

Define the action of G^* on D by

$$(h_+, h_-)\colon (x,\, y) \longmapsto ({}^\tau h_+\, x\, h_-^{-1},\, h_+\, y\, h_-^{-1}). \tag{6.28}$$

PROPOSITION 6.17. *The action* (6.28) *of $G^* \subset D \times D$ is admissible.*

PROOF. Note that if there are two commuting Poisson actions $H \times \mathcal{M} \to \mathcal{M}$, $H' \times \mathcal{M} \to \mathcal{M}$, their combination defines a Poisson action of the direct product $H \times H'$ (equipped with the product Poisson structure, as in (4.1)). In our present setting the twisted double is a principal homogeneous space with respect to the group $D_{({}^\tau r_d, -{}^\tau r_d)}$ acting by left translations and a principal homogeneous space with respect to the group $D_{(-r_d, r_d)}$ acting by right translations(we changed the sign of the r-matrix for the "right" group in order to convert right action into left one). The group G^* is embedded into $D \times D$ by means of the mapping

$$i^\tau\colon h \longmapsto ({}^\tau h_+, h_+, h_-, h_-).$$

In order to prove admissibility of the action of G^*, we shall use Theorem 5.13. Since in our case the tangent Lie bialgebra of $D \times D$ is $(\mathfrak{d} \oplus \mathfrak{d}, \mathfrak{d}_{{}^\tau r_\mathfrak{d}} \oplus \mathfrak{d}_{-r_\mathfrak{d}})$, our assertion immediately follows from the following lemma.

LEMMA 6.18. *The annihilator of $i^\tau(\mathfrak{g}^*) \subset \mathfrak{d} \oplus \mathfrak{d}$ is a Lie subalgebra in* $(\mathfrak{d} \oplus \mathfrak{d}, \mathfrak{d}_{{}^\tau r_\mathfrak{d}} \oplus \mathfrak{d}_{-r_\mathfrak{d}})$.

PROOF. An element $\bigl(Y_1,\, Y_2, Z_1,\, Z_2\bigr) \in (\mathfrak{d} \oplus \mathfrak{d})^* \simeq \mathfrak{d} \oplus \mathfrak{d}$ belongs to the annihilator of $i^\tau(\mathfrak{g}^*) \subset \mathfrak{d} \oplus \mathfrak{d}$ if and only if

$$r_-({}^{\tau^{-1}} Y_1 - Y_2) + r_+(Z_1 - Z_2) = 0;$$

equivalently

$$\mathfrak{g}_r^\perp = \left\{(({}^\tau\xi, \xi) + ({}^\tau\eta_+, \eta_-),\, (\xi', \xi') + (\eta'_+, \eta'_-))\right\};$$
$$\xi,\, \xi' \in \mathfrak{g}, \quad \eta,\, \eta' \in \mathfrak{g}^*, \quad \eta_- + \eta'_+ = 0.$$

Since ${}^{\tau}\mathfrak{g}$, ${}^{\tau}\mathfrak{g}^* \subset \mathfrak{d}_{{}^{\tau}r_d}$ and $\mathfrak{g}^{\delta}$, $\mathfrak{g}^* \subset \mathfrak{d}_{-r_{\mathfrak{d}}}$ are Lie subalgebras (in the second case this is true up to an anti-automorphism associated with the change of sign of r), it is enough to check that the equalities

$$r_-\eta_1 = -r_+\eta_1', \quad r_-\eta_2 = -r_+\eta_2'$$

imply that

$$r_-\big([\eta_1,\, \eta_2]_r\big) = -r_+\big(-[\eta_1',\, \eta_2']_r\big).$$

This implication immediately follows from the Yang–Baxter identity:

$$r_-\big([\eta_1,\, \eta_2]_r\big) = [r_-\eta_1,\, r_-\eta_2] = [r_+\eta_1',\, r_+\eta_2'] = r_+\big([\eta_1',\, \eta_2']_r\big).$$

Note that the subgroup $(G,\, e) \subset D$ is a cross-section of the action (6.28). Thus the quotient space D/G^* may be identified with G. The projection map $s\colon D \to G$ is given by

$$s\colon (x,\, y) \longmapsto {}^{\tau}y_+^{-1}xy_-. \tag{6.29}$$

As in Theorem 6.7, free Hamiltonians h_φ, $\varphi \in {}^{\tau}I(G)$ are invariant with respect to the action (6.28) and, moreover, $h_\varphi = s^*\varphi$.

PROPOSITION 6.19. *The quotient Poisson structure on G is given by*

$$\{\varphi, \psi\}_{\mathrm{red}} = \tfrac{1}{2}\langle rX, Y\rangle - \tfrac{1}{2}\langle rX', Y'\rangle + \\ + \tfrac{1}{4}\langle (r^3 - r)\cdot (X - {}^{\tau}X'),\, (Y - {}^{\tau}Y')\rangle, \tag{6.30}$$

where $X = \nabla\varphi,\ X' = \nabla'\varphi,\ Y = \nabla\psi,\ Y' = \nabla'\psi$.

PROOF. Choose $\varphi,\, \psi \in C^\infty(G)$, and let $H_\varphi = s^*\varphi$, $H_\psi = s^*\psi$. Let $X = \nabla\varphi$, $X' = \nabla'\varphi$, $\nabla\varphi Y' = \nabla'\psi$. It is easy to verify that the restriction of the gradient of H_φ to the cross-section $(G,\, e) \subset D$ are given by the formula

$$\nabla H_\varphi = (X,\, X_+' - {}^{\tau^{-1}}X_-), \quad \nabla' H_\varphi = (X',\, X_+' - {}^{\tau^{-1}}X_-)). \tag{6.31}$$

Similar formulae hold for the restrictions of the gradients of H_ψ. Computing ${}^{\tau}r_{\mathfrak{d}}X$, $r_{\mathfrak{d}}X'$ and substituting and substituting these expressions into (6.23), we get, after a number of cancellations, (6.30). The "unwanted" terms in the reduced bracket which contain $r^3 - r$ arise from the inner products $\langle X_+, Y_-\rangle = \langle r_+X, r_-Y\rangle$; note that $r_\pm^* = -r_\mp$, but in general case operators $r_\pm$ are not idempotent and hence computation of such inner products gives terms which are nonlinear in r. Computation shows that all such terms may be combined into a single aggregate $\frac{1}{4}\langle (r^3 - r)\cdot (X - {}^{\tau}X'),\, (Y - {}^{\tau}Y')\rangle$.

If $r^3 - r = 0$, and in particular if $r = P_+ - P_-$ is the difference of complementary projection operators, the reduced bracket coincides with the Sklyanin bracket. In the general case let us note that if φ is invariant with respect to twisted conjugations, $\nabla\varphi = {}^{\tau}\nabla'\varphi$ and hence

$$\langle (r^3 - r)\cdot (\nabla\varphi - {}^{\tau}\nabla'\varphi),\, \nabla\psi - {}^{\tau}\nabla'\psi\rangle$$

is identically zero for any $\psi \in C^\infty(G)$. Thus in the study of Lax equations the unwanted terms may be safely discarded.

6.7. Difference Lax Equations.

The most important special case of Theorem 6.8 is connected with Lax equations on one-dimensional lattice. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a factorizable Lie bialgebra and G the corresponding factorizable Poisson group. Put

$$\mathbb{G} = G^N, \quad \mathfrak{G} = \bigoplus^N \mathfrak{g}, \quad \mathfrak{G}^* = \bigoplus^N \mathfrak{g}^*.$$

It is convenient to consider the elements of $\mathbb{G}$ and $\mathfrak{G}$ as functions on $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ with values in G and $\mathfrak{g}$, respectively. The inner product in $\mathfrak{G}$ is given by the natural formula

$$\langle X, Y\rangle = \sum_n \langle X_n, Y_n\rangle. \tag{6.32}$$

Extend $r \in \operatorname{End}\mathfrak{g}$ to a linear operator in $\mathfrak{G}$ by the formula

$$r(X_0 \dots X_{N-1}) = (rX_0 \dots rX_{N-1}). \tag{6.33}$$

Clearly, $(\mathfrak{G}, \mathfrak{G}^*)$ is also a factorizable Lie bialgebra. We equip $\mathbb{G}$ with the product Poisson structure (4.32). Obviously, $\mathbb{G}$ is a Poisson Lie group and its tangent Lie bialgebra is $\mathfrak{G}$, $\mathfrak{G}^*$. An element of $\mathbb{G}$ will be denoted by $L = (L_0 \dots L_{N-1})$.

Define the mappings ψ_m, $m \in \mathbb{Z}$, $T\colon \mathbb{G} \to G$, by the formulae

$$\psi_0(L) = 1, \quad \psi_m(L) = \overset{\curvearrowleft}{\prod_{0\le k\le m-1}} L_{k \bmod N}, \tag{6.34}$$

$$T(L) = \overset{\curvearrowleft}{\prod_{0\le n\le N-1}} L_k. \tag{6.35}$$

The functions ψ_m satisfy the system of linear difference equations

$$\psi_{m+1} = L_m\psi_m, \quad m = 1 \dots N, \tag{6.36}$$

and $T(L)$ is the monodromy matrix for this system.

Proposition 6.20. *The mapping $T\colon \mathbb{G} \to G\colon L \mapsto T(L)$ is Poisson.*

We have already noted that this property of Sklyanin bracket was the main motivation in the theory of Poisson Lie groups.

Let $\tau \in \operatorname{Aut}\mathbb{G}$ be the cyclic permutation, $({}^\tau g)_m = g_{m+1 \bmod N}$. The corresponding automorphism of the Lie algebra $\mathbb{G}$ preserves the inner product and commutes with r. It is easy to see that twisted conjugations in $\mathbb{G}$ sending L to ${}^\tau g L g^{-1}$ coincide with gauge transformations for the linear system (6.36), i. e., with transformation of L induced by left translations $\psi_m \mapsto g_m\psi_m$, $g \in \mathbb{G}$, in the space of its solutions. Thus we can use Theorem 6.8 to construct generalized Lax equations on $\mathbb{G}$.

We have already seen in Theorem 6.1 that the elements $L, L' \in \mathbb{G}$ lie on the same gauge orbit in $\mathbb{G}$ if and only if their monodromy matrices $T(L)$, $T(L')$ are conjugate in G.

REMARK 6.21. Proposition 6.13 implies that gauge orbits are Poisson submanifolds in $\mathbb{G}$; one can show that if the length of the lattice N is odd, generic symplectic leaves of $\mathbb{G}$ are open submanifolds of gauge orbits.

COROLLARY 6.22. *Gauge invariant functions on* $\mathbb{G}$ *have the form* $h_\varphi = \varphi \circ T$, *where* $\varphi \in I(G)$ *and* $T \colon \mathbb{G} \to G$ *is the monodromy mapping* (6.35).

The next assertion is a specialization of Theorem 6.8 for the case of lattice Lax equations.

THEOREM 6.23. (*i*) *Functions* h_φ, $\varphi \in I(G)$, *are in involution with each other with respect to the Sklyanin bracket on* $\mathbb{G}$.

(*ii*) *The Hamiltonian equation of motion on* $\mathbb{G}$ *with Hamiltonian* h_φ *has the form*

$$\frac{dL_m}{dt} = L_m M_m^\pm - M_{m+1}^\pm L_m; \tag{6.37}$$

where $M_m^\pm$ *are given by*

$$M_m^\pm = r_\pm\bigl(\psi_m \nabla\varphi\bigl(T(L)\bigr)\psi_m^{-1}\bigr). \tag{6.38}$$

(*iii*) *The monodromy matrix satisfies the* Novikov equation

$$\frac{dT}{dt} = TM_\pm - M_\pm T, \quad M_\pm = r_\pm\bigl(\nabla\varphi(T)\bigr). \tag{6.39}$$

The Hamiltonian flow of h_φ *is factorized over* G *by means of the monodromy map* $T \colon \mathbb{G} \to G$.

(*iv*) *Let* $\bigl(g_m(t)\bigr)_\pm$ *be the solution of the factorization problem* (2.26) *with the l.h.s.*

$$g_m(t) = {}^0\psi_m \exp t\nabla\varphi\bigl(T(L^0)\bigr)\,{}^0\psi_m^{-1}, \quad {}^0\psi_m = \psi_m(L^0). \tag{6.40}$$

The integral curve of (6.37) *emanating from* $L^0 = (L_0^0 \dots L_{N-1}^0)$ *is given by*

$$L_m(t) = g_{m+1}(t)_\pm^{-1} L_m^0 g_m(t)_\pm. \tag{6.41}$$

The only formula which still has to be checked is the formula for the gradient of the gauge invariant function h_φ.

LEMMA 6.24. *The right gradient of* h_φ *is given by* $\nabla' h_\varphi(L) = (X_0', \dots, X_{N-1})$, *where*

$$X_m'(L) = \psi_m(L)\nabla\varphi(T(L))\psi_m(L)^{-1}. \tag{6.42}$$

PROOF. For simplicity we assume that G is a matrix group. Variation of coefficients in the auxiliary linear system for ψ gives the relation

$$\delta\psi_{m+1} = \delta L_m \psi_m + L_m \delta\psi_m. \tag{6.43}$$

Put $C_m = \psi_m^{-1}\delta\psi_m$; formula (6.43) implies that

$$C_{m+1} - C_m = \psi_{m+1}^{-1}\delta L_m \psi_m,$$

and hence

$$C_m = \sum_{0 \leqslant k \leqslant m} \psi_{k+1}^{-1}\delta L_k \psi_k.$$

If the variation of the potential δL_k is generated by right translations $L_k \mapsto L_k e^{t\xi_k}$, we have

$$\left(\frac{d}{dt}\right)_{t=0} h_\varphi(Le^{t\xi}) = \left\langle \nabla\varphi(T(L)), \sum_{0\leqslant k\leqslant N-1} \psi_k^{-1}\xi_k\psi_k \right\rangle.$$

Taking into account the invariance of the inner product, this formula is equivalent to (6.42).

Remark 6.25. The use of cyclic permutation, which leads to difference Lax equations, is just one example of the use of Theorem 6.8. Here is another useful application.

Let $\mathfrak{G} = \mathcal{L}(\mathfrak{gl}\,(n))$ be the loop algebra equipped with the trigonometric r-matrix (4.22). It is easy to see that the trigonometric r-matrix commutes with the inner automorphisms $\operatorname{Ad} h$ of the loop algebra, where $h \in H \subset GL(n)$ is a diagonal matrix (i. e., an element of the standard Cartan subgroup of $GL(n)$). Let us fix h and consider twisted conjugations acting on the associated loop group,

$$L \longmapsto {}^h x \cdot L \cdot x^{-1}, \quad \text{where} \quad {}^h x = \operatorname{Ad} h \cdot x.$$

Their invariants form a commutative subalgebra in the Poisson algebra of functions on the corresponding loop group (with respect to the Sklyanin bracket).

Proposition 6.26. *The algebra of twisted invariants is generated by functions*

$$L \longmapsto \operatorname{Res} \varphi(\lambda) \operatorname{tr} hL(\lambda)^n. \tag{6.44}$$

In the theory of Quantum Groups such functionals arise naturally because of the "quantum corrections" associated with the square of the antipode. For this reason functionals (6.44) are sometimes called *quantum traces.*

We leave it to the reader to formulate for this case an analogue of Theorem 6.8.

6.8. Poisson Lie Groups and Central Extensions

The twisted Poisson bracket (6.26) associated with an automorphism of the tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ looks rather complicated. In order to understand this formula let us first study the behaviour of Lie bialgebras under central extensions.

Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a factorizable Lie bialgebra. A linear operator $\partial \in \operatorname{End}\mathfrak{g}$ is called a derivation of $(\mathfrak{g}, \mathfrak{g}^*)$ if ∂ is a derivation of $\mathfrak{g}$ which is skew with respect to the inner product in $\mathfrak{g}$ and commutes with $r \in \operatorname{End}\mathfrak{g}$. Formula

$$\omega(X, Y) = \langle X, \partial Y\rangle \tag{6.45}$$

defines a 2-cocycle on $\mathfrak{g}$; thus we get an embedding $\operatorname{Der}(\mathfrak{g}, \mathfrak{g}^*) \hookrightarrow C^2(\mathfrak{g})$.

EXAMPLE 6.27. Let $\mathfrak{g} = \mathcal{L}(\mathfrak{a})$ be the loop algebra of a simple Lie algebra $\mathfrak{a}$ with the standard Lie bialgebra structure (4.22), and $\partial = \lambda\frac{\partial}{\partial\lambda}$ the derivative with respect to the loop parameter. Then $\partial \in \mathrm{Der}(\mathfrak{g}, \mathfrak{g}^*)$; the group of classes of all derivations modulo inner derivations $[\mathrm{Der}(\mathfrak{g}, \mathfrak{g}^*)]$ is isomorphic to the second cohomology group $H^2(\mathfrak{g}) \simeq H^3(\mathfrak{a})$ (this group is one-dimensional).

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R} \cdot c$ be the central extension of $\mathfrak{g}$ associated with the cocycle ω, and $\widehat{\mathfrak{g}}^* = \mathfrak{g}^* \rtimes \mathbb{R} \cdot \partial$ the semidirect product of $\mathfrak{g}^*$ and the abelian algebra $\mathbb{R}$ with the Lie bracket

$$[f + \alpha\partial,\, g + \beta\partial] = [f,\, g]_* + \alpha\partial(rg) - \beta\partial(rf). \tag{6.46}$$

PROPOSITION 6.28. *$(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}^*)$ is a Lie bialgebra.*

We shall now describe the double of $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}^*)$; for simplicity we shall assume that the r-matrix satisfies the following extra condition:

$$\partial - \partial \circ r^2 = 0. \tag{6.47}$$

(It clearly holds for standard r-matrices on loop algebras.) Put

$$\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}, \quad \widehat{\mathfrak{d}} = \mathfrak{d} \oplus \mathbb{R} \cdot c \oplus \mathbb{R} \cdot \partial$$

We define the inner product on $\widehat{\mathfrak{d}}$ by the formula

$$\langle\langle (X_1, Y_1, \alpha_1, \beta_1),\, (X_2, Y_2, \alpha_2, \beta_2) \rangle\rangle = \langle X_1, X_2 \rangle - \langle Y_1, Y_2 \rangle + \alpha_1\beta_2 + \beta_1\alpha_2.$$

Extend the derivation ∂ to $\mathfrak{d}$ by setting $\widehat{\partial}(X, Y) = (\partial X, -\partial Y)$ and define the 2-cocycle on $\mathfrak{d}$, which gives the 'central' component of the commutator in $\widehat{\mathfrak{d}}$, by

$$\omega_{\mathfrak{d}}(a, b) = \langle\langle a, \widehat{\partial} b \rangle\rangle.$$

PROPOSITION 6.29. *Let us assume that r satisfies condition* (6.47)*. Then the algebra $\widehat{\mathfrak{d}}$ is isomorphic to the double of the Lie bialgebra $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}^*)$.*

PROOF. We have to define embeddings $\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}^* \hookrightarrow \widehat{\mathfrak{d}}$ which make $(\widehat{\mathfrak{d}}, \widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}^*)$ a Manin triple. These embeddings are given by

$$(X, \alpha) \longmapsto (X, X, , \alpha c), \quad (f, \beta) \longmapsto (r_+ f, r_- f, \beta\partial);$$

using condition (6.47) it is easy to see that the embedded subalgebras are isotropic and complementary in $\widehat{\mathfrak{d}}$.

Let $\Gamma \simeq \mathbb{R}^\times$ be the group of automorphisms of $(\mathfrak{g}, \mathfrak{g}^*)$ which is generated by the derivation ∂. The Lie group which corresponds to $\widehat{\mathfrak{g}}^*$ is the semidirect product $\widehat{G}^* = G^* \rtimes \Gamma$. Let us introduce the exponential parametrization of Γ setting $\tau_e = \exp e\partial$; thus e is a coordinate function on Γ and, for each value of e, τ_e is an automorphism of $(\mathfrak{g}, \mathfrak{g}^*)$.

PROPOSITION 6.30. *The coordinate e is a Casimir function of the Poisson structure on $\widehat{G}^*$.*

Thus e plays the rôle of central charge and the Poisson structure on $\widehat{G}^*$ is in fact a *family* of Poisson brackets on G^*, parameterized by the values of e.

THEOREM 6.31. *Under the identification of G^* and G given by the factorization map the Poisson bracket on the "slice" $G^* \times \{e\}$ is given by*

$$\begin{aligned}(6.48)\quad \{\varphi, \psi\}_e = \tfrac{1}{2}\langle r(\nabla\varphi), \nabla\psi\rangle + \tfrac{1}{2}\left\langle r(\nabla'\varphi), \nabla'\psi\right\rangle \\ - \left\langle r_+({}^{\tau_e}\nabla\varphi), \nabla'\psi\right\rangle - \left\langle r_-(\nabla'\varphi), {}^{\tau_e}\nabla\varphi\right\rangle.\end{aligned}$$

Remarkably, the Poisson structure on $G^* \times \{e\}$ depends only on the automorphism $\tau = \exp(e\partial)$. Hence formula (6.48) may be applied when τ is any automorphism which need not have a logarithm, i. e. arise from a derivation of the Lie bialgebra. The bracket (6.48) is practically identical to the twisted bracket (6.26) which we studied in Section 6.5. Usually the group of outer automorphisms of a Lie bialgebra is wider than its infinitesimal analogue, the group $[\mathrm{Der}(\mathfrak{g}, \mathfrak{g}^*)]$ of outer derivations. Hence the Poisson structure (5.17) may be twisted even when $[\mathrm{Der}(\mathfrak{g}, \mathfrak{g}^*)] = 0$, e. g., for finite-dimensional semisimple Lie algebras[3].

EXERCISE 6.32. Show that formal linearization of the bracket (6.48) at the unit element gives the Lie–Poisson bracket of the central extension of $\mathfrak{g}$. In particular, for loop algebras we obtain the Schwinger bracket (7.11) which will be discussed in the next lecture.

One of important examples of deformation of a quadratic Poisson structure is associated with the central extension of the Lie algebra $\mathfrak{A}$ of formal pseudodifferential operators. This central extension was discovered by O. Kravchenko and B.Khesin [**KK**]; it is associated with the outer derivation of $\mathfrak{A}$, $D = \operatorname{ad}\log\partial_x$. Recall that there is an associative product in $\mathfrak{A}$ given by

$$(6.49)\qquad a \circ b = \sum_{n=0}^{\infty} \frac{1}{n!}\,\partial_\xi^n a\, \partial_x^n b.$$

We may formally adjoin to $\mathfrak{A}$ the element $\log\xi$; since $\partial_\xi \log\xi = \xi^{-1}$, $\partial_x \log\xi = 0$, the products $a \circ \log\xi$ and $\log\xi \circ a$ are well defined for any $a \in \mathfrak{A}$.

EXERCISE 6.33. Show that

$$Da = \log\xi \circ a - a \circ \log\xi$$

is a well defined element of $\mathfrak{A}$ and that D is a derivation of $\mathfrak{A}$.

The bilinear form of D is a nontrivial 2-cocycle on $\mathfrak{A}$; one can show that its restriction to the subalgebra $\mathfrak{A}_-$ of formal integral operators remains nontrivial. Clearly, we have $\exp\alpha\log\partial_x = \partial_x^\alpha$; hence the elements of the group Γ generated by D are *fractional derivations*. For $\alpha \in \mathbb{N}$ we have simply $\exp\alpha\log\partial_x = \partial_x^n$. The geometry of the group of formal integral operators and of its central extension is thoroughly studied in the paper of Khesin and Zakharevich [**KZ**].

[3]This observation applies to Quantum Groups as well; quantization of the twisted bracket (6.26) on the loop group gives a convenient realization of the quantum universal enveloping algebra with nonzero central charge [**RS2**].

6.8.1. Nonlocal Poisson Brackets on the Lattice. If the Lie algebra $\mathfrak{G} = \bigoplus^N \mathfrak{g}$ is connected with a periodic one-dimensional lattice, formula (6.26) gives a nonlocal Poisson bracket on the lattice, that is, dynamical variables associated with neighbouring points do not commute with each other. It is useful to write down this bracket in tensor notation.

PROPOSITION 6.34. *Let* $\tau \in \operatorname{Aut}\mathfrak{G}$ *be the cyclic permutation. The bracket* (6.26) *on* $\mathbb{G} = G^N$ *has the form*[4]

$$\{L_m^1, L_n^2\} = \tfrac{1}{2} r L_m^1 L_n^2 \delta_{mn} + \tfrac{1}{2} L_m^1 L_n^2 r \delta_{mn} - \\ - L_n^1 r_+ L_m^2 \delta_{m+1,n} - L_m^2 r_- L_n^1 \delta_{m,n+1}. \tag{6.50}$$

The bracket (6.50) is characterized by two important properties:

(1) This bracket is *covariant* with respect to gauge transformations on the lattice. In other words, the gauge action
$$G^N \times \mathbb{G} \longrightarrow \mathbb{G} \colon L_n \longmapsto g_n L_n g_{n+1}^{-1}$$
is a Poisson mapping.

(2) The monodromy map $T \colon \mathbb{G} \longrightarrow G$ is Poisson (the target group G is equipped with the dual Poisson structure (5.18)).

The covariance with respect to the gauge action immediately follows from the origin of the twisted bracket (6.50) (reduction of the twisted double $G^N \times G^N$ with respect to the diagonal subgroup). The second assertion generalizes the simple multiplicativity property to the case of non-local Poisson brackets; importantly, the Poisson bracket relations between the first and the last matrices L_1 and L_N are non-trivial, which is the key point of the proof.

[4] For typographic reasons, the indices 1,2 which label different tensor factors have raised up.

LECTURE 7

Zero Curvature Equations and Current Algebras

7.1. Introduction

As we already mentioned in Lecture 6, integrable partial differential equations (with two independent variables) arise as "zero curvature equations‘"

$$\partial_t L = \partial_x M + [L, M], \tag{7.1}$$

i. e. as consistency conditions for an auxiliary linear system

$$\begin{aligned}\partial_x \psi &= L\psi,\\ \partial_t \psi &= M\psi.\end{aligned}$$

The phase space of zero curvature equations may be identified with the dual of an appropriate Lie algebra with linear Poisson structure. Still, some aspects of the theory of zero curvature equations are closely related to the theory Poisson Lie groups. In this lecture we shall give a general outline of this theory.

To understand the main idea of our approach, let us look at the formula

$$\frac{dL}{dt} = -\operatorname{ad}^* M \cdot L, \tag{7.2}$$

which gives the form of a generalized Lax equation for an arbitrary Lie algebra (Theorem 2.5). In order to make (7.2) coincide with a zero curvature equation (7.2), our basic Lie algebra $\mathfrak{G}$ should consist of functions of x with values in an appropriate "little" Lie algebra $\mathfrak{g}$ which parameterizes the "intrinsic" degrees of freedom of our system "in a single point". Moreover we need that

$$\operatorname{ad}^* M \cdot L = [M, L] - \partial_x M. \tag{7.3}$$

A Lie algebra whose coadjoint representation is given by (7.3) is easy to find: this is precisely the central extension of the loop algebra. Following our general scheme, we must describe coadjoint invariants for this new Lie algebra and equip it with the structure of a Lie dialgebra with the help of an appropriate r-matrix. We shall see that if the "little" Lie algebra $\mathfrak{g}$ is finite-dimensional, the ring of its coadjoint invariants is finitely generated. This is of course not sufficient in order to construct a complete set of integrals for a nonlinear PDE. A natural remedy is to introduce one more parameter into our Lie algebra, i.e., to consider functions of *two* variables with values in a matrix Lie algebra. The second parameter naturally becomes a spectral parameter in the auxiliary linear problem. Its introduction immediately supplies us with a natural set of r-matrices.

7.2. Current Algebras and Their Central Extensions

We shall start with central extensions of ordinary loop algebras. Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate inner product $\langle\cdot\,,\,\cdot\rangle$ which we use to identify $\mathfrak{g}$ with its dual space. Let $\mathfrak{G} = C^\infty(S^1;\,\mathfrak{g})$ be the Lie algebra of smooth functions on the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with values in $\mathfrak{g}$ and with the pointwise commutator. An invariant inner product in $\mathfrak{G}$ is given by

$$\langle X,\,Y\rangle = \int_0^{2\pi} dx \langle X(x),\,Y(x)\rangle. \tag{7.4}$$

Note that our choice of topology in $\mathfrak{G}$ is different from the topology of the Lie algebra of polynomial loops $\mathfrak{g}[\lambda,\lambda^{-1}]$ which we studied in chapter 3; the role of the loop parameter x is now also totally different. To avoid any confusion it is useful to change also the name of our basic Lie algebra. In the sequel, we shall call it *current algebra* (with values in $\mathfrak{g}$).

Proposition 7.1. *Formula*

$$\omega(X,\,Y) = \int_0^{2\pi} dx \langle X(x),\,\partial_x Y(x)\rangle \tag{7.5}$$

defines a 2*-cocycle on* $\mathfrak{G}$ ("the Maurer–Cartan cocycle")*; in other words,*

$$\omega\big([X,\,Y],\,Z\big) + \omega\big([Y,\,Z],\,X\big) + \omega\big([Z,\,X],\,Y\big) = 0. \tag{7.6}$$

The proof is obvious. Note that in order to check (7.6) one needs to integrate by parts; it is therefore important that the operator ∂/∂_x is skew in the space of periodic functions.

Let $\widehat{\mathfrak{G}} = \mathfrak{G} \dot{+} \mathbb{C}$ be the central extension of $\mathfrak{G}$, associated with the cocycle (7.5)[1]. By definition, elements of $\widehat{\mathfrak{G}}$ are pairs (X, c), $X \in \mathfrak{G}$, $c \in \mathbb{C}$; their Lie bracket is given by

$$\big[(X,\,c),\,(Y,\,c')\big] = \big([X,\,Y],\,\omega(X,\,Y)\big). \tag{7.7}$$

Notice that $\mathfrak{c} = \{(0,c); c \in \mathbb{C}\} \subset \widehat{\mathfrak{G}}$ is central in $\widehat{\mathfrak{G}}$; the original Lie algebra $\mathfrak{G}$ may be identified with the quotient algebra $\widehat{\mathfrak{G}}/\mathfrak{c}$. The inner product (7.4) on $\mathfrak{G}$ defines an embedding of $\mathfrak{G}$ in its dual space. Its image under this embedding consists of smooth linear functionals $\mathfrak{G}$. In the sequel we shall mainly deal only with smooth functionals (a noteworthy exception is Proposition 7.19 below where smoothness is violated because of the boundary conditions). By an abuse of language we shall simply assume that the spaces

[1]Central extensions of current algebras were first introduced by J.Schwinger and are widely used in Quantum Field Theory under the name of "current algebras with the Schwinger term". These central extensions also arised in the study of Kac–Moody algebras. Traditionally the term "Kac–Moody algebras" (of affine type) is reserved precisely for central extensions of loop algebras $\mathcal{L}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[\lambda,\,\lambda^{-1}]$).

$\mathfrak{G}$ and $\mathfrak{G}^*$ are identified. This convention implies that the dual of $\widehat{\mathfrak{G}}$ may be identified with $\mathfrak{G}\dot{+}\mathbb{C}$.

PROPOSITION 7.2. *The coadjoint representation of* $\mathfrak{G}$ *in* $\widehat{\mathfrak{G}}^* \simeq \mathfrak{G} \dot{+} \mathbb{C}$ *is given by*

$$\operatorname{ad}^* M \cdot (L,\, e) = \big([M,\, L] + e\,\partial_x M,\, 0\big). \tag{7.8}$$

Note that the coadjoint representation is trivial on the center and hence it may be regarded as a representation of the quotient algebra $\mathfrak{G} \simeq \widehat{\mathfrak{G}}/\mathfrak{c}$.

PROOF. We have

$$\begin{aligned}\big\langle (L,\, e),\, \operatorname{ad} M \cdot (X,\, c)\big\rangle = \\ \big\langle (L,\, e),\, \big([M,\, X],\, \omega(M,\, X)\big)\big\rangle = \\ -\big([M,\, L],\, X\big) - e(\partial_x M,\, X).\end{aligned}$$

Since $\operatorname{ad}^* M = -(\operatorname{ad} M)^*$, formula (7.8) follows immediately.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For simplicity we shall assume that G is a matrix group (semisimple or reductive[2]); the general case does not lead to any difficulties, but the formulations become more bulky. Let $\mathcal{G} = C^\infty(S^1,\, G)$ be the group of smooth functions on the circle with values in G and with pointwise multiplication. $\mathcal{G}$ is usually called a current group. A description of the central extension of $\mathcal{G}$, which corresponds to the central extension (7.7) of its Lie algebra is rather complicated; one can show that the fiber bundle $\widehat{\mathcal{G}} \longrightarrow \mathcal{G}$ is topologically non-trivial. Luckily, since the center acts trivially in the coadjoint representation, we may regard this latter as a representation of the original current group $\mathcal{G}$.

The construction of Lie groups associated with the central extension of current algebras is nontrivial. In these lectures we shall not deal with these groups; one of their most important applications in the theory of Integrable Systems is Sato's theory of dressing transformations. The reader may find a beautiful exposition of this theory as well as an explicit global description of the central extensions of the current groups in [**PS**].

PROPOSITION 7.3. *The coadjoint representation of* $\mathcal{G}$ *in* $\widehat{\mathfrak{G}}^* \simeq \mathfrak{G} \dot{+} \mathbb{C}$ *is given by*

$$\operatorname{Ad}^* g \cdot (L,\, e) = (gLg^{-1} + e\partial_x g g^{-1},\, e). \tag{7.9}$$

PROOF. Clearly, the r.h.s. of (7.9) defines a linear represetation K of $\mathcal{G}$ in $\widehat{\mathfrak{G}}^*$. In order to check that K coincides with the coadjoint representation, it is enough to observe that the operators of the infinitesimal representation

$$K(X) = \big(\tfrac{d}{dt}\big)_{t=0} K(e^{tX}),\ X \in \mathfrak{G},$$

coincide with the operators $\operatorname{ad}^* X$ given by (7.8).

[2]We need reductive groups in order not to drop out the $GL(n)$ case.

Proposition 7.3 has an important geometric interpretation. Let us consider an auxiliary linear equation on the line with periodic potential

$$e\,\partial_x\psi = L\psi, \quad \psi \in C^\infty(\mathbb{R}, G). \tag{7.10}$$

THEOREM 7.4. *Coadjoint representation operators* $\mathrm{Ad}^* g$, $g \in \mathcal{G}$ *leave invariant the hyperplanes* $e = const$ *in* $\widehat{\mathfrak{G}}$*; their restriction to any such hyperplane with* $e = const \neq 0$ *coincides with gauge transformations of the potential* L *in the linear differential equation* (7.10) *which are induced by the natural action of* $\mathcal{G}$ *by left multiplication* $\psi \longmapsto g\cdot\psi$ *in the space of its solutions.*

Classification of coadjoint orbits in $\widehat{\mathfrak{G}}$ is practically equivalent to the classical Floquet theorem. Let $\psi_L \in C^\infty(\mathbb{R}, G)$ be the fundamental solution of (7.10) normalized by the condition $\psi_L(0) = id$. The matrix $T_L = \psi_L(2\pi)$ is called the *monodromy matrix* of equation (7.10).

THEOREM 7.5 (Floquet). *Potentials* $L, L' \in C^\infty(S^1;\mathfrak{g})$ *lie on the same coadjoint orbit in* $\widehat{\mathfrak{G}}^*$ *(with fixed* $e \neq 0$*) if and only if the corresponding monodromy matrices* T_L, $T_{L'}$ *are conjugate in* G.[3]

PROOF. The proof is completely similar to the proof of the "difference" Floquet theorem in section 6.7.

COROLLARY 7.6. *Let* φ *be a central function on* G*. Functionals* $L \longmapsto \varphi(T_L)$ *on* $\widehat{\mathfrak{G}}^*$ *are gauge invariant and generate the ring of Casimir functions of* $\widehat{\mathfrak{G}}^*$ *on each hyperplane* $e = const \neq 0$.

Hamiltonian mechanics in the space $\widehat{\mathfrak{G}}^*$ (equipped, as usual, with the Lie–Poisson bracket) may be constructed with the help of the elementary variational calculus. Let $\varphi\,[L]$be a smooth functional of L, let $\nabla\varphi = \mathrm{grad}\,\varphi\,[L] \in \mathfrak{G}$ be its Frechet derivative defined by

$$\left(\tfrac{d}{ds}\right)_{s=0}\varphi\,[L + s\eta] = \int_0^{2\pi}\langle\nabla\varphi\,(x)\,,\,\eta\,(x)\rangle dx, \quad \eta \in \mathfrak{G}.$$

The Lie–Poisson bracket of two such functionals φ_1, φ_2 is given by

$$\{\varphi_1, \varphi_2\}\,[L] = \langle\Lambda_L\nabla\varphi_1,\,\nabla\varphi_2\rangle, \quad \text{where} \quad \Lambda_L = e\,\partial_x + \mathrm{ad}\,L.$$

Physicists usually write this bracket using the "field theory notation". Fix a basis $\{e_a\}$ in $\mathfrak{g}$ and let f_{ab}^c be the corresponding structure constants and $K_{ab} = \langle e_a, e_b\rangle$ the associated Gram matrix. Let us define the "evaluation functionals" $\varphi_{a,x}$ on the current algebra by setting $\varphi_{a,x}[L] = \langle L(x), e_a\rangle$.

EXERCISE 7.7. The Lie–Poisson bracket of evaluation functionals is given by

$$\{\varphi_{a,x},\,\varphi_{b,y}\}\,[L] = \sum_c f_{ab}^c\,L_c(x)\,\delta(x-y) + e\,K_{ab}\delta'(x-y).$$

[3]An important observation on the connection between the Floquet theory and classification of coadjoint orbits in $\widehat{\mathfrak{G}}^*$ was made by I.Frenkel in the end of 1970's; his paper **[F]** on orbits method for affine Lie algebras has been published only a few years later.

Physicists usually do not distinguish the evaluation functionals and their values on a test function and write simply

$$\{L_a(x),\, L_b(y)\} = \sum_c f_{ab}^c L_c(x)\,\delta(x-y) + e\,\delta_{ab}\delta'(x-y). \tag{7.11}$$

In physical literature formula (7.11) is commonly used as the definition of the Poisson bracket relations in the "current algebra with Schwinger term".

For the comparison of this formula with formulas arising in the theory of Poisson groups let us rewrite it in tensor notation Let us replace "scalar" evaluation functionals by vector ones with values in the Lie algebra $\mathfrak{g}$; thus we set $\varphi_x[L] = L(x)$; the Poisson bracket of two such functionals is an element of $\mathfrak{g}\otimes\mathfrak{g}$. Let $\{e^a\}$ be the dual basis of $\mathfrak{g}$, $\langle e_a, e^b\rangle = \delta_{ab}$. Set $t = \sum K_{a,b}e^a\otimes e^b$; this is the so called *tensor Casimir element* of $\mathfrak{g}$. Making again no distinction between functionals and their values we get

$$\left\{L(x) \overset{\otimes}{,} L(y)\right\} = [t,\, L(x)\otimes I - I\otimes L(y)]\,\delta(x-y) + e\,t\,\delta'(x-y). \tag{7.12}$$

PROPOSITION 7.8. *The Hamiltonian equation of motion on $\widehat{\mathfrak{G}}^*$ with Hamiltonian φ is equivalent to the following differential equation for L:*

$$\frac{\partial L}{\partial t} = -\,[X_\varphi, L] - e\frac{\partial X_\varphi}{\partial x}. \tag{7.13}$$

Hence Hamiltonian equations on $\widehat{\mathfrak{G}}^*$ automatically have the form of zero curvature equations for a $\mathfrak{g}$-valued connection form

$$Ldx + X_\varphi dt.$$

We can now try to use the Lie algebra $\widehat{\mathfrak{G}}$ to produce integrable PDE's. Our preliminary observations look quite encouraging:

(1) The description of coadjoint orbits in $\widehat{\mathfrak{G}}^*$ has automatically lead us to the auxiliary linear problem (7.10).
(2) Hamiltonian equation of motion in $\widehat{\mathfrak{G}}$ have the form of zero curvature equations.

However, there also two important difficulties:

(1) Algebra $\widehat{\mathfrak{G}}$ has got only a finite number of independent Casimir functions (a set of its generators is provided for instance by the coefficients of the characteristic polynomial $\det(T_L - \mu)$).
(2) Casimir functions are non-local functionals of the potential.

In order to construct an integrable PDE we certainly need to have an *infinite* number of independent conservation laws; in applications such conservation laws are usually expressed by integrals of *local densities* which are polynomial (or may be rational) functions of the matrix coefficients of L and of their derivatives over x. A natural way to overcome these difficulties is suggested by the auxiliary linear problem (7.10) iteself: in order to restore the potential from the monodromy matrix or from scattering data, one needs to know the monodromy *for all values of energy*; in other words, we must include into the differential equation (7.10) a spectral parameter. Algebraically, this means that we have to modify the choice of our initial Lie algebra.

7.3. Double Loop Algebras

As in the previous section, we set $\mathfrak{G} = C^{\infty}\left(S^1; \mathfrak{g}\right)$. Let $\mathbf{g} = \mathfrak{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ denote the Lie algebra of Laurent polynomials with values in $\mathfrak{G}$. (In other words, $\mathbf{g}$ *is a double loop algebra*[4] $\mathfrak{g}$; the "space" variable x is distinguished from the spectral variable λ by the choice of the central extension.) In the Lie algebra $\mathbf{g}$ we define the inner product

$$\langle X, Y \rangle = \operatorname{Res}_{\lambda=0} \int \langle X(x, \lambda),\, Y(x, \lambda) \rangle \, dx. \tag{7.14}$$

Let the 2-cocycle ω on $\mathbf{g}$ be given by

$$\omega(X, Y) = \operatorname{Res}_{\lambda=0} \int \left\langle X(x, \lambda), \tfrac{dY(x,\lambda)}{dx} \right\rangle dx. \tag{7.15}$$

Let $\widehat{\mathbf{g}}$ be the central extention of $\mathbf{g}$ defined by this cocycle. As before, we identify the dual space $\widehat{\mathbf{g}}$ with $\mathbf{g} \oplus \mathbb{C}$. The coadjoint action is still given by (7.8). We see that the coadjoint action of the Lie algebra $\widehat{\mathbf{g}}$ is given exactly by the infinitesimal gauge transformations associated with the auxiliary linear problem (7.10), where this time $L \in \mathbf{g}$ is a Laurent polynoimial in λ with coefficients in $\mathfrak{G}$. To put it another way, λ plays the role of the spectral parameter in the auxiliary linear problem. We remark that the monodromy matrix T_L is a well-defined function of λ (with values in $G = GL(n)$), holomorphic in $\mathbb{C} \setminus \{0\}$.

Our choice of the basic Lie algebra easily provides the second key element of the general construction, the classical r-matrix. Let

$$\mathbf{g}_+ = \mathfrak{G} \otimes \mathbb{C}[\lambda], \ \mathbf{g}_- = \mathfrak{G} \otimes \lambda^{-1}\mathbb{C}\left[\lambda^{-1}\right]. \tag{7.16}$$

Clearly, $\mathbf{g} = \mathbf{g}_+ \dot{+} \mathbf{g}_-$ as a linear space. As usual, Le

$$r = P_+ - P_-, \tag{7.17}$$

where P_+, P_- are the projection operators onto $\mathbf{g}_+$ and $\mathbf{g}_-$, respectively. Let the r-bracket on $\mathbf{g}$ be defined by $[X, Y]_r = \frac{1}{2}([rX, Y] + [X, rY])$. In a more general way, let $r \in \operatorname{End} \mathfrak{g}$ –be any solution of the modified Yang-Baxter equation on $\mathfrak{g}$; extend it to $\mathfrak{G} = C^{\infty}(S^1, \mathfrak{g})$ by

$$(rX)(x) = r\big(X(x)\big). \tag{7.18}$$

This gives the Lie algebra $\mathbf{g}_r$ which is not yet exactly what we need to be able to use the Theorem 2.5, since we need a Lie dialgebra structure on the central extension of $\mathbf{g}$ rather than on $\mathbf{g}$ itself. It turns out that the algebra $\mathbf{g}_r$

[4] Double loop algebra should not be confused with the double *of* a loop algebra which we discussed in chapter 5. In the present setting "double loop algebra" means "algebra of functions of two variables with values in a Lie algebra".

has in turn to be replaced by its central extension where the corresponding 2-cocycle is derived from the cocycle on $\mathbf{g}$ in a canonical way.

PROPOSITION 7.9. *Let* $(\mathfrak{g}, r)$ *be a Lie dialgebra and let* Ω *be a 2-cocycle on* $\mathfrak{g}$. *Suppose that the operator* $r \in \operatorname{End} \mathfrak{g}$ *satisfies the modified Yang-Baxter equation* (2.21). *Then*

$$\Omega_r(X, Y) = \tfrac{1}{2}\big(\Omega(X, rY) + \Omega(rX, Y)\big) \tag{7.19}$$

is a cocycle on $\mathfrak{g}_r$.

PROOF. First assume that r satisfies the «usual» Yang-Baxter (2.20), that is, $r\big([X, Y]_r\big) = \frac{1}{2}[rX, rY]$. Then

$$\begin{aligned}2\Omega_r\big([X, Y]_r, Z\big) = \Omega\big([X, Y]_r, rZ\big) + \Omega\big(r\big([X, Y]_r\big), Z\big) = \\ = \tfrac{1}{2}\Omega\big([rX, rY], Z\big) + \Omega\big([X, Y]_r, rZ\big).\end{aligned}$$

Since Ω is closed and skew-symmetric, we have

$$\begin{aligned}&\Omega\big([rX, rY], Z\big) + c.p. = \\ &\quad = -\Omega\big([rY, Z], rX\big) - \Omega\big([Z, rX], rY\big) - c.p. = \\ &\quad\quad = -2\Omega\big([X, Y]_r, rZ\big) - c.p.\end{aligned}$$

therefore

$$d\Omega_r(X,Y,Z) = \Omega_r\big([X, Y]_r, Z\big) + \Omega_r\big([Y, Z]_r, X\big) + \Omega_r\big([Z, X]_r, Y\big) = 0.$$

If r satisfies the modified Yang-Baxter equation (2.21), then the expression for $d\Omega_r$ contains additional terms $\Omega([X,Y],Z) + c.p.$, which vanish because Ω is closed.

When $\Omega = \omega$ is the Maurer-Cartan cocycle, the operator $\partial/\partial x$ is *essentially skew-self-adjoint* on its domain of definition $D = C^\infty(R/\mathbb{Z}; \mathfrak{g})$ and the operator r leaves this domain invariant. (Technically this is a crucial fact since the proof of the Jacobi identity requires integration by parts.)

Proposition 7.9 allows us to equip the central extension of the current algebra with the the structure of a Lie dialgebra. More precisely, we extend the operator $r \in \operatorname{End}\mathfrak{g}$, given by (7.18) on $\widehat{\mathfrak{g}} = \mathfrak{g} \dot{+} \mathbb{C}$ by, for instance,

$$\widehat{r}(X, c) = (rX, c). \tag{7.20}$$

Clearly, the Lie algebra structure defined in $\widehat{\mathfrak{G}}$ by the operator (7.20) is the central extension of $\mathfrak{G}_r$, defined by the cocycle ω_r.

Theorem 2.5 provides a family of commuting Hamiltonians on $\widehat{\mathfrak{G}}^*$ and the induced equations of motion have the form of zero-curvature equation.

The Lie–Poisson bracket for $\mathfrak{G}_r$ is given by

$$\{\varphi_1, \varphi_2\}(L) = \int_0^1 dx\big(L, [\text{grad}\ \varphi_1(L), \text{grad}\ \varphi_2(L)]_r\big) \tag{7.21}$$

This type of bracket is said to be *ultra-local.*[5] The Lie–Poisson bracket of $\widehat{\mathfrak{G}}_r$ differs from (7.21) by the term $\omega_r(\mathrm{grad}\ \varphi_1, \mathrm{grad}\ \varphi_2)$. Clearly, this bracket is ultra-local if and only if $\omega_r = 0$. This case is particularly important for applications.

PROPOSITION 7.10. *The following properties are equivalent: (i)* $\omega_r = 0$ *(ii)* $r = -r^*$. *Thus the Lie–Poisson bracket of* $\widehat{\mathfrak{G}}_r$ *is ultra-local if and only if its r-matrix is skew-symmetric.*

PROOF. The operator (7.18)commutes with derivation, so $\omega_r = 0$ is equivalent to $r + r^* = 0$.

If $\omega_r = 0$, the algebra $\mathfrak{G}_r$ is essentially the same as $\widehat{\mathfrak{G}}_r$. The simplest family of orbits of $\mathfrak{G}_r$ is given by the following proposition.

PROPOSITION 7.11. *Let* $\mathcal{O}$ *be an orbit of* $\mathfrak{g}_r$. *Then the loop space* $\Omega(\mathcal{O})$, *that is the space of smooth mappings* $\mathbb{R}/\mathbb{Z} \to \mathcal{O}$ *is an orbit of* $\mathfrak{G}_r$.

In a more general way, one can vary the orbit $\mathcal{O}$ as the point varies. Instead of $\Omega(\mathcal{O})$ we get sections of the fibration $E \to S^1$ with orbits as fibers.

EXERCISE 7.12. (1) Let $\mathfrak{g} = \mathfrak{su}\,(2)$; then the matrices $s \in \mathfrak{g}$,

$$s = i\begin{pmatrix} s_3 & s_1 + is_2 \\ s_1 - is_2 & -s_3 \end{pmatrix}, \quad s_j \in \mathbb{R}, \quad s_1^2 + s_2^2 + s_3^2 = 1, \tag{7.22}$$

make up a coadjoint orbit $S^2 \subset \mathfrak{su}\,(2)$. Let $r \in \mathrm{End}\,\mathcal{L}(\mathfrak{g})$ be the standard r-matrix on the loop algebra, which is skew-symmetric with respect to the scalar product

$$\langle X, Y\rangle = -\mathrm{Res}_{\lambda=0}\,\mathrm{tr}\,X(\lambda)Y(\lambda).$$

Check that $\mathcal{O}_H = \left\{\lambda^{-1}\ s, s \in S^2\right\} \subset \mathcal{L}(\mathfrak{g}) \simeq \mathcal{L}(\mathfrak{g}^*)$ is a coadjoint orbit of the subalgebra $\mathcal{L}(\mathfrak{g})_+ \subset \mathcal{L}(\mathfrak{g})_r$. The corresponding orbit $\mathbf{O}_H \subset \mathbf{g}$ of $\mathfrak{G}_r$ is parameterized by triples of 2π-periodic functions s_j, $j = 1, 2, 3$, satisfying (7.22). The linear differential operator associated to this orbit is

$$\frac{d}{dx} - \lambda^{-1} is\,(x)\,. \tag{7.23}$$

(2) Let

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

the matrices

$$U = i\begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix} + i\lambda\sigma, \quad u \in \mathbb{C},$$

[5]By definition, a Poisson bracket on a current algebra is ultra-local if its Poisson operator is a multiplication operator in the x variable. More general Poisson operators may involve differentiation with respect to x; non-local Poisson brackets may also be given by integral operators.

fill a coadjoint orbit of the subalgebra $\mathcal{L}(\mathfrak{g})_- \subset \mathcal{L}(\mathfrak{g})_r$. The corresponding orbit $\mathbf{O}_S \subset \mathbf{g}$ is parameterized by pairs of complex conjugate functions

$u, \bar{u} \in C^\infty(S^1, \mathbb{C})$; The linear differential operator associated to this orbit is

$$\frac{d}{dx} - U;$$

The dynamic systems related to these orbits are respectively the Heisenberg magnet and the Nonlinear Schroedinger equation.

7.4. Monodromy Mapping

Previous examples (see Exercise 7.12) show that our construction gives rise to meaningful auxiliary linear operators. The next step will be define the related Hamiltonians. We shall be interested in the Hamiltonians which are spectral invariants of the auxiliary linear operator, that is its monodromy matrix invariants. The monodromy matrix T_U of the differential equation (7.10) is holomorphic in $\mathbb{C}P_1 \setminus (\{0\} \cup \{\infty\})$; any conjugation invariant function of the monodromy matrix gives rise to a zero-curvature equation on the coadjoint orbits of $\mathbf{g}_r$. The mapping $\mathbf{T}\colon U \rightsquigarrow T_U$ is the *direct spectral transform,* related to (7.10). We can view $\mathbf{T}$ as a map from $\mathbf{g}$ to the group of matrix-valued functions which are regular in the punctured Riemann sphere. If the Poisson structure is *ultra-local* (that is, the r-matrix is skew-symmetric with respect to the scalar product in $\mathbf{g}$), the spectral transform $\mathbf{T}$ has a special property: its image has a natural Poisson structure which makes $\mathbf{T}$ a Poisson map. This Poisson structure is precisely the Sklyanin bracket associated with the given r-matrix.

For simplicity, in our study of the Poisson properties of the monodromy map we shall assume that $\mathfrak{g}$ is a matrix Lie algebra; let G be the corresponding simply-connected group and $r \in \operatorname{End}\mathfrak{g}$ the skew-symmetric linear operator which satisfies the modified Yang-Baxter equation (2.21). Let $\mathfrak{G} = C^\infty(S^1, \mathfrak{g})$; we equip $\mathfrak{G}$ with the inner product (7.5); the Poisson bracket of the functionals φ_1, φ_2 on $\mathfrak{G}^* \simeq \mathfrak{G}$ is given by

$$\{\varphi_1, \varphi_2\}_r [L] = \int_0^{2\pi} \langle [\operatorname{grad}\varphi_1 [L] (x), \operatorname{grad}\varphi_2 [L] (x)]_r , L (x)\rangle dx. \tag{7.24}$$

For $L \in \mathfrak{G}$ let ψ_L be the fundamental solution of (7.10), normalized by $\psi_L (0) = id$, and let $T_L \in G$ be the corresponding monodromy matrix. Chose a smooth function $\varphi \in C^\infty (G)$ and let $h_\varphi [L] = \varphi (T_L)$. Let $\nabla\varphi, \nabla'\varphi \in \mathfrak{g}$ be the *left and right* gradients of φ on G, defined by

$$\left(\tfrac{d}{ds}\right)_{s=0} \varphi(e^{sX} x) = \langle X, \nabla\varphi (x)\rangle,\ \left(\tfrac{d}{ds}\right)_{s=0} \varphi(x e^{sX}) = \langle X, \nabla'\varphi (x)\rangle, X \in \mathfrak{g}.$$

Lemma 7.13. *The Frechet derivative of the functional h_φ is given by*

$$\operatorname{grad} h_\varphi [L] (x) = \psi (x) \nabla'\varphi (T_L) \psi (x)^{-1} . \tag{7.25}$$

COROLLARY 7.14. *The Frechet derivative satisfies the differential equation*

$$\frac{dX}{dx} = [L, X] \tag{7.26}$$

with the boundary condition

$$X\left(0\right) = \nabla' \varphi\left(T_L\right), X\left(2\pi\right) = \nabla \varphi\left(T_L\right). \tag{7.27}$$

PROOF. Taking the variation in the both sides of (7.10) we get

$$\partial_x \delta\psi_L = L\delta\psi_L + \left(\delta L\right)\psi_L. \tag{7.28}$$

Let $\delta\psi_L = \psi_L Y$, where Y is an unknown function $Y \in C^{\infty}\left(\mathbb{R}, \mathfrak{g}\right)$, $Y\left(0\right) = 0$; from (8.36) we get $\partial_x Y = \psi^{-1}\delta L\psi$, so that

$$Y\left(x\right) = \int_0^x \psi_L^{-1}\left(y\right)\delta L\left(y\right)\psi_L\left(y\right)\,dy.$$

Since $T_L = \psi_L\left(2\pi\right)$, we find

$$T_L^{-1}\delta T_L = \int_0^{2\pi} \psi_L^{-1}\cdot\delta L\cdot\psi_L\,dy.$$

We get

$$\begin{aligned}\delta\varphi\left(T_L\right) = \langle\nabla'\varphi(T_L), T_L^{-1}\delta T_L\rangle &= \\ &= \int_0^{2\pi}\langle\psi_L\left(y\right)\nabla'\varphi\left(T_L\right)\psi_L^{-1}\left(y\right), \delta L(y)\rangle\,dy,\end{aligned}$$

which implies (7.25). Taking the derivatives on both sides of (7.25) we get (7.26).

PROPOSITION 7.15. *The Poisson bracket of the functionals $h_{\varphi_1}, h_{\varphi_2}$ on $\mathfrak{G}_r^*$ is given by*

$$\left\{h_{\varphi_1}, h_{\varphi_2}\right\}\left[L\right] = h_{\left\{\varphi_1, \varphi_2\right\}}\left[L\right], \tag{7.29}$$

where the Poisson bracket of the functions φ_1, φ_2 on G is given by

$$\left\{\varphi_1, \varphi_2\right\}_G = \frac{1}{2}\left(\langle r\left(\nabla\varphi_1\right), \nabla\varphi_2\rangle - \langle r\left(\nabla'\varphi_1\right), \nabla'\varphi_2\rangle\right). \tag{7.30}$$

PROOF. Let $X_i = \operatorname{grad} h_{\varphi_i}$, $i = 1, 2$. We have:

$$\begin{aligned}\left\{h_{\varphi_1}, h_{\varphi_2}\right\}\left[L\right] = \int_0^{2\pi}\langle\left[X_1, X_2\right]_r, L\rangle\,dx &= \\ = \tfrac{1}{2}\int_0^{2\pi}\langle\left[rX_1, X_2\right] + \left[X_1, rX_2\right], L\rangle\,dx &= \\ = \tfrac{1}{2}\int_0^{2\pi}\langle\left[L, X_2\right], rX_1\rangle + \langle\left[L, X_1\right], rX_2\rangle\,dx &= \\ = \tfrac{1}{2}\int_0^{2\pi}\frac{d}{dx}\langle rX_1, X_2\rangle\,dx,&\end{aligned}$$

where we used the definition of the r-bracket, the invariance of the scalar product, the differential equation (4.31) for the gradients X_i and, finally, the skew-symmetry of r. Evaluating the last integral and taking into account the boundary conditions (7.27) for X_i, we obtain (7.29).

The bracket (7.30) is precisely the Sklyanin bracket on G. The preceding computation also gives the following result.

THEOREM 7.16. *Let G be the Poisson group equipped with the Sklyanin bracket* (7.30). *Then the monodromy map $\mathbf{T}\colon \mathfrak{G}_r^* \to G\colon L \rightsquigarrow T_L$ is a Poisson map.*

Our proof was carried out for the loop algebra $\mathfrak{G} = C^\infty(S^1, \mathfrak{g})$; is is easily extended to the case of double loop algebra: it is enough to replace the finite-dimensional Lie algebra $\mathfrak{g}$ by its loop algebra $\mathcal{L}(\mathfrak{g})$; instead of smooth functions $\varphi_1, \varphi_2 \in C^\infty(G)$ one has to consider *smooth functionals* on the corresponding loop group; its left and right gradients are replaced by left and right variational derivatives, and so on. Spectral invariants of the auxiliary linear problem correspond to *central functionals* on the loop group. One example of such functionals is provided by *evaluating functionals* $H_{n,w}[L] = \operatorname{tr} T_L^n(w), w \in \mathbb{C}P_1 \setminus (\{0\} \cup \{\infty\})$.

EXERCISE 7.17. Compute the variational derivative of $H_{n,w}$ with respect to L.

A serious drawback of the functionals $H_{n,w}$ is their *non-locality.* In the next section we describe *local* functionals.

REMARK 7.18. In the non-ultralocal case (i. e., if $r \neq -r^*$) we can still use Theorem 2.5 to produce Poisson-commuting functionals with respect to the Lie–Poisson bracket of the Lie algebra $\widehat{\mathfrak{G}}_r$. However, Poisson brackets of *arbitrary* functionals of the monodromy matrix in general do not make sense any more. Indeed, by lemma 4.39 the gradients of functionals $\Phi(T_L)$ are functions on the line with values in $\mathfrak{g}$, which satisfy the quasi-periodicity condition

$$X(x + 2\pi) = TX(x)T^{-1}. \tag{7.31}$$

The Lie–Poisson bracket of the algebra $\widehat{\mathfrak{G}}_r$ is a bilinear form of the gradients of the functions

$$\{\varphi_1, \varphi_2\} = (\widehat{H}\, d\varphi_1, d\varphi_2); \tag{7.32}$$

by virtue of the definition of the cocycle ω_r the operator $\widehat{H}$ is given by

$$\widehat{H} = \tfrac{1}{2}\partial_x \circ (r + r^*) + \tfrac{1}{2} ad\, L(x). \tag{7.33}$$

The Poisson bracket of two functionals of the form $\Phi \circ T_L$ is well defined if the operator (7.33) is essentially skew-self-adjoint in the space of smooth functions satisfying (7.31). As is easily seen, this is the case only if the operator $r + r^*$ commutes with $\operatorname{Ad} T$, that is, proportional to the identity on every simple component of the algebra $\mathfrak{g}$. We get therefore

PROPOSITION 7.19. *Let $\mathfrak{g}$ be a simple Lie algebra. The following statements are equivalent:*

(i) For any smooth function $\Phi \in C^\infty(G)$ the functional $\Phi \circ \mathbf{T}$ on $\widehat{\mathfrak{G}}_r$ is smooth.

(ii) $r + r^*$ is a scalar operator.

EXAMPLE 7.20. Let $\mathfrak{g}$ be an affine Lie algebra with standard grading by powers of λ and inner product $\widetilde{K}_n$ defined by (3.2). Let r be the standard r-matrix associated with the decomposition (3.3) and $r^*_{(n)}$ its adjoint with respect to the inner product $\widetilde{K}_n$. Let P_k be the projection operator onto the subspace $\mathfrak{g} \cdot \lambda^k \subset \mathcal{L}(\mathfrak{g})$ parallel to the graded complement. We have

$$
\begin{aligned}
r + r^*_{(n)} &= 0, \ n = -1, \\
r + r^*_{(n)} &= 2\sum_{k=0}^{n} P_k, \ n \geq 0, \\
r + r^*_{(n)} &= -2\sum_{k=1}^{|n|-1} P_{-k}, \ n < 0.
\end{aligned}
\tag{7.34}
$$

Thus in this case functionals depending on the monodromy matrix are non-smooth whenever $r \neq r^*$.

Let us stress that the troubles with smoothness condition do not arise for central functions on G. For such functions the gradient of the functional $\Phi \circ T$, $\Phi \in I(G)$ is a periodic function, $X(x + 2\pi) = X(x)$ and there are no troubles with the domain of the Poisson operator.

7.5. Local Conservation Laws

Unlike the functionals $H_{n,w}$, local conservation laws are related to the *asymptotic* expansion of the monodromy matrix in singular points, that is in $\lambda = 0, \infty$. This leads to several complications:

(1) Local functionals are not defined on the whole of the double loop algebra.
(2) Their variational derivatives do not belong to the original loop algebra but rather to its completion consisting of formal Laurent series (in powers of the local parameter in the pole of the Lax operator); moreover, the formal series for these derivatives are usually divergent.

One needs therefore to verify directly their involutivity; the factorisation problem related to local Hamiltonians encounters serious difficulties.

All proofs in this section are valid if the basic Lie algebra $\mathfrak{g}$ is semi-simple (or reductive) with a fixed scalar product. To simplify our notation we shall limit ourselves to the case of $\mathfrak{g} = \mathfrak{gl}(n)$, the full matrix algebra. (The extension of all proofs to the general case is a useful exercice!) As before, we set $\mathfrak{G} = C^\infty(S^1;\mathfrak{g})$, $\mathbf{g} = \mathfrak{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. Assume that the potential $L(x,\lambda)$ in the auxiliary linear problem (7.10) is a Laurent polynomial,

$$L = \sum_{-N}^{M} U_k \lambda^k, \quad U_k \in C^\infty\left(S^1, \mathfrak{g}\right).$$

Let $J_\infty = U_M, J_0 = U_{-N}$ be its higher and lower coefficients.

DEFINITION 7.21. L is called *regular* if

(i) the matrices $J_0(x), J_\infty(x)$ are semi-simple,
(ii) the centralizers of the matrices $J_0(x), J_\infty(x)$ in $\mathfrak{g}$ are conjugate for all $x \in S^1$.

As we have seen, Laurent polynomials with given order of poles at zero and infinity make up a Poisson subspace with respect to the r-bracket. It is easily checked that the regularity condition holds simultaneously for all elements of the coadjoint orbit in this space; in this way it characterizes the phase space of the problem. With a regular Lax operator one can associate two families of local Hamiltonians, one for each pole of the potential on the Riemann sphere. Let us describe the family associated with the pole at infinity. After a suitable gauge transformation (which does not depend on the spectral parameter) we may assume that the leading coefficient $J_\infty(x)$ satisfies a stronger condition:

(ii)$'$ The centraliser of J_∞ in $\mathfrak{g}$ is a fixed subalgebra $\mathfrak{g}^{J_\infty} \subset \mathfrak{g}$ which does not depend on x.

(By construction, local Hamiltonians are invariant under gauge transformations, so this condition does not imply any further restriction.) Let $\mathfrak{g}_{J_\infty} \subset \mathfrak{g}^{J_\infty}$ be the commutant of $\mathfrak{g}^{J_\infty}$,

$$\mathfrak{g}_{J_\infty} = \left\{X \in \mathfrak{g}; [X,Y] = 0 \text{ for all } Y \in \mathfrak{g}^{J_\infty}\right\}.$$

THEOREM 7.22 (On the normal form at infinity). *There is a formal gauge transformation*

$$\Phi^\infty = Id + \sum_{m=1}^{\infty} \Phi_m \lambda^{-m}, \Phi_m \in C^\infty\left(S^1, \mathrm{Mat}\,(n)\right),$$

which reduces the differential operator $\partial_x - L$ to normal form,

$$(\Phi^\infty)^{-1} \circ \left(\frac{d}{dx} - L\right) \circ \Phi^\infty = \frac{d}{dx} - D^\infty, \tag{7.35}$$

where

$$D^\infty = \sum_{m=-M}^{\infty} D^\infty_m \lambda^{-m},\ D^\infty_m \in C^\infty\left(S^1,\ \mathfrak{g}^{J_\infty}\right),\ D^\infty_{-M} = J_\infty.$$

The coefficients of the formal series D^∞ are polynomials in the coefficients of the operator L and their derivatives.[6]

As already noted, in order to deal with formal gauge transformations we have to replace the original polynomial loop algebra with its completion $\mathbf{g}^\infty$, which is obtained by localisation of the original algebra at $\lambda = \infty$,

$$\mathbf{g}^\infty = \mathfrak{G} \otimes \mathbb{C}((\lambda^{-1}));$$

associated to this completion of the Lie algebra is a corresponding formal group, Φ_∞ being one of its elements.

REMARK 7.23. Φ_∞, is called a *formal Baker-Akhiezer function* for the operator $\partial_x - L$; the name is due to the fact that for finite-gap potentials L the function Φ_∞ is equal to the expansion of the exact Baker-Akhiezer function (see [**Kr**]) for $\partial_x - L$ in powers of local parameter at infinity (cf. Proposition 7.39 below).

SKETCH OF A PROOF. The intertwining relation (7.35) is equivalent to the differential equation

$$\left(\frac{d}{dx} - L\right)\Phi^\infty = -\Phi^\infty D^\infty, \tag{7.36}$$

which can be solved recursively in powers of the local parameter λ^{-1}. The first nontrivial coefficients of the expansion Φ_1, D_{-M+1} satisfy

$$J_\infty \Phi_1^\infty - \Phi_1^\infty J_\infty = D_{-M+1}^\infty - U_{M-1}. \tag{7.37}$$

This equation for Φ_1 has a solution if and only if its right-hand side lies in the image of the operator $\operatorname{ad} J_\infty \in \operatorname{End} \mathfrak{g}$. Since J_∞ is semi-simple, we have a direct decomposition

$$\mathfrak{g} = \operatorname{Im} \operatorname{ad} J_\infty \dot{+} \operatorname{Ker} \operatorname{ad} J_\infty; \tag{7.38}$$

In virtue of (ii$'$) the subspaces $\operatorname{Im}\operatorname{ad} J_\infty$ and $\operatorname{Ker}\operatorname{ad} J_\infty = \mathfrak{g}^{J_\infty}$ do not depend on x. It follows that the coefficient $D_{-M+1}^\infty \in \mathfrak{g}^{J_\infty}$ is uniquely determined by the solvability condition of (7.37) and

$$\Phi_1^\infty = (\operatorname{ad} J_\infty)^{-1}\left(D_{-M+1}^\infty - U_{M-1}\right).$$

Suppose that the coefficients $\Phi_1^\infty, \dots, \Phi_m^\infty, D_{-M+1}^\infty, \dots, D_{-M+m}^\infty$ are already determined; then for Φ_{m+1}^∞ we get a relation of the form

$$\operatorname{ad} J_\infty \cdot \Phi_{m+1}^\infty = -F_m\left(U, \Phi_1^\infty, \dots, \Phi_m^\infty, D_{-M+1}^\infty, \dots, D_{-M+m}^\infty\right), \tag{7.39}$$

where F_m depends on the potential L and on the previously determined coefficients and their derivatives. Equation (7.39) enables us to compute D_{-M+m+1}^∞ and Φ_{m+1}^∞.

[6]In typical applications J_0, J_∞ are regular matrices with simple spectrum. In this case $\mathfrak{g}^{J_0} = \mathfrak{g}_{J_0}$ and $\mathfrak{g}^{J_\infty} = \mathfrak{g}_{J_\infty}$ are abelian. Thus Theorem 7.22 means that the potential in the auxiliary linear problem may be diagonalized by a formal gauge transformation. If the spectrum of J_0, J_∞ is not simple, the potential may be reduced only to a *block-diagonal form.*

REMARK 7.24. Equation (7.39) does not determine the coefficients $\Phi_m^\infty, D_m^\infty$ uniquely, since we have to fix a choice of the inverse operator $(ad\, J_\infty)^{-1}$. One can show that this ambiguity corresponds to the freedom of making formal gauge transformations of the form

$$\begin{aligned}\frac{d}{dx} - D^\infty \quad &\rightsquigarrow \quad \exp(-\phi) \circ \left(\frac{d}{dx} - D^\infty\right) \circ \exp\phi, \\ \phi \quad &= \quad Id + \sum_{m=1}^{\infty} \phi_m z^{-m}, \phi_m \in \mathfrak{g}^{J_\infty}.\end{aligned} \tag{7.40}$$

The ambiguity can be dealt away by fixing a gauge condition

$$P_{\mathfrak{g}^{J_\infty}}\, \Phi_\infty = 0, \tag{7.41}$$

where $P_{\mathfrak{g}^{J_\infty}}$ is the projection operator onto the subalgebra $\mathfrak{g}^{J_\infty} = \operatorname{Ker} \operatorname{ad} J_\infty$ in the decomposition (7.38).

For $\alpha \in \mathfrak{g}_{J_\infty} \otimes \mathbb{C}\left[\lambda, \lambda^{-1}\right]$ let us set

$$H_\alpha^\infty [L] = \operatorname{Res}_{\lambda=0} \int_0^{2\pi} \langle \alpha(\lambda), D^\infty(x, \lambda) \rangle \, dx. \tag{7.42}$$

THEOREM 7.25. *(i) The functionals H_α^∞ are independent of the ambiguity in the definition of the normal form.*

(ii) All the H_α^∞ Poisson commute with respect to the bracket (7.24) on $\mathbf{g}_r^*$.

(iii) Hamiltonian equation of motion defined by the Hamiltonian H_α^∞ on $\mathbf{g}_r^*$ is of the zero-curvature type.

LEMMA 7.26. *Gauge transformations (7.40) leave the Hamiltonian density $\langle \alpha(\lambda), D^\infty(x, \lambda) \rangle$ invariant up to a total derivative.*

PROOF. Gauge transformations (7.40) convert D^∞ into $e^{-\phi} D^\infty e^\phi - e^{-\phi} \partial_x e^\phi$. We have:

$$e^{-\phi}\partial_x e^\phi = \frac{e^{-\operatorname{ad}\phi} - Id}{-\operatorname{ad}\phi} \cdot \partial_x \phi = \left(Id - \frac{1}{2}\operatorname{ad}\phi + \frac{1}{3!}(\operatorname{ad}\phi)^2 + \ldots\right) \cdot \partial_x \phi.$$

Hence

$$\begin{aligned}\langle \alpha \exp(-\phi), \partial_x(\exp\phi) \rangle &= \left\langle \alpha, \frac{e^{-\operatorname{ad}\phi} - Id}{-\operatorname{ad}\phi} \cdot \partial_x\phi \right\rangle = \\ &= \left\langle \frac{e^{\operatorname{ad}\phi} - Id}{\operatorname{ad}\phi} \cdot \alpha, \partial_x \phi \right\rangle = \langle \alpha, \partial_x\phi \rangle = \partial_x \langle \alpha, \phi \rangle;\end{aligned}$$

we used the invariance of the scalar product, the assumption $\alpha \in \mathfrak{g}_{J_\infty}$, which guarantees that α commutes with ϕ and finally we used $\partial_x \alpha = 0$.

REMARK 7.27. It will be useful to note that the assumption of periodicity of their coefficients of the Lax operator played no role in the proof of Theorem 7.22 on the normal form. (The proof shows, however, that for periodic coefficients, the formal gauge transformation that reduces the Lax operator to normal form also has periodic coefficients.) This condition may therefore be substantially weakened. Still, it cannot be discarded completely: as we have seen, local Hamiltonian densities are defined up to total derivatives. Hence we have to deal with a class of functions for which integrals of total derivatives can be neglected. Apart from periodic functions, such are, for instance, Schwartz functions or almost periodic functions (in this case the integral should be replaced by the average).

LEMMA 7.28. *The Frechet derivative of H_α^∞ is*

$$\operatorname{grad} H_\alpha^\infty = \Phi^\infty \alpha \left(\Phi^\infty\right)^{-1}, \tag{7.43}$$

where Φ^∞ is the formal Baker-Akhiezer function.

SKETCH OF THE PROOF. Under gauge transformations (7.40) Φ_∞ transforms via

$$\Phi_\infty \rightsquigarrow \Phi_\infty \cdot \exp\phi;$$

this implies that the right-hand side of (7.43) does not depend on the ambiguity in the formal Baker-Akhiezer function. Taking the variation on both sides of (7.36), we have:

$$\delta D^\infty = \left(\Phi^\infty\right)^{-1} \delta U \Phi^\infty + \left[D^\infty, \left(\Phi^\infty\right)^{-1} \delta\Phi^\infty\right] - \partial_x \left(\left(\Phi^\infty\right)^{-1} \delta\Phi^\infty\right).$$

Hence,

$$\begin{aligned}&\delta H_\alpha^\infty = \\ &\operatorname{Res}_{\lambda=0} \int_0^{2\pi} \left\{\left\langle \Phi^\infty \alpha \left(\Phi^\infty\right)^{-1}, \delta L\right\rangle + \left\langle \partial_x \alpha - \left[D^\infty, \alpha\right], \left(\Phi^\infty\right)^{-1} \delta\Phi^\infty\right\rangle\right\} dx,\end{aligned}$$

where we used the invariance of the scalar product and performed integration by parts; the second term give zero contribution since $\partial_x\alpha = [D^\infty, \alpha] = 0$. □

Note that the gradient (7.43) is a well-defined Laurent series in powers of λ^{-1}, containing a *finite* nimber of termes of positive powers of λ; hence there is a well-defined projection $M_+ = P_+(\operatorname{grad} H_\alpha^\infty)$.

COROLLARY 7.29. *The Frechet derivative $X = \operatorname{grad} H_\alpha^\infty$ satisfies the differential equation*

$$\partial_x X = [U, X]\,. \tag{7.44}$$

Indeed, from (7.43) and (7.36) it follows that

$$\begin{aligned}\partial_x X = &\left(U\Phi^\infty - \Phi^\infty D^\infty\right) \alpha \left(\Phi^\infty\right)^{-1} \\ &- \Phi^\infty \alpha \left(\Phi^\infty\right)^{-1} \left(U\Phi^\infty - \Phi^\infty D^\infty\right) \left(\Phi^\infty\right)^{-1} \\ &= [U, X] - \Phi^\infty \left[D^\infty, \alpha\right] \left(\Phi^\infty\right)^{-1} = [U, X]\,.\end{aligned}$$

We note that (7.44) can be written as

$$ad^*_{\hat{\mathfrak{g}}} \operatorname{grad} H_\alpha^\infty [U] \cdot U = 0, \tag{7.45}$$

where $\mathrm{ad}^*_{\widehat{\mathbf{g}}}$ is the coadjoint action of $\widehat{\mathbf{g}}$ (the central extension of the algebra $\mathbf{g}$); this is precisely the characteristic property of the Casimir functions (cf. Proposition 2.2). The functionals H^{∞}_{α} are not "true" Casimir functions: in fact, they are defined only in the subspace of Lax operators with a fixed leading coefficient subject to the regularity condition. Still, the property (7.45) is sufficient for the Proof of theorem 2.5; this implies the last assertion of 7.25.

REMARK 7.30. Instead of affine Lie algebras one can deal with the algebras of rational functions; in this case L may have poles at arbitrary points of the Riemann sphere. Under suitable regularity conditions, to every pole of L one can associate a formal Baker-Akhiezer function.

The following is a converse to Lemma 7.29.

PROPOSITION 7.31. *Let $X \in \mathfrak{G}^{\infty}$ be a solution of* (7.44)*, whose coefficients are differential polynomials of the coefficients of L. Then $X = \Phi\,\alpha\,\Phi^{-1}$ for some $\alpha \in \mathfrak{g}_{J_\infty}$.*

PROOF. Let us perform a gauge transformation $X \to \widetilde{X} = \mathrm{Ad}\,\Phi^{-1}\cdot X$, $\Phi = g_0\,\Phi_\infty$. The function $\widetilde{X}$ satisfies the differential equation

$$\partial_x \widetilde{X} = [D^{\infty},\, X],$$

which gives

$$\widetilde{X}(x) = P\,\exp\,\mathrm{ad}\int_0^x D^{\infty}(y)\,dy\cdot\widetilde{X}(0). \tag{7.46}$$

The function (7.46) is a differential polynomial if and only if $\widetilde{X} \in \mathfrak{g}_{J_\infty}$.

Local Hamiltonians related to *different* poles of the Lax operator Poisson commute as well. The proof is slightly more subtle, since the gradients lie in different completions of the initial loop algebra; still, the basic argument of Theorem 2.5 remains valid. The involutivity can also be checked directly. For this note that, as before, the equation of motion defined by the Hamiltonian h^0_β has the form

$$\partial_t\, L = -\partial_x\, M_- + [L,\, M_-], \quad M_- = r_-(\mathrm{grad}\ h^0_\beta), \tag{7.47}$$

where the operator M_- contains finitely many terms. Let Φ_∞ be the formal eigenfunction of the operator $\partial_x - L$ normalized by the condition (7.41).

LEMMA 7.32. *Equation* (7.47) *is equivalent to a system of equations for the intertwining operator* Φ

$$\begin{aligned} \partial_t\Phi_\infty &= M_-\Phi_\infty - \Phi_\infty\,B,\\ B &= P_{\mathfrak{g}^{J_\infty}}(\Phi_\infty^{-1}M_-\Phi_\infty). \end{aligned} \tag{7.48}$$

The proof follows directly from the uniqueness of the intertwining operator.

COROLLARY 7.33. *The operator* $D^{\infty} = \Phi_{\infty}^{-1} L\, \Phi_{\infty} - \Phi_{\infty}^{-1} \partial_x \Phi_{\infty}$ *satisfies the equation*

$$\partial_t D_{\infty} = \partial_x B + [B,\, D_{\infty}]. \tag{7.49}$$

From (7.49) we conclude that for any $\alpha \in {}^{+}C$

$$\partial_t(\alpha,\, D^{\infty}) = \partial_x(\alpha,\, B).$$

Thus the Lie derivative of the Hamiltonian density h_{α}^{∞} with respect to the flow of the Hamiltonian h_{β}^{0}, is a total derivative. Finally, this implies $\{h_{\alpha}^{\infty},\, h_{\beta}^{0}\} = 0$.

7.6. Dressing Transformations

Formula (7.43) for the Frechet derivatives of local Hamiltonians makes it clear that in general it is impossible to use Factorization Theorem (Theorem 2.8) to solve zero curvature equations. Indeed, $\operatorname{grad} H_{\alpha}^{\infty}[U]$ is a *formal* series in local parameter λ^{-1}; thus it is even impossible to define the 1-parameter subgroup $\exp t \operatorname{grad} H_{\alpha}^{\infty}[U]$. This observation reflects real and deep analytic difficulties which arise in the solution of initial value problems for integrable PDE's with arbitrary initial conditions. In order to get around these difficulties let us introduce the following draft definition.

DEFINITION 7.34. A differential operator $\partial_x - L$ is called *strongly regular at zero (resp. at infinity),* if L satisfies the conditions of definition 7.21 and, moreover, its formal Baker–Akhiezer function at zero (resp. at infinity) is given by a convergent series.

At first glance, this definition provokes of course strong doubts. Its appears to be reasonable due to an *a priori* rather unexpected circumstance: strongly regular operators form a homogeneous space with respect to a natural action of the loop group on the space of first order differential operators. This action is known as *dressing transformations.* Commuting Hamiltonian flows of local Hamiltonians are naturally included in the group of dressing transformations as its maximal commutative subgroup. Examples of strongly regular operators include soliton and finite-band potentials; on the other hand it is very difficult or impossible to to give a complete characteristic of strongly regular operators in local terms (i. e., to determine the precise class of initial data in $\mathbf{g}$ which give rise to strongly regular solutions of zero curvature equations).

Dressing transformations were discovered by Zakharov and Shabat as a tool to proliferate solutions of integrable non-linear equations starting with trivial ("vacuum") ones [**ZS**]. (At first they did not notice the composition law because they they always started with the trivial solution and did not consider dressing on arbitrary background.) Another and a completely different definition was proposed by Sato and his school (an exposition of Sato's theory may be found in [**PS**] and in the paper [**SW**]). A difficult question which arises in the theory of dressing transformations consists in taking into account boundary conditions. In our exposition in this chapter we used periodic boundary conditions. This choice is very convenient technically, since it reduces the description of coadjoint orbits and of the Casimir functions to Floquet theory. In general, dressing transformations do *not* preserve periodicity[7].

One more interesting question is related to the Poisson properties of dressing transformations. Since dressing is defined by means of the matrix Riemann problem and, on the other hand, this same Riemann problem determines, via the associated classical r-matrix, the Poisson structure of integrable equations, it is clear that there should exist some relation between the two topics. However, the simplest hypothesis appears to be false: dressing transformations do *not* preserve Poisson brackets on the phase spaces of integrable equations of motion. The exact statement is more subtle: the dressing group is a a Poisson group and dressing transformations define a *Poisson action* of this group on the phase space [**STS2**]. We shall see that the relevant group in this case is the dual Poisson group G^* (and not the group G equipped with the Sklyanin bracket, as one might guess at first glance).

We start with the basic definitions. Let $\mathbb{G}$ be the double loop group which consists of functions

$$g\colon \mathbb{R} \times \mathbb{C}P_1 \setminus (\{0\} \cup \{\infty\}) \longrightarrow GL\,(n)$$

which are holomorphic with respect to the second argument. Its Lie algebra $\mathbf{g}$ consists of functions

$$U\colon \mathbb{R} \times \mathbb{C}P_1 \setminus (\{0\} \cup \{\infty\}) \longrightarrow \mathfrak{gl}\,(n)\,,$$

holomorphic with respect to the second argument. We shall refer to elements of $\mathbb{G}$ as *wave functions*. Let us define the mapping

$$p\colon \mathbb{G} \to \mathbf{g}$$

by

$$p\colon \psi \longmapsto U_\psi = \partial_x \psi \cdot \psi^{-1}.$$

The group $\mathbb{G}$ acts on itself by left multiplications; we have

$$g \cdot U_\psi \overset{\text{def}}{=} U_{g\psi} = gU_\psi g^{-1} + \partial_x g \cdot g^{-1}; \tag{7.50}$$

in other words, left multiplications on $\mathbb{G}$ induce gauge transformations on the set of "potentials" U. Conversely, let us associate with $U \in \mathbf{g}$ the fundamental solution ψ_U of the differential equation

$$\frac{d\psi}{dx} = U(x,z)\psi, \tag{7.51}$$

[7]Dressing transformations according to Sato are connected with the matrix Riemann problem in the punctured sphere. One can consider another version of the theory which is based on the *Riemann problem in the half-plane*; under certain additional conditions this problem allows to define dressing action in the class of rapidly decreasing functions on the line.

on the line with initial condition $\psi_U(0) = Id$. The mapping

$$\psi\colon \mathbf{g} \to \mathbb{G}\colon U \mapsto \psi_U$$

is a right inverse of p. Let $\mathcal{G} \subset \mathbb{G}$ be the subgroup of functions which do not depend on the first argument $x \in \mathbb{R}$.

LEMMA 7.35. *(i) $\mathcal{G} \subset \mathbb{G}$ is the isotropy subgroup of* 0 *(i.e., of zero potential on the line). (ii) The conjugate subgroup $\mathcal{G}^U = \psi_U \mathcal{G} \psi_U^{-1} \subset \mathbb{G}$ is the isotropy subgroup of $U \in \mathbf{g}$.*

Let $\mathbb{G}_+ \subset \mathbb{G}$ be the subgroup of functions which are holomorphic in $\mathbb{C}P_1 \setminus \{\infty\}$ with respect to the second argument and $\mathbb{G}_- \subset \mathbb{G}$ the subgroup of functions which are holomorphic in $\mathbb{C}P_1 \setminus \{0\}$ and satisfy the normalization condition $g_-(\infty) = Id$. *Factorization problem* in $\mathbb{G}$ consists in representing an element $g \in \mathbb{G}$ as a product of two factors, $g = g_+ g_-^{-1}$, $g_\pm \in \mathbb{G}_\pm$; the first argument $x \in \mathbb{R}$ is regarded as a parameter.

THEOREM 7.36. *Formula*

$$\mathrm{Dr}_{(x,y)}\psi = \left(\psi x y^{-1}\psi^{-1}\right)_+^{-1} \psi\, x = \left(\psi x y^{-1}\psi^{-1}\right)_-^{-1} \psi\, y \tag{7.52}$$

defines a right *action* $\mathrm{Dr}: (\mathcal{G} \times \mathcal{G}) \times \mathbb{G} \longrightarrow \mathbb{G}$.

The existence of two versions of the dressing formula in (7.52), which is in fact one of its key features, is due to the fact that $\psi x y^{-1} \psi^{-1} \in \mathcal{G}^{U_\psi}$ and hence both factors $\left(\psi x y^{-1}\psi^{-1}\right)_\pm$ define the same gauge transformation of U_ψ. The geometric origin of the dressing action becomes clear from the comparison with the dressing action for factorizable Poisson Lie groups which we described in sections 5.5, 5.6 (see the diagram (5.13) and formula (5.15)).

Under reasonable conditions, the factorization problem in (7.52) is solvable on an open dense subset and hence the diagonal subgroup in $D(K)$ may be identified with a "big cell" in the quotient space $K_r \backslash D(K)$. This situation is completely similar, for instance, with the description of fractional linear transformations on the plane.

Returning to formula (7.52) let us notice that the diagonal subgroup $\mathcal{G}^\delta \subset D(\mathcal{G})$ acts on wave functions via $\mathrm{Dr}(g,g)\colon \psi \mapsto \psi g$; this action amounts to a simple change of normalization and does not affect the potential $U = \partial_x \psi \psi^{-1}$. On the other hand, the subgroup $\mathcal{G}_r = \mathcal{G}_+ \times \mathcal{G}_-$ preserves the normalization condition $\psi(0) = Id$. Hence we may define the dressing action on the space of potentials $\mathcal{G}_r \times \mathbf{g} \to \mathbf{g}$ by means of the commutative diagram

$$\begin{array}{ccc} \mathcal{G}_r \times \mathbb{G} & \xrightarrow{\mathrm{Dr}} & \mathbb{G} \\ \Big\uparrow{\scriptstyle id\times\psi} & & \Big\uparrow{\scriptstyle \psi} \\ \mathcal{G}_r \times \mathbf{g} & \xrightarrow{\mathrm{Dr}} & \mathbf{g} \end{array} \tag{7.53}$$

Let $\mathbf{g}_{M,N} \subset \mathbf{g}$ be the subspace of Laurent polynomials for which the pole order at zero (resp., at infinity) does not exceed M (resp., N).

PROPOSITION 7.37. *Dressing transformations on* $\mathbf{g}$ *preserve* $\mathbf{g}_{M,N}$.

Sketch of a proof. Let us compare the two equivalent formulae for the dressing which follow from (7.52); the first one shows that dressing does not increase the order of pole at zero, and the second one shows that it does not increase the order of pole at infinity.

The argument above explains the key idea of the dressing method (of course, the same idea is implicit in Theorem 2.8). A slightly more accurate argument shows that *dressing preserves symplectic leaves of the r-bracket* in $\mathbf{g}_{M,N} \subset \mathbf{g} \simeq \mathbf{g}_r^*$ (here r is the standard r-matrix associated with the factorization problem in $\mathbb{G}$).

EXERCISE 7.38. Prove this assertion using formula (2.28).

PROPOSITION 7.39. *Dressing transformations preserve strong regularity.*

SKETCH OF A PROOF. Formal Baker–Akhiezer functions of dressed operator at zero and at infinity are given by

$$\Phi_0^g = \left(\psi g_+ g_-^{-1} \psi^{-1}\right)_+^{-1} \Phi_0,$$
$$\Phi_\infty^g = \left(\psi g_+ g_-^{-1} \psi^{-1}\right)_-^{-1} \Phi_\infty.$$

It is clear that the dressing factors $\left(\psi g_+ g_-^{-1} \psi^{-1}\right)_+^{-1}$, $\left(\psi g_+ g_-^{-1} \psi^{-1}\right)_-^{-1}$ may be expanded into convergent series in local parameter at zero (resp., at infinity) and hence the same is true for the "dressed" wave functions

In practical examples dressing is usually applied to *trivial* or "free" Lax operators. Let us assume that the highest coefficient of a Lax operator at infinity is a diagonal matrix with simple spectrum. By definition, a free Lax operator has the form

$$L_{free} = \frac{d}{dx} - D(\lambda),$$

where $D(z)$ is a constant diagonal matrix whose coefficients are polynomial in λ.

PROPOSITION 7.40. *Assume that L is obtained from L_{free} by dressing, $L = L_{free}^g$. Let $L(t)$ be the integral curve emanating from L of the Hamiltonian equation of motion with Hamiltonian H_α given by (7.42). We have*

$$L\left(t\right) = g_\pm \left(t\right)^{-1} \circ L \circ g_\pm \left(t\right),$$

where we regard $g_\pm\left(t\right)$ as multiplication operators on the line and $g_+\left(t, x\right)$, $g_-\left(t, x\right)$ are solutions of the factorization problem

$$g_+\left(t, x\right) g_-\left(t, x\right)^{-1} = \psi_{free}\left(x\right) \exp t\alpha\left(\lambda\right) \cdot g \cdot \psi_{free}\left(x\right)^{-1},$$
$$\psi_{free}\left(x\right) = \exp x D(\lambda).$$

Thus Theorem 2.8 remains valid for strongly regular potentials; the Hamiltonian flows associated with zero curvature equations are give by the action of an abelian subgroup of the "big" dressing group (in the example above the maximal abelian subgroup consists of diagonal loops which are holomorphic at infinity).

Note that the action of the dressing group on L_{free} is not effective: indeed, since by assumption the free wave function ψ_{free} is regular in the open plane, the dressing action of the subgroup $\mathcal{G}_+$ amounts to a change of its normalization. Thus the orbit of the trivial solution is a *homogeneous space* of the loop group (In the case of the KdV equation this is the famous *infinite-dimensional Grassmannian* introduced by Sato [**PS, SW**]).

7.7. Dressing Transformations as a Poisson Group Action

The main goal of this section is the study of Poisson properties of dressing transformations defined in Section 7.6). We shall assume that G is a factorizable Poisson group and that the factorization problem which enters the definition of dressing transformations is defined with the help of the classical r-matrix which also determines the Poisson structures on G and on G^*. Recall that dressing transformations act in the space of solutions of the auxiliary linear problem and are given by

$$\begin{aligned}(7.54)\quad \mathrm{Dr}_{(x_+,-)}\psi = \left(\psi x_+ x_-^{-1}\psi^{-1}\right)_+^{-1}\psi\, x_+ = \\ = \left(\psi x_+ x_-^{-1}\psi^{-1}\right)_-^{-1}\psi\, x_- \quad (x_+, x_-) \in G^* \subset G\times G.\end{aligned}$$

The infinitesimal dressing action on a wave function $\psi(x)$ is given by

$$\mathrm{dr}_{(X_+,X_-)}\cdot\psi(x) = \left(\psi(x)X\psi(x)^{-1}\right)_\pm \psi(x) - \psi(x)X_\pm. \tag{7.55}$$

The action of the diagonal subgroup $G^\delta \subset G\times G$ amounts to a simple change in the normalization condition on ψ, while the action of the complementary subgroup $G^* \subset G\times G$ preserves normalization and may descends to the phase space of the zero curvature system. Hence effectively the group of dressing transformations is the *dual* group G^*. As we shall see, infinitesimal dressing transformations may be defined with the help of a nonabelian moment mapping; moreover, the moment mapping associated with the action of G^* coincides with the monodromy matrix for the auxiliary linear problem.

Our next definition applies in the general setting.

LEMMA 7.41. *Let $T(L)$ be the monodromy matrix of the linear problem*

$$\partial_x\psi(x) = L(x)\psi(x), \quad x\in[0,2\pi]. \tag{7.56}$$

Then

$$\{\psi(x)_1,\, T(L)_2\} = T(L)_2\psi(x)_2^{-1}[r,\, \psi(x)_1\psi(x)_2]. \tag{7.57}$$

PROOF. the value of the wave function in this formula is considered as a functional of L. The check of (7.57) is a good exercise on the use of tensor formalism. Of course, the computation is completely parallel to the proof of Proposition 7.15 above; nevertheless, we reproduce it in full. Taking variations of the coefficients of the linear problem as in lemma 7.25, we get:

$$\psi(x)^{-1}\delta\psi(x) = \int_0^x \psi(y)^{-1}\delta L(y)\,\psi(y)\,dy,$$

which gives

$$\{\psi_1(x),\, T_2(L)\} = \tag{7.58}$$
$$= \int_0^x \int_0^{2\pi} \psi_1(x) T_2(L) \psi_1(y)^{-1} \psi_2(z)^{-1} \{L_1(y),\, L_2(z)\}\, \psi_1(y)\psi_2(z)\, dy\, dz.$$

the Poisson bracket of potentials is ultralocal, i. e., in tensor notation,

$$\{L_1(y),\, L_2(z)\} = [r_{12},\, L_1(y) + L_2(y)]\, \delta(y-z);$$

This allows to remove one integration. Using the relations $L\psi = \partial_y \psi$, $\psi^{-1} L = -\partial_y \psi^{-1}$, it is easy to check that the integrand in (7.58) is a total derivative,

$$\{\psi_1(x),\, T_2(L)\} = \int_0^x \psi_1(x) T_2(L) \frac{d}{dy} \left(\psi_1(y)^{-1}\psi_2(y)^{-1} r_{12} \psi_1(y)\psi_2(y)\right) dy;$$

Computing the integral, we immediately get (7.57).

Taking “the trace with respect to the second space” in (7.57), we get

$$\mathrm{tr}_2 X_2 T(L)_2^{-1} \{\psi(x)_1,\, T(L)_2\} = \left(\psi(x) X \psi(x)^{-1}\right)_\pm \psi(x) - \psi(x) X_\pm,$$

which coincides with the r.h.s. of (7.55). Thus we have checked the following assertion:

PROPOSITION 7.42. *The infinitesimal dressing action of* $\mathfrak{g}^*$ *in the space of solutions of the auxiliary linear problem* (7.56) *is defined by the nonabelian moment map* T*, the monodromy map of* (7.56)*.*

As we know, the monodromy map $T\colon C^\infty(S^1, \mathfrak{g})_r \longrightarrow G$ is Poisson (with respect to the Sklyanin bracket in G); hence our moment map is automatically G-equivariant. As a corollary we get

THEOREM 7.43. *The dressing action of* G^* *in the space of solutions of the auxiliary linear problem* (7.56) *and in the space of potentials* $C^\infty(S^1, \mathfrak{g})_r$ *is a Poisson action.*

Theorem 7.43 has been proved in **[STS2]** with the help of the theory of double. The direct proof reproduced above is due to Babelon and Bernard **[BB1, BB2]**. .

LECTURE 8

Virasoro Algebra and Schroedinger Operators on the Circle

A key point in our construction of zero curvature equations from current algebras is the existence for these algebras of a nontrivial central extension. There is one more infinite-dimensional Lie algebra, the algebra of vector fields on the line or on the circle, which admits a nontrivial central extension. This central extension is called the *Virasoro algebra.* Its applications to nonlinear equations are less universal than in the case of current algebras: they are essentially restricted to the case of the KdV equation and its versions. We shall now briefly describe the corresponding results.

8.1. Virasoro Algebra and its Coadjoint Orbits

Let $\operatorname{Vect} S^1$ be the Lie algebra of vector fields on the circle. Its elements are linear differential operators on the line with periodic coefficients of the form $\xi_f = f\,\partial_x,\ f \in C^\infty(S^1)$, with the Lie bracket

$$[\xi_f,\, \xi_g] = \xi_{w(f,g)}, \quad \text{where} \quad w(f,\, g) = f'g - g'f. \tag{8.1}$$

The dual space of smooth linear functionals on $\operatorname{Vect} S^1$ consists of quadratic differentials $F_u = u\,dx^2,\ u \in C^\infty(S^1)$; the action of a linear functional F_u on a vector field is given by the coupling

$$F_u(\xi_f) = \int \langle u\,dx^2, \xi_f\rangle = \int u\,f\,dx. \tag{8.2}$$

The Lie group which corresponds to $\operatorname{Vect} S^1$ is the group $\operatorname{Diff} S^1$ of diffeomorphisms of the circle; its adjoint and coadjoint representations correspond to the standard change of variables for a vector field and for a quadratic differential, respectively. We have

$$\operatorname{Ad}^*\phi\cdot F_u = (\phi^{-1})^* F_u = \phi'(x)^{-2}u(\phi^{-1}(x))\,dx^2. \tag{8.3}$$

THEOREM 8.1 (Gelfand–Fuchs). *The second cohomology group of* $\operatorname{Vect} S^1$ *is one-dimensional; it is generated by the 2-cocycle*

$$\Omega(\xi_f,\, \xi_g) = \int f'''g\,dx. \tag{8.4}$$

DEFINITION 8.2. The central extension of $\operatorname{Vect} S^1$ associated with the cocycle (8.4) is called the *Virasoro algebra.*

Commutation relations in the Virasoro algebra are frequently written with the help of a standard basis in the complexified Lie algebra of vector fields,

$$\xi_k = ie^{ikx}\partial_x,\ k \in \mathbb{Z};$$

REMARK 8.3. Vector fields ξ_0, ξ_1, ξ_{-1} generate the Lie algebra of projective transformations of the circle, which is isomorphic to $\mathfrak{su}\,(1,1)$; since this algebra is simple, the restriction of the Gelfand–Fuchs cocycle to this algebra is trivial. It is convenient to modify the cocycle (8.4) in such a way that this restriction is identically zero. The modified cocycle is given by

$$\widehat{\Omega}(\xi_f,\, \xi_g) = \int (f'''g - f'g)\, dx. \tag{8.5}$$

With these conventions the commutation relations in the Virasoro algebra take the form

$$[\xi_k,\, \xi_l] = (k-l)\xi_{k+l} + c\,(k^3 - k)\delta_{k+l,0}. \tag{8.6}$$

As usual, the coadjoint representation of the Virasoro algebra is in fact a representation of the quotient algebra $\mathrm{Vect}(S^1)$ and integrates to the group $\mathrm{Diff}\, S^1$.

PROPOSITION 8.4. *The coadjoint representation of* $\mathrm{Diff}\, S^1$ *on* $\widehat{\mathrm{Vir}}^* \simeq \mathrm{Vect}(S^1)^* \dot{+} \mathbb{R}$ *is given by*

$$\mathrm{Ad}^*\,\phi \cdot (F_u,\, e) = \left((\phi^{-1})^* F_u + e\, S(\phi^{-1}) dx^2,\, e\right), \tag{8.7}$$

where

$$S(\phi) = \frac{\phi'''}{\phi'} - \frac{3}{2}\frac{(\phi'')^2}{(\phi')^2} \tag{8.8}$$

is the Schwarz derivative of ϕ.

It is useful to note that the Schwarz derivative of a function ϕ is identically zero if and only if ϕ is a fractional linear transformation; thus the additional term in (8.7) vanishes precisely on the projective subgroup $PSU(1,1) \subset \mathrm{Diff}\, S^1$.

As in the case of current algebras, the complicatedly looking formula (8.7) is related to the *change of variables in an auxiliary linear problem* Namely, consider the Schroedinger equation with periodic potential on the line,

$$H_u\psi = -\psi'' + u\psi = 0. \tag{8.9}$$

Let us denote by Ω_α the space of degree α densities on the line. We shall regard the Schroedinger operator as a mapping

$$H_u\colon \Omega_{-1/2} \longrightarrow \Omega_{3/2},$$

i. e., we assume that under the change of variables ϕ the wave function transforms according to the rule

$$\psi \longmapsto \phi^*\psi = (\phi')^{-\frac{1}{2}}\psi \circ \phi, \tag{8.10}$$

and $H_u\psi$ acquires an extra square of derivative. This transformation law is in fact the only one possible in order to preserve the form of the Schroedinger equation.

REMARK 8.5. The "twisting" of the module of functions with the help of degree $-\frac{1}{2}$-densities $dx^{-1/2}$ is closely connected with the replacement of the cocycle (8.4) with an equivalent one (8.5).

LEMMA 8.6. *Under the change of variables ϕ the potential in the Schroedinger equation goes to $(\phi')^2 u(\phi(x)) - \frac{1}{2}S(\phi)$.*

Now in order to prove that the transformation law described in lemma 8.6 coincides with the coadjoint representation of the Virasoro group it is enough to check that the infinitesimal transformation coincides with the coadjoint representation of the Virasoro algebra. This easy exercise is left to the reader.

The classification of coadjoint orbits for the Virasoro algebra is more subtle than in the case of current algebras (where it is given by the Floquet theorem). In order to formulate the corresponding theorem we must start with the projective geometry of the space of wave functions, i. e., solutions of the Schroedinger equation (at zero energy level).

According to the elementary theory, for a given u the space $V = V_u$ of solutions of the Schroedinger equation is 2-dimensional and for any two solutions ϕ, ψ their wronskian $W = \phi\psi' - \phi'\psi$ is constant. Any $w \in V$ may be regarded as a non-degenerate quasi-periodic plane curve (the non-degeneracy condition means that $w \wedge w'$ is nowhere zero). There exists a matrix $M \in SL(2, \mathbb{R})$ (the monodromy matrix) such that, writing elements of V as row vectors $w = (\phi, \psi)$,

$$w(x + 2\pi n) = w(x)M^n, \quad n \in \mathbb{Z}.$$

It is useful to pass to the corresponding projective curve with values in $\mathbb{R}P_1 \simeq S^1$.

THEOREM 8.7 ([**OT**]). *(i) Any pair of linearly independent solutions of the Schroedinger equation defines a non-degenerate*[1] *quasi-periodic projective curve $\gamma : \mathbb{R} \to \mathbb{R}P_1$ such that $\gamma(x + 2\pi) = \gamma(x)M$. (ii) Conversely, any such curve uniquely defines a second ordre differential on the line with periodic potential; any two projective curves associated with a given Schroedinger equation are related by a global projective transformation.*

PROOF. For each $y \in \mathbb{R}$ there exists a solution ψ_y of the Schroedinger equation which vanishes at y; this solution is unique up to a scaling factor. Thus there is a canonical map $\gamma : y \mapsto \mathbb{R}\cdot\psi_y \in P(V)$ which assigns to y a one-dimensional subspace in V. Choosing a basis φ, ψ in V we identify it with $\mathbb{R}^2$ and get a curve with values in $\mathbb{R}P_1$ defined uniquely up to a projective

[1] A parameterized curve $\gamma : R \to \mathbb{R}P_1$ is called nondegenerate if its velocity is nowhere zero.

transformation. Conversely, any non-degenerate curve can be uniquely lifted to a non-degenerate curve in $w : \mathbb{R} \to \mathbb{R}^2$ such that its wronskian is equal to 1. Differentiating the relation $w \wedge w' = 1$, we see that the acceleration vector w'' is collinear with w; in other words, both components of w satisfy a second order differential equation. $\square$

Without restricting the generality we may fix an affine coordinate on $\mathbb{C}P_1$ in such a way that ∞ corresponds to the zeros of the second coordinate ψ of the point on the plane curve; with this choice γ is replaced with the affine curve $x \mapsto \eta(x) = \phi(x)/\psi(x)$. The potential u may be restored from η by the formula

$$u = \tfrac{1}{2} S(\eta),$$

where S is the Schwarzian derivative.

Taken a bit more abstractly, the quasiperiodicity condition $\gamma(x + 2\pi) = \gamma(x)M$ may be regarded as a glueing condition for a S^1-bundle over $R/2\pi\mathbb{Z} \simeq S^1$. Any such bundle is topologically trivial and the degree is the only topological invariant for its sections.

We may now pass to the classification of the coadjoint orbits for the Virasoro algebra.

Exercise 8.8. Show that the under the change of variables the monodromy matrix of the Schroedinger equation transforms by conjugation.

Corollary 8.9. *The eigenvalues of the monodromy matrix are invariant with respect to the coadjoint action* (8.7)*.*

Theorem 8.10. *Two periodic potentials lie on the same orbit of* $\mathrm{Diff}(S^1)$ *if and only if (i) the corresponding monodromy matrices are conjugate in* $SL(2, \mathbb{R})$ *and (ii) the associated projective curves have the same degree.*

The subtle point of the classification theorem, as compared to the Floquet theorem, is the presence of an additional discrete invariant. In applications to integrable systems we need only a more rough information, since in this case the group of diffeomorphisms of the circle is replaced with the group of its *formal diffeomorphisms* which is more easy to deal with.

8.2. Formal Diffeomorphisms of the Circle

Let $\mathcal{D} = \mathrm{Vect}\, S^1 \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ be the loop algebra with values in $\mathrm{Vect}\, S^1$. Let us extend to $\mathcal{D}$ the cocycle (8.4) setting

$$\Omega(\xi_f, \xi_g) = \mathrm{Res}_{\lambda=0} \lambda^{-1} \int f'''(x, \lambda) g(x, \lambda)\, dx. \tag{8.11}$$

We define the splitting of $\mathcal{D}$ into complementary subalgebras and the associated r-matrix in the usual way,

$$\mathcal{D} = \mathcal{D}_+ \dot{+} \mathcal{D}_-, \quad \mathcal{D}_+ = \bigoplus_{n \geqslant 0} \mathrm{Vect}\, S^1 \otimes \lambda^n,\ \mathcal{D}_- = \bigoplus_{n \geqslant 1} \mathrm{Vect}\, S^1 \otimes \lambda^{-n},$$

Formula

$$\Omega_r(\xi_f, \xi_g) = \tfrac{1}{2} \left(\Omega(r\xi_f, \xi_g) + \Omega(\xi_f, r\xi_g) \right)$$

defines a 2-cocycle on $\mathcal{D}_r$. This allows to apply our general scheme to the affine version of the Virasoro algebra in complete analogy with the case of current algebras.

As in section 7.5, in order to construct local conservation laws we must complete the algebra $\mathcal{D}$ adjoining to it formal series. Let us consider, for example, the conservation laws associated with formal series in λ^{-1}. Let $\mathcal{F} = C^\infty(S^1) \otimes \mathbb{C}((\lambda^{-1}))$ be the ring of formal series with coefficients in $C^\infty(S^1)$. We can associate with $\mathcal{D}_-$ the Lie group $\mathbb{D}_-$ of *formal diffeomorphisms of the circle* of the form

$$\varphi\colon x \longmapsto x + y(x), \quad \text{where} \quad y(x) = \sum_{n=1}^{\infty} \lambda^{-n} y_n(x),\ y_n \in C^\infty(S^1)$$

(in other words we assume that λ^{-1} is an "infinitesimal parameter"). The formal change of variables in the ring $\mathcal{F}$ is defined by

$$\varphi^* u(x,\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_x^n u(x,\lambda) y(x,\lambda)^n.$$

We can now define the formal coadjoint action of $\mathbb{D}_-$ on $\mathcal{F}$ by

$$\operatorname{Ad}^* \varphi \cdot (u\, dx^2)(x) = (\varphi'(x))^{-2} u(\varphi^{-1}(x)) dx^2 - \tfrac{1}{2} S(\varphi^{-1}(x)) dx^2. \tag{8.12}$$

(Note that the jacobean $\varphi'(x)$ and the Schwarzian derivative are well defined formal series in λ^{-1}.)

THEOREM 8.11 (On normal form). *Let $u \in \mathcal{F}$ be a formal series of the form*

$$u = -\lambda^N + \sum_{-\infty}^{N-1} u_n \lambda^n, \quad u_n \in C^\infty(S^1).$$

There exists a formal diffeomorphism $\varphi \in \mathbb{D}_-$ such that

$$\operatorname{Ad}^* \varphi \cdot u\, dx^2 = d(\lambda)\, dx^2, \quad \textit{where} \quad d(\lambda) = -\lambda^N + \sum_{-\infty}^{N-1} d_n \lambda^n, \quad \partial_x d_n = 0. \tag{8.13}$$

PROOF. Substitute the series for inverse diffeomorphism

$$\varphi^{-1}\colon x \longmapsto x + \sum_{n=1}^{\infty} \lambda^{-n} z_n(x)$$

into (8.12). Equating the coefficients of different powers of λ yields recursive relations

$$z_n' = F_n(z_1, \dots, z_{n-1}, d_{-N+1}, \dots, d_{-N+n-1}) + d_{-N+n},$$

where F_n is a polynomial depending on the coefficients of u and on their derivatives. The coefficient d_{-N+n} is uniquely determined from the solvability condition of this equation in the class of periodic functions. □

Let α be a formal series, $\alpha \in \mathbb{C}((\lambda^{-1}))$. Put

$$h_\alpha[u] = \mathrm{Res}_{\lambda=0}\alpha(\lambda)d(\lambda). \tag{8.14}$$

Since the normal form of $u\,dx^2$ is in fact not just a formal series, but rather a weight 2 density with constant coefficients, i. e., an element of $\Omega_2 \otimes \mathbb{C}((\lambda^{-1}))$, definition (8.14) should we written more accurately as

$$h_\alpha[u] = \mathrm{Res}_{\lambda=0}\langle d(\lambda)dx^2, \alpha(\lambda)\partial_x\rangle. \tag{8.15}$$

Moreover, since the transformation law for potentials includes the Schwarzian derivative, the normal form is in fact an element of the *extended* dual space $\Omega_2 \otimes \mathbb{C}((\lambda^{-1})) \dot{+} \mathbb{R}$, i. e., a pair $(d(\lambda)\,dx^2, e)$; with the chosen normalization we must set $e = -\frac{1}{2}$. This remark will be used in our following assertion.

PROPOSITION 8.12. *The Frechet derivative of $h_\alpha[u]$ is given by*

$$\nabla h_\alpha = \mathrm{Ad}\,\varphi(x, \lambda) \cdot \alpha(\lambda)\partial_x. \tag{8.16}$$

PROOF. Formal variation of (8.15) with respect to u gives

$$\delta(d(\lambda)dx^2) = \mathrm{Ad}^*(\varphi)\delta u\,dx^2 + \mathrm{ad}^*\,X(d(\lambda)\,dx^2), \quad \text{where} \quad X = \delta\varphi\,\varphi^{-1}.$$

We have

$$\begin{aligned}\langle \mathrm{ad}^*\,X \cdot \big(d(\lambda)\,dx^2, e\big)\,,\,(\alpha(\lambda)\partial_x\rangle, c)\rangle &= \\ &= \langle d(\lambda)\,dx^2,\,[X, \partial_x]\rangle + e\Omega(X, \partial_x) = \\ &= d(\lambda)\int X'(x)\,dx + e\int X'''\,dx = 0,\end{aligned}$$

and hence the variation of φ drops out. This yields

$$\delta h_\alpha = \mathrm{Res}_{\lambda=0}\langle \delta d(\lambda)dx^2, \alpha(\lambda)\partial_x\rangle = \mathrm{Res}_{\lambda=0}\langle \delta u,\,\mathrm{Ad}\,\varphi \cdot \alpha\partial_x\rangle.$$

Equation (8.16) is equivalent to invariance of h_α. Let us now assume that the potential u is *polynomial* in λ and hence belongs to the dual space of $\mathcal{D}_-$. As usual, we get

PROPOSITION 8.13. *Hamiltonians* (8.15) *are in involution with respect to the Lie–Poisson bracket of $\mathcal{D}_-$.*

The formal change of variables (8.13) reduces the Schroedinger equation to an equation with constant coefficients. Alternatively, a formal diagonalization of the Schroedinger operator is provided by formal gauge transformations. Both ways yield equivalent results.

PROPOSITION 8.14. *Let us suppose again, like in Theorem 8.11, that $u \in \mathcal{F}$ is a formal series of the form*

$$u = -\lambda^N + \sum_{-\infty}^{N-1} u_n\lambda^n, \quad u_n \in C^\infty(S^1), \quad \lambda = k^2.$$

There exist formal series $\chi(x,k) = x + \sum_{n=1}^{\infty} \chi_n(x) k^{-n}$, $\varkappa(k) = 1 + \sum_{n=1}^{\infty} \varkappa_n k^{-n}$, $\chi_n \in C^{\infty}(S^1)$, $\partial_x \varkappa_n^{\pm} = 0$ *such that*

$$\psi_{\pm}(k,x) = \frac{1}{\sqrt{\chi'(x)}} e^{\pm i k^N \chi(x,k)}, \tag{8.17}$$

is a formal eigenfunction of the Schroedinger operator,

$$-\psi''_{\pm}(k,x) + u(k^2,x)\psi_{\pm}(k,x) = k^{2N}\varkappa(k)^2\psi_{\pm}(k,x).$$

The series (8.17) is the usual quasi-classical or, equivalently, high-energy expansion of the wave function of the Schroedinger operator. To prove its existence substitute the formal series

$$\psi_{\pm}(k,x) = \exp\left\{k^N\chi(x,k) - \tfrac{1}{2}\log\chi'(x,k)\right\}$$

into the Schroedinger equation. As usual, the Riccati-type substitution leads to the recurrence relations of the form

$$\chi'_n = F_n(u, \chi_1, \ldots, \chi_{n-1}, \varkappa_1, \ldots, \varkappa_{n-1}) + \varkappa_n;$$

constant coefficients $\varkappa_n$ are successively determined from the solvability condition of these relations in the class of periodic functions. As a mater of fact, $\psi_{\pm}$ coincide with the formal Baker-Akhiezer functions for the Schroedinger equations (which may be constructed starting with the first order matrix differential equation associated with the Schroedinger equation). On the other hand, it is obvious from (8.17) that the phase function $\chi(x,k)$ coincides with the formal diffeomorphism from theorem 7.22. This means that both the Virasoro algebra and the current algebra lead to the same conservation laws.

The simplest coadjoint orbit of $\mathcal{D}_-$ consists of potentials of the form $u(x,\lambda) = -\lambda + u(x)$; the associated nonlinear equations is the Korteweg–de Vries equation and its hierarchy. The Poisson structure in the space of such potentials associated with the r-bracket coincides with the Lie–Poisson bracket of the Virasoro algebra and is given by

$$\{\Phi_1,\, \Phi_2\}\,[u] = \int \Lambda_u(\nabla\Phi_1)\,\nabla\Phi_2\,dx,$$

where Λ_u is the Poisson operator,

$$\Lambda_u = -\frac{1}{2}\frac{d^3}{dx^3} + u\frac{d}{dx} + \frac{d}{dx}u.$$

This is the so called *second Hamiltonian structure* for the KdV equation.

Remark 8.15. In a more general way, we can consider Schroedinger operators whose potential has a polynomial dependence on λ (the so called energy dependent potential, see for instance [**Fo**]).

An interesting question which is suggested by the projective treatment of Schroedinger equations, which we outlined in Theorem 8.10, is the possibility to lift the Poisson structure to the space of wave functions (or, more precisely, to its projectivization). There exists an interesting analogue of the KdV equation which holds for the ratio $\eta(x) = \varphi(x)/\psi(x)$. This is the so called *Schwarz–KdV equation*

$$\eta_t = S(\eta)\eta_x. \tag{8.18}$$

Its characteristic property is its invariance with respect to fractional linear transformations: if $\eta(x)$ is a solution of (8.18), the same is true for

$$\widetilde{\eta}(x) = \frac{a\eta(x) + c}{b\eta(x) + d}$$

for any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2).$$

Moreover, if η is its solution, then $S(\eta)$ is a solution of the ordinary KdV equation, $u_t = u_{xxx} + 6uu_x$. The Poisson structure for this equation is surprisingly nontrivial [**MS**]. We shall describe it in Section 8.3.

8.3. Poisson Groups and Differential Galois Theory of the Schroedinger Equation on the Circle

Poisson geometry of the space of Schroedinger equations with periodic potential on the line provides another interesting application of the Poisson groups philosophy. Here again, the basic Poisson brackets, namely, the Lie–Poisson brackets of the Virasoro algebra, are non-ultralocal, and this causes troubles in an attempt to compute the Poisson brackets for wave functions and for the monodromy. Two important hints which allow to settle the problem come from the projective geometry approach to the Schroedinger equations which we already described in Section 8 (Theorem 8.10) and from differential Galois theory.

Speaking informally, differential Galois theory applies to the description of the 'algebras of observables' associated with an auxiliary linear problem. Let us assume, for concreteness, that our Lax operator is a differential operator of order n,

$$L = \partial_x^n + u_{n-2}\partial_x^{n-2} + \cdots + u_0.$$

Natural observables for the associated integrable systems (with two independent variables t, x) are local functionals

$$F[u] = \int F(u, \partial_x u, \partial_x^2 u, \dots)\, dx,$$

where F is a polynomial (or, more generally, a rational) function of $u = (u_0, \dots, u_{n-2})$ and of its derivatives. We can identify the observable $F[u]$ and the corresponding density; in other words, our basic algebra of observables is identified with the *differential field* $\mathbb{C}\langle u\rangle$. (Algebraically speaking, $\mathbb{C}\langle u\rangle$ is the algebra of rational functions of an infinite set of variables $u, u', u'', \dots$ equipped with a formal derivation operator $\partial = \partial/\partial x$ such that $\partial u^{(n)} = u^{(n+1)}$. However, it is more convenient to think of $\mathbb{C}\langle u\rangle$ as of a true functional space.) For a given u, the space $V = V_u$ of solutions of the auxiliary linear problem $L\psi = 0$ is n-dimensional. Let us choose a basis $\psi = \{\psi_1, \ldots, \psi_n\}$ in V and associate with it a bigger differential field $\mathbb{C}\langle\psi\rangle$. There is a natural action of the group $G = GL(n)$ on $\mathbb{C}\langle\psi\rangle$ induced by the linear transformations in V; algebraically, the group G may be characterized as the group of automorphisms of the algebra $\mathbb{C}\langle\psi\rangle$ which commute with the derivation ∂. Since the coefficients of L are rational functions of ψ and of its derivatives, we have an obvious inclusion $\mathbb{C}\langle u\rangle \subset \mathbb{C}\langle\psi\rangle$. In fact, this subfield coincides with the subfield of G-invariants in $\mathbb{C}\langle\psi\rangle$, i. e.,

$$\mathbb{C}\langle u\rangle = \mathbb{C}\langle\psi\rangle^G.$$

Various subgroups of G give rise to intermediate subfields of $\mathbb{C}\langle\psi\rangle$ and there is a natural bijection between all such subfields and the subgroups of G. The group G is called the *differential Galois group* of our linear differential equation. (Below we shall speak simply of Galois group, for short.)

From now on we shall assume that $n = 2$ and restrict ourselves to the case of Schroedinger operators. As we already mentioned, the space $\mathcal{H}$ of Schroedinger operators is the phase space of the KdV equation. We equip it with the Lie–Poisson bracket of the Virasoro algebra as in Section 8.1. Along with the Korteweg–de Vries equation

$$u_t = u_{xxx} + 6uu_x$$

there exist quite a few closely related equations of the form $u_t = u_{xxx} + F(u, u_x, u_{xx})$; the most famous of these "equations of the KdV type" is the modified KdV equation

$$v_t = v_{xxx} - 6v^2 v_x.$$

Another one is the Schwarz-KdV equation

$$\eta_t = S(\eta)\eta_x$$

which we already mentioned in the previous section. Various equations of KdV type are related by "differential substitutions". The famous example of such substitution is the Miura map

$$u = v' - v^2$$

which maps solutions of the modified KdV equation into solutions of the ordinary KdV equation. At least some part of these equations can be understood in the framework of a beautiful Galois theory picture proposed by G.Wilson [**W**].

According to the elementary theory, for a given u the space $V = V_u$ of solutions of the Schroedinger equation is 2-dimensional and for any ϕ, ψ their wronskian $W = \phi\psi' - \phi'\psi$ is constant. We write (ϕ, ψ) as a row vector; the group $G = SL(2)$ acts naturally on V by right multiplications. We denote by $\mathcal{V}$ the functional space of all wave functions for all Schroedinger operators $H_u \in \mathcal{H}$. The natural idea explored in [**W**] is the possibility to lift the KdV flows a originally defined on $\mathcal{H}$ to the bigger space $\mathcal{V}$. Speaking more formally, we shall consider the differential field $\mathbb{C}\langle\phi, \psi\rangle$ generated by the solutions; the phase space of the original KdV equation may be identified with the differential field $\mathbb{C}\langle u\rangle$. Clearly, $\mathbb{C}\langle\phi,\psi\rangle\supset\mathbb{C}\langle u\rangle$; as a matter of fact, $\mathbb{C}\langle u\rangle$ is isomorphic to the differential subfield of G-invariants and hence $\mathbb{C}\langle\phi,\psi\rangle\supset\mathbb{C}\langle u\rangle$ is a differential Galois extension with differential Galois group $G=SL(2)$.

Various subgroups of G give rise to intermediate differential fields. In particular, for $Z=\{\pm I\}$ the associated subfield of invariants is naturally isomorphic to $\mathbb{C}\langle\eta\rangle$, where $\eta=\phi/\psi$; since Z is the center of G, the extension $\mathbb{C}\langle\eta\rangle\supset\mathbb{C}\langle u\rangle$ is again a Galois extension with Galois group $PSL(2)=SL(2)/Z$. Let $B=HN\subset G$ be the standard Borel subgroup consisting of upper triangular matrices and N,H its subgroups of strictly upper triangular and diagonal matrices. They give rise to the following tower of differential extensions:

(8.19)
$$\begin{tikzcd}
& \mathbb{C}\langle\eta\rangle & \\
\mathbb{C}\langle\eta\rangle^{H} \arrow[ur, hook] & & \mathbb{C}\langle\eta\rangle^{N} \arrow[ul, hook'] \\
& \mathbb{C}\langle\eta\rangle^{B} \arrow[ul, hook'] \arrow[ur, hook] \arrow[uu, hook] & \\
& \mathbb{C}\langle\eta\rangle^{G} \arrow[u, hook] &
\end{tikzcd}$$

The subfields of invariants in this tower admit a simple description: We have $\mathbb{C}\langle\eta\rangle^N\simeq\mathbb{C}\langle\theta\rangle$, where $\theta:=\eta'$; in a similar way, the subalgebra of B-invariants is generated by $v:=\eta''/\eta'=\theta'/\theta$, the subalgebra of H-invariants is generated by $\rho:=\eta'/\eta$ and the subalgebra of G-invariants is generated by $u=v'-v^2$; moreover, we have $u=\frac{1}{2}S(\eta)$, where S is the Schwarzian derivative,

$$S(\eta)=\frac{\eta'''}{\eta'}-\tfrac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2.$$

Recall that the crucial property of the Schwarzian derivative is its invariance under projective transformations

$$\eta\mapsto\frac{a\eta+c}{b\eta+d}$$

induced by the right action of G in the space of wave functions.

A natural family of evolution equations of the KdV type associated with the tower (8.27) is represented on the commutative diagram on Fig. 1. We see in particular that the mKdV equation is associated with the subfield $\mathbb{C}\langle\phi,\psi\rangle^B$ of the Borel subgroup invriants.

So far, this Galois theory picture does not include the Hamiltonian structure for the dynamical equations listed in Fig. 1. In order to understand the

Figure 1.

$$
\begin{tikzcd}
 & \boxed{\eta_t = S(\eta)\eta_x} \arrow[dl, "\rho=\eta'/\eta"'] \arrow[dr, "\theta=\eta'"] \arrow[dd, "v=\eta''/\eta'"] \arrow[ddd, bend right, "u=S(\eta)"'] & \\
\boxed{\rho_t = \rho_{xxx} - \frac{3}{2}(\rho_x^2/\rho)_x - \frac{1}{2}(\rho^3)_x} \arrow[dr, "v=\rho+\rho'\rho^{-1}"'] & & \boxed{\theta_t = \theta_{xxx} - \frac{3}{2}(\theta_x^2/\theta)_x} \arrow[dl, "v=\theta'/\theta"] \\
 & \boxed{v_t = v_{xxx} - 6v^2 v_x} \arrow[d, "u=v'-v^2"] & \\
 & \boxed{u_t = u_{xxx} + 6uu_x} &
\end{tikzcd}
$$

extension tower dynamically we now want to equip $\mathcal{V}$ with a Poisson structure or its substitute. One option, explored in [**W**], is to look at symplectic forms rather than on Poisson brackets, because they may be naturally pulled back from $\mathcal{H}$ to $\mathcal{V}$ (at the expense of becoming degenerate). A closer look at the situation reveals yet another difficulty: the relevant 'variational' 2-form is an integral of a density whose differential is not identically zero; rather it is a closed form on the circle and hence its contribution disappears only if we may discard 'total derivatives'. For functionals which depend on wave functions, which are quasiperiodic, this is certainly illegitimate. This is the same kind of obstruction which we already encountered in Section 7.4 in the form of violation of the Jacobi identity (see Remark 7.18).

At this point we abandon Wilson's paper [**W**] and look for an alternative [**MS**]. Instead of using the symplectic form, we shall try to keep to the Poisson structure. Of course, Poisson brackets cannot be pulled back, and hence we have to *guess* a Poisson structure on the extended algebra and then check its consistency with the original bracket. Our strategy will be based on the projective point of view which we outlined in Section 8 above. Our main object will be the space of projective curves introduced in Theorem 8.7. However, it is natural to start with the much bigger space $\mathcal{W}$ of all quasi-periodic plane curves,

$$\mathcal{W} = \{(w = (\phi, \psi), M) \mid w(x + 2\pi) = w(x)M\}\,.$$

The space $\mathcal{W}$ contains the set $\mathcal{W}'$ of all non-degenerate plane curves with non-zero wronskian as an open subset. Let $\mathcal{C} := C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^\times)$ be the *scaling group* which acts on $\mathcal{W}$ via

$$f \cdot (w\,, M) = (fw\,, M). \tag{8.20}$$

Clearly, $\mathcal{C}$ acts freely on $\mathcal{W}'$ and the quotient may be identified with $\mathcal{V}$. The action of the linear group $G = SL(2)$ on $\mathcal{W}$ is via

$$g\colon w \mapsto w \cdot g,\ M \mapsto g^{-1}Mg.$$

The key condition which we use to restrict the choice of the Poisson structure on $\mathcal{W}$ is its *covariance* with respect to the group action. This condition puts us in the framework of Poisson group theory, as it allows both $\mathcal{C}$ and G to carry nontrivial Poisson structures, although it does not presume any *a priori* choice of these structures. As it happens, the covariance condition together with the natural constraint on the wronskian make their choice almost completely canonical. (In particular, the Poisson bracket on G is fixed up to scaling and conjugation; it is of the standard "quastriangular" type and the case of zero bracket is excluded.)

We want to find the most general Poisson structure on $\mathcal{W}$ which is covariant with respect to the right action of $G = SL(2)$ and to the action of the scaling group $\mathcal{C}$. This structure appears to be partially rigid. It is convenient to describe this Poisson structure by giving the Poisson brackets of the 'evaluation functionals' which assign to wave functions ϕ, ψ their values at the running point $x \in \mathbb{R}$. The covariance with respect to the local scaling group implies that these brackets are quadratic and local, i.e., depend only on the values of ϕ, ψ at the given points.

LEMMA 8.16. *Assume that the Poisson bracket on $\mathcal{W}$ is covariant with respect to the action of $\mathcal{C}$. Then the Poisson structure on $\mathcal{C}$ is trivial and, writing $w = (\phi, \psi)$, the bracket of evaluation functionals has the form*

$$\begin{aligned} \{\phi(x), \phi(y)\} &= A(x,y)\phi(x)\phi(y), \\ \{\psi(x), \psi(y)\} &= D(x,y)\psi(x)\psi(y), \\ \{\phi(x), \psi(y)\} &= B(x,y)\phi(x)\psi(y) + C(x,y)\phi(y)\psi(x). \end{aligned} \tag{8.21}$$

It is natural to assume that the bracket (8.21) is translation invariant, i.e., the structure functions depend only on the difference $x - y$. Using tensor notation, we can write these Poisson brackets in the following condensed form:

$$\{w_1(x), w_2(y)\} = w_1(x)w_2(y)R(x,y), \tag{8.22}$$

where $w(x) = (\phi(x), \psi(x))$ and we write the tensor product $w_1(x)w_2(y)$ as a row vector of length 4; the matrix $R(x,y) \in \mathrm{Mat}(4)$ is given by

$$R(x,y) = \begin{pmatrix} A(x-y) & 0 & 0 & 0 \\ 0 & B(x-y) & -C(y-x) & 0 \\ 0 & C(x-y) & -B(y-x) & 0 \\ 0 & 0 & 0 & D(x-y) \end{pmatrix}.$$

Poisson bracket relations of this type are called *exchange algebra relations.* They were first studied in [**B**] (for a special choice of R).

It is convenient to drop temporarily the Jacobi identity condition and to consider all (generalized) Poisson brackets which are covariant with respect to the Galois group action.

LEMMA 8.17. *Let us assume that the Poisson bracket* (8.22) *is right-G-invariant; then the exchange matrix has the structure*

$$R_0(x,y) = a(x-y)I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c(x-y) & -c(x-y) & 0 \\ 0 & c(x-y) & -c(x-y) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{8.23}$$

where a and c are arbitrary odd functions.

LEMMA 8.18. *Fix an arbitrary r-matrix $r \in \mathfrak{g}\wedge\mathfrak{g}$ and equip G with the corresponding Sklyanin bracket* (4.32). *Let us assume that the Poisson bracket* (8.22) *is right-G-covariant; then the exchange matrix has the structure*

$$R_r(x,y) = R_0(x,y) + r, \tag{8.24}$$

where we write $r \in \mathfrak{g}\wedge\mathfrak{g} \subset \mathrm{Mat}(2)\otimes\mathrm{Mat}(2)$ as a 4×4-matrix in the standard way.

For $\mathfrak{g} = \mathfrak{sl}(2)$ the classical Yang–Baxter equation does not impose any restrictions on the choice of r; indeed, it amounts to the requirement that the Schouten bracket $[r,r] \in \mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g}$ should be ad $\mathfrak{g}$-invariant, but for $\mathfrak{g} = \mathfrak{sl}(2)$ we have $\wedge^3\mathfrak{g} \simeq \mathbb{C}$. Let h, e, f be the standard generators of $\mathfrak{sl}(2)$. Up to the natural equivalence there exist three types of classical r-matrices:

(a) $r = 0$;
(b) $r = h\wedge f$
(c) $r = \epsilon\, e\wedge f$, where ϵ is a scaling parameter.

They correspond to three types of G-orbits in $\mathfrak{g}$ (zero orbit, the conic orbit consisting of nilpotent elements and the orbit of a regular semisimple element, respectively). Case (a) gives trivial bracket; case (c) is generic; case (b) (the so called triangular r-matrix) is degenerate. The Schouten tensor $[r,r]$ is identically zero in cases (a) and (b). In case (c) we have $[r,r] = -\epsilon^2 \neq 0$ (in this case our Lie bialgebra is factorizable)

Let us describe explicitly the Poisson structure on $G = SL(2)$ associated with various r-matrices. Recall that the Sklyanin bracket is completely specified by the Poisson bracket relations for matrix coefficients. In usual tensor notation we have

$$\{g_1,\, g_2\} = [r,\, g_1\, g_2], \tag{8.25}$$

where in the r.h.s. we regard $r \in \mathfrak{g}\wedge\mathfrak{g}$ and $g_1\, g_2 = g\otimes g$ as elements of $\mathrm{Mat}(2)\otimes\mathrm{Mat}(2) \simeq \mathrm{Mat}(4)$ and compute the commutator in $\mathrm{Mat}(4)$.

The standard Poisson bracket on G which corresponds to case (c) is given by the following set of relations for the matrix coefficients of $g = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$, we have

$$\begin{aligned} &\{\alpha,\beta\} = \epsilon\alpha\beta, \quad &&\{\alpha,\gamma\} = \epsilon\alpha\gamma,\\ &\{\beta,\delta\} = \epsilon\beta\delta, &&\{\gamma,\delta\} = \epsilon\gamma\delta,\\ &\{\beta,\gamma\} = 0, &&\{\alpha,\delta\} = 2\epsilon\beta\gamma. \end{aligned} \tag{8.26}$$

Notice that $\det g = \alpha\delta - \beta\gamma$ is a Casimir function and hence the Poisson bracket is well defined on the coordinate ring of $SL(2)$ and even of $PSL(2)$.)

In the sequel we shall be mainly concerned with the standard bracket (8.26). We shall see that the covariance condition together with the wronskian constraint fix the Poisson structure on G uniquely up to scaling and conjugation; in particular, r-matrices of types (a) and (b) are excluded.

We return now to the analysis of the exchange matrix. Since R_r in (8.24) is the sum of 2 terms, the Schouten bracket $[r, r]$ gives an extra term to the Jacobi identity for the corresponding exchange bracket.

LEMMA 8.19. *The exchange bracket* (8.22) *with exchange matrix* (8.24) *satisfies the Jacobi identity if and only if*

$$c(x-y)c(y-z) + c(y-z)c(z-x) + c(z-x)c(x-y) = 0 \tag{8.27}$$

in cases (a) and (b) and

$$c(x-y)c(y-z) + c(y-z)c(z-x) + c(z-x)c(x-y) = -\epsilon^2 \tag{8.28}$$

in case (c).

To solve equation (8.28), let us put $c(x) = \epsilon\, C(x)$ and express C as a Cayley transform,

$$C(x) = \frac{f(x)+1}{f(x)-1};$$

then (8.28) immediately yields for f the standard 2-cocycle relation

$$f(x-y)f(y-z)f(z-x) = 1.$$

The obvious solution is thus $C_\lambda(x-y) = \coth \lambda(x-y)$, where λ is a parameter. Setting $\lambda \to \infty$, we obtain a particular solution $C(x-y) = \operatorname{sign}(x-y)$. We shall see that this special solution is the only one which is compatible with the constraint $W = 1$. The solution of the degenerate equation (8.27) is $c(x) = 1/x$.

So far, the most general Poisson structure on $\mathcal{W}$ still contains functional moduli and a free parameter. As is easy to check, the Poisson brackets for the ratio $\eta = \phi/\psi$ do not depend on a:

PROPOSITION 8.20. *We have*

$$\{\eta(x), \eta(y)\} = \epsilon\left(\eta(x)^2 - \eta(y)^2\right) - c(x-y)\left(\eta(x) - \eta(y)\right)^2. \tag{8.29}$$

Formula (8.29) defines a family of G-covariant Poisson brackets on the space of projective curves. However, in order to establish a connection between these brackets and Schroedinger operators we must take into account the wronskian constraint which restricts the choice of c. The second structure function a drops out after projectivization and is not restricted by the Jacobi identity. We shall see, however, that the wronskian constraint suggests a natural way to choose a as well. Our next proposition describes the basic Poisson bracket relations for the wronskian:

PROPOSITION 8.21. *We have*

$$\{W(x), \phi(y)\} = (c(x-y) - 2a(x,y))W(x)\phi(y) - c'(x-y)\phi(x)[\phi(x)\psi(y) - \psi(x)\phi(y)]. \tag{8.30}$$

By symmetry, a similar formula holds for $\{W(x), \psi(y)\}$.

Formula (8.30) immediately leads to the following crucial observation:

PROPOSITION 8.22. *The constraint $W = 1$ is compatible with the Poisson brackets for scaling invariant η if and only if the last term in* (8.30) *is identically zero; this is possible if and only if $C'(x-y)$ is a multiple of $\delta(x-y)$, i.e., if $C(x-y)$ is a multiple of* $\operatorname{sign}(x-y)$.

It is important that the wronskian constraint excludes the possibility that $\epsilon = 0$ and hence the corresponding Poisson structure on G is conjugate to the standard one (case (c)). From now on, without restricting the generality, we fix $\epsilon = 1$.

PROPOSITION 8.23. *Let us assume that* $c(x-y) = \operatorname{sign}(x-y)$; *then the Poisson bracket relations for the wronskian are given by:*

$$\{W(x), W(y)\} = (\operatorname{sign}(x-y) - 2a(x,y))W(x)W(y), \tag{8.31}$$

or, equivalently

$$\{\log W(x), \log W(y)\} = (\operatorname{sign}(x-y) - 2a(x,y)). \tag{8.32}$$

Formulae (8.30) and (8.31) suggest the following distinguished choice of a:

PROPOSITION 8.24. *Assume that a is so chosen that*

$$\operatorname{sign}(x-y) - 2a(x,y) = \delta'(x-y).$$

(In other words, $a(x,y)$ is the distribution kernel of the operator $\frac{1}{2}\left(\partial^{-1} - \partial\right)$.) Then: (i) The logarithms of wronskians form a Heisenberg Lie algebra, the central extension of the abelian Lie algebra of $\mathcal{C}$. (ii) Let $\mathcal{C}' = \mathcal{C}/\mathbb{C}^*$ be the quotient of the scaling group over the subgroup of constants; $\log W$ *is the moment map for the action of $\mathcal{C}'$ on $\mathcal{W}$.*

Recall that according to the general theory the Poisson bracket relations for the moment map may reproduce the commutation relations for a central extension of the original Lie algebra. This is precisely what happens in the present case.

With this choice of a and C the Poisson geometry of the space $\mathcal{V}$ of wave functions becomes finally quite transparent: $\mathcal{V}$ arises as a result of

Hamiltonian reduction with respect to $\mathcal{C}$ over the zero level of the associated moment map. The constraint set $\log W = 0$ is (almost) non-degenerate (i.e., this is a second class constraint, according to Dirac). The projective invariants commute with the wronskian and hence their Poisson brackets are not affected by the constraint.

The description of the Poisson structure on $\mathcal{V}$ is completed by the Poisson brackets for the monodromy.

Proposition 8.25. *The Poisson covariant brackets for the monodromy have the form*

$$\begin{aligned} \{w(x)_1, M_2\} &= w(x)_1\big[M_2 r_+ - r_- M_2\big], \\ \{M_1, M_2\} &= M_1 M_2 r + r M_1 M_2 - M_2 r_+ M_1 - M_1 r_- M_2. \end{aligned} \tag{8.33}$$

The Poisson bracket for the monodromy is precisely the Poisson bracket of the dual group G^* described in (5.17), (5.18). In other words, the 'forgetting map' $\mu : (w, M) \mapsto M$ is a Poisson morphism from $\mathcal{W}$ into the dual group G^*. This mapping is of special importance.

Proposition 8.26. *The mapping μ is the non-abelian moment map associated with the right action of G on $\mathcal{W}$.*

Let us now list the Poisson bracket relations in the differential algebra $\mathbb{C}\langle\eta\rangle$ and its various subalgebras which correspond to different admissible subgroups of G.

Theorem 8.27. *(i) All arrows in the commutative diagram*

$$\begin{tikzcd}
& \mathbb{C}\langle\eta\rangle & \\
\mathbb{C}\langle\eta\rangle^H \arrow[ur, hook] & & \mathbb{C}\langle\eta\rangle^N \arrow[ul, hook'] \\
& \mathbb{C}\langle\eta\rangle^B \arrow[ul, hook'] \arrow[ur, hook] \arrow[uu, hook] & \\
& \mathbb{C}\langle\eta\rangle^G \arrow[u, hook] &
\end{tikzcd}$$

are Poisson morphisms.

(ii) The basic Poisson bracket relations in $\mathbb{C}\langle\eta\rangle$ are given by

$$\{\eta(x), \eta(y)\} = \eta(x)^2 - \eta(y)^2 - \operatorname{sign}(x-y)\left(\eta(x) - \eta(y)\right)^2. \tag{8.34}$$

(iii) We have $\mathbb{C}\langle\eta\rangle^N \simeq \mathbb{C}\langle\theta\rangle$, where $\theta := \eta'$; moreover,

$$\{\theta(x), \theta(y)\} = 2\operatorname{sign}(x-y)\theta(x)\theta(y). \tag{8.35}$$

(iv) The subalgebra of B-invariants is generated by $v := \frac{1}{2}\eta''/\eta' = \frac{1}{2}\theta'/\theta$; we have:

$$\{v(x), v(y)\} = \tfrac{1}{2}\delta'(x-y). \tag{8.36}$$

(v) The subalgebra of G-invariants is generated by $u = \frac{1}{2}S(\eta) = v' - v^2$; we have:

$$\{u(x), u(y)\} = \tfrac{1}{2}\delta'''(x-y) + \delta'(x-y)\big[u(x)+u(y)\big]. \tag{8.37}$$

Formula (8.37) reproduces the standard Virasoro algebra; in other words, the Poisson algebra (8.34) constructed from general covariance principles is indeed an extension of the Poisson–Virasoro algebra.

REMARK 8.28. The Poisson bracket relations (8.35) – (8.37) listed above are particularly simple, since their r.h.s. is algebraic. Because the basic Poisson bracket relations (8.34) are nonlocal, this need not always be the case. This is what happens in the case of H-invariants:

PROPOSITION 8.29. *(i) The differential subalgebra of H-invariants in $\mathbb{C}\langle\eta\rangle$ is generated by $\rho = \eta'/\eta$. (ii) The Poisson brackets for ρ have the form*

$$\{\rho(x),\, \rho(y)\} = 2\rho(x)\rho(y)\left[\sinh\int_x^y \rho(s)\,ds + \operatorname{sign}(x-y)\cosh\int_x^y \rho(s)\,ds\right].$$

It is well known that the standard KdV equation is generated with respect to the Virasoro bracket by the Hamiltonian

$$H = \int u^2\,dx. \tag{8.38}$$

The Hamiltonians of all higher KdV equations are associated with trace identities for H_u and hence are G-invariant; they generate a system of compatible commuting flows on all levels of the extension tower. We conclude our discussion by getting back to the commutative diagram on Fig.1 which relates different dynamical system generated by this Hamiltonian on different levels of the extension tower. Our main result can be now stated as follows:

THEOREM 8.30. *The commutative diagram represented on Fig.1 is compatible with the dynamics induced by the KdV Hamiltonians. Dynamical flows on each level of the extension tower associated with these Hamiltonians factorize over those lying on a lower level. All arrows in this diagram are Poisson mappings.*

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