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Plücker environments, wiring and tiling diagrams, and weakly separated set-systems

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Abstract. For the ordered set $[n]$ of n elements, we consider the class $\mathcal{B}_n$ of bases B of tropical Plücker functions on $2^{[n]}$ such that B can be obtained by a series of mutations (flips) from the basis formed by the intervals in $[n]$. We show that these bases are representable by special wiring diagrams and by certain arrangements generalizing rhombus tilings on the n-zonogon. Based on the generalized tiling representation, we then prove that each maximal weakly separated set-system on $[n]$ belongs to $\mathcal{B}_n$, thus answering affirmatively a conjecture due to Leclerc and Zelevinsky.

Keywords: Plücker relations, octahedron recurrence, wiring diagram, rhombus tiling, TP-mutation, weakly separated sets

AMS Subject Classification 05C75, 05E99

1 Introduction

For a positive integer n, let $[n]$ denote the ordered set of elements $1, 2, \ldots, n$. In this paper we consider a certain "class" $\mathcal{B}_n \subseteq 2^{2^{[n]}}$. The collections (set-systems) $B \subseteq 2^{[n]}$ constituting $\mathcal{B}_n$ have equal cardinalities $|B|$, and for some pairs of collections, one can be obtained from the other by a single "mutation" (or "flip") that consists in exchanging a pair of elements of a very special form in these collections. The class we deal with arises, in particular, in a study of bases of so-called tropical Plücker functions (this seems to be the simplest source; one more source will be indicated later). For this reason, we may liberally call $\mathcal{B}_n$ along with mutations on it a *Plücker environment.*

More precisely, let f be a real-valued function on the subsets of $[n]$, or on the Boolean cube $2^{[n]}$. Following [1], f is said to be a *tropical Plücker function*, or a *TP-function* for short, if it satisfies

$$f(Xik) + f(Xj) = \max\{f(Xij) + f(Xk),\ f(Xi) + f(Xjk)\} \tag{1.1}$$

for any triple $i < j < k$ in $[n]$ and any subset $X \subseteq [n] - \{i, j, k\}$. Throughout, for brevity we write $Xi' \ldots j'$ instead of $X \cup \{i'\} \cup \ldots \cup \{j'\}$. The set of TP-functions on $2^{[n]}$ is denoted by $\mathcal{TP}_n$.

Definition. A subset $B \subseteq 2^{[n]}$ is called a *TP-basis*, or simply a *basis*, if the restriction map $res : \mathcal{TP}_n \to \mathbb{R}^B$ is a bijection. In other words, each TP-function is determined by its values on B, and moreover, values on B can be chosen arbitrarily.

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Such a basis does exist and the simplest instance is the set $\mathcal{I}_n$ of all intervals $\{p, p+1, \ldots, q\}$ in $[n]$ (including the empty set); see, e.g., [2]. In particular, the dimension of the polyhedral conic complex $\mathcal{TP}_n$ is equal to $|\mathcal{I}_n| = \binom{n+1}{2} + 1$. The basis $\mathcal{I}_n$ is called *standard*.

(Note that the notion of a TP-function is extended to other domains, of which most popular are an *integer box* $\mathbf{B}^{n,a} := \{x \in \mathbb{Z}^{[n]} \colon 0 \le x \le a\}$ for $a \in \mathbb{Z}^{[n]}$ and a *hyper-simplex* $\Delta^{n,m} = \{S \subseteq [n] \colon |S| = m\}$ for $m \in \mathbb{Z}$ (in the later case, (1.1) should be replaced by a relation on quadruples $i < j < k < \ell$). Aspects involving TP-functions are studied in [1, 5, 7, 8, 9, 10, 11] and some other works. Instances of TP-bases for a hyper-simplex are pointed out in [8], and [2] constructs a TP-basis for a "truncated integer box" $\{x \in \mathbf{B}^{n,a} \colon m \le x_1 + \ldots + x_n \le m'\}$, where $0 \le m \le m' \le n$. The domains different from Boolean cubes are beyond this paper, though main results presented here can be extended to the case of an integer box and some others, see Section 9.)

One can see that for a basis B, the collection $\{[n] - X \colon X \in B\}$ forms a basis as well, called the *complementary basis* of B and denoted by co-B. An important instance is the collection co-$\mathcal{I}_n$ of co-intervals in $[n]$.

Once we are given a basis B (e.g., the standard one), we can produce more bases by making a series of elementary transformations relying on (1.1). More precisely, suppose there is a cortege (X, i, j, k) such that the four sets occurring in the right hand side of (1.1) and one set $Y \in \{Xj, Xik\}$ in the left hand side belong to B. Then the replacement in B of Y by the other set Y' in the left hand side results in a basis B' as well (and we can further transform the latter basis in a similar way). The basis B' is said to be obtained from B by the *flip* (or *mutation*) with respect to X, i, j, k. When Xj is replaced by Xik (thus increasing the total size of sets in the basis by 1), the flip is called *raising*. When Xik is replaced by Xj, the flip is called *lowering*. We write $j \rightsquigarrow ik$ and $ik \rightsquigarrow j$ for such flips. The standard basis $\mathcal{I}_n$ does not admit lowering flips, whereas its complementary basis co-$\mathcal{I}_n$ does not admit raising flips.

We distinguish between two sorts of flip (mutations), which inspire consideration of two classes of bases.

Definitions. For a TP-basis B and a cortege (X, i, j, k) as above, the flip (mutation) $j \rightsquigarrow ik$ or $ik \rightsquigarrow j$ is called *strong* if both sets X and $Xijk$ belong to B as well, and *weak* otherwise. (The former (latter) is also called the flip in the presence of six (res. four) "witnesses", in terminology of [7].) A basis is called *normal* (by terminology in [2]) if it can be obtained by a series of *strong* flips starting from $\mathcal{I}_n$. A basis is called *semi-normal* if it can be obtained by a series of *any* flips starting from $\mathcal{I}_n$.

Leclerc and Zelevinsky [7] showed that the normal bases (in our terminology) are exactly the collections $C \subseteq [n]$ of maximum possible size $|C|$ that possess the strong separation property (defined later). Also the class of normal bases admits a nice "graphical" characterization, even for a natural generalization to the integer boxes (see [2, 4]): such bases one-to-one correspond to the rhombus tilings on the related *zonogon*.

Let $\mathcal{B}_n$ denote the set of semi-normal TP-bases for the Boolean cube $2^{[n]}$; this set (together with weak flips on its members) is just the Plücker environment of our interest mentioned at the beginning. The first goal of this paper is to characterize $\mathcal{B}_n$. We give two characterizations for semi-normal bases: via a bijection to special collections of

n curves on the zonogon, that we call *proper wirings*, and via a bijection to certain graphical arrangements, called *generalized tilings* or, briefly, *g-tilings* (in fact, these characterizations are interrelated via planar duality). We associate to a proper wiring W (a g-tiling T) a certain collection of subsets of $[n]$ called its *spectrum*. (We shall see that W (resp. T) is determined by its spectrum.)

Note that it is still open at present whether there exists a non-semi-normal (or "wild") basis; we conjecture that there is none.

The characterization of semi-normal bases via generalized tilings helps us to answer one conjecture of Leclerc and Zelevinsky concerning weakly separated set-systems; this is the second goal of our work. Recall some definitions from [7]. Hereinafter for sets A, B, we write $A - B$ for $A \setminus B = \{e \colon A \ni e \notin B\}$. Let $X, Y \subseteq [n]$. We write $X \prec Y$ if $Y - X \neq \emptyset$ and $i < j$ for any $i \in X - Y$ and $j \in Y - X$ (which slightly differs from the meaning of $\prec$ in [7]); note that this relation need not be transitive. We write $X \rhd Y$ if $Y - X$ has a (unique) bipartition $\{Y_1, Y_2\}$ such that $Y_1, Y_2, X - Y \neq \emptyset$ and $Y_1 \prec X - Y \prec Y_2$.

Definitions. Sets $X, Y \subseteq [n]$ are called: (a) *strongly separated* if either $X \prec Y$ or $Y \prec X$, and (b) *weakly separated* if either $X \prec Y$, or $Y \prec X$, or $X \rhd Y$ and $|X| \geq |Y|$, or $Y \rhd X$ and $|Y| \geq |X|$. Accordingly, a collection $C \subseteq 2^{[n]}$ is called strongly (weakly) separated if any two members of C are strongly (resp. weakly) separated.

(As is seen from a discussion in [7], an interest in studying weakly separated collections is inspired, in particular, by the problem of characterizing all families of quasicommuting quantum flag minors, which in turn comes from exploration of Lusztig's canonical bases for certain quantum groups. It is proved in [7] that, in an $n \times n$ generic q-matrix, the flag minors with column sets $I, J \subseteq [n]$ quasicommute if and only if the sets I, J are weakly separated. See also [6].)

Important properties shown in [7] are that any weakly separated collection $C \subseteq 2^{[n]}$ has cardinality at most $\binom{n+1}{2} + 1$ and that the set of such collections is closed under weak flips (which are defined as for TP-bases above). Let $\mathcal{C}_n$ denote the set of *largest* weakly separated collections in $[n]$, i.e., having size $\binom{n+1}{2} + 1$. It turns into a poset by regarding C as being less than C' if C is obtained from C' by a series of weak lowering flips. This poset contains $\mathcal{I}_n$ and co-$\mathcal{I}$ as minimal and maximal elements, respectively, and it is conjectured in [7, Conjecture 1.8] that there are no other minimal and maximal elements in it. This would imply that $\mathcal{C}_n$ coincides with $\mathcal{B}_n$. We prove this conjecture.

The main results in this paper are summarized as follows.

Theorem A (main) *For $B \subseteq 2^{[n]}$, the following statements are equivalent:*

(i) *B is a semi-standard TP-basis;*

(ii) *B is the spectrum of a proper wiring;*

(iii) *B is the spectrum of a generalized tiling;*

(iv) *B is a largest weakly separated collection.*

Another important problem is extendability a given subset of $2^{2^{[n]}}$ into a basis. We solve this problem in another paper [3].

The paper is organized as follows. Section 2 contains basic definitions and states involved in Theorem A. It introduces proper wirings and claims the equivalence of (i) and (ii) in the above theorem (Theorem 2.1). Then it introduces generalized tilings and claims the equivalence of (i) and (iii) (Theorem 2.2). Section 3 describes some "elementary" properties of generalized tilings that will be used later. The combined proof of Theorems 2.1 and 2.2 consists of four stages and is lasted throughout Sections 4–7. In fact, generalized tilings are the central objects of treatment in the paper; we take advantages from their nice visualization and structural features and all implications that we explicitly prove involve just generalized tilings. Implication (i)→(iii) in Theorem A is proved in Section 4, (iii)→(i) in Section 5, (iii)→(ii) in Section 6, and (ii)→(iii) in Section 7. The above-mentioned Leclerc-Zelevinsky's conjecture is proved in Section 8 by showing implication (iv)→(iii). This completes the proof of Theorem A, taking into account that (i)→(iv) was established in [7]. The concluding Section 9 discusses two generalizations of our theorems: to an integer box and to an arbitrary permutation of $[n]$.

Acknowledgement. We thank Alexander Postnikov who drew our attention to paper [7] and a possible relation between TP-bases and weakly-separated set-systems. One of the authors (GK) thanks RIMS (Kyoto University) for hospitality where during his visiting professorship a part of this work was completed.

2 Wirings and tilings

Throughout the paper we assume that $n > 1$. Both a special wiring diagram and a generalized tiling diagram that we introduce in this section live within a zonogon, which is defined as follows.

In the upper half-plane $\mathbb{R} \times \mathbb{R}_+$, take n non-colinear vectors $\xi_1, \ldots, \xi_n$ so that:

(2.1) (i) $\xi_1, \ldots, \xi_n$ follow in this order clockwise around $(0,0)$, and (ii) all integer combinations of these vectors are different.

Then the set

$$Z = Z_n := \{\lambda_1\xi_1 + \ldots + \lambda_n\xi_n \colon \lambda_i \in \mathbb{R},\ 0 \le \lambda_i \le 1,\ i = 1, \ldots, n\}$$

is a $2n$-gone. Moreover, Z is a *zonogon*, as it is the sum of n line-segments $\{\lambda\xi_i \colon 1 \le \lambda \le 1\}$, $i = 1, \ldots, n$. Also it is the image by a linear projection π of the solid cube $conv(2^{[n]})$ into the plane $\mathbb{R}^2$, defined by $\pi(x) = x_1\xi_1 + \ldots + x_n\xi_n$. The boundary $bd(Z)$ of Z consists of two parts: the *left boundary* $lbd(Z)$ formed by the points (vertices) $p_i := \xi_1 + \ldots + \xi_i$ ($i = 0, \ldots, n$) connected by the line-segments $p_{i-1}p_i := p_{i-1} + \{\lambda\xi_i \colon 0 \le \lambda \le 1\}$, and the *right boundary* $rbd(Z)$ formed by the points $p'_i := \xi_i + \ldots + \xi_n$ ($i = 0, \ldots, n$) connected by the segments $p'_ip'_{i-1}$. So $p_0 = p'_n$ is the minimal vertex and $p_n = p'_0$ is the maximal vertex of Z. We orient each segment $p_{i-1}p_i$ from p_{i-1} to p_i and orient each segment $p'_ip'_{i-1}$ from p'_i to p'_{i-1}. Let s_i (resp. s'_i) denote the median point in the segment $p_{i-1}p_i$ (resp. $p'_ip'_{i-1}$).

2.1 Wiring diagrams

A *special wiring diagram*, also called a *W-diagram* or a *wiring* for brevity, is an ordered collection W of n wires $w_1, \ldots, w_n$ satisfying three axioms below. A *wire* w_i is a continuous injective map of the segment $[0,1]$ into Z (or the curve in the plane represented by this map) such that $w_i(0) = s_i$, $w_i(1) = s'_i$, and $w_i(\lambda)$ lies in the interior of Z for $0 < \lambda < 1$. We say that w_i begins at s_i and ends at s'_i, and orient w_i from s_i to s'_i. The diagram W is considered up to a homeomorphism of Z stable on $bd(Z)$ (and up to parameterizations of the wires). Clearly any two wires have at least one point in common. Axioms (W1)–(W3) specify W as follows.

(W1) No three wires w_i, w_j, w_k have a common point, i.e., there are no $\lambda, \lambda', \lambda''$ such that $w_i(\lambda) = w_j(\lambda') = w_k(\lambda'')$. If two wires w_i, w_j intersect at a point v, then they *cross* (not touch) at v (i.e., when passing v, the wire w_i goes from one connected component of $Z - w_j$ to the other one).

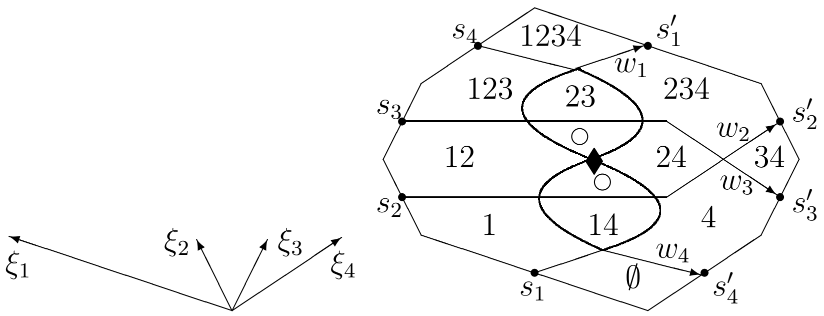
(W2) Any two wires w_i, w_j intersect at a finite number of points and these common points follow in opposed orders along the wires, i.e., if $w_i(\lambda_q) = w_j(\lambda'_q)$ for $q = 1, \ldots, r$ and if $\lambda_1 < \ldots < \lambda_r$, then $\lambda'_1 > \ldots > \lambda'_r$.

Note that the number $r = r_{ij}$ of common points of w_i, w_j is odd; assuming that $i < j$, we denote these points as $x_{ij}(1), \ldots, x_{ij}(r)$ following the direction of w_i from $w_i(0)$ to $w_i(1)$. When $r > 1$, the (bounded) region in the plane surrounded by the pieces of w_i, w_j between $x_{ij}(q)$ and $x_{ij}(q+1)$ (where $q = 1, \ldots, r-1$) is denoted by $L_{ij}(q)$ and called the q-th *lens* for i, j. The points $x_{ij}(q)$ and $x_{ij}(q+1)$ are regarded as the *upper* and *lower* points of $L_{ij}(q)$, respectively. When q is odd (even), we say that $L_{ij}(q)$ is an *odd* (resp. *even*) lens. The points $x_{ij}(q)$ with q even are of especial interest: at this point the wire with the smaller number (namely, w_i) is crossed by the wire with the bigger number (namely, w_j) *from right to left* (w.r.t. the direction of w_i), whereas the crossing behavior at the points $x_{ij}(q)$ with q odd is different. We call such a point *orientation-reversing*, or *black* (for reasons that will be clear later). Also we say that $x_{ij}(q)$ with q even is the *root* of the lenses $L_{ij}(q-1)$ and $L_{ij}(q)$.

The wiring W is associated, in a natural way, with a planar directed graph G_W embedded in Z. The vertices of G_W are the points p_i, p'_i, s_i, s'_i and the intersection points of wires. The edges of G_W are the corresponding directed line-segments in $bd(Z)$ and the pieces of wires between neighboring points of intersection with other wires or with the boundary, which are directed according to the direction of wires. We say that an edge contained in a wire w_i has *color* i, or is an *i-edge*. Let $\mathcal{F}_W$ be the set of (inner, or bounded) faces of G_W. Here each face F is considered as the closure of a maximal connected component in $Z - \cup(w \in W)$. We say that a face F is *cyclic* if its boundary $bd(F)$ is a directed cycle in G_W.

(W3) There is a bijection ϕ between the set $\mathcal{L}(W)$ of lenses in W and the set $\mathcal{F}^{cyc}_W$ of cyclic faces in G_W. Moreover, for each lens L, $\phi(L)$ is the (unique) face lying in L and containing its root.

We say that W is *proper* if none of cyclic faces is a whole lens, i.e., for each lens $L \in \mathcal{L}(W)$, there is at least one wire going across L. An instance of proper wirings for $n = 4$ is illustrated in the picture; here the cyclic faces are marked by circles and the black rhombus indicates the black point.



$B_W = \{\emptyset, 1, 4, 12, 14, 23, 24, 34, 123, 234, 1234\}$

Now we associate to W a set-system $B_W \subseteq 2^{[n]}$ as follows. For each *non-cyclic* face F, let $X(F)$ be the set of elements $i \in [n]$ such that F lies *on the left* from the wire w_i, i.e., F and the maximal point p_n lie in the same of the two connected components of $Z - w_i$. We define

$$B_W := \{X \subseteq [n] \colon X = X(F) \text{ for some } F \in \mathcal{F}_W - \mathcal{F}_W^{cyc}\},$$

referring to it as the *effective spectrum*, or simply the *spectrum* of W. Sometimes it will also be useful to consider the *full spectrum* $\widehat{B}_W$ consisting of all sets $X(F)$, $F \in \mathcal{F}_W$. (In fact, when W is proper, all sets in $\widehat{B}_W$ are different; see Lemma 7.2. When W is not proper, there are different faces F, F' with $X(F) = X(F')$. We can turn W into a proper wiring W' by getting rid, step by step, of lenses forming faces (by making a series of Reidemeister moves of type II, namely, $()$ $\to$ $)($ operations). This preserves the effective spectrum: $B_{W'} = B_W$, whereas the full spectrum may decrease.)

Note that when any two wires intersect at exactly one point (i.e., when no black points exist), B_W is a normal basis, and conversely, any normal basis is obtained in this way (see [2]).

Our main result on wirings is the following

Theorem 2.1 *For any proper wiring W (obeying (W1)–(W3)), the spectrum B_W is a semi-normal TP-basis. Conversely, for any semi-normal TP-basis B, there exists a proper wiring W such that $B_W = B$.*

This theorem will be obtained in Sections 6–7.

2.2 Generalized tilings

When it is not confusing, we identify a subset $X \subseteq [n]$ with the corresponding vertex of the n-cube and with the point $\sum_{i \in X} \xi_i$ in the zonogon Z. Due to (2.1)(ii), all points X are different (concerning Z).

Assuming that the vectors ξ_i have the same Euclidean norm, a *rhombus tiling diagram* is defined to be a subdivision T of Z into rhombi of the form $x + \{\lambda\xi_i + \lambda'\xi_j \colon 0 \leq$

$\lambda, \lambda' \leq 1\}$ for some $i < j$ and a point x in Z, i.e., the rhombi are pairwise non-overlapping (have no common interior points) and their union is Z. From (2.1)(ii) it follows that for i, j, x as above, x represents a subset in $[n] - \{i, j\}$. The diagram T is also regarded as a directed planar graph whose vertices and edges are the vertices and side segments of the rhombi, respectively. An edge connecting X and Xi is directed from the former to the latter. It is shown in [2, 4] that the vertex set of T forms a normal basis and that each normal basis is obtained in this way.

It makes no difference whether we take vectors $\xi_1, \ldots, \xi_n$ with equal or arbitrary norms (subject to (2.1)); to simplify technical details and visualization, throughout the paper we will assume that these vectors have *unit height*, i.e., each ξ_i is of the form $(x, 1)$. Then we obtain a subdivision T of Z into parallelograms of height 2, and for convenience we refer to T as a *tiling* and to its elements as *tiles*. A tile τ defined by X, i, j (with $i < j$) is called an *ij-tile* at X and denoted by $\tau(X; i, j)$. According to a natural visualization of τ, its vertices X, Xi, Xj, Xij are called the *bottom, left, right, top* vertices of τ and denoted by $b(\tau)$, $\ell(\tau)$, $r(\tau)$, $t(\tau)$, respectively. Also we will say that: for a point (subset) $Y \subseteq [n]$, $|Y|$ is the *height* of Y; the set of vertices of tiles in T having height h form h-th *level*; and a point Y *lies on the right* from a point Y' if Y, Y' have the same height and $\sum_{i \in Y} \xi_i \geq \sum_{i \in Y'} \xi_i$.

In a *generalized tiling*, or a *g-tiling*, the union of tiles is again Z but some tiles may overlap. It is a collection T of tiles which is partitioned into two subcollections T^w and T^b, of *white* and *black* tiles (say), respectively, obeying axioms (T1)–(T4) below. When $T^b = \emptyset$, we will obtain a tiling as before, for convenience referring to it as a *pure tiling.* Let V_T and E_T denote the sets of vertices and edges, respectively, occurring in tiles of T, not counting multiplicities. For a vertex $v \in V_T$, the set of edges incident with v is denoted by $E_T(v)$, and the set of tiles having a vertex at v is denoted by $F_T(v)$.

(T1) All tiles are contained in Z. Each boundary edge of Z belongs to exactly one tile. Each edge in E_T not contained in $bd(Z)$ belongs to exactly two tiles. All tiles in T are different (in the sense that no two coincide in the plane).

(T2) Any two white tiles having a common edge do not overlap (in the sense that they have no common interior point). If a white tile and a black tile share an edge, then these tiles do overlap. No two black tiles share an edge.

(T3) Let τ be a black tile. None of $b(\tau), t(\tau)$ is a vertex of another black tile. All edges in $E_T(b(\tau))$ *leave* $b(\tau)$ (i.e., are directed from $b(\tau)$). All edges in $E_T(t(\tau))$ *enter* $t(\tau)$ (i.e., are directed to $t(\tau)$).

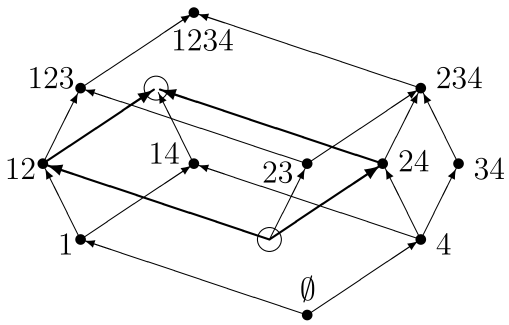
We distinguish between three sorts of vertices by saying that $v \in V_T$ is: (a) a *terminal* vertex if it is the bottom or top vertex of some black tile; (b) an *ordinary* vertex if all tiles in $F_T(v)$ are white; and (c) a *mixed* vertex otherwise (i.e. v is the left or right vertex of some black tile). Note that a mixed vertex may belong, as the left or right vertex, to several black tiles.

Each tile $\tau \in T$ is associated, in a natural way, to a square in the solid n-cube $conv(2^{[n]})$, denoted by $\sigma(\tau)$: if $\tau = \tau(X; i, j)$ then $\sigma(\tau)$ spans the vertices (corresponding to) X, Xi, Xj, Xij in the cube. In view of (T1), the interiors of these squares are disjoint, and $\cup(\sigma(\tau)\colon \tau \in T)$ forms a 2-dimensional surface, denoted by D_T, whose boundary is the preimage by π of the boundary of Z; the vertices in $bd(D_T)$ correspond to the *principal intervals* $\emptyset$, $[q]$ and $[q..n]$ for $q = 1, \ldots, n$. (For $1 \le p \le r \le n$, we denote the interval $\{p, p+1, \ldots, r\}$ by $[p..r]$). The last axiom is:

(T4) D_T is a disc (i.e., is homeomorphic to $\{x \in \mathbb{R}^2 \colon x_1^2 + x_2^2 \le 1\}$).

The *reversed* g-tiling T^{rev} of a g-tiling T is formed by replacing each tile $\tau(X; i, j)$ of T by the tile $\tau([n] - Xij; i, j)$ (or by changing the orientation of all edges in E_T, in particular, in $bd(Z)$). Clearly (T1)–(T4) remain valid for T^{rev}.

The *effective spectrum*, or simply the *spectrum*, of a g-tiling T is the collection B_T of (subsets of $[n]$ represented by) *non-terminal* vertices in T. The *full spectrum* $\widehat{B}_T$ is formed by all vertices in T. An example of g-tilings for $n = 4$ is drawn in the picture, where the unique black tile is indicated by thick lines and the terminal vertices are surrounded by circles (this is related to the wiring shown on the previous picture).



$$B_T = \{\emptyset, 1, 4, 12, 14, 23, 24, 34, 123, 234, 1234\}$$

Our main result on g-tilings is the following

Theorem 2.2 *For any generalized tiling T (obeying (T1)–(T4)), the spectrum B_T is a semi-normal TP-basis. Conversely, for any semi-normal TP-basis B, there exists a generalized tiling T such that $B_T = B$.*

(In particular, the cardinalities of the spectra of all g-tilings on Z_n are the same and equal to $\binom{n+1}{2} + 1$.) The first part of this theorem will be proved in Section 5, and the second one in Section 4.

We will explain in Section 7 that for each semi-normal basis B, there are precisely one proper wiring W and precisely one g-tiling T such that $B_W = B_T = B$ (see Theorem 7.5); this is similar to the one-to-one correspondence between the normal bases and pure tilings.

3 Elementary properties of generalized tilings

In this section we give additional definitions and notation and demonstrate several corollaries from axioms (T1)–(T4) which will be used later on. Let T be a g-tiling on $Z = Z_n$.

1. An edge e of G_T is called *black* if there is a black tile containing e (as a side edge); otherwise e is called *white*. The sets of white and black edges incident with a vertex v are denoted by $E^w_T(v)$ and $E^b_T(v)$, respectively. For a vertex v of a tile τ, let $C(\tau,v)$ denote the minimal cone at v containing τ (i.e., generated by the pair of edges of τ incident to v), and let $\alpha(\tau,v)$ denote the angle of this cone taken with sign $+$ if τ is white, and $-$ if τ is black. The sum $\sum(\alpha(\tau,v) \colon \tau \in F_T(v))$ is called the *(full) rotation angle* at v and denoted by $\rho(v)=\rho_T(v)$. We observe from (T1)–(T3) that terminal vertices behave as follows.

Corollary 3.1 *Let v be a terminal vertex belonging to a black ij-tile τ. Then:*

(i) *v is not connected by edge with another terminal vertex (whence $|E^b_T(v)|=2$);*

(ii) *$|E_T(v)|\ge 3$ (whence $E^w_T(v)\ne\emptyset$);*

(iii) *each edge $e\in E^w_T(v)$ lies in the cone $C(\tau,v)$ (whence e is a q-edge for some $i<q<j$);*

(iv) *$\rho(v)=0$;*

(v) *v does not belong to the boundary of Z (whence any tile containing a boundary edge of Z is white).*

Indeed, since each edge of G_T belongs to some tile, at least one of its end vertices has both entering and leaving edges, and therefore (by (T3)), this vertex cannot be terminal (yielding (i)). Next, if $|E_T(v)|=2$, then $F_T(v)$ would consist only of the tile τ and its white copy; this is not the case by (T1) (yielding (ii)). Next, assume that $v=t(\tau)$. Then v is the top vertex of all tiles in $F_T(v)$ (by (T3)). This together with the facts that all tiles in $F_T(v)-\{\tau\}$ are white and that any two white tiles sharing an edge do not overlap (by (T2)) implies (iii) and (iv). When $v=b(\tau)$, the argument is similar. Finally, v cannot be a boundary vertex p_k or p'_k for $k\ne 0,n$ since the latter vertices have both entering and leaving edges. In case $v=p_0$, the tile τ would contain both boundary edges p_0p_1 and $p_0p'_{n-1}$ (in view of (iii)). But then the white tile sharing with τ the edge $(r(\tau),t(\tau))$ would trespass the boundary of Z. The case $v=p_n$ is impossible for a similar reason (yielding (v)).

Note that (iii) in this corollary implies that

(3.1) if a black ij-tile τ and a (white) tile τ' share an edge e, then τ' is either an iq-tile or a qj-tile for some $i<q<j$; also $\tau'\subset C(\tau,v)$ and $\tau\subset C(\tau',v')$, where v and v' are the terminal and non-terminal ends of e, respectively.

2. The following important lemma specifies the rotation angle at non-terminal vertices.

Lemma 3.2 *Let $v\in V_T$ be a non-terminal vertex.*

(i) *If v belongs to $bd(Z)$, then $\rho(v)$ is positive and equals the angle between the boundary edges incident to v.*

(ii) *If v is inner (i.e., not in $bd(Z)$), then $\rho(v)=2\pi$.*

Proof (i) For $v\in bd(Z)$, let e,e' be the boundary edges incident to v, where e,Z,e' follow clockwise around v. Consider the maximal sequence $e=e_0,\tau_1,e_1,\dots,\tau_r,e_r$ of edges in $E_T(v)$ and tiles in $F_T(v)$ such that for $q = 1, \dots, r$, e_{q-1}, e_q are distinct edges of the tile τ_q, and $\tau_q \neq \tau_{q+1}$ (when $q < n$). Using (3.1), one can see that all tiles in this sequence are different and give the whole $F_T(v)$; also $e_r = e'$ and the tiles τ_1, τ_r are white. For each q, the ray at v containing e_q is obtained by rotating the ray at v containing e_{q-1} by the angle $\alpha(\tau_q, v)$ (where the rotation is clockwise if the angle is positive). So the sum of angles over this sequence amounts to $\rho(v)$ and is equal to the angle of e, e'.

To show (ii), let $V := V_T$ and $E := E_T$. Also denote the set of terminal vertices by V^t, and the set of *inner* non-terminal vertices by $\widehat{V}$. Since the boundary of Z contains $2n$ vertices and by (i),

$$|V| = |V^t| + |\widehat{V}| + 2n \quad \text{and} \quad \sum\nolimits_{v \in V \cap bd(Z)} \rho(v) = \pi \cdot 2n - 2\pi = 2\pi(n-1) \tag{3.2}$$

Let $\Sigma := \sum(\rho(v) \colon v \in V)$ and $\widehat{\Sigma} := \sum(\rho(v) \colon v \in \widehat{V})$. The contribution to Σ from each white (black) tile is 2π (resp. -2π). Therefore, $\Sigma = 2\pi(|T^w| - |T^b|)$. On the other hand, in view of Corollary 3.1(iv) and the second relation in (3.2), $\Sigma = \widehat{\Sigma} + 2\pi(n-1)$. Then

$$\widehat{\Sigma} = 2\pi(|T^w| - |T^b| - n + 1). \tag{3.3}$$

Considering G_T as a planar graph properly embedded in the disc D_T and applying Euler formula to it, we have $|V| + |T| = |E| + 1$. Each tile has four edges, the number of boundary edges is $2n$, and each inner edge belongs to two tiles; therefore, $|E| = 2n + (4|T| - 2n)/2 = 2|T| + n$. Then $|V|$ is expressed as

$$|V| = |E| - |T| + 1 = 2|T| + n - |T| + 1 = |T| + n + 1. \tag{3.4}$$

Also $|V| = |\widehat{V}| + 2|T^b| + 2n$ (using the first equality in (3.2) and the equality $|V^t| = 2|T^b|$). This and (3.4) give

$$|\widehat{V}| = |V| - 2|T^b| - 2n = (|T| + n + 1) - 2|T^b| - 2n = |T^w| - |T^b| - n + 1.$$

Comparing this with (3.3), we obtain $\widehat{\Sigma} = 2\pi|\widehat{V}|$. Now the desired equality $\rho(v) = 2\pi$ for each vertex $v \in \widehat{V}$ follows from the fact that $\rho(v)$ equals $2\pi \cdot d$ for some integer $d \geq 1$. The latter is shown as follows. Let us begin with a white tile $\tau_1 \in F_T(v)$ and its edges $e_0, e_1 \in E_T(v)$, in this order clockwise, and form a sequence $e_0, \tau_1, e_1, \dots, \tau_r, e_r, \dots$ similar to that in (i) above, until we return to the initial edge e_0. Let R_q be the ray at v containing e_q. Since $\alpha(\tau_q) > 0$ when τ_q is white, and $\alpha(\tau_q) + \alpha(\tau_{q+1}) > 0$ when τ_q is white and τ_{q+1} is black (cf. (3.1)), the current ray $R_\bullet$ must make at least one turn clockwise before it returns to the initial ray R_0. If it happens that the sequence uses not all tiles in $F_T(v)$, we start with a new white tile to form a next sequence (for which the corresponding ray makes at least one turn clockwise as well), and so one. Thus, $d \geq 1$, as required (implying $d = 1$). ∎

Remark 1 If we postulate property (ii) in Lemma 3.2 as an axiom and add it to axioms (T1)–(T3), then we can eliminate axiom (T4); in other words, (ii) and (T4) are equivalent subject to (T1)–(T3). Indeed, reversing reasonings in the above proof, one can conclude that $\widehat{\Sigma} = 2\pi|\widehat{V}|$ implies $|V| + |T| = |E| + 1$. The latter is possible only if D_T is a disc. (Indeed, if D_T forms a regular surface with g handles and c cross-caps, from which an open disc is removed, then Euler formula is modified as $|V| + |T| = |E| + 1 - 2g - c$. Also $|V|$ decreases when some vertices merge.)

3. Considering the sequence of rotations of the edge ray $R_\bullet$ around a non-terminal vertex v (like in the proof of Lemma 3.2), one can see that the sets $E_T^w(v)$ and $E_T^b(v)$ are arranged as follows.

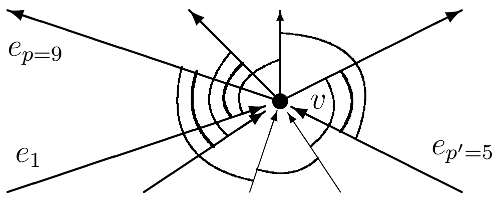
(3.5) For an ordinary or mixed vertex $v \in V_T$, let $E_T(v)$ consists of edges $e_1, \dots, e_p$ following counterclockwise around v, and let e_1 enter and e_p leave v. Then:

(i) there is $1 \le p' < p$ such that $e_1, \dots, e_{p'}$ enter v and $e_{p'+1}, \dots, e_p$ leave v;

(ii) if v is the right vertex of r black tiles and the left vertex of r' black tiles, then $r + r' < \min\{p', p - p'\}$ and the black edges incident to v are exactly $e_{p-r+1}, \dots, e_p, e_1, \dots, e_r$ and $e_{p'-r'+1}, \dots, e_{p'+r'}$;

(iii) the black tiles in $F_T(v)$ have the following pairs of edges incident to v: $\{e_{p-r+1}, e_1\}, \dots, \{e_p, e_r\}$ and $\{e_{p'-r'+1}, e_{p'+1}\}, \dots, \{e_{p'}, e_{p'+r'}\}$, while the white tiles in $F_T(v)$ have the following pairs of edges incident to v: (a) $\{e_{r+1}, e_{r+2}\}, \dots, \{e_{p'-r'-1}, e_{p'-r'}\}$; (b) $\{e_{p'+r'+1}, e_{p'+r'+2}\}, \dots, \{e_{p-r-1}, e_{p-r}\}$; (c) $\{e_{p-r}, e_1\}, \dots, \{e_p, e_{r+1}\}$; (d) $\{e_{p'-r'}, e_{p'+1}\}, \dots, \{e_{p'}, e_{p'+r'+1}\}$.

(If v is ordinary, then $r = r' = 0$ and each (white) tile in $F_T(v)$ meets a pair of consecutive edges e_q, e_{q+1} or e_p, e_1.) The case with $p = 9$, $p' = 5$, $r = 2$, $r' = 1$ is illustrated in the picture; here the black edges are drawn in bold and the white (black) tiles at v are indicated by thin (bold) arcs.



Note that (3.5) implies the following property (which will be used, in particular, in Subsection 4.3):

(3.6) for a tile $\tau \in T$ and a vertex $v \in \{\ell(\tau), r(\tau)\}$, let e, e' be the edges of τ entering and leaving v, respectively, and suppose that there is an edge $\widetilde{e} \ne e, e'$ incident to v and contained in $C(\tau, v)$; then $\widetilde{e}$ is black; furthermore: (a) e' is black if $\widetilde{e}$ enters v; (b) e is black if $\widetilde{e}$ leaves v.

4. We will often use the fact (implied by (2.1)(ii)) that for any g-tiling T,

(3.7) the graph $G_T = (V_T, E_T)$ is *graded* for each color $i \in [n]$, which means that for any closed path P in G_T, the numbers of forward i-edges and backward i-edges in P are equal.

Hereinafter, speaking of a path in a directed graph, we mean is a sequence $P = (\widetilde{v}_0, \widetilde{e}_1, \widetilde{v}_1, \dots, \widetilde{e}_r, \widetilde{v}_r)$ in which each $\widetilde{e}_p$ is an edge connecting vertices $\widetilde{v}_{p-1}, \widetilde{v}_p$; an edge $\widetilde{e}_p$ is called *forward* if it is directed from $\widetilde{v}_{p-1}$ to $\widetilde{v}_p$ (denoted as $\widetilde{e}_p = (\widetilde{v}_{p-1}, \widetilde{v}_p)$), and *backward* otherwise (when $\widetilde{e}_p = (\widetilde{v}_p, \widetilde{v}_{p-1})$. The path P is called: *closed* if $v_0 = v_r$, *directed* if all its edges are forward, and *simple* if all vertices $v_0, \dots, v_r$ are different. P^{rev} denotes the reversed path $(\widetilde{v}_r, \widetilde{e}_r, \widetilde{v}_{r-1}, \dots, \widetilde{e}_1, \widetilde{v}_0)$.

4 From semi-normal bases to generalized tilings

In this section we prove the second assertion in Theorem 2.2, namely, the inclusion

$$\mathcal{B}_n \subseteq \mathcal{BT}_n, \tag{4.1}$$

where $\mathcal{B}_n$ is the set of semi-normal bases in $2^{[n]}$ and $\mathcal{BT}_n$ denotes the collection of the sets B_T generated by g-tilings on Z_n. The proof falls into three parts, given in Subsections 4.1–4.3.

4.1 Flips in g-tilings

Let T be a g-tiling. By an *M-configuration* in T we mean a quintuple of vertices of the form Xi, Xj, Xk, Xij, Xjk with $i < j < k$ (as it resembles the letter "M"), which is denoted as $CM(X; i, j, k)$. By a *W-configuration* in T we mean a quintuple of vertices Xi, Xk, Xij, Xik, Xjk with $i < j < k$ (as resembling "W"), briefly denoted as $CW(X; i, j, k)$. A configuration is called *feasible* if all five vertices are non-terminal, i.e., they belong to B_T.

We know that any normal basis B (in particular, $B = \mathcal{I}_n$) is expressed as B_T for some pure tiling T, and therefore, $B \in \mathcal{BT}_n$. Thus, to conclude with (4.1), it suffices to show the following assertion, which says that the set of g-tilings is closed under transformations analogous to flips for semi-normal bases.

Proposition 4.1 *Let a g-tiling T contain five non-terminal vertices Xi, Xk, Xij, Xjk, Y, where $i < j < k$ and $Y \in \{Xik, Xj\}$. Then there exists a g-tiling T' such that $B_{T'}$ is obtained from B_T by replacing Y by the other member of $\{Xik, Xj\}$.*

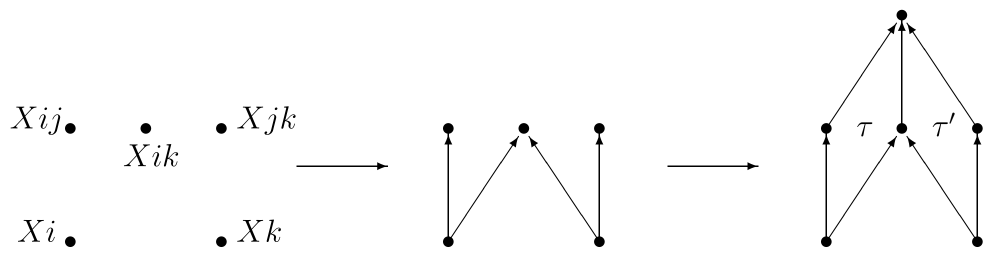
Proof We may assume that $Y = Xik$, i.e., that we deal with the feasible W-configuration $CW(X; i, j, k)$ (since an M-configuration in T turns into a W-configurations in the reversed g-tiling T^{rev}). We rely on the following two facts which will be proved Subsections 4.2 and 4.3.

(4.2) Any pair of non-terminal vertices $X', X'i'$ in T is connected by edge.

(Therefore, for T as above, E_T contains edges (Xi, Xij), (Xi, Xik), (Xk, Xik) and (Xk, Xjk). Note that vertices $X', X'i'$ need not be connected by edge if some of them is terminal; e.g., in the picture before the statement of Theorem 2.2, the vertices with $X' = \emptyset$ and $i' = 2$ are not connected.)

(4.3) T contains the jk-tile τ with $b(\tau) = Xi$ and the ij-tile τ' with $b(\tau') = Xk$.

Then $\ell(\tau) = Xij$, $r(\tau) = \ell(\tau') = Xik$, $r(\tau') = Xk$, and $t(\tau) = t(\tau') = Xijk$. Since the vertices Xi, Xk are non-terminal, both tiles τ, τ' are white. See the picture.



Assuming that (4.2) and (4.3) are valid, we argue as follows. First of all we observe that

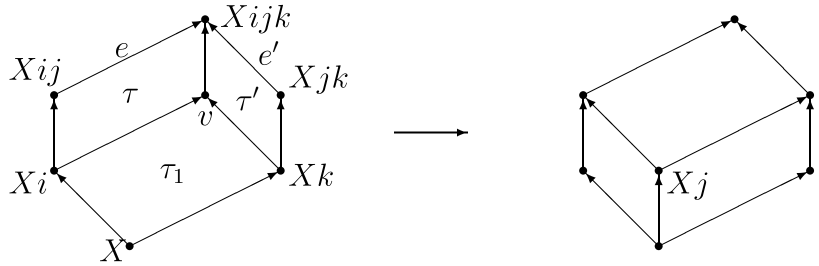
(4.4) the vertex $v := Xik$ is ordinary.

Indeed, since both vertices Xi, Xik are non-terminal, the edge (Xi, Xik) cannot belong to a black tile. So the edge (Xi, Xik), which belongs to the white tile τ and enters v, is white. Also the edge $(Xik, Xijk)$ of τ that leaves v is white (for if it belongs to a black tile $\overline{\tau}$, then $\overline{\tau}$ should have $v' := Xijk$ as its top vertex, but then the cone of $\overline{\tau}$ at v' cannot simultaneously contain both edges $(Xij, Xijk)$ and $(Xjk, Xijk)$, contrary to Corollary 3.1(iii)). Now one can conclude from (3.5) that there is no black tile having its left or right vertex at v. So v is ordinary.

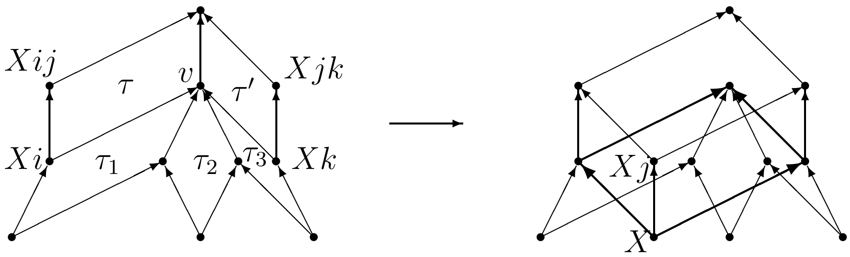
Let $e_0, \ldots, e_q$ be the sequence of edges entering v in the counterclockwise order; then $e_0 = (Xi, Xik)$ and $e_q = (Xk, Xik)$. Since v is ordinary, each pair e_{p-1}, e_p ($p = 1, \ldots, q$) belongs to a white tile τ_p. Two cases are possible.

Case 1: The edges $e := (Xij, Xijk)$ and $e' := (Xjk, Xijk)$ do not belong to the same black tile. Consider two subcases.

(a) Let $q = 1$. We replace in T the tiles τ, τ', τ_1 by three new white tiles: $\tau(X; i, j)$, $\tau(X; j, k)$ and $\tau(Xj; i, k)$ (so the vertex v is replaced by Xj). See the picture.

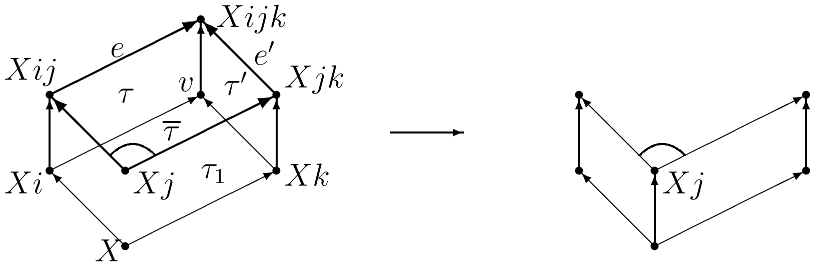


(b) Let $q > 1$. We remove the tiles τ, τ' and add four new tiles: the white tiles $\tau(X; i, j)$, $\tau(X; j, k)$, $\tau(Xj; i, k)$ (as before) and the black tile $\tau(X; i, k)$ (so v becomes terminal). See the picture for $q = 3$; here the added black tile is drawn in bold.

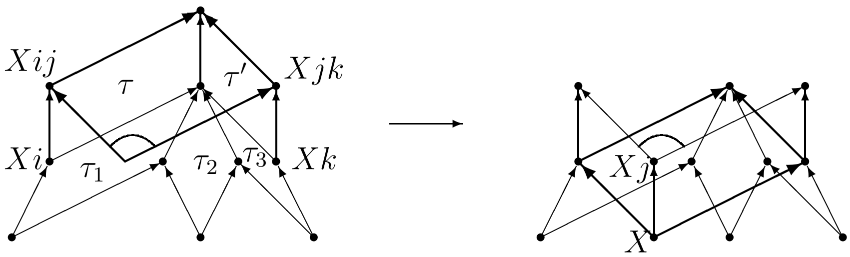


Case 2: Both edges e and e' belong to a black tile $\overline{\tau}$ (which is $\tau(Xj; i, k)$). We act as in Case 1 with the only difference that $\overline{\tau}$ is removed from T and the white ik-tile at Xj (which is a copy of $\overline{\tau}$) is not added. Then the vertex $Xijk$ vanishes, v either vanishes or becomes terminal, and Xj becomes non-terminal. See the picture; here (a') and (b') concern the subcases $q = 1$ and $q > 1$, respectively, and the arc above the vertex Xj indicates the bottom cone of $\overline{\tau}$ in which some white edges (not indicated) are located.

(a')



(b')



Let T' be the resulting collection of tiles. It is routine to check that in all cases the transformation of T into T' maintains the conditions on tiles and edges involved in axioms (T1)–(T3) at the vertices Xi, Xk, Xij, Xjk, as well as at the vertices Xik and $Xijk$ when the last ones do not vanish. Also the conditions continue to hold at the vertex X in Cases 1(a) and 2(a') (with $q = 1$), and at the vertex Xj in Case 2 (when the terminal vertex Xj becomes non-terminal). A less trivial task is to verify for T' the correctness at Xj in Case 1 and at X in Cases 1(b) and 2(b'). We assert that

(4.5) (i) V_T does not contain Xj in Case 1; and (ii) V_T does not contain X in Cases 1(b) and 2(b').

Then these vertices (in the corresponding cases) are indeed new in the arising T', and now the required properties for them become evident by the construction. Note that this implies (T4) as well. We will prove (4.5) in Subsection 4.3.

Thus, assuming validity of (4.2), (4.3), (4.5), we can conclude that T' is a g-tiling and that $B_{T'} = (B_T - \{Xik\}) \cup \{Xj\}$, as required. ∎

Remark 2 Adopting terminology for set-systems, we say that for the g-tilings T, T' as in the proof of Proposition 4.1, T' is obtained from T by the *lowering flip* w.r.t. the feasible W-configuration $CW(X; i, j, k)$. One can see that Xi, Xj, Xk, Xij, Xjk are non-terminal vertices in $G_{T'}$; so they form a feasible M-configuration for T'. Moreover, it is not difficult to check that the corresponding lowering flip applied to the reverse of T' results in the g-tiling T^{rev}. Equivalently: the *raising flip* of T' w.r.t. the configuration $CM(X; i, j, k)$ returns the initial T. An important consequence of this fact will be demonstrated in Section 7 (see Theorem 7.5).

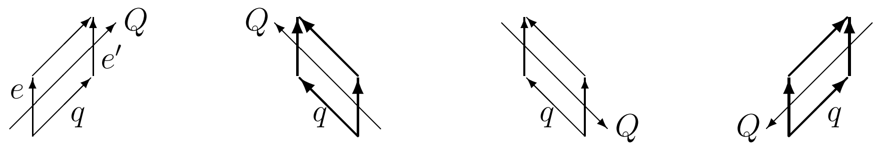
4.2 Strips in a g-tiling

In this subsection we show property (4.3). For this purpose, we introduce the following notion (which will be extensively used subsequently as well).

Definition. For $i \in [n]$, an *i-strip* (or a *dual i-path*) in a g-tiling T is a maximal sequence $Q = (e_0, \tau_1, e_1, \dots, \tau_r, e_r)$ of edges and tiles in it such that: (a) $\tau_1, \dots, \tau_r$ are different tiles, each being an iq- or qi-tile for some q, and (b) for $p = 1, \dots, r$, e_{p-1} and e_p are the opposite i-edges of τ_p.

(Recall that speaking of an $i'j'$-tile, we assume that $i' < j'$.) Clearly Q is determined uniquely (up to reversing it and up to shifting cyclically when $e_0 = e_r$) by any of its edges or tiles. Also, unless $e_0 = e_r$, one of e_0, e_r lies on the left boundary, and the other on the right boundary of Z; we default assume that Q is directed so that $e_0 \in \ell bd(Z)$. In this case, going along Q, step by step, and using (T2), one can see that

(4.6) for consecutive elements e, τ, e' in an i-strip Q: (a) if τ is either a white iq-tile or a black qi-tile (for some q), then e leaves $b(\tau)$ and e' enters $t(\tau)$; and (b) if τ is either a white qi-tile or a black iq-tile, then e enters $t(\tau)$ and e' leaves $b(\tau)$ (see the picture where the i-edges e, e' are drawn vertically).



Let v_p (resp. v'_p) be the beginning (resp. end) vertex of an edge e_p in Q. Define the *right boundary* of Q to be the path $R_Q = (v_0, a_1, v_1, \dots, a_r, v_r)$, where a_p is the edge of τ_p connecting v_{p-1}, v_p. The *left boundary* L_Q of Q is defined in a similar way (regarding the vertices v'_p). From (4.6) it follows that

(4.7) for an i-strip Q, the forward edges of R_Q are exactly those edges in it that belong to either a white iq-tile or a black qi-tile in Q, and similarly for the forward edges of L_Q.

For $I \subseteq [n]$, we call a *maximal alternating I-subpath* in R_Q a maximal subsequence P of *consecutive* elements in R_Q such that each $a_p \in P$ is a q-edge with $q \in I$, and in each pair a_p, a_{p+1}, one edge is forward and the other is backward in R_Q (i.e., exactly one of the tiles τ_p, τ_{p+1} is black). A maximal alternating I-subpath in L_Q is defined in a similar way. The following fact is of importance.

Lemma 4.2 *A strip Q cannot be cyclic, i.e., its first and last edges are different.*

Proof For a contradiction, suppose that some i-strip $Q = (e_0, \tau_1, e_1, \dots, \tau_r, e_r)$ is cyclic ($e_0 = e_r$). One may assume that (4.6) holds for Q (otherwise reverse Q). Take the right boundary $R_Q = (a_1, \dots, a_r = a_0)$ of Q. For $q \in [n]$, let α_q (β_q) denote the number of forward (resp. backward) q-edges in R_Q. Since G_T is graded, $\alpha_q = \beta_q$ (cf. (3.7)).

Assume that R_T contains a q-edge with $q > i$. Put $I^> := [i+1..n]$ and consider a maximal alternating $I^>$-subpath in R_Q (regarding Q up to shifting cyclically and taking indices modulo r). Using (3.1), we observe that if a_p is an edge in P such that τ_p is black, then the edges a_{p-1}, a_{p+1} are contained in P as well; also both tiles τ_{p-1}, τ_{p+1} are white. This together with (4.7) implies that the difference Δ_P between the number of forward edges and the number of backward edges in P is equal to 0 or 1, and that $\Delta_P = 0$ is possible only if P coincides with the whole R_Q (having equal numbers of forward and backward edges). On the other hand, the sum of numbers Δ_P over the maximal alternating $I^>$-subpaths must be equal to $\sum_{q>i}(\alpha_q - \beta_q) = 0$.

So R_Q is an alternating $I^>$-cycle. To see that this is impossible, notice that if a_{p-1}, a_p, a_{p+1} are q'-, q-, and q''-edges, respectively, and if the tile τ_p is black, then (3.1) implies that $q', q'' < q$. Therefore, taking the maximum q such that R_Q contains a q-edge, we obtain $\alpha_q = 0$ and $\beta_q > 0$; a contradiction. Thus, R_Q has no q-edges with $q > i$ at all.

Similarly, considering maximal alternating $I^<$-subpaths in R_Q for $I^< := [i-1]$ and using (3.1) and (4.7), we conclude that R_Q has no q-edge with $q < i$. Thus, a cyclic i-strip is impossible. ∎

Corollary 4.3 *For a g-tiling T and each $i \in [n]$, there is a unique i-strip Q_i. It contains all i-edges of T, begins at the edge $p_{i-1}p_i$ and ends at the edge $p'_ip'_{i-1}$ of $bd(Z)$.*

Using strip techniques, we are now able to prove property (4.3) in the assumption that (4.2) is valid (the latter will be shown in the next subsection).

Proof of (4.3) Let X, i, j, k as in the hypotheses of Proposition 4.1 (with $Y = Xik$). We consider the part Q of the j-strip between the j-edges $e := (Xi, Xij)$ and $e' := (Xk, Xjk)$ (these edges exist by (4.2) and Q exists by Corollary 4.3). Suppose that Q begins at e and ends at e' and consider the right boundary $R_Q = (a_1, \dots, a_r)$ of Q. This is a (not necessarily directed) path from Xi to Xk. Comparing R_T with the path $\widetilde{P}$ from Xi to Xk formed by the forward k-edge (Xi, Xik) and the backward i-edge (Xik, Xk), we have (since G_T is graded):

$$\alpha_q - \beta_q = \begin{cases} -1 & \text{for } q = i, \\ 1 & \text{for } q = k, \\ 0 & \text{otherwise,} \end{cases} \tag{4.8}$$

where α_q (β_q) is the number of forward (resp. backward) q-edges in R_Q. We show that $\alpha_i = 0$, $\beta_i = 1$, $\alpha_k = 1$, $\beta_k = 0$ and $\alpha_q = \beta_q = 0$ for $q \neq i, k$, by arguing as in the proof of Lemma 4.2.

Let $P_1, \ldots, P_d$ be the maximal alternating $J^>$-subpaths in R_Q, where $J^> := \{j+1, \ldots, n\}$. Since each subpath P_h begins and ends with forward edges, we have $\Delta_{P_h} = 1$. Then $\Delta_1 + \ldots + \Delta_{P_d} = \sum_{q>j}(\alpha_q - \beta_q) = \alpha_k - \beta_k = 1$ (cf. (4.8)) implies $d = 1$. Moreover, $|P_1| = 1$. For if $|P_1| > 1$, then P_1 contains a backward edge (whose tile in Q is black), and taking the maximum q such that P_1 contains a q-edge, we obtain $\alpha_q = 0$ and $\beta_q > 0$, which is impossible. Hence P_1 consists of a unique forward edge, and now (4.8) implies that it is a k-edge.

Similarly, there is only one maximal alternating $J^<$-subpath P' in R_Q, where $J^< := \{1, \ldots, j-1\}$, and P' consists of a unique backward i-edge.

Thus, $R_Q = (a_1, a_2)$, and one of a_1, a_2 is a forward k-edge, while the other is a backward i-edge in R_Q. If $R_Q = \widetilde{P}$ (i.e., a_1 is a k-edge), then the tiles in Q are as required in (4.3). And the case when a_1 is an i-edge is impossible, since in this case the first tile τ in Q uses the edge $a_1 = (X, Xi)$ entering $v' := Xi$ and the edge $e = (Xi, Xij)$ leaving v', and the cone $C(\tau, v')$ contains the white edge (Xi, Xik), contrary to (3.5).

Now suppose that Q goes from e' to e. Then R_Q begins at Xk and ends at Xi. Define the numbers α_q, β_q as before. Then $\sum_{q>j}(\alpha_q - \beta_q)$ (equal to the numbers of maximal alternating $J^>$-subpaths in R_Q) is nonnegative. But a similar value for the path reverse to $\widetilde{P}$ (also going from Xk to Xi) equals -1, due to the k-edge (Xi, Xik) which is backward in this path; a contradiction. ∎

4.3 Strip contractions

The remaining properties (4.2) and (4.5) are proved by induction on n, relying on a natural contracting operation on g-tilings (also important for purposes of Section 8).

Let T be a g-tiling on Z_n and $i \in [n]$. We partition T into three subsets T_i^0, T_i^-, T_i^+ conisting, respectively, of all $i*$- and $*i$-tiles, of the tiles $\tau(X; i', j')$ with $i', j' \neq i$ and $i \notin X$, and of the tiles $\tau(X; i', j')$ with $i', j' \neq i$ and $i \in X$. Then T_i^0 is the set of tiles occurring in the i-strip Q_i, and the tiles in T_i^- are vertex disjoint from those in T_i^+.

Definition. The *i-contraction* of T is the collection T/i obtained by removing the members of T_i^0, keeping the members of T_i^-, and replacing each $\tau(X; i', j') \in T_i^+$ by $\tau(X - \{i\}; i', j')$. The image of $\tau \in T$ in T/i is denoted by τ/i, regarding it as the "void tile" $\{\emptyset\}$ if $\tau \in T_i^0$.

The tiles of T/i live within the zonogon generated by the vectors ξ_q, $q \in [n] - \{i\}$ (and cover this zonogon). The regions $D_{T_i^-}$ and $D_{T_i^+}$ of the disc D_T are simply connected, as they arise when the interior of (the image of) the strip Q_i is removed from D_T. The shape $D_{T/i}$ is obtained as the union of $D_{T_i^-}$ and $D_{T_i^+} - \epsilon_i$, where ϵ_i is the i-th unit base vector in $\mathbb{R}^n$. In other words, D_i^+ is shifted by $-\epsilon_i$ and the (image of) the left boundary L_{Q_i} of Q_i in it merges with (the image of) R_{Q_i} in $D_{T_i^-}$. In general, $D_{T_i^-}$ and $D_{T_i^+} - \epsilon_i$ may intersect at some other points, and therefore, $D_{T/i}$ need not be a disc (this happens when G_T contains two vertices X, Xi not connected by edge, or equivalently, such that $X \notin R_{Q_i}$ and $Xi \notin L_{Q_i}$).

For our purposes it suffices to deal with the case $i = n$. We take advantages from the important property that T/n is a feasible g-tiling, i.e., obeys (T1)–(T4). Instead of a direct proof of that property (in which verification of axiom (T4) is rather tiresome), we prefer to appeal to explanations in Section 7 where a similar property is obtained on the language of wirings along the way; see Corollary 7.4 in Remark 4. (More precisely, Sections 6 and 7 establish a bijection β of the g-tilings to the proper wirings. One shows that removing w_n from a proper n-wiring $W=(w_1,\dots,w_n)$ results in a proper $(n-1)$-wiring W'. It turns out that the g-tiling $\beta^{-1}(W')$ is just $(\beta^{-1}(W))/n$, yielding the result.)

We will use several facts, which are exposed in (i)–(iv) below.

(i) In light of explanations above, (T4) for T/n implies that

(4.9) any two vertices of the form $X', X'n$ in G_T are connected by edge (which is an n-edge, and therefore, $X'\in R_{Q_n}$ and $X'n\in L_{Q_n}$).

(ii) Clearly a tile $\tau\in T_n^-\cup T_n^+$ contains a boundary edge of Z_n if and only if τ/n contains a boundary edge of Z_{n-1}. Also tiles $\tau,\tau'\in T_n^-$ sharing an edge do not overlap if and only if $\tau/n,\tau'/n$ do so, and similarly for T_n^+. These facts imply that the white (black) tiles of $\tau\in T_n^-\cup T_n^+$ produce white (resp. black) tiles of T/n.

(iii) For an $i'j'$-tile $\tau\in T$ with $i',j'\ne n$, a vertex $v\in\{t(\tau),b(\tau)\}$ cannot occur in (the boundary of) Q_n. Indeed, all edges in $E_T(v)$ are q-edges with $q\le j'<n$, while each vertex occurring in Q_n is incident with an n-edge. Also for a vertex X' not occurring in Q_n, the local tile structure of T at X' is equivalent to that of T/n at $X'-\{n\}$. Therefore,

(4.10) if X' is a non-terminal vertex for T, then $X'-\{n\}$ is such for T/n.

Now we are ready to prove (4.2) and (4.5).

Proof of (4.2) Let $X', X'i'$ be non-terminal vertices for T. If $i'=n$ then these vertices are connected by edge in G_T, by (4.9). Now let $i'\ne n$. Then $G_{T/n}$ contains the vertices $\widetilde X, \widetilde Xi'$, where $\widetilde X := X'-\{n\}$, and these vertices are non-terminal, by (4.10). So by induction these vertices are connected by an edge e in $G_{T/n}$. Let τ' be a tile in T/n containing e. Then the tile $\tau\in T_n^-\cup T_n^+$ such that $\tau'=\tau/n$ has an edge connecting X' and $X'i'$, as required. ∎

Proof of (4.5) We use notation as in the proof of Proposition 4.1 and consider three possible cases.

(A) Let $k<n$ and $n\notin X$. Then all tiles in T containing the vertex $v=Xik$ are tiles in T/n, and Xi, Xk, Xik, Xij, Xjk are vertices for T/n forming a feasible W-configuration in it (as they are non-terminal, by (4.10)). By induction $G_{T/n}$ contains as a vertex neither Xj in Case 1, nor X in Cases 1(b) and 2(b'). Then the same is true for G_T, as required.

(B) Let $k<n$ and $n\in X$. The argument is similar to that in (A) (taking into account that all vertices X' for T that we deal with contain the element n, and the corresponding vertices for T/n are obtained by removing this element).

(C) Let $k=n$. First we consider Case 1 and show (i) in (4.5). Suppose G_T contains the vertex Xj. Then G_T contains the n-edge $\widetilde e=(Xj, Xjn)$, by (4.9). This edge lies in the cone of τ' at Xjn (where τ' is the white ij-tile with $b(\tau') = Xn$, $r(\tau') = Xjn$ and $t(\tau') = Xijn$). By (3.6), the edge $e' = (Xjn, Xijn)$ is black (since $\widetilde{e}$ enters Xjn). Therefore, the vertex $Xijn$ is terminal, and moreover, it is the top vertex of a black tile $\widetilde{\tau}$ (since $Xijn$ has entering edges). The fact that the cone of $\widetilde{\tau}$ must contain the n-edge $e = (Xij, Xijn)$ implies that e is just the left-to-top edge of $\widetilde{\tau}$. So both e, e' are edges of the same black tile $\widetilde{\tau}$, which is not the case.

Now we consider Cases 1(b) and 2(b') and show (ii) by arguing in a similar way. Suppose G_T contains the vertex X. Then G_T contains the n-edge $\widetilde{e} = (X, Xn)$, by (4.9). This edge lies in the cone of τ_q at Xn (where, according to notation in the proof of Proposition 4.1, τ_q is the white tile in T with $r(\tau_q) = Xn$ and $t(\tau') = Xin$). By (3.6), the edge $e'' := (Xn, Xin)$ of τ_q is black (since $\widetilde{e}$ enters Xn). But each of the end vertices Xn, Xin of e'' has both entering and leaving edges, and therefore, it cannot be terminal; a contradiction. ▮

This completes the proof of inclusion (4.1).

5 From generalized tilings to semi-normal bases

In this section we complete the proof of Theorem 2.2 by proving the first assertion in it, namely, we show the inclusion

$$\mathcal{BT}_n \subseteq \mathcal{B}_n. \tag{5.1}$$

This together with the reverse inclusion (4.1) will give $\mathcal{BT}_n = \mathcal{B}_n$, as required.

Let T be a g-tiling. We have to prove that B_T is a semi-normal basis.

If T has no black tile, then B_T is already a normal basis, and we are done. So assume $T^b \neq \emptyset$. Our aim is to show the existence of a feasible W-configuration $CW(X; i, j, k)$ for T (formed by non-terminal vertices Xi, Xk, Xij, Xjk, Xik, where $i < j < k$). Then we can transform T into a g-tiling T' as in Proposition 4.1, i.e., with $B_{T'} = (B_T - \{Xik\}) \cup \{Xj\}$. Under such a *lowering flip* (concerning g-tilings), the sum of sizes of the sets involved in $B_\bullet$ decreases. Then the required relation $B_T \in \mathcal{B}_n$ follows by induction on $\sum(|X'|\colon X' \in B_T)$ (this sort of induction is typical when one deals with tilings or related objects, cf. [2, 4, 7]).

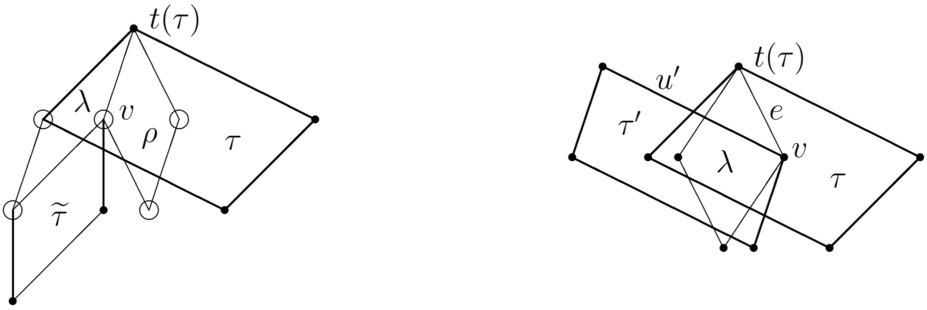
In what follows by the *height* $h(v)$ of a vertex $v \in V_T$ we mean the size of the corresponding subset of $[n]$. The height $h(\tau)$ of a tile $\tau \in T$ is defined to be the height of its left vertex; then $h(\tau) = h(r(\tau)) = h(b(\tau)) + 1 = h(t(\tau)) - 1$. The height of a W-configuration $CW(X; i, j, k)$ is defined to be $|X| + 2$.

In fact, we are able to show the following sharper version of the desired property.

Proposition 5.1 *Let $h \in [n]$. If a g-tiling T has a black tile of height h, then there exists a feasible W-configuration $CW(X; i, j, k)$ of the same height h. Moreover, such a $CW(X; i, j, k)$ can be chosen so that $Xijk$ is the top vertex of some black tile (of height h).*

(Note that, by the equivalence of (iii) and (iv) in Theorem A from the Introduction, this proposition answers affirmatively Conjecture 5.5 in [7], even a strengthening of it.)

Proof Let τ be a black tile of height h. Denote by $M(\tau)$ the set of vertices v such that there is a white edge from v to $t(\tau)$. This set is nonempty (by Corollary 3.1(ii)) and each vertex in it is non-terminal. Suppose that some $v \in M(\tau)$ is ordinary, and let λ and ρ be the (white) tiles sharing the edge $(v, t(\tau))$ and such that $v = r(\lambda) = \ell(\rho)$. Then the five vertices $b(\lambda), b(\rho), \ell(\lambda), v, r(\rho)$ form a W-configuration of height h (since $h(v) = h(\tau) = h$). Moreover, this configuration is feasible. Indeed, the vertices $\ell(\lambda), v, r(\rho)$ are non-terminal (since each has an entering edge and a leaving edge). And the tile $\widetilde{\tau}$ that shares the edge $(b(\lambda), v)$ with λ has v as its top vertex (taking into account that $\widetilde{\tau}$ is white and overlaps neither λ nor ρ since v is ordinary); then $b(\lambda)$ is the left vertex of $\widetilde{\tau}$, and therefore $b(\lambda)$ is non-terminal. The vertex $b(\rho)$ is non-terminal for a similar reason. For an illustration, see the left fragment on the picture.



We assert that a black tile τ of height h whose set $M(\tau)$ contains an ordinary vertex does exist (yielding the result).

Suppose this is not so. We construct an alternating sequence of white and black edges as follows. Choose a black tile τ of height h and a vertex $v \in M(\tau)$. Let e be the white edge $(v, t(\tau))$. Since v is mixed (by the supposition), there is a black tile τ' (of height h) such that either (a) $v = r(\tau')$ or (b) $v = \ell(\tau')$. We say that τ' *lies on the left from* τ in the former case, and *lies on the right from* τ in the latter case. Let u' be the right-top edge $(r(\tau'), t(\tau'))$ of τ' in case (a), and the left-top edge $(\ell(\tau'), t(\tau'))$ in case (b). Case (a) is illustrated on the right fragment of the picture above.

Repeat the procedure for τ': choose $v' \in M(\tau')$ (which is mixed again by the supposition); put $e' := (v', t(\tau'))$; choose a black tile τ'' such that either (a) $v' = r(\tau'')$ or (b) $v' = \ell(\tau'')$; and define u' to be be the edge $(r(\tau''), t(\tau''))$ in case (a), and the edge $(\ell(\tau''), t(\tau''))$ in case (b). Repeat the procedure for τ'', and so on. Sooner or later we must return to a black tile that has occurred earlier in the process. Then we obtain an alternating cycle of white and black edges.

More precisely, there appear a cyclic sequence of different black tiles $\tau_1, \ldots, \tau_{r-1}, \tau_r = \tau_0$ of height h and an alternating sequence of white and black edges $C = (e_0 = e_r, u_1, e_1, \ldots, u_r = u_0)$ (forming a cycle in G_T) with the following properties, for $q = 1, \ldots, r$: (a) e_q is the edge $(v_q, t(\tau_q))$ for some $v_q \in M(\tau_q)$; (b) τ_{q+1} is a black tile whose right of left vertex is v_q; and (c) $u_{q+1} = (r(\tau_{q+1}), t(\tau_{q+1}))$ when $r(\tau_{q+1}) = v_q$, and $u_{q+1} = (\ell(\tau_{q+1}), t(\tau_{q+1}))$ when $\ell(\tau_{q+1}) = v_q$, where the indices are taken modulo r. We consider C up renumbering the indices cyclically and assume that τ_q is an $i_q k_q$-tile, that e_q is a j_q-edge, and that u_q is a p_q-edge. Then $i_q < j_q < k_q$, $p_q = i_q$ if τ_q lies on the left from τ_{q-1}, and $p_q = k_q$ if τ_q lies on the right from τ_{q-1}. Note that in the former (latter) case the vertex v_q lies on the left (resp. right) from v_{q-1} in the horizontal line at height h in Z. This implies that there exists a q such that τ_q lies on the left from τ_{q-1}, and there exists a q' such that τ'_q lies on the right from $\tau_{q'-1}$.

To come to a contradiction, consider a maximal subsequence Q of (cyclically) consecutive tiles in which each but first tile lies on the left from the previous one; one may assume that $Q = (\tau_1, \tau_2, \dots, \tau_d)$. Then τ_1 lies on the right from τ_0, and therefore u_1 is the left-top edge of τ_1, whence $p_1 = k_1$. Also we observe that

$$k_1 \geq k_2 \geq \ldots \geq k_d. \tag{5.2}$$

Indeed, for $1 \leq q < d$, let λ be the (white) tile containing the edge e_q and such that $r(\lambda) = v_q$. This tile lies in the cone of τ_q at $t(\tau_q)$. So λ is an $i'k'$-tile with $i_q \leq i' < k' \leq k_q$, and therefore the (bottom-right) edge $\widetilde{e}$ of λ entering v_q has color $k' \leq k_q$. Using (3.5)(iii), we observe that $\widetilde{e}$ lies in the cone of the black tile τ_{q+1} at v_q (taking into account that $v_q = r(\tau_{q+1})$). This implies that the the bottom-right edge $e' = (b(\lambda), v_q)$ of τ_{q+1} has color at most k'. Since u_{q+1} is parallel to e', we obtain $k_{q+1} \leq k' \leq k_q$, as required.

By (5.2), we have $j_q < k_q \leq k_1 = p_1$ for all $q = 1, \dots, d$. Also if a tile $\tau_{q'}$ lies on the right from the previous tile $\tau_{q'-1}$, then $u_{q'}$ is the left-top edge of $\tau_{q'}$, whence $j_{q'} < k_{q'} = p_{q'}$. Thus, the maximum of $p_1, \dots, p_r$ is strictly greater than the maximum of $j_1, \dots, j_r$. This is impossible since all $u_1, \dots, u_r$ are forward edges, all $e_1 \dots, e_r$ are backward edges in C, and the graph G_T is graded. ∎

This completes the proof of Theorem 2.2.

Remark 3 For black tiles $\tau, \tau' \in T^b$, let us denote $\tau' \prec^\diamond \tau$ if there is a white edge $(v, t(\tau))$ such that v is the right or left vertex of τ'. The proof of Proposition 5.1 gives the following additional result.

Corollary 5.2 *The relation $\prec^\diamond$ determines a partial order on T^b.*

Similarly, the relation $\prec_\diamond$ determines a partial order on T^b, where for $\tau, \tau' \in T^b$, we write $\tau' \prec_\diamond \tau$ if there is a white edge $(b(\tau), v)$ such that v is the right or left vertex of τ'. It is possible that the graph on T^b induced by $\prec^\diamond$ (or $\prec_\diamond$) is a forest, but we do not go in our analysis so far.

6 From generalized tilings to proper wirings

In this section we show the following

Proposition 6.1 *For a g-tiling T on Z_n, there exists a proper wiring W on Z_n such that $B_W = B_T$.*

This and the converse assertion established in the next section will imply that the collection $\mathcal{BT}_n$ of the sets B_T generated by g-tilings on Z_n coincides with the collection $\mathcal{BW}_n$ of the sets B_W generated by proper wirings W on Z_n, and then Theorem 2.1 will follow from Theorem 2.2.

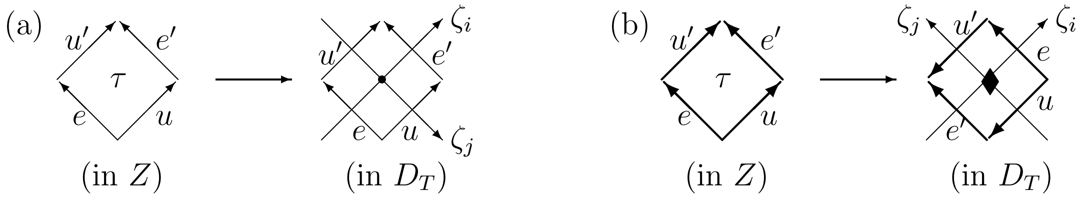
Proof Like we have identified each subset in $[n]$ with the corresponding points in the solid cube $conv(2^{[n]})$ and in the zonogon $Z = Z_n$, we may identify each tile τ in T with the corresponding square $\sigma(\tau)$, and each edge in E_T with the corresponding line-segment in the disc D_T; we will use similar notation for related objects in Z and D_T, specifying when needed which of them we deal with. Then the graph $G_T = (V_T, E_T)$ turns into a planar directed graph properly embedded in D_T.

The desired wiring W is constructed by drawing curves on D_T related to strips (dual paths) in G_T. More precisely, for $i \in [n]$, take the i-strip $Q_i = (e_0, \tau_1, e_1, \dots, \tau_r, e_r)$ (defined in Section 4), considering it as a sequence of edges and squares in D_T. (Recall that, when dealing with T, Q_i contains all $i*$- and $*i$-tiles in T (by Lemma 4.2), e_0 is the edge $p_{i-1}p_i$ on the left boundary $\ell bd(Z)$, and e_r is the edge $p'_i p'_{i-1}$ on $rbd(Z)$.) For $q = 1, \dots, r$, draw the line-segment on the square τ_q connecting the median points of the edges e_{r-1} and e_r. This segment meets the central point of τ_q, denoted by $c(\tau_q)$. The concatenation of these segments gives the desired (piece-wise linear) curve ζ_i corresponding to Q_i; we direct ζ_i according to the direction of Q_i.

Let γ be a homeomorphic map of D_T to Z such that each boundary edge of D_T is linearly mapped to the corresponding edge of $bd(Z)$. To simplify notation (and when no confuse can arise), we identify points of Z and D_T by use of γ. Then the curves $\zeta_1, \dots, \zeta_n$ turn into "wires" on T, forming the desired wiring W (each "wire" ζ_i begins at the median point s_i of $p_{i-1}p_i$ on $\ell bd(Z)$ and ends at the medial point s'_i of $p'_i p'_{i-1}$ on $rbd(Z)$). Axiom (W1) for W is obvious.

To verify the other axioms, we first explain how the graphs G_T and G_W on D_T are related to each other (which is a sort of planar duality). The vertices of G_W are the central points $c(\tau)$ of squares τ (where corresponding pairs of wires are crossed) and the points s_i, s'_i. Each vertex v of G_T corresponds to the face of G_W where v is contained, denoted by v^* (and this is a one-to-one correspondence). The edges of color i in G_W (which are the pieces of ζ_i in its subdivision by the central points of squares lying on ζ_i) one-to-one correspond to the i-edges of G_T. More precisely, if an i-edge $e \in E_T$ belongs to squares τ, τ' and if τ, e, τ' occur in this order in the i-strip, then the i-edge of G_W corresponding to e, denoted by e^*, is the piece of ζ_i between $c(\tau)$ to $c(\tau')$, and this e^* is directed from $c(\tau)$ to $c(\tau')$. Observe that e crosses e^* *from right to left* on the disc. (We assume that the clockwise orientations on D_T and Z are agreeable by γ.) The first and last pieces of ζ_i correspond to the boundary i-edges $p_{i-1}p_i$ and $p'_i p'_{i-1}$ of G_T, respectively.

Consider an ij-tile $\tau \in T$, and let e, e' be its i-edges, and u, u' its j-edges, where e, u leave $b(\tau)$ and e', u' enter $t(\tau)$. We know that: (a) if τ is white, then e occurs in Q_i before e', while u occurs in Q_j after u', and (b) if τ is black, then e occurs in Q_i after e', while u occurs in Q_j before u'. In each case, in the disc D_T, both e, e' cross the wire ζ_i from right to left (w.r.t. the direction of ζ_i), and similarly both u, u' cross ζ_j from right to left. Also it is not difficult to realize (using (T1),(T2)) that when τ is white, the orientation of the tile τ in Z coincides with that of the square τ in D_T, whereas when τ is black, the clockwise orientation of τ in Z turns in the counterclockwise orientation of τ in D_T (justifying the term "orientation-reversing" for the vertex $c(\tau)$ of G_W). One can conclude from these facts that: in case (a), ζ_j crosses ζ_i at $c(\tau)$ from left to right, and therefore, the vertex $c(\tau)$ of G_W is white, and in case (b), ζ_j crosses ζ_i at $c(\tau)$ from right to left, and therefore, the vertex $c(\tau)$ is black. (Both cases are illustrated in the picture.) So the white (black) tiles of T generate the white (resp. black) vertices of G_W.



Consider a vertex v of G_T and an edge $e \in E_T(v)$. Then the edge e^* belongs to the boundary of the face v^* of G_W. As mentioned above, e crosses e^* from right to left on D_T. This implies that e^* is directed clockwise around v^* if e leaves v, and counterclockwise if e enters v. In view of axiom (T3), we obtain that

(6.1) the terminal vertices of G_T and only these generate cyclic faces of G_W; moreover, for $\tau \in T^b$, the boundary cycle of $(t(\tau))^*$ is directed counterclockwise, while that of $(b(\tau))^*$ is directed clockwise.

(We use the fact that any non-terminal vertex $v \neq p_0, p_n$ has both entering and leaving edges, and therefore the boundary of the face v^* has edges in both directions. When $v = p_0, p_n$, a similar fact for v^* is valid as well.)

Next, for each $i \in [n]$, removing from D_T the interior of the i-strip Q_i (i.e., the relative interiors of all edges and tiles in it) results in two closed regions Ω_1, Ω_2, the former containing the vertex $\emptyset$, and the latter the vertex $[n]$ (regarding the vertices as subsets of $[n]$). The fact that all edges in Q_i (which are the i-edges of G_T) go from Ω_1 to Ω_2 implies that each vertex v of G_T occurring in Ω_1 (in Ω_2) is a subset X not containing (resp. containing) the element i. Then $i \notin X(v^*)$ (resp. $i \in X(v^*)$. Thus, we obtain $B_W = B_T$.

A less trivial task is to show validity of (W2) for W. One can see that (W2) is equivalent to the following assertion.

Claim Let wires ζ_i, ζ_j with $i < j$ intersect at a white point x. Then the part ζ of ζ_i from x to s'_i and the part ζ' of ζ_j from x to s'_j have no other common points.

Proof of the Claim Suppose this is not so and let y be the common point of ζ, ζ' closest to x in ζ. Then y is black (since x is white). Therefore, the ij-tile τ such that $x = c(\tau)$ is white, and the ij-tile τ' such that $y = c(\tau')$ is black. Also in both strips Q_i and Q_j, the tile τ occurs earlier than τ'. One can see (cf. (4.6)) that in the strip Q_i, the edge succeeding τ is $(r(\tau), t(\tau))$ and the edge preceding τ' is $(r(\tau'), t(\tau'))$, whereas in the strip Q_j, the edge succeeding τ is $(b(\tau), r(\tau))$ and the edge preceding τ' is $(b(\tau'), r(\tau'))$. So the right boundary of Q_i passes the vertices $r(\tau)$ and $r(\tau')$ (in this order), and similarly for the left boundary of Q_j.

Consider the part R of R_{Q_i} from $r(\tau)$ to $r(\tau')$ and the part L of L_{Q_j} from $r(\tau)$ to $r(\tau')$. For $q \in [n]$, let α_q, α'_q, β_q, β'_q be the numbers of q-edges that are forward in R, forward in L, backward in R, and backward in L, respectively. Since G_T is graded, we have $(*)$ $\alpha_q - \beta_q = \alpha'_q - \beta'_q$.

Now we argue in a similar spirit as in the proof of Lemma 4.2. Define $I := \{i+1, \ldots, j-1\}$, $\Delta := \sum_{q\in I}(\alpha_q - \beta_q)$, and $\Delta' := \sum_{q\in I}(\alpha'_q - \beta'_q)$. We assert that $\Delta > 0$ and $\Delta' < 0$, which leads to a contradiction with $(*)$ above.

To see $\Delta > 0$, consider a q-edge e in R with $q \in I$, and let τ^e denote the tile in Q_i containing e. Since $q > i$, τ^e is white if e is forward, and τ^e is black if e is backward in R (cf. (4.7)). Using this, one can see that:

(i) for a q-edge $e \in R$ such that $q \in I$ and τ^e is black, the next edge e' in R_{Q_i} is a forward q'-edge in R with $q' \in I$ (since the fact that τ^e is a black iq-tile implies that $\tau^{e'}$ is a white $q'q$-tile with $i < q' < q$, in view of (3.1)); a similar property holds for the previous edge (for e) in R_{Q_i};

(ii) the last edge e of R is a forward q-edge with $q \in I$ (since the tile τ^e shares an edge with the black ij-tile τ');

(iii) if the first edge e of R is backward, then it is a q-edge with $q \notin I$ (since τ^e is black and shares an edge with the ij-tile τ).

These properties show that the first and last edges of any maximal alternating I-subpath P in R are forward, and therefore P contributes $+1$ to Δ. Also at least one such P exists, by (ii). So $\Delta > 0$, as required.

The inequality $\Delta' < 0$ is shown in a similar fashion, by considering L and swapping "forward" and "backward" in the above reasonings (due to replacing $q > i$ by $q < j$). More precisely, for a q-edge e in L with $q \in I$, the tile τ^e in Q_j containing e is black if e is forward, and white if e is backward (in view of $q < j$ and (4.7)). This implies that:

(i') for a q-edge $e \in L$ such that $q \in I$ and τ^e is black, the next edge e' in L_{Q_j} is a backward q'-edge in L with $q' \in I$; and similarly for the previous edge in L_{Q_j};

(ii') the last edge e of L is a backward q-edge with $q \in I$;

(iii') if the first edge e of R is forward, then it is a q-edge with $q \notin I$.

Then the first and last edges of any maximal alternating I-subpath P in L are backward, and therefore P contributes -1 to Δ'. Also at least one such P exists, by (ii'). So $\Delta' < 0$, contrary to $\Delta = \Delta'$. ▮

Thus, (W2) is valid. Considering lenses formed by a pair of wires and using (6.1) and (W2), one can easily obtain (W3). Finally, since $|E_T(v)| \geq 3$ holds for each terminal vertex in G_T (by Corollary 3.1(ii)), each cyclic face in G_W is surrounded by at least three edges, and therefore this face cannot be a lens. So the wiring W is proper.

This completes the proof of Proposition 6.1. ▮▮

7 From proper wirings to generalized tilings

In this section we complete the proof of Theorem 2.1 by showing the property converse to that in Proposition 6.1.

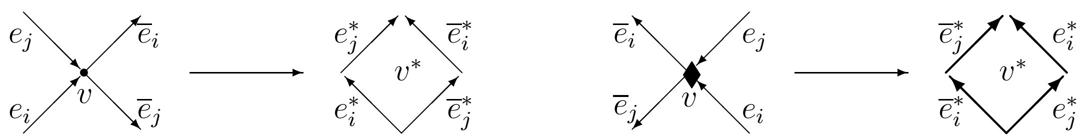
Proposition 7.1 *For a proper wiring W on $Z = Z_n$, there exists a g-tiling T on Z*

such that $B_T = B_W$.

Proof The construction of the desired T is converse, in a sense, to that described in the proof of Proposition 6.1; it combines planar duality techniques and geometric arrangements. We use definitions and notation from Section 2.

We associate to each (inner) face F of the graph G_W the point (viz. the subset) $X(F)$ in the zonogon, also denoted as F^*. These points are just the vertices of tiles in T. The edges concerning T are defined as follows. Let faces $F, F' \in \mathcal{F}_W$ have a common edge e formed by a piece of a wire w_i, and let F lie on the right from w_i according to the direction of this wire (and F' lies on the left from w_i). Then the vertices F^*, F'^* are connected by edge e^* going from F^* to F'^*. Note that in view of the evident relation $X(F') = X(F) \cup \{i\}$, the direction of e^* matches the edge direction for g-tilings.

The tiles in T one-to-one correspond to the intersection points of wires in W. More precisely, let v be a common point of wires w_i, w_j with $i < j$. Then the vertex v of G_W has four incident edges $e_i, \overline{e}_i, e_j, \overline{e}_j$ such that: $e_i, \overline{e}_i \subset w_i$; $e_j, \overline{e}_j \subset w_j$; e_i, e_j enter v; and $\overline{e}_i, \overline{e}_j$ leave v. Also one can see that for the four faces F containing v, the subsets $X(F)$ are of the form X, Xi, Xj, Xij for some $X \subset [n]$. The tile surrounded by the edges $e_i^*, \overline{e}_i^*, e_j^*, \overline{e}_j^*$ connecting these subsets (regarded as points) is just the ij-tile in T corresponding to v, denoted as v^*. Observe that the edges $e_i, e_j, \overline{e}_i, \overline{e}_j$ follow in this order counterclockwise around v if v is black (orientation-reversing), and clockwise otherwise (when v is "orientation-respecting"). The tile v^* is regarded as black in T if v is black, and white otherwise. Both cases are illustrated in the picture where the right fragment concerns the orientation-reversing case.



Next we examine properties of the obtained collection T of tiles. The first and second conditions in (T2) (concerning overlapping and non-overlapping tiles with a common edge) follow from the above construction and explanations.

1) Consider an i-edge $e = (u, v)$ in G_W (a piece of the wire w_i). If $u \neq s_i$ and $v \neq s'_i$, then the dual edge e^* belongs to exactly two tiles, namely, u^* and v^*. If $u = s_i$, then e^* belongs to the unique tile v^*. Furthermore, for the faces $F, F' \in \mathcal{F}_W$ containing e, the sets $X(F), X(F')$ are the principal intervals $[i-1]$ and $[i]$ (letting $[0] := \emptyset$). This implies that e^* is the boundary edge $p_{i-1}p_i$ of Z, and this edge belongs to a unique tile in T (which is, obviously, white). Considering $v = s'_i$, we obtain a similar property for the edges in $rbd(Z)$. This gives the first and second condition in (T1).

The proper wiring W possesses the following important property, which will be proved later (see Lemma 7.2): $(*)$ each face F in G_W has at most one i-edge for each i, and all sets $X(F)$ among $F \in \mathcal{F}_W$ are different. This implies that T has no tile copies, yielding the third condition in (T1). Also property $(*)$ and the planarity of G_W imply validity of axiom (T4).

2) For a face $F \in \mathcal{F}_W$, let $E(F)$ denote the set of its edges not contained in $bd(Z)$. By the construction and explanations above,

(7.1) the edges in $E(F)$ one-to-one correspond to the edges incident to the vertex $v = F^*$ of G_T; moreover, for $e \in E(F)$, the corresponding edge e^* enters v if e is directed counterclockwise (around F), and leaves v otherwise.

This implies that v has both entering and leaving edges if and only if F is non-cyclic, unless $v = p_0$ or p_n. (Here we also use an easy observation that if F contains an edge $p_{i-1}p_i$ or $p'_ip'_{i-1}$ for some $1 < i < n-1$, then $E(F)$ has edges in both directions.)

Consider a cyclic face $F \in \mathcal{F}_W^{cyc}$, and let $C = (v_0, e_1, v_1, \dots, e_r, v_r = v_0)$ be its boundary cycle, where for $p = 1, \dots, r$, the edge e_p goes from v_{p-1} to v_p. Denote the color of e_p by i_p. Suppose C is directed clockwise. Then for each p, we have $i_{p-1} < i_p$ if v_p is white, and $i_{p-1} > i_p$ if v_p is black (taking the indices modulo r). Hence C contains at least one black point (for otherwise we would have $i_0 < \dots < i_r = i_0$). Moreover,

(7.2) C contains exactly one black point.

Indeed, let v_p be black. Then v_p is the root of the (even) lens L of wires $w_{i_{p-1}}$ and w_{i_p} such that $F \subseteq L$. (This lens is obtained when we take the the part w' of $w_{i_{p-1}}$ from $s_{i_{p-1}}$ to v_p and the part w'' of w_{i_p} from v_p to s'_{i_p}, the former containing the edge e_{p-1} and the latter containing e_p. In view of $i_{p-1} > i_p$, w' must cross w_{i_p} at some point $x \neq v_p$, and x cannot lie on the part of w_{i_p} from s_{i_p} to v_p, by axiom (W2). So x lies on w'', and L arises when we choose as x the last point of w' meeting $w'' - \{v_p\}$.) By axiom (W3), L is bijective to F. The existence of another black vertex in C would cause the appearance of another lens bijective to F, which is impossible. So (7.2) is valid. This implies that the vertex F^* (which has only leaving edges, by (7.1)) is the bottom vertex of exactly one black tile. When C is directed counterclockwise, we have $i_{p-1} > i_p$ if v_p is white, and $i_{p-1} < i_p$ if v_p is black, implying (7.2) again, which in turn implies that F^* is the top vertex of exactly one black tile. Thus, T obeys (T3).

3) If a cyclic face F and another face F' in G_W have a common edge $e = (u, v)$, then F' is non-cyclic. Indeed, the edge e' preceding e in the boundary cycle of F enters the vertex u. The wire in W passing through e' leaves u by an edge e''. Obviously, e'' belongs to F'. Since the edges e, e'' of F' have the same beginning vertex, F' is non-cyclic. Hence the cyclic faces in G_W are pairwise disjoint, implying that no pair of black tiles in T share an edge (the third condition in (T2)).

Thus, T is a g-tiling. If a face F of G_W lies on the left from a wire w_i, then the vertices F^* and $[n]$ occur in the same region when the interior of the i-strip is removed from the disc D_T. This implies that the sets $X(F)$, $F \in \mathcal{F}_W$, are just the vertices of T. Then $B_T = B_W$, as required. ▮

It remains to show the following (cf. $(*)$ in the above proof).

Lemma 7.2 *Let W be a proper wiring. Then:*

(i) for each face F in G_W, all edges surrounding F belong to different wires;

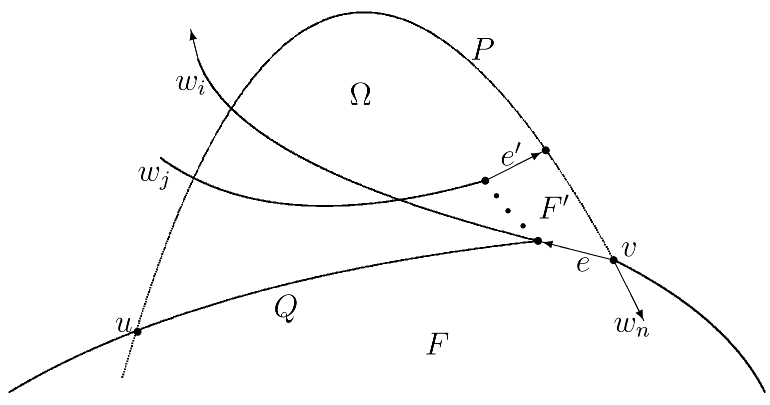
(ii) there are no different faces $F, F' \in \mathcal{F}_W$ such that $X(F) = X(F')$.

Proof Suppose that a face F contains two i-edges e, e' for some i. One may assume that the wire w_i meets e earlier than e' and that w_i does not meet F between these edges. One can see that e, e' have the same direction in the boundary of F, and the face $F' \neq F$ containing e is different from the face $F'' \neq F$ containing e'. Then $X(F') = X(F'')$. Therefore, (i) follows from (ii).

To show (ii), we use induction on n (the assertion is obvious if $n = 2$). Let $W' := (w_1, \ldots, w_{n-1})$. Clearly W' obeys axioms (W1),(W2). One can see that if none of cyclic faces of $G_{W'}$ is separated by w_n, then (W3) is valid for W' as well. So suppose that some cyclic face F of $G_{W'}$ is separated by w_n. Let v be a black vertex in F (such a vertex exists, for otherwise the edge colors along the boundary cycle of F would be monotone increasing or monotone decreasing, which is impossible). Then there is a lens L for W' such that $F \subseteq L$ and v is the root of L. Take the face $\widetilde{F}$ of G_W such that $v \in \widetilde{F} \supset F$. By (W3) for W, $\widetilde{F}$ is cyclic (since L and v continue to be a lens and its root when w_n is added to W'). Assume that the boundary cycle C of $\widetilde{F}$ is directed clockwise (when C is directed counterclockwise, the argument is similar). Since $\widetilde{F} \neq F$, C should contain consecutive edges $e' = (u', u), e'' = (u, u'')$ with colors n and $i < n$, respectively. Then the wire w_i crosses w_n at u from left to right. This implies that the vertex u is black and there is an *in*-lens L' rooted at u and containing $\widetilde{F}$. But $\widetilde{F}$ is bijective to L; a contradiction.

Thus, W' is a wiring (obeying (W1)–(W3)). We first prove (ii) in the assumption that W' is proper. Then by induction all sets $X(F)$, $F \in \mathcal{F}_{W'}$, are different. Suppose there are faces $\widetilde{F}, \widetilde{F}' \in \mathcal{F}_W$ such that $X(\widetilde{F}) = X(\widetilde{F}')$. Let F and F' be the faces for W' containing $\widetilde{F}$ and $\widetilde{F}'$, respectively. Then $X(F) = X(\widetilde{F}) - \{n\}$ and $X(F') = X(\widetilde{F}') - \{n\}$. This implies $X(F) = X(F')$, and therefore $F = F'$. Furthermore, w_n separates F at most twice (for otherwise $F = \widetilde{F} \cup \widetilde{F}'$, whence $X(\widetilde{F}) \neq X(\widetilde{F}')$).

Then there exist two points u, v on w_n such that: (a) u occurs in w_n earlier than v, and (b) the piece P of w_n from u to v lies outside F except for the points u, v where P and F meet. In particular, u, v are vertices of G_W. Let Q be the part of the boundary of F between u and v such that the simply connected region Ω bounded by P and Q is disjoint from the interior of F. We choose such u, v so that w_n does not meet the interior of Q (which is always possible). First we examine the case when P goes clockwise around Ω; see the picture.



Let e be the edge in E_W contained in Q and incident to v; then e has color $i < n$. Take the maximal connected piece Q' of w_i lying in Ω and containing e. Since w_i does not meet the interior of F, the end x of Q' different from v lies on P. Then Q and the piece P' of P from x to v form an *in*-lens L for W. Since P' is directed from x to v, Q' must be directed from v to x (by (W2)); in particular, e leaves v. So w_n crosses w_i at v from right to left, and therefore, the vertex v is black and is the root of L. Let F' be face in G_W lying in L and containing v (which is cyclic), and let C be its boundary cycle. Since W is proper, $F' \neq L$, whence $C \neq P' \cup Q'$. Then C contains an edge e' with color $j \neq i, n$ (one can take as e' the edge of C that either succeeds e or precedes the last edge on P). Take the maximal connected piece R of w_j, from a point y to a point z say, that lies in Ω and contains e'. It is not difficult to realize that y occurs in P earlier than z. This violates (W2) for w_j, w_n.

When P goes counterclockwise around Ω, a contradiction is shown in a similar way. (In this case, we take as e the edge on Q incident to u; one shows that e enters e, whence the vertex u is black.)

Finally, we assert that the wiring W' is always proper. Indeed, suppose this is not so and consider an "empty" ij-lens L (where $i < j$), i.e., forming a face in $G_{W'}$. One may assume that L is an odd lens with lower vertex u and upper vertex v (the case of an even "empty" lens is examined in a similar way). Then v is black (the root of L), and the boundary of L is formed by the piece P of w_i from u to v and the piece Q of w_j from v to u (giving the edges in $G_{W'}$ connecting u and v). Consider the (cyclic) face F in G_W lying in L and containing v, and let C be its boundary cycle. Since W is proper, $F \neq L$. Then, besides the i-edge (in P) entering v, say, $e = (x, v)$, and the j-edge (in Q) leaving v, say, $e' = (v, y)$, the cycle C contains some n-edge e''. Note that e'' cannot connect two points on P or two points on Q, for otherwise there would appear an "empty" *in*- or *jn*-lens. This implies that e'' goes from y to x (respecting the direction on C). But then w_n crosses w_j from right to left, whence the vertex y is black. Thus, the face F contains two black vertices, contradicting (7.2). ∎

Propositions 6.1 and 7.1 imply the desired equality $\mathcal{BT}_n = \mathcal{BW}_n$, and now Theorem 2.1 follows from Theorem 2.2. Analyzing the transformation of a g-tiling into a proper wiring described in Section 6 and the converse transformation described above (and using reasonings above), one can conclude that the composition of such transformations returns the initial g-tiling (or the initial proper wiring). This implies the following (where, as before, B_T and $\widehat{B}_T$ stand for the effective and full spectra of a g-tiling T, respectively, and similarly for wirings).

Theorem 7.3 *There is a bijection β of the set $\mathbf{T}_n$ of g-tilings to the set $\mathbf{W}_n$ of proper wirings on Z_n such that $B_T = B_{\beta(T)}$ holds for each $T \in \mathbf{T}_n$. Furthermore, for each proper wiring W, all subsets $X(F) \subseteq [n]$ determined by the faces F for W (forming $\widehat{B}_W$) are different, and one holds $\widehat{B}_W = \widehat{B}_T$, where $T = \beta^{-1}(W)$.*

We conclude this section with several remarks and additional results.

Remark 4 As is shown in the proof of Lemma 7.2, for any proper wiring $W = (w_1, \ldots, w_n)$, the set $W' = (w_1, \ldots, w_{n-1})$ forms a proper wiring as well (concerning the zonogon Z_{n-1}. Clearly a similar result takes place when we remove the wire w_1. As a generalization, we obtain that for any $1 \leq i < j \leq n$, the set $(w_i, \ldots, w_j)$ forms a proper wiring on the corresponding subzonogon. One can see that removing w_n from

W corresponds to the contracting operation concerning n in the g-tiling $\beta^{-1}(W)$, and this results in the set of tiles determined by W' (via β^{-1}). This gives the following important result to which we have appealed in Section 4.

Corollary 7.4 *For a g-tiling T on Z_n, its n-contraction T/n is a (feasible) g-tiling on Z_{n-1}.*

Remark 5 Properties of g-tilings and proper wirings established during the proofs of Theorems 2.1 and 2.2 enables us to obtain the following result saying that these objects are determined by their spectra.

Theorem 7.5 *For each semi-normal basis B, there are a unique g-tiling T and a unique proper wiring W such that $B = B_T = B_W$.*

Proof Due to Theorem 7.3, it suffices to prove this uniqueness property for g-tilings. We apply induction on $h(B) := \sum(|X|\colon X \in B)$. Suppose there are different g-tiles T, T' with $B_T = B_{T'} =: B$. This is impossible when none of T, T' has black tiles. Indeed, the vertices of G_T and $G_{T'}$ (which are the sets in B) are the same and they determine the edges of these graphs, by (4.2). So $G_T = G_{T'}$. This graph is planar and subdivides Z_n into little parallelograms, which are just the tiles in T and the tiles in T'. Then $T = T'$. Now let T (say) have a black tile. By Proposition 5.1, T has a feasible W-configuration $CW(X; i, j, k)$, and we can make the corresponding lowering flip for T, obtaining a g-tiling $\widetilde{T}$ with $B_{\widetilde{T}} = (B - \{Xik\}) \cup \{Xj\}$. Since $B_T = B_{T'}$, $CW(X; i, j, k)$ is a feasible W-configuration for T' as well, and making the corresponding lowering flip for T', we obtain a g-tiling $\widetilde{T}'$ with $B_{\widetilde{T}'} = B_{\widetilde{T}}$. We have $h(B_{\widetilde{T}}) < h(B)$, whence, by induction, $\widetilde{T} = \widetilde{T}'$. But the raising flip in $\widetilde{T}$ w.r.t. the (feasible) M-configuration $CM(X; i, j, k)$ returns T, as mentioned in Remark 2 in Subsection 4.1. Hence $T = T'$; a contradiction. ∎

8 Weakly separated set-systems

The goal of this section is to prove the following theorem answering Leclerc–Zelevinsky's conjecture mentioned in the Introduction.

Theorem 8.1 *Any largest weakly separated collection $C \subseteq 2^{[n]}$ is a semi-normal TP-basis.*

For brevity we will abbreviate "weakly separated collection" as "ws-collection". Recall that we say that a ws-collection $C \subseteq 2^{[n]}$ is *largest* if its cardinality $|S|$ is maximum among all ws-collections in $2^{[n]}$; this maximum is equal to $\binom{n+1}{2} + 1$ [7]. An important example is the set $\mathcal{I}_n$ of intervals in $[n]$ (including the empty set). Also it was shown in [7] that a (lowering or raising) flip in a ws-collection produces again a ws-collection. Note that a six tuple $Xi, Xj, Xk, Xij, Xik, Xjk$ (where $i < j < k$ and $X \cap \{i, j, k\} = \emptyset$) is not a ws-collection; therefore, the cardinality of a ws-collection preserves under flips. Due to these facts, the set $\mathcal{C}_n$ of largest ws-collections includes $\mathcal{B}_n$ (the set of semi-normal bases for $\mathcal{TP}_n$). Theorem 8.1 says that the converse inclusion takes place as well. As a result, we will conclude with the following

Corollary 8.2 $\mathcal{C}_n = \mathcal{B}_n$.

In view of Theorem 2.2, to obtain Theorem 8.1, it suffices to show the following

Theorem 8.3 *Any $C \in \mathcal{C}_n$ is the spectrum B_T of some g-tiling T on Z_n.*

In the proof of this proposition we will use some additional (relatively simple) facts established in [7]. Let $C \in \mathcal{C}_n$. To construct the desired tiling for C, we consider the projection C' of C into $2^{[n-1]}$, i.e., the collection of subsets $X \subseteq [n-1]$ such that either $X \in C$ or $Xn \in C$ or both. Partition C' into three subcollections M, N, S, where

$$M := \{X : X \in C \not\ni Xn\}, \quad N := \{X : Xn \in C \not\ni X\}, \quad S := \{X : X, Xn \in C\}.$$

Also for $q = 0, \dots, n-1$, define

$$C'_q := \{X \in C' : |X| = q\}, \quad M_q := M \cap C'_q, \quad N_q := N \cap C'_q,$$

referring to C'_q as the q-th *layer* in C'. It is shown in [7] that

(8.1) for each $q = 0, \dots, n-1$, $S \cap C'_q$ contains exactly one element.

We call S the *separator* in C' and denote its elements by $S_0, \dots, S_{n-1}$, where $|S_q| = q$. Property (8.1) implies that $|C'| = |C| - |S| = \binom{n+1}{2} + 1 - n = \binom{n}{2} + 1$, and as is shown in [7],

(8.2) C' is a ws-collection, and therefore it is a *largest* ws-collection in $2^{[n-1]}$.

Two more observations in [7] are:

(8.3) (i) $S_0 \prec S_1 \prec \dots \prec S_{n-1}$;

(ii) for each $q = 0, \dots, n-1$ and any $Y \in M_q$ and $Y' \in N_q$, one hold $Y \prec S_q$ and $S_q \prec Y'$.

Now we start proving Theorem 8.3. This is led by two inductions. The first induction is by n. The result is trivial for $n \le 2$. Let $n > 2$ and assume by induction that there is a g-tiling T' on Z_{n-1} such that $B_{T'} = S'$. Our aim is to transform T' into a g-tiling on Z_n whose spectrum is S. This is performed by constructing a certain path P in the graph $G_{T'}$ that goes through the separator S and, further, by expanding P into an n-strip. The expansion operation that we apply is converse to the n-contraction operation developed in Subsection 4.3. This operation deserves to be discussed as an independent topic, and for this reason we interrupt our proof for a while.

8.1 n-contraction and n-expansion

First of all we examine, in more details, the n-contraction operation in an arbitrary g-tiling T on Z_n. Let P be the reverse path to the right boundary R_Q of the n-strip Q. It possesses a number of important features, as follows:

(8.4) For the path $P = (v_0, e_1, v_1, \dots, e_r, v_r)$ as above and the colors $i_1, \dots, i_r$ of its edges $e_1, \dots, e_r$, respectively, the following hold:

(i) P begins at the minimal point p_0 of Z_n and ends at p_{n-1};

(ii) none of $v_0, \dots, v_r$ is the top or bottom vertex of a black ij-tile with $i, j < n$;

(iii) P has no pair of consecutive backward edges;

(iv) if $e_q = (v_{q-1}, v_q)$ and $e_{q+1} = (v_{q+1}, v_q)$ (i.e., e_q is forward and e_{q+1} is backward in P), then $i_q > i_{q+1}$;

(v) if $e_q = (v_q, v_{q-1})$ and $e_{q+1} = (v_q, v_{q+1})$ (i.e., e_q is backward and e_{q+1} is forward in P), then $i_q < i_{q+1}$.

Indeed, the first and last edges of Q are $p_{n-1}p_n$ and $p_0p'_{n-1}$, yielding (i). Property (ii) follows from the facts that each vertex v_q has an incident n-edge (which belongs to Q) and that all edges incident to the top or bottom vertex of a black ij-tile have colors between i and j (see Corollary 3.1(iii)). The forward (backward) edges of P are the backward (resp. forward) edges of R_Q. Therefore, each forward (backward) edge e_q of P belongs to a white (resp. black) i_qn-tile, taking into account the maximality of color n; cf. (4.7). Then for any two consecutive edges e_q, e_{q+1}, at least one of them is forward, yielding (iii) (for otherwise the vertex v_q is terminal and has both entering and leaving edges, which is impossible). Next, let τ be the i_qn-tile (in Q) containing e_q, and τ' the $i_{q+1}n$-tile containing e_{q+1}. If e_q is forward and e_{q+1} is backward in P, then τ' is black, v_q is the left vertex of τ', and the i_q-edge e opposite to e_q in τ enters the top vertex of τ'. Since e lies in the cone of τ' at $t(\tau')$, we have $i_{q+1} < i_q < n$, as required is (iv). And if e_q is backward and e_{q+1} is forward, then τ is black and v_q is its bottom vertex. Since e_{q+1} lies in the cone of τ at $b(\tau)$, we have $i_q < i_{q+1} < n$, as required in (v).

Recall that the n-contraction operation applied to T shrinks the n-strip in such a way that R_Q merge with the left boundary L_Q of Q. From (8.4)(ii) it follows that in the resulting g-tiling T' on Z_{n-1}, the path P as above no longer contains terminal vertices at all.

Next we describe the converse operation that transforms a pair consisting of an arbitrary g-tiling T' on Z_{n-1} and a certain path in $G_{T'}$ into a g-tiling on Z_n. To explain the construction, we first consider an arbitrary simple path P in $G_{T'}$ which begins at p_0, ends at the maximal point p_{n-1} of Z_{n-1}, and may contain backward edges. Since the graph $G_{T'}$ is planar (and has a natural embedding in the disc $D_{T'}$), the path P subdivides $G_{T'}$ into two connected subgraphs $G' = G'_P$ and $G'' = G''_P$ such that: $G' \cup G'' = G_{T'}$, $G' \cap G'' = P$, G' contains $\ell bd(Z_{n-1})$, and G'' contains $rbd(Z_{n-1})$; we call G' (G'') the *left* (resp. *right*) *subgraph* w.r.t. P. Then each tile of T becomes a face of one of G', G'' (and the inner faces of G', G'' are only such), and for an edge e of P not in $bd(Z_{n-1})$, the two tiles sharing e occur in different subgraphs. So T is partitioned into two subsets, one being the set of faces of G', and the other of G''.

The *n-expansion operation* for (T', P) makes G', G'' disconnected by cutting $G_{T'}$ along P and then glue them by adding the corresponding n-strip. More precisely, we shift the vertices of G'' by the vector ξ_n, i.e., each vertex X in it changes to Xn; this induces the corresponding shift of edges and tiles in G''. The vertices of G' preserve. So each vertex X occurring in the path P produces two vertices, namely, X and Xn. As a result, for each edge $e = (X, Xi)$ of P, there appears its copy $\widetilde{e} = (Xn, Xin)$ in the shifted G''; we connect e and $\widetilde{e}$ by the corresponding (new) in-tile, namely, by $\tau(X; i, n)$. This added tile is colored white if e is a forward edge of P, and black if e is backward. The colors of all old tiles preserve.

We refer to the resulting set T of tiles, with the partition into white and black ones, as the *n-expansion of T' along P*. Since the right boundary of the shifted G'' becomes the part of $rbd(Z_n)$ from the point p'_{n-1} ($=\{n\}$) to p_n ($=[n]$), it follows that the union of the tiles in T is Z_n. Also it is easy to see that the shape D_T in $conv(2^{[n]})$ associated to T is again a disc (as required in axiom (T4)), and that T obeys axiom (T1). The path P generates the n-strip Q for T (consisting of the added $*n$-tiles and the edges of the form (X, Xn)), and we observe that $R_Q = P^{-1}$ and that L_Q is the shift of P^{-1} by ξ_n. Therefore, the n-contraction operation applied to T returns T'.

To ensure validity of the remaining axioms (T2) and (T3), we have to impose additional conditions on the path P. In fact, they are similar to those exposed in (8.4). Moreover, these conditions are necessary and sufficient.

Lemma 8.4 *Let $P = (p_0 = v_0, e_1, v_1, \dots, e_r, v_r = p_{n-1})$ be a simple path in $G_{T'}$. Then the following are equivalent:*

(i) the n-expansion T of T' along P is a (feasible) g-tiling on Z_n;

(ii) P contains no terminal vertices for T' and satisfies (8.4)(iii),(iv),(v).

Proof Let P be as in (ii). We have to verify axioms (T2),(T3) for T. Let $P' = (v'_0, e'_1, v'_1, \dots, e'_r, v'_r)$ and $P'' = (v''_0, e''_1, v''_1, \dots, e''_r, v''_r)$ be the copies of P in the graphs G' and G'' (taken apart), respectively. It suffices to check conditions in (T2),(T3) for objects involving elements of P', P'' (since for any vertex of G' not in P', the structure of its incident edges and tiles, as well as white/black coloring of tiles, is inherited from $G_{T'}$, and similarly for G'').

Consider a vertex v_q with $1 \le q < r$. Let E^L_q (E^R_q) denote the set of edges in $E_{T'}(v_q)$ lying on the left (resp. right) when we move along P and pass through e_q, v_q, e_{q+1}; we include e_q, e_{q+1} in both E^L_q and E^R_q. Let F^L_q (F^R_q) denote the set of tiles in $F_{T'}(v_q)$ of which both edges incident to v_q belong to E^L_q (resp. E^R_q). Note that each tile $\tau \in F_{T'}(v_q)$ must occur in either F^L_q or F^R_q, i.e., τ is not separated by e_q or e_{q+1} (taking into account that all vertices of P are non-terminal, and therefore the edges of P are white, and considering the behavior of edges and tiles at a non-terminal vertex exhibited in (3.5)). By the construction of G', G'', any two tiles of T' that share an edge not in P are faces of the same graph among G', G'', and if a tile $\tau \in T'$ has an edge contained in $\ell bd(Z_{n-1}) - P$ (resp. $rbd(Z_{n-1}) - P$), then τ is a face of G' (resp. G''). Using these observations, one can conclude that

(8.5) for $1 \le q < r$, E^L_q and F^L_q are entirely contained in G', while E^R_q and F^R_q are entirely contained in G''.

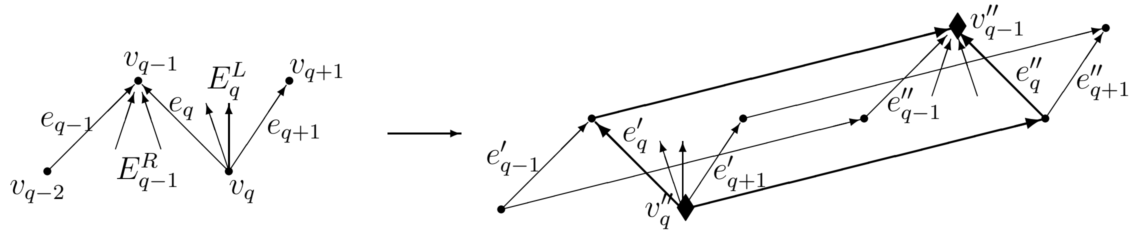
For $q = 1, \dots, r$, let τ^L_q (τ^R_q) denote the tile in T' (if exists) that contains the edge e_q and lies on the left (resp. right) when we traverse e_q from v_{q-1} to v_q. By (8.5), τ^L_q

is in G' and τ_q^R is in G''. Also each of τ_q^L, τ_q^R is white. Let τ_q be the $i_q n$-tile in T that was added to connect the edges e'_q and e''_q. Then

$$e'_q = (b(\tau_q), \ell(\tau_q)) \quad \text{and} \quad e''_q = (r(\tau_q), t(\tau_q)). \tag{8.6}$$

Suppose that e_q is forward in P. Then τ_q is white. Since e_q is directed from v_{q-1} to v_q and τ_q^L lies on the left from e_q when moving from v_{q-1} to v_q, e_q belongs to the right boundary of τ_q^L. This and (8.6) imply that τ_q and τ_q^L do not overlap. In its turn, τ_q^R contains e_q in its left boundary; this together with (8.6) implies that τ_q and the shifted τ_q^R (sharing the edge e''_q) do not overlap as well. Now suppose that e_q is backward in P. Then τ_q is black. Since e_q is directed from v_q to v_{q-1} and τ_q^L lies on the left from e_q when moving from v_{q-1} to v_q, e_q belongs to the left boundary of τ_q^L. This implies that τ_q and τ_q^L overlap. Similarly, τ_q and τ_q^R overlap. Thus, (T2) holds for τ_q, τ_q^L and for τ_q, τ_q^R, as required. Also the non-existence of pairs of consecutive reverse edges in P implies that no black tiles in T share an edge.

To verify (T3), consider a black tile τ_q. Then $1 < q < r$, the edges e_{q-1}, e_{q+1} are forward, and e_q is backward in P. Also $i_{q-1}, i_{q+1} > i_q$ (by (8.4)(iv),(v)). Observe that the set E_{q-1}^R consists of the edges in $E_{T'}(v_{q-1})$ that enter v_{q-1} and have color j such that $i_q \le j \le i_{q-1}$ (including e_{q-1}, e_q). All these edges are white (as is seen from (3.5)). The second copies of these edges (shifted by ξ_n) plus the n-edge (v'_{q-1}, v''_{q-1}) are exactly those edges of G_T that are incident to the top vertex v''_{q-1} of τ_q. It its turn, the set E_q^L consists of the edges in $E_{T'}(v_q)$ that leave v_q and have color j such that $i_q \le j \le i_{q+1}$, and these edges are white. Exactly these edges plus the n-edge (v'_q, v''_q) form the set of edges of G_T incident to $b(\tau_q)$. (See the picture.) This gives (T3) for T.



Thus, (ii) implies (i) in the lemma. The converse implication (i)→(ii) follows from (8.4) and the fact (mentioned earlier) that for the n-expansion T of T' along P, the n-contraction operation applied to T produces T', and under this operation the n-strip for T shrinks into P^{-1}. This completes the proof of the lemma. ∎

Let us call a path P as in (ii) of Lemma 8.4 *legal*. It is the concatenation of $P_1, \ldots, P_{n-1}$, where P_p is the maximal subpath of P whose edges connect levels $p-1$ and p, i.e., are of the form (X, Xi) with $|X| = p-1$. We refer to P_p as p-th *segment* of P and say that this segment is *ordinary* if it has only one edge, and *zigzag* otherwise. The beginning vertices of segments together with p_{n-1} are called *critical* in P (so there is exactly one critical vertex in each level); these vertices will play an important role in what follows. Note that the critical vertices of a legal path $P = (v_0, e_1, v_1, \ldots, e_r, v_r)$ are $v_0 = p_0$, $v_r = p_{n-1}$ and the intermediate vertices v_q such that e_q enters and e_{q+1} leaves v_q. We distinguish between two sorts of non-critical vertices v_q by saying that v_q is a $\vee$-*vertex* if both e_q, e_{q+1} leave v_q, and a $\wedge$-*vertex* if both e_q, e_{q+1} enter v_q. Observe that

(8.7) regarding a vertex of P as a subset X of $[n-1]$, the following hold: (a) if X is critical, then both X, Xn are in B_T; (b) if X is a $\wedge$-vertex, then $X \in B_T$ and $Xn \notin B_T$; and (c) if X is a $\vee$-vertex, then $X \notin B_T$ and $Xn \in B_T$ (where T is the n-expansion of T' along P).

Indeed, from the proof of Lemma 8.4 one can see that: if X is critical, then both vertices X, X_T of G_T have entering and leaving edges, so they are non-terminal; if X is a $\wedge$-vertex, then Xn is terminal while X is not; and if X is a $\vee$-vertex, then X is terminal while Xn is not (see the above picture).

It follows that

(8.8) $B_T = B' \cup B''$, where B' consists of all non-terminal vertices X in G'_P that are not $\vee$-vertices in P, and B'' consists of all Xn such that X is a non-terminal vertex in G''_P that is not a $\wedge$-vertex of P.

We also notice that

(8.9) for each $p = 1, \dots, n-1$, the graph H_p induced by the set of white edges connecting levels $p-1$ and p in $G_{T'}$ is a forest; therefore, any legal path for T' is determined by the set of its critical vertices.

This follows from the fact that for a path in H_p formed by three edges e, e', e'' (in this order), the edges e and e'' do not intersect; equivalently: the colors i, i', i'' of these edges (respectively) satisfy either $i, i'' < i'$ or $i, i'' > i'$. (To see the latter, suppose, for instance, that $i < i' < i''$ and that e and e' have the same beginning vertex, i.e., $e = (X, Xi)$ and $e' = (X, Xi')$ for some X. Then e'' enters Xi'. Using (3.5), one can see that there exists a white $\widetilde{ii'}$-tile τ such that $i \le \widetilde{i} < i'$, $b(\tau) = X$ and $r(\tau) = Xi'$, and there exists a white $\widehat{i'i}$-tile ρ such that $i' < \widehat{i} \le i''$, $t(\rho) = Xi'$ and $r(\rho) = X$. But such τ and ρ overlap, contrary to (T2).)

Finally, since the n-strip in a g-tiling on Z_n is determined uniquely, we can conclude with the following

Corollary 8.5 *The correspondence $(T', P) \mapsto T$, where T' is a g-tiling on Z_{n-1}, P is a legal path for T', and T is the n-expansion of T' along P, gives a bijection between the set of such pairs (T', P) and the set of g-tilings on Z_n.*

8.2 Drawing a legal path through the separator

We continue the proof of Theorem 8.3, using terminology, notation and results from above. Our aim is to show the following

Proposition 8.6 *Let M, N, S satisfy (8.3). Then there exists a legal path P for T' such that:*

(i) S is the set of critical vertices of P;

(ii) M is the set of non-terminal vertices of the left subgraph G'_P that are neither critical nor $\vee$-vertices of P;

(iii) N is the set of non-terminal vertices of the right subgraph G''_P that are neither critical nor $\wedge$-vertices of P.

Then, by reasonings in the previous subsection (cf. (8.8) and Lemma 8.4), the n-expansion of T' along P gives the desired g-tiling T on Z_n: its spectrum B_T is $M \cup \{Xn\colon X \in N\} \cup \{X, Xn\colon X \in S\}$. Therefore, $B_T = C$, yielding Theorem 8.3.

Like the proof of inclusion (5.1) in Section 5, we prove Proposition 8.6 by induction on the total size $\sum(|X|\colon X \in B_{T'})$ of the spectrum of T' (which is the second induction in the proof of Theorem 8.3). The idea is as follows. If T has a feasible W-configurations (defined in Section 4), we choose one of them and make the corresponding lowering flip, obtaining a g-tiling $\widetilde{T}$ on Z_{n-1} whose spectrum has a smaller total size. We also transform M, N, S into appropriate sets $\widetilde{M}, \widetilde{N}, \widetilde{S}$ for $\widetilde{T}$ maintaining property (8.3). By induction there exists a legal path $\widetilde{P}$ as required in the proposition for $\widetilde{T}, \widetilde{M}, \widetilde{N}, \widetilde{S}$. Then we show that $\widetilde{P}$ can be transformed into the desired legal path for T', M, N, S.

When T' has no feasible W-configuration, T' is the standard tiling (i.e., $B_{T'} = \mathcal{I}_{n-1}$), in which case the result is proved directly (giving a base for the induction). Moreover, this can be easily done for an arbitrary pure tiling T', by arguing as follows. We know that the spectrum $B_{T'}$ (formed by all vertices of $G_{T'}$ since $T'^b = \emptyset$) is a normal basis, so $B_{T'}$ is a strongly separated collection, by a result in [7]. For $p = 1, \dots, n-1$, consider the graph (forest) H_p as in (8.9). Observe that any tile $\tau \in T'$ containing an edge in H_p either has the top vertex in level p or the bottom vertex in level $p-1$ (and therefore, τ contains two edges in H_p). Also such tiles τ are pairwise non-overlapping and cover the region Ω_p in Z_{n-1} between the horizontal lines $(\mathbb{R}, p-1)$ and $(\mathbb{R}, p)$. These facts imply that H_p is connected, i.e., it is a tree; moreover, H_p is embedded in Ω_p without intersection of non-adjacent edges. So any two vertices u, v of H_p are connected by a unique path $P(u, v)$ in H_p. The concatenation of the paths $P(S_0, S_1), P(S_1, S_2), \dots, P(S_{n-2}, S_{n-1})$ is the (unique) legal path going through the separator S. Finally, if $X^1, \dots, X^d$ are the vertices in the same level p ($1 \le p < n-1$) ordered from left to right, then $X^1 \prec \cdots \prec X^d$. (This follows from the transitivity of $\prec$ for strongly separated collections and from the fact that consecutive vertices X^q, X^{q+1} are connected in H_p by a 2-edge path (X^q, e, Y, e', X^{q+1}) such that both e, e' leave Y and their colors i, i' (respectively) satisfy $i < i'$, implying $X^q \prec X^{q+1}$, in view of $X^q = Yi$ and $X^{q+1} = Yi'$.) Now (i)–(iii) in Proposition 8.6 easily follow from (8.3).

In a general case, we will use the following observation from [7]:

(8.10) if sets $A, A', A'' \subseteq [n']$ are weakly separated, and if $|A| \le |A'| \le |A''|$, $A \prec A'$ and $A' \prec A''$, then $A \prec A''$.

Another property important to us is the following.

Lemma 8.7 *Let $Y \subseteq [n']$ be weakly separated from each of $Xk, Xij, Xik, Xjk \subseteq [n']$ and different from Xjk, where $i < j < k$ and $X \cap \{i, j, k\} = \emptyset$. Let $|Y| \ge |X| + 2$ and $Xik \prec Y$. Then $Xjk \prec Y$.*

(Recall that when writing $A \prec B$ or $A \rhd A$, we assume that $A \neq B$.)

Proof Note that the obvious relations $Xij \prec Xik$ and $Xk \prec Xik$ together with $Xik \prec Y$ imply $Xij, Xk \prec Y$, by (8.10). In order to obtain $Xjk \prec Y$, we show that none of the relations $Y \rhd Xjk$, $Y \prec Xjk$, and $Xjk \rhd Y$ is possible (taking into account that Xjk and Y are weakly separated and $Xjk \neq Y$). Let $Y' := Y - Xjk$ and $Z := Xjk - Y$.

(i) Suppose $Y \rhd Xjk$. Let $\{Z_1, Z_2\}$ be the partition of Z such that $Z_1 \prec Y' \prec Z_2$. Since $|Y'| \geq 2$, Y' contains an element $\alpha \neq i$. Take $\beta \in Z_2$; then $\alpha < \beta$. We have: $\alpha \in Y - Xij$, $\alpha \in Y - Xik$, and β belongs to at least one of $Xij - Y$ and $Xik - Y$. But $\beta \in Xij - Y$ contradicts $Xij \prec Y$, and $\beta \in Xik - Y$ contradicts $Xik \prec Y$.

(ii) Suppose $Y \prec Xjk$. If Y' contains an element $\alpha \neq i$, we come to a contradiction by taking an element $\beta \in Z$ (satisfying $\alpha < \beta$, in view of $Y' \prec Z$) and arguing as in (i). So $Y' = \{i\}$, whence $|Y| \geq |X| + 2$ holds with equality. Then Z consists of a single element β, and we have $i < \beta$ and $Y = Xijk - \{\beta\}$. Therefore, $Y - Xik = \{j\}$ and $Xik - Y = \{\beta\}$, implying $\beta < j$, in view of $Xij \prec Y$. Now considering the pair Y, Xk, we have $i \in Y - Xk$, $\beta \in Xk - Y$ and $i < \beta$. This is impossible since $Xk \prec Y$.

(iii) Suppose $Xjk \rhd Y$. Partition Y' into subsets Y_1, Y_2 such that $Y_1 \prec Z \prec Y_2$. We have $|Z| \geq 2$ and $i \notin Z$. Therefore, Z contains an element $\beta \neq i, j$. Then $\beta \in Xik - Y$ and $Y_1 \prec \{\beta\}$, whence the relation $Xik \prec Y$ is possible only if $Y_1 = \{i\}$. Now for the pair Y, Xk, we have $i \in Y - Xk$, $\beta \in Xk - Y$ and $i < \beta$, contrary to $Xk \prec Y$. ∎

Now we finish the proof of Proposition 8.6. We choose a feasible W-configuration $K = CW(X; i, j, k)$ such that, besides the five sets Xi, Xk, Xij, Xik, Xjk,

(8.11) the spectrum $B_{T'}$ contains the set X; equivalently: the edges (Xi, Xik) and (Xk, Xik) belong to the same (white) tile, namely, $\tau(X; i, k)$

(see Case 1(a) in the proof of Proposition 4.1). Such a property holds if K has the minimum possible height h $(=|X| + 1)$. Indeed, by Proposition 5.1, there exists no black tile of height $< h$. If the edges (Xi, Xik) and (Xk, Xik) do not belong to the same tile (which is of height $h - 1$), then the vertex Xik has an entering q-edge e with $i < q < k$ (i.e., lying between (Xi, Xik) and (Xk, Xik)). The beginning vertex Y of e has only one leaving edge (namely, e) and at least two entering edges. Now by repeatedly applying a simple descending procedure (similar to searching a hexagon in [4]), we are able to find a vertex Z of height $|Z| < h - 1$ that has exactly one leaving edge and exactly two entering edges. Then the three (white) tiles sharing Z form a hexagon containing a required W-configuration $CW(X'; i', j', k')$ (where the lowest tile is $\tau(X'; i', k')$).

We make the lowering flip for K. The resulting g-tiling $\widetilde{T}$ on Z_{n-1} has the spectrum $B_{\widetilde{T}} = (B_{T'} - \{Xik\}) \cup \{Xj\}$. Our aim is to assign sets $\widetilde{M}, \widetilde{N}, \widetilde{S}$ for $\widetilde{T}$ in a due way and to draw a legal path $\widetilde{P}$ through $\widetilde{S}$ and, finally, to transform $\widetilde{P}$ into the desired legal path P through the separator S in $G_{T'}$. We consider two possible cases, letting $h := |Xik|$.

Case 1: Xik belongs to S, i.e., $Xik = S_h$. The set Xik vanishes in $B_{\widetilde{T}}$, and we assign $\widetilde{S}$ by replacing S_h by $Xjk =: \widetilde{S}_h$ and keeping the members in the other levels: $\widetilde{S}_q := S_q$ for $q \neq h$. Accordingly, we put $\widetilde{M}_q := M_q$ and $\widetilde{N}_q := N_q$ for the levels where the spectrum preserves, i.e., for $q \neq h, h-1$, and put $\widetilde{M}_h := M_h$ and $\widetilde{N}_h := N_h - \{Xjk\}$. Under these assignments and by (8.3), we obtain:

(i) $\widetilde{S}_0 \prec \cdots \prec \widetilde{S}_{n-1}$, and

(ii) for $q = 0, \ldots, h-2, h, \ldots, n-1$ and any $Y \in \widetilde{M}_p$ and $Y' \in \widetilde{N}_p$, one hold $Y \prec \widetilde{S}_q$ and $\widetilde{S}_q \prec Y'$.

Here: $\widetilde{S}_{h-1} \prec \widetilde{S}_h$ $(=Xjk)$ follows from $S_{h-1} \prec S_h = Xik \prec Xjk$ (by (8.10)); $Y \prec \widetilde{S}_h$ for $Y \in \widetilde{M}_h$ follows from $Y \prec Xik \prec Xjk$; and the relations $\widetilde{S}_h \prec \widetilde{S}_{h+1}$ and $\widetilde{S}_h \prec Y'$ for $Y' \in \widetilde{N}_h$ follow from Lemma 8.7.

As to level $h-1$, we include in $\widetilde{M}_{h-1}$ ($\widetilde{N}_{h-1}$) all members of M_{h-1} (resp. N_{h-1}). A less trivial task is to assign a due inclusion for the new element Xj appeared in level $h-1$. A priori, the following four cases are possible for the separating element S_{h-1}: (a) $S_{h-1} = Xk$; (b) $S_{h-1} \preceq Xi$; (c) $S_{h-1} \succ Xk$; and (d) $Xi \prec S_{h-1} \prec Xk$. In reality, cases (c) and (d) cannot occur.

Indeed, suppose $S_{h-1} \succ Xk$. A routine examination (in spirit of one in the proof of Lemma 8.7) shows that this relation together with $S_{h-1} \prec S_h = Xik$ and $|S_{h-1}| = |Xik|$ is possible only if $Xk - S_{h-1} = \{\alpha\}$ and $S_{h-1} - Xk = \{i\}$ for some $\alpha < i$, i.e., S_{h-1} is viewed as $Xik - \{\alpha\}$. This means that $G_{T'}$ contains the α-edge e going from S_{h-1} to Xik (since these are non-terminal vertices and by (4.2)). In view of $\alpha < i$, e lies strictly inside the cone of τ at $\ell(\tau) = Xik$, where τ is the ij-tile in T' containing the edges (Xk, Xik) and $(Xik, Xijk)$. But this contradicts the fact that the vertex Xik is ordinary (cf. (4.4) and (3.6)). Next suppose $Xi \prec S_{h-1} \prec Xk$. One can check that this is possible only if either (i) $S_{h-1} = X\beta$ or (ii) $S_{h-1} = Xik - \{\beta\}$ for some $i < \beta < k$. Then $G_{T'}$ contains the β-edge $(X, X\beta)$ (lying in the cone of the bottom vertex of $\tau = \tau(X; i, k)$) in case (i), and the β-edge $(Xik - \{\beta\}, Xik)$ (lying in the cone of τ at $t(\tau) = Xik$) in case (ii). Clearly none of these situations is possible. (This is where we essentially use the choice of a configuration so as to obey (8.11).)

In case (a), we have $Xj \prec S_{h-1}$ and attribute Xj to $\widetilde{M}_{h-1}$. In case (b), we have $S_{h-1} \prec Xj$ (in view of $Xi \prec Xj$) and attribute Xj to $\widetilde{N}_{h-1}$.

Thus, (8.3) holds for the constructed $\widetilde{M}, \widetilde{N}, \widetilde{S}$, and by induction there exists a legal path $\widetilde{P}$ in $G_{\widetilde{T}}$ satisfying (i)–(iii) in Proposition 8.6 w.r.t. $\widetilde{M}, \widetilde{N}, \widetilde{S}$. It remains to explain how to transform $\widetilde{P}$ into the desired P. We will use the following observation:

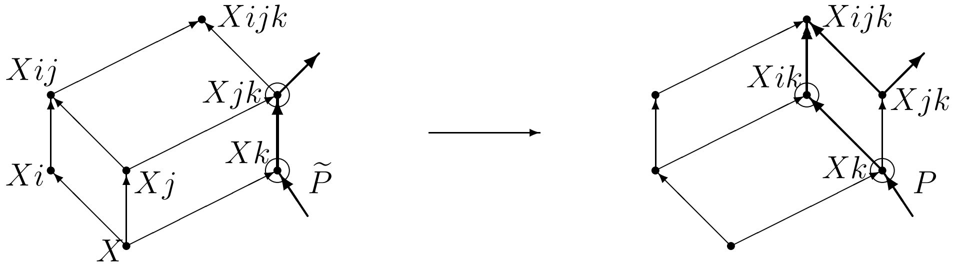
(8.12) the vertex $Xijk$ of $G_{T'}$ is non-terminal.

Indeed, suppose $Xijk$ is terminal; then it is the top vertex of some black pq-tile $\tau = \tau(X'; p, q)$. Since the edge $(Xik, Xijk)$ lies strictly inside the cone of τ at $Xijk$, we have $p < j < q$, and the vertex Xik is expressed as $X'pq - \{j\}$. Take a white edge $e = (X', X'\alpha)$ leaving the bottom vertex of τ; then $p < \alpha < q$ (such an edge exists since $|E_{T'}(b(\tau))| \geq 3$). The vertex $X'\alpha$ is non-terminal and $|X'\alpha| = |Xik|$. We have $j \in X'$, $Xik - X'\alpha = \{p, q\}$ and $X'\alpha - Xik = \{j, \alpha\}$ (or $=\{j\}$ if $j = \alpha$). Since $p < j, \alpha < q$, we obtain $X'\alpha \rhd Xik$. This is impossible since $Xik = S_h$ is comparable by $\prec$ with all non-terminal vertices in level h.

Now we consider cases (a),(b) as above.

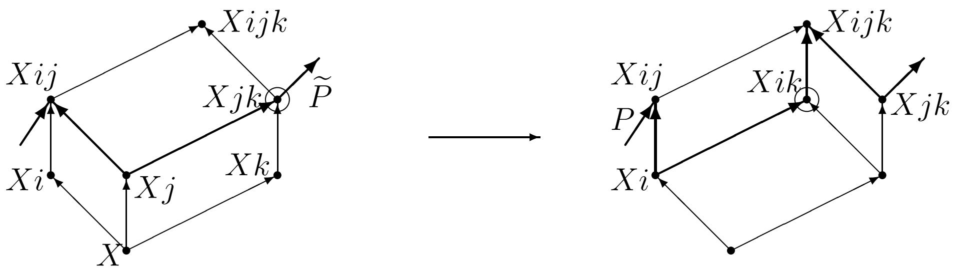
Subcase 1a: $S_{h-1} = Xk$. Then the h-th segment $\widetilde{P}_h$ of $\widetilde{P}$ is ordinary; it goes from $\widetilde{S}_{h-1} = S_{h-1}$ to $\widetilde{S}_h = Xjk$ through the edge (Xk, Xjk). The beginning part of $\widetilde{P}$, from

p_0 to $\widetilde{S}_{h-1}$, does not pass the vertex Xj, since in the subgraph $\widetilde{H}_{h-1}$ (defined as in (8.9) for $\widetilde{T}$) this vertex has only one incident edge, namely, (X, Xj). We replace in $\widetilde{P}$ the segment $\widetilde{P}_h$ by the 3-edge path $(Xk, Xik, Xijk, Xjk)$ (denoting a path by the sequence of its vertices). Let P be the resulting path; if the edge $(Xjk, Xijk)$ occurs in P twice (which happens when the segment $\widetilde{P}_{h+1}$ passes $Xijk$), we remove both occurrences, making P simple. Since the vertex $Xijk$ is non-terminal (by (8.12)), all vertices of P are such. Moreover, comparing P and $\widetilde{P}$, one can see that P is a legal path in $G_{T'}$ whose set of critical vertices is just S. The segment structure of P differs from that of $\widetilde{P}$ by only two segments, namely, by h-th and $(h+1)$-th segments whose last and first vertices, respectively, has changed from Xjk to Xik. See the picture below where (fragments of) $\widetilde{P}$ and P are drawn in bold and critical vertices are indicated with circles. Comparing $\widetilde{P}, \widetilde{M}, \widetilde{N}, \widetilde{S}$ with P, M, N, S and using some observations above, it is routine to check that the latter quadruple satisfies (i)–(iii) in Proposition 8.6.



Subcase 1b: $S_{h-1} \preceq Xi$. We assert that the segment $\widetilde{P}_h$ (from $\widetilde{S}_{h-1} = S_{h-1}$ to $\widetilde{S}_h = Xjk$) contains the edge (Xj, Xjk), and therefore, it also contains (Xj, Xij). For suppose the contrary. Then the concatenation Q of the path (Xi, Xij, Xj, Xjk) and the reverse of $\widetilde{P}_h$ is a simple (zigzag) path from Xi to S_{h-1} in $\widetilde{H}_h$. By explanations in the previous subsection, any connected component of $\widetilde{H}_h$ has a planar layout, i.e., non-adjacent edges in it (regarded as line-segments) do not intersect. This implies that for the sequence $e_1, \dots, e_d$ of edges of Q (where $e_1 = (Xi, Xij)$, $e_2 = (Xj, Xij)$, and d is even), the sequence $i_1, \dots, i_d$ of its colors has alternating signs by comparison, namely, $i_1 = j > i_2 = i < i_3 > i_4 < \dots > i_{d-1} < i_d$. Now since $S_{h-1} - Xi$ (resp. $Xi - S_{h-1}$) consists of the colors i_q with q odd (resp. even), one can conclude that $S_{h-1} \preceq Xi$ is impossible.

Thus, $\widetilde{P}$ contains the subpath $R = (Xij, Xj, Xjk)$ (or $R' = (Xi, Xij, Xj, Xjk)$). We replace R by the 4-edge path $(Xij, Xi, Xik, Xijk, Xjk)$ (resp. replace R' by the 3-edge path $(Xi, Xik, Xijk, Xjk)$), obtaining the desired legal path P in $G_{T'}$ as required in the proposition. See the picture below. (Like Subcase 1a, if the edge $(Xjk, Xijk)$ occurs in P twice, we remove both occurrences from it.)



Case 2: $Xik \notin S$. This case is essentially simpler. We put $\widetilde{S} := S$, $\widetilde{M}_q := M_q$ and $\widetilde{N}_q := N_q$ for $q \neq h-1, h$. The sets $\widetilde{M}_h, \widetilde{N}_h$ are obtained from M_h, N_h by removing Xik from the corresponding set. To define $\widetilde{M}_{h-1}, \widetilde{N}_{h-1}$, we use the fact (shown in Case 1) that $Xi \prec S_{h-1} \prec Xk$ is impossible. When $S_{h-1} \preceq Xi$, the new element Xj in level $h-1$ is added to $\widetilde{N}_q$ (since $S_{h-1} \prec Xj$, in view of $Xi \prec Xj$); so $\widetilde{M}_{h-1} := M_{h-1}$ and $\widetilde{N}_{h-1} := N_{h-1} \cup \{Xj\}$. When $S_{h-1} \succeq Xk$, Xj is added to $\widetilde{M}_{h-1}$.

The corresponding legal path $\widetilde{P}$ in $G_{\widetilde{T}}$ (existing by induction) produces a legal path P in $G_{T'}$ in a natural way. If $\widetilde{P}$ does not pass the vertex Xj, then $P := \widetilde{P}$. And if Xj is in $\widetilde{P}$, then $\widetilde{P}$ contains the subpath $Q = (Xij, Xj, Xjk)$ (for Xj would be critical if $\widetilde{P}$ passed through X, Xj, Xij or X, Xj, Xjk). We replace Q by the path (Xij, Xi, Xik, Xk, Xjk). The resulting path P is as required.

This completes the proof of Proposition 8.6, yielding Theorem 8.1. ∎

9 Generalizations

In this concluding section we outline two generalizations, omitting proofs (which are close to the corresponding proofs in this paper or in [2]). They will be discussed in full, with details and related topics, in a separate paper.

A. The obtained relationships between semi-normal bases, proper wirings and generalized tilings are extendable to the case of an integer n-box $\mathbf{B}^{n,a} = \{x \in \mathbb{Z}^{[n]} \colon 0 \leq x \leq a\}$, where $a \in \mathbb{Z}^n_+$. Recall that a function f on $\mathbf{B}^{n,a}$ is a TP-function if it satisfies

$$\begin{aligned} & f(x+1_i+1_k) + f(x+1_j) \\ & \quad = \max\{f(x+1_i+1_j) + f(x+1_k),\ f(x+1_i) + f(x+1_j+1_k)\} \end{aligned} \tag{9.1}$$

for any x and $1 \leq i < j < k \leq n$, provided that all six vectors occurring as arguments in this relation belong to $\mathbf{B}^{n,a}$, where 1_q denotes q-th unit base vector. In this case the standard basis of the TP-functions consists of the vectors x such that $x_i, x_j > 0$ for $i < j$ implies $x_q = a_q$ for $q = i+1, \ldots, j-1$ (see [2]); such vectors are called *fuzzy-intervals*. Normal and semi-normal bases are corresponding collections of integer vectors in $\mathbf{B}^{n,a}$, defined by a direct analogy with the Boolean case.

The semi-normal bases in the box case admit representations via natural generalizations of proper wiring and g-tiling diagrams concerning the Boolean case, which are defined as follows.

The zonogon for a given a is the set $Z_{n,a} := \{\lambda_1\xi_1 + \ldots + \lambda_n\xi_n \colon \lambda_i \in \mathbb{R},\ 0 \leq \lambda_i \leq a_i,\ i = 1, \ldots, n\}$, where the vectors ξ_i are chosen as above. For each $i \in [n]$ and $q = 0, 1, \ldots, a_i$, define the point $p_{i,q} := a_1\xi_1 + \ldots + a_{i-1}\xi_{i-1} + q\xi_i$ (on the left boundary of $Z_{n,a}$) and the point $p'_{i,q} := a_n\xi_n + \ldots a_{i+1}\xi_{i+1} + q\xi_i$ (on the right boundary). These points are regarded as the vertices on the boundary of $Z_{n,a}$, and the edges in it are the directed line-segments $p_{i,q-1}p_{i,q}$ and $p'_{i,q}p'_{i,q-1}$ When $q \geq 1$, we define $s_{i,q}$ ($s'_{i,q}$) to be the median point on the edge $p_{i,q-1}p_{i,q}$ (resp. $p'_{i,q}p'_{i,q-1}$).

A generalized tiling T on $Z = Z_{n,a}$ is defined by essentially the same axioms (T1)–(T4) from Subsection 2.2. A wiring W on Z consists of wires $w_{i,q}$ going from $s_{i,q}$ to

$s'_{i,q}$, $i=1,\dots,n$, $q=1,\dots,a_i$. It is also defined by the same axioms (W1)–(W3) from Subsection 2.1.

Note that for any i and $1\le q<q'\le a_i$, the point $s'_{i,q}$ occurs earlier than $s'_{i,q'}$ in the right boundary of Z (beginning at p_0), which corresponds to the order of $s_{i,q}, s_{i,q'}$ in the left boundary of Z. This and axiom (W2) imply that the wires $w:=w_{i,q}$ and $w':=w_{i,q'}$ are always disjoint. Indeed, suppose that w and w' meet and take the first point x of w' that belongs to w. Let Ω_0,Ω_1 be the connected components of $Z-(P\cup P')$, where P is the part of w from x to $s'_{i,q}$, P' is the part of w' from $s_{i,q'}$ to x, and Ω_0 contains p_0. Then the end point $s'_{i,q'}$ of w' is in Ω_1. Furthermore, when the wire w' crosses w at x, it enters the region Ω_0. Therefore, the part of w' from x to $s'_{i,q'}$ must intersect $P\cup P'$ at some point $y\ne x$. But $y\in P$ is impossible by (W2) and $y\in P'$ is impossible because w' is not self-intersecting.

Like the Boolean case, for a g-tiling T, the set (spectrum) B_T consists of non-terminal vertices (= n-vectors) for T. For a wiring W and an (inner) face F of its associated planar graph, let $x(F)$ denote the n-vector whose i-th entry is the number of wires $w_{i,q}$ such that F lies on the left from $w_{i,q}$. Then B_W is defined to be the collection of vectors $x(F)$ over all non-cyclic faces F.

Theorems 2.1 and 2.2 remain valid for these extended settings (where Z_n is replaced by $Z_{n,a}$), and proving methods are essentially the same as those in Sections 4–7, with minor refinements on some steps. (In particular, instead of a unique dual i-path (i-strip) for each i, we now deal with a_i dual i-paths, each connecting a boundary edge $p_{i,q-1}p_{i,q}$ to $p'_{i,q}p'_{i,q-1}$, which does not cause additional difficulty in the proof.)

B. The second generalization involves an arbitrary permutation ω on $[n]$. (In fact, so far we have dealt with the longest permutation ω_0, where $\omega_0(i)=n+1-i$). For $i,j\in[n]$, we write $i\prec_\omega j$ if $i<j$ and $\omega(i)<\omega(j)$. This relation is transitive and gives a partial order on $[n]$. Let $\mathcal{X}_\omega\subseteq 2^{[n]}$ be the set (lattice) of ideals X of $([n],\prec_\omega)$, i.e., $i\prec_\omega j$ and $j\in X$ implies $i\in X$. In particular, $\mathcal{X}_\omega$ is closed under taking a union or intersection of its members. Below we specify settings and outline how results concerning ω_0 can be extended to ω.

(i) By a TP-function for ω, or an ω-*TP-function*, we mean a function f defined on the set $\mathcal{X}_\omega$ (rather than $2^{[n]}$) and satisfying (1.1) when all six sets in it belong to $\mathcal{X}_\omega$. Note that $Xi, Xk, Xij, Xjk\in\mathcal{X}_\omega$ implies that each of $X, Xj, Xik, Xijk$ is in $\mathcal{X}_\omega$ as well (since each of the latter is obtained as the intersection or union of a pair among the former). The notion of TP-basis is extended to the set $\mathcal{TP}_\omega$ of ω-TP-functions in a natural way. It turns out that the role of standard basis is now played by the set $\mathcal{I}_\omega$ of ω-*dense* sets $X\in\mathcal{X}_\omega$, which means that there are no triples $i<j<k$ such that $i,k\in X\not\ni j$ and each of the sets $X-\{i\}$, $X-\{k\}$ and $(X-\{i,k\})\cup\{j\}$ belongs to $\mathcal{X}_\omega$. In particular, $\mathcal{I}_\omega$ contains the sets $[i]$, $\{i'\colon i'\preceq_\omega i\}$ and $\{i'\colon \omega(i')\le\omega(i)\}$ for each $i\in[n]$; when $\omega=\omega_0$, $\mathcal{I}_\omega$ turns into the set $\mathcal{I}_n$ of intervals in $[n]$. (It is rather easy to prove that any ω-TP-function is determined by its values on $\mathcal{I}_\omega$; this is done by exactly the same method as applied in [2] to show a similar fact for $\mathcal{TP}_n$ and $\mathcal{I}_n$. The fact that the restriction map $\mathcal{TP}_\omega\to\mathbb{R}^{\mathcal{I}_\omega}$ is surjective (which is more intricate) can be shown by extending a flow approach developed in [2] for the cases of TP-functions on Boolean cubes and integer boxes.) Normal and semi-normal bases for the ω-TP-functions are

defined via flips from the standard basis $\mathcal{I}_\omega$, by analogy with those for ω_0.

(ii) Instead of the zonogon Z_n, we now should consider the region in the plane bounded by two paths: the left boundary of Z_n and the path P_ω formed by the points $p'_{i,\omega} := \xi_{\omega(i+1)}+\ldots+\xi_{\omega(n)}$ $(i=0,\ldots,n)$ connected by the (directed) segments $p'_{i,\omega}p'_{i-1,\omega}$. We denote this region as Z_ω and call it the ω-*deformation* of the zonogon Z_n, or, liberally, the ω-*zonogon*. A wiring for ω is a collection W of wires $w_1,\ldots,w_n$ in Z_ω satisfying axioms (W1)–(W3) and such that each w_i begins at the point s_i (as before) and ends at the median point $s'_{i,\omega}$ of $p'_{i,\omega}p'_{i-1,\omega}$ (a wire w_i degenerates into a point if the boundary edges $p_{i-1}p_i$ and $p'_{i,\omega}p'_{i-1,\omega}$ coincide). Note that if $i \prec_\omega j$ then $s'_{i,\omega}$ occurs earlier than $s'_{j,\omega}$ in the right boundary P_ω of Z_ω, and therefore, the wires w_i and w_j does not meet (as explained in part A above). This implies that all sets in the full spectrum of W belong to $\mathcal{X}_\omega$.

In its turn, a generalized tiling T for ω is defined in the same way as for ω_0, with the only differences that now the union of tiles in T is Z_ω and that the corresponding shape D_T is required to be simply connected (then D_T is homeomorphic to Z_ω). (Depending on ω, points $p_{i,\omega}$ and $p'_{i,\omega}$ for some i may coincide, so D_T need not be a disc in general.) The constructions and arguments in Sections 6,7, based on planar duality, can be transferred without essential changes to the ω case, giving a natural one-to-one correspondence between the g-tilings and proper wirings for ω. (In particular, the fact that i-th wire w_i in a proper wiring W for ω turns into the i-strip Q_i in the corresponding g-tiling T (which begins with the i-edge $p_{i-1}p_i$ in $\ell bd(Z_\omega)$ and ends with the i-edge $p'_{i,\omega}p'_{i-1,\omega}$ in $rbd(Z_\omega)$) implies that all vertices of G_T represent sets in $\mathcal{X}_\omega$.) The arguments in Sections 4,5 continue to work in the ω case as well. As a result, we obtain direct generalizations of Theorems 2.1 and 2.2 to an arbitrary permutation ω.

Remark 7 In fact, the generalization in part A is a special case of the one in part B. More precisely, given $a \in \mathbb{Z}^n_+$, define $\overline{a}_i := a_1+\ldots+a_i$, $i=0,\ldots,n$ (letting $\overline{a}_0 := 0$). Let us form a permutation ω' on $[\overline{a}_n]$ as follows: for $i=1,\ldots,n$ and $q=1,\ldots,a_i$,

$$\omega'(\overline{a}_{i-1}+q) := \overline{a}_n - \overline{a}_i + q,$$

i.e., ω' permutes the blocks $B_1,\ldots,B_n$, where $B_i := \{\overline{a}_{i-1}+1,\ldots,\overline{a}_i\}$, according to the permutation ω_0 on $[n]$, and preserves the order of elements within each block. Then there is a one-to-one correspondence between the vectors $x \in \mathbf{B}^{n,a}$ and the ideals X of $([\overline{a}_n], \prec_{\omega'})$, namely: $X \cap B_i$ consists of the first x_i elements of B_i, for each i. Under this correspondence, (9.1) is equivalent to (1.1). Although the shape of the zonogon $Z_{n,a}$ looks somewhat different compared with $Z_{\omega'}$ (since the generating vectors $\xi_\bullet$ for different elements in a block are non-colinear), it is easy to see that the wirings for the former and the latter are, in fact, the same. (This implies an equivalence of the g-tilings for these two cases, which is not seen immediately.) So the integer box case is reduced, in all aspects we deal with, to the permutation one.

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