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Gröbner Basis, Mordell-Weil Lattices and Deformation of Singularities

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Abstract

We call a section of an elliptic surface to be everywhere integral if it is disjoint from the zero-section. The set of everywhere integral sections of an elliptic surface is always a finite set. We pose the basic problem about this set when the base curve is $\mathbf{P}^1$. In the case of a rational elliptic surface, we obtain a complete answer, described in terms of the root lattice E_8 and its roots. Our results are related to some problems in Gröbner basis, Mordell-Weil lattices and deformation of singularities, which have served as the motivation and idea of proof as well.

1 Introduction

Let S be a smooth projective surface having an elliptic fibration $f : S \to C$ with the zero-section O over a curve C, and let E be the generic fibre of f which is an elliptic curve over the function field $K = k(C)$ (k is a base field of any characteristic). Assume that S has at least one singular fibre. Then the group $M = E(K)$ of K-rational points is finitely generated (Mordell-Weil theorem). It can be identified with the group of sections of f. For each P in $E(K)$, we denote by (P) the image curve of the corresponding section $C \to S$; the curve (P) may be also called a "section" without confusion.

An element P of M is called *everywhere integral* ([15]) if (P) is disjoint from the zero-section (O). Let $\mathcal{P}$ be the set of all everywhere integral sections $P \in M$:

$$\mathcal{P} = \{P \in M | (P) \cap (O) = \emptyset\} \tag{1.1}$$

Theorem 1.1 *The set $\mathcal{P}$ is a finite subset of the Mordell-Weil group M.*

Proof By the height formula [10, Theorem 8.6], we have for any $P \in M$

$$\langle P, P \rangle = 2\chi + 2(PO) - \sum_{w \in R_f} contr_w(P), \tag{1.2}$$

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where the notation is as follows: χ is the arithmetic genus of S (a positive integer), (PO) is the intersection number of two irreducible curves (P) and (O) on S, and $contr_w(P)$ is the local contribution at w (a non-negative rational number); the summation is over the set R_f of the points $w \in C$ with $f^{-1}(w)$ reducible. If P belongs to the set $\mathcal{P}$, then it follows that $\langle P, P\rangle \leq 2\chi$. Thus $\mathcal{P}$ forms a set of points with bounded height in M, and hence it is a finite set. (Recall that, by the theory of Mordell-Weil lattices ([10]), the height pairing is positive-definite on M modulo torsion.) *q.e.d.*

Now consider the case: $C = \mathbf{P}^1$, $K = k(t)$. For the sake of simplicity, we assume in the following that the base field k is algebraically closed. Suppose that E/K is given by a generalized Weierstrass equation:

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{1.3}$$

and O is the point at infinity $(x : y : 1) = (0 : 1 : 0)$. Without loss of generality, we assume that the coefficients a_ν are polynomials in t and "minimal" in the sense that if, for some $l \in k[t]$, a_ν is divisible by l^ν for all ν, then l must be a constant (i.e. $l \in k$). Then we have

$$\deg a_\nu \leq \nu\chi \quad (\nu = 1, 2, 3, 4, 6) \tag{1.4}$$

where χ is the arithmetic genus of S, which is known to be characterized as the smallest integer satisfying the the above condition.

Lemma 1.2 *Let $P \in M = E(K)$. Then $P = (x, y)$ belongs to the set $\mathcal{P}$ if and only if x, y are polynomials in t such that*

$$\deg(x) \leq 2\chi, \quad \deg(y) \leq 3\chi. \tag{1.5}$$

Proof See the proof of [15, Theorem 2]. *q.e.d.*

Let

$$P = (x, y) : \begin{cases} x = x_0 + x_1t + \cdots + x_{2\chi}t^{2\chi} \\ y = y_0 + y_1t + \cdots + y_{3\chi}t^{3\chi}, \end{cases} \tag{1.6}$$

and let

$$z = z(P) = (x_0, x_1, \cdots, x_{2\chi}, y_0, y_1, \cdots, y_{3\chi}). \tag{1.7}$$

Then, substituting (1.6) into (1.3), we obtain a polynomial identity in t:

$$y^2 + \cdots - (x^3 + \cdots + a_6) = \phi_0 + \phi_1t + \cdots + \phi_{6\chi}t^{6\chi}. \tag{1.8}$$

Let us denote by I the ideal generated by the coefficients ϕ_d of t^d in the polynomial ring R:

$$I := (\phi_0, \ldots, \phi_{6\chi}) \subset R := k[x_0, x_1, \cdots, x_{2\chi}, y_0, y_1, \cdots, y_{3\chi}]. \tag{1.9}$$

We call I the *defining ideal* of $\mathcal{P}$. Obviously we have

$$P = (x, y) \in \mathcal{P} \Longleftrightarrow z = z(P) \in V(I) \subset \mathbf{A}^{5\chi+2} \tag{1.10}$$

with $V(I)$ denoting, as usual, the affine scheme of common zeroes of I in the affine space. The map $P \mapsto z(P)$ defines a bijection from $\mathcal{P}$ to the reduced part $V(I)_{red}$ of $V(I)$, and in particular, we have

$$n := \#\mathcal{P} = \#V(I)_{red} \tag{1.11}$$

Note that $V(I)_{red} = V(\sqrt{I})$ where $\sqrt{I}$ denotes the radical of I.

Now we consider the (irredundant) primary decomposition of the ideal I:

$$I = \mathbf{q}_1 \cap \cdots \cap \mathbf{q}_n \tag{1.12}$$

and the associated prime decomposition of the radical $\sqrt{I}$:

$$\sqrt{I} = \mathbf{p}_1 \cap \cdots \cap \mathbf{p}_n. \tag{1.13}$$

Here each $\mathbf{q}_i$ is a primary ideal in the polynomial ring R and $\mathbf{p}_i = \sqrt{\mathbf{q}_i}$ is a prime ideal. In fact, $\mathbf{p}_i$ is the maximal ideal of the point $z(P) \in V(I)$ defined by (1.7) for the corresponding $P = P_i \in \mathcal{P}$. Let us call

$$\mu(P_i) := \dim_k R/\mathbf{q_i} \tag{1.14}$$

the *multiplicity* of $P_i \in \mathcal{P}$ (cf. [3, Ch.4], [8, Ch.4].)

We study the following question:

Question 1.3 *Given an elliptic surface S over $\mathbf{P}^1$ of arithmetic genus χ, with the generic fibre E given by (1.3) and (1.4) as above, what are (i) the number of everywhere integral sections : $n = \#\mathcal{P}$, (ii) the linear dimension:* $\dim_k R/I$, *and (iii) the multiplicity $\mu(P_i) = \dim_k R/\mathbf{q}_i$ for each $i \leq n$?*

Note that, by the Chinese Remainder theorem, we have

$$\dim_k R/I = \sum_{i=1}^{n} \dim_k R/\mathbf{q}_i = \sum_{i=1}^{n} \mu(P_i). \tag{1.15}$$

Hence (ii) will follow from (iii).

Before going further, we present an explicit example.

Example 1.4 *Let $E/k(t)$ be the elliptic curve*

$$y^2 = x^3 + t^5 + 1. \tag{1.16}$$

Here we assume k has characteristic 0 or $p > 5$. Then (i) the number of everywhere integral sections $n = \#\mathcal{P}$ is equal to 240. (ii) The linear dimension $\dim_k R/I$ *is equal to 240, too. (iii) For all $P \in \mathcal{P}$, the multiplicity $\mu(P)$ is equal to 1.*

Proof Let us show that $\dim_k R/I_\lambda = 240$ by a direct computation using the method of Gröbner basis. To simplify the notation, we replace the ideal

$$I \subset R = k[x_0, x_1, x_2, y_0, y_1, y_2, y_3]$$

by a similar ideal

$$I' \subset R' = k[u, x_0, x_1, y_0, y_1, y_2]$$

by letting $x_2 = u^2, y_3 = u^3$. (Note that $x_2^3 - y_3^2$ is contained in I.) The Gröbner basis method yields a "shape basis" of I', i.e. a set of generators of I' of the form:

$$I' = (\Psi_{240}(u), x_i - \phi_i(u), y_j - \psi_j(u) | i = 0, 1, j = 0, 1, 2)$$

where Ψ, ϕ_i, ψ_j are polynomials of u and Ψ is a separable polynomial of degree 240. (The explicit form of the polynomial Ψ can be found in [12] or [14] if desired.) Therefore we have

$$\dim_k R/I = \dim_k R'/I' = \dim k[u]/(\Psi(u)) = 240.$$

Moreover the k-algebra $R/I \cong k[u]/(\Psi(u))$ is isomorphic to a direct sum of 240 copies of k, which shows that $I = \sqrt{I}$ and the primary decomposition of I is given by the 240 maximal ideals corresponding to the 240 roots of the polynomial $\Psi(u)$. In other words, $\mathcal{P}$ consists of $n = 240$ elements and $\mu(P) = 1$ for each P. *q.e.d.*

In this paper, we give a complete answer to Question 1.3 in the case $\chi = 1$, i.e. where S is a rational elliptic surface. The main result (Theorems 2.1) will be stated in the next section, whose proof will be given in §4. In §3 we study the behavior of the 240 roots in the E_8-frame of a rational elliptic surface under specialization and obtain a basic theorem (Theorem 3.4). As a by-product, we obtain a simple proof of the fact that the Mordell-Weil group M is generated by the set $\mathcal{P}$ of everywhere integral sections (Theorem 3.5), whose known proof depends on some case-by-case checking ([9]). In §5, a few examples are given as illustration of our main result. Actually rational elliptic surfaces are classified by Oguiso-Shioda [9] in terms of the trivial lattice and Mordell-Weil lattice. For each type, we have determined the data $n, m(P)(P \in \mathcal{P})$ appearing in Theorem 2.1, but the results will be given elsewhere. In the final section §6, we discuss some open questions in case $\chi > 1$.

As for the title of this paper, Gröbner basis computation is useful, as the above example shows, in dealing with Question 1.3 when S or E is explicitly given. We have profitably used the software "Risa/asir" (developped by the authors of [8]) for some numerical experiments and for direct verification of our results based on the theory of Mordell-Weil lattices and geometry of elliptic surfaces. The idea from deformation of singularities (cf. [12]) is disguised as the specialization arguments in the proof of our main results.

Convention: Throughout the paper, we keep the notation of §1; we sometimes write $\mathcal{P}_S, I_S, \ldots$ to specify the dependence of $\mathcal{P}, I, \ldots$ on the elliptic surface S under consideration. We continue to assume that k is algebraically closed.

2 Answer in case $\chi = 1$

To state our main results, let us first recall some basic facts on rational elliptic surfaces, fixing the notation (cf. [9], [10, §10]).

Let $N = \mathrm{NS}(S)$ denote the Néron-Severi lattice of an elliptic surface S with a section. Let U be the rank two unimodular sublattice of N spanned by the classes of the zero-section (O) and any fibre F. Let $V = U^{\perp}$ be the orthogonal complement of U in N, which is called the *frame* of S; we have $N = U \oplus V$. If S is a rational elliptic surface (RES), the frame V is a negative-definite even unimodular lattice of rank 8, and hence it is isomorphic to E_8^-, the opposite lattice of the root lattice E_8 (cf. [2, Ch.4]).

$$\mathrm{NS}(S) = U \oplus V, \quad V \cong E_8^-. \tag{2.1}$$

Thus we call the frame V of a RES as the E_8-frame.

Let $\mathcal{D} = \mathcal{D}_S \subset V$ be the subset of "roots" in V:

$$\mathcal{D} = \{cl(D) \in V | D^2 = -2\}. \tag{2.2}$$

By the above, it forms a root system of type E_8. In particular, we have

$$\#\mathcal{D} = 240. \tag{2.3}$$

For any $P \in \mathcal{P} = \mathcal{P}_S$, we set

$$D(P) := (P) - (O) - F. \tag{2.4}$$

Then we have $D(P) \perp U$ and $D(P)^2 = -2$, hence $D(P) \in \mathcal{D}$. (N.B. Here and in what follows, we sometimes write $D \in \mathcal{D}$ by abbreviating $cl(D) \in \mathcal{D}$, where $cl(D)$ denotes the class of a divisor D in N. We write $D_1 \equiv D_2$ if $cl(D_1) = cl(D_2)$ in N.)

On the other hand, each reducible fibre $f^{-1}(v)(v \in R_f)$ is decomposed as a sum of its irreducible components with positive integer coefficients :

$$f^{-1}(v) = \Theta_{v,0} + \sum_{i=1}^{m_v - 1} k_{v,i}\Theta_{v,i} \tag{2.5}$$

where $\Theta_{v,0}$ is the unique component intersecting the zero-section (O) and where m_v denotes the number of the irreducible components. Let T_v denote the sublattice of N generated by $\Theta_{v,i}(i = 1, \ldots m_v - 1)$. It is known (see [6]) that each $\Theta_{v,i}$ has self-intersection number -2 (i.e. $\Theta_{v,i} \in \mathcal{D}$) and T_v is a (negative) root lattice of ADE-type determined by the type of the reducible fibre. Let T be the sublattice of the E_8-frame V defined by

$$T = \oplus_{v \in R_f} T_v \subset V \cong E_8^- \tag{2.6}$$

which is called the *trivial lattice* of S.

Now our main result is the following:

Theorem 2.1 *Assume that S is a rational elliptic surface. Then (i) the number of everywhere integral sections $n = \#\mathcal{P}$ is bounded by 240:*

$$0 \leq n \leq 240, \tag{2.7}$$

and we have

$$n = 240 \Longleftrightarrow T = 0. \tag{2.8}$$

(ii)

$$\dim_k R/I = 240 - \nu(T) \tag{2.9}$$

where $\nu(T)$ is the number of roots in the trivial lattice T.

(iii) For each $i \leq n$, the multiplicity $\mu(P_i)$ (see (1.14)) is equal to the combinatorial multiplicity $m(P_i)$ to be defined below. In other words, we have

$$\mu(P) = m(P) \text{ for all } P \in \mathcal{P}. \tag{2.10}$$

Definition 2.2 For any $P \in \mathcal{P}$, let $R_f(P)$ denote the subset of $v \in R_f$ such that (P) intersects some non-identity component $\Theta_{v,i}(i \neq 0)$ of $f^{-1}(v)$. The *root graph associated with P*, denoted by $\Delta(P)$, is the connected graph with the vertices

$$D(P), \Theta_{v,i} \ (v \in R_f(P), i \neq 0), \tag{2.11}$$

where two vertices α, β are connected by an edge iff the intersection number $\alpha \cdot \beta = 1$. By a *distinguished root* of $\Delta(P)$, we mean a linear combination of the vertices of the form:

$$D = D(P) + \sum_{v,i} n_{v,i}\Theta_{v,i} \ (n_{v,i} \in \mathbf{Z}, \geq 0) \tag{2.12}$$

satisfying $D^2 = -2$. Further we denote by $m(P)$ the number of distinguished roots in the root graph $\Delta(P)$, and call it the *combinatorial multiplicity* of P.

The proof will be given in §4, after we establish the relationship of the two sets $\mathcal{P}$ and $\mathcal{D}$ for a given RES (Theorem 3.4) in the next section..

3 Relationship of $\mathcal{P}$ and $\mathcal{D}$

For a rational elliptic surface, the Mordell-Weil group $M = E(K)$ is isomorphic to the quotient group of the Néron-Severi group N by the subgroup $U \oplus T$, hence to the quotient group V/T:

$$M \cong N/(U \oplus T) \cong V/T \tag{3.1}$$

where V and $T = \oplus T_v$ are defined before in §2 (see [9], [10]).

Now we study the relation of $\mathcal{P}$ and $\mathcal{D}$, by restricting the natural projection $\pi : V \to V/T \cong M$, to the set of the roots $\mathcal{D} \subset V$:

$$\pi : \mathcal{D} \to M. \tag{3.2}$$

Lemma 3.1 *Assume $T = 0$. Then the Mordell-Weil lattice M is isomorphic to E_8, and $\mathcal{P}$ is equal to the set of sections $P \in M$ of height $\langle P, P\rangle = 2$. In this case, the map π gives a bijection: $\mathcal{D} \to \mathcal{P}$. The inverse map $\mathcal{P} \to \mathcal{D}$ is given by $P \mapsto D(P)$.*

Proof If $T = 0$, the rational elliptic surface $f : S \to \mathbf{P}^1$ has no reducible fibres, and hence $M \cong E_8$ (see [10, §10] or [9]). Now the height formula (1.1) says that for any $P \in M$

$$\langle P, P \rangle = 2 + 2(PO)$$

where (PO) is the intersection number of (P) and (O). Hence P has height 2 iff $(PO) = 0$, i.e. iff $P \in \mathcal{P}$.

As the set of roots in E_8, both $\mathcal{P}$ and $\mathcal{D}$ have the same cardinality 240. Thus the map $P \mapsto D(P)$ gives a bijection $\mathcal{P} \to \mathcal{D}$, and it is clear that $\pi(D(P)) = P$ for any P. Hence the assertion. *q.e.d.*

Lemma 3.2 *Suppose S is any rational elliptic surface. Let $\tilde{S}$ be a generic rational elliptic surface (see §4.1), and we consider a smooth specialization $\tilde{S} \to S$ preserving the elliptic fibration and the zero-section. Then it induces an isomorphism of the Néron-Severi lattices*

$$\sigma : \mathrm{NS}(\tilde{S}) \xrightarrow{\sim} \mathrm{NS}(S), \tag{3.3}$$

which gives rise to a bijection $\mathcal{D}_{\tilde{S}} \to \mathcal{D}_S$.

Proof In general, a specialization of smooth projective surfaces $\tilde{S} \to S$ induces an injective homomorphism $\mathrm{NS}(\tilde{S}) \hookrightarrow \mathrm{NS}(S)$ preserving the intersection pairings. In the case of RES, it gives a lattice isomorphism of $\mathrm{NS}(\tilde{S})$ onto $\mathrm{NS}(S)$ in view of (2.1), which preserves the sublattices U, V by assumption. It is obvious that the set of roots $\mathcal{D}$ in V, (2.2), is also preserved, proving the last assertion. *q.e.d.*

(N.B. This result may be called the *conservation law* of the E_8-roots on RES under specialization or deformation: cf. [12])

Lemma 3.3 *For any $D \in \mathcal{D}_S$, $\pi(D) = P$ belongs to $\mathcal{P}_S$ unless $\pi(D) = O$. In this case, we have*

$$D \equiv D(P) + \gamma \quad (\gamma \in T) \tag{3.4}$$

where γ is a linear combination of $\Theta_{v,i} (v \in R_f, i > 0)$ with non-negative integer coefficients.

Proof Fix $D \in \mathcal{D}_S$, and assume that $\pi(D) = P \neq O$. We claim that $P \in \mathcal{P}_S$.

We may suppose that S is in the situation described in Lemma 3.2. Then there exists some $\tilde{D} \in \mathcal{D}_{\tilde{S}}$ such that $\sigma(\tilde{D}) = D$. Applying Lemma 3.1 to $\tilde{S}$ (which obviously has $T = 0$), we have

$$\tilde{D} = D(\tilde{P}) := (\tilde{P}) - (\tilde{O}) - \tilde{F} \tag{3.5}$$

for some $\tilde{P} \in \mathcal{P}_{\tilde{S}}$, where $\tilde{O}$ (or $\tilde{F}$) denotes the zero-section (or a fibre) of $\tilde{S}$.

Suppose that, under the specialization, the irreducible curve $\tilde{\Gamma} := (\tilde{P})$ on $\tilde{S}$ reduces to an effective divisor on S:

$$\Gamma = \sum_j \Gamma_j$$

with the irreducible components Γ_j. By the conservation of intersection numbers, we have

$$1 = (\tilde{\Gamma}\tilde{F}) = (\Gamma F) = \sum_j (\Gamma_j F)$$

with each $(\Gamma_j F) \geq 0$. Hence there exists a unique Γ_j, say $j = 1$, such that

$$(\Gamma_1 F) = 1, \quad (\Gamma_j F) = 0 \ (j \neq 1).$$

Then Γ_1 is a section of S, i.e. $\Gamma_1 = (P_1)$ for some $P_1 \in M$, and all other Γ_j are contained in fibres. Obviously P_1 is equal to $P = \pi(D)$.

Next, in the intersection number relation:

$$0 = (\tilde{\Gamma}(\tilde{O})) = (\Gamma(O)) = (PO) + \sum_{j>1} (\Gamma_j(O)),$$

observe that $(PO) \geq 0$ (because $P \neq O$ by assumption) and $(\Gamma_j O) \geq 0$. Hence we have $(PO) = 0$ and $(\Gamma_j O) = 0$. The former implies that $P \in \mathcal{P}_S$, as claimed, while the latter implies that the other components $\Gamma_j (j > 1)$, if any, are among the non-identity components $\Theta_{v,i} (i > 0)$ of reducible fibres. Therefore $\tilde{D}$ specializes via σ to the following:

$$D^* = (P) - (O) - (F) + \gamma, \quad \gamma = \sum_{v,i>0} m_{v,i}\Theta_{v,i} \in T \tag{3.6}$$

where $m_{v,i}$ are some non-negative integers. On the other hand, since $\sigma(\tilde{D}) = D$, we have $D \equiv D^*$. This proves Lemma 3.3. *q.e.d.*

Theorem 3.4 *For any rational elliptic surface S with a section, let $\mathcal{D}$ be the set of roots in the E_8-frame. Then the map $\pi : \mathcal{D} \to \mathcal{P} \cup \{O\}$ is a surjective map unless $T = 0$, and $\mathcal{D}$ is decomposed into the disjoint union:*

$$\mathcal{D} = \pi^{-1}(O) \bigsqcup \bigsqcup_{P \in \mathcal{P}} \pi^{-1}(P). \tag{3.7}$$

The inverse image $\pi^{-1}(O)$ is the set of roots in T (it is empty if $T = 0$). For any $P \in \mathcal{P}$, we have

$$\pi^{-1}(P) = \{D \in \mathcal{D} \mid D \equiv D(P) + \sum_{v,i>0} m_{v,i}\Theta_{v,i} \ (m_{v,i} \geq 0)\} \tag{3.8}$$

which is equal to the set of distinguished roots in the root graph $\Delta(P)$ defined in §2. In particular, its cardinality is equal to the combinatorial multiplicity of P:

$$m(P) = \#\pi^{-1}(P), \tag{3.9}$$

and

$$\sum_{P \in \mathcal{P}} m(P) = 240 - \nu(T). \tag{3.10}$$

Proof This is clear by Lemma 3.1 and 3.3. The decomposition (3.7) of $\mathcal{D}$ is just the union of the inverse images of π, and counting the cardinality gives the relation (3.10). *q.e.d.*

As a by-product of the above proof, we obtain a conceptual proof of the following fact (see [9, Theorem 2.5], [10, Theorem 10.8]), which has been proven by using the classification of RES plus some case-by-case checking:

Theorem 3.5 *For any rational elliptic surface with a section (defined over an algebraically closed field of arbitrary characterisitic), the Mordell-Weil group is generated by the set $\mathcal{P}$ of sections P which are disjoint from the zero-section.*

Proof It is well-known that the root lattice E_8 is generated by a basis consisting of eight roots (see e.g. [2]). Hence the E_8-frame V is generated by the set $\mathcal{D}$ of roots. Since we have $M \cong V/T$ by (3.1), M is generated by $\pi(\mathcal{D})$, hence by $\mathcal{P}$ by the first part of Lemma 3.3. *q.e.d.*

4 Proof of Theorem 2.1

4.1 The case $T = 0$

First we consider the case $T = 0$. By Lemma 3.1, Theorem 2.1 reduces to the following statement:

Theorem 4.1 *Assume that S is a rational elliptic surface with $T = 0$. Then we have*

$$n = \dim_k R/I = 240, \quad \mu(P) = m(P) = 1 \text{ for all } P \in \mathcal{P}. \tag{4.1}$$

Proof It suffices to prove the equality:

$$\dim R/I = 240 \tag{4.2}$$

in the statement (4.1). In fact, we already know that $n = \#\mathcal{P} = 240$ and that $m(P) = 1$ for each $P \in \mathcal{P}$. The latter holds, because the root graph $\Delta(P)$ consists of the vertex $D(P)$ alone as $T = 0$. In view of the Chinese Remainder equality (1.15), we see that the claim (4.2) is equivalent to the following:

$$\mu(P) = 1 \text{ for all } P \in \mathcal{P}. \tag{4.3}$$

Thus we proceed as follows to show (4.2) (see Lemma 4.3).

First we write down a "universal" rational elliptic surface. In view of the condition (1.4) for $\chi = 1$, we let S_λ denote the elliptic surface defined by the Weierstrass equation (1.3) where we set

$$\lambda = (a_{i,j})\ (i \le 6, i \ne 5, j \le i), \quad a_i(t) = \sum_{j=0}^{i} a_{i,j} t^j \quad (i = 1, 2, 3, 4, 6). \tag{4.4}$$

Let

$$\Lambda = \{\lambda | S_\lambda \text{ is a RES}\} \tag{4.5}$$

and

$$\Lambda_0 = \{\lambda \in \Lambda | S_\lambda \text{ is a RES without reducible fibres }\}. \tag{4.6}$$

In characteristic different from 2 and 3, one can choose $a_i(t) = 0$ $(i = 1, 2, 3)$ (i.e. $a_{i,j} = 0$ for $i = 1, 2, 3$ and all j) without loss of generality. In any case, Λ is open in an affine space of suitable dimension, and Λ_0 is an open subset of Λ.

We denote by $\mathcal{P}_\lambda$ and I_λ the set of everywhere integral sections $\mathcal{P}$ of S_λ and its defining ideal, and by $V(I_\lambda)$ the 0-dimensional affine scheme defined as in §1.

Lemma 4.2 *Assume $\chi = 1$. Then $\{V(I_\lambda)|\lambda \in \Lambda\}$ forms a flat family over Λ.*

Proof (I owe this remark to Takeshi Saito.) For any χ, the ideal I_λ is generated by $6\chi + 1$ elements by definition, while the number of variables x_i, y_j is $(2\chi + 1) + (3\chi + 1) = 5\chi + 2$ (§1). Hence, if $\chi = 1$, $V(I_\lambda)$ is a complete intersection, and the flatness follows from [4, Ch.IV]. *q.e.d.*

Lemma 4.3 *Under the same assumption, $\{V(I_\lambda)|\lambda \in \Lambda_0\}$ forms a finite flat family over Λ_0.*

Proof For any (geometric) point $\lambda \in \Lambda_0$, $V(I_\lambda)$ consists of 240 points by (1.10) and Lemma 3.1. The affine coordinates of these points in the ambient affine space of $V(I_\lambda)$ are given by $z(P_m)$ $(1 \le m \le 240)$ (see (1.7)), if we set $\mathcal{P}_\lambda = \{P_m\ (1 \le m \le 240)\}$.

Now fix any $\lambda \in \Lambda_0$. Let $\tilde{\lambda}$ be a generic point of Λ_0, and let $z(\tilde{P})$ be a generic point of $\tilde{V} := V(I_{\tilde{\lambda}})$. Take any specialization $\sigma : \tilde{\lambda} \to \lambda$, and any specialization $\tilde{\sigma}$ of $z(\tilde{P})$ over σ. Since $\tilde{V}$ is specialized to $V(I_\lambda)$, bijectively as the point sets consisting of 240 points, the point $z(\tilde{P})$ must specialize to one of $z(P_m)'s$, which are obviously finite. This is the case for any choice of specialization $\tilde{\sigma}$, and hence the family in question is a proper family (cf. [5, Ch.II] or [17, Ch.VII]). Since it is a family of 0-dimensional schemes, the assertion follows. *q.e.d.*

Lemma 4.4 *(i) The dimension $\dim_k R/I_\lambda$ is constant for any $\lambda \in \Lambda_0(k)$.*
(ii) The constant value is equal to 240.

Proof The claim (i) follows from a general result for finite flat morphisms (see e.g. [7, Prop.8, Lect.6]). Thus, to prove (ii), it suffices to check it at one point $\lambda \in \Lambda_0(k)$. For instance, take λ corresponding to the rational elliptic surface $y^2 = x^3 + t^5 + 1$ treated in Example 1.4 in §1. This proves Theorem 4.1.

4.2 General case

Now we prove Theorem 2.1 in general.

For any $\lambda \in \Lambda$, let $\mathcal{D}_\lambda$ denote the set of roots in the E_8-frame (2.1) on S_λ.

Let $\tilde{\lambda}$ be a generic point of Λ_0, and let $\mathcal{P}_{\tilde{\lambda}} = \{\tilde{P}_i \ (1 \leq i \leq 240)\}$. The set $\mathcal{D}_{\tilde{\lambda}}$ consists of $D(\tilde{P}_i)$'s by Lemma 3.1.

Take any point $\lambda \in \Lambda(k)$ and any specialization $\sigma : \tilde{\lambda} \to \lambda$. By Lemma 3.2, $\mathcal{D}_{\tilde{\lambda}}$ is mapped bijectively to $\mathcal{D}_\lambda$ under the specialization, and, by Lemma 3.3, each $D(\tilde{P}_i)$ is mapped either to some element of T or to an element of the form (3.4) for some $P \in \mathcal{P}_\lambda$. For a fixed $P \in \mathcal{P}_\lambda$, the number of $\tilde{P}_i$'s corresponding to P in the above sense is equal to the multiplicity $\mu(P)$, because each $\tilde{P}_i$ has multiplicity 1 by (4.2) which has just been established above. Comparing this with the decomposition (3.7) of $\mathcal{D} = \mathcal{D}_\lambda$ in Theorem 3.4, we conclude that $\mu(P) = m(P)$ for each $P \in \mathcal{P}_\lambda$. This proves the claim (iii) of Theorem 2.1.

Next, to prove (ii), we combine (1.15) with (iii) just proven above:

$$\dim_k R/I_\lambda = \sum_{P \in \mathcal{P}} \mu(P) = \sum_{P \in \mathcal{P}} m(P)$$

By (3.10) in Theorem 3.4, this implies

$$\dim_k R/I_\lambda = 240 - \nu(T).$$

Thus we have proven the claim (ii) of Theorem 2.1.

The claim (i) is obvious: we have

$$n = \#\mathcal{P} \leq dim_k R/I \leq 240,$$

where the first inequality holds by (1.10) and the second one from (ii) above. The assertion (2.8) follows from (ii). This completes the proof of Theorem 2.1. *q.e.d.*

4.3 Further information in a special case (cf. [11], [12])

The idea of the above proof is adapted from our previous work [11, §8] and [12], treating a slightly less general family which admits a singular fibre of type II (a cuspidal cubic). We remark here that, if we restrict our attention to that family, everything in the above proof becomes clearer and more explicit.

Namely we consider

$$E_\lambda : y^2 = x^3 + x(p_0 + p_1 t + p_2 t^2 + p_3 t^3) + q_0 + q_1 t + q_2 t^2 + q_3 t^3 + t^5 \quad (4.7)$$

where

$$\lambda = (p_0, p_1, p_2, p_3, q_0, q_1, q_2, q_3) \in \mathbf{A}^8.$$

Assume that λ is generic (i.e. $p_0, \ldots, q_3$ are algebraically independent) over $\mathbf{Q}$, and let k be the algebraic closure of $k_0 := \mathbf{Q}(\lambda) = \mathbf{Q}(p_0, \ldots, q_3)$. Then the elliptic surface S_λ is a RES without reducible fibres and $M_\lambda = E_\lambda(k(t))$ is

isomorphic to the root lattice E_8. Take a basis $\{P_1, \ldots . P_8\}$ forming the Dynkin diagram of type E_8, and let $u_i = sp_\infty(P_i) \in k$, where

$$sp_\infty : E_\lambda(k(t)) \longrightarrow k \tag{4.8}$$

denotes the specialization homomorphism: for any P, $sp_\infty(P)$ is defined as the unique intersection point of the section (P) and the singular fibre of type II $f^{-1}(\infty)$.

By the *fundamental theorems for the algebraic equations of type* E_8 ([11, Theorems 8.3, 8.4, 8.5]), we have the following results:
(i) $\mathcal{K} = \mathbf{Q}(u_1, \ldots, u_8)$ is the splitting field of $E_\lambda/\mathbf{Q}(\lambda)(t)$, i.e. we have $E_\lambda(\mathcal{K}(t)) = E_\lambda(k(t))$ and $\mathcal{K}$ is the smallest extension of $\mathbf{Q}(\lambda)$ with this property.
(ii) $\mathcal{K}/\mathbf{Q}(\lambda)$ is a Galois extension with Galois group $W(E_8)$ (the Weyl group of type E_8).
(iii) $W(E_8)$ acts on the polynomial ring $\mathbf{Q}[u_1, \ldots, u_8]$, and the ring of invariants is equal to $\mathbf{Q}[\lambda] := \mathbf{Q}[p_0, \ldots, q_3]$. In other words, $\{p_0, \ldots, q_3\}$ forms a set of fundamental invariants of $W(E_8)$ (of weight $20, 14, 8, 2, 30, 24, 18, 12$ respectively).
(iv) The minimal polynomial $\Phi(X)$ of u_1 over $\mathbf{Q}(\lambda)$ splits completely in $\mathcal{K}$ and it has coefficients in $\mathbf{Q}[\lambda]$:

$$\Phi(X, \lambda) = \prod_{i=1}^{240} (X - u_i) \in \mathbf{Q}[\lambda][X], \tag{4.9}$$

where each root u_i is $\mathbf{Z}$-linear combination of $u_1, \ldots, u_8$. The 240 u_i form a root system of type E_8.
(v) For each $i \leq 240$, there is a section $P_i \in E_\lambda(k(t))$ of the form:

$$P_i = (\frac{1}{u_i^2}t^2 + at + b, \frac{1}{u_i^3}t^3 + ct^2 + dt + e), \ sp_\infty(P_i) = u_i \tag{4.10}$$

where the coefficients a, b, c, d, e belong to $\mathbf{Q}(\lambda)(u_i) \cap \mathbf{Q}[u_1, \ldots, u_8]$.

Let $u := (u_1, \ldots, u_8) \in \mathbf{A}^8$. Then it follows from (iii) above that the map $\phi : u \mapsto \lambda = \phi(u)$ defines a finite ramified Galois covering $\mathbf{A}^8 \to \mathbf{A}^8$ with Galois group $W(E_8)$, which is unramified on the open set $U \subset \mathbf{A}^8$ where the "discriminant" $\delta(\lambda)$ (cf. [1]) does not vanish:

$$\delta(\lambda) = \Phi(0, \lambda) = \prod_{i=1}^{240} u_i. \tag{4.11}$$

Furthermore $S_u := S_{\phi(\lambda)}$ defines a *smooth* family of rational elliptic surfaces parametrized by the affine space $\mathbf{A}^8$ upstairs (see [12, Prop.4.3] and references given there).

Now we consider specializing the generic point of the affine space upstairs $u = (u_1, \ldots, u_8)$ to some $u^0 = (u_1^0, \ldots, u_8^0)$. It induces a unique specialization $\lambda = \phi(u) \to \lambda^0 = \phi(u^0)$ in the affine space downstairs. By (v) above, we can write

each P_i as $P_i(u)$ with its coefficients of t lying in $\mathbf{Q}(\lambda)(u_i) \cap \mathbf{Q}[1/u_i, u_1, \ldots, u_8]$. Hence, as far as $u_i^0 \neq 0$, P_i has a unique specialization P_i^0 with $sp_\infty(P_i^0) = u_i^0$.

Thus, if $\delta(\lambda^0) \neq 0$, $P_i \to P_i^0$ gives a bijection of the set of 240 roots in the MWL M_λ to that in M_{λ^0}. (N.B. The map $u_i \to u_i^0$ is not necessarily injective even if we assume $\delta(\lambda^0) \neq 0$. See [11, p.685] for such an example.)

On the other hand, if $\delta(\lambda^0) = 0$, then there exist some i such that $u_i^0 = 0$. In this case, P_i must specialize to O in M_{λ^0}. The number ν of such i's is equal to $\nu(T)$, the number of roots in the trivial lattice $T \subset \mathrm{NS}(S_{\lambda^0})$. In other words, the multiplicity of the factor X in the polynomial $\Phi(X, \lambda^0)$ is equal to $\nu(T)$. If we set $\mathcal{P}_{\lambda^0} = \{Q_1, \ldots, Q_n\}$, then we have

$$\Phi(X, \lambda^0) = \prod_{i=1}^{240}(X - u_i^0) = X^\nu \prod_{j=1}^{n}(X - sp_\infty(Q_j))^{m(Q_j)}. \tag{4.12}$$

Thus, for a fixed $u^0 = u_i^0$, the multiplicity of $(X - u^0)$ in $\Phi(X, \lambda^0)$ is equal to the sum of $m(Q_j)$'s such that $sp_\infty(Q_j) = u^0$.

5 Examples

By [9], the Mordell-Weil lattice (abbreviated as MWL) of a rational elliptic surface is classified into 74 types by the triple $\{T, L, M\}$, where (i) $T = \sum_v T_v$ is the trivial lattice (2.6), with the opposite sign, embedded in E_8, (ii) L is the narrow MWL $E(K)^0$ which is isomorphic to the orthogonal complement of T in E_8, and (iii) M is the MWL $E(K)$ which is the direct sum of the dual lattice of L and the torsion group T'/T, where T' is the primitive closure of T in E_8.

For each type $\{T, L, M\}$, we have determined the set $\mathcal{P} \subset M$, $n = \#\mathcal{P}$, and the combinatorial multiplicities $m(P)$ for each $P \in \mathcal{P}$. The summary will be reported elsewhere.

Here we illustrate our results with a few classical examples. Examples in §5.1 are the prototype of the present work treated in the earlier paper [12]. Next §5.2 shows more complicated new features, dealing with the familiar Legendre curve.

5.1 Cases of higher Mordell-Weil rank (cf. [12, §5])

For a rational elliptic surface, the rank $r = \mathrm{rk}M$ is bounded by 8 and the higher MW-rank cases correspond to the cases of smaller $\mathrm{rk}T$. The first four cases in [9] are the following (where $\mathrm{rk}T \leq 2$): $(i) T = 0, L = M = E_8, (ii) T = A_1, L = E_7, M = E_7^*, (iii) T = A_2, L = E_6, M = E_6^*, (iv) T = A_1^{\oplus 2}, L = D_6, M = D_6^*$.

The set $\mathcal{P}$ of everywhere integral sections in M consists of the roots in the root lattice L and the minimal vectors of $M = L^*$ (the dual lattice of L) for the first three cases. Thus $n = \#\mathcal{P}$ is equal to the number $\nu(L)$ of the roots in L, plus the number of minimal vectors in case (ii) or (iii):

$$(i)\ n = 240,\ (ii)\ n = 126 + 56 = 182,\ (iii)\ n = 72 + 54 = 126.$$

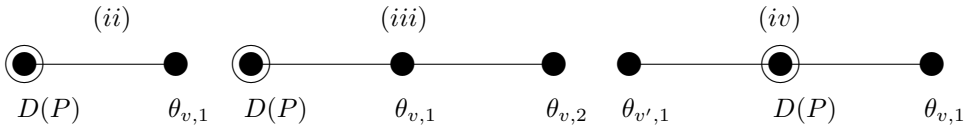

Figure 1: Root graph $\Delta(P)$

If $P \in \mathcal{P}$ is a root of L, then the multiplicity $m(P)$ is 1, because the root graph consists of the single vertex $D(P)$. On the other hand, if P is a minimal vector of $M = L^*$, then the multiplicity $m(P)$ is equal to $m(P) = 2$ in case (ii) and $m(P) = 3$ in case (iii), because then the root graph $\Delta(P)$ is given, respectively, by Figure 1. Here the root $D(P)$ is denoted by the encircled vertex and other roots $\Theta_{v,i}$ in (2.11) by the black vertices. (We write θ for Θ in the following.)

In case (iv), the set $\mathcal{P}$ consists of 60 roots of $L = D_6$, 12 minimal vectors of height $\langle P, P\rangle = 1$ in $M = D_6^*$, plus 64 $Q \in M$ with height $\langle Q, Q\rangle = 3/2$. We have $m(P) = 4$ and $m(Q) = 2$, as shown by Figure 1 (iv) or (ii) respectively. Compare [12, §5].

In each case, check the identity:

$$\begin{aligned} 126 \cdot 1 + 56 \cdot 2 &= 238 = 240 - 2, \quad 2 = \nu(A_1) && (5.1)\\ 72 \cdot 1 + 54 \cdot 3 &= 234 = 240 - 6, \quad 6 = \nu(A_2) && (5.2)\\ 60 \cdot 1 + 64 \cdot 2 + 12 \cdot 4 &= 236 = 240 - 4, \quad 4 = \nu(A_1^{\oplus 2}) && (5.3) \end{aligned}$$

5.2 The Legendre surface

Let E be defined by the Legendre form:

$$E : y^2 = x(x-1)(x-t). \tag{5.4}$$

Let $K = k(t)$ where k is any field of characteristic $\neq 2$. The elliptic surface defined by this equation is obviously a rational surface, since the function field $K(E) = k(t, x, y)$ is equal to $k(x, y)$.

There are two singular fibres of type I_2 at $t = 0, 1$ and one of type I_2^* at $t = \infty$. The trivial sublattice $T = A_1^{\oplus 2} \oplus D_6$ is of index 4 in E_8, and the Mordell-Weil group is $M = E_8/T \cong (\mathbf{Z}/2\mathbf{Z})^2$, a torsion group of order 4. More explicitly, we have

$$E(K) = \{O, P_1 = (0, 0), P_2 = (1, 0), P_3 = (t, 0)\} \tag{5.5}$$

Thus $\mathcal{P}$ consists of three 2-torsions $\{P_1, P_2, P_3\}$ and $n = \#\mathcal{P} = 3$. Figure 2 shows how each section (P_j) intersects the irreducible components $\theta_{v,i}(v =$

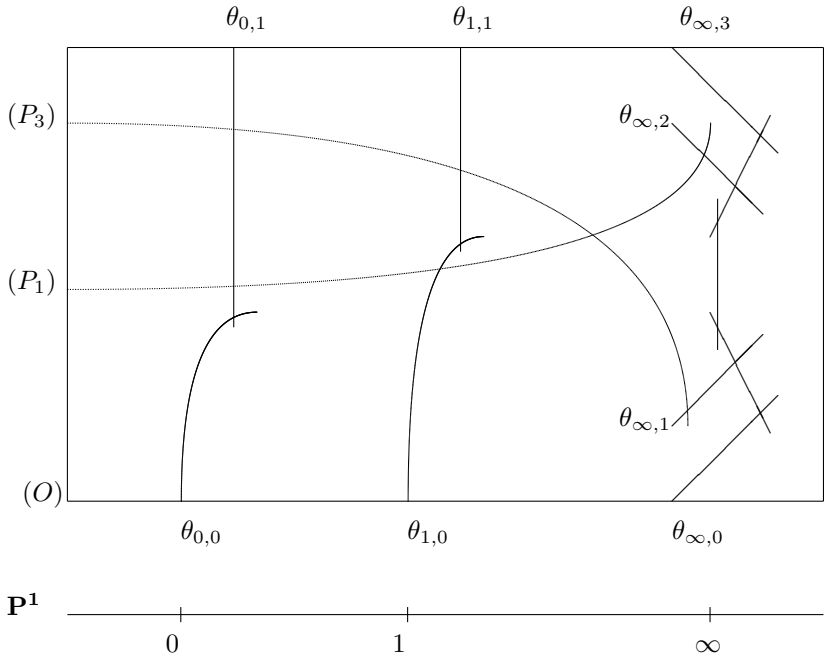

Figure 2: Legendre elliptic surface

$0, 1, \infty$) of three singular fibres. (N.B. Two different sections do not intersect. The picture is not correct in that (P_1) and (P_3) look as if they intersect.)

We can determine their (combinatorial) multiplicities as follows:

$$m(P_1) = 64,\ m(P_2) = 64,\ m(P_3) = 48 \tag{5.6}$$

Indeed the root graph $\Delta(P)$ for $P = P_1$ is shown by Figure 3 (and similarly for $P = P_2$), while $\Delta(P)$ for $P = P_3$ is as in Figure 4.

Then, by counting the number of distinguished roots in the root graph $\Delta(P)$, (5.3) can be verified. For instance, to show that $m(P_1) = 64$, consider first the distinguished roots $\xi = D(P) + \cdots$ not containing the vertex $\theta_{0,1}$ in Figure 3. Thus we seek for the number of "positive roots" in the Dynkin diagram of type E_7 whose coefficient of $D(P)$ is 1. As is well-known (see [1]), there exist 33 positive roots in the Dynkin diagram of type E_7 containing the left vertex $D(P)$, but one of them is of the form $2D(P) + \cdots$. Hence we have exactly 32 ξ of the required form. Then, considering $\xi + \theta_{0,1}$ for each such ξ, we obtain another set of 32 distinguished roots. In this way, we check that the number of distinguished roots in the root graph $\Delta(P_1)$ is equal to $2 \cdot 32$, i.e. $m(P_1) = 64$.

Incidentally, it should be remarked that the root graph of an everywhere

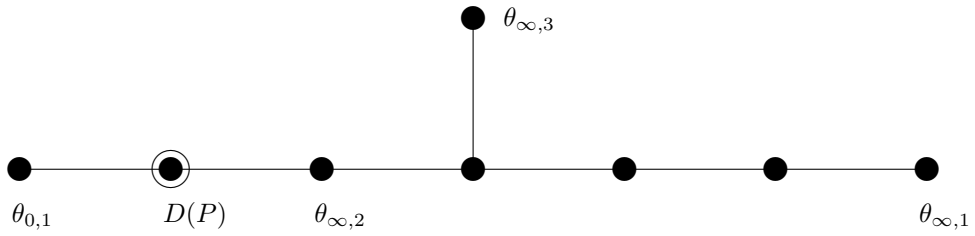

Figure 3: Root graph $\Delta(P)$ for $P = P_1$

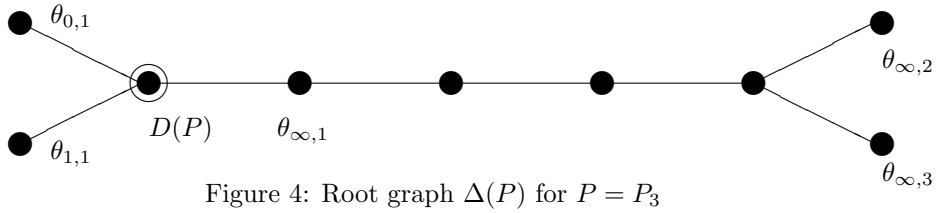

Figure 4: Root graph $\Delta(P)$ for $P = P_3$

integral section P is a visual counterpart of the height formula for P. For instance, the height formula (1.2) for $P = P_i$ above is:

$$\begin{aligned} \langle P_1, P_1 \rangle &= 2 + 0 - 6/4 - 1/2 - 0 && (5.7) \\ \langle P_2, P_2 \rangle &= 2 + 0 - 6/4 - 0 - 1/2 && (5.8) \\ \langle P_3, P_3 \rangle &= 2 + 0 - 1 - 1/2 - 1/2 && (5.9) \end{aligned}$$

where the local contribution terms $contr_v(P)$ (see [10, p.229]) on the right hand side are written in the order of $v = \infty, 0, 1$.

Now Theorem 2.1 implies that, if I denotes the defining ideal of $\mathcal{P}$, then the primary decomposition of I is of the form $I = \mathbf{q}_1 \cap \mathbf{q}_2 \cap \mathbf{q}_3$, with $\mathbf{q}_i$ corresponding to $P_i (i = 1, 2, 3)$, and we have

$$\dim_k R/\mathbf{q}_i = 64 (i = 1, 2), \quad \dim_k R/\mathbf{q}_3 = 48, \quad \dim_k R/I = 176. \qquad (5.10)$$

As mentioned before, Gröbner basis computation allows one to make a direct verification of such a result.

6 Open questions

When the arithmetic genus χ is greater than 1, Question 1.3 remains open. Let us pose a few more specific questions here.

We use the same notation as in §1. In particular, $\mathcal{P}$ denotes the set of everywhere integral sections (1.1) on a given elliptic surface S over $\mathbf{P}^1$ of arithmetic genus χ, and I denotes its defining ideal (1.9).

Question 6.1 *Assume that $P \in \mathcal{P}$ has height $\langle P, P \rangle = 2\chi$. Is the multiplicity $\mu(P)$ equal to 1?*

The assumption is equivalent to saying that $P \in \mathcal{P}$ belongs to the narrow Mordell-Weil lattice, or that the sections (P) and (O) intersect the same irreducible component for every reducible fibre. Question 6.1 is true if $\chi = 1$ by Theorem 2.1, since the assumption implies that the combinatorial multiplicity $m(P) = 1$.

In particular, we ask:

Question 6.2 *Assume that the trivial lattice $T = 0$, or equivalently, there are no reducible fibres. Then is it true that $I = \sqrt{I}$?*

Next consider the case $\chi = 2$, i.e. S is an elliptic K3 surface.

Question 6.3 *What is the maximum cardinality $n = \#\mathcal{P}$ when S varies among elliptic K3 surfaces?*

Question 6.4 *Assume $\chi = 2$. Can one give some combinatorial description of the multiplicity $\mu(P)$ for $P \in \mathcal{P}$?*

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