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Abstract

We give several equivalent combinatorial descriptions of the space of states for the box-ball systems, and connect certain partition functions for these models with the q-weight multiplicities of the tensor product of the fundamental representations of the Lie algebra $\mathfrak{gl}(n)$. As an application, we give an elementary proof of the special case $t = 1$ of the Haglund–Haiman–Loehr formula. Also, we propose a new class of combinatorial statistics that naturally generalize the so-called energy statistics.

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1 Introduction

The purpose of the present paper is two-fold. First of all, we would like to give an introduction to the beautiful Combinatorics related with Box-Ball Systems, and secondly, to relate the latter with the "Classical Combinatorics" revolving around transportation matrices, tabloids, the Lascoux–Schützenberger statistics charge, Macdonald polynomials, [30],[34], Haglund–Haiman–Loehr's formula [8], and so on. As a result of our investigations, we will prove that two statistics naturally appearing in the context of Box-Ball systems, namely *energy function* and *tau-function*, have nice combinatorial properties. More precisely, the statistics energy E is an example of a generalized machonian statistics [21], Section 2, whereas the statistics τ related with Kostka–Macdonald polynomials, see Section 5.2 of the present paper.

Box-Ball Systems (BBS for short) were invented by Takahashi–Satsuma [42, 41] as a wide class of discrete integrable soliton system. In the simplest case, BBS are described by simple combinatorial procedures using box and balls. Despite its simple outlook, it is known that the BBS have various remarkably deep properties;

- Time evolution of the BBS coincides with isomorphism of the crystal bases [10, 5]. Thus the BBS possesses quantum integrability.
- BBS are ultradiscrete (or tropical) limit of the usual soliton systems [43, 28]. Thus the BBS possesses classical integrability at the same time.
- Inverse scattering formalism of the BBS coincides with the rigged configuration bijection originating in completeness problem of the Bethe states [27, 37].

Let us say a few words about the main results of our paper.

- In the case of statistics *tau*, our main result can be formulated as a computation of the corresponding partition function for the BBS in terms of the values of the Kostka–Macdonald polynomials at $t = 1$.
- In the case of the statistics *energy*, our result can be formulated as an interpretation of the corresponding partition function for the BBS as the q-weight multiplicity in the tensor product of the fundamental representations of the Lie algebra $\mathfrak{gl}(n)$. We *expect* that the same statement is valid for the BBS corresponding to the tensor product of rectangular representations.

  We are reminded that a *q-analogue of the multiplicity* of a highest weight λ in the tensor product $\bigotimes_{a=1}^{L} V_{s_a\omega_{r_a}}$ of the highest weight $s_a\,\omega_{r_a}$, $a = 1, \ldots, L$, irreducible representations $V_{s_a\omega_{r_a}}$ of the Lie algebra $\mathfrak{gl}(n)$ is defined as

$$q\text{-Mult}\,[V_\lambda \;:\; \bigotimes_{a=1}^{L} V_{s_a\omega_{r_a}}] = \sum_{\eta} K_{\eta,R}\; K_{\eta,\lambda}(q),$$

  where $K_{\eta,R}$ stands for the parabolic Kostka number corresponding to the sequence of rectangles $R := \{(s_a^{r_a})\}_{a=1,\ldots,L}$, see e.g. [21], [24].

- We give several equivalent descriptions of *paths* which appear in the description of partition functions for BBS: in terms of transportation matrices, tabloids, plane partitions. We expect that such interpretations may be helpful for better understanding connections of the BBS and other integrable models such as melting crystals [33], q-difference Toda lattices [6], ...

Our result about connections of the energy partition functions for BBS and q-weight multiplicities suggests a deep hidden connections between partition functions for the BBS and characters of the Demazure modules, solutions to the q-difference Toda equations, cf.[6].

As an interesting open problem we want to give raise a question about an interpretation of the sums $\sum_\eta K_{\eta,R}\, K_{\eta,\lambda}(q,t)$, where $K_{\eta,\lambda}(q,t)$ denotes the Kostka–Macdonald polynomials [30], as *refined partition functions* for the BBS corresponding to the tensor product of rectangular representations $R = \{(s^{r_a}_a)\}_{1\le a\le n}$. See Conjecture 5.18. In other words, one can ask: what is a meaning of the second statistics (see [8]) in the Kashiwara theory [18] of crystal bases (of type A) ?

Organization of the present paper is as follows. In Section 2, we review necessary facts from the Kirillov–Reshetikhin crystals. Especially we explain an explicit algorithm to compute the combinatorial R and the energy function. In section 3, we introduce several combinatorial descriptions of paths. Then we define several statistics on paths such as Haglund's statistics, energy statistics $\bar{E}$ and tau statistics $\tau^{r,s}$. In Section 4, we collect necessary facts from the BBS which will be used in the next section. In Section 5, we present our main result (Theorem 5.6) as well as several relating conjectures. We conjecture that $\tau^{r,s}$ gives independent statistics depending on one parameter r although they all give rise to the unique generating function up to constant shift of power. In Section 6, we show that the energy statistics $\bar{E}$ belong to the class of statistics $\tau^{r,s}$ (Theorem 6.3). Therefore $\tau^{r,s}$ gives a natural extension of the energy statistics $\bar{E}$.

2 Kirillov–Reshetikhin crystal

2.1 $A^{(1)}_n$ type crystal

Let $W^{(r)}_s$ be a $U_q(\mathfrak{g})$ Kirillov–Reshetikhin module, where we shall consider the case $\mathfrak{g} = A^{(1)}_n$. The module $W^{(r)}_s$ is indexed by a Dynkin node $r \in I = \{1, 2, \ldots, n\}$ and $s \in \mathbb{Z}_{>0}$. As a $U_q(A_n)$-module, $W^{(r)}_s$ is isomorphic to the irreducible module corresponding to the partition (s^r). For arbitrary r and s, the module $W^{(r)}_s$ is known to have crystal bases [18, 17], which we denote by $B^{r,s}$. As the set, $B^{r,s}$ is consisting of all column strict semi-standard Young tableaux of depth r and width s over the alphabet $\{1, 2, \ldots, n+1\}$.

For the algebra A_n, let P be the weight lattice, $\{\Lambda_i \in P | i \in I\}$ be the fundamental roots, $\{\alpha_i \in P | i \in I\}$ be the simple roots, and $\{h_i \in \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) | i \in I\}$ be the simple coroots. As a type A_n crystal, $B = B^{r,s}$ is equipped with the Kashiwara

operators $e_i, f_i : B \longrightarrow B \cup \{0\}$ and $\mathrm{wt} : B \longrightarrow P$ $(i \in I)$ satisfying

$$\begin{aligned} &f_i(b) = b' \Longleftrightarrow e_i(b') = b \quad \text{if } b, b' \in B, \\ &\mathrm{wt}\,(f_i(b)) = \mathrm{wt}(b) - \alpha_i \quad \text{if } f_i(b) \in B, \\ &\langle h_i, \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b). \end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ is the natural pairing and we set $\varepsilon_i(b) = \max\{m \geq 0 \mid \tilde{e}_i^m b \neq 0\}$ and $\varphi_i(b) = \max\{m \geq 0 \mid \tilde{f}_i^m b \neq 0\}$. Actions of the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ for $i \in I$ coincide with the one described in [19]. Since we do not use explicit forms of these operators, we omit the details. See [32] for complements of this section. Note that in our case A_n, we have $P = \mathbb{Z}^{n+1}$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ where ϵ_i is the i-th canonical unit vector of $\mathbb{Z}^{n+1}$. We also remark that $\mathrm{wt}(b) = (\lambda_1, \cdots, \lambda_{n+1})$ is the weight of b, i.e., λ_i counts the number of letters i contained in tableau b.

For two crystals B and B', one can define the tensor product $B \otimes B' = \{b \otimes b' \mid b \in B, b' \in B'\}$. The actions of the Kashiwara operators on tensor product have simple form. Namely, the operators $\tilde{e}_i, \tilde{f}_i$ act on $B \otimes B'$ by

$$\begin{aligned} \tilde{e}_i(b \otimes b') &= \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b') \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases} \\ \tilde{f}_i(b \otimes b') &= \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'), \end{cases} \end{aligned}$$

and $\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$. We assume that $0 \otimes b'$ and $b \otimes 0$ as 0. Then it is known that there is a unique crystal isomorphism $R : B^{r,s} \otimes B^{r',s'} \xrightarrow{\sim} B^{r',s'} \otimes B^{r,s}$. We call this map (classical) combinatorial R and usually write the map R simply by $\simeq$.

Let us consider the affinization of the crystal B. As the set, it is

$$\mathrm{Aff}(B) = \{b[d] \,|\, b \in B,\, d \in \mathbb{Z}\}. \tag{1}$$

There is also explicit algorithm for actions of the affine Kashiwara operators $\tilde{e}_0$, $\tilde{f}_0$ in terms of the promotion operator [40]. For the tensor product $b[d] \otimes b'[d'] \in \mathrm{Aff}(B) \otimes \mathrm{Aff}(B')$, we can lift the (classical) combinatorial R to affine case as follows:

$$b[d] \otimes b'[d'] \overset{R}{\simeq} \tilde{b}'[d' - H(b \otimes b')] \otimes \tilde{b}[d + H(b \otimes b')], \tag{2}$$

where $b \otimes b' \simeq \tilde{b}' \otimes \tilde{b}$ is the isomorphism of (classical) combinatorial R. The function $H(b \otimes b')$ is called the energy function. We will give explicit forms of the combinatorial R and energy function in the next section.

2.2 Combinatorial R and energy function

We give explicit description of the combinatorial R-matrix (combinatorial R for short) and energy function on $B^{r,s} \otimes B^{r',s'}$. To begin with we define few terminologies about Young tableaux. Denote rows of a Young tableaux Y by $y_1, y_2, \ldots y_r$ from

the top to bottom. Then row word $row(Y)$ is defined by concatenating rows as $row(Y) = y_r y_{r-1} \dots y_1$. Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two partitions. We define concatenation of x and y by the partition $(x_1 + y_1, x_2 + y_2, \dots)$.

Proposition 2.1 ([40]) *$b \otimes b' \in B^{r,s} \otimes B^{r',s'}$ is mapped to $\tilde{b}' \otimes \tilde{b} \in B^{r',s'} \otimes B^{r,s}$ under the combinatorial R, i.e.,*

$$b \otimes b' \overset{R}{\simeq} \tilde{b}' \otimes \tilde{b}, \tag{3}$$

if and only if

$$(b' \leftarrow row(b)) = (\tilde{b} \leftarrow row(\tilde{b}')). \tag{4}$$

Moreover, the energy function $H(b \otimes b')$ is given by the number of nodes of $(b' \leftarrow row(b))$ outside the concatenation of partitions (s^r) and $(s'^{r'})$.

We define another normalization of the energy function $\bar{H}$ such that for $b \otimes b' \in B^{r,s} \otimes B^{r',s'}$,

$$\bar{H}(b \otimes b) := \min(r, r') \cdot \min(s, s') - H(b \otimes b). \tag{5}$$

For special cases of $B^{1,s} \otimes B^{1,s'}$, the function H is called unwinding number and $\bar{H}$ is called winding number in [31]. Explicit values for the case $b \otimes b' \in B^{1,1} \otimes B^{1,1}$ are given by

$$H(b \otimes b') = \chi(b < b'), \qquad \bar{H}(b \otimes b') = \chi(b \geq b'), \tag{6}$$

where $\chi(\text{True}) = 1$ and $\chi(\text{False}) = 0$.

In order to describe the algorithm for finding $\tilde{b}$ and $\tilde{b}'$ from the data $(b' \leftarrow row(b))$, we introduce a terminology. Let Y be a tableau, and Y' be a subset of Y such that Y' is also a tableau. Consider the set theoretic subtraction $\theta = Y \setminus Y'$. If the number of nodes contained in θ is r and if the number of nodes of θ contained in each row is always 0 or 1, then θ is called vertical r-strip.

Given a tableau $Y = (b' \leftarrow row(b))$, let Y' be the upper left part of Y whose shape is (s^r). We assign numbers from 1 to $r's'$ for each node contained in $\theta = Y \setminus Y'$ by the following procedure. Let θ_1 be the vertical r'-strip of θ as upper as possible. For each node in θ_1, we assign numbers 1 through r' from the bottom to top. Next we consider $\theta \setminus \theta_1$, and find the vertical r' strip θ_2 by the same way. Continue this procedure until all nodes of θ are assigned numbers up to $r's'$. Then we apply inverse bumping procedure according to the labeling of nodes in θ. Denote by u_1 the integer which is ejected when we apply inverse bumping procedure starting from the node with label 1. Denote by Y_1 the tableau such that $(Y_1 \leftarrow u_1) = Y$. Next we apply inverse bumping procedure starting from the node of Y_1 labeled by 2, and obtain the integer u_2 and tableau Y_2. We do this procedure until we obtain $u_{r's'}$ and $Y_{r's'}$. Finally, we have

$$\tilde{b}' = (\emptyset \leftarrow u_{r's'} u_{r's'-1} \cdots u_1), \qquad \tilde{b} = Y_{r's'}. \tag{7}$$

Example 2.2 Consider the following tensor product:

$$b \otimes b' = \begin{array}{|c|c|c|}\hline 1&1&4\\\hline 2&3&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \in B^{2,3} \otimes B^{3,2}.$$

From b, we have $row(b) = 235114$, hence we have

$$\left(\begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \leftarrow 235114\right) = \begin{array}{|c|c|c|c|}\hline 1&1&3&4_3\\\hline 2&2&5\\\hline 3_6&3_2\\\hline 4_5&4_1\\\hline 5_4\\\hline\end{array}\quad .$$

Here subscripts of each node indicate the order of inverse bumping procedure. For example, we start from the node 4_1 and obtain

$$\left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&3&5\\\hline 3&4\\\hline 4\\\hline 5\\\hline\end{array} \leftarrow 1\right) = \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline 2&2&5\\\hline 3&3\\\hline 4&4\\\hline 5\\\hline\end{array}\quad , \qquad \text{therefore,} \qquad Y_1 = \begin{array}{|c|c|c|c|}\hline 1&2&3&4_3\\\hline 2&3&5\\\hline 3_6&4_2\\\hline 4_5\\\hline 5_4\\\hline\end{array}\quad , \qquad u_1 = 1.$$

Next we start from the node 4_2 of Y_1. Continuing in this way, we obtain $u_6u_5\cdots u_1 = 321421$ and $Y_6 = \begin{array}{|c|c|c|}\hline 3&3&4\\\hline 4&5&5\\\hline\end{array}$. Since $(\emptyset \leftarrow 321421) = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&4\\\hline\end{array}$, we obtain

$$\begin{array}{|c|c|c|}\hline 1&1&4\\\hline 2&3&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \simeq \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&4\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 3&3&4\\\hline 4&5&5\\\hline\end{array}, \qquad H\left(\begin{array}{|c|c|c|}\hline 1&1&4\\\hline 2&3&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array}\right) = 3.$$

Note that the energy function is derived from the concatenation of shapes of b and b', i.e., $\begin{array}{|c|c|c|c|c|}\hline &&&&\\\hline &&&&\\\hline &&\\\hline\end{array}$.

3 Combinatorics on the set of paths

3.1 Combinatorics

3.1.1 Transportation matrices and tabloids

Let n be a positive integer, $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots .\beta_n)$ be two compositions of the same size. Denote by $M_n(\alpha, \beta)$ the set of matrices $M = (m_{i,j})_{1\le i,j\le n}$ such that

$$m_{i,j} \in \mathbb{Z}_{\ge 0}, \quad \sum_j m_{i,j} = \alpha_i, \quad \sum_i m_{i,j} = \beta_j. \tag{8}$$

Remind that a tabloid of shape α and weight β is a filling of the shape α by the numbers $1, 2, \cdots, n$ in such a way that the number of i's appearing in the filling is equal to β_i. It is clear that the number of tabloids of shape α and weight β is equal to the multinomial coefficient

$$\frac{(\beta_1 + \beta_2 + \cdots + \beta_n)!}{\beta_1!\beta_2!\cdots\beta_n!}.$$

A row (column) weakly strict tabloid of shape α and weight β is a filling of the shape α by numbers $1, 2, \cdots, n$ such that the numbers along each row (column) are weakly increasing and β_i is equal to the number of i's appearing in the filling.

Example 3.1 Take $\alpha = (2, 1, 3, 1)$, $\beta = (1, 3, 0, 2, 1)$, then

$$\begin{array}{|c|c|}\hline 4 & 5 \\ \hline 1 \\ \cline{1-1} 2 & 2 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\ .$$

is a row weakly strict tabloid of shape α and weight β.

We denote by $\mathrm{Tab}(\alpha, \beta)$ the set of all row weakly strict tabloids of shape α and weight β. It is well-known that there exists a bijection between the sets $M_n(\alpha, \beta)$ and $\mathrm{Tab}(\alpha, \beta)$. Namely, given a matrix $m = (m_{ij}) \in M_n(\alpha, \beta)$, we fill the row α_i of the shape α by the numbers $1^{m_{i1}}, 2^{m_{i2}}, \cdots, n^{m_{in}}$. For example, let

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}\ .$$

Then the corresponding row weakly strict tabloid is $\begin{array}{|c|c|}\hline 4 & 5 \\ \hline 1 \\ \cline{1-1} 2 & 2 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$. To each tabloid T, one can associate the reading word, namely the word obtained by reading the filling of tabloid T from the right to the left starting from the top row. For example, for the tabloid T displayed above, $w(T) = 5414222$.

If weight μ of a tabloid T is a partition, we define the charge $c(T)$ of tabloid T to be the charge $c(w(T))$ of the reading word. See page 242 of [30] for the definition of the Lascoux–Schützenberger charge [29].

Example 3.2 Take standard tabloid

$$T = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline 1 \\ \cline{1-1} 3 & 6 & 7 \\ \hline 2 \\ \cline{1-1}\end{array}\ ,$$

then

$$w(T) = 5_3 4_2 1_0 7_4 6_3 3_1 2_0$$

and therefore $c(T) = 3 + 2 + 0 + 4 + 3 + 1 + 0 = 13$.

3.1.2 Plane partitions

Let λ be a partition. *A plane partition of shape* λ is a tabloid π of shape λ such that the numbers in each row and each column are weakly decreasing. For example,

$$\pi = \begin{array}{|c|c|c|}\hline 7 & 5 & 4 \\ \hline 7 & 4 & 4 \\ \hline 3 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array},$$

is a plane partitions of shape $(3, 3, 2, 1)$.

A plane partition π has a three-dimensional diagram, consisting of the points (i, j, k) with integer coordinates such that $(i, j) \in \lambda$ and $1 \leq k \leq \pi(i, j)$, where $\pi(i, j) \in \pi$ is the number that is located in the box $(i, j) \in \lambda$. By definition, the size $|\pi|$ of a plane partition π of shape λ is $|\pi| = \sum_{(i,j)\in\lambda} \pi(i, j)$.

Let α and β be two compositions of the same size. Denote by $\mathcal{PP}(\alpha, \beta)$ the set of plane partitions π such that

$$\sum_{k\geq 0} \pi(i, i + k) = \sum_{j\geq k} \alpha_j, \quad \sum_{k\geq 0} \pi(i + k, i) = \sum_{j\geq k} \beta_j.$$

Finally, let us remind two classical results

(A) (P. MacMahon, see e.g. [30], page 81) Let l, m, n be three positive integers, and B be the box with side-lengths $l.m, n$. Then

$$\sum_{\pi\subset B} q^{|\pi|} = \prod_{(i,j,k)\in B} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}.$$

(B) (Robinson–Schensted–Knuth, see e.g. [34], Chapter 3) There are bijections

$$\mathcal{M}(\alpha, \beta) \overset{1:1}{\longleftrightarrow} \mathcal{PP}(\alpha, \beta) \overset{1:1}{\longleftrightarrow} \mathrm{Tab}(\alpha, \beta).$$

3.2 Paths

3.2.1 $B^{1,1}$ type paths

Let α be a partition of size n. A path p of type $B^{1,1}$ and weight $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a sequence of positive integers $a_1 a_2 \cdots a_n$ such that $\alpha_i = \#\{j | a_j = i\}$. We denote by $\mathcal{P}(\alpha)$ the set of all paths of type $B^{1,1}$ and weight α. A path p is called a highest weight path if the sequence $a_1 a_2 \cdots a_n$ satisfies the Yamanouchi condition. We denote by $\mathcal{P}_+(\alpha)$ the set of all $B^{1,1}$ type highest paths with weight α. It is well known that the total number of $B^{1,1}$ type paths of weight $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is equal to the multinomial coefficient

$$\frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)!}{\alpha_1!\alpha_2! \cdots \alpha_n!},$$

and there are bijections

$$\mathcal{P}(\alpha) \xleftrightarrow{1:1} \mathrm{Mat}_n(\alpha, 1^n) \xleftrightarrow{1:1} \mathrm{Tab}(\alpha, 1^n).$$

Let us describe the general prescription to get the corresponding tabloid from a given path. Let the path $a_1 a_2 \cdots a_n \in \mathcal{P}(\alpha)$, we recursively add letters to the tabloid according to a_1, a_2, $\cdots$, a_n as follows. Starting from the empty tabloid, assume that we have done up to a_{i-1} and have gotten a tabloid $T^{(i-1)}$. Then we add the letter i to the right of the a_i-th row of $T^{(i-1)}$ and get $T^{(i)}$. For example, the path $p = 4221343$ can be related to the following transportation matrix and row strict tabloid

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad T = \begin{array}{|c|c|}\hline 4 & \multicolumn{1}{c}{} \\ \hline 2 & 3 \\ \hline 5 & 7 \\ \hline 1 & 6 \\ \hline \end{array}\,.$$

3.2.2 General rectangular paths

More generally, we define path to be an arbitrary element of tensor product of crystals $B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$. Recall that for type $A_n^{(1)}$ case, $B^{r,s}$ is, as the set, consisting of semi-standard tableaux over alphabet $\{1, 2, \cdots n+1\}$, and tensor product of crystals $B \otimes B'$ is, as a set, cartesian product of two sets B and B'. Crystal graph structure on the set $B \otimes B'$ is is given according to [19]. Weight $\lambda = (\lambda_1, \cdots, \lambda_{n+1})$ of a path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$ is given by

$$\lambda_i = \text{total number of letters } i \text{ contained in tableaux } B^{r_1,s_1}, \cdots, B^{r_L,s_L}. \qquad (9)$$

For example,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 & 5 \\ \hline \end{array} \in B^{3,2} \otimes B^{2,2} \otimes B^{2,2}$$

is a path of rectangular shape $R = ((2^3), (2^2), (2^2))$, and its weight is $\lambda = (4, 3, 3, 2, 2)$. Note that the number of standard (i.e., weight of (1^N)) rectangular shape $R = \{(r_a^{s_a})_{a=1,2,\cdots,L}\}$ paths is equal to the generalized multinomial coefficient

$$\begin{bmatrix} N \\ R_1, \cdots, R_L \end{bmatrix} = \frac{N!}{\prod_a H_{R_n}(1)}, \qquad (10)$$

where $N = \sum_a r_a s_a$, and for any diagram λ, $H_\lambda(q)$ denotes the hook polynomial (see definition on page 45 of [30]) corresponding to diagram λ.

Comments. Summarizing, one has the following (equivalent) combinatorial descriptions of the set of (crystal) paths of type $B^{1,s_1} \otimes B^{1,s_2} \otimes \cdots \otimes B^{1,s_n}$ and weight $\alpha = (\alpha_1, \ldots, \alpha_L)$ as the set of

(*a*) transportation matrices $\mathcal{M}_L(\alpha, s)$,
(*b*) row weakly increasing tabloids $\mathcal{T}(\alpha, s)$,
(*c*) plane partitions $\mathcal{PP}(\alpha, s)$.

For the given path $b_1 \otimes b_2 \otimes \cdots \otimes b_n \in B^{1,s_1} \otimes B^{1,s_2} \otimes \cdots \otimes B^{1,s_n}$, the corresponding element in $\mathcal{T}(\alpha, s)$ is determined as follows. Staring from the empty tabloid, we recursively add letters to the tabloid according to b_1, b_2, $\cdots$, b_n as follows. Assume we have done up to b_{i-1} and have gotten the tabloid $T^{(i-1)}$. Denote the number of k contained in b_i by x_k. Then, for all k, we add letters i for x_k times to the right of the k-th row of $T^{(i-1)}$ and get $T^{(i)}$.

Example 3.3 Consider the path

$$\boxed{3}\boxed{3} \otimes \boxed{1}\boxed{1}\boxed{2}\boxed{3} \otimes \boxed{1}\boxed{3} \otimes \boxed{2}\boxed{3}\boxed{3}$$

of type $B^{1,2} \otimes B^{1,4} \otimes B^{1,2} \otimes B^{1,3}$ and weight $\alpha = (3, 2, 6)$. The corresponding tabloid and transportation matrix are

$$T = \begin{array}{|c|c|c|}\hline 2&2&3\\\hline 2&4\\\cline{1-2} 1&1&2&3&4&4\\\cline{1-6}\end{array}\,, \qquad M = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}.$$

To find the plane partition which corresponds to the tabloid T (or matrix M), one can apply the Robinson–Schensted–Knuth algorithm [26] to the multi-permutation

$$w := \begin{pmatrix} 1\,1\,1\,2\,2\,3\,3\,3\,3\,3\,3 \\ 2\,2\,3\,3\,2\,4\,1\,1\,2\,3\,4\,4 \end{pmatrix}$$

which corresponds to the tabloid T. One has

$$w \xleftrightarrow{RSK} \left(\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&2&2&3&4&4\\\hline 2&2&4\\\cline{1-3} 3\\\cline{1-1}\end{array}\,,\; \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&3&3&3\\\hline 2&3&3\\\cline{1-3} 3\\\cline{1-1}\end{array} \right).$$

Finally, the plane partition we are looking for, can be obtained from the pair of semi-standard Young tableaux displayed above by gluing the Gelfand–Tsetlin patterns that correspond to the Young tableaux in question:

$$\left(\begin{array}{ccccccc} 7 & 3 & 1 & 0 \\ & 5 & 2 & 1 \\ & & 4 & 2 \\ & & & 2 \end{array} \,,\; \begin{array}{ccccccc} 7 & 3 & 1 & 0 \\ & 7 & 3 & 1 \\ & & 4 & 1 \\ & & & 3 \end{array} \right) \longleftrightarrow \begin{array}{ccccccccccc} 7 & & 7 & & 4 & & 3 \\ & 5 & & 3 & & 3 & & 1 \\ & & 4 & & 2 & & 1 & & 1 \\ & & & 2 & & 2 & & 1 & & 0 \\ & & & & & & & & & & 11 \\ & & & & & & 2 & & 6 & & 8 & & 11 \end{array}$$

The result is a plane partition from the set $\mathcal{PP}((2, 4, 2, 3), (3, 2, 6, 0))$.

Remark 3.4 For the reader's convenience, let us recall the way to get the corresponding Gelfand–Tsetlin patterns from given semi-standard tableaux. By looking contents of semi-standard tableau T, one can define a sequence of partitions

$$\emptyset = \mu^{(0)} \subset \mu^{(1)} \subset \cdots \subset \mu^{(n)} = \mu \tag{11}$$

such that each skew diagram $\mu^{(i)} \setminus \mu^{(i-1)}$ $(1 \leq i \leq n)$ is a horizontal strip, see e.g., Chapter I of [30]. Starting from the sequence of partitions (11), one can define the corresponding Gelfand–Tsetlin pattern $x := x(T)$ by the following rule

$$x^{(i)}(T) = \mathrm{shape}(\mu^{(i)}), \qquad (1 \leq i \leq n). \tag{12}$$

It is known that thus obtained x indeed satisfies the defining properties of the Gelfand–Tsetlin patterns.

Remark 3.5 One of the most remarkable property of the BBS is that the second Young tableau[1] (of weight β) obtained by means of the Robinson–Schensted–Knuth algorithm, is conserved under the dynamics of the BBS [4] (see also [44, 1] for the other connections between the simplest BBS and the RSK algorithm). Nowadays, conserved quantities and linearization parameters (or angle variables) of the BBS are completely determined in the most general settings [27], and, surprisingly enough, they are elegantly described by the so-called (unrestricted) rigged configurations [20, 23, 22, 25, 38, 2].

3.3 Statistics on the set of paths

3.3.1 Energy statistics

For a path $b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$, let us define elements $b_j^{(i)} \in B^{r_j,s_j}$ for $i < j$ by the following isomorphisms of the combinatorial R;

$$\begin{aligned} & b_1 \otimes b_2 \otimes \cdots \otimes b_{i-1} \otimes b_i \otimes \cdots \otimes b_{j-1} \otimes b_j \otimes \cdots \\ \simeq\ & b_1 \otimes b_2 \otimes \cdots \otimes b_{i-1} \otimes b_i \otimes \cdots \otimes b_j^{(j-1)} \otimes b'_{j-1} \otimes \cdots \\ \simeq\ & \cdots \\ \simeq\ & b_1 \otimes b_2 \otimes \cdots \otimes b_{i-1} \otimes b_j^{(i)} \otimes \cdots \otimes b'_{j-2} \otimes b'_{j-1} \otimes \cdots , \end{aligned} \tag{13}$$

where we have written $b_k \otimes b_j^{(k+1)} \simeq b_j^{(k)} \otimes b'_k$ assuming that $b_j^{(j)} = b_j$.

For a given path $p = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$, define statistics $\bar{E}(p)$ by

$$\bar{E}(p) = \sum_{i<j} \bar{H}(b_i \otimes b_j^{(i+1)}). \tag{14}$$

Define the statistics $\mathrm{maj}(p)$ by

$$\mathrm{maj}(p) = \sum_{i<j} H(b_i \otimes b_j^{(i+1)}). \tag{15}$$

[1] Equivalently the upper part of the corresponding plane partition.

For example, consider a path $a = a_1 \otimes a_2 \otimes \cdots \otimes a_L \in (B^{1,1})^{\otimes L}$. In this case, we have $a_j^{(i)} = a_i$, since the combinatorial R act on $B^{1,1} \otimes B^{1,1}$ as identity. Therefore, we have

$$\mathrm{maj}(a) = \sum_{i=1}^{L-1} (L-i)\chi(a_i < a_{i+1}). \tag{16}$$

Define another statistics *tau* as follows.

Definition 3.6 *For the path $p \in B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$, define $\tau^{r,s}$ by*

$$\tau^{r,s}(p) = \mathrm{maj}(u_s^{(r)} \otimes p), \tag{17}$$

where $u_s^{(r)}$ is the highest element of $B^{r,s}$.

We use abbreviation τ for the statistics $\tau^{1,1}$ on $B^{1,1}$ type paths $a \in (B^{1,1})^{\otimes L}$, i.e.,

$$\tau(a) = \mathrm{maj}(1 \otimes a) = \mathrm{maj}(a) + L\ (1 - \delta_{1,a_1}), \tag{18}$$

where a_1 denotes the first letter of the path a. This τ is a special case of the tau functions for the box-ball systems [28, 35] which originate as ultradiscrete limit of the tau functions for the KP hierarchy [16].

Definition 3.7 *For composition $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$, write $\mu_{[i]} = \sum_{j=1}^{i} \mu_j$ with convention $\mu_{[0]} = 0$. Then we define a generalization of τ by*

$$\tau_\mu(a) = \sum_{i=1}^{n} \tau(a_{[i]}), \tag{19}$$

where

$$a_{[i]} = a_{\mu_{[i-1]}+1} \otimes a_{\mu_{[i-1]}+2} \otimes \cdots \otimes a_{\mu_{[i]}} \in (B^{1,1})^{\otimes \mu_i}. \tag{20}$$

Note that we have $a = a_{[1]} \otimes a_{[2]} \otimes \cdots \otimes a_{[n]}$, i.e., the path a is partitioned according to μ. It is convenient to identify τ_μ as statistics on a tabloid of shape μ whose reading word coincides with the partitioned path according to μ. For example, to path $p = abcdefgh$ and the composition $\mu = (3, 2, 3)$ one associates the tabloid

c	b	a
e	d	
h	g	f

.

3.3.2 Statistics charge

Let p be a path of type $B^{1,1}$, denote by $T(p)$ the corresponding row strict tabloid and by $w(T(p))$ its reading word. Define the charge of a path p to be the charge of tabloid $T(p)$, i.e. the charge of the reading word $w(T(p))$. For example, take $p = 4221343$. Then $w(T(p)) = 4327561$, and therefore, $c(p) = 3 + 2 + 1 + 4 + 3 + 3 + 0 = 16$.

If μ is a composition, define $c_\mu(p) = \sum_i\ c(p_{[i]})$, where

$$p_{[i]} = p_{\mu_{[i-1]}+1}\ p_{\mu_{[i-1]}+2}\ \cdots\ p_{\mu_{[i]}},$$

cf. Definition 3.3.

Lemma 3.8 *One has*

$$\tau_\mu(p) + c_\mu(p) = \sum_i \left(\binom{m_i+1}{2} - \mu_i \ \delta_{1,p_1^{[i]}} \right),$$

where $p_1^{[i]}$ *denotes the first letter of the path* $p_{[i]}$.

Proof follows from two simple observations that

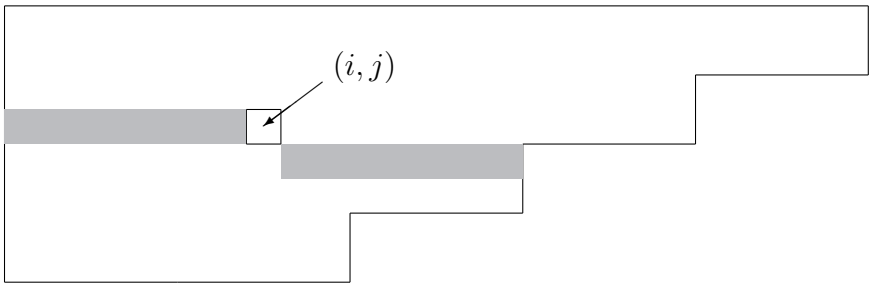
$$\tau(p) + c(1 \otimes p) = \binom{L}{2}, \qquad c(1 \otimes p) - c(p) = L \ \delta_{1,p_1},$$

where L denotes the length of path p.

Comments. It follows from [28] that on the set of semi-standard Young tableaux, i.e., on the set of highest weight paths, the statistics tau coincides with statistics *cocharge*. Therefore, one can consider the statistics tau as a natural extension of the statistics cocharge from the set of semi-standard tableaux to the set of tabloids, or on the set of transportation matrices.

3.3.3 Haglund's statistics

Tableaux language description For a given path $a = a_1 \otimes a_2 \otimes \cdots \otimes a_L \in (B^{1,1})^{\otimes L}$, associate tabloid t of shape μ whose reading word coincides with a. This correspondence is the same as those used in the above explanation of τ_μ. Denote the cell at the i-th row, j-th column (we denote the coordinate by (i,j)) of the tabloid t by t_{ij}. Attacking region of the cell at (i,j) is all cells (i,k) with $k < j$ or $(i+1,k)$ with $k > j$. In the following diagram, gray zonal regions are the attacking regions of the cell (i,j).



Follow [8], define $|\mathrm{Inv}_{ij}|$ by

$$|\mathrm{Inv}_{ij}| = \#\{(k,l) \in \text{ attacking region for } (i,j) \,|\, t_{kl} > t_{ij}\}. \tag{21}$$

Then we define

$$|\mathrm{Inv}_\mu(a)| = \sum_{(i,j)\in\mu} |\mathrm{Inv}_{ij}|. \tag{22}$$

If we have $t_{(i-1)j} < t_{ij}$, then the cell (i,j) is called by *descent*. Then define

$$\mathrm{Des}_\mu(a) = \sum_{\text{all descent } (i,j)} (\mu_i - j). \tag{23}$$

Note that $(\mu_i - j)$ is the arm length of the cell (i,j).

Path language description Consider two paths $a^{(1)}, a^{(2)} \in (B^{1,1})^{\otimes\mu}$. We denote by $a^{(1)} \otimes a^{(2)} = a_1 \otimes a_2 \otimes \cdots \otimes a_{2\mu}$. Then we define

$$\mathrm{Inv}_{(\mu,\mu)}(a^{(1)}, a^{(2)}) = \sum_{k=1}^{\mu} \sum_{i=k+1}^{k+\mu-1} \chi(a_k < a_i). \tag{24}$$

For more general cases $a^{(1)} \in (B^{1,1})^{\otimes\mu_1}$ and $a^{(2)} \in (B^{1,1})^{\otimes\mu_2}$ satisfying $\mu_1 > \mu_2$, we define

$$\mathrm{Inv}_{(\mu_1,\mu_2)}(a^{(1)}, a^{(2)}) := \mathrm{Inv}_{(\mu_1,\mu_1)}(a^{(1)}, 1^{\otimes(\mu_1-\mu_2)} \otimes a^{(2)}). \tag{25}$$

Then the above definition of $|\mathrm{Inv}_\mu(a)|$ is equivalent to

$$|\mathrm{Inv}_\mu(a)| = \sum_{i=1}^{n-1} \mathrm{Inv}_{(\mu_i,\mu_{i+1})}. \tag{26}$$

For example, consider the following tabloid ($a = 2312133212$);

$$t = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 1 & 2 & 1 & 3 & 2 \\ \hline 2 & 1 & 2 & 3 \\ \cline{1-4} \end{array}, \qquad |\mathrm{Inv}_{(6,4)}(a)| = 10.$$

We associate the paths $a^{(1)} = 231213$ and $a^{(2)} = 1^{\otimes 2}3212$. Then

$$\begin{aligned} |\mathrm{Inv}_{(a^{(1)},a^{(2)})}| &= \chi(a_1 < a_2) + \chi(a_1 < a_3) + \chi(a_1 < a_4) + \chi(a_1 < a_5) + \chi(a_1 < a_6) + \\ &\quad \chi(a_2 < a_3) + \chi(a_2 < a_4) + \chi(a_2 < a_5) + \chi(a_2 < a_6) + \chi(a_2 < a_7) + \\ &\quad \chi(a_3 < a_4) + \chi(a_3 < a_5) + \chi(a_3 < a_6) + \chi(a_3 < a_7) + \chi(a_3 < a_8) + \\ &\quad \chi(a_4 < a_5) + \chi(a_4 < a_6) + \chi(a_4 < a_7) + \chi(a_4 < a_8) + \chi(a_4 < a_9) + \\ &\quad \chi(a_5 < a_6) + \chi(a_5 < a_7) + \chi(a_5 < a_8) + \chi(a_5 < a_9) + \chi(a_5 < a_{10}) + \\ &\quad \chi(a_6 < a_7) + \chi(a_6 < a_8) + \chi(a_6 < a_9) + \chi(a_6 < a_{10}) + \chi(a_6 < a_{11}) \\ &= (1+0+1+0+1) + (0+0+0+0+0) + (1+0+1+0+0) + \\ &\quad (0+1+0+0+1) + (1+0+0+1+1) + (0+0+0+0+0) = 10. \end{aligned}$$

Consider two paths $a^{(1)} \in (B^{1,1})^{\otimes\mu_1}$ and $a^{(2)} \in (B^{1,1})^{\otimes\mu_2}$ satisfying $\mu_1 \geq \mu_2$. Denote $a = a^{(1)} \otimes a^{(2)}$. Then define

$$\mathrm{Des}_{(\mu_1,\mu_2)}(a) = \sum_{k=\mu_1-\mu_2+1}^{\mu_1} (k - (\mu_1 - \mu_2) - 1)\chi(a_k < a_{k+\mu_2}). \tag{27}$$

For the tableau T of shape μ corresponding to the path a, we define

$$\mathrm{Des}_\mu(T) = \sum_{i=1}^{n} \mathrm{Des}_{(\mu_i,\mu_{i+1})}(a_{[i]} \otimes a_{[i+1]}). \tag{28}$$

Definition 3.9 ([7]) *For a path a, statistics maj_μ is defined by*

$$\mathrm{maj}_\mu(a) = \sum_{i=1}^{\mu_1} \mathrm{maj}(t_{1,i} \otimes t_{2,i} \otimes \cdots \otimes t_{\mu'_i,i}). \tag{29}$$

and $\mathrm{inv}_\mu(a)$ is defined by

$$\mathrm{inv}_\mu(a) = |\mathrm{Inv}_\mu(a)| - \mathrm{Des}_\mu(a). \tag{30}$$

If we associate to a given path p with shape μ tabloid T, we sometimes write $\mathrm{maj}_\mu(p) = \mathrm{maj}(T)$ and $\mathrm{inv}_\mu(p) = \mathrm{inv}(T)$.

Example 3.10 For highest weight paths of weight $(2,2,2)$ and shape $(4,2)$, the following is a list of the corresponding tabloids associated with data in the form (maj, inv):

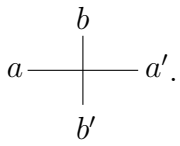
$\begin{array}{\|c\|c\|c\|c\|}\hline 2&2&1&1\\\hline 3&3\\\cline{1-2}\end{array}$ $(2,4)$	$\begin{array}{\|c\|c\|c\|c\|}\hline 2&1&2&1\\\hline 3&3\\\cline{1-2}\end{array}$ $(2,3)$	$\begin{array}{\|c\|c\|c\|c\|}\hline 3&1&2&1\\\hline 3&2\\\cline{1-2}\end{array}$ $(1,5)$
$\begin{array}{\|c\|c\|c\|c\|}\hline 1&3&2&1\\\hline 3&2\\\cline{1-2}\end{array}$ $(1,4)$	$\begin{array}{\|c\|c\|c\|c\|}\hline 3&2&1&1\\\hline 3&2\\\cline{1-2}\end{array}$ $(0,6)$	

4 Box-ball system

In this section, we summarize basic facts about the box-ball system which will be used in the next section. For our purpose, it is convenient to express the isomorphism of the combinatorial R

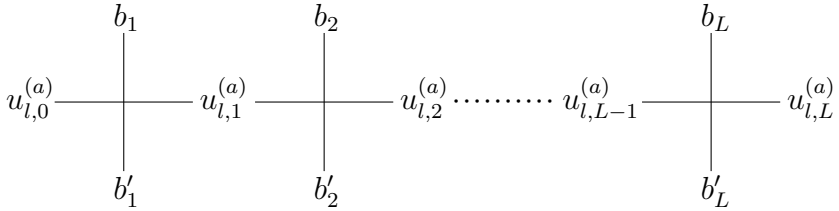
$$a \otimes b \simeq b' \otimes a' \tag{31}$$

by the following vertex diagram:



Successive applications of the combinatorial R is depicted by concatenating these vertices.

Following [10, 5], we define time evolution of the box-ball system $T_l^{(a)}$. Let $u_{l,0}^{(a)} = u_l^{(a)} \in B^{a,l}$ be the highest element and $b_i \in B^{r_i,s_i}$. Here the highest element $u_l^{(a)} \in B^{a,l}$ is the tableau whose i-th row is occupied by integers i. For example, $u_4^{(3)} = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2&2&2\\\hline 3&3&3&3\\\hline\end{array}$. Define $u_{l,j}^{(a)}$ and $b'_i \in B^{r_i,s_i}$ by the following diagram.



(32)

$u_{l,j}^{(a)}$ are usually called *carrier* and we set $u_{l,0}^{(a)} := u_l^{(a)}$. Then we define operator $T_l^{(a)}$ by

$$T_l^{(a)}(b) = b' = b'_1 \otimes b'_2 \otimes \cdots \otimes b'_L. \tag{33}$$

Recently [36, 37], operators $T_l^{(a)}$ have used to derive crystal theoretical meaning of the rigged configuration bijection.

It is known ([27] Theorem 2.7) that there exists some $l \in \mathbb{Z}_{>0}$ such that

$$T_l^{(a)} = T_{l+1}^{(a)} = T_{l+2}^{(a)} = \cdots (=: T_\infty^{(a)}). \tag{34}$$

If the corresponding path is $b \in (B^{1,1})^{\otimes L}$, we have the following combinatorial description of the box-ball system [42, 41]. We regard $\boxed{1} \in B^{1,1}$ as an empty box of capacity 1, and $\boxed{i} \in B^{1,1}$ as a ball of label (or internal degree of freedom) i contained in the box. Then we have:

Proposition 4.1 ([10]) *For a path $b \in (B^{1,1})^{\otimes L}$ of type $A_n^{(1)}$, $T_\infty^{(1)}(b)$ is given by the following procedure.*

1. *Move every ball only once.*
2. *Move the leftmost ball with label $n+1$ to the nearest right empty box.*
3. *Move the leftmost ball with label $n+1$ among the rest to its nearest right empty box.*
4. *Repeat this procedure until all of the balls with label $n+1$ are moved.*
5. *Do the same procedure 2–4 for the balls with label n.*
6. *Repeat this procedure successively until all of the balls with label 2 are moved.*

There are extensions [14, 15] of this box and ball algorithm corresponding to generalizations of the box-ball systems with respect to each affine Lie algebra [13, 12]. Using this box and ball interpretation, our statistics $\tau(b)$ admits the following interpretation.

Theorem 4.2 ([28] Theorem 7.4) *For a path $b \in (B^{1,1})^{\otimes L}$ of type $A_n^{(1)}$, $\tau(b)$ coincides with number of all balls $2, \cdots, n+1$ contained in paths b, $T_\infty^{(1)}(b)$, $\cdots$, $(T_\infty^{(1)})^{L-1}(b)$.*

Example 4.3 Consider the path $p = a \otimes b$ where $a = 4312111$, $b = 4321111$. Then we have

4	3	1	2	1	1	1
1	1	4	1	3	2	1
1	1	1	4	1	1	3
1	1	1	1	4	1	1
1	1	1	1	1	4	1
1	1	1	1	1	1	4
1	1	1	1	1	1	1

4	3	2	1	1	1	1
1	1	1	4	3	2	1
1	1	1	1	1	1	4
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1

Here the left and right tables correspond to a and b, respectively. Rows of left (resp. right) table represent a, $T_\infty^{(1)}(a)$, $\cdots$, $(T_\infty^{(1)})^L(a)$ (resp., those for b) from top to bottom. Note that we omit all frames of tableaux of $B^{1,1}$ and symbols for tensor product. Counting letters 2, 3 and 4 in each table, we have $\tau(a) = 11$, $\tau(b) = 7$ and $\tau_{(7,7)}(p) = 11+7 = 18$. Meanings of the above two dynamics corresponding to paths a and b are summarized as follows:

(a) Dynamics of the path a. In the first row, there are two solitons (length two soliton 43 and length one soliton 2), and in the second row, there are also two solitons (length one soliton 4 and length two soliton 32). This is scattering of two solitons. After the scattering, soliton 4 propagates at velocity one and soliton 32 propagates at velocity two without scattering.

(b) Dynamics of the path b. This shows free propagation of one soliton of length three 432 at velocity three.

5 Main results

5.1 Haglund–Haiman–Loehr formula

Let $\tilde{H}_\mu(x;q,t)$ be the (integral form) Macdonald polynomials where x stands for infinitely many variables $x_1, x_2, \cdots$. Here $\tilde{H}_\mu(x;q,t)$ is obtained by simple plethystic substitution (see, e.g., section 2 of [9]) from the original definition of the Macdonald polynomials [30]. Schur function expansion of $\tilde{H}_\mu(x;q,t)$ is given by

$$\tilde{H}_\mu(x;q,t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t) s_\lambda(x), \tag{35}$$

where $\tilde{K}_{\lambda,\mu}(q,t)$ is transformation of the Kostka–Macdonald polynomials:

$$\tilde{K}_{\lambda,\mu}(q,t) = t^{n(\mu)} K_{\lambda,\mu}(q,t^{-1}). \tag{36}$$

Here we have used notation $n(\mu) = \sum_i (i-1)\mu_i$. Then the celebrated Haglund–Haiman–Loehr (HHL) formula is as follows.

Theorem 5.1 ([8]) *Let $\sigma : \mu \to \mathbb{Z}_{>0}$ be the filling of the Young diagram μ by positive integers $\mathbb{Z}_{>0}$, and define $x^\sigma = \prod_{u\in\mu} x_{\sigma(u)}$. Then the Macdonald polynomial $\tilde{H}_\mu(x;q,t)$ have the following explicit formula:*

$$\tilde{H}_\mu(x;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_{>0}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma. \tag{37}$$

From the HHL formula, we can show the following formula.

Proposition 5.2 *For any partition μ and composition α of the same size, one has*

$$\sum_{p\in\mathcal{P}(\alpha)} q^{\mathrm{inv}_\mu(p)} t^{\mathrm{maj}_\mu(p)} = \sum_{\eta\vdash|\mu|} K_{\eta,\alpha} \tilde{K}_{\eta,\mu}(q,t), \tag{38}$$

where η runs over all partitions of size $|\mu|$.

Proof. Indeed, [8], if two fillings σ and σ' belong to the same s_∞ orbit, then $\mathrm{Inv}(\sigma) = \mathrm{Inv}(\sigma')$, $\mathrm{Des}(\sigma) = \mathrm{Des}(\sigma')$. □

Corollary 5.3 *The Macdonald polynomial $\tilde{H}_\mu(x;q,t)$ have the following expansion in terms of the monomial symmetric functions $m_\lambda(x)$:*

$$\tilde{H}_\mu(x;q,t) = \sum_{\lambda \vdash |\mu|} \left(\sum_{p \in \mathcal{P}(\lambda)} q^{\mathrm{inv}_\mu(p)} t^{\mathrm{maj}_\mu(p)} \right) m_\lambda(x), \tag{39}$$

where λ runs over all partitions of size $|\mu|$.

To find combinatorial interpretation of the Kostka–Macdonald polynomials $\tilde{K}_{\lambda,\mu}(q,t)$ remains significant open problem. Among many important partial results about this problem, we would like to mention the following theorem also due to Haglund–Haiman–Loehr:

Theorem 5.4 ([8] Proposition 9.2) *If $\mu_1 \leq 2$, we have*

$$\tilde{K}_{\lambda,\mu}(q,t) = \sum_{p \in \mathcal{P}_+(\lambda)} q^{\mathrm{inv}_\mu(p)} t^{\mathrm{maj}_\mu(p)}. \tag{40}$$

It is interesting to compare this formula with the formula obtained by S. Fishel [3], see also [24].

Concerning validity of the formula Eq.(40), we give the following conjecture.

Conjecture 5.5 *Explicit formula for the Kostka–Macdonald polynomials*

$$\tilde{K}_{\lambda,\mu}(q,t) = \sum_{p \in \mathcal{P}_+(\lambda)} q^{\mathrm{inv}_\mu(p)} t^{\mathrm{maj}_\mu(p)}. \tag{41}$$

is valid if and only if at least one of the following two conditions is satisfied.

(i) $\mu_1 \leq 3$ *and* $\mu_2 \leq 2$.

(ii) λ *is hook shape.*

5.2 Generating function of tau functions

Our main result is an elementary proof for special case $t = 1$ of the formula Eq.(38) in the following form.

Theorem 5.6 *Let α be a composition and μ be a partition of the same size. Then,*

$$\sum_{p \in \mathcal{P}(\alpha)} q^{\mathrm{maj}_\mu(p)} = \sum_{\eta \vdash |\mu|} K_{\eta,\alpha}\ K_{\eta,\mu}(q,1). \tag{42}$$

Conjecture 5.7 *Let α be a composition and μ be a partition of the same size. Then,*

$$q^{-\sum_{i \geq 2} \alpha_i} \sum_{p \in \mathcal{P}(\alpha)} q^{\tau_\mu(p)} = \sum_{\eta \vdash |\mu|} K_{\eta,\alpha} \tilde{K}_{\eta,\mu}(q,1). \tag{43}$$

Here $\sum_{i\geq 2}\alpha_i$ is equal to the number of letters other than 1 contained in each path $p \in \mathcal{P}(\alpha)$.

Example 5.8 Let us consider case $\alpha = (4,1,1)$ and $\mu = (4,2)$. The following is a list of paths p and the corresponding value of tau function $\tau_{(4,2)}(p)$. For example, the top left corner $\boxed{111123\ \ 3}$ means $p = \boxed{1}\otimes\boxed{1}\otimes\boxed{1}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{3}$ and $\tau_{(4,2)}(p) = 3$.

111123 3	111132 2	111213 2	111231 3	111312 2	111321 3
112113 3	112131 4	112311 3	113112 3	113121 4	113211 2
121113 4	121131 5	121311 4	123111 5	131112 4	131121 5
131211 4	132111 3	211113 5	211131 6	211311 5	213111 6
231111 7	311112 5	311121 6	311211 5	312111 6	321111 4

Summing up, LHS of Eq.(43) is

$$q^{-2}\sum_{p\in\mathcal{P}((4,1,1))} q^{\tau_{(4,2)}(p)} = q^5 + 4q^4 + 7q^3 + 7q^2 + 7q + 4.$$

In the RHS of Eq.(43), nontrivial contributions come from the following 4 terms:

$$\begin{aligned}
& K_{(6),(4,1,1)}\tilde{K}_{(6),(4,2)}(q,t) + K_{(5,1),(4,1,1)}\tilde{K}_{(5,1),(4,2)}(q,t) \\
& +K_{(4,2),(4,1,1)}\tilde{K}_{(4,2),(4,2)}(q,t) + K_{(4,1,1),(4,1,1)}\tilde{K}_{(4,1,1),(4,2)}(q,t) \\
= \ & 1\cdot(1) + 2\cdot(q^3+q^2+qt+q+t) \\
& +1\cdot(2q^4+q^3t+q^3+2q^2t+q^2+qt+t^2) \\
& +1\cdot(q^5+q^4t+q^4+2q^3t+q^3+2q^2t+qt^2+qt) \\
= \ & q^5+(t+3)q^4+(3t+4)q^3+(4t+3)q^2+\left(t^2+4t+2\right)q+t^2+2t+1.
\end{aligned}$$

By setting $t = 1$, we get Eq.(43).

Proof of Theorem 5.6

Definition 5.9 *Let μ be a composition and T be a tabloid of size $|\mu|$. Denote by $T^{(i)}$ the part of T which is filled by numbers from the interval $[\mu_{[i-1]+1}, \mu_{[i]}]$. Then*

$$c_\mu(T) = \sum_{i\geq 1} c(T^{(i)}). \tag{44}$$

Lemma 5.10

$$\sum_{\eta} K_{\eta,\alpha}K_{\eta,\mu}(1,t) = \sum_{T}\ t^{c_\mu(T)}, \tag{45}$$

where the sum in the right hand side runs over standard tabloids T of shape α.

Proof. Recall the following three formulas from [30], Chapter VI.
Formula 1.

$$K_{\lambda,\mu}(q,t) = K_{\lambda,\mu'}(t^{-1},q^{-1})t^{n(\mu)}q^{n(\mu')}, \tag{46}$$

and thus

$$\sum_{\eta} K_{\eta,\lambda}K_{\eta,\mu}(q,t) = \sum_{\eta} K_{\eta,\lambda}K_{\eta,\mu'}(t^{-1},q^{-1})t^{n(\mu)}q^{n(\mu')}. \tag{47}$$

As a corollary of the formulas above,

$$\sum_{\eta} K_{\eta,\lambda}K_{\eta,\mu}(q,1) = \sum_{\eta} K_{\eta,\lambda}K_{\eta,\mu'}(1,q^{-1})q^{n(\mu')}. \tag{48}$$

Formula 2.

$$J_{\mu}(x;1,t) = (t,t)_{\mu'}e_{\mu'}(x). \tag{49}$$

Formula 3.

$$(t,t)_r e_r(x) = \sum_{\lambda\vdash r} \frac{t^{n(\lambda')}(t,t)_r}{H_{\lambda}(t)} S_{\lambda}(x,t) = \sum_{\lambda\vdash r} K_{\lambda,(1^r)}(t)S_{\lambda}(x,t). \tag{50}$$

Therefore,

$$J_{\mu}(x;1,t) = \prod_{i\geq 1}\left(\sum_{\lambda^{(i)}\vdash\mu'_i} K_{\lambda^{(i)},(1^{\mu'_i})}(t)S_{\lambda^{(i)}}(x,t)\right), \tag{51}$$

and after the plethystic change of variables $X \longmapsto \frac{X}{1-t}$, we obtain

$$\tilde{J}_{\mu}(x;1,t) = \prod_{i\geq 1}\left(\sum_{\lambda^{(i)}\vdash\mu_i} K_{\lambda^{(i)},(1^{\mu'_i})}(t)s_{\lambda^{(i)}}(x)\right) \tag{52}$$

Claim 5.11

$$\sum_{\lambda^{(i)}\vdash\mu'_i} K_{\lambda^{(i)},(1^{\mu'_i})}(t)s_{\lambda^{(i)}}(x) = \sum_{T} t^{\tilde{c}(T)}\ x^{Sh(T)}, \tag{53}$$

where the second sum runs over all standard tabloids T of the size r, and $\tilde{c}(T)$ denotes either the charge of T, or the value of tau function on the path corresponding to tabloid.

In the case $\tilde{c}(T) = c(T)$ the charge of tabloid T, this result is due to Lascoux–Schützenberger; in the case of $\tilde{c}(T) = n(\mu') - \tau(T)$ this statement is a corollary of Theorem 7.4 and Corollary 6.13 from [28], where identification of tau function and cocharge is given.

Corollary 5.12

$$\tilde{J}_{\mu}(x;1,t) = \sum_{\lambda\vdash|\mu|}\sum_{T} t^{\bar{c}_{\mu'}(T)}m_{\lambda}(x), \tag{54}$$

where the second sum runs over the set of standard tabloids of shape λ, and $\bar{c}_{\mu'}(T) = \sum \bar{c}(T^{(i)})$.

On the other hand,

$$\begin{aligned}\tilde{J}_\mu(x;1,t) &= \sum_\eta K_{\eta,\mu}(1,t)s_\eta(x)\\ &= \sum_\eta K_{\eta,\mu}(1,t)\sum_\lambda K_{\eta,\lambda}m_\lambda(x)\\ &= \sum_\lambda\left(\sum_\eta K_{\eta,\mu}(1,t)K_{\eta,\lambda}\right)m_\lambda(x),\end{aligned} \tag{55}$$

and therefore

$$\sum_\eta K_{\eta,\mu}(1,t)K_{\eta,\lambda} = \sum_T t^{\tilde{c}_{\mu'}(T)}, \tag{56}$$

where the sum in the right hand side runs over the set of standard λ-tabloids.

Finally,

$$\sum_\eta\ K_{\eta,\alpha}K_{\eta,\mu}(q,1) = \sum_\eta\ K_{\eta,\alpha}K_{\eta,\mu'}(1,q^{-1})\ q^{n(\mu')} = \sum_T\ q^{n(\mu')-\tilde{c}_\mu(T)}, \tag{57}$$

where the the third sum runs over the set of all standard α-tabloids.

It remains to observe that according to Lemma 3.4,

$$\sum_T\ n(\mu') - \tilde{c}_\mu(T) = \sum_i\ \left(\binom{\mu_i}{2} - c(p_{[i]})\right) = \tau_\mu - \sum_i\ \mu_i\ (1-\delta_{1,p_1^{[i]}}) = \mathrm{maj}_\mu(p).$$

□

5.3 Comments on generalizations of Section 5.2

In order to clarify nature of tau statistics, we consider possible generalizations of the results in Section 5.2.

5.3.1 Regularization map and parabolic Kostka polynomials

The main objective of this Section is to give an interpretation of the energy statistics partition function for the BBS as the value of a certain parabolic Kostka polynomial, see e.g., [21, 24]. This observation allows to write a fermionic formula for the parabolic Kostka polynomials in question, see e.g. [21], as well as appears to be useful in the study of the BBS, see e.g. [28].

Definition 5.13 *Let p be a path of type $\bigotimes_i B^{r_i,s_i}$ and weight λ, define regularization $\tilde{p} = \mathrm{reg}(p)$ of the path p to be*

$$\mathrm{reg}(p) = (1\cdots n-1)^{\lambda_n}\cdots(1\cdots i)^{\lambda_{i+1}}\cdots 1^{\lambda_2}p, \tag{58}$$

where $(1\cdots i) := 1\otimes\cdots\otimes i$, and we have omitted all symbols $\otimes$.

Let $\tilde{T}$ be semi-standard Young tableau (i.e., highest weight) corresponding to the regularized path $\tilde{p}$.

Lemma 5.14 ([28], Lemma 7.2) *Assume that all $r_i = 1$, then*

$$\tau(p) = \tau(\tilde{p}) + \text{const} = \bar{c}(\tilde{T}) + \text{Const}, \tag{59}$$

where

$$\text{Const} = L(L-\mu_1) - \binom{L}{2} + \sum_{a=1}^{n} a\binom{\mu_a}{2} + \sum_{1\le a<b\le n} a\mu_a\mu_b,$$

and $L = \sum_a\ m_a$.

Example 5.15 Let $p_1 = 3332221$, then

$$\tilde{p_1} = 1212121113332221.$$

We have $\tau(p_1) = 7$, $\tau(\tilde{p}_1) = 46$, and $\tau(\tilde{p}_1) - \tau(p_1) = 39$. On the other hand, let $p_2 = 3223123$, then

$$\tilde{p_2} = 1212121113223123,$$

and $\tau(p_2) = 14$, $\tau(\tilde{p}_2) = 53$, so that $\tau(\tilde{p}_2) - \tau(p_2) = 39$, as expected. Time evolution of p_2 is

3	2	2	3	1	2	3
1	1	1	2	3	1	2
1	1	1	1	2	3	1
1	1	1	1	1	2	3
1	1	1	1	1	1	2
1	1	1	1	1	1	1
1	1	1	1	1	1	1

Corollary 5.16

$$\sum_{\eta} K_{\eta,\mu}K_{\eta,\lambda}(q) = q^{C_1}K_{\Lambda,(\kappa,\lambda)}(q^{-1}), \tag{60}$$

where C_1 is a constant and partitions Λ and κ are defined as follows; $\Lambda = (\Lambda_1, \cdots, \Lambda_n)$, and $\Lambda_i = \sum_{a\ge i}\mu_a$, $i = 1, \cdots, n$; $\kappa = (\kappa_1, \cdots, \kappa_n)$, $\kappa_i = \sum_{a\ge i+1}$, $i = 1, \cdots, n-1$.

Proof. Indeed, we have

$$\text{LHS} = \sum_{p\in\mathcal{P}(\lambda)} q^{\tau_\mu(p)} = q^{C_2}\sum_{\tilde{T}} q^{\bar{c}(\tilde{T})} = q^{C_3}K_{\Lambda,(\kappa,\lambda)}(q^{-1}), \tag{61}$$

where summation in the third term runs over all Littlewood–Richardson tableaux of shape Λ and weight (κ, λ). □

Conjecture 5.17 *Let $R_1 = (s_i{}^{r_i})_{1,2,\cdots}$ be a sequence of rectangles, then*

$$\sum_{\eta} K_{\eta,R_1}K_{\eta,R_2}(q) = q^{C}K_{\Lambda,(\kappa,R_2)}(q^{-1}), \tag{62}$$

where Λ denotes partition $\left(\sum_{a\ge i} r_a\right)^{s_i}$, $\kappa = \left(\sum_{a\ge i+1} r_a\right)^{s_i}$.

It's well-known that parabolic Kostka polynomials $K_{\lambda,R}(q)$, where $R = \{(r_a^{s_a}\}_{a=1,2,\cdots}$ is a dominant (i.e., $r_1 \geq r_2 \geq \cdots$) sequence of rectangular shape partitions, satisfy the so-called duality theorem

$$K_{\lambda,R}(q) = K_{\lambda',R'}(q^{-1})q^{n(R)}, \tag{63}$$

where R' denotes a dominant rearrangement of the sequence of rectangular shape partitions $\{(s_a^{r_a})\}_{a=1,2,\cdots}$ and

$$n(R) = \sum_{1\leq a<b} \min(r_a, r_b)\min(s_a, s_b). \tag{64}$$

Questions 1. Can one define a set $\mathcal{P}(R_1, R_2)$ of paths π of rectangular type $R_2 = \bigotimes_a (r_a^{s_a})$ and rectangular weight $R_1 = \bigotimes_b (\mu_b^{\eta_b})$, and energy function $e(\pi)$ such that

$$KK_{R_1,R_2}(q) := \sum_{\pi} q^{e(\pi)} = \sum_{\eta} K_{\eta,R_1} K_{\eta,R_2}(q). \tag{65}$$

Questions 2. Are there some "physical interpretations" of the duality theorems

$$K_{\lambda,R}(q) = q^{n(R)} K_{\lambda',R'}(q) \tag{66}$$

and

$$KK_{R_1,R_2}(q) = q^{n(R_2)} KK_{R_1',R_2'}(q^{-1}) \tag{67}$$

for polynomials $K_{\lambda,R}(q)$ and $KK_{R_1,R_2}(q)$?

5.3.2 Generating functions of generalized tau statistics

Let us consider generating functions of the generalized tau statistics:

$$\tau^{r,s}(p) := \mathrm{maj}(u_s^{(r)} \otimes p), \tag{68}$$

where $u_s^{(r)} \in B^{r,s}$ is the highest element and $p \in B^{r_1,s_1} \otimes \cdots \otimes B^{r_L \cdot s_L}$. Let η be a partition and R be a sequence of rectangles, denote by $K_{\eta,R}(q)$ the corresponding parabolic Kostka polynomial. For any partition λ, let $K_{\eta,\lambda} := K_{\eta,\lambda}(1)$ be the corresponding Kostka number. Then we have:

Conjecture 5.18 *Let $R = \{(r_i, s_i)_{i=1}^L\}$ be a sequence of rectangles. Denote by $\mathcal{P}_R$ the set of all paths (including non-highest elements) corresponding to the tensor product of crystals $B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$. Then,*

$$\sum_{p\in\mathcal{P}_R,\, \mathrm{wt}(p)=\lambda} q^{\tau^{r,s}(p)} = q^{\phi_r} \sum_{\eta} K_{\eta,R}(q^{-1}) K_{\eta,\lambda}, \tag{69}$$

where $s \geq \max_i\{s_i\}$, $\phi_r \in \mathbb{Z}$.

In other words, the statistics $\tau^{r,s}$ defines the essentially unique class of polynomials although its definition depends on choice of $u_s^{(r)}$. As we see in the following example, this is not obvious from definition of $\tau^{r,s}$.

Example 5.19 Let us consider the case $\lambda = (4,3,3,2,2)$ and $B^{3,2} \otimes B^{2,2} \otimes B^{2,2}$. Then we have total of 759 paths, and by direct computations, we have the following summation over all 759 paths.

$$\begin{aligned}
\sum q^{\tau^{1,s}(p)} &= q^{10} + 8q^9 + 33q^8 + 89q^7 + 161q^6 + 198q^5 + 163q^4 + 82q^3 + 24q^2 \\
\sum q^{\tau^{2,s}(p)} &= q^{13} + 8q^{12} + 33q^{11} + 89q^{10} + 161q^9 + 198q^8 + 163q^7 + 82q^6 + 24q^5 \\
\sum q^{\tau^{3,s}(p)} &= q^{12} + 8q^{11} + 33q^{10} + 89q^9 + 161q^8 + 198q^7 + 163q^6 + 82q^5 + 24q^4 \\
\sum q^{\tau^{4,s}(p)} &= q^{10} + 8q^9 + 33q^8 + 89q^7 + 161q^6 + 198q^5 + 163q^4 + 82q^3 + 24q^2 \\
\sum q^{\tau^{r,s}(p)} &= q^8 + 8q^7 + 33q^6 + 89q^5 + 161q^4 + 198q^3 + 163q^2 + 82q + 24,
\end{aligned}$$

where $s = 2, \cdots, 5$ and, in the last expression, $5 \le r \le 10$. However, if we look at specific paths, for example,

$$b_1 = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&1\\\hline 3&5\\\hline\end{array}, \qquad b_2 = \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 5&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array},$$

$$b_3 = \begin{array}{|c|c|}\hline 1&2\\\hline 2&4\\\hline 3&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&1\\\hline 4&5\\\hline\end{array}, \qquad b_4 = \begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline 4&4\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&1\\\hline 2&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&1\\\hline 3&5\\\hline\end{array}.$$

Then we have

	$\tau^{1,5}$	$\tau^{2,5}$	$\tau^{3,5}$	$\tau^{4,5}$	$\tau^{5,5}$	$\tau^{6,5}$	$\tau^{7,5}$	$\tau^{8,5}$	$\tau^{9,5}$	$\tau^{10,5}$
b_1	5	7	9	6	3	3	3	3	3	3
b_2	4	9	8	8	2	2	2	2	2	2
b_3	7	7	6	3	2	2	2	2	2	2
b_4	9	9	5	3	3	3	3	3	3	3

In particular, dependences of $\tau^{r,s}(b)$ on r are different for each b.

Remark 5.20 We use the same notations of Conjecture 5.18. Then, as we will see in Corollary 6.4 below, we have $\phi_r = \phi_{l(\lambda)}$ for all $r \ge l(\lambda)$, where $l(\lambda)$ is length of weight λ.

5.4 Generating functions related with the energy statistics on the set of rectangular paths

Let us consider generating function with respect to $\bar{E}$ statistics. This is a more traditional problem compared with it for $\tau^{r,s}$. Indeed, special cases of this problem as well as its restriction to the set of highest weight paths were considered by several authors, see, e.g., [31, 11, 32, 39] and references therein.

Conjecture 5.21 *Let $R = \{(r_i, s_i)_{i=1}^L\}$ is a sequence of rectangles. Denote by $\mathcal{P}_R$ the set of all paths corresponding to tensor product of crystals $B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$. Then,*

$$\sum_{p \in \mathcal{P}_R,\, \mathrm{wt}(p)=\lambda} q^{\bar{E}(p)} = \sum_{\eta \vdash |\lambda|} K_{\eta,R}(q) K_{\eta,\lambda}. \tag{70}$$

Example 5.22 Let us consider the case $\lambda = (4, 6, 3, 1)$ and $R = \{(2, 2), (2, 2), (3, 2)\}$, i.e., $B^{2,2} \otimes B^{2,2} \otimes B^{3,2}$. Then we have the following nine paths:

p	$\bar{E}(p)$	p	$\bar{E}(p)$
$\begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&2\\\hline 2&3\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&2\\\hline 2&3\\\hline 3&4\\\hline\end{array}$	10	$\begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 2&2\\\hline 3&3\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline 3&4\\\hline\end{array}$	9
$\begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 2&2\\\hline 3&4\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline\end{array}$	10	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 2&3\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&2\\\hline 2&3\\\hline 3&4\\\hline\end{array}$	11
$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 2&3\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&2\\\hline 2&3\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline 3&4\\\hline\end{array}$	10	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 2&3\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&2\\\hline 2&4\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline\end{array}$	11
$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 2&4\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&2\\\hline 2&3\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline\end{array}$	11	$\begin{array}{\|c\|c\|}\hline 2&2\\\hline 3&3\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline 3&4\\\hline\end{array}$	12
$\begin{array}{\|c\|c\|}\hline 2&2\\\hline 3&4\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline\end{array} \otimes \begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline\end{array}$	12		

Therefore, the LHS of Eq.(70) is

$$q^9 + 3q^{10} + 3q^{11} + 2q^{12}.$$

On the other hand, non-zero contributions for RHS of Eq.(70) comes from

η	$K_{\eta,R}(q)$	$K_{\eta,\lambda}(q)$
$(6, 4, 3, 1)$	$q^9 + q^{10}$	1
$(6, 5, 2, 1)$	$q^{10} + q^{11}$	q
$(6, 4, 4)$	q^{10}	q
$(6, 5, 3)$	q^{11}	$q + q^2$
$(6, 6, 2)$	q^{12}	$q^2 + q^3$

Summing up,

$$\begin{aligned} \text{RHS} &= (q^9 + q^{10}) \cdot 1 + (q^{10} + q^{11}) \cdot 1 + (q^{10}) \cdot 1 + (q^{11}) \cdot 2 + q^{12} \cdot 2 \\ &= q^9 + 3q^{10} + 3q^{11} + 2q^{12}, \end{aligned}$$

which coincides with the LHS.

6 Discussion: $\tau^{r,s}$ and $\bar{E}$

So far in this paper, we have considered several statistics including generalized tau statistics $\tau^{r,s}$ and more traditional one $\bar{E}$. Let us investigate several further aspects of these two statistics. Our main results in this section are (i) to show that $\bar{E}$ belong to the class of statistics $\tau^{r,s}$ and (ii) to show that $\tau^{r,s}$ stabilize when we increase the value of r. The following proposition will be a key property.

Proposition 6.1 *Let $b_{i,j}$ be the integer at the i-th row, j-th column of the tableau representation of $b \in B^{r',s'}$, and let the highest element of $B^{r,s}$ be $u^{r,s}$. Then we have*

$$H(u^{r,s} \otimes b) = 0 \tag{71}$$

if $r \geq b_{r',s'}$ and $s \geq s'$.

Note that $b_{r',s'}$ is the largest integer in the tableau representation of b.

Proof. According to the algorithm presented in Proposition 2.1, we have to compute the insertion $b \leftarrow row(u^{r,s})$. This is worked out in Lemma 6.2 below, and shape of the resulting tableau coincides with the concatenation of two tableaux b and $u^{r,s}$. Hence $H(u^{r,s} \otimes b) = 0$ due to Proposition 2.1. □

Lemma 6.2 *Under the same assumptions of Proposition 6.1, the insertion*

$$b \leftarrow \underbrace{rr\cdots r}_{s} \cdots \underbrace{22\cdots 2}_{s}\underbrace{11\cdots 1}_{s}. \tag{72}$$

gives the concatenation of tableaux $u^{r,s}$ and b, i.e.,

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & \cdots & 1 & b_{1,1} & b_{1,2} & \cdots & b_{1,s'} \\ \hline
2 & 2 & \cdots & 2 & b_{2,1} & b_{2,2} & \cdots & b_{2,s'} \\ \hline
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline
r' & r' & \cdots & r' & b_{r',1} & b_{r',2} & \cdots & b_{r',s'} \\ \hline
\cdots & \cdots & \cdots & \cdots & & & & \\ \cline{1-4}
r & r & \cdots & r & & & & \\ \cline{1-4}
\end{array}. \tag{73}$$

Proof. This insertion procedure can be divided into two steps. First, we show

$$\tilde{b} := (b \leftarrow \underbrace{rr\cdots r}_{s} \cdots \underbrace{\bar{\delta}\bar{\delta}\cdots\bar{\delta}}_{s}\underbrace{\delta\delta\cdots\delta}_{s}) = \underbrace{\begin{array}{|c|c|c|c|}\hline b_{1,1} & b_{1,2} & \cdots & b_{1,s'} \\ \hline b_{2,1} & b_{2,2} & \cdots & b_{2,s'} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline b_{r',1} & b_{r',2} & \cdots & b_{r',s'} \\ \hline \end{array}}_{s'} \underbrace{\begin{array}{|c|c|c|c|}\hline \delta & \delta & \cdots & \delta \\ \hline \bar{\delta} & \bar{\delta} & \cdots & \bar{\delta} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline r & r & r & r \\ \hline \end{array}}_{s} \tag{74}$$

where $\delta = r - r' + 1$ and $\bar{\delta} = \delta + 1$. Note that we have $\delta \geq 1$ since (i) by the semi-standard property of b, we have $b_{r',s'} \geq r'$ and (ii) by the assumption $r \geq b_{r',s'}$, thus $r - r' \geq 0$. Again from the assumption $r \geq b_{r',s'}$, we have

$$r - i \geq b_{r'-i,s'} \qquad (0 \leq \forall i < r') \tag{75}$$

by the semi-standard property $b_{k-1,s'} < b_{k,s'}$. Consider the insertion $(b \leftarrow rr\cdots r)$. From Eq.(75), we have $r \geq r - (r' - 1) \geq b_{r'-(r'-1),s'} = b_{1,s'}$. Thus the first row of

$(b \leftarrow rr\cdots r)$ is $b_{1,1}b_{1,2}\cdots b_{1,s'}rr\cdots r$, and the remaining rows are identical to the corresponding rows of b. If $r=1$, this finishes the proof (i.e., by $r \geq b_{r',s'}$ we have $b_{r',s'}=1$), therefore let us consider the case $r>1$. Assume that we have, for some $r \geq k > \delta$ and $\bar{k}=k+1$,

$$b^\circ := (b \leftarrow \underbrace{rr\cdots r}_{s}\cdots\underbrace{\bar{k}\bar{k}\cdots\bar{k}}_{s}\underbrace{kk\cdots k}_{s}) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline b_{1,1} & b_{1,2} & \cdots & b_{1,s'} & k & k & \cdots & k \\ \hline b_{2,1} & b_{2,2} & \cdots & b_{2,s'} & \bar{k} & \bar{k} & \cdots & \bar{k} \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline b_{*,1} & b_{*,2} & \cdots & b_{*,s} & r & r & \cdots & r \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdots & \cdots & \cdots & \cdots \\ \hline b_{r',1} & b_{r',2} & \cdots & b_{r',s'} \\ \hline \end{array} \tag{76}$$

where $*=r'-k+1$. We insert $(k-1)$ into this b°. From Eq.(75) and the assumption $k>\delta$, we have $k>\delta=r-(r'-1)\geq b_{r'-(r'-1),s'}=b_{1,s'}$, thus $k-1\geq b_{1,s'}$. Therefore inserted $(k-1)$ bumps k at $(s'+1)$-th column of b°. As the next step, we have to insert k to the second row of b°. By the similar reasoning, it bumps $(k+1)$ at $(s'+1)$-th column. In this way, insertion of $(k-1)$ causes downward shift of $(s'+1)$-th column of b° and addition of $(k-1)$ to the first row of $(s'+1)$-th column. Similarly, we see that the second insertion of $(k-1)$ causes shift and addition of $(k-1)$ to the $(s'+2)$-th column. Continuing in this way, we see that successive insertions $(b^\circ \leftarrow (k-1)(k-1)\cdots(k-1))$ gives the tableau b° with replacement k by $k-1$. By induction, we can show Eq.(74).

Next, we consider the insertion of $(\delta-1)^s=(\delta-1)(\delta-1)\cdots(\delta-1)$ into $\tilde{b}$:

$$\tilde{\tilde{b}} := (\tilde{b} \leftarrow (\delta-1)^s). \tag{77}$$

We denote the i-th row of $\tilde{b}$ (resp. $\tilde{\tilde{b}}$) by $\tilde{b}_i$ (resp. $\tilde{\tilde{b}}_i$). Although we independently repeat insertions of $(\delta-1)$ for s times, we can argue more systematically as follows. Let us consider the first row of $\tilde{b}$:

$$\tilde{b}_1 = b_{1,1}b_{1,2}\cdots b_{1,s'}\underbrace{\delta\delta\cdots\delta}_{s}. \tag{78}$$

As we saw in the last paragraph, we have $b_{1,s'}\leq\delta$ from Eq.(75). Suppose there are k_1 letters δ in the first row of b, i.e., $b_{1,s'-k_1+1}=b_{1,s'-k_1+2}=\cdots=b_{1,s'}=\delta$. Thus there are $(s+k_1)$ letters δ in the first row of $\tilde{b}$. After inserting $(\delta-1)^s$, precisely s letters δ are bumped and go down to the second row, and the first row becomes

$$\tilde{\tilde{b}}_1 = b_{1,1}b_{1,2}\cdots b_{1,s'-k_1}\underbrace{(\delta-1)(\delta-1)\cdots(\delta-1)}_{s}\underbrace{\delta\delta\cdots\delta}_{k_1}. \tag{79}$$

In particular, k_1 letters δ on the right of the first row of b are precisely reproduced in the right part of $\tilde{\tilde{b}}_1$.

Now we have to consider the insertion of s letters δ which are bumped from $\tilde{b}_1$ into the second row of $\tilde{b}$:

$$\tilde{b}_2 = b_{2,1}b_{2,2}\cdots b_{2,s'}\underbrace{(\delta+1)(\delta+1)\cdots(\delta+1)}_{s}. \tag{80}$$

Again, from Eq.(75), we have $b_{2,s'} \le \delta+1$, and suppose that there are k_2 letters $(\delta+1)$ in the second row of b, i.e., $b_{2,s'-k_2+1} = b_{2,s'-k_2+2} = \cdots = b_{2,s'} = \delta+1$. After inserting δ^s, precisely s letters $(\delta+1)$ are bumped and go down to the third row, and the second row becomes

$$\tilde{\tilde{b}}_2 = b_{2,1}b_{2,2}\cdots b_{2,s'-k_2}\underbrace{\delta\delta\cdots\delta}_{s}\underbrace{(\delta+1)(\delta+1)\cdots(\delta+1)}_{k_2}. \tag{81}$$

Again, k_2 letters $(\delta+1)$ on the right of the second row of b are precisely reproduced in the right part of $\tilde{\tilde{b}}_2$.

As we see in the above discussions, this procedure can be continued recursively. Let the number of $\delta+i$ contained in the $(i+1)$-th row of b be k_{i+1}. Then $\tilde{\tilde{b}}_{i+1}$ is

$$b_{i+1,1}b_{i+1,2}\cdots b_{i+1,s'-k_{i+1}}\underbrace{(\delta+i-1)(\delta+i-1)\cdots(\delta+i-1)}_{s}\underbrace{(\delta+i)(\delta+i)\cdots(\delta+i)}_{k_{i+1}} \tag{82}$$

and the right k_{i+1} letters $(\delta+i)$ are copy of those originally contained in the right of $(i+1)$-th row of b. In this way, $\tilde{\tilde{b}}$ contains copy of the letters in b.

As we have investigated insertion of $(\delta-1)^s$, let us consider the insertion of $(\delta-2)^s = (\delta-2)(\delta-2)\cdots(\delta-2)$ into $\tilde{\tilde{b}}$. Consider the row $\tilde{\tilde{b}}_1$. Suppose that there are l_1 letters $(\delta-1)$ within the first row of b, i.e., $b_{s'-(k_1+l_1)+1} = \cdots = b_{s'-k_1} = \delta-1$. Thus there are total of l_1+s letters $\delta-1$ in the row $\tilde{\tilde{b}}_1$. Therefore, after insertion of $(\delta-2)^s$, the row $\tilde{\tilde{b}}_1$ becomes

$$b_{1,1}b_{1,2}\cdots b_{1,s'-(k_1+l_1)}\underbrace{(\delta-2)(\delta-2)\cdots(\delta-2)}_{s}\underbrace{(\delta-1)(\delta-1)\cdots(\delta-1)}_{l_1}\underbrace{\delta\delta\cdots\delta}_{k_1}. \tag{83}$$

In particular, l_1 letters $(\delta-1)$ and k_1 letters δ are copy of the corresponding letters of the first row of b. As the result, s letters $(\delta-1)$ are bumped from $\tilde{\tilde{b}}_1$. Therefore, for the row $\tilde{\tilde{b}}_2$, we have to insert $(\delta-1)^s$, and again obtain copy of the letters δ and $(\delta+1)$ contained in the second row of b. We can continue this procedure until the bottom row of $\tilde{\tilde{b}}$. Therefore each row of the resulting tableau $(\tilde{\tilde{b}} \leftarrow (\delta-2)^s)$ contains copy of at most two species of letters in the original b. We can recursively continue insertions of $(\delta-3)^s$, $(\delta-4)^s$, $\cdots$, 1^s, and each insertion generates copy of part of letters of b. Finally we get the result Eq.(73). □

The following theorem shows that the statistics $\bar{E}$ essentially belong to the class of statistics $\tau^{r,s}$.

Theorem 6.3 *Let $p = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$, and denote by N the largest integer contained in tableau representation of p. If $r \geq N$ and $s \geq \max_i\{s_i\}$, we have*

$$\bar{E}(p) = C - \tau^{r,s}(p), \tag{84}$$

$$C = \sum_{i<j} \min(r_i, r_j) \cdot \min(s_i, s_j). \tag{85}$$

Proof. Recall the definition of $\tau^{r,s}$

$$\tau^{r,s}(p) = \mathrm{maj}(u^{r,s} \otimes p), \tag{86}$$

where maj is defined by Eq.(15) and $u^{r,s}$ is the highest element of $B^{r,s}$. Within the definition of $\tau^{r,s}$, let us first consider the energy functions involving $u^{r,s}$ and next consider the remaining ones. As for the terms involving $u^{r,s}$, we have

$$H(u^{r,s} \otimes b_j^{(1)}) = 0 \qquad (1 \leq \forall j \leq L) \tag{87}$$

by Proposition 6.1. On the other hand, for the remaining contributions, recall that $b_i \in B^{r_i,s_i}$ and $b_j^{(i+1)} \in B^{r_j,s_j}$. Then from the definition of normalizations, we have

$$\bar{H}(b_i \otimes b_j^{(i+1)}) = \min(r_i, r_j) \cdot \min(s_i, s_j) - H(b_i \otimes b_j^{(i+1)}). \tag{88}$$

Combining both contributions, we obtain the sought relation. □

Corollary 6.4 *Let $p = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$, and denote by N the largest integer contained in tableau representation of p, and define $S = \max_i\{s_i\}$. Then we have*

$$\tau^{N,S}(p) = \tau^{r,s}(p) \tag{89}$$

for all $r \geq N$ and $s \geq S$.

Therefore Conjecture 5.18 means that we can define at most N independent statistics $\tau^{r,s}(p)$, where N is the largest integer contained in the path p, and these statistics have essentially unique generating function.

Let us remark physical interpretation of $\tau^{r,s}$. For the paths (including non-highest elements) of shape $B^{1,s_1} \otimes B^{1,s_2} \otimes \cdots \otimes B^{1,s_L}$ and statistics $\tau^{1,S}$ ($S = \max_i\{s_i\}$), there is a straightforward generalization of Theorem 4.2, see Section 4.1 of [28]. Under the same assumptions, $\tau^{1,S}$ is identified with cocharge of the unrestricted rigged configurations. It will be an interesting problem to find a physical interpretation of the more general $\tau^{r,s}(p)$.

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