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Abstract

We consider some special reductions of generic Darboux-Crum dressing formulas and of their difference versions. As a matter of fact we obtain some new formulas for Darboux-Pöschl-Teller potentials, their difference deformations and the related eigenfunctions.

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1 Darboux-Pöschl-Teller equation and Darboux-Crum dressing formula

1.1 How to get DPT equation and its solutions from Wronskian determinants.

Darboux-Pöschl-Teller equation, (we often use the abbreviation DPT equation for it), reads

$$-y'' + \left(\frac{n(n+1)\alpha^2}{\cos^2 \alpha x} + \frac{m(m+1)\alpha^2}{\sin^2 \alpha x} \right) y = \lambda y. \tag{1}$$

Here $\alpha > 0$ and m, n are some positive integers. DPT equation is obviously invariant with respect to the action of each of the maps: $m \to -m-1, \quad n \to -n-1$. The integrability of (1) was proved in 1882 by Darboux.
In [7] he proposed two independent methods for solving (1). First, taking $\sin^2(\alpha x)$ as a new independent variable, he reduced (1) to Gauss hypergeometric equation and expressed the solutions in terms of the hypergeometric functions. Darboux also remarked that the functions $y(m,n) := \cos^{n+1}(\alpha x) \sin^{m+1}(\alpha x)$ represent the particular solution of (1) with $\lambda = \alpha^2(m+n+2)^2$. Taking these solutions as generating functions of some sequence of Darboux transformations connecting the potentials with different values of m and n, he obtained the general solution of the same equation in terms of elementary functions. Comparison of these two forms of solution provides the nontrivial case of reduction of the Gauss hypergeometric function to elementary functions. Almost 50 years later both trigonometric DPT equation, considered here, and its hyperbolic version, (obtained by replacing α by $i\alpha$) were independently solved by Pöschl and Teller [22], (in a context of their studies of quantum theory of the two atomic molecules), using the Frobenius series approach. They also arrived to a hypergeometric representation for the solutions to the DPT equation ignoring somehow the elementary function solutions trivially following from the results of Darboux. In the later literature, (see [8, 9, 12] for further references and results), the DPT equation was mainly called the Pöschl-Teller equation. We are somehow convinced that the name of Darboux should be added to label of equation which he solved first.[1]

We present here some new representation for the solution to the DPT equation by means of Wronskian determinants. In order to explain the related result let us recall the following statement known as Darboux-Crum dressing formula.

In a sequel $W(f_1, f_2, \ldots, f_m)$ denotes a Wronskian determinant of m functions $f_j(x)$ i.e. $W(f_1, f_2, \ldots, f_m) = \det ||\partial_x^{j-1} f_k(x)||; \quad j, k = 1, \ldots, m$.

Theorem 1.1 *Suppose that f_j, f, $\quad j = 1 \ldots m$ satisfy the Schrödinger equa-*

[1] In fact Darboux not only solved the DPT equation but he also introduced and solved in a very elegant way its 4-integers elliptic extension [5], rediscovered 100 years later by Verdier and Treibich in a context of considering special reductions of the generic finite gap potentials described by Its-Matveev formula.

tions

$$-f_j'' + v(x)f_j = E_j f_j, \quad -f'' + v(x)f = Ef,$$

Then the function ψ defined by the formula

$$\psi(x,E) =: \frac{W(f_1, f_2, \ldots, f_m, f)}{W(f_1, f_2, \ldots, f_m)}, \tag{2}$$

satisfies the Schrödinger equation

$$-\psi'' + u\psi = E\psi, \quad u =: v - 2\partial_x^2 \log W(f_1, f_2, \ldots, f_m). \tag{3}$$

In the case $m = 1$, this proposition was found and proved by Darboux [7], and in general case by Crum [4].
Surprisingly the link between the later formula and DPT potentials was explained only in 2002 [12].
In order to explain the connection between Darboux-Crum dressing theorem and DPT equation we have to answer the following question. How to choose the numbers $E_j, j = 1, \ldots, m$, and the functions $f_1, \ldots, f_m$, satisfying $f_j'' = E_j f_j$ in order to get (1) from the Schrödinger equation (3) with $v = 0$? After finding the reply to this question, obviously, the general solution to (1) will be given by (2). The precise description of the corresponding choice of $E_j, f_j(x)$ is given by the following statement first proved in [12]. In this section we give partially different and simpler proof of the same result.

Theorem 1.2 *Let $v(x) = 0$; m, n, $(m \geq n)$, are the same non negative integers as in (1), $d =: m - n$, $\alpha > 0$ and E_p, $f_p(x)$, $p = 1, \ldots m$ are defined as follows:*

$$E_p =: (\alpha a_p)^2, \quad f_p(x) = \sin \alpha a_p x, \quad a_p = p, \quad d \neq 0 \quad 1 \leq p \leq m-n,$$
$$a_{m-n+j} = m - n + 2j, \quad n \neq 0, \quad 1 \leq j \leq n. \tag{4}$$

[2] *Then the following Wronskian representation for the DPT potential $u_{mn}(x)$ holds:*

$$u_{mn}(x) := \frac{n(n+1)\alpha^2}{\cos^2 \alpha x} + \frac{m(m+1)\alpha^2}{\sin^2 \alpha x} = -2\partial_x^2 \log W(f_1, f_2, \ldots, f_m). \tag{5}$$

The general solution to (1) is given by (2). Moreover, in this case the Wronskian $W(f_1, \ldots, f_m)(m, n, \alpha, x) \equiv W_{mn}$ can be explicitly computed:

$$W_{m0}(\alpha) = (-2\alpha)^{\frac{m(m-1)}{2}} \prod_{k=0}^{m-1} k! \cdot \sin^{\frac{m(m+1)}{2}}(\alpha x), \quad W_{nn}(\alpha) = W_{n0}(2\alpha). \tag{6}$$

$$W_{mn} = c_{mn} \cdot \sin^{\frac{m(m+1)}{2}}(\alpha x) \ \cos^{\frac{n(n+1)}{2}}(\alpha x), \tag{7}$$

$$c_{mn} = (-2\alpha)^{\frac{m(m-1)}{2}} \cdot 2^{n^2} \prod_{p=0}^{d-1} p! \prod_{k=1}^{n} \frac{(d+2k-1)!}{(2k-1)!}(k-1)!, \qquad m > n,$$

[2]In particular, for the case $m = n$ we have $a_j = 2j, \quad j = 1, \ldots n$.

$$c_{mm} = (-2\alpha)^{\frac{m(m-1)}{2}} \cdot 2^{m^2} \prod_{k=1}^{m-1} k!, \qquad m \geq 2.$$

Example

$$W(\sin x, \sin 2x, \sin 4x, \sin 6x, \sin 8x) = 2^{24} \cdot 5 \cdot 7 \cdot 9 \cos^6(x) \sin^{15}(x)$$

$$= 5284823040 \cos^6(x) \sin^{15}(x).$$

Hint of the Proof: The crucial part of the proof is to demonstrate the formulas (6), (7) which can be done in two steps. First, one have to establish the formula (6), (See [12] for details), which is also enough for calculate W_{nn} because of the relation $W_{n0}(2\alpha) = W_{nn}(\alpha)$. Next part of proof sketched below is different from that of [12].
Denote by g_j the functions

$$g_j := \sin(\alpha j x), \quad j \in \mathbb{Z}.$$

Then it is easy to see that

$$W_{mn} = W(g_1, g_2, \dots, g_{m-n}, g_{m-n+2}, g_{m-n+4}, \dots, g_{m+n}).$$

We define the sequence of Wronskians $W_{m-k,n}$ of order $m-k$ by the formulas:

$$W_{m-1,n} = W(g_1, g_2, \dots, g_{m-n-1}, g_{m-n+1}, g_{m-n+3}, \dots, g_{m+n-1}),$$

$$W_{m-k,n} = W(g_1, g_2, \dots, g_{m-n-k}, g_{m-n-k+2}, g_{m-n-k+4}, \dots, g_{m-k+n}),$$

$$W_{n,n} = W(g_2, g_4, \dots, g_{2n}).$$

With these notations it is easy to prove the recursion relation

$$W_{m-k,n} = g_1^{m-k}(-2\alpha)^{m-k-1}(m-k-1-n)!$$
$$\cdot \prod_{j=1}^{n}(m-k-1-n+2j) \cdot W_{m-k-1,n}, \quad 0 \leq k \leq m-n-1. \tag{8}$$

Applying the recursion relation (8), $(m-n)$ times (for $0 \leq k \leq m-n-1$) we obtain:

$$W_{mn} = [\sin(\alpha x)]^{(m-n)(m+n+1)/2}(-2\alpha)^{(m-n)(m+n-1)/2}$$
$$\cdot \prod_{j=0}^{m-n-1} j! \prod_{k=1}^{m-n} \prod_{j=1}^{n}(k+2j-1) \cdot W_{nn}. \tag{9}$$

Substituting the RHS of (6) in the RHS of (9) we get (7) which completes the proof. □

1.2 The action of the KdV hierarchy on the DPT potentials

The advantage of the Wronskian representation for the DPT potentials is that it allows to describe immediately the action of KdV and higher flows of the KdV hierarchy on those potentials taken as initial data. The j-th equation of the KdV hierarchy can be obtained as a compatibility condition of (3) and the following evolution equation:

$$\partial_{t_j} f = c_j \partial_x^{2j+1} f - \frac{(2j+1)c_j}{2} u(x,t_j)\partial_x^{2j-1} f + \sum_{p=0}^{2j-2} u_p(x,t_j)\partial_x^p f, \quad c_j \in \mathbb{R}.^3 \tag{10}$$

In particular for $j = 1$, $c_j = -4$ and $u_0 = 3u_x$ compatibility of (1) and (10) implies that v satisfies the KdV equation:

$$u_t = 6\,u\,u_x - u_{xxx}. \tag{11}$$

Then the following formula describes the solution to the j-th KdV equation satisfying the initial condition $u(x, 0) = u_{mn}(x)$:

$$u(x,t_j) = -2\partial_x^2 \log W(f_1, f_2, \ldots, f_m). \tag{12}$$

$$f_p(x,t) = \sin[\alpha p x + (-1)^j c_j \alpha^{2j+1} p^{2j+1} t_j], \;\; p = 1, \ldots, d, \;\; d = m - n,$$

$$f_{d+l}(x,t) = \sin[\alpha(d+2l)x + (-1)^j c_j \alpha^{2j+1}(d+2l)^{2j+1} t_j], \;\; l = 1, \ldots, n, \;\; m \geq n,$$

In particular, the solution of (11) with the initial condition (5) is given by the formula (12) with

$$f_p(x,t) := \sin(\alpha p x + 4\alpha^3 p^3 t), \;\; p = 1, \ldots, d, \;\; d = m - n,$$

$$f_{d+l}(x,t) := \sin[\alpha(d+2l)(x + 4\alpha^2(d+2l)^2 t], \;\; l = 1, \ldots, n, \;\; m \geq n.$$

1.3 Differential identities

Formulas (6) and (7) lead to some nontrivial identities. Let $\widehat{D}$ be a differential operator defined as follows,

$$(\widehat{D} f)(x) = \left(\prod_{l=2}^{m} \partial_{x_l}^{l-1} \right) f(x_1, x_2, \ldots, x_m) \Big|_{x_1 = x_2 = \ldots = x_m = x}. \tag{13}$$

Let D_m be the following determinant: $D_m = \det a_{j,k}$, $\;a_{j,k} = \sin jx_k$, with $1 \leq j, \;\; k \leq m$.
Clearly, we have,

$$\widehat{D} D_n = \det \partial_x^{j-1} \sin kx = W(\sin x, \sin 2x, \ldots, \sin nx) = W_{n0}. \tag{14}$$

[3] In general the choice of the real constants c_j is arbitrary: it just fixes normalization of the KdV hierarchy.

The determinant D_n can be easily computed [23]):

$$\det \| \sin(jx_k) \| = 2^{n(n-1)} \prod_{j=1}^{n} \sin x_k \prod_{1\le l<k\le n} \sin \frac{x_l + x_k}{2} \sin \frac{x_l - x_k}{2}$$

$$= 2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n} \sin x_k \prod_{n\ge k>j\ge 1} (\cos x_k - \cos x_j), \quad j,k = 1,\dots n. \tag{15}$$

Now, substituting this expression for D_n in (14) and taking into account the formula (7) for W_{n0}, we get the following identity :

$$\left(\prod_{l=2}^{n} \partial_{x_l}^{l-1}\right) \left[\left(\prod_{k=1}^{n} \sin x_k\right) \cdot \left(\prod_{m\ge k>j\ge 1} (\cos x_j - \cos x_k)\right)\right] \Bigg|_{x_1=x_2=\ldots=x_n=x}$$

$$= \prod_{k=1}^{n-1} k! \cdot \sin^{\frac{n(n+1)}{2}} x\,, \qquad n \ge 2. \tag{16}$$

Further examples of the related differential identities can be found in [12].

2 Casorati determinant approach to the DDPT-I model

Here we discuss a family of integrable deformations of the DPT-model, representing the special case of the difference equation[4]

$$v(x)f(x+h) + f(x-h) = \lambda f(x), \quad h > 0. \tag{17}$$

In the sequel we use the following abbreviations
$c(x) := \cos \alpha x$, $s(x) := \sin \alpha x$, $\alpha > 0$.
Below we consider the special potentials $v = v_{mn}(x, h, \alpha)$, defined as follows :

$$v_{mn}(x) := \frac{c(x+(n+1)h)c(x-nh)s(x+(m+1)h)s(x-mh)}{c(x)c(x+h)s(x)s(x+h)} \tag{18}$$

$$= v_{n0}(x)v_{0m}(x). \tag{19}$$

Equation (17) with $v(x) = v_{mn}(x)$ is called below DDPT-I equation.
It has been shown in [20, 21] that DDPT-I model is integrable and its global solution can be expressed by means of the elementary functions.
In the limit $h \to 0$, DDPT-1 equation reduces to the differential DPT equation. The cases $n = 0$, or $m = 0$ were studied in the recent papers [24, 10, 26, 25] using different tools and leading to the formulas different from ours and having more complicated combinatorial structure. The Lattice specialization of the case m=0, corresponding to the choice $h = 1, x = j, j \in \mathbb{Z}$ was studied in [26, 25].

[4] It is important to mention that multiplying any solution of (17) by any h-periodic function of x we obtain again the solution of (17). Therefore, contrary to the lattice case the space of solutions of (17) is infinite dimensional.

Remark 2.1 *Similarly to the DPT equation, the potential $v_{nm}(x)$, is invariant under the action of each of the mappings $m \to -m-1$, and $n \to -n-1$.*

2.1 Multiple difference Darboux transform and DDPT-I model

First, we recall a simplest case of Matveev's dressing theorem [16] for the case of second order functional difference equation. For further generalizations and non Abelian and non stationary extensions see [16].

In the sequel the notation $\delta_n(x) = \delta_n[f_1, f_2, \ldots, f_n](x)$ will be used for the following Casorati determinant:

$$\delta_n(x) = \det A, \quad A_{ij} := f_j(x - (n+1-2i)h), \quad i, j = 1, \ldots, n) \tag{20}$$

We also use below the following notations

$$\phi_n(x) := \frac{\delta_{n+1}[f, f_1, f_2, \ldots, f_n](x)}{\delta_n[f_1, f_2, \ldots, f_n](x+h)}, \; n \geq 1, \; \phi_0(x) := f(x), \tag{21}$$

$$\Phi_j(x) = \phi_{j-1}(x)|_{f(x)=f_j(x)}, \; j = 1, 2, \ldots, n.$$

$$\kappa_j(x) = \frac{\Phi_j(x-h)}{\Phi_j(x+h)}, \tag{22}$$

where f_j, are some fixed solutions to

$$v(x) f_j(x+h) + f_j(x-h) = \lambda_j f_j(x), \tag{23}$$

and f denotes any solution of (17). The function $\phi_n(x)$ defined above is called the N-fold, (difference), Darboux transform of $f(x)$, generated by $f_1, f_2, \ldots, f_n$.

Theorem 2.1 *The function ϕ_n represents the general solution to the functional difference equation*

$$v_n(x)\phi_n(x+h) + \phi_n(x-h) = \lambda \phi_n(x), \tag{24}$$

$$v_n(x) = v(x+nh)\,\frac{\delta_n(x-h)\delta_n(x+2h)}{\delta_n(x)\delta_n(x+h)}. \tag{25}$$

In other words, Darboux transformation maps (17) into equation of the same form, with the same value of spectral parameter λ, but with a new potential constructed in terms of the initial potential $v(x)$ and a fixed solutions $f_j(x)$ of (23). [5] The function $\phi_n(x)$ can be also represented in a following factorized form

$$\phi_n(x) = (T^{-1} - \kappa_n(x)T)\ldots(T^{-1} - \kappa_1(x)T)f(x, \lambda),$$

[5] In fact Darboux established the similar property for more special case of the Sturm-Liouville equation for the case $m = 1$. Its extensions to the case of linear and nonlinear PDE of any order and their difference and non Abelian versions were proposed by Matveev in 1979 [15, 16, 17].

where T is the shift operator : $T^{\pm k}f(x) = f(x \pm kh)$. By this reason it is often called the N-fold Darboux transformation of $f(x)$. Below the theorem 2.1 will be applied to the case $v(x) = 1$ in order to show that the potentials $v_{mn}(x)$ allow a natural representation in a form of the ratio of four Casorati determinants. To prove this we need to calculate some special Casorati determinants.

2.2 Casorati addition formula for sine functions

Let $m = n+d$ where n and d be some non negative integers, α a real parameter. Suppose that the functions $f_j(x)$ are defined as in (4). Then they satisfy to (23), with $v = 1$ and $\lambda_j = 2\cos(\alpha a_j h)$.

We define the determinant $\delta_m(x, n)$ by the formula :

$$\delta_m(x,n) = \det A, \quad A_{ij} := f_j((x-(m+1-2i)h)), \quad i,j = 1\dots m. \tag{26}$$

Theorem 2.2 [6] *The determinant $\delta_m(x, n)$ is equal to the following product :*
a. $m \geq 2$, $1 \leq n \leq$ *m-1,*

$$\delta_m(x,n) = c_{mn} \cdot \prod_{k=-m+1}^{m-1} \sin^{[\frac{m+1-|k|}{2}]} \alpha(x+kh) \prod_{k=-n+1}^{n-1} \cos^{[\frac{n+1-|k|}{2}]} \alpha(x+kh),$$

$$c_{mn} = 2^{\frac{n(n+1)}{2}}(-4)^{\frac{m(m-1)}{2}} \prod_{l=0}^{m-n-1}\prod_{j=1}^{n} \sin\alpha(l+2j)h \cdot \prod_{j=1}^{m-n} \sin^{m-n-j}(\alpha jh) \cdot \prod_{j=1}^{n} \sin^{n-j}(2\alpha jh). \tag{27}$$

b. $m \geq 2$, $n= 0$

$$\delta_m(x,0) = (-4)^{\frac{m(m-1)}{2}} \prod_{j=1}^{m-1} \sin^{m-j}(\alpha jh) \prod_{k=-m+1}^{m-1} \sin^{[\frac{m+1-|k|}{2}]} \alpha(x+kh). \tag{28}$$

c. $m \geq 2$, $n = m$

$$\delta_m(x,m,\alpha) = \delta_m(x,0,2\alpha). \tag{29}$$

Proof : We present here only a hint of the proof. The complete proof can be found in a forthcoming paper [14]. The idea of the proof is first, to establish a recursion relation of the form

$$\delta_{m-k}(x,n) = 2^{2(m-k-1)}(-1)^{m-k-1} \prod_{j=1}^{m-k} \sin(\alpha(x-(m-k+1-2j)h)) \cdot \prod_{j=1}^{m-k-n+1} \sin(\alpha jh)$$

$$\cdot \prod_{j=1}^{n} \sin(\alpha(m-k-n-1+2j)h) \cdot \delta_{m-k-1}(x,n),$$

[6] Here we use the standard notation $[x]$ for the integer part of x, and $|x|$ for its absolute value.

for $0 \leq k \leq m-n-1$.
Repeated application of this recursion relation leads to the formula :

$$\delta_m(x,n) = \prod_{k=n+1}^{m} 2^{2(k-1)}(-1)^{k-1} \prod_{j=1}^{k} \sin\alpha(x-(k+1-2j)h) \prod_{j=1}^{n} \sin\alpha(k-n-1+2j)h$$

$$\cdot \prod_{k=n+2}^{m} \prod_{j=1}^{k-n-1} \sin(\alpha jh) \cdot \delta_n(x,n). \tag{30}$$

But it is obvious that $\delta_n(x,n,\alpha) = \delta_n(x,0,2\alpha)$. So the computation of $\delta_m(x,n)$ is reduced to calculate $\delta_n(x,0)$, which is trivial to do, using the formula (15) already mentioned above.

Therefore, combining (30) and (15), we get (27) which achieves the proof.□

2.3 Applications to DDPT-I model

The integrability of the DDPT-I was proved "à la Darboux" in [20, 21]. exploring the factorized representation (26) of the general solution and the following statement.

Proposition 2.1 *The function*

$$F_1(x,m,n) = \prod_{k=0}^{n} c(x-kh) \prod_{j=0}^{m} s(x-jh), \tag{31}$$

satisfies the DDPT-I equation with $\lambda = 2\cos\alpha h(n+m+2)$.

Here, we present the following new result :

Theorem 2.3 *The DPT potentials* $v_{mn}(x)$ *defined in (18) can be represented as a ratio of four Casorati determinants :*

$$v_{mn}(x) = \frac{\delta_m(x-h,n)\delta_m(x+2h,n)}{\delta_m(x+h,n)\delta_m(x,n)}. \tag{32}$$

The general solution of the DDPT-I equation is given by the formula

$$\psi(x,\lambda) = \frac{\delta_{m+1}[f,f_1,\ldots,f_m](x,n)}{\delta_m[f_1,\ldots,f_m](x+h,n)}, \tag{33}$$

where f *satisfies* $f(x+h)+f(x-h) = \lambda f$ *and* $f_j(x)$ *are defined in (4).*

Hint of the proof : Using formula (27), we obtain that

$$\frac{\delta_{m+1}(x,n+1)}{\delta_m(x+h,n)} = k_{mn} \cdot F_1(x,m,n),$$

$$k_{mn} = 2^{2m+n+1}(-1)^m \prod_{l=0}^{m-n-1} \sin(\alpha h(l+2n+2)) \prod_{j=1}^{n} \sin(2\alpha jh) \tag{34}$$

where k_{mn} does not depend on x. This proves formula (32). [7]

Now, (33) becomes obvious from (21, 24).
□
Thus, we got the new proof of the integrability of the DDPT-I model. Indeed, the representation (32) for DDPT-I potential has the same advantage as the Wronskian representation of the DPT potentials : it allows to obtain automatically the solution to the Cauchy problem for the difference KdV hierarchy with DDPT-I potentials taken as initial data. Hear we skip further discussion of this point for respect the volume limitations.

3 Casorati determinant representation of DDPT-II model

We consider in this section another kind of difference deformations of DPT potentials connected with a difference equation:

$$w(x)f(x+h) + f(x-h) + b(x)f(x) = \lambda f(x). \tag{35}$$

Let $w = w_{mn}$ and $b = b_{mn}$ be the potentials given by :

$$w_{mn}(x) = \frac{c(x-nh/2)c(x-(n-1)h/2)c(x+(n+1)h/2)c(x+(n+2)h/2)}{c(x)c(x+h)(c(x+h/2))^2} \cdot \frac{s(x-mh/2)s(x-(m-1)h/2)s(x+(m+1)h/2)s(x+(m+2)h/2)}{s(x)s(x+h)(s(x+h/2))^2}, \tag{36}$$

and,

$$b_{mn}(x) = -\frac{2s(mh/2)s((m+1)h/2)c(nh/2)c((n+1)h/2)}{c(x-h/2)c(x+h/2)} - \frac{2c(mh/2)c((m+1)h/2)s(nh)s(n+1)h/2)}{s(x-h/2)s(x+h/2)}. \tag{37}$$

We call (35) with $w(x) = w_{mn}(x)$ and $b(x) = b_{mn}(x)$ the DDPT-II equation (difference Darboux-Pöschl-Teller-II equation).
In (36), (37), m, n are non negative integers, α is an arbitrary scaling parameter, $c(x) := \cos(\alpha x)$, $s(x) := \sin(\alpha x)$.
Integrability of the DDPT-II equation by means of the elementary functions was first established in [13] in a way similar to [20, 21].

Passing to the limit $h \to 0$ we restore the differential DPT equation, (see [13] for details).

Similarly to the case of the DPT potentials and the DDPT-I model, DDPT-II equation is invariant under the action of each of the maps $m \to -m-1$, and $n \to -n-1$.

[7] This proves also the Proposition 2.1. (31) in a way which is shorter and easier with respect to [20, 21].

3.1 Modified difference Darboux transformation and DDPT-II equation

Here again we use some special case of the results proved in [16, 17].

Difference Darboux transformation $\psi_1(x)$ of an arbitrary solution $f(x,\lambda)$ of (35), generated by the fixed solution $f_1(x)$ of the same equation (35) with $\lambda = \lambda_1$, is defined by the formula

$$\psi_1(x) = \frac{\begin{vmatrix} f(x-h/2) & f_1(x-h/2) \\ f(x+h/2) & f_1(x+h/2) \end{vmatrix}}{f_1(x)},$$

$$\psi_1(x) = f(x-h/2) - \hat{\sigma}_1(x) f(x+h/2)$$

$$\hat{\sigma}_1(x) = \frac{f_1(x-h/2)}{f_1(x+h/2)},$$

$$\hat{\sigma}_1(x,h) = \sigma_1(x,h/2)$$

Theorem 3.1 .
$\psi_1(x)$ represents a general solution to the following equation

$$w_1(x)\psi_1(x+h) + \psi_1(x-h) + \hat{b}_1(x)\psi_1(x) = \lambda\psi_1(x), \tag{38}$$

$$w_1(x) = \frac{f_1(x+3h/2) f_1(x-h/2)}{(f_1(x+h/2))^2} v(x+h/2), \tag{39}$$

$$b_1(x) = b(x-h/2) + \frac{f_1(x-3h/2)}{f_1(x-h/2)} - \frac{f_1(x-h/2)}{f_1(x+h/2)}. \tag{40}$$

Proof : see for instance [16, 17].
Darboux transform maps (35) into equation of the same form, with the same value of spectral parameter λ, but with a new potential constructed in terms of the initial potential $v(x)$ and a fixed solution $f_1(x)$ of (35).[8]

3.2 Multiple difference Darboux transform 2

We use the following notations.

$$d_m(x) := d_m[f_1, f_2, \ldots, f_m](x) =$$

$$\begin{vmatrix} f_1(x-(m-1)h/2) & f_2(x-(m-1)h/2) & \ldots & f_m(x-(m-1)h/2) \\ f_1(x-(m-3)h/2) & f_2(x-(m-3)h/2) & \ldots & f_m(x-(m-3)h/2) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x+(m-1)h/2) & f_2(x+(m-1)h/2) & \ldots & f_m(x+(m-1)h/2) \end{vmatrix}, \tag{41}$$

$$c_m(x) := c_m[f_1, \ldots, f_m](x) :=$$

[8] Contrary to (35), equation (17) is not covariant with respect to the action of the Darboux transformation described in this section, since it does not preserve the coefficient b(x).

$$:= \begin{vmatrix} f_1(x-(m-1)h/2) & f_2(x-(m-1)h/2) & \dots & f_m(x-(m-1)h/2) \\ f_1(x-(m-5)h/2) & f_2(x-(m-5)h/2) & \dots & f_m(x-(m-5)h/2) \\ f_1(x-(m-7)h/2) & f_2(x-(m-7)h/2) & \dots & f_m(x-(m-7)h/2) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x+(m+1)h/2) & f_2(x+(m+1)h/2) & \dots & f_n(x+(m+1)h/2) \end{vmatrix}. \tag{42}$$

The last determinant has "almost Casoratian" structure, i.e. strictly Casoratian structure for the last $m-1$ rows, and a double shift of the arguments of f_j between the first row and the second row.

Theorem 3.2 *The function* $\psi_m(f)(x)$,

$$\psi_m(f)(x) := \frac{d_{m+1}[f, f_1, ..., f_m](x)}{d_m[f_1, ..., f_m](x+h/2)}, \tag{43}$$

gives a general solution of the following equation

$$w_m(x)\psi_m(f)(x+h) + \psi_n(f)(x-h) + b_m(x)\psi_m(f)(x) = \lambda\psi_n(f)(x), \tag{44}$$

$$w_m(x) := \frac{d_m(x-h/2)d_m(x+3h/2)}{d_m(x+h/2)^2} w(x+mh/2), \quad m \geq 1, \tag{45}$$

$$b_m(x) := b(x-mh/2) + \frac{c_m(x-3h/2)}{d_m(x-h/2)} - \frac{c_m(x-h/2)}{d_m(x+h/2)}, \quad m \geq 2, \tag{46}$$

Formula (45) is well known (see [16, 17]). By contrast, we were not able to find (46) in the literature. Here we omitted somehow the proof of (46) for brevity. □

Similarly to the case of DDPT-I model, the function $\psi_n(x)$ can be also represented in a following factorized form

$$\phi_m(x) = (U^{-1} - \kappa_n(x)U)\dots(U^{-1} - \kappa_1(x)U)f(x, \lambda),$$

where U is the shift operator $U^{\pm k}f(x) = f(x \pm kh/2)$ and $\kappa_j(x)$ is defined by the iterative way : $\Phi_0(x) = f(x)$, $\Phi_j(x) = \phi_{j-1}(x)|_{f(x)=f_j(x)}$, $\kappa_j(x) = \frac{\Phi_j(x-h/2)}{\Phi_j(x+h/2)}$, $j = 1 \dots m$.

3.3 Explicit formulas for some special determinants

Here we show that taking $w(x) = 1$, $b(x) = 0$ and defining the functions $f_1, \dots, f_n$ as in (4) we get from the formulas (45), (46), (40) the same potentials w_{mn}, b_{mn} as defined by (36) and (37)).

Let n and d be some non negative integers, $m = n+d$, h, α be some real parameter. The main technical tools we use are explicit formulas for the determinants $D_{mn}(x), C_{mn}(x)$ defined as follows.

We denote D_{mn} the determinant:

$$\begin{aligned} D_{mn} &= D_{mn}(x,h) = D_{mn}[f_1,\dots,f_m](x) \\ &:= \det\left(f_j((x-(m+1-2i)h/2))\right)_{i,j=1,\dots,m}). \end{aligned} \tag{47}$$

It is clear that

$$D_{mn}(x,h) = \delta_{mn}(x,h/2), \tag{48}$$

where δ_{mn} is defined by (20).
$C_{mn}(x,h) = c_m(f_1,\dots,f_m)$, where $f_j(x)$ are defined in (4). The determinant $C_{mn}(x,h)$ can be also explicitly computed:

Theorem 3.3 *1.* $m \geq 2$, $1 \leq n \leq m-1$,

$$C_{mn} = a_{mn} \cdot \prod_{k=-m+1}^{m-1} \sin^{[\frac{m+1-|k|}{2}]} \alpha(x+(k+2)h/2) \prod_{k=-n+1}^{n-1} \cos^{[\frac{n+1-|k|}{2}]} \alpha\alpha(x+(k+2)h/2)$$

$$\cdot \frac{\sin\alpha(x-(m-1)h/2)}{\sin(\alpha h)\sin\alpha(x+h)\cos\alpha(x+h)} \cdot (\sin(\alpha(m+n+1)h/2)\cos\alpha(x-(n-2)h/2) +$$

$$\sin(\alpha(m-n-1)h/2)\cos\alpha(x+(n+2)h/2));$$

$$a_{mn} = 2^{\frac{n(n+1)}{2}}(-4)^{\frac{m(m-1)}{2}} \prod_{l=0}^{m-n-1}\prod_{j=1}^{n} \sin\alpha(l+2j)h/2$$

$$\cdot \prod_{j=1}^{m-n} \sin^{m-n-j}(\alpha jh/2) \cdot \prod_{j=1}^{n} \sin^{n-j}(\alpha jh); \tag{49}$$

2. $m \geq 2$, $n = 0$,

$$C_{m0} = (-1)^{\frac{m(m-1)}{2}}(2)^{\frac{m(m-3)}{2}+1} \prod_{j=1}^{m-1} \sin^{m-j}(\alpha jh/2) \prod_{k=-m+1}^{m-1} \sin^{[\frac{m+1-|k|}{2}]} \alpha(2x+(k+2)h)$$

$$\cdot \frac{\sin\alpha(x-(m-1)h/2)}{\sin(\alpha h)\sin\alpha(x+h)} \cdot (\sin(\alpha mh/2)\cos(\alpha h/2)); \tag{50}$$

3. $m \geq 2,$, $n = m$,

$$C_{mm} = (-1)^{\frac{m(m-1)}{2}}(2)^{m(m-1)} \prod_{j=1}^{m-1} \sin^{m-j}(\alpha jh) \cdot \prod_{k=-m+1}^{m-1} \sin^{[\frac{m+1-|k|}{2}]} \alpha(2x+(k+2)h)$$

$$\cdot \frac{\sin\alpha(x-(m-1)h/2)}{\sin(\alpha h)\sin\alpha(x+h)\cos\alpha(x+h)} \cdot (\sin(\alpha(2m+1)h/2)\cos(\alpha(x-(m-2)h/2)$$

$$-\sin(\alpha h/2)\cos\alpha(x+(m+2)h/2)). \tag{51}$$

Proof: It is enough to check the following relation :

$$C_{mn}(x) = \frac{\sin\alpha(x-(m-1)h/2)}{\sin(\alpha h)\cos\alpha(x+h)\sin\alpha(x+h)}$$
$$\cdot(\sin(\alpha(m+n+1)h/2)\cos\alpha(x-(n-2)h/2)+$$
$$\sin(\alpha(m-n-1)h/2)\cos\alpha(x+(n+2)h/2))D_{mn}(x+h), \tag{52}$$

The later formula can be proved by using the recursion relation:

$$C_{m+1n+1}(x) = \frac{D_{m+1n+1}(x)}{D_{mn}(x+h/2)}D_{mn}(x+3h/2) + \frac{C_{mn}(x-h/2)}{D_{m,n}(x+h/2)}D_{m+1n+1}(x+h). \tag{53}$$

We postpone further details for a more detailed publication. □

3.4 Getting DDPT-II model from multiple Darboux dressing formulas

Integrability of DDPT-II model was first proved in [13] using the approach similar to [20, 21]. Below we present a different and shorter proof.

Theorem 3.4 *The DPT potentials $w_{mn}(x)$ and $b_{mn}(x)$ defined by (36), (37), can be expressed in terms of Casorati determinants in the following way :*

$$w_{mn} = \frac{D_{mn}(x-h/2)D_{mn}(x+3h/2)}{(D_{mn}(x+h/2))^2}, \tag{54}$$

$$b_{mn}(x) = \frac{C_{mn}(x-3h/2)}{D_{mn}(x-h/2)} - \frac{C_{mn}(x-h/2)}{D_{mn}(x+h/2)}. \tag{55}$$

The general solution of the DDPT-II equation can be written as

$$\phi(x,\lambda) = \frac{D_{m+1,n+1}[f,f_1,\dots,f_m](x)}{D_{mn}[f_1,\dots,f_m](x+h/2)}. \tag{56}$$

Hint of the Proof : It is sufficient to replace D_{mn} and C_{mn} by their expressions (47), (49).
Using the explicit formula of D_{mn}, it is easy to see that[9] :

$$\frac{D_{m+1,n+1}(x)}{D_{mn}(x+h/2)} = p_{mn}\cdot\prod_{k=0}^{m}\sin(x-kh/2)\prod_{k=0}^{n}\cos(x-kh/2) \equiv p_{mn}\cdot\Phi_1(x,m,n),$$

$$p_{mn} := 2^{2m+n+1}(-1)^m\prod_{l=1}^{m-n}\sin(\alpha(m+n+2-l)h/2))\cdot\prod_{j=1}^{n}\sin(\alpha jh).$$

[9]It was shown in [13] in a different and longer way that $\Phi(x,m,n)$ solves DDPT-II equation with $\lambda = 2\cos(\alpha h(m+n+2))$. Here, we get the same result as a trivial consequence of generic Casorati determinant formalism described above.

Formula above shows that (54) really holds.

Using explicit formula for D_{mn} and (49), we can also easily check that (55) holds. □

As a corollary, we get an another proof of the integrability of the DDPT-II equation.

Concluding Remarks

- Comparison of the results of the section 1 with [1, 2, 3] shows that the DPT trigonometric potentials produce the subfamily of two dimensional Huygens potentials $V(r,\varphi)$ via a formula

$$V(r,\varphi) = \frac{1}{r^2}\left[\frac{m(m+1)}{\sin^2\varphi} + \frac{n(n+1)}{\cos^2\varphi}\right],$$

  since the generic 2D Huygens potentials in polar coordinates are described by the formula [10]

$$V(r,\varphi) = -\frac{2}{r^2}W(\chi_1,\ldots,\chi_m),$$

$$\chi_j = \sin(k_j\varphi+\delta_j), \quad j=1,\ldots,m; \quad k_j \in \mathbb{Z}, \quad \delta_j \in \mathbb{R}.$$

  Therefore, taking $k_j = a_j$, where a_j are the same as in the Theorem 1.2 and, taking $\delta_j = 0$, we get the aforementioned link between the DPT potentials and Huygens potentials.

- We have shown also that a scepticism expressed in [25] (p. 371), concerning the use of generic Casorati determinants formulas for studying discrete DPT potentials was not justified since the case considered in [25] corresponds to a special reduction of DDPT-I or DDPT-II equations namely to the case $n=0$, and $x \in \mathbb{Z}$, $h=1$: we can easily recover all the results obtained in [13, 20, 21]

- Hyperbolic case of DDPT-I and DDPT-II models is quite similar: we have to replace α by $i\alpha$. Of course the related spectral properties are different but the algebraic structure of the main formulas remains the same. We postpone the detailed discussion of the hyperbolic DPPT-I and DDPT-II equations for more detailed publication. The related bound states eingenfunctions were described in [13, 20, 21].

- It is interesting to mention that the same expression as the determinant (15) , (up to a different constant normalization factor), recently appeared in [27] (Th. A, Th. 5.1) as a density of probability measure describing the asymptotic distribution of Frobenius roots on the m -dimensional abelian varieties over finite fields F_α when $\alpha \to \infty$.

[10] In [1, 2, 3] the formula below was written in slightly different but strictly equivalent form, using the wronskian of $m+1$ cosine functions.

- Hirota or Sato-like formulas for the solutions of some of the considered equations, (also providing the solutions by means of the elementary functions), were obtained via IST approach by Kirillov and van Diejen,- first, for the DPT equation [8, 9] and, next, for DDPT-I equation with $n = 0$ [10] where the solutions in terms of q-hypergeometric functions were also have been constructed. For the $n \neq 0$ or for DDPT-II model Sato like formulas for the solutions still are not constructed. We claim that it is possible to obtain these formulas via taking an appropriate limit of some finite-gap Baker-Akhiezer functions. We postpone the detailed explanation for future publications.

- The content of the section 1 is obviously connected with the usual (trigonometric) BC_1 two-particle quantum Calogero-Moser system. The results of the sections 2 and 3 are obviously relevant to the trigonometric case of some two particles quantum difference BC_1 Calogero-Moser systems,(also known as quantum relativistic, (QRCM), Calogero-Moser systems), although we avoided in the sections 2 and 3 to reduce our L-operators to QRCM hermitian hamiltonians by well known similarity transformations. See [11] for the related definitions and detailed comments.

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