RIMS-1662

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March 2009

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Abstract

There are two distinct approaches to the study of initial value problem of the periodic box-ball systems. One way is the rigged configuration approach due to Kuniba–Takagi–Takenouchi and another way is the 10-elimination approach due to Mada–Idzumi–Tokihiro. In this paper, we describe precisely interrelations between these two approaches.

Mathematics Subject Classification (2000) 17B37, 37K15, 05E15.
Key words and phrases: crystal basis, periodic box-ball system, combinatorics.

1 Introduction

The main purpose of the present paper is to compare two approaches to the study of dynamics of the periodic box-ball systems (PBBS for short) of type $A_1^{(1)}$ due to Kuniba–Takagi–Takenouchi [13] (KTT for short) and Mada–Idzumi–Tokihiro [14] (MIT for short). Our main result states that

$$\text{KTT} \approx \text{MIT}.$$

The approach developed in [13] is based on the theory of the rigged configurations (RC for short — for details concerning the RC-bijection, see e.g. [8, 15]), whereas the approach developed by [14] is based on the 10-elimination procedure. To be more specific, our main result describes precisely interrelations between the 10-elimination algorithm and the RC-bijection in the case under consideration.

Brief history. The box-ball systems (BBS for short) have been introduced by Takahashi–Satsuma in 1990 [18, 17] during their study of cellular automaton and attempts to construct examples of those which have a solitonical nature. The periodic version of BBS, PBBS, was then introduced in [23, 22]. From that time the BBS were extensively studied since many deep and unexpected connections with different branches of mathematics and mathematical physics were discovered. Among those are connections with the theory of crystal base [3, 2], combinatorics [20, 1, 9], the Riemann theta functions [10, 11], tropical algebraic geometry [4, 5], and also with the theory of discrete KP and Toda type integrable systems [19, 21, 12].

The organization of the present paper is as follows. In Section 2, we briefly recall the 10-elimination procedure. In Section 3, we recall the rigged configuration and initial value problem of the PBBS in terms of the RC-bijection. In Section 4, we give a precise interpretation of the 10-elimination in terms of the RC-approach (Theorem 4.3).

2 10-elimination

A state of the PBBS is given by a sequence of integers 0 and 1 on a circle. We usually cut the circle at a suitable position to be determined below and treat it as a sequence on a straight line which we call a path. In this paper, we always assume that within each path, the number of letters 0 is equal to or bigger than the number of letters 1. We will call such paths as positive weight paths. Due to the second paragraph of Section 3.3 of [13], this assumption does not result in loss of generality. Originally, a path is identified with a sequence of capacity one boxes, and 0 stands for a vacant box and 1 stands for a ball within the box. Given a path p, we introduce 10-elimination procedure and notion of 0-solitons [14]. Input of the 10-elimination procedure is a path p and output is a finite sequence of paths $E^0(p) := p$, $E^1(p)$, $E^2(p)$, $\cdots$ defined recursively as follows.

Suppose that we have constructed $E^{k-1}(p)$. We give coordinate 1, 2, 3, $\cdots$, to each letter in $E^{k-1}(p)$ from left to right. The 10-pair is a neighboring pair of 10

where 1 at location j and 0 at location $j+1$ for some j. Then $E^k(p)$ is obtained by erasing all 10-pairs of $E^{k-1}(p)$. If $E^k(p)$ does not contain letter 1 for some k, stop the procedure. Let the number of 10-pairs in $E^k(p)$ be e_k. By definition, we have $e_{k-1} \geq e_k$. Then we denote distinct lengths of rows of the transposed diagram ${}^t(e_1, e_2, \cdots)$ by L_j. We assume that $L_1 > L_2 > L_3 > \cdots > L_s$ and denote the multiplicity of L_j by m_{L_j}, i.e.,

$$ {}^t(e_1, e_2, \cdots) = (L_1^{m_{L_1}}, L_2^{m_{L_2}}, \cdots, L_s^{m_{L_s}}). \tag{1} $$

Position of the cut of the original circle is determined by the following condition. Note that to each letter in $E^k(p)$, we can specify the original position of p from which the letter is originated. For a 10-pair in some $E^k(p)$, we join the corresponding 1 and 0 in p by arc in this direction. Since the path is a positive weight path, we can always choose a suitable cyclic shift so that no such arc cross the left or the right end of p. In the sequel we assume that the path p is cut with this property.

Given a path p, we draw all possible arcs according to the procedure in the last paragraph. Then one step time evolution of the path p is obtained by replacing all connected 1 and 0 by 0 and 1, respectively, and leaving non-connected letters unchanged. We denote the resulting time evolved path by $T_\infty(p)$.

Finally, we introduce a notion of 0-solitons. As we have seen, 10-pairs of $E^{k-1}(p)$ are erased in $E^k(p)$. In the following diagram, $m(>0)$ 10-pairs between X and Y are erased:

$$ E^{k-1}(p) = \cdots X(10)^m Y \cdots \xrightarrow{\text{10-elimination}} E^k(p) = \cdots XY \cdots . $$

Under this setting,

(a) if $XY = 11, 01, 00$, then there are m 0-solitons at position X of $E^k(p)$,

(b) if $XY = 10$, then there are $(m-1)$ 0-solitons at position X of $E^k(p)$.

We can grasp the meaning of the 0-soliton if we consider the inverse procedure to get $E^{k-1}(p)$ from $E^k(p)$: 0-solitons give information about how many extra 10-pairs should be inserted between XY to get $E^{k-1}(p)$. Here, notice that if $XY = 10$, it is guaranteed that there is at least one 0-soliton between XY in $E^{k-1}(p)$ so that we have no need to specify it. Based on this observation, we see that there are precisely m_{L_j} 0-solitons at $E^{L_j}(p)$. Denote the positions of 0-solitons of $E^{L_j}(p)$ by

$$ x_1^{(j)} \leq x_2^{(j)} \leq \cdots \leq x_{m_{L_j}}^{(j)}. \tag{2} $$

As we will see, the data L_j combined with $x_i^{(j)}$ provide sufficient information to solve initial value problem of the PBBS.

Example 2.1 Let us take the length 32 path

$$ p = 00111011100100011110001101000000. $$

Then 10-elimination procedure goes as follows.

$$
\begin{aligned}
E^0(p) &= 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0,\\
E^1(p) &= 0\ 0\ 1\ 1 \quad 1\ 1 \quad 0 \quad 0\ 0\ 1\ 1\ 1 \quad 0\ 0\ 1 \quad 0\ 0\ 0\ 0\ 0,\\
E^2(p) &= 0\ 0\ 1\ 1 \quad 1 \quad 0\ 0\ 1\ 1 \quad 0 \quad 0\ 0\ 0\ 0,\\
E^3(p) &= 0\ 0\ 1\ 1 \quad 0\ 1 \quad 0\ 0\ 0\ 0,\\
E^4(p) &= 0\ 0\ 1 \quad 0\ 0\ 0,\\
E^5(p) &= 0\ 0 \quad 0\ 0.
\end{aligned}
$$

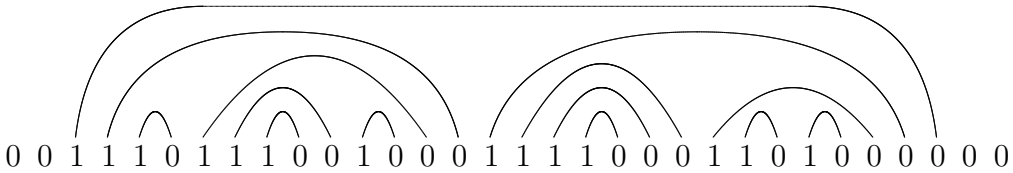
From these data, we obtain the following data:

$$(e_0, e_1, e_2, e_3, e_4) = (6, 3, 2, 2, 1), \tag{3}$$

$${}^t(e_0, e_1, e_2, e_3, e_4) = (L_1^{m_{L_1}}, L_2^{m_{L_2}}, L_3^{m_{L_3}}, L_4^{m_{L_4}}) = (5^1, 4^1, 2^1, 1^3), \tag{4}$$

$$\{x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, x_1^{(4)}, x_2^{(4)}, x_3^{(4)}\} = \{2, 3, 10, 4, 7, 15\}. \tag{5}$$

We draw arcs on p as follows:



We reverse all connected 10-pairs and obtain

$$T_\infty(p) = 000001000110111000011100101111100$$

as the one step time evolution. □

3 Rigged configurations and the KTT theorem

3.1 Rigged configuration

Let $X = \{x_1 < x_2 < \cdots < x_N\}$ be an ordered set, and α be a composition of size N. A standard tabloid of shape α is a filling of the shape α by elements of the ordered set X such that the elements in each row are strictly increasing. Let ν be a partition. Then $m_i(\nu)$ is the number of occurrences of i in ν and $\ell(\nu)$ is the length of ν.

Definition 3.1 *A path of type $B_1^{\otimes L}$ (= path of length L for short) is a sequence $p = a_1 a_2 \cdots a_L$ where $a_i \in \{0, 1\}$ for all $1 \leq i \leq L$. To each path $p = a_1 \cdots a_L$ we associate a two row standard Young tabloid according to the following rule: we put i to the first row if $a_i = 0$, and to the second row otherwise.* □

Our first goal is to remind a construction of the rigged configuration bijection in the special case of $A_1^{(1)}$. An input of the RC-bijection is a standard tabloid T of shape (λ_1, λ_2). The output of the RC-bijection is a rigged partition, i.e., a partition $\nu = (\nu_1, \nu_2, \cdots)$ of size λ_2 together with a collection of integer numbers

$$\{J_{\alpha,i} \,|\, 1 \leq i \leq \nu_1 \text{ s.t. } m_i(\nu) \neq 0, 1 \leq \alpha \leq m_i(\nu)\} \tag{6}$$

such that

$$-i \le J_{\alpha,i} \le P_i(\nu) := L - 2Q_i(\nu), \tag{7}$$

and certain additional restrictions that are not important for our considerations. Here $Q_i(\nu) := \sum_a \min(i, \nu_a)$ represents number of boxes contained in the left i columns of diagram ν. The integers $J_{\alpha,i}$ are called the **riggings** and $P_i(\nu)$ are called the **vacancy numbers**.

Let us briefly remind a construction of a map from the set of two row standard tabloids to the set of the rigged partitions. The construction runs as follows. Let $\boxed{b_1}\boxed{\cdots}\boxed{\cdots}\boxed{b_{\lambda_2}}$ be the second row of a given two row standard tabloid. The first step of our construction is to consider tabloid $\boxed{b_1}$ and define the corresponding rigged partition to be $(\boxed{\ }\, J_{1,1})$ where $J_{1,1} = b_1 - 2$. The next step is to consider tabloid $\boxed{b_1}\boxed{b_2}$ and define the corresponding rigged partition to be $(\boxed{\ }\boxed{\ }\, J_{1,2})$ where $J_{1,2} = b_2 - 4$ if $b_2 - b_1 = 1$, and $\left(\begin{array}{|c|}\hline\ \\\hline\end{array}\begin{array}{c} J_{2,1} \\ J_{1,1}\end{array}\right)$ where $J_{1,1} = b_1 - 2$ and $J_{2,1} = b_2 - 4$ if $b_2 - b_1 > 1$. We emphasize that in the case $b_2 - b_1 = 1$, the first row of the configuration $\nu := \begin{array}{|c|}\hline\ \\\hline\end{array}$ is singular, i.e., it contains the rigging with the maximal possible value.

We proceed further by induction. Assume that tabloid $\boxed{b_1}\boxed{\cdots}\boxed{\cdots}\boxed{b_k}$ $(1 \le k < \lambda_2)$, corresponds to the rigged partition

$$\{\tilde{\nu}, \tilde{J}_{\alpha,i} \,|\, 1 \le \alpha \le m_i(\tilde{\nu})\}, \qquad |\,\tilde{\nu}\,| = k. \tag{8}$$

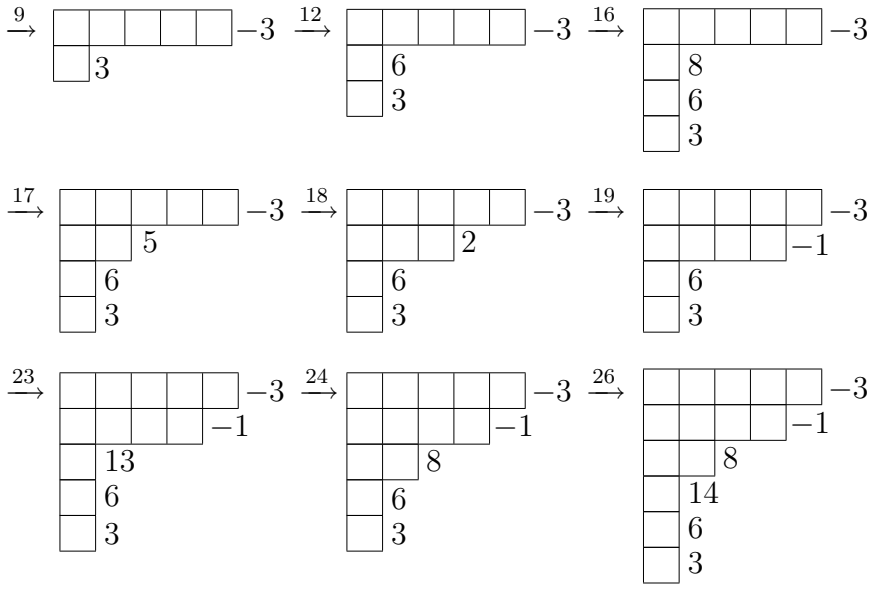
The next step is to describe the rigged partition $\{\nu, J_{\alpha,i}\}$ that corresponds to tabloid $\boxed{b_1}\boxed{\cdots}\boxed{\cdots}\boxed{b_k}\boxed{b_{k+1}}$. If $b_{k+1} - b_k > 1$, or equivalently $a_{b_{k+1}-1} = 0$, then $\nu = (\tilde{\nu}, 1)$ and $J_{\alpha,i} = \tilde{J}_{\alpha,i}$ if $i > 1$, or $i = 1$ and $\alpha \le m_1(\tilde{\nu})$, and $J_{m_1(\nu),1} = b_{k+1} - 2(\ell(\tilde{\nu}) + 1) = \tilde{J}_{m_1(\tilde{\nu})} + b_{k+1} - b_k - 2$. In the case $b_{k+1} - b_k = 1$, there should exist non empty singular strings, i.e., rows whose riggings are equal to the corresponding vacancy numbers. Take a singular string, say $\tilde{\nu}_a$, of the longest length. Then the corresponding rigged partition is defined to be $\{\nu, J_{\alpha,i}\}$, where $\nu_i = \tilde{\nu}_i$ if $i \ne a$, and $\nu_a = \tilde{\nu_a} + 1$. Moreover, $J_{\alpha,i} = \tilde{J}_{\alpha,i}$ if $i \ne a$, $J_{m_{\nu_a}(\nu),\nu_a} = b_{k+1} - 2\sum_j \min(\nu_a, \nu_j)$.

Example 3.2 Let us consider the path p treated in Example 2.1. The corresponding tabloid is

1	2	6	10	11	13	14	15	20	21	22	25	27	28	29	30	31	32
3	4	5	7	8	9	12	16	17	18	19	23	24	26				

In other words, the second row of the tabloid records positions of occurrences of letter 1 in p. Corresponding to the second row of the tabloid, the RC-bijection proceeds as follows:

$$\emptyset \xrightarrow{3} \boxed{\ }\, 1 \xrightarrow{4} \boxed{\ }\boxed{\ }\, 0 \xrightarrow{5} \boxed{\ }\boxed{\ }\boxed{\ }\, -1 \xrightarrow{7} \begin{array}{|c|c|c|}\hline\ & \ & \ \\\hline\ \\\cline{1-1}\end{array}\, -1 \xrightarrow{8} \begin{array}{|c|c|c|c|}\hline\ & \ & \ & \ \\\hline\ \\\cline{1-1}\end{array}\, -2$$
$$\qquad\qquad\qquad



In the above diagrams, we attach each rigging on the right of the corresponding row of the partition. The last rigged configuration in the above sequence gives the output of RC-bijection applied to p. □

3.2 Kuniba–Takagi–Takenouchi's construction

3.2.1 Crystals B_l

Let B_l be the classical crystal [6] corresponding to the l-fold symmetric tensor representation of $U'_q(A_1^{(1)})$. As the set, $B_l = \{(x_0, x_1) \in \mathbb{Z}^2_{\geq 0} \,|\, x_0 + x_1 = l\}$. Note that in usual notation, say that used in [13], our 0 and 1 are denoted by 1 and 2, respectively. We sometimes represent elements of B_l by semi-standard Young tableaux (without frame for simplicity). More precisely, the element (x_0, x_1) is also represented by $\overbrace{0\cdots0}^{x_0}\overbrace{1\cdots1}^{x_1}$. In this notation we usually denote $(1,0)$ and $(0,1)$ simply by 0 and 1, respectively. We can define tensor product $B_l \otimes B_k$. As the set, it is the product of the two sets and it is known that we can define algebraic structure on them. Then we can define the map $R : B_l \otimes B_1 \to B_1 \otimes B_l$ which is compatible with the algebraic structure. Explicitly, it is given by

$$(x_0, x_1) \otimes 0 \longmapsto \begin{cases} 0 \otimes (l, 0) & \text{if } (x_0, x_1) = (l, 0) \\ 1 \otimes (x_0 + 1, x_1 - 1) & \text{otherwise,} \end{cases} \tag{9}$$

$$(x_0, x_1) \otimes 1 \longmapsto \begin{cases} 1 \otimes (0, l) & \text{if } (x_0, x_1) = (0, l) \\ 0 \otimes (x_0 - 1, x_1 + 1) & \text{otherwise.} \end{cases} \tag{10}$$

R is a bijection and called the combinatorial R matrix. We write the relation $R(u \otimes b) = b' \otimes u'$ simply as $u \otimes b \simeq b' \otimes u'$, and similarly for any consequent relation of the form $a \otimes u \otimes b \otimes c \simeq a \otimes b' \otimes u' \otimes c$. When we consider elements of $p \in B_1^{\otimes L}$, we sometimes omit symbols $\otimes$ and identify them with paths.

3.2.2 Time evolutions

Given a path $p = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B_1^{\otimes L}$, we define the time evolution operators T_l for $l \in \mathbb{Z}_{>0}$ of the PBBS in the following way. Take $u_l := (l, 0) \in B_l$ and define $v_l \in B_l$ by

$$u_l \otimes p \simeq p_l^* \otimes v_l, \qquad (p_l^* \in B_1^{\otimes L}). \tag{11}$$

In practice, we can compute the map R step by step as follows:

$$\begin{aligned} u_l \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_L &\simeq b_1^* \otimes u_l^{(1)} \otimes b_2 \otimes \cdots \otimes b_L \simeq b_1^* \otimes b_2^* \otimes u_l^{(2)} \otimes b_3 \otimes \cdots \otimes b_L \\ &\simeq \cdots \simeq b_1^* \otimes b_2^* \otimes \cdots \otimes b_L^* \otimes u_l^{(L)}, \end{aligned} \tag{12}$$

where $u_l^{(i)} \in B_l$ and $u_l^{(L)} = v_l$. This element v_l depends on the path p. Then, according to Section 2 of [13], we have

$$v_l \otimes p \simeq p_l^{**} \otimes v_l. \tag{13}$$

Using this non-trivial relation, we define

$$T_l(p) := p_l^{**}. \tag{14}$$

Since combinatorial R matrix acts trivially on $B_1 \otimes B_1$, T_1 is just the cyclic shift operator: $T_1(b_1 \otimes \cdots \otimes b_{L-1} \otimes b_L) = b_L \otimes b_1 \otimes \cdots \otimes b_{L-1}$. The commutativity $T_l T_k = T_k T_l$ for all $l, k \in \mathbb{Z}_{>0}$ holds due to the Yang–Baxter relation ([13] Theorem 2.2). In general, there is sufficiently large N so that $T_N = T_{N+1} = \cdots =: T_\infty$. This definition of T_∞ coincides with the definition in Section 2 (Example 2.7 of [13]).

3.2.3 Action variable

Consider the path $p = b_1 \otimes \cdots \otimes b_L$ of weight $\lambda = (\lambda_1, \lambda_2)$, where $\lambda_i = \#\{a \,|\, b_a = i\}$ satisfying $\lambda_1 \geq \lambda_2$. Then we can always find d $(1 \leq d \leq L)$ such that $T_1^d(p) = b'_1 \otimes \cdots \otimes b'_L$ has the following property: for any $1 \leq i \leq L$,

$$\#\{a \,|\, b'_a = 1, a \in [i, L]\} \leq \#\{a \,|\, b'_a = 0, a \in [i, L]\}. \tag{15}$$

Let the rigged configuration corresponding to this $T_1^d(p)$ be (ν, J). We define the *action variable* of the path p by ν. From the condition (15), we see that the configuration part of the RC data corresponding to $T_1^d(p)^{\otimes N}$ is $\overbrace{\nu \cup \nu \cup \cdots \cup \nu}^{N}$. Due to Corollary 3.5 of [13], ν is conserved under arbitrary time evolution. In the following, for the given action variable ν, we put

$$H = \{i \in \mathbb{Z}_{\geq 1} \,|\, m_i \neq 0\} = \{i_1 < i_2 < \cdots < i_s\}. \tag{16}$$

3.2.4 Angle variable

Let us define angle variable according to [13]. In order to explain motivation of the construction, let us mention the following simple property.

Lemma 3.3 *Let p be a path of length L satisfying the condition (15), and $(\nu, J_{\alpha,i})$ be the corresponding rigged configuration. Then the rigged configuration corresponding to the path $\pi_N = \overbrace{p \otimes p \otimes \cdots \otimes p}^{N}$ is equal to*

$$\nu(\pi_N) = \nu \cup \nu \cup \cdots \cup \nu, \tag{17}$$

$$J(\pi_N) = \coprod_{a=1}^{N} \{J_{k+am_i,i} \,|\, i \in H, 1 \le k \le m_i\}, \tag{18}$$

where $J_{k+am_i,i} := J_{k,i} + aP_i(\nu)$. □

Motivated by this lemma, define $\bar{\mathcal{J}}$ as follows:

$$\bar{\mathcal{J}} = \bar{\mathcal{J}}(\nu) = \mathcal{J}_{i_1} \times \mathcal{J}_{i_2} \times \cdots \times \mathcal{J}_{i_s}, \tag{19}$$

$$\mathcal{J}_i = \{(J_{k,i})_{k\in\mathbb{Z}} \,|\, J_{k,i} \in \mathbb{Z}, J_{k,i} \le J_{k+1,i}, J_{k+m_i,i} = J_{k,i} + P_i(\nu) \text{ for all } k\}. \tag{20}$$

Definition 3.4 *For $j \in \mathbb{Z}_{\ge 1}$, define the map $\sigma_j : \mathcal{J}_i \longrightarrow \mathcal{J}_i$ by*

$$\sigma_j : (J_{k,i})_{k\in\mathbb{Z}} \longmapsto (J'_{k,i})_{k\in\mathbb{Z}}, \qquad J'_{k,i} = J_{k+\delta_{i,j},i} + 2\min(i,j). \tag{21}$$

We extend σ_j to the map $\bar{\mathcal{J}} \longrightarrow \bar{\mathcal{J}}$ by $\sigma_j(\bar{\mathcal{J}}) = \sigma_j(\bar{\mathcal{J}}_{i_1}) \times \sigma_j(\bar{\mathcal{J}}_{i_2}) \times \cdots \times \sigma_j(\bar{\mathcal{J}}_{i_s})$. □

The operators σ_j are the key to define angle variable. By definition, we have $\sigma_j\sigma_k = \sigma_k\sigma_j$ for any $j, k \in \mathbb{Z}_{\ge 1}$. Therefore the set $\mathcal{A} := \{\sigma_{i_1}^{n_1}\sigma_{i_2}^{n_2}\cdots\sigma_{i_s}^{n_s} \,|\, n_1, n_2, \cdots, n_s \in \mathbb{Z}\}$ is abelian group.

Definition 3.5 (1) *Define equivalence relation $\simeq$ for elements of $\bar{\mathcal{J}}$ by $J \simeq K$ $(J, K \in \bar{\mathcal{J}})$ if there exist $\sigma \in \mathcal{A}$ such that $J = \sigma(K)$.*
(2) *Define the set $\mathcal{J} = \mathcal{J}(\nu)$ by*

$$\mathcal{J} = \bar{\mathcal{J}} / \simeq . \tag{22}$$

An element of $\mathcal{J}$ is called angle variable. □

Now we shall explain how to compute the operators σ_k based on explicit example. To begin with, we have to embed a sequence of the riggings $(J_{k,i})_{1\le k\le m_i}$ attached to length i rows of ν into the set $\mathcal{J}_i$ by the map

$$\iota : (J_{k,i})_{1\le k\le m_i} \longrightarrow (J_{k,i})_{k\in\mathbb{Z}} \tag{23}$$

using periodicity $J_{k+m_i,i} = J_{k,i} + P_i(\nu)$.

Example 3.6 Consider the following length $L = 19$ paths:

$$q = 0011010001110100011, \qquad q' = 0001110100011001101. \tag{24}$$

Notice that $q = T_1^6(q')$. The rigged configurations corresponding to q and q' are

$$q \longmapsto (\mu, J) = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline & \\ \cline{1-1} & \\ \cline{1-1}\end{array}\begin{array}{l}0\\3\\0\\6\\2\end{array} \quad , \qquad q' \longmapsto (\mu, J') = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline & \\ \cline{1-1} & \\ \cline{1-1}\end{array}\begin{array}{l}0\\3\\3\\9\\4\end{array} \quad .$$

We compute actions of σ_2 and σ_1 on the rigged configuration (μ, J) corresponding to q. Since $\mu = (3, 2, 2, 1, 1)$, we have $Q_3(\mu) = 9$, $Q_2(\mu) = 8$ and $Q_1(\mu) = 5$. Therefore the vacancy numbers are $P_3(\mu) = 1$, $P_2(\mu) = 3$ and $P_1(\mu) = 9$. We would like to explain the following computations:

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline & \\ \cline{1-1} & \\ \cline{1-1}\end{array}\begin{array}{l}0\\3\\0\\6\\2\end{array} \overset{\sigma_2}{\longmapsto} \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline & \\ \cline{1-1} & \\ \cline{1-1}\end{array}\begin{array}{l}4\\7\\7\\8\\4\end{array} \overset{\sigma_1}{\longmapsto} \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline & \\ \cline{1-1} & \\ \cline{1-1}\end{array}\begin{array}{l}6\\9\\9\\15\\10\end{array} = 6 + \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline & \\ \cline{1-1} & \\ \cline{1-1}\end{array}\begin{array}{l}0\\3\\3\\9\\4\end{array}$$

• Action of σ_2 is divided into the following two parts. (a) For the riggings associated to length 2 rows, i.e., $(0, 3)$ we embed them into $\mathcal{J}_2$ by ι as follows:

$$(0, 3) \longmapsto (\cdots, -6, -3, -3, 0, \underline{0, 3}, 3, 6, 6, 9, \cdots). \tag{25}$$

Here the original $(0, 3)$ is regarded as $(J_{1,2}, J_{2,2})$ which is enlarged with periodicity $P_2(\mu) = 3$. As a result of σ_2, we take riggings $(J_{2,2}, J_{3,2}) = (3, 3)$. (b) For the riggings associated to length i rows, we further add $2\min(i, 2)$.

• Similarly, action of σ_1 is computed as follows. (a) For the riggings associated to length 1 rows, i.e., $(4, 8)$ we embed them into $\mathcal{J}_1$ by ι as follows:

$$(4, 8) \longmapsto (\cdots, -14, -10, -5, -1, \underline{4, 8}, 13, 17, 22, 26, \cdots). \tag{26}$$

Here the original $(4, 8)$ is regarded as $(J_{1,1}, J_{2,1})$ which is enlarged with periodicity $P_1(\mu) = 9$. As a result of σ_1, we take riggings $(J_{2,1}, J_{3,1}) = (8, 13)$. (b) For the riggings associated to length i rows, we further add $2\min(i, 1)$.

To summarize, we have $\sigma_1\sigma_2(J) = 6 + J'$. This computation has the following interpretation [16]. (a) 6 corresponds to the exponent in $T_1^6(q') = q$. (b) As for $\sigma_1\sigma_2$, we see that in order to get q from q', we have to move six letters 001101 form right of q' to left. These six letters contains two solitons of length 2 and 1, which explains the form of composition $\sigma_1\sigma_2$. □

3.2.5 Direct and inverse scattering transform

Recall that we can choose an integer d such that $p' = T_1^d(p)$ satisfies condition (15). We define direct scattering transform Φ by

$$\begin{array}{rccccc} \Phi: & \mathcal{P}(\nu) & \longrightarrow \mathbb{Z} \times \tilde{\mathcal{P}}(\nu) & \longrightarrow & \bar{\mathcal{J}}(\nu) & \longrightarrow & \mathcal{J}(\nu) \\ & p & \longmapsto (d, p') & \longmapsto \iota(J) + d & \longmapsto & [\iota(J) + d] \end{array} \tag{27}$$

where $\tilde{\mathcal{P}}(\nu)$ is the subset of $\mathcal{P}(\nu)$ consisting of all paths satisfying condition (15). In [13], it was shown that the inverse map Φ^{-1} is well-defined.

Theorem 3.7 ([13], Theorem 3.12) *Define T_l on $\mathcal{J}(\nu)$ by*

$$T_l : (J_{k,i})_{k\in\mathbb{Z},i\in H} \longmapsto (J_{k,i} + \min(i,l))_{k\in\mathbb{Z},i\in H}. \tag{28}$$

Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}(\nu) & \xrightarrow{\Phi} & \mathcal{J}(\nu) \\ {\scriptstyle T_l}\downarrow & & \downarrow{\scriptstyle T_l} \\ \mathcal{P}(\nu) & \xrightarrow{\Phi} & \mathcal{J}(\nu) \end{array} \tag{29}$$

□

4 Interpretation of 10-elimination in terms of the rigged configurations

Let p be a path of type $B_1^{\otimes L}$.

Theorem 4.1 *Let $x_i^{(j)}$ be a position of the i-th 0-soliton located on the $E^{L_j}(p)$. Let $T(p)$ be a two-row tabloid corresponding to the path p, and $J_1, \cdots, J_g$ be the riggings corresponding to tabloid $T(p)$ under the RC-bijection. Then,*

$$x_i^{(j)} = J_{i,L_j} + L_j. \tag{30}$$

Proof. We use induction on the length of path. Let the configuration corresponding to length L path under the RC-map be ν_L. The case $L = 1$ is clear. Let $L > 1$. Let π be a path of length $L + 1$. Then one can distinguish two cases: $\pi = p0$ or $\pi = p1$. In the first case, the positions of 0-solitons as well as RC data remain the same.

In the case $\pi = p1$ we have to distinguish further two cases: $\pi = p'01$ and $\pi = p''11$. In the case $\pi = p'01$, the rigged configuration corresponding to the path π can be obtained from that corresponding to the path $\pi' = p'0$ of length L by creating the new length 1 singular string: The rigging corresponding to the new singular string is equal to

$$J_{m_1(\nu_{L+1}),1} = L + 1 - 2(\ell(\nu_L) + 1). \tag{31}$$

Therefore

$$J_{m_1(\nu_{L+1}),1} + 1 = L - 2\ell(\nu_L). \tag{32}$$

But it is easily seen that exactly one new 0-soliton appears in the position $L-2\ell(\nu_L)$. In the case $\pi = p'11$ we observe that the position of all 0-soliton do not change. Now we have to check that all numbers $J_{i,L_j} + L_j$ do not change as well. Indeed, according to the RC-algorithm, we have to add one box to the singular string in the highest position. After creating the new singular string, its rigging becomes equal to the old one minus 1. But the length of the new string becomes bigger by exactly 1. That means that the modified riggings (i.e., sum of length of row and the corresponding rigging) do not change. ■

Let $X = \{x_0 < x_1 < \cdots < x_N < \cdots\}$ be an ordered set. For any $x \in X$, $x = x_i$, define $x^{\pm} = x_{i\pm1} \in X$, if $i \geq 1$, and $x_0^- = x_0$. Let α be a composition. A standard X-tabloid of the shape α is a filling of the diagram corresponding to the composition α by the elements from the totally ordered set $X \setminus \{x_0\}$ such that the elements along each row are strictly increasing, and the all elements of the filling in question are distinct. An element $x \in T$ is called descent (resp. acsent) if either $x \neq x_1$ and the element x^+ belongs to T, and located in the tabloid T below (resp. above) the element x, or $x = x_1$ and it does't belong to the first row of the tabloid T. We denote by $Des(T) = \{x^+ \,|\, x \in T \text{ is a descent}\}$ (resp. $Acs(T) = \{x^- \,|\, x \in T \text{ is an acsent}\}$). Let $d(T) = \#\,|\,Des(T)\,|$ (resp. $a(T) = \#\,|\,Acs(T)\,|$).

To continue, let us recall that under the rigged configuration bijection to a (semi) standard tabloid T of shape $\alpha = (\alpha_1, \ldots, \alpha_r)$ one associates a collection of partitions $(\nu^{(1)}, \ldots, \nu^{(r-1)})$ such that $|\nu^{(k)}| = \sum_{j \geq k+1} \alpha_j$, $k = 1, \ldots, r-1$, and a set of non-negative integers (riggings) $\{J_{i,j}^{(k)}\}$, $\quad j = \nu_1^{(k)}, \nu_2^{(k)}, \ldots, \quad 1 \leq i \leq \#|s, j = \nu_s^{(k)}|$, with certain constraints, see e.g. [8] for details. Let T be a (semi) standard tabloid, we will call the partition $\nu^{(1)}$ which appears after applying the RC-bijection to T, to be the first configuration corresponding to the tabloid T.

Our next goal is to define recursively a sequence of standard tabloids $T_0 = T$, $T_1, T_2, \cdots$ to be used to define the shape of the first configuration corresponding to a tabloid T under the RC-bijection. Thus let us describe how to obtain tabloid T_i from that T_{i-1}, $i \geq 1$. Namely, consider the descent set $Des(T_{i-1})$, and denote by T_i the tabloid obtained from that T_{i-1} by deleting all entries of T_{i-1} which are equal to either x or x^- for all $x \in Des(T_{i-1})$.

Theorem 4.2 ([7]) *Let T be a standard tabloid, denote by ν the shape of the first configuration corresponding to tabloid T under the RC-bijection. Then*

$$\nu_i' = \#\,|\,Des(T_{i-1})\,|\,, \qquad i = 1, 2, \cdots, \tag{33}$$

where ν' is the transposition of ν. □

In the case when a tabloid T has two rows and corresponds to a positive weight path, one can use a similar construction by using the sets $Acs(T_{i-1})$ instead of those

$Des(T_{i-1})$. It is easy to see that in the $A_1^{(1)}$ case, for positive weight paths we also have

$$\nu_i' = \#\,|\,Acs(T_{i-1})\,|\,, \qquad i \geq 1. \tag{34}$$

As a corollary, we see that in the case under consideration, the 10-elimination algorithm produces the same shape as the RC-algorithm does:

$$\nu = (L_1^{m_{L_1}}, L_2^{m_{L_2}}, \cdots, L_s^{m_{L_s}}), \tag{35}$$

where ν is the configuration associated with the given path under RC-algorithm, and $L_i^{m_{L_i}}$ $(i = 1, 2, \cdots, s)$ are determined by the 10-elimination algorithm.

To summarize, we obtain the following main theorem of our paper.

Theorem 4.3 *Let p be a $B_1^{\otimes L}$ path of $A_1^{(1)}$. Then we have*

$$(\nu, J) = \{(L_i, x_\alpha^{(i)} - L_i)\}_{1 \leq i \leq s, 1 \leq \alpha \leq m_{L_i}} \tag{36}$$

where (ν, J) is the rigged configuration associated with the given path under RC-algorithm, and $L_i^{m_{L_i}}$, $x_\alpha^{(i)}$ are the parameters determined by the 10-elimination algorithm. □

Example 4.4 Consider the length 32 path p treated in Example 2.1. Let us compare (4) and (5) with the result of Example 3.2. Indeed, (4) coincides with the partition $(5, 4, 2, 1, 1, 1)$ obtained in Example 3.2. As for (30), we compute

$$\{x_1^{(1)} - L_1, x_1^{(2)} - L_2, x_1^{(3)} - L_3, x_1^{(4)} - L_4, x_2^{(4)} - L_4, x_3^{(4)} - L_4\} = \{-3, -1, 8, 14, 6, 3\}$$

which coincides with the riggings obtained in Example 3.2.

We now demonstrate how the rigged configuration essentially obtained by 10-elimination will yield the solution of the initial value problem. As an example, we compute $T_\infty^{10000}(p)$. Under T_∞^{10000}, the riggings behave as

$$T_\infty^{10000}: \quad \begin{array}{|c|c|c|c|c|l} \hline & & & & & -3 \\ \hline & & & & & -1 \\ \hline & & & & 8 \\ \hline & & & & 14 \\ \hline & & & & 6 \\ \hline & & & & 3 \\ \hline \end{array} \quad \longmapsto \quad \begin{array}{|c|c|c|c|c|l} \hline & & & & & 49997 \\ \hline & & & & & 39999 \\ \hline & & & & 20008 \\ \hline & & & & 10014 \\ \hline & & & & 10006 \\ \hline & & & & 10003 \\ \hline \end{array}$$

To save space, we hereafter abbreviate such computation as

$$(-3, -1, 8, 14, 6, 3) \xrightarrow{T_\infty^{10000}} (49997, 39999, 20008, 10014, 10006, 10003). \tag{37}$$

The remaining task is to find a proper representative in the equivalence class corresponding to the rigging vector on the right hand side so that we can apply the rigged configuration bijection. To begin with, recall that the action of the cyclic shift operator T_1^d on the rigging vector simply means constant shift $d = (d, d, \cdots, d)$

mod L, where L is the length of the system. Then we can always choose a suitable element $\sigma \in \mathcal{A}$ and a constant vector d so that the riggings $J' = \sigma(J) - d$ satisfy

$$0 \leq J'_{1,i} \leq J'_{2,i} \leq \cdots \leq J'_{m_i,i} \leq P_i(\nu), \qquad \forall i \in H. \tag{38}$$

Let us explain the method based on the example under consideration. In this case, we have $L = 32$, $\nu = (5,4,2,1,1,1)$, and thus $P_5(\nu) = 4$, $P_4(\nu) = 6$, $P_2(\nu) = 14$ and $P_1(\nu) = 20$. Remind that by acting σ_k, the riggings corresponding to rows with length strictly bigger than k gains uniform shift by $2 \cdot k$. From this property, we can adjust the riggings successively from longer rows to shorter ones. In the above example, suppose that we apply σ_4^N for some integer N. Then the rigging corresponding to length 5 row gains $N \times (2 \cdot 4) = 8N$ whereas the rigging corresponding to length 4 row gains $N \times (P_4(\nu) + 2 \cdot 4) = 14N$ (remind that $m_4(\nu) = 1$). By choosing $N = 1667$, we get

$$\text{RHS of (37)} \xrightarrow{\sigma_4^{1667}} (63333, 63337, 26676, 13348, 13340, 13337). \tag{39}$$

Similarly, from $P_2(\nu) = 14$, we get

$$\text{RHS of (39)} \xrightarrow{\sigma_4^{2619}} (73809, 73813, 73818, 18586, 18578, 18575). \tag{40}$$

Finally, from $P_1(\nu) = 20$ and $m_1(\nu) = 3$, we get

$$\begin{aligned}\text{RHS of (40)} \xrightarrow{\sigma_1^{3\cdot 2762}} &(90381, 90385, 90390, 90398, 90390, 90387) \\ &= 90381 + (0,4,9,17,9,6).\end{aligned}$$

Such adjustment is always possible because the vacancy numbers corresponding to all rows of ν except for the longest rows are strictly positive (Lemma 3.9 of [13]). On the rigged configuration (ν, J'), $J' = (0,4,9,17,9,6)$, we can apply the RC-bijection (see Appendix A to [13]), or equivalently the explicit formula (42) to get the corresponding path p':

$$p' = 00000111011100100001100011101100.$$

Since $90381 \equiv 13 \pmod{32}$, we have

$$T_1^{90381}(p') = T_1^{13}(p') = 11000111011000000011101110010000,$$

which reconstructs the computation in the final part of [14]. □

Remark 4.5 There are explicit formula for the initial value problem of the PBBS in terms of the ultradiscrete (or tropical) Riemann theta function [10, 11]. The formulae are direct consequence of the formula (42) bellow and are logically independent to the main claims of [13]. These formulae are parametrized by the rigged configurations and thus can be completely determined by the data L_j and $x_i^{(j)}$ derived from the 10-elimination procedure. □

Remark 4.6 All considerations in this section can be used for linear systems as well as the BBS. Let p be an arbitrary linear path. Then we can embed p into semi-infinite system by

$$p \hookrightarrow p \otimes 1 \otimes 1 \otimes 1 \otimes \cdots . \tag{41}$$

Then we can use Theorem 4.3 and considerations following it without any changes.

In [12, 15] complete solution of the initial value problem of the BBS was derived for all $\bigotimes_i B_{l_i}$ type paths of $A_n^{(1)}$. We quote here the formula in the special case of $B_1^{\otimes L}$ of $A_1^{(1)}$. Let the path be $p = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B_1^{\otimes L}$ and denote the element b_k by $(1 - x(k), x(k))$, i.e., if $b_k = 1$ then $x(k) = 1$ and $x(k) = 0$ otherwise. Denote the corresponding rigged configuration by $(\nu, J) = (\nu_i, J_i)_{i=1}^g$. Here we do not assume that all ν_i, $i = 1, 2, \ldots$ are distinct. Then

$$x(k) = \tau_0(k) - \tau_0(k-1) - \tau_1(k) + \tau_1(k-1), \tag{42}$$

$$\tau_r(k) = -\min_{n \in \{0,1\}^g} \left\{ \sum_{i=1}^{g} (J_i + r\nu_i - k) n_i + \sum_{i,j=1}^{g} \min(\nu_i, \nu_j) n_i n_j \right\} \quad (r = 0, 1), \tag{43}$$

where $n = (n_1, n_2, \cdots, n_g)$. As for the time evolution, the rigged configuration corresponding to $T_l(p \otimes 1 \otimes 1 \otimes \cdots)$ is $(\nu_i, J_i + \min(\nu_i, l))_{i=1}^g$ and we can plug this into (42) directly. Note that these formulae are parametrized by the rigged configurations. Therefore these formulae are completely determined by the data L_j and $x_i^{(j)}$ derived from the 10-elimination procedure. □

Acknowledgements: The work of RS is supported by the Core Research for Evolutional Science and Technology of Japan Science and Technology Agency.

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