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On the structure of higher order simple-pole type operators in exact WKB analysis

By

Takahiro KAWAI, Tatsuya KOIKE and Yoshitsugu TAKEI

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
KYOTO UNIVERSITY, Kyoto, Japan

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Takahiro Kawai
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-8502 JAPAN

Tatsuya Koike
Department of Mathematics
Graduate School of Science
Kobe University
Kobe, 657-8501 JAPAN

and

Yoshitsugu Takei
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-8502 JAPAN

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1 Introduction

The purpose of this paper is to introduce the notion of higher order simple-pole type operators and to relate the operators to second order simple-pole type operators, which have been studied by Olver ([O]), Koike ([Ko1], [Ko2], [Ko3]) and others. The operators we are to deal with are, in an intuitive description, higher order linear ordinary differential operators with a large parameter η whose coefficients may have simple poles at the origin of $\mathbf{C}$. Although we give a precise definition of such operators in Section 2, here we present a simple and illuminating example of a third order simple-pole type operator so that the reader may envisage what they look like:

$$P_{\mathrm{BNR}'} = \frac{d^3}{dx^3} + \frac{3}{2x}\frac{d^2}{dx^2} + \frac{3}{4x}\eta^2\frac{d}{dx} + \frac{i}{4x}\eta^3 \tag{1.1}$$

is the example, where the suffix BNR' indicates that this operator is closely tied up with the celebrated Berk-Nevins-Roberts operator ([BNR]). The concrete WKB analysis of this operator together with the explanation of its relationship with the Berk-Nevins-Roberts operator will be given in Section 3 as an illustration of the power of our main result (Theorem 2.1), namely the decomposition theorem for a higher order simple-pole type operator. It states, in plain language, that a higher order simple-pole type operator is, in a neighborhood of the origin, a second order simple-pole type operator multiplied by innocent factors. Here an innocent factor means a differential operator irrelevant to the Stokes geometry of the higher order operator in question near the origin. To exemplify the content of Theorem 2.1 we show how $P_{\mathrm{BNR}'}$ is decomposed à la Theorem 2.1:

(1.2)
$$P_{\mathrm{BNR}'} = \left(\frac{d}{dx} - \eta q(x,\eta)\right)\left(\frac{d^2}{dx^2} + \eta a_1(x,\eta)\frac{d}{dx} + \eta^2 a_2(x,\eta)\right),$$

where

(1.3) $xq(x,\eta) = \sum_{k\geq 0} xq_k(x)\eta^{-k}$ and $xa_j(x,\eta) = \sum_{k\geq 0} xa_{j,k}(x)\eta^{-k}$ $(i = 1, 2)$ are symbols of microdifferential operator (cf. e.g. [K^3], [AKY] for the basic facts about microdifferential operators),

(1.4) q_k $(k \neq 1)$ is holomorphic on a neighborhood U of the origin,

(1.5) xq_1 is holomorphic on U and the residue of q_1 at the origin is -1,

(1.6) $a_{1,k} = q_k \quad (k \neq 1)$,

(1.7) $xa_{1,1}$ and $xa_{2,k}$ are holomorphic on U.

The conditions (1.6) and (1.7) guarantee that the first factor of the right-hand side of (1.2) is, in essence, an operator studied in [Ko1] from the viewpoint of the exact WKB analysis, while (1.4) means that the first factor has no relevance to the Stokes geometry in question. Thus the connection formula for WKB solutions of $P_{\mathrm{BNR}'}$ near the origin is obtained from the results for the second order operators. Now it will be easy for the reader to surmise the core part of Theorem 2.1 in the general context; it is a counterpart of the decomposition theorem for a higher order operator near a simple turning point (versus a simple pole) that was obtained in [AKT2] (see also [AKKoT1]) . Our main result in this paper, i.e., Theorem 2.1 asserts that a simple-pole type operator P of order m (≥ 2) can be expressed as

$$P = QR, \tag{1.8}$$

where Q is an innocent operator of order $(m - 2)$ and R is a second order simple-pole type operator. As this paper is the first one of a series of articles on simple-pole type operators, we restrict our consideration to the simplest operators of the sort. There are, however, several higher order operators which may deserve the name "simple-pole type operators" but that are not covered by Theorem 2.1. Hence we briefly discuss in Section 4 what kind of generalizations of Definition 2.1 below may be imagined.

2 Decomposition theorem for simple-pole type operators

Let us first give the definition of a (higher order) simple-pole type operator.

Definition 2.1. *Let U be an open neighborhood of the origin $x = 0$ of $\mathbf{C}$, and let $A_{j,k}$ $(j = 1, 2, \cdots, m;\ k = 0, 1, 2, \cdots)$ be meromorphic functions on U which satisfy the following conditions* (2.1) $\sim$ (2.4)*:*

(2.1) $A_{1,k}$ $(k \neq 1)$ is holomorphic on U,

(2.2) $xA_{1,1}$ and $xA_{j,k}$ $(j = 2, 3, \cdots, m;\ k \geq 0)$ are holomorphic on U,

(2.3) there exists a constant C_K for each compact set K in U for which
$$\sup_{x \in K} |xA_{j,k}(x)| \leq C_K^k k!$$
holds for every $k \geq 0$ and $j = 1, 2, \cdots, m$,

(2.4) for $\alpha_j \underset{\text{def}}{=} \operatorname*{Res}_{x=0} A_{j,0} \quad (j = 2, 3, \cdots, m)$ we find

(2.4.a) $\alpha_2 \neq 0, \alpha_m \neq 0$

and

(2.4.b) $f(\zeta) \underset{\text{def}}{=} \sum_{j=2}^{m} \alpha_j \zeta^{m-j} = 0$ has $(m-2)$ mutually different roots.

Then the following m-th order differential operator P with a large parameter η is called **a simple-pole type operator** (with its singularity at the origin)

$$P = \frac{d^m}{dx^m} + \eta A_1(x, \eta) \frac{d^{m-1}}{dx^{m-1}} + \cdots + \eta^m A_m(x, \eta), \tag{2.5}$$

where

$$A_j(x, \eta) = \sum_{k \geq 0} A_{j,k}(x) \eta^{-k}. \tag{2.6}$$

Remark 2.1. (i) It is evident that $P_{\mathrm{BNR}'}$ given in (1.1) is one of the simplest examples of simple-pole type operators; actually only finitely many $A_{j,k}$'s survive in $P_{\mathrm{BNR}'}$.
(ii) When $m = 2$, it is customary to rewrite the equation $P\psi = 0$ into the Schrödinger form by using the gauge transformation

$$\varphi = \exp\left(\frac{1}{2}\eta \int^x A_1(x, \eta) dx\right) \psi; \tag{2.7}$$

then the equation that φ solves is

$$\frac{d^2\varphi}{dx^2} = \eta^2 Q(x, \eta) \varphi \tag{2.8}$$

with

$$Q = -A_2(x, \eta) + \frac{1}{4} A_1(x, \eta)^2 + \frac{1}{2} \eta^{-1} \frac{dA_1(x, \eta)}{dx}. \tag{2.9}$$

Writing

$$Q = \sum_{k\geq 0} Q_k(x)\eta^{-k}, \tag{2.10}$$

we find

$$Q_0 = -A_{2,0} + \frac{1}{4}A_{1,0}^2, \tag{2.11}$$

$$Q_1 = -A_{2,1} + \frac{1}{2}A_{1,0}A_{1,1} + \frac{1}{2}\frac{dA_{1,0}}{dx}, \tag{2.12}$$

$$Q_2 = -A_{2,2} + \frac{1}{4}\left(2A_{1,0}A_{1,2} + A_{1,1}^2\right) + \frac{1}{2}\frac{dA_{1,1}}{dx}, \tag{2.13}$$

$$Q_k = -A_{2,k} + \frac{1}{4}\Big(\sum_{p+q=k} A_{1,p}A_{1,q}\Big) + \frac{1}{2}\frac{dA_{1,k-1}}{dx} \quad (k \geq 3). \tag{2.14}$$

Then (2.2) and (2.1) entail Q_0 has a simple pole at the origin with residue $-\alpha_2$, which is different from 0 by the assumption (2.4.a). Assumptions (2.1) and (2.2) again guarantee that Q_1 is with a simple pole at the origin. Similarly Q_2 is with at most a double pole at the origin. An interesting feature of Q_k ($k \geq 3$) is that it is with at most a simple pole at the origin thanks to the assumption (2.1). Its important consequence is that the invariant λ that Koike ([Ko1]) uses is a genuine constant (versus an infinite series of η^{-1} that is independent of x). This particular property of the potential Q will be discussed in Section 4.

We also note that the regularity of $A_{1,0}$ implies

$$\int^x A_{1,0}(x)dx \tag{2.15}$$

converges to a finite value as x tends to 0. This fact guarantees that the gauge transformation (2.7) does not conflict with the basic property of a Borel transformed WKB solution of a second order simple-pole type equation to the effect that its singularities converge to finite points (versus points at infinity) as x tends to the origin; actually Koike was partly motivated by this geometric property of the Borel transformed WKB solution in developing his theory on simple-pole type operators ([Ko1], [Ko2]).

(iii) The condition (2.3) guarantees that $x\bigl(\sum_{k\geq 0} A_{j,k}(x)\eta^{-k}\bigr)$ is a symbol of a microdifferential operator $x\bigl(\sum_{k\geq 0} A_{j,k}(\partial/\partial y)^{-k}\bigr)$. This aspect of our theory will be discussed in our subsequent article; in this paper we concentrate our attention primarily upon the formal aspect of the theory.

One point worth emphasizing concerning Definition 2.1 is that characteristic properties (2.1) $\sim$ (2.4) of a simple-pole type operator are stable under the decomposition given in the following

Theorem 2.1. *Let P be an m-th $(m \geq 3)$ order simple-pole type operator of the form* (2.5)*. Then there exist an open neighborhood V of the origin, meromorphic functions $q_k(x)$ defined on V and an $(m-1)$-st order simple-pole type operator R defined on V which satisfy the following relations* (2.16) $\sim$ (2.19) *if we let $q(x,\eta)$ denote $\sum_{k\geq 0} q_k\eta^{-k}$ and define a differential operator Q by $d/dx - \eta q(x,\eta)$:*

(2.16) $P = QR$,

(2.17) *q_k $(k \neq 1)$ is holomorphic on V,*

(2.18) *xq_1 is holomorphic on V, and the residue of q_1 at the origin is -1,* (2.19) *for each compact set K in V there exists a constant M_K for which the following relation holds for $k \geq 2$:* $\sup_K |q_k(x)| \leq M_K^k k!$.

Remark 2.2. (i) Since q_0 is holomorphic on V, the highest degree part $d/dx - \eta q_0(x)$ of the operator Q has nothing to do with the Stokes geometry of P near the origin.
(ii) Since the operator R is again a simple-pole type operator, Theorem 2.1 applies to R. Hence by repeating the above decomposition we eventually arrive at a second order simple-pole type operator R_2; we can find first order differential operators $Q_1, \cdots, Q_{m-2}$ and a second order simple-pole type operator R_2 which satisfy

$$P = Q_1 Q_2 \cdots Q_{m-2} R_2, \tag{2.20}$$

where Q_j has the form

$$\frac{d}{dx} - \eta q_j(x, \eta) \tag{2.21}$$

with $q_j(x, \eta) = \sum_{k \geq 0} q_{j,k}(x) \eta^{-k}$ for $\{q_{j,k}\}_k$ satisfying conditions (2.17) $\sim$(2.19).

Proof of Theorem 2.1. Let us write down the required simple-pole type operator R as

$$\frac{d^{m-1}}{dx^{m-1}} + \eta a_1(x, \eta) \frac{d^{m-2}}{dx^{m-2}} + \cdots + \eta^{m-1} a_{m-1}(x, \eta) \tag{2.22}$$

with

$$a_j(x, \eta) = \sum_{k \geq 0} a_{j,k}(x) \eta^{-k}, \tag{2.23}$$

and try to find $\{a_{j,k}\}$ together with $\{q_k\}$ so that (2.16) is satisfied. Then the comparison of coefficients of like orders of differentiation in

(2.16) entails the following relations

$$(2.24)\qquad \begin{cases} A_1 = a_1 - q & (2.24.1) \\ A_2 = a_2 - qa_1 + \eta^{-1}a_1' & (2.24.2) \\ A_3 = a_3 - qa_2 + \eta^{-1}a_2' & (2.24.3) \\ \quad\vdots & \quad\vdots \\ A_{m-1} = a_{m-1} - qa_{m-2} + \eta^{-1}a_{m-2}' & (2.24.m-1) \\ A_m \quad = \qquad\quad - qa_{m-1} + \eta^{-1}a_{m-1}'. & (2.24.m) \end{cases}$$

Here, and in what follows, a_1' etc. respectively designate da_1/dx etc. In what follows we try to construct $a_{j,k}$ and q_k by comparing the coefficients of like powers of η^{-1} in (2.24). First the comparison of the top degree part, i.e., the degree 0 part of (2.24) results in the following relations.

$$(2.25)\qquad \begin{cases} A_{1,0} = a_{1,0} - q_0 & (2.25.1) \\ A_{2,0} = a_{2,0} - q_0 a_{1,0} & (2.25.2) \\ A_{3,0} = a_{3,0} - q_0 a_{2,0} & (2.25.3) \\ \quad\vdots & \quad\vdots \\ A_{m-1,0} = a_{m-1,0} - q_0 a_{m-2,0} & (2.25.m-1) \\ A_{m,0} \quad = \qquad\quad - q_0 a_{m-1,0}. & (2.25.m) \end{cases}$$

Solving the equations (2.25) for $a_{j,0}$ $(1 \le j \le m-1)$ and q_0, we find

$$(2.26)\qquad a_{j,0} = \sum_{l=0}^{j} A_{l,0} q_0^{j-l} \quad (j = 1, 2, \cdots, m-1)$$

and

$$(2.27)\qquad q_0^m + A_{1,0} q_0^{m-1} + A_{2,0} q_0^{m-2} + \cdots + A_{m,0} = 0.$$

To find the required holomorphic function q_0, let us consider the func-

tion
(2.28)
$$F(x,\zeta) = x\zeta^m + xA_{1,0}(x)\zeta^{m-1} + xA_{2,0}(x)\zeta^{m-2} + \cdots + xA_{m,0}(x).$$
Then it follows from (2.1) and (2.4) that the equation

$$F(0,\zeta) = \alpha_2\zeta^{m-2} + \cdots + \alpha_m = 0 \tag{2.29}$$

has $(m-2)$ mutually different roots. Let us choose and fix one of them, say ζ_0. We note

$$\zeta_0 \neq 0 \tag{2.30}$$

by the assumption (2.4.a). For the ζ_0 thus chosen we find

$$\frac{\partial F}{\partial \zeta}(x,\zeta)\Big|_{(x,\zeta)=(0,\zeta_0)} \neq 0, \tag{2.31}$$

as every root of (2.29) is simple. Thus the implicit function theorem tells us that the equation $F(x,\zeta) = 0$ has a unique analytic solution $\zeta(x)$ such that $\zeta(0) = \zeta_0$. Denoting the domain of definition of $\zeta(x)$ by V $(\subset U)$, we can choose $\zeta(x)$ as the required $q_0(x)$. Then it follows from (2.1) and (2.25.1) that

$$a_{1,0} = A_{1,0} + q_0 \tag{2.32}$$

is also holomorphic on V. Similarly we find

(2.33) $xa_{j,0}$ $(j = 2, 3, \cdots, m-1)$ is holomorphic on V.

Concerning the residue of $a_{j,0}$ at the origin, we find by (2.4.a) and (2.25.2)

$$\operatorname*{Res}_{x=0} a_{2,0} = \operatorname*{Res}_{x=0} A_{2,0} \neq 0. \tag{2.34}$$

Since we have

$$F(x,\zeta) = \big(\zeta - q_0(x)\big)\big(x\zeta^{m-1} + xa_{1,0}(x)\zeta^{m-2} + xa_{2,0}(x)\zeta^{m-3} + \cdots + xa_{m-1,0}(x)\big), \tag{2.35}$$

we set $\zeta = x = 0$ in (2.35) to find

$$-q_0(0) \operatorname*{Res}_{x=0} a_{m-1,0} = \operatorname*{Res}_{x=0} A_{m,0} \neq 0. \tag{2.36}$$

Note that $q_0(0) = \zeta_0 \neq 0$ by (2.30). Furthermore we find

$$F(0,\zeta) = (\zeta - \zeta_0)\Big((\operatorname*{Res}_{x=0} a_{2,0})\zeta^{m-3} + \cdots + \operatorname*{Res}_{x=0} a_{m-1,0}\Big) = f(\zeta), \tag{2.37}$$

and hence (2.4.b) entails that

$$\big(\operatorname*{Res}_{x=0} a_{2,0}\big)\zeta^{m-3} + \cdots + \operatorname*{Res}_{x=0} a_{m-1,0} = 0 \tag{2.38}$$

has $(m-3)$ mutually different roots. Thus the functions $\{a_{j,0}\}_{j=1,2,\cdots,m-1}$ meet the requirement (2.4) that the top degree part of a simple-pole type operator should satisfy.

Now we study the degree (-1) (in η) part of (2.24). We are then to find $\{a_{j,1}\}_{1\leq j\leq m-1}$ and q_1 that satisfy

(2.39)

$$\begin{cases} A_{1,1} = a_{1,1} \qquad\qquad - q_1 & (2.39.1) \\ A_{2,1} = a_{2,1} - q_0 a_{1,1} - q_1 a_{1,0} + a'_{1,0} & (2.39.2) \\ A_{3,1} = a_{3,1} - q_0 a_{2,1} - q_1 a_{2,0} + a'_{2,0} & (2.39.3) \\ \quad\vdots & \quad\vdots \\ A_{m-1,1} = a_{m-1,1} - q_0 a_{m-2,1} - q_1 a_{m-2,0} + a'_{m-2,0} & (2.39.m-1) \\ A_{m,1} \quad = \qquad - q_0 a_{m-1,1} - q_1 a_{m-1,0} + a'_{m-1,0}. & (2.39.m) \end{cases}$$

Let us first rewrite (2.39) in a matrix form for the sake of clarity, i.e.,

$$C \begin{pmatrix} q_1 \\ a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m-1,1} \end{pmatrix} = \begin{pmatrix} -A_{1,1} \\ a'_{1,0} - A_{2,1} \\ a'_{2,0} - A_{3,1} \\ \vdots \\ a'_{m-1,0} - A_{m,1} \end{pmatrix}, \tag{2.40}$$

where

$$C = \begin{pmatrix} 1 & -1 & & 0 & \\ a_{1,0} & q_0 & -1 & & \\ a_{2,0} & 0 & q_0 & -1 & \\ \vdots & & & \ddots & -1 \\ a_{m-1,0} & & 0 & & q_0 \end{pmatrix}. \tag{2.41}$$

Then Cramer's formula tells us

$$q_1 = \frac{\det \tilde{C}}{\det C}, \tag{2.42}$$

where

$$\tilde{C} = \begin{pmatrix} -A_{1,1} & -1 & & 0 & \\ a'_{1,0} - A_{2,1} & q_0 & -1 & & \\ a'_{2,0} - A_{3,1} & 0 & q_0 & -1 & \\ \vdots & & & \ddots & -1 \\ a'_{m-1,0} - A_{m,1} & & 0 & & q_0 \end{pmatrix}. \tag{2.43}$$

Similarly we can find $\{a_{j,1}\}_{1\le j\le m-1}$ by Cramer's formula. As we see below $\det C$ is not identically 0, the existence of q_1 and $\{a_{j,1}\}_{1\le j\le m-1}$ as meromorphic functions is thus obvious. To show that they satisfy required conditions, we need to find some concrete expression of $\det C$ and $\det \tilde{C}$. Actually a straightforward computation shows the following

$$\det C = q_0^{m-1} + a_{1,0}q_0^{m-2} + \cdots + a_{m-1,0} \tag{2.44}$$

and

$$\det \tilde{C} = -A_{1,1}q_0^{m-1} + (a'_{1,0} - A_{2,1})q_0^{m-2} + (a'_{2,0} - A_{3,1})q_0^{m-3} + \cdots + a'_{m-1,0} - A_{m,1}. \tag{2.45}$$

Since q_0 and $a_{1,0}$ are holomorphic on V, (2.33) entails that $x \det C$ is holomorphic on V and that

(2.46)

$$\gamma \underset{\text{def}}{=} \operatorname*{Res}_{x=0} \det C$$

$$= \big(\operatorname*{Res}_{x=0} a_{2,0}\big) q_0(0)^{m-3} + \big(\operatorname*{Res}_{x=0} a_{3,0}\big) q_0(0)^{m-4} + \cdots + \operatorname*{Res}_{x=0} a_{m-1,0}.$$

Similarly we find

$$\det \tilde{C} = -\frac{\gamma}{x^2} + O\Big(\frac{1}{x}\Big). \tag{2.47}$$

These results are crucially important to confirm the required properties of q_1, and the confirmed properties of q_1 are effectively employed to show the simple-pole character of $\{a_{j,1}\}_{1\le j\le m-1}$ in our reasoning below.

We first note that (2.46) and (2.37) imply

$$\gamma = f'(\zeta_0) = \frac{\partial F}{\partial \zeta}(0, \zeta_0). \tag{2.48}$$

Then (2.31) guarantees

$$\gamma \neq 0. \tag{2.49}$$

Hence by shrinking V if necessary we may assume

(2.50) $x \det C \neq 0$ on V.

Then it follows from (2.42), (2.46), and (2.47) that

$$x q_1(x) = \frac{x^2 \det \tilde{C}}{x \det C} = \frac{-\gamma + O(x)}{\gamma + O(x)}. \tag{2.51}$$

Thus we find that $xq_1(x)$ is holomorphic on V. Furthermore the right-most side of (2.51) means

$$\operatorname*{Res}_{x=0} q_1 = -1. \tag{2.52}$$

Then (2.39.1) implies that

(2.53) $xa_{1,1}$ is holomorphic on V.

Since q_0 and $a_{1,0}$ are holomorphic on V, we conclude from (2.39.2) that

(2.54) $xa_{2,1}$ is holomorphic on V.

A similar assertion about $a_{j,1}$ ($j \geq 3$) is somewhat more subtle; the right-hand side of (2.39.3) might appear to contain a double pole originating from

$$-q_1 a_{2,0} + a'_{2,0}, \tag{2.55}$$

but we infer from (2.52) that the double poles in (2.55) cancel each other out. Thus (2.39.3) entails that

(2.56) $xa_{3,1}$ is holomorphic on V.

The same reasoning applies to (2.39.j) ($4 \leq j \leq m-1$) and we conclude that

(2.57) $xa_{j,1}$ ($4 \leq j \leq m-1$) is holomorphic on V.

Thus q_1 and $\{a_{j,1}\}_{1 \leq j \leq m-1}$ constructed by Cramer's formula enjoy all the required properties.

The treatment of the degree $(-k)$ ($k \geq 2$) part of (2.24) is basically the same as that in the case of $k = 1$. There is, however, a tiny difference; we assume that the coefficient $A_1(x,\eta) = \sum_{k \geq 0} A_{1,k}(x)\eta^{-k}$ of a simple-pole type operator P should satisfy (2.1), and accordingly it should be satisfied also by $a_1(x,\eta) = \sum_{k \geq 0} a_{1,k}(x)\eta^{-k}$ that appears in the simple-pole type operator R which we are now constructing. This means that, while $a_{1,1}$ is with a simple pole, $a_{1,k}$ ($k \neq 1$) is holomorphic near the origin. Hence to make the induction on k run smoothly, we use the induction starting with $k = 2$. Now the degree (-2) part of (2.24) is as follows:

(2.58)

$$\begin{cases} A_{1,2} = a_{1,2} - q_2 & (2.58.1) \\ A_{2,2} = a_{2,2} - \sum_{k=0}^{2} q_{2-k} a_{1,k} + a'_{1,1} & (2.58.2) \\ A_{3,2} = a_{3,2} - \sum_{k=0}^{2} q_{2-k} a_{2,k} + a'_{2,1} & (2.58.3) \\ \vdots & \vdots \\ A_{m-1,2} = a_{m-1,2} - \sum_{k=0}^{2} q_{2-k} a_{m-2,k} + a'_{m-2,1} & (2.58.m-1) \\ A_{m,2} \quad = \qquad - \sum_{k=0}^{2} q_{2-k}, a_{m-1,k} + a'_{m-1,1}. & (2.58.m) \end{cases}$$

In parallel with the case $k = 1$, we rewrite (2.58) in a matrix form (2.59) below, again using the matrix C given by (2.41):

$$C \begin{pmatrix} q_2 \\ a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m-1,2} \end{pmatrix} = \begin{pmatrix} \mathfrak{a}_{0,2} \\ \mathfrak{a}_{1,2} \\ \mathfrak{a}_{2,2} \\ \vdots \\ \mathfrak{a}_{m-1,2} \end{pmatrix}, \tag{2.59}$$

where

$$\mathfrak{a}_{0,2} = -A_{1,2} \tag{2.60}$$

and

$$\mathfrak{a}_{j,2} = a'_{j,1} - q_1 a_{j,1} - A_{j+1,2} \quad (1 \leq j \leq m-1). \tag{2.61}$$

Here we note that (2.52) together with the simple-pole character of $a_{j,1}$ guarantees that

(2.62) $a'_{j,1} - q_1 a_{j,1}$ is with a simple pole at the origin.

This means that, if we set

$$\tilde{C}_2 = \begin{pmatrix} \mathfrak{a}_{0,2} & -1 & & 0 & \\ \mathfrak{a}_{1,2} & q_0 & -1 & & \\ \mathfrak{a}_{2,2} & 0 & q_0 & -1 & \\ \vdots & & & \ddots & -1 \\ \mathfrak{a}_{m-1,2} & & 0 & & q_0 \end{pmatrix}, \tag{2.63}$$

$$\det \tilde{C}_2 = \mathfrak{a}_{0,2} q_0^{m-1} + \mathfrak{a}_{1,2} q_0^{m-2} + \mathfrak{a}_{2,2} q_0^{m-3} + \cdots + \mathfrak{a}_{m-1,2} \tag{2.64}$$

has at most a simple pole at the origin. Therefore the simple-pole character of $\det C$ guarantees that

$$q_2 = \frac{\det \tilde{C}_2}{\det C} \tag{2.65}$$

is holomorphic on V. Then it follows from (2.58.1) together with the holomorphy of $A_{1,2}$ that $a_{1,2}$ is also holomorphic. To confirm the simple-pole character of $a_{2,2}$, it suffices to note that

$$-(q_2 a_{1,0} + q_1 a_{1,1} + q_0 a_{1,2}) + a'_{1,1} \tag{2.66}$$

is again free from double poles. Using the holomorphy of q_0 and q_2 together with (2.52), we can confirm, with the help of the induction on j, that $a_{j,2}$ $(2 \le j \le m-1)$ is with a simple pole at the origin. Thus we find the required q_2 and $\{a_{j,2}\}_{1 \le j \le m-1}$. We now use the induction on k; we assume that q_k and $\{a_{j,k}\}_{1 \le j \le m-1}$ have been appropriately found for $k \le k_0 - 1$, with $k_0 \ge 3$. Then the degree $(-k_0)$ part of (2.24) is as follows:

(2.67)
$$\begin{cases} A_{1,k_0} = a_{1,k_0} - q_{k_0} & (2.67.1) \\ A_{2,k_0} = a_{2,k_0} - \sum_{k=0}^{k_0} q_{k_0-k} a_{1,k} + a'_{1,k_0-1} & (2.67.2) \\ A_{3,k_0} = a_{3,k_0} - \sum_{k=0}^{k_0} q_{k_0-k} a_{2,k} + a'_{2,k_0-1} & (2.67.3) \\ \quad \vdots & \vdots \\ A_{m-1,k_0} = a_{m-1,k_0} - \sum_{k=0}^{k_0} q_{k_0-k} a_{m-2,k} + a'_{m-2,k_0-1} & (2.67.m-1) \\ A_{m,k_0} \quad = \qquad - \sum_{k=0}^{k_0} q_{k_0-k} a_{m-1,k} + a'_{m-1,k_0-1}. & (2.67.m) \end{cases}$$

By letting $\mathfrak{a}_{0,k_0}$ and $\mathfrak{a}_{j,k_0}$ $(1 \le j \le m-1)$ denote respectively

(2.68)
$$-A_{1,k_0}$$

and

(2.69)
$$a'_{j,k_0-1} - \sum_{k=1}^{k_0-1} q_{k_0-k} a_{j,k} - A_{j+1,k_0},$$

we can rewrite (2.67) in a matrix form:

(2.70)
$$C \begin{pmatrix} q_{k_0} \\ a_{1,k_0} \\ a_{2,k_0} \\ \vdots \\ a_{m-1,k_0} \end{pmatrix} = \begin{pmatrix} \mathfrak{a}_{0,k_0} \\ \mathfrak{a}_{1,k_0} \\ \mathfrak{a}_{2,k_0} \\ \vdots \\ \mathfrak{a}_{m-1,k_0} \end{pmatrix}.$$

Here we observe that

(2.71) $\mathfrak{a}_{j,k_0}$ $(1 \le j \le m-1)$ is free from double poles despite the existence of potentially harmful terms

$$a'_{j,k_0-1} - q_1 a_{j,k_0-1}, \tag{2.72}$$

because a_{j,k_0-1} is with a simple pole at the origin and the residue of q_1 there is -1. In parenthesis we note that terms in $\mathfrak{a}_{j,k_0}$ $(1 \le j \le m-1)$ other than (2.72) are at most with simple poles at the origin by the holomorphy of q_k $(k \ne 1)$ on V. Defining a matrix $\tilde{C}_{k_0}$ by

$$\begin{pmatrix} \mathfrak{a}_{0,k_0} & -1 & & 0 & \\ \mathfrak{a}_{1,k_0} & q_0 & -1 & & \\ \mathfrak{a}_{2,k_0} & 0 & q_0 & -1 & \\ \vdots & & & \ddots & -1 \\ \mathfrak{a}_{m-1,k_0} & & 0 & & q_0 \end{pmatrix}, \tag{2.73}$$

we infer from (2.71) that

$$\det \tilde{C}_{k_0} = \mathfrak{a}_{0,k_0} q_0^{m-1} + \mathfrak{a}_{1,k_0} q_0^{m-2} + \mathfrak{a}_{2,k_0} q_0^{m-3} + \cdots + \mathfrak{a}_{m-1,k_0} \tag{2.74}$$

is with at most a simple pole at the origin. Hence we conclude

$$q_{k_0} = \frac{\det \tilde{C}_{k_0}}{\det C} \tag{2.75}$$

is holomorphic on V. Then a_{1,k_0} is holomorphic by (2.67.1). The simple-pole character of a_{j,k_0} $(2 \le j \le m-1)$ can be immediately confirmed with the induction on j applied to (2.67). Thus we have found q_{k_0} and $\{a_{j,k_0}\}_{1 \le j \le m-1}$ with the required analytic properties.

Once the construction of these functions is completed we can confirm the estimation (2.19) for $\{q_k\}_k$ and (2.3) for $\{a_{j,k}\}_k$ by the same reasoning used in [AKT1, Appendix §A.1]. For the sake of completeness we repeat the core part of [AKT1].

In what follows we fix an arbitrary point x_0 in $V - \{0\}$, and we let $D(r)$ denote a closed disc centered at x_0 and with radius r. Let r_0 be a positive number such that

$$D(r_0) \subset V - \{0\}. \tag{2.76}$$

By (2.3) we may assume the following:

(2.77) There exists a constant M for which

$$\sup_{\substack{D(r_1) \\ j=1,2,\cdots,m}} \left|A_{j,k}(x)\right| \le k! M^k (r_0 - r_1)^{-k}$$

holds for any $r_1 < r_0$.

We then prove the following by the induction on k:

(2.78) There exists a constant C for which

$$\sup_{\substack{D(r_1) \\ j=1,2,\cdots,m-1}} \{\left|q_k\right|, \left|a_{j,k}\right|\} \le k! C^k (r_0 - r_1)^{-k}$$

holds.

The validity of (2.78) for $k = 0$ is clear. In view of (2.74) and (2.75), it suffices to show

$$\sup_{\substack{D(r_1) \\ j=1,2,\cdots,m-1}} \left|\mathfrak{a}_{j,k_0}\right| \le k_0! C^{k_0} (r_0 - r_1)^{-k_0} \tag{2.79}$$

holds on the condition that (2.79) holds for $k \le k_0 - 1$. To show (2.79) we follow the reasoning in [AKT1]. To dominate $\left|da_{j,k}/dx\right|$ we use the following device: for each positive number r that is smaller than r_0 we use the induction hypothesis by defining

$$r_1 = r + \frac{r_0 - r}{k_0}. \tag{2.80}$$

Then we have

$$r_0 - r_1 = \Big(1 - \frac{1}{k_0}\Big)(r_0 - r). \tag{2.81}$$

Hence Cauchy's formula entails

$$\sup_{D(r)}\left|\frac{da_{j,k_0-1}}{dx}\right| \le (k_0-1)!C^{k_0-1}(r_0-r_1)^{-k_0+1}\frac{k_0}{r_0-r} \tag{2.82}$$
$$\le k_0!C^{k_0-1}\Big(1-\frac{1}{k_0}\Big)^{-k_0+1}(r_0-r)^{-k_0}$$
$$\le k_0!C^{k_0-1}e(r_0-r)^{-k_0},$$

where $e = 2.718\cdots$. Since

$$\sum_{k=1}^{k_0-1}(k_0-k)!k! \le 4(k_0-1)!, \tag{2.83}$$

we obtain

$$\sup_{\substack{D(r)\\ j=1,2,\cdots,m-1}} \left|\mathfrak{a}_{j,k_0}\right| \le k_0!C^{k_0}\Big(eC^{-1}+4{k_0}^{-1}+\Big(\frac{M}{C}\Big)^{k_0}\Big)(r_0-r)^{-k_0}. \tag{2.84}$$

Hence by choosing C sufficiently large we find

$$\sup_{\substack{D(r)\\ j=1,2,\cdots,m-1}} \left|\mathfrak{a}_{j,k_0}\right| \le k_0!C^{k_0}(r_0-r)^{-k_0}. \tag{2.85}$$

Since $D(r)$ is a closed disc centered at x_0 in $V-\{0\}$, and x_0 is an arbitrary point in $V-\{0\}$, we can confirm (2.19) for $\{q_k\}_k$ and (2.3) for $\{a_{j,k}\}_k$. Note that it suffices to dominate $a_{j,k}$, a function with a simple pole at the origin, on an arbitrary compact set in $V-\{0\}$ to dominate $xa_{j,k}$ on V. Thus we have completed the proof of Theorem 2.1.

□

3 An example — $P_{\mathrm{BNR}'}$

In this section we discuss how to describe the Stokes geometry of $P_{\mathrm{BNR}'}$ given by (1.1). As mentioned in the introduction, the operator $P_{\mathrm{BNR}'}$

is closely tied up with the BNR operator

$$P_{\text{BNR}} = \frac{d^3}{dz^3} + 3\eta^2 \frac{d}{dz} + 2iz\eta^3. \tag{3.1}$$

Actually by applying a singular coordinate transformation

$$x = z^2 \tag{3.2}$$

to P_{BNR}, we find

$$\begin{aligned} P_{\text{BNR}'} &= (8z^3)^{-1} P_{\text{BNR}} \\ &= \frac{d^3}{dx^3} + \frac{3}{2x}\frac{d^2}{dx^2} + \frac{3\eta^2}{4x}\frac{d}{dx} + \frac{i\eta^3}{4x}. \end{aligned} \tag{3.3}$$

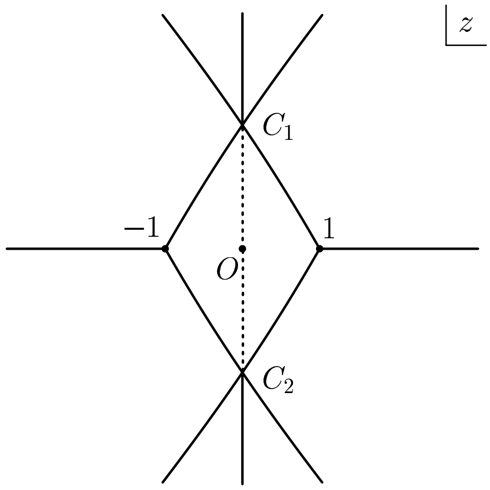

Figure 3.1.

As is now well-known, the Stokes geometry of the BNR operator is an intriguing one; as is seen in Figure 3.1, it has a virtual turning point $z = 0$ besides two simple turning points $z = \pm 1$, and the Stokes curve emanating from the virtual turning point, i.e., the so-called new Stokes curve, is inert on its dotted portion $[C_1, C_2]$ in the sense that no Stokes phenomena are observed there among WKB solutions of the BNR equation. See [AKT2] for the Stokes geometry of the BNR equation; we note that a virtual turning point is called a new turning point in [AKT2]. Concerning the Stokes geometry of $P_{\mathrm{BNR}'}$, we can find Figure 3.2 with the help of a computer. In Figure 3.2 Stokes curves emanating from the turning point $x = 1$ is given by

$$\mathrm{Im} \int_1^x \big(\xi_1(x) - \xi_2(x)\big) dx = 0, \tag{3.4}$$

where $\xi_1(x)$ and $\xi_2(x)$ are solutions of the characteristic equation of $P_{\mathrm{BNR}'}$, i.e.,

$$\xi^3 + \frac{3}{4x}\xi + \frac{i}{4x} = 0 \tag{3.5}$$

which coincide at $x = 1$, while the Stokes curve near the origin is given by

$$\mathrm{Im} \int_0^x \big(\xi_\alpha(x) - \xi_\beta(x)\big) dx = 0, \tag{3.6}$$

where $\xi_\alpha(x)$ and $\xi_\beta(x)$ are solutions of (3.5) that satisfy

$$\xi_\alpha(x), \xi_\beta(x) = O(1/\sqrt{x}\,) \tag{3.7}$$

near $x = 0$. Otherwise stated, we regard the origin as a kind of turning points from which Stokes curves emanate. This recipe is a straightforward generalization of T. Koike's recipe for obtaining the Stokes geometry for second order simple-pole type operators, and its validity in our case is guaranteed by the decomposition theorem (Theorem 2.1) for higher order simple-pole type operators. Geometrically speaking, we may understand the point $x = 1$ (resp., $x = C$) to be the image of simple turning points $z = \pm 1$ (resp., crossing points C_1 and C_2 in Figure 3.1) under the transformation (3.2). In parenthesis, we note that there is no virtual turning point in the Stokes geometry of $xP_{\mathrm{BNR}'}$. (See Appendix for the proof.) Analytically speaking, however, we are somewhat baffled; no Stokes phenomena are observed along the

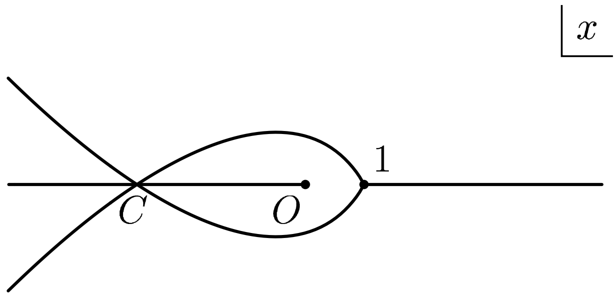

Figure 3.2.

segment $[C_1, C_2]$ in Figure 3.1, while along its image $[C, O]$ in Figure 3.2 we normally expect some Stokes phenomena. On the other hand, a computer-assisted study ([AKKoT3]) indicates that no Stokes phenomena should be observed across the segment $[C, O]$ in Figure 3.2. The method employed in [AKKoT3] is the steepest descent one; some subtle cancellation of integrals along some paths of steepest descent results in the above indication despite some change of the topography of relevant paths of steepest descent. This indication is analytically confirmed by the combination of Theorem 2.1 and the results of Koike ([Ko1], [Ko2], [Ko3]) for the second order simple-pole type operators. Actually Koike's results applied to the second factor in the right-hand side of (1.2) read as follows: when WKB solutions are analytically continued across a Stokes curve emanating from O, the relevant Stokes multiplier is

$$2i\cos(\pi c) \tag{3.8}$$

with

$$c = \operatorname*{Res}_{x=0} a_{1,1}. \tag{3.9}$$

On the other hand the decomposition (1.2) entails

$$a_{1,1} = \frac{3}{2x} + q_1. \tag{3.10}$$

Hence (1.5) implies

$$c = \frac{1}{2}, \tag{3.11}$$

showing that the quantity in (3.8) is 0! This means that the portion $[C, O]$ of the Stokes curves in Figure 3.2 is actually inert, and it is cleanly consistent with Figure 3.1.

Remark 3.1. We were very keen to know what kind of mechanism lay behind the subtly beautiful Stokes geometry of $P_{\mathrm{BNR}'}$. It was the main motivation of the experimental study done in [AKKoT3].

4 Discussions and concluding remarks

Our eventual purpose of this paper and subsequent ones is to find an appropriate class of higher order simple-pole type operators and develop the exact WKB analysis of operators in the class. Theorem 2.1 indicates that we will be able to obtain a good class of operators if we find a class of operators that is stable under the successive applications of division by innocent operators. In order to use in the discussion that follows we summarize Theorem 2.1 using some symbols.

Let (S_0) designate the totality of operators satisfying the conditions (2.1) $\sim$ (2.4) and let (F) designate the totality of operators that has the form

$$\frac{d}{dx} - \eta q(x, \eta), \tag{4.1}$$

where q satisfies the conditions (2.17) $\sim$ (2.19). Then Theorem 2.1 may be rephrased as follows:

Theorem 4.1. (i) *For each operator P in (S_0) we can find an operator Q in (F) that factorizes P, that is,*

$$P = QR \tag{4.2}$$

holds for some operator R *in* (S_0) *(possibly with smaller domain of definition).*
(ii) *If* P *in* (S_0) *is of the second order, then* P *can be reduced by an appropriate gauge transformation* (2.7) *to a Schrödinger operator studied in* [Ko3].

Here we note that the potential V of the Schrödinger operator studied in [Ko3] satisfies the following condition (4.3).

$$V(x,\eta) = \frac{V_0(x)}{x} + \frac{V_1(x)}{x}\eta^{-1} + \sum_{k\geq 2} \frac{V_k(x)}{x^2}\eta^{-k}, \tag{4.3}$$

with $\{V_k\}$ satisfying the conditions (4.4) $\sim$ (4.7) below:

(4.4) Each V_k is holomorphic on a neighborhood U of the origin of $\mathbf{C}$,

(4.5) $V_0(0) \neq 0$,

(4.6) $V_k(0) = 0 \quad \text{for} \quad k \geq 3$,

(4.7) for each compact set K in U there exists a constant C_K for which
$$\sup_K |V_k(x)| \leq C_K^k k!$$
holds.

In what follows we denote by (C_0) the totality of a potential $V(x,\eta)$ that satisfies the condition (4.3).

In view of Theorem 4.1 one may naturally be tempted to find out an appropriate triplet that may work as a substitute of $((S_0),(F),(C_0))$. One candidate is the triplet $((L),(\tilde{F}),(\tilde{C}))$ given in the discussion of Case 1 below. In what follows we omit the conditions on the growth order of the coefficients (like (4.7)) and concentrate our attention on the formal aspect of the problem (except in Case 3).

Case 1. The large class (L) of simple-pole type operators

Let P be an operator of the following form:

$$\frac{d^m}{dx^m}+\eta A_1(x,\eta)\frac{d^{m-1}}{dx^{m-1}}+\eta^2 A_2(x,\eta)\frac{d^{m-2}}{dx^{m-2}}+\cdots+\eta^m A_m(x,\eta), \tag{4.8}$$

where

$$A_j(x,\eta)=\sum_{k\geq 0} A_{j,k}(x)\eta^{-k} \tag{4.9}$$

with $A_{j,k}(x)$ meromorphic on a neighborhood U of the origin of $\mathbf{C}$.

Then we say P belongs to the class (L) if the following conditions (4.10) $\sim$ (4.12) are satisfied:

(4.10) for any j and k, $a_{j,k} \underset{\text{def}}{=} x^j A_{j,k}$ is holomorphic on U,

(4.11) the discriminant of the polynomial

$$F(x,\theta) \underset{\text{def}}{=} \theta^m + a_{1,0}(x)\theta^{m-1}+a_{2,0}\theta^{m-2}+\cdots+a_{m,0}(x)$$

in θ has a simple zero at the origin,

(4.12) $F=0$ has $(m-2)$ simple roots as an equation in θ.

The class $(\tilde{F})$ of first order operators and the class $(\tilde{C})$ of potentials are introduced below in conjunction with the class (L).

A first order operator $\tilde{Q}$ of the form

$$\frac{d}{dx}+\eta\,\frac{p(x,\eta)}{x} \tag{4.13}$$

is, by definition, in the class $(\tilde{F})$ if

$$p(x,\eta)=\sum_{k\geq 0} p_k(x)\eta^{-k} \tag{4.14}$$

with $p_k(x)$ being holomorphic on U, and a potential $V(x,\eta)$ is in $(\tilde{C})$ if it satisfies the following conditions:

$$V(x,\eta) = \frac{V_0(x)}{x} + \frac{V_1(x)}{x^2}\eta^{-1} + \sum_{k\geq 2} \frac{V_k(x)}{x^2}\eta^{-k}, \tag{4.15}$$

where

(4.16) $V_k(x)$ $(k \geq 0)$ is holomorphic on U

and

(4.17) $V_0(0) \neq 0$.

We then find the following

Theorem 4.2. (i) *For each operator P in (L) we can find an operator $\tilde{Q}$ in $(\tilde{F})$ so that*

$$P = \tilde{Q}R \tag{4.18}$$

holds for some operator R in (L) (possibly with smaller domain of definition).
(ii) *If P in (L) is of the second order, then P can be reduced by an appropriate gauge transformation* (2.7) *to a Schrödinger operator with its potential in the class $(\tilde{C})$.*

The proof of Theorem 4.2 is similar to that of Theorem 2.1, and this result is probably the cleanest one from the algebraic viewpoint. Furthermore the definition of class (L) nicely fits in with our naive intuition that a simple pole plays the role of a turning point as it appears through the confluence of a (simple) turning point and a singular point. (Cf. (4.11).) Still we do not seriously study operators in class (L) because of the following three obstacles.

Obstacle 1. The structure of a Schrödinger operator with its potential in the class $(\tilde{C})$ is difficult to analyze. As a matter of fact, we do not know what kind of connection formula we can expect even for the operator whose potential is

$$V(x,\eta) = \frac{1}{x} + \frac{\alpha}{x^2}\eta^{-1} \tag{4.19}$$

with α non-zero constant.

We believe analysis of such an operator is worth doing, separately from other obstacles described below.

Obstacle 2. For a second order operator P that is in class (L), the integral (2.15) is a divergent one in general. Thus in view of Remark 2.1 we are not sure whether an operator in (L) has all the properties that we imagine a simple-pole type operator to have.

Obstacle 3. Even if Obstacle 1 should be removed, we would be unable to obtain enough information about the (semi-)global structure of WKB solutions of the equation defined by a higher order operator in class (L). The trouble is that an operator $\tilde{Q}$ in $(\tilde{F})$ is associated with Stokes curves that flow toward the origin which may cross a Stokes curve of the operator R that emanates from the origin. In general we have to seek for virtual turning points which are to resolve the troubles around the crossing point. (See [AKT2] for the details.) Our computer-assisted study indicates that this is almost a formidable task, because the Stokes curves of $\tilde{Q}$ normally flow toward the origin in a spiral and cross a Stokes curve of R infinitely many times.

The class (M) discussed in Case 2 below is designed to circumvent Obstacles 1 and 2.

Case 2. The medium class (M) of simple-pole type operators

Let P be an operator of the form (4.8) whose coefficients satisfy the condition (4.10). Then we say P is in class (M) if the following conditions (4.20) $\sim$ (4.23) are satisfied:

(4.20) $$a_{m-1,0}(0) = a_{m,0}(0) = a_{m,1}(0) = 0,$$

(4.21) $$(da_{m,0}/dx)(0) \neq 0,$$

(4.22) $$a_{m-2,0}(0) \neq 0,$$

(4.23) the equation
$$\theta^{m-2} + a_{1,0}(0)\theta^{m-3} + \cdots + a_{m-2,0}(0) = 0$$
has mutually distinct $(m-2)$ roots which are all different from 0.

In conjunction with this class of operators we introduce the following class (C) of potentials: let $V(x, \eta)$ be a potential of the following form

(4.24) $$\frac{V_0(x)}{x} + \frac{V_1(x)}{x}\eta^{-1} + \sum_{k \geq 2} \frac{V_k(x)}{x^2}\eta^{-k},$$

where

(4.25) $V_k(x)$ $(k \geq 0)$ is holomorphic on U

and

(4.26) $$V_0(0) \neq 0.$$

Then we say V is in class (C). Concerning operators in class (M) we find the following

Theorem 4.3. (i) *For each operator P in (M) we can find an operator $\tilde{Q}$ in $(\tilde{F})$ so that*

(4.27) $$P = \tilde{Q}R$$

holds for some operator R in (M) (possibly with smaller domain of definition).

(ii) *If P in (M) is of the second order, then P can be reduced by an appropriate gauge transformation* (2.7) *to a Schrödinger operator with its potential in the class (C).*

The proof of Theorem 4.3 is again similar to that of Theorem 2.1. Obstacle 2 observed in Case 1 disappears this time. The trouble in analyzing a Schrödinger operator with its potential in (C) is that its canonical form contains an infinite series ([Ko3]); the canonical form is

$$-\frac{d^2}{dx^2}+\eta^2\left(\frac{1}{x}+\eta^{-2}\frac{\lambda}{x^2}\right), \tag{4.28}$$

with

$$\lambda=\sum_{k\geq 0}V_{k+2}(0)\eta^{-k}. \tag{4.29}$$

Note that λ is a genuine constant when V is in (C_0). Hence the difficulty in analyzing a Schrödinger operator with its potential in (C) should be enormous. This anticipation, however, does not make us feel intimidated but rather encouraged; a recent study ([AKT3]) of the Weber equation with its constant being an infinite series strongly suggests us that the use of microdifferential operators ([AKY], [K^3]) should be effective in analyzing such a canonical equation. But the problem with class (M) is that Obstacle 3 persists. Thus our target in our subsequent work should be the triplet $((S),(F),(C))$ to be described in Case 3 below.

Case 3. The class (S) of amenable simple-pole type operators

Let P be an operator of the form

$$\frac{d^m}{dx^m}+\eta A_1(x,\eta)\frac{d^{m-1}}{dx^{m-1}}+\cdots+\eta^m A_m(x,\eta), \tag{4.30}$$

where

$$A_j(x,\eta) = \sum_{k\geq 0} A_{j,k}(x)\eta^{-k} \tag{4.31}$$

with $A_{j,k}$ being a meromorphic function on a neighborhood U of the origin in $\mathbf{C}$. Then we say P is in class (S) if the following conditions (4.32) $\sim$ (4.35) are satisfied:

(4.32) $A_{1,0}$ is holomorphic on U,

(4.33) $xA_{1,k}$ $(k \geq 1)$ and $xA_{j,k}$ $(j = 2, 3, \cdots, m;\ k \geq 0)$ are holomorphic on U,

(4.34) for each compact set K in U there exists a constant C_K for which

$$\sup_{x\in K} |xA_{j,k}| \leq C_K^k k!$$

holds for every k and $j = 1, 2, \cdots, m$,

(4.35) for $\alpha_j \underset{\text{def}}{=} \operatorname*{Res}_{x=0} A_{j,0}$ $(j = 2, 3, \cdots, m)$ we find

$$\alpha_2 \neq 0, \quad \alpha_m \neq 0 \tag{4.35.a}$$

and

$$f(\zeta) \underset{\text{def}}{=} \sum_{j=2}^{m} \alpha_j \zeta^{m-j} = 0 \tag{4.36}$$

has $(m-2)$ mutually different roots.

As the reader readily finds, an operator P in (S) belongs to (S_0) if $A_{1,k}$ $(k \geq 2)$ is holomorphic on U. Hence in order to prove Theorem 4.4 below we only need to replace the sentence below (2.65) and that below (2.75) respectively by the following sentences:

"Then it follows from (2.58.1) together with the simple-pole character of $A_{1,2}$ that $a_{1,2}$ is also with a simple pole at the origin."

and

“Then a_{1,k_0} is with a simple pole at the origin by (2.67.1).”

Keeping other parts of the proof of Theorem 2.1 intact, we obtain

Theorem 4.4. (i) *For each operator P in (S) we can find an operator Q in (F) that factorizes P, that is*

$$P = QR \tag{4.37}$$

holds for some operator R in (S) (possibly with smaller domain of definition).
(ii) *If P in (S) is of the second order, then P can be reduced by an appropriate gauge transformation* (2.7) *to a Schrödinger operator with its potential in class (C).*

This theorem will be the starting point of the second part of this series of papers (in preparation).

Remark 4.1. In view of examples where several (possibly infinitely many) simple turning points with different characteristic values sit on the same point ([AKKoT2]), one might imagine such a situation could also be managed for simple-pole type operators. But analysis of such operators would be very difficult, because the characteristic value associated with any simple-pole should be infinity if properly defined. (See the paragraph below (A.12) in Appendix.) Hence, if two simple-poles sit on the same point (e.g. the product of two second order simple-pole type operators), then the situation should be regarded as a rather degenerate one.

Appendix. Bicharacteristic strips of $xP_{\mathrm{BNR}'}$

The purpose of this appendix is to confirm there exists no virtual turning point for $xP_{\mathrm{BNR}'}$ (or $4xP_{\mathrm{BNR}'}$ for convenience). A virtual turning point is, by definition, the x-component of a self-intersection point of a bicharacteristic curve of the Borel transform of the operator in question. As a virtual turning point is usuallly considered for operators with holomorphic coefficients, we start with the operator $P = 4xP_{\text{BNR}'}$ instead of $P_{\text{BNR}'}$ itself; that is, we study the bicharacteristic strips of the operator

$$\tilde{P} = 4x\frac{\partial^3}{\partial x^3} + 3\frac{\partial^3}{\partial x \partial y^2} + i\frac{\partial^3}{\partial y^3} + 6\frac{\partial^2}{\partial x^2}. \tag{A.1}$$

Then the associated Hamilton-Jacobi equation is

$$\begin{cases} \dfrac{dx}{dt} = 12x\xi^2 + 3\eta^2 \\ \dfrac{dy}{dt} = 6\xi\eta + 3i\eta^2 \\ \dfrac{d\xi}{dt} = -4\xi^3 \\ \dfrac{d\eta}{dt} = 0, \end{cases} \tag{A.2}$$

and hence the bicharacteristic strip of $\tilde{P}$ emanating from $(x, y; \xi, \eta) = (1, y_0; -i/2, 1)$ in $T^*\mathbf{C}^2_{(x,y)}$ is described as follows:

$$\begin{cases} x(t) = -3(2t-1) - 2i(2t-1)^{3/2} \\ y(t) = 3(2t-1)^{1/2} + \dfrac{3i}{2}(2t-1) - \dfrac{3i}{2} + y_0 \\ \xi(t) = \dfrac{1}{2(2t-1)^{1/2}} \\ \eta(t) = 1, \end{cases} \tag{A.3}$$

where the branch of $(2t-1)^{1/2}$ is chosen so that

(A.4) $$(2t-1)^{1/2}\big|_{t=0} = i.$$

It is clear that for any point (x_0, ξ_0) satisfying

(A.5) $$4x_0\xi_0^3 + 3\xi_0 + i = 0,$$

we can find t_0 so that

(A.6) $$(x(t_0), \xi(t_0)) = (x_0, \xi_0)$$

holds. This means that any point in the characteristic variety of $\tilde{P}$ can be reached by the bicharacteristic strip given by (A.3) with an appropriate choice of y_0. Thus it suffices for us to use only the particular bicharacteristic strip given by (A.3) in seeking for a virtual turning point of P. Otherwise stated, what we have to check is whether or not the relation $t = t'$ follows from the following simultaneous equations:

(A.7) $$\begin{cases} x(t) = x(t') & \text{(A.7.1)} \\ y(t) = y(t'). & \text{(A.7.2)} \end{cases}$$

For the sake of simplicity of notations we set $s = (2t-1)^{1/2}$ and $s' = (2t'-1)^{1/2}$. Then it follows from (A.7.2) together with the assumption $t \neq t'$ that

(A.8) $$s + s' = 2i.$$

Similarly (A.7.1) and the assumption $t \neq t'$ imply

(A.9) $$3(s+s') + 2i\{(s+s')^2 - ss'\} = 0.$$

Combining (A.8) and (A.9) we find

(A.10) $$s = s' = i.$$

This contradicts the assumption $t \neq t'$. Thus we have confirmed that the operator $P = 4xP_{\mathrm{BNR}'}$ does not have a virtual turning point.

Using this opportunity, we give a geometric explanation of why a simple pole in $P_{\mathrm{BNR}'}$ plays a role of a turning point from the analytic viewpoint, as Koike ([Ko1], [Ko2]) has found. In the traditional WKB analysis the notion of a turning point is defined in terms of a characteristic equation, that is, a turning point is, by definition, a zero of the discriminant of the following polynomial in ξ:

$$p = 4x\xi^3 + 3\xi + i. \tag{A.11}$$

Hence only the point $x = 1$ is the turning point of P in the traditional approach. But in the exact WKB analysis a turning point should be a point where two "cognate" singularities coalesce in the Borel plane. From the viewpoint of microlocal analysis, the most reasonable criterion for judging whether two singularities are cognate or not is given by checking whether they are relevant to the same bicharacteristic strip. Not only a traditional turning point but also a virtual turning point are defined in this way. (See [AKT2], [AHKKoNSSY] and references cited there.) In the case of the operator $P_{\mathrm{BNR}'}$ we observe by (A.3) that there exist two points $y = y_+(x)$ and $y = y_-(x)$ in the Borel plane which coalesce at $x = 0$. Note that $\xi(t)$ is multi-valued near $t = 1/2$; we find that this fact and the existence of mutually nearby points $y = y_\pm(x)$ near $x = 0$ are the two sides of the same coin, in view of the relation

$$\frac{dy}{dt}\Big/\frac{dx}{dt} = -\xi(t), \tag{A.12}$$

which follows from (A.3). As is clear from (A.3), $y = y_+(x)$ and $y = y_-(x)$ are cognate in the sense that both originate from the same bicharacteristic strip. Thus $x = 0$ should play the role of a turning point, that is, it should be relevant to the Stokes phenomena. We note that ξ becomes infinite at $t = 1/2$. This means that, unless we use the compactification in the ξ-space, the traditional definition of a turning point is not applicable to this case. But the picture of the bicharacteristic strip of P described above indicates that a simple pole might be regarded as a virtual turning point in the extended sense, i.e., in the sense that two distinct but cognate singularities in the Borel plane coalesce at the pole.

References

[AHKKoNSSY] T. Aoki, N. Honda, T. Kawai, T. Koike, Y. Nishikawa, S. Sasaki, A. Shudo and Y. Takei: Virtual turning points — A gift of microlocal analysis to the exact WKB analysis, Algebraic Analysis of Differential Equations, Springer-Verlag, 2008, pp.29–43.

[AKY] T. Aoki, K. Kataoka and S. Yamazaki: Hyperfunctions · FBI Transformation · Pseudo-differential Operators of Infinite Order, Kyouritsu, Tokyo, 2004. (In Japanese)

[AKKoT1] T. Aoki, T. Kawai, T. Koike and Y. Takei: On the exact WKB analysis of operators admitting infinitely many phases, Adv. Math., **181** (2004), 165–189.

[AKKoT2] T. Aoki, T. Kawai, T. Koike and Y. Takei: On global aspects of exact WKB analysis of operators admitting infinitely many phases, Contemporary Math., **373** (2005), 11–47.

[AKKoT3] ______: A fresh glimpse into the Stokes geometry of Berk-Nevins-Roberts equation through a singular coordinate transformation, RIMS-Kôkyûroku **1431** (2005), 1–13.

[AKT1] T Aoki, T. Kawai and T. Takei: The Bender-Wu analysis and the Voros theory, Special Functions, Springer-Verlag, 1991, pp.1–29.

[AKT2] ______: New turning points in the exact WKB analysis for higher-order ordinary differential equations, Analyse algébrique des perturbations singulières I, Hermann, 1994, pp.69–84.

[AKT3] ______: The Bender-Wu analysis and the Voros theory. II, preprint (RIMS-1616)), 2007. To appear in Advanced Studies in Pure Mathematics.

[BNR] H. K. Berk, W. M. Nevins and K. V. Roberts: New Stokes' line in WKB theory, J. Math. Phys., **23** (1982), 988–1002.

[K^3] K. Kashiwara, T. Kawai and T. Kimura: Foundations of Algebraic Analysis, Princeton University Press, 1986. Japanese edition was published by Kinokuniya in 1980.

[Ko1] T. Koike: On a regular singular point in the exact WKB analysis, Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear, Kyoto University Press, 2000, pp.39–54.

[Ko2] ______: On the exact WKB analysis of second order linear ordinary differential equations with simple poles, Publ. RIMS, Kyoto Univ., **36** (2000), 297–319.

[Ko3] ______: On a connection problem of simple pole type operators of second order in exact WKB analysis, RIMS Kôkyûroku **1433** (2005), 9–26.

[O] F. W. J. Olver: Asymptotics and Special Functions, A. K. Peters Ltd., 1997. Original edition was published by Academic Press in 1974.