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Enriques surfaces covered by Jacobian Kummer surfaces

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1 Introduction

A $K3$ surface is a simply connected compact complex surface whose canonical bundle is trivial. Every Enriques surface appears as a quotient of a $K3$ surface by a fixed-point-free (shortly, free) involution. Theoretically, to consider an Enriques surface is equivalent to consider the pair of the covering $K3$ surface and the free involution. For example, the period map for Enriques surfaces is constructed under this description. But the properties of free involutions on a *fixed* $K3$ surface are rather unclear to us. The existence is already a special property, their geometric realizations and the isomorphism classes of the quotient Enriques surfaces are other problems.

For a fixed $K3$ surface X, two quotient Enriques surfaces are isomorphic if and only if the two free involutions are conjugate in $\mathrm{Aut}(X)$. In [16] it is shown that the number of the conjugacy classes of free involutions (and more generally, of finite subgroups) are finite. There this number, i.e., the number of isomorphism classes of quotient Enriques surfaces, is computed for $K3$ surfaces with Picard number $\rho = 11$ or for Kummer surfaces associated with the product of two elliptic curves whose periods are very general.

The aim of this paper is to study fixed-point-free involutions on surfaces studied in [11, 12]. Let C be a smooth projective curve of genus 2. Its Jacobian variety $J(C)$ is the abelian surface parametrizing divisor classes on C of degree 0. The quotient surface $J(C)/\{\pm 1_{J(C)}\}$ has 16 nodes and can be embedded into $\mathbb{P}^3$ as a quartic hypersurface. We call it the *Kummer quartic surface* associated with C and denote by $\overline{Km}(J(C)) =: \overline{X}$. The minimal desingularization $Km(J(C)) =: X$ of $\overline{Km}(J(C))$ is called the *Jacobian Kummer surface* associated with C, which is a $K3$ surface. X is *Picard-general* if the Picard number of X equals 17, the minimum possible value. In what follows, X will always be a Picard-general Jacobian Kummer surface except for Sections 2 and 4.

In [13], Mukai observed that there exist three kinds of free involutions on X.

- *A switch* associated with *an even theta characteristic* β.
- *A Hutchinson-Göpel (shortly HG) involution* associated with *a Göpel tetrad* G.
- *A Hutchinson-Weber (shortly HW) involution* associated with *a Weber hexad* W.

Essentially these automorphisms date back more than a century, but their freeness are found only recently in comparison. Mukai studied HG involutions in connection with the numerically reflective involutions of Enriques surfaces. Also he conjectured that these are the all free involutions on X. In this paper we prove the following theorem and confirm the conjecture.

Theorem 1.1. On a Picard-general Jacobian Kummer surface X, there are exactly $31 = 10 + 15 + 6$ free involutions up to conjugacies in $\mathrm{Aut}(X)$. 10 are switches, 15 are HG involutions and 6 are HW involutions.

In [12], Kondo proved that $\mathrm{Aut}(X)$ is generated by 32 translations and switches, 32 projections and correlations, 60 HG involutions, and 192 Keum's automorphisms. One point of the proof was that 192 Keum's automorphisms did not correspond in one-to-one way to the 192 facets of the polyhedral introduced by Borcherds and Kondo. Moreover they had infinite order while the others had order 2. In this respect, it can be expected that there exist 192 *involutions* which correspond in one-to-one way to the 192 facets of the polyhedral and together with the $32 + 32 + 60$ involutions they generate $\mathrm{Aut}(X)$. In fact, the HW involutions work well.

Theorem 1.2. $\mathrm{Aut}(X)$ is generated by the following *involutions*: translations, switches, projections, correlations, HG involutions and HW involutions.

This is a biproduct of the proof of Theorem 1.1.

The proof of Theorem 1.1 is given in the following way. In Section 2 we introduce an invariant of a free involution, called *a patching subgroup*, which is a subgroup of $A_{NS(X)} = NS(X)^*/NS(X)$. This subgroup appears naturally in the light of Nikulin's theory of lattices [15]. Under some condition, we can show the invariance of the patching subgroup under conjugations. Section 3, Proposition 3.4 shows conversely two free involutions are conjugate if their patching subgroups are the same, when X is a Picard-general Jacobian Kummer surface. Simultaneously we see that X has no more than 31 non-isomorphic Enriques quotients. These two Sections reduce the proof of Theorem 1.1 to concrete computations of patching subgroups of free involutions itemized above. The occurence of 31 distinct patching subgroups shows Theorem 1.1. The computations are worked out in Sections 5-7. The result shows that the generators of patching subgroups are expressed in terms of the classical notions. It is summarized as follows.

In the switch case, let β be an even theta characteristic and σ_β be the switch. β corresponds to a pair of Rosenhain subgroups R_1, R_2. Then the patching subgroup Γ_{σ_β} is cyclic of order 4 and generated by

$$H/4 + \sum_{\alpha \in R_1} N_\alpha/2.$$

Of course we obtain the same group after replacing R_1 by R_2 in this case.

In the HG involution case, let G be a Göpel tetrad and σ_G be the HG involution. Then the

patching subgroup Γ_{σ_G} is 2-elementary abelian of order 4 and generated by

$$H/2 \text{ and } \sum_{\alpha \in G} N_\alpha/2.$$

We remark that this result of HG involution case also follows from the computations of [13].

In the HW involution case, let W be a Weber hexad and σ_W be the HW involution. Then the patching subgroup Γ_{σ_W} is cyclic of order 4 and generated by

$$H/4 + \sum_{\alpha \in W} N_\alpha/2.$$

The divisors $H, N_\alpha \in NS(X)$ and also the classical notions appeared here will be defined in Section 4, where we recall the basic properties of Jacobian Kummer surfaces. After fixing the basis of $A_{NS(X)}$, we can easily check that there are appearing 31 distinct patching subgroups.

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Notations: We refer the readers to [15] for the basic properties of the finite quadratic form (A_L, b_L, q_L) associated with an even nondegenerate lattice L. By definition, A_L is the finite abelian group L^*/L, $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ is the symmetric bilinear form and $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$ is the quadratic form, both naturally induced from that of L. Usually we denote finite forms by (A_L, q_L), omitting b_L, or only by A_L.

The hyperbolic plane is denoted by U, the root lattices A_l, D_m, E_n are considered to be negative definite. The rank one lattice $\langle 2n \rangle$ is also used in this paper. On finite forms, $u(2)$ is the associated form of the lattice $U(2)$, $\langle 1/2n \rangle$ is that of $\langle 2n \rangle$. The set of generators $\{e, f\}$ of $u(2)$ satisfying

$$q(e) = q(f) = 0, b(e, f) = 1/2$$

is called the standard generator.

For a lattice T and $k = \mathbb{Q}, \mathbb{C}$ we denote the scalar extension by T_k. If T is a lattice and $T_{\mathbb{C}}$ is equipped with a Hodge structure, then $\mathrm{Aut}_{\mathrm{Hodge}}(T)$ is a subgroup of $O(T)$ whose elements preserve the Hodge decomposition.

2 The method of counting

In this section X is any $K3$ surface. Let σ be a free involution on X. The (-1)-eigenspace of the action of σ on $NS(X)$ is denoted by K. Then it is well-known that K is negative

definite, contains no (-2)-element and the primitive hull of $K \oplus T_X$ in $H^2(X, \mathbb{Z})$ is isometric to $U \oplus U(2) \oplus E_8(2) =: N$. We choose a marking $\phi : \overline{K \oplus T_X} \to N$ for this isometry.

The nonzero global holomorphic 2-form ω_X on X determines via ϕ a point in $\mathcal{D}(N)/O(N)$, which is the period of the Enriques surface $Y := X/\sigma$, where

$$\mathcal{D}(N) := \{\mathbb{C}\omega \in \mathbb{P}_{\text{lines}}(N_{\mathbb{C}}) | \omega \in N \otimes \mathbb{C}, \omega \cdot \omega = 0, \omega \cdot \overline{\omega} > 0\}$$

is the (two copies of) bounded symmetric domain of type IV associated to lattice N of signature $(2, 10)$. Obviously this period is independent of the choice of ϕ and the Torelli theorem of Enriques surfaces says that this point determines the isomorphism class of Y uniquely.

Conversely given a primitive embedding $\phi : T_X \to N$ such that the orthogonal complement K is free from (-2)-elements, by the surjectivity there exists an Enriques surface Y whose period is exactly $[\phi(\mathbb{C}\omega_X)]$. If $\rho(X) \geq 12$ then [10, Theorem 1] shows that X is isomorphic to the universal double cover of Y. Even if $\rho(X) \leq 11$ the same holds, whose proof is in [17].

Thus we have shown

Proposition 2.1. There is a one-to-one correspondence between the sets

$$\{\text{Enriques quotients of } X\}/(\text{isomorphisms})$$

and

$$\left\{ \begin{array}{l} \text{Primitive embeddings } \phi : T_X \to N \\ \text{such that } K = T_X^{\perp} \text{contains no } (-2)-\text{elements} \end{array} \right\} /(\text{Hodge isometries of } N),$$

where for each ϕ we equip N with a Hodge structure induced from that of T_X by ϕ.

In the following, we identify $\overline{K \oplus T_X}$ with N by ϕ. By [15], there are subgroups $\Gamma_K \subset A_K$ and $\Gamma_{T_X} \subset A_{T_X}$ and a sign-reversing isometry $\varphi : \Gamma_K \xrightarrow{\sim} \Gamma_{T_X}$ such that N is the sublattice of $K_{\mathbb{Q}} \oplus T_{X,\mathbb{Q}}$ generated by K, T_X and $\{(x, \varphi(x)) | x \in \Gamma_K\}$.

Definition 2.2. The *patching subgroup* Γ_σ of the free involution σ is the inverse image of Γ_{T_X} by the natural sign-reversing isometry $A_{NS(X)} \xrightarrow{\sim} A_{T_X}$.

Under a condition, Γ_σ is an invariant of a conjugacy class which is very computable.

Proposition 2.3. If $\mathrm{Aut}_{\mathrm{Hodge}}(T_X) = \{\pm \mathrm{id}\}$, then Γ_σ depends only on the isomorphism class of the quotient Enriques surface.

Proof. By Proposition 2.1, conjugate free involutions induce on N an isometric Hodge structure. Any Hodge isometry of N preserves $K = \omega_X^{\perp}$ and hence T_X. Thus it induces $\pm \mathrm{id}$ on T_X and preserves the subgroup Γ_{T_X}. □

Remark 2.4. The condition above is weak. It is true if $\rho(X)$ is odd, see [12, p597], or even if $\rho(X)$ is even, it is true if X is very general in the period domain ([16, Proposition 3.1]).

In general there are free involutions not conjugate each other but with the same Γ_σ. However in the Picard-general Jacobian Kummer case, Γ_σ completely classifies free involutions. This will be shown in the next section.

The computation of Γ_σ is done by

Lemma 2.5. Let σ, K as above. Then

$$\Gamma_\sigma = \{[x] \in NS(X)^*/NS(X) | \exists [y] \in K^*/K,\ x - y \in NS(X)\}.$$

Proof. Let $\rho : A_{NS(X)} \to A_{T_X}$ be the canonical isomorphism. Then $\rho([x]) = [z]$ is equivalent to $x + z \in H^2(X, \mathbb{Z})$. Since $\Gamma_{T_X} = \{[z] \in T_X^*/T_X | \exists [y] \in K^*/K,\ y + z \in N\}$,

$$\begin{aligned} \Gamma_\sigma &= \{[x] | \rho([x]) \in \Gamma_{T_X}\} \\ &= \{[x] \in NS(X)^*/NS(X) | \exists [y] \in K^*/K,\ x - y \in NS(X)\}. \end{aligned}$$

This is what we need. □

3 Invariants of free involutions

Let C be a genus 2 curve, $J(C)$ its Jacobian and $Km(J(C)) = X$ the associated Jacobian Kummer surface as in the Introduction. As is well-known, $J(C)$ contains C as a theta divisor:

$$\Theta = \{[p - p_0] | p \in C\} \subset J(C), \quad p_0 \in C.$$

Hence $\operatorname{rank} NS(J(C)) \geq 1$ and $\operatorname{rank} NS(X) \geq 17$ holds. When we have the equality, we call X *Picard-general*. In this case, since $T_{J(C)} = U^{\oplus 2} \oplus \langle -2 \rangle$ we have $T_X = U(2)^{\oplus 2} \oplus \langle -4 \rangle$ and $NS(X) = U \oplus D_4^{\oplus 2} \oplus D_7$.

For simplicity, we put $T := T_X$. Suppose we are given a primitive embedding of T into N such that the orthogonal complement is free from (-2)-elements, as in Proposition 2.1. First we determine the orthogonal complement.

Proposition 3.1. The lattice $K = T_X^\perp$ is isometric to $E_7(2)$.

Proof. Consider the unique embedding of N into the abstract $K3$ lattice L. The orthogonal complement is denoted by M, $M \simeq U(2) \oplus E_8(2)$. By [15], we have the following isomorphism of discriminant quadratic forms:

$$-q_K \simeq (q_M \oplus q_T|_{\Gamma^\perp})/\Gamma \tag{3.1}$$

where Γ is the pushout (i.e. the graph) of a sign-reversing isometry of subgroups $\Gamma_M \subset A_M$ and $\Gamma_T \subset A_T$.

For a finite quadratic form (A, q), we denote the quadratic form induced on the 2-torsion subgroup $A_2 = \{x \in A | 2x = 0\}$ by (A_2, q_2). Note that even if q is nondegenerate, q_2 may be degenerate.

Put $\#\Gamma = 2^a$. In our equality (3.1), A_M is already 2-elementary and Γ_T is contained in $(A_T)_2$. This shows $a \leq 5 = l_2(A_T)$, where l_2 denotes the number of minimal generators of the 2-Sylow subgroup of A_T.

Also it follows

$$((A_M \oplus A_T)_2|_{\Gamma^\perp})/\Gamma \subset (A_M \oplus A_T|_{\Gamma^\perp})/\Gamma = A_K, \tag{3.2}$$

since Γ is 2-elementary. $(A_M \oplus A_T)_2$ has a radical of order 2 contained in $(A_T)_2$. Since Γ is a graph, this radical is not contained in Γ. This shows that $\#((A_M \oplus A_T)_2|_{\Gamma^\perp}) = 2^{15-a}$. Thus the order of the left-hand-side of (3.2) is 2^{15-2a}. Since K is of rank 7, we have $15 - 2a \leq 7$ and hence $a = 4, 5$.

We show that if $a = 5$ then K contains (-2)-elements and contradicts vthe assumption. For this, first note that in this case $\Gamma_T = (A_T)_2$ is uniquely determined and the embedding of Γ_T in $A_M \simeq u(2)^{\oplus 5}$ is unique up to isomorphism by Witt's theorem. So we can compute q_K directly and get $q_K \simeq u(2)^{\oplus 2} \oplus \langle 1/4 \rangle$. From this we see that there are inclusions $K \subset K' \subset \langle -1 \rangle^{\oplus 7}$ such that K' is an even lattice, $[K' : K] = 4$ and $[\langle -1 \rangle^{\oplus 7} : K'] = 2$. By the definition of D_7, $K' \simeq D_7$. Consider the Dynkin diagram of D_7 and take a subgraph isomorphic to A_6 with vertices $e_1, \cdots e_6$ in this order. Put $f_0 = 0, f_j = e_1 + \cdots + e_j$, $1 \leq j \leq 6$. Any difference of two of these seven elements have self-intersection (-2). If K has no (-2)-elements, then $\{f_j\}_{0 \leq j \leq 6}$ cannot be in the same residue class of K'/K. Then we must have $[K' : K] \geq 7$ and contradiction.

Thus we obtain $a = 4$. From (3.2), we see that $\#(A_K)_2 \geq 2^7$. It follows that $K(1/2)$ is an integral (may be odd) lattice and $\det K(1/2) = -2$. By assumption, the minimal norm of the positive definite lattice $K(-1/2)$ is greater than 1. It follows from [3, p400, Table 15.8] that $K(1/2) \simeq (\langle -2 \rangle^\perp \text{ in } E_8) \simeq E_7$. □

The following nature of the lattice $K = E_7(2)$ will be used.

Lemma 3.2. The canonical homomorphism $\sigma : O(K) \to O(q_K)$ is surjective.

Proof. The same property for the lattice $E_8(2)$ is known by [1]. We reduce the lemma to this case. Firstly, we know the orders of the two groups. By [2], $\#O(E_7(2)) = \#O(E_7) = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$. On the other hand, we can easily compute the order of $O(q_K)$ as $\#O(q_K) = \#O(u(2)^{\oplus 2} \oplus \langle -1/4 \rangle) = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$ (c.f. Lemma 3.3). Thus it is enough to show that σ is injective.

We take a (-4)-element r of $E_8(2)$ and identify K with $r^\perp$. Obviously $g \in \ker \sigma$ can be extended to an isometry $\overline{g}$ of $E_8(2)$ by defining $\overline{g}(r) = r$. It is clear that $\overline{g}$ acts on the discriminant $A_{E_8(2)}$ trivially. It follows from [1, Proposition 1.7] that $\overline{g} = \pm \mathrm{id}$. Since $\overline{g}(r) = r$, $\overline{g} = \mathrm{id}$. □

Because $(A_K, q_K) \simeq u(2)^{\oplus 3} \oplus \langle 1/4 \rangle$, the next lemma is also used.

Lemma 3.3. Let $(A, q) = u(2)^m \oplus \langle 1/4 \rangle$ be a finite quadratic form. Then the action of $O(q)$ on A decomposes A into 6 orbits. If we denote the standard generator of one of $u(2)$ by $\{e, f\}$ and that of $\langle 1/4 \rangle$ by $\{g\}$, they are as in the following table.

a representative	length	square
0	1	0
$2g$	1	1
e	$2^{2m}-1$	0
$e+f$	$2^{2m}-1$	1
g	$2^{2m}+2^m$	$1/4$
$e+f+g$	$2^{2m}-2^m$	$-3/4$

In this table, for a representative x, the length is $\#(O(q)\cdot x)$ and the square is $q(x)\in\mathbb{Q}/2\mathbb{Z}$.

The proof is given by induction on m and we omit it.

Now we are going to describe the latter set of Proposition 2.1, i.e., we classify the Hodge structures on N induced from embeddings $T\subset N$ as in the proposition. We recall Definition 2.2, there is an isomorphism

$$q_N \simeq (q_K\oplus q_T)|_{\Gamma^\perp}/\Gamma, \tag{3.3}$$

where Γ is the pushout of a sign-reversing isometry of subgroups $\Gamma_K\subset A_K$ and $\Gamma_T\subset A_T$. By Proposition 2.3, Γ_T is an invariant of the Hodge structure.

We will prove the converse. Namely, suppose we have two embeddings $T\subset N_1$ and $T\subset N_2$ whose orthogonal complements are denoted by K_i. For each embedding we have $(\Gamma_i,\Gamma_{K,i}\Gamma_{T,i})$ and the equality (3.3). What we want to show is

$$(*):\Gamma_{T,1}=\Gamma_{T,2}\Rightarrow N_1\simeq N_2(\text{Hodge isometry}).$$

The argument goes as follows. Assume we could find an isometry $\sigma_K: A_{K_1}\to A_{K_2}$ such that the following commutes.

$$\begin{array}{ccc} A_{K_1}\supset\Gamma_{K,1} & \xrightarrow{\sim} & \Gamma_{T,1}\subset A_T \\ {\scriptstyle\sigma_K}\downarrow \quad {\scriptstyle\sigma_K}\downarrow & & \downarrow{\scriptstyle\mathrm{id}} \quad \downarrow{\scriptstyle\mathrm{id}} \\ A_{K_2}\supset\Gamma_{K,2} & \xrightarrow{\sim} & \Gamma_{T,2}\subset A_T \end{array}$$

Then by Lemma 3.2 we can lift σ_K to $\sigma'_K: K_1\xrightarrow{\sim}K_2$ and the pair $(\sigma'_K,\mathrm{id}_T)$ can be lifted to an Hodge isometry $N_1\xrightarrow{\sim}N_2$. Thus it is enough to find σ_K.

By Proposition 3.1, $[N:K\oplus T]=\#\Gamma=4$. Thus there are two possibilities of underlying groups of $\Gamma_K\simeq\Gamma_T$. We consider each case separately.

First we consider the case $\Gamma_{T,i}\simeq\mathbb{Z}/4\mathbb{Z}$. The square of the generator $g_T\in\Gamma_{T,1}=\Gamma_{T,2}$ is independent of the choice and there are two possibilities, $q_T(g_T)=-1/4$ or $3/4$. Let $(g_{K,i},g_T)\in\Gamma_i$. We have $q_{K_i}(g_{K,i})=1/4$ or $-3/4$ respectively. By Lemma 3.3, in these cases we can find σ_K and $(*)$ is proved. We find easily that there are 10 subgroups Γ_T satisfying $q_T(g_T)=-1/4$. Also there are 6 with $q_T(g_T)=3/4$.

Second we consider the case $\Gamma_T \simeq \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. The argument becomes slightly complicated, but the conclusion is the same. To prove $(*)$ in this case, first we show that Γ_i always contains a particular element. Here, for a clear argument, we take generators g_i and g' of $\langle 1/4\rangle \subset A_{K_i}$ and $\langle -1/4\rangle \subset A_T$ respectively. We denote an element of $A_{K_i} \oplus A_T$ by

$$(x, y; z, w) \in A_{K_i} \oplus A_T \ ; \ x \in u(2)^{\oplus 3}, y \in \langle 1/4\rangle, z \in u(2)^{\oplus 2}, w \in \langle -1/4\rangle.$$

Then the claim is that

$$(0, 2g_i; 0, 2g') \in \Gamma_i.$$

In fact, since Γ_i is contained in $(A_{K_i} \oplus A_T)_2$, the radical element $(0, 2g_i; 0, 0)$ of $(A_{K_i} \oplus A_T)_2$ is in $\Gamma_i^{\perp}$. Hence its residue class $(0, 2g_i; 0, 0) + \Gamma_i$ determines an element of A_{N_i} by the isomorphism (3.3). It is nonzero because $q_{K_i}(2g_i) = 1$. Since A_{N_i} is nondegenerate, there exists an element $(x, y; z, w) + \Gamma_i \in A_{N_i}$ with $(0, 2g_i; 0, 0) \cdot (x, y; z, w) = 1/2$. It follows $y = \pm g_i$. Further since $(q_{K_i} \oplus q_T)(x, y; z, w) \in \mathbb{Z}$, it follows $w = \pm g'$, i.e., there exists an element in $\Gamma_i^{\perp}$ of the form $(x, \pm g_i; z, \pm g')$. Since the residue class of this element is of order 2 in A_{N_i}, we have that $(0, 2g_i; 0, 2g') \in \Gamma_i$.

Let $\Gamma_{T,1} = \Gamma_{T,2} = \langle 2g', \alpha\rangle$. Replacing α by $\alpha + 2g'$ if necessary, we can assume $q_T(\alpha) = 0$. Let $(\beta_i, \alpha) \in \Gamma_i$, $q_{K_i}(\beta_i) = 0$. By Lemma 3.3, we can find $\sigma_K : A_{K_1} \xrightarrow{\sim} A_{K_2}$ which takes β_1 to β_2. This σ_K must take $2g_1$ to $2g_2$, so we have now proved $(*)$. There are 15 possible Γ_T in this case.

In summary, we have obtained the following.

Proposition 3.4. Let X be a Picard-general Jacobian Kummer surface. Then free involutions σ_1, σ_2 are conjugate if and only if the patching subgroups $\Gamma_{\sigma_1}, \Gamma_{\sigma_2}$ coincide. There exist (at most) $31 = 10 + 15 + 6$ free involutions.

The existence of 31 free involutions is assured by concrete constructions in the following sections.

4 The $(16)_6$ configuration on a Jacobian Kummer surface

In this section we recall and prepare notations concerning the divisors on Jacobian Kummer surfaces. The content of this section is known, references are [11, 12, 4].

The index set. Let C be a smooth projective curve of genus 2. It is a double cover of $\mathbb{P}^1$ which ramifies at 6 Weierstrass points $\{p_1, \cdots, p_6\} \subset C$. Here we should notice the linear equivalence

$$p_i + p_j + p_k - p_l - p_m - p_n \sim 0$$

for an arbitrary permutation $\{i, j, k, l, m, n\}$ of $\{1, \cdots, 6\}$. The set of theta characteristics of C is by definition

$$S(C) = \{D \in \mathrm{Pic}(C) | 2D \sim K_C\}.$$

They are divided into odd theta characteristics $\{[p_i]|i = 1, \cdots, 6\}$ and even ones $\{[p_i + p_j - p_k]|i, j, k$ are distinct each other$\}$. There are 16 theta characteristics.

The Jacobian variety $J(C)$ consists of divisor classes of degree 0 on C. We denote by $J(C)_2$ the set of sixteen 2-torsion points of $J(C)$. Then

$$J(C)_2 = \{0\} \cup \{[p_i - p_j]|i \neq j\}.$$

These $16 + 16 = 32$ divisor classes naturally correspond to partitions of the set $\{1, \cdots 6\}$ into two subsets in the following way.

$$[p_i] \in S(C) \longleftrightarrow \{i\} \cup \{i\}^c.$$
$$[p_i + p_j - p_k] \in S(C) \longleftrightarrow \{i, j, k\} \cup \{i, j, k\}^c.$$
$$[p_i - p_j] \in J(C)_2 \longleftrightarrow \{i, j\} \cup \{i, j\}^c.$$
$$0 \in J(C)_2 \longleftrightarrow \emptyset \cup \{1, \cdots, 6\},$$

where the complement is taken in the set $\{1, \cdots, 6\}$. We denote these partitions by exhibiting one of the subsets, surrounded by [ ]. For example, $p_1 - p_2$ corresponds to $[12] = [3456]$, $p_1 + p_2 - p_3$ corresponds to $[123] = [456]$, etc. $[\emptyset]$ is denoted by $[0]$. In this notation, we see that the symmetric difference of subsets α, β of $\{1, \cdots, 6\}$ corresponds to addition or difference in $\mathrm{Div}(C)$ as follows.

$$[\alpha \ominus \beta] = [\alpha] - [\beta] \text{ if } [\alpha], [\beta] \in S(C),$$
$$[\alpha \ominus \beta] = [\alpha] + [\beta] \text{ otherwise.}$$

When we use a partition $[\alpha]$ as an index, [ ] will be omitted.

The $(16)_6$ configuration. The sixteen theta divisors on $J(C)$ corresponding to $\beta \in S(C)$ are

$$\Theta_\beta = \{[p - \beta] \in J(C)|p \in C\}.$$

The sixteen nodes $\{n_\alpha \in \overline{X}|\alpha \in J(C)_2\}$ on $\overline{X} = J(C)/\{\pm 1\}$ are the images of $\alpha \in J(C)_2$. On the minimal desingularization X of $\overline{X}$, n_α is blown up to give a smooth rational curve N_α on X. The tropes $\overline{T_\beta} \subset \overline{X}$ and $T_\beta \subset X$ are the strict transforms of Θ_β. Hence we obtain 32 (-2)-curves $\{N_\alpha, T_\beta\}_{\alpha,\beta}$ on X. The incidence relation between these divisors is called the $(16)_6$ configuration. It is given explicitly by

$$(N_\alpha, N_{\alpha'}) = -2\delta_{\alpha,\alpha'}, \qquad (T_\beta, T_{\beta'}) = -2\delta_{\beta,\beta'},$$
$$(N_\alpha, T_\beta) = 1 \Leftrightarrow \alpha + \beta \in \{[1], [2], [3], [4], [5], [6]\}.$$

A permutation of the set $\{N_\alpha, T_\beta\}_{\alpha,\beta}$ which preserves the incidence relation above is called an automorphism. Nikulin [14] showed that the automorphism group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5 \rtimes \mathfrak{S}_6$, where $(\mathbb{Z}/2\mathbb{Z})^5$ consists of automorphisms induced from translations by elements of $J(C)_2 \cup S(C)$

and $\mathfrak{S}_6$ acts on the index set $\{1, \cdots, 6\}$. We took our notations as above because this $\mathfrak{S}_6$-action is best seen.

Translations with respect to $\alpha \in J(C)_2$ are geometrically realized on $J(C)$. They induce automorphisms t_α of X. These are the *translations* in the classical terms. In the next section we will see that translations with respect to $\beta \in S(C)$ are also geometrically realized by $\sigma_\beta \in \mathrm{Aut}(X)$. These σ_β are the *switches*. On the other hand, in general the action of $\mathfrak{S}_6$ cannot be lifted to an automorphism of X.

Remark: In [11] and [12], the notations are a little different. To adjust notations of [12] to ours, first we regard p_0 of [12] as our p_6. Then the correspondence is as in below.

[12]	N_0	N_i	N_{ij}	T_0	T_i	T_{ij}
ours	N_0	N_{i6}	N_{ij}	T_6	T_i	T_{ij6}

Lemma 4.1 ([12])**.** For $\beta \in S(C)$, let $\Lambda(\beta) := \{\alpha \in J(C)_2 | (N_\alpha, T_\beta) = 1\}$. Then the divisor class of

$$H = 2T_\beta + \sum_{\alpha \in \Lambda(\beta)} N_\alpha$$

is independent of β and coincides with the pullback of the hyperplane section by the morphism $X \to \overline{X} \subset \mathbb{P}^3$.

Lemma 4.2 ([12])**.** Assume that X is Picard-general.

1. $NS(X)$ is generated over $\mathbb{Z}$ by $\{N_\alpha, T_\beta\}_{\alpha,\beta}$.
2. $\{H, N_\alpha\}_\alpha$ is an orthogonal basis of $NS(X)_\mathbb{Q}$ over $\mathbb{Q}$.
3. A generator set of the discriminant group $A_{NS(X)}$ is given by

$$\mathsf{e}_1 = (N_{26} + N_{12} + N_{36} + N_{13})/2, \mathsf{f}_1 = (N_{16} + N_{12} + N_{46} + N_{24})/2,$$
$$\mathsf{e}_2 = (N_{26} + N_{12} + N_{46} + N_{14})/2, \mathsf{f}_2 = (N_{16} + N_{12} + N_{36} + N_{23})/2,$$
$$\mathsf{g} = H/4 + (N_0 + N_{16} + N_{26} + N_{12})/2.$$

Special sets of nodes. Lastly we mention several special sets of nodes of $\overline{X}$. See also [4]. We identify the set of nodes with $J(C)_2$ which is a 4-dimensional vector space over $\mathbb{F}_2$. We have then the symplectic bilinear form

$$([\alpha], [\alpha']) \mapsto \#(\alpha \cap \alpha') \mod 2.$$

A 2-dimensional subspace G is called *Göpel* if it is totally isotropic. The translations of Göpel subgroups are called *Göpel tetrads*. There are 60 Göpel tetrads. A 2-dimensional subspace R which is not Göpel is called *Rosenhain* and its translations *Rosenhain tetrads*. There are 80 Rosenhain tetrads. A *Weber hexad* is a 6-set which can be written as the symmetric difference

of a Göpel tetrad and a Rosenhain tetrad. It can be shown that any Weber hexad is of one of the following forms

$$\{0, ij, jk, kl, lm, mi\} \text{ or } \{ij, jk, ki, il, jm, kn\} \tag{4.1}$$

according to whether it contains 0 or not. There are 192 Weber hexads.

In the following sections, we introduce automorphisms using these special sets.

5 Switches

Switches are one kind of automorphisms found by F. Klein [8]. The freeness in even cases is an easy consequence of the description of [9], although it is implicit there. Let $\beta \in S(C)$. For a smooth point $\overline{a} \in \overline{X}$, which means that the preimage of $\overline{a}$ in $J(C)$ is $\{a, -a\}$, the divisors $t_a(\Theta_\beta)$ and $t_{-a}(\Theta_\beta)$ intersect at two points, which is of the form

$$t_a(\Theta_\beta) \cap t_{-a}(\Theta_\beta) = \{b, -b\}.$$

The switch is defined by $\sigma_\beta : \overline{a} \mapsto \overline{b}$.

More precisely, these switches are defined as the composite of the Gauss map

$$G : \mathbb{P}^3 \supset \overline{X} \dashrightarrow \overline{X}^* \subset (\mathbb{P}^3)^*,$$

which maps a smooth point $\overline{a}$ to $T_{\overline{a}}\overline{X}$, and the projective linear isomorphism

$$F_\beta : \overline{X}^* \to \overline{X},$$

defined for each β. See [9].

σ_β is a birational involution of $\overline{X}$. Hence it induces an involution of X, which we denote by the same σ_β. We can easily check that σ_β interchanges N_α with $T_{\alpha+\beta}$ for $\forall \alpha \in J(C)_2$.

Proposition 5.1. For an even theta characteristic β, σ_β is a free involution on X.

Proof. Suppose a smooth point $\overline{a} \in \overline{X}$ is a fixed point of σ_β. This is equivalent to $t_a(\Theta_\beta) \cap t_{-a}(\Theta_\beta) = \{a, -a\}$ and it is necessary that $a \in t_a(\Theta_\beta)$, $0 \in \Theta_\beta$. This is untrue if β is even.

On the other hand, the divisor N_α is disjoint from $T_{\alpha+\beta}$, so σ_β has no fixed points. □

Remark: (1) The proof above does not use the assumption of being Picard-general. Thus switches for even theta characteristics are always free involutions.
(2) The fixed point set of a switch for an odd theta characteristic is a curve of genus 5, named after Humbert.

Let σ_β be a free switch. In the following computation, we take the case $\beta = [123]$ for simplicity. We can obtain the result for other cases by the action of $\mathfrak{S}_6$. Let K be the (-1)-eigenspace of the action of σ_{123} on $NS(X)$ as in Section 2.

Proposition 5.2. For Picard-general X, K is generated over $\mathbb{Z}$ by the following elements.

$$f = N_{15} - T_{146},\ e_2 = T_{145} - N_{16},\ e_3 = N_{45} - T_6,\ e_4 = T_{123} - N_0,$$
$$e_5 = N_{12} - T_3,\ e_6 = T_{124} - N_{34},\ e_7 = N_{24} - T_{134},$$
$$e_1 = -(1/2)(f + 2e_2 + 3e_3 + 4e_4 + 3e_5 + 2e_6 + e_7).$$

Proof. We can check that $\{f, e_2, \cdots, e_7\}$ spans a sublattice of K isomorphic to $A_7(2)$. We now show $e_1 \in NS(X)$. Modulo $NS(X)$,

$$\begin{aligned} e_1 &\equiv (f + e_3 + e_5 + e_7)/2 \\ &\equiv (N_{15} + N_{45} + N_{12} + N_{24})/2 + (T_{146} + T_6 + T_3 + T_{134})/2 \\ &= (N_{15} + N_{45} + N_{12} + N_{24})/2 + (\frac{H}{2} \cdot 4 - \frac{1}{2} \sum_{\alpha \in \Lambda([146]) \cup \Lambda([6]) \cup \Lambda([3]) \cup \Lambda([134])} N_\alpha)/2 \\ &= (N_{15} + N_{45} + N_{12} + N_{24})/2 + H - \frac{1}{4} \sum_{\alpha \in J(C)_2 - \{[15],[45],[12],[24]\}} 2N_\alpha \\ &\equiv \sum_{\alpha \in J(C)_2} N_\alpha/2. \end{aligned}$$

The blow up $\widehat{J(C)}$ of $J(C)$ at points of $J(C)_2$ is the double cover of X branched exactly over $\cup_\alpha N_\alpha$. Thus $e_1 \in NS(X)$ follows.

Then it is easy to check that $\{e_1, e_2, \cdots, e_7\}$ spans a sublattice of K isomorphic to $E_7(2)$. By Proposition 3.1, they coincide. □

Proposition 5.3. The patching subgroup of σ_{123} is the cyclic group generated by the element $[x = H/4 + (N_0 + N_{12} + N_{23} + N_{31})/2] \in A_{NS(X)}$.

Proof. The facts $x \in NS(X)^*$ and $y := -(e_1 + e_5 + e_7)/4 + e_5/2 + e_6/2 \in K^*$ are easily checked. We use Lemma 2.5. We first check $x - y \in NS(X)$. This is because

$$\begin{aligned} y &= (1/8)(f + 2e_2 + 3e_3 + 4e_4 + 5e_5 + 6e_6 - e_7) \\ &= H/4 - (N_{14} + N_{24} + N_{34} + N_{56})/2 \end{aligned}$$

and

$$\begin{aligned} x - y &= (1/2)(N_0 + N_{12} + N_{23} + N_{31} + N_{14} + N_{24} + N_{34} + N_{56}) \\ &\equiv T_{123} - T_4 \equiv 0. \end{aligned}$$

Thus $[x] \in \Gamma_{\sigma_{123}}$. Then since $[x]$ is of order 4 in $A_{NS(X)}$ and $\#\Gamma_{\sigma_{123}} = 4$, $\Gamma_{\sigma_{123}}$ is generated by $[x]$. □

Observation: In the expression of $[x]$, $\{n_0, n_{12}, n_{23}, n_{31}\}$ is a Rosenhain subgroup defined in Section 4. The class of $-x$ can be written as $[H/4+(N_0+N_{45}+N_{56}+N_{64})/2]$, where $\{n_0, n_{45}, n_{56}, n_{64}\}$

is also a Rosenhain subgroup. In general, for an even theta characteristic β, the 6-set $\Lambda(\beta)$ (see Lemma 4.1) can be uniquely written in the form $R_1 \ominus R_2$ where R_i are Rosenhain subgroups. In our case $\beta = [123]$, $R_1 = \{n_0, n_{12}, n_{23}, n_{31}\}$ and $R_2 = \{n_0, n_{45}, n_{56}, n_{64}\}$.

Proposition 5.4. The patching subgroup of σ_β for general β is generated by $[H/4+(\sum_{\alpha\in R} N_\alpha)/2]$ where R is one of the two Rosenhain subgroups corresponding to β.

Proof. When $\beta = [123]$, this is Proposition 5.3. Since the action of $\mathfrak{S}_6$ is compatible with the observation above, the general case follows. □

By Proposition 5.4, we can write down the generator of the patching subgroup of the switch σ_β for all β. We use the notations of Lemma 4.2.

β	[123]	[124]	[125]	[126]	[134]
	$\mathsf{e}_1+\mathsf{f}_2+\mathsf{g}$	$\mathsf{e}_2+\mathsf{f}_1+\mathsf{g}$	$\mathsf{e}_1+\mathsf{f}_1+\mathsf{e}_2+\mathsf{f}_2+\mathsf{g}$	g	$\mathsf{f}_1+\mathsf{f}_2+\mathsf{g}$

β	[135]	[136]	[145]	[146]	[156]
	$\mathsf{f}_1+\mathsf{g}$	$\mathsf{e}_1+\mathsf{g}$	$\mathsf{f}_2+\mathsf{g}$	$\mathsf{e}_2+\mathsf{g}$	$\mathsf{e}_1+\mathsf{e}_2+\mathsf{g}$

Since all these are distinct each other, we see that the ten switches are not conjugate each other in $\mathrm{Aut}(X)$ if X is Picard-general.

6 Hutchinson's involutions associated with Göpel tetrads

These automorphisms appear in [7]. The generic freeness is found by J. H. Keum in [10]. We briefly recall the construction. Let G be a Göpel tetrad. If we choose G as the reference points of the homogeneous coordinates of $\mathbb{P}^3$, the equation of $\overline{X}$ becomes

$$A(x^2t^2+y^2z^2)+B(y^2t^2+z^2x^2)+C(z^2t^2+x^2y^2)+Dxyzt$$
$$+E(yt+zx)(zt+xy)+G(zt+xy)(xt+yz)+H(xt+yz)(yt+zx)=0,$$

for suitable scalars $A, \cdots, H$. σ_G is the Cremona involution

$$(x,y,z,t)\mapsto(1/x,1/y,1/z,1/t).$$

For a translation $t = t_\alpha$, we have $\sigma_{t(G)} = t\sigma_G t$, so that we can restrict ourselves to the case G is a Göpel subgroup. But any Göpel subgroup is of the form $\{n_0, n_{ij}, n_{kl}, n_{mn}\}$ hence up to $\mathfrak{S}_6$ we can assume $G_0 = \{n_0, n_{12}, n_{34}, n_{56}\}$. By [11], the induced action of σ_{G_0} on $NS(X)$ is given by

$$N_\alpha \leftrightarrow H - N_0 - N_{12} - N_{34} - N_{56} + N_\alpha, \quad \text{for } \alpha = [0], [12], [34], [56]$$
$$T_1 \leftrightarrow T_2,\ T_3 \leftrightarrow T_4,\ T_5 \leftrightarrow T_6,$$
$$T_{134} \leftrightarrow T_{234},\ T_{123} \leftrightarrow T_{124},\ T_{125} \leftrightarrow T_{126}.$$

Proposition 6.1. The (-1)-eigenspace K of σ_{G_0} is generated over $\mathbb{Z}$ by the following elements.

$$g = N_0 + N_{12} + N_{34} + N_{56} - H, e_5 = T_1 - T_2,\ e_1 = T_3 - T_4,\ e_7 = T_5 - T_6,$$
$$f = T_{134} - T_{234},\ e_3 = T_{123} - T_{124},\ h = T_{125} - T_{126},$$
$$e_2 = (1/2)(f + g + h - e_3), e_4 = (1/2)(f - e_1 - e_3 - e_5),$$
$$e_6 = (1/2)(f + h - e_5 - e_7).$$

Proof. $e_1, e_3, e_5, e_7, f, g, h \in K$ generate a sublattice of K isomorphic to $A_1(2)^{\oplus 7}$. It is easy to see that $e_2, e_4, e_6 \in NS(X)$. For example, modulo $NS(X)$,

$$\begin{aligned} e_2 &\equiv (1/2)(H + N_0 + N_{12} + N_{34} + N_{56} \\ &\quad + T_{123} + T_{124} + T_{125} + T_{126} + T_{134} + T_{234}) \\ &= 2H + N_0 - (1/2) \sum_{\alpha \in J(C)_2} N_\alpha. \end{aligned}$$

and as in Section 5 $e_2 \in NS(X)$. e_4, e_6 are similar.

Then we see that $e_1, \cdots, e_7$ span a lattice isomorphic to $E_7(2)$. □

Proposition 6.2. The patching subgroup of σ_{G_0} is 2-elementary abelian and generated by

$$x = (N_0 + N_{12} + N_{34} + N_{56})/2, \text{ and } y = H/2.$$

Proof. This proposition is proved in the same way as Proposition 5.3. The corresponding element in K^*/K is $x' = (e_1 + e_3)/2, y' = (e_1 + e_5 + e_7)/2$ and we can check $x - x', y - y' \in NS(X)$. Then we use Lemma 2.5. □

By the $\mathfrak{S}_6$-symmetry, we obtain the following.

Proposition 6.3. For any Göpel subgroup G, we have $\Gamma_{\sigma_G} = \langle H/2, (1/2) \sum_{\alpha \in G} N_\alpha \rangle$.

More generally, ussing the translation relation $\sigma_{t(G)} = t\sigma_G t$, the generator above is valid for any Göpel tetrad.

There are 15 Göpel subgroups. Under the notations of Lemmas 4.1 and 4.2, we deduce the following table.

The tetrad	Patching element corresponding to x
$[0]+[12]+[34]+[56]$	$\mathsf{e}_1+\mathsf{f}_1+\mathsf{e}_2+\mathsf{f}_2$
$[0]+[12]+[35]+[46]$	$\mathsf{f}_1+\mathsf{e}_2$
$[0]+[12]+[36]+[45]$	$\mathsf{e}_1+\mathsf{f}_2$
$[0]+[13]+[24]+[56]$	$\mathsf{e}_1+\mathsf{f}_1+2\mathsf{g}$
$[0]+[13]+[25]+[46]$	$\mathsf{e}_1+\mathsf{f}_1+\mathsf{f}_2+2\mathsf{g}$
$[0]+[13]+[26]+[45]$	f_2
$[0]+[14]+[23]+[56]$	$\mathsf{e}_2+\mathsf{f}_2+2\mathsf{g}$
$[0]+[14]+[25]+[36]$	$\mathsf{f}_1+\mathsf{e}_2+\mathsf{f}_2+2\mathsf{g}$
$[0]+[14]+[26]+[35]$	f_1
$[0]+[15]+[23]+[46]$	$\mathsf{e}_1+\mathsf{e}_2+\mathsf{f}_2+2\mathsf{g}$
$[0]+[15]+[24]+[36]$	$\mathsf{e}_1+\mathsf{f}_1+\mathsf{e}_2+2\mathsf{g}$
$[0]+[15]+[26]+[34]$	$\mathsf{f}_1+\mathsf{f}_2$
$[0]+[16]+[23]+[45]$	e_1
$[0]+[16]+[24]+[35]$	e_2
$[0]+[16]+[25]+[34]$	$\mathsf{e}_1+\mathsf{e}_2$

Since all these are distinct each other, we see that the 15 Hutchinson involutions are not conjugate each other in $\mathrm{Aut}(X)$ if X is Picard-general.

Remark: In [13] it is shown that if (C, G) is bielliptic, then the involution σ_G cannot be defined.

7 Hutchinson's involutions associated with Weber hexads

These automorphisms appear in [5, 6]. The freeness is found in [4]. We fix a Weber hexad W. Then the linear system $L = |\mathcal{O}_{\overline{X}}(2) - W|$ with the assigned base points at W defines an another quartic model $\overline{X}_W$ of X in $\mathbb{P}^4$,

$$\overline{X}_W : \ s_1 + \cdots + s_5 = 0, \ \lambda_1/s_1 + \cdots + \lambda_5/s_5 = 0,$$

where λ_i are nonzero constants.

σ_W is the Cremona involution

$$\sigma_W : (s_1, \cdots, s_5) \mapsto (\lambda_1/s_1, \cdots, \lambda_5/s_5),$$

preserving $\overline{X}_W$. It is free if X is Picard-general [4]. For any translation $t = t_\alpha$, we have $\sigma_{t(W)} = t\sigma_W t$ as in the Hutchinson case. Hence we can assume that the Weber hexad doesn't contain n_0. Then recalling (4.1) in Section 4, we have only one Weber hexad $W_0 = \{n_{12}, n_{23}, n_{31}, n_{14}, n_{25}, n_{36}\}$ up to the action of $\mathfrak{S}_6$. In the following we discuss this case.

Lemma 7.1 ([4])**.** σ_{W_0} interchanges the following 10 pairs of smooth rational curves.

$$(N_0, T_{123}), (N_{56}, T_1), (N_{46}, T_2), (N_{45}, T_3), (N_{15}, T_{124}),$$
$$(N_{16}, T_{134}), (N_{24}, T_{125}), (N_{26}, T_{146}), (N_{34}, T_{136}), (N_{35}, T_{236}).$$

Proposition 7.2. The (-1)-eigenspace K of σ_{W_0} is generated over $\mathbb{Z}$ by the following elements.

$$e_1 = T_2 - N_{46},\ e_2 = N_{15} - T_{124},\ e_3 = T_1 - N_{56},$$
$$e_4 = N_0 - T_{123},\ e_5 = T_3 - N_{45},\ e_6 = N_{34} - T_{136},$$
$$e_7 = N_{23} - N_{56} - N_{34} - N_{24} - T_{134} - T_{124}.$$

Proof. By computing the determinant, we can see that 10 divisors

$$N_0 + T_{123}, \cdots, N_{35} + T_{236} \tag{7.1}$$

from Lemma 7.1 span over $\mathbb{Q}$ the invariant sublattice. $e_1, \cdots, e_6 \in K$ is easy. $e_7 \in K$ follows from the fact that e_7 is orthogonal to all of the divisors in (7.1). Then $e_1, \cdots, e_7$ spans the lattice $E_7(2) \simeq K$. □

Remark: The action of σ_W on $NS(X)$ is very complicated, but essentially we can write down this action using the proposition above. In fact we find the following.

Let W be a general Weber hexad. The "degree 1 part" W_1 of W is the set

$$\{\beta \in S(C) | (T_\beta, \sum_{\alpha \in W} N_\alpha) = 1\}.$$

W_1 consists of 6 elements. We have a natural bijection $\mu : W \to W_1$ defined by $(N_\alpha, T_{\mu(\alpha)}) = 1$. On the other hand, for $\alpha \notin W$, we have the unique decomposition

$$W = G \ominus R,\ G \cap R = \{n_\alpha\}.$$

Let $R^\perp$ be the 2-dimensional affine subspace of $J(C)_2$ which is orthogonal to R and contains n_α. Then $R \ominus R^\perp$ is a Rosenhain hexad, i.e., $R \ominus R^\perp$ is of the form $\Lambda(\beta)$ for some $\beta \in S(C)$. This defines a bijection $\mu' : J(C)_2 - W \to S(C) - W_1$, $\alpha \mapsto \mu'(\alpha) = \beta$. Using these data, the action of σ_W is as follows.

$$\sigma_W(N_\alpha) = 3H - (\sum_{\alpha \in J(C)_2} N_\alpha)/2 - (\sum_{\alpha \in W} N_\alpha) - T_{\mu(\alpha)},\ \text{if } \alpha \in W.$$
$$\sigma_W(N_\alpha) = T_{\mu'(\alpha)} \text{ if } \alpha \notin W.$$
$$\sigma_W(H) = 9H - (\sum_{\alpha \in J(C)_2} N_\alpha) - 4(\sum_{\alpha \in W} N_\alpha).$$

Proposition 7.3. The patching subgroup of σ_{W_0} is cyclic and generated by

$$x = (3/4)H + (1/2)(N_{12} + N_{23} + N_{31} + N_{14} + N_{25} + N_{36}).$$

Proof. The corresponding element in K^*/K is

$$y = \frac{1}{4}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_4 + \frac{3}{4}e_5 + \frac{1}{4}e_7,$$

and we check $x - y \in NS(X)$. □

By the $\mathfrak{S}_6$-symmetry and the translation relation, we obtain

Proposition 7.4. For general W, the patching subgroup of σ_W is

$$\Gamma_{\sigma_W} = \langle (3/4)H + (\sum_{\alpha \in W} N_\alpha)/2 \rangle.$$

There are 12 Weber hexads modulo translations. One more relation is hidden in the remark above. For $\alpha \notin W$, we have the unique decomposition $W = G \ominus R$, $G \cap R = \{n_\alpha\}$. Let $R^\perp$ be the orthogonal complement of R at n_α and let $W_\alpha^\perp$ be the Weber hexad $G \ominus R^\perp$. Then σ_W and $\sigma_{W_\alpha^\perp}$ are conjugate, related by $\sigma_{W_\alpha^\perp} = \sigma_{\mu'(\alpha)} \sigma_W \sigma_{\mu'(\alpha)}$. Modulo this relation, we have 6 Weber hexads. Under the notations of Lemmas 4.1 and 4.2, their patching subgroups are as follows.

Weber hexad	patchings
$[12]+[23]+[31]+[14]+[25]+[36]$	$\mathsf{e}_1+\mathsf{f}_1+\mathsf{e}_2+\mathsf{g}$
$[12]+[13]+[23]+[24]+[15]+[36]$	$\mathsf{f}_1+\mathsf{e}_2+\mathsf{f}_2+\mathsf{g}$
$[23]+[13]+[12]+[34]+[25]+[16]$	$\mathsf{e}_2+\mathsf{f}_2+\mathsf{g}$
$[24]+[23]+[34]+[14]+[25]+[36]$	$\mathsf{e}_1+\mathsf{f}_1+\mathsf{f}_2+\mathsf{g}$
$[25]+[23]+[35]+[54]+[21]+[36]$	$\mathsf{e}_1+\mathsf{e}_2+\mathsf{f}_2+\mathsf{g}$
$[26]+[23]+[36]+[64]+[25]+[13]$	$\mathsf{e}_1+\mathsf{f}_1+\mathsf{g}$

Thus we see that there are 6 HW involutions that are not conjugate each other in $\mathrm{Aut}(X)$ if X is Picard-general.

Remark: (1) In a forthcoming paper we will be able to determine when σ_W is not free.

(2) The 6 conjugacy classes of HW involutions are naturally "dual" to the 6 Weierstrass points, in the sense that the $\mathfrak{S}_6$ action on both factors through an outer automorphism. Details are as follows. There are 20 Weber hexads W which don't contain n_0 and conjugate each other. Writing W uniquely as $W = G \ominus R$ with $G \cap R = \{n_0\}$, we can associate with such W the Göpel subgroup G. But a Göpel subgroup $G = \{n_0, n_{ij}, n_{kl}, n_{mn}\}$ is determined just by the "syntheme" $(ij)(kl)(mn) \in \mathfrak{S}_6$. Thus we obtain 20 synthemes from the conjugacy class. The fact is that there appear only 10 synthemes, and the synthemes not appearing here form a "total", which is the classical description of the dual of the 6-set $\{1, \cdots, 6\}$.

(3) The method of this paper is applicable to the case of Picard-general quartic Hessian surfaces of [4]. In this case we have exactly one Enriques quotient.

Proof of Theorem 1.2: Let N' be the group generated by 16 translations t_α, 16 switches σ_β, 16 projections p_α, 16 correlations p_β, 60 HG involutions σ_G, 192 HW involutions σ_W. The theorem follows from the following lemma as in [12, Lemma 7.3].

Lemma 7.5. Let ϕ be an isometry of $NS(X)$ that preserves the ample cone. Then there exists a $g \in N'$ such that $g\phi \in \mathrm{Aut}(D')$.

Proof. Let $w' = 2H - \sum N_\alpha/2$ be the projection of the Weyl vector w. Let $v = \phi(w')$ and let $g \in N'$ be an element that attains the minimum $min\{(g(v), w')|g \in N'\}$. If r is the Leech root corresponding to [12, Lemma 4.6, Case (0),(1),(2)], then as in [12] we have $(r', g(v)) > 0$.

If r is the Leech root corresponding to [12, Lemma 4.6, Case (3)], then it corresponds to some Weber hexad W and [12, Remark 6.3, (1)] can be rewritten as

$$4r' = 3H - 2\sum_{\alpha \in W} N_\alpha.$$

Using Proposition 7.2 and its Remark, we have

$$\sigma_W(w') = w' + 8r'.$$

Thus, we have

$$(g(v), w') \leq (g(v), \sigma_W^{-1}(w')) = (g(v), w') + 8(g(v), r'),$$
$$(g(v), r') > 0$$

Hence $g(v) \in D'$. □

Remark: Unfortunately, $\mathrm{Aut}(X)$ cannot be generated only by the subset

$$S = \{t_\alpha, \sigma_\beta, \sigma_G, \sigma_W | \alpha \in J(C)_2, \beta \in S(C), G : \text{Göpel tetrad}, W : \text{Weber hexad}\},$$

introduced in this paper. It is easy to see that for any element φ written as a product of elements in S, we have $(w', \varphi(w')) \in 4\mathbb{Z}$. But the projection p_α have $(w', p_\alpha(w')) = 26$.

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Abstract

This paper classifies Enriques surfaces whose $K3$-cover is a fixed Picard-general Jacobian Kummer surface. There are exactly 31 such surfaces. We describe the free involutions which give these Enriques surfaces explicitly. As a biproduct, we show that $\mathrm{Aut}(X)$ is generated by elements of order 2, which is an improvement of the theorem of S. Kondo.