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Abstract

Let D be an acyclic digraph. The competition graph of D is a graph which has the same vertex set as D and has an edge between x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of G is the smallest number of such isolated vertices.

A hole of a graph is a cycle of length at least 4 as an induced subgraph. In 2005, Kim [5] conjectured that the competition number of a graph with h holes is at most $h + 1$. Though Kim *et al.* [7] and Li and Chang [8] showed that her conjecture is true when the holes do not overlap much, it still remains open for the case where the holes share edges in an arbitrary way. In order to share an edge, a graph must have at least two holes and so it is natural to start with a graph with exactly two holes. In this paper, the conjecture is true for such a graph.

Keywords: competition graph; competition number; hole

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1 Introduction

Suppose D is an acyclic digraph (for all undefined graph-theoretical terms, see [1] and [13]). The *competition graph* of D, denoted by $C(D)$, has the same vertex set as D and has an edge between vertices x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. Roberts [12] observed that, for any graph G, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* $k(G)$ of a graph G to be the smallest number k such that G together with k isolated vertices added is the competition graph of an acyclic digraph.

The notion of competition graph was introduced by Cohen [3] as a means of determining the smallest dimension of ecological phase space. Since then, various variations have been defined and studied by many authors (see [4, 9] for surveys). Besides an application to ecology, the concept of competition graph can be applied to a variety of fields, as summarized in [11].

Roberts [12] observed that characterization of competition graph is equivalent to computation of competition number. It does not seem to be easy in general to compute $k(G)$ for a graph G, as Opsut [10] showed that the computation of the competition number of a graph is an NP-hard problem (see [4, 6] for graphs whose competition numbers are known). It has been one of important research problems in the study of competition graphs to characterize a graph by its competition number. Cho and Kim [2] and Kim [5] studied the competition number of a graph with exactly one hole. A cycle of length at least 4 of a graph as an induced subgraph is called a *hole* of the graph and a graph without holes is called a *chordal graph*. As Roberts [12] showed that the competition number of a chordal graph is at most 1, the competition number of a graph with 0 hole is at most 1. Cho and Kim [2] showed that the competition number of a graph with exactly 1 hole is at most 2.

Theorem 1.1 (Cho and Kim [2])**.** *Let G be a graph with exactly one hole. Then the competition number of G is at most 2.*

Kim [5] conjectured that the competition number of a graph with h holes is at most $h + 1$ from these results. Recently, Li and Chang [8] showed that her conjecture is true for graphs, all of whose holes are independent. In a graph G, a hole C is *independent* if the following two conditions hold for any other hole C' of G,

1. C and C' have at most two common vertices.
2. If C and C' have two common vertices, then they have one common edge and C is of length at least 5.

Theorem 1.2 (Li and Chang [8])**.** *Suppose that G is a graph with exactly h holes, all of which are independent. Then $k(G) \leq h + 1$.*

After then, Kim, Lee, and Sano [7] generalized the above theorem to the following theorem.

Theorem 1.3 (Kim *et al.* [7])**.** *Let $C_1, \ldots, C_h$ be the holes of a graph G. Suppose that*

(a) *any pairs of $C_1, \ldots, C_h$ shares at most one edge, and*

(b) *if C_i and C_j share an edge, then both C_i and C_j have length at least 5.*

Then $k(G) \leq h + 1$.

Thus, it is natural to ask if the bound holds when the holes share the arbitrary many edges. In this paper, we show that the answer is yes for a graph G with exactly two holes. Our main theorem is as follows.

Theorem 1.4. *Let G be a graph with exactly two holes. Then the competition number of G is at most 3.*

This paper is organized as follows. In Section 2, we investigate some properties of graphs with holes. In Section 3, we give a proof of Theorem 1.4.

2 Preliminaries

A set S of vertices of a graph G is called a *clique* of G if the subgraph of G induced by S is a complete graph. A set S of vertices of a graph G is called a *vertex cut* of G if the number of connected components of $G - S$ is greater than that of G.

Cho and Kim [2] showed that for a chordal graph G, we can construct an acyclic digraph D with the vertices of indegree 0 as many as the number of the vertices of a clique so that the competition graph of D is G with one more isolated vertex:

Lemma 2.1 ([2])**.** *If X is a clique of a chordal graph G, then there exists an acyclic digraph D such that $C(D) = G \cup I_1$, and the vertices of X have only outgoing arcs in D.*

Theorem 2.2. *Let G be a graph and k be a nonnegative integer. Suppose that G has a subgraph G_1 with $k(G_1) \leq k$ and a chordal subgraph G_2 such that $E(G_1) \cup E(G_2) = E(G)$ and $X := V(G_1) \cap V(G_2)$ is a clique of G_2. Then $k(G) \leq k + 1$.*

Proof. Since $k(G_1) \leq k$, there exists an acyclic digraph D_1 such that $C(D_1) = G_1 \cup I_k$ where I_k is a set of k isolated vertices with $I_k \cap V(G) = \emptyset$. Since X is a clique of a chordal graph G_2, there exists an acyclic digraph D_2 such that $C(D_2) = G_2 \cup \{a\}$ where a is an isolated vertex not in $V(G) \cup I_k$ and the vertices in X have only outgoing arcs in D_2 by Lemma 2.1. Now we define a digraph D as follows: $V(D) = V(D_1) \cup V(D_2)$ and $A(D) = A(D_1) \cup A(D_2)$.

Suppose that there is an edge in $E(C(D))$ but not in $E(C(D_1)) \cup E(C(D_2))$. Then there exist an arc (u, x) in D_1 and an arc (v, x) in D_2 for some $x \in X$. However, this is impossible since every vertex in X has indegree 0 in D_2. Thus $E(C(D)) \subset E(C(D_1)) \cup E(C(D_2))$. It is obvious that $E(C(D)) \supset E(C(D_1)) \cup E(C(D_2))$ since $E(C(D)) \supset E(C(D_i))$ for $i = 1, 2$. Thus $E(C(D)) = E(C(D_1)) \cup E(C(D_2)) = E(G_1) \cup E(G_2) = E(G)$.

Moreover, since D_1 and D_2 are acyclic, $V(G_1) \cap V(G_2) = X$, and each vertex in X has only outgoing arcs in D_2, it is true that D is also acyclic. Hence $C(D) = G \cup I_k \cup \{a\}$ and so $k(G) \leq k + 1$. □

Lemma 2.3 ([7]). *Let G be a graph and C be a hole of G. Suppose that v is a vertex not on C that is adjacent to two non-adjacent vertices x and y of C. Then exactly one of the following is true:*

(1) *v is adjacent to all the vertices of C;*

(2) *v is on a hole C^* different from C such that there are at least two common edges of C and C^* and all the common edges are contained in exactly one of the (x, y)-sections of C.*

For a graph G and a hole C of G, we denote by X_C the set of vertices which are adjacent to all vetices of C. Given a walk W of a graph G, we denote by W^{-1} the walk represented by the reverse of vertex sequence of W and denote the length of W by $|W|$. For a graph G and a hole C of G, we call a walk (resp. path) W a *C-avoiding walk* (resp. *C-avoiding path*) if one of the following hold:

- $|W| \geq 2$ and none of the internal vertices of W are in $V(C) \cup X_C$;
- $|W| = 1$ and one of the two vertices of W is not in $V(C) \cup X_C$.

Lemma 2.4. *Let G be a graph and $C = v_0v_1 \cdots v_{m-1}v_0$ be a hole of G. Suppose that there exists a vertex v satisfying the following properties:*

- *v is not on any hole of G.*
- *v is adjacent to v_i for some $i \in \{0, \ldots, m-1\}$.*
- *There is a C-avoiding path from v to a vertex on C other than v_i.*

Let v_j be a vertex with the smallest $|i-j|$ such that there is a C-avoiding (v, v_j)-path and P be the shortest among C-avoiding (v, v_j)-paths. Then v_i is adjacent to every internal vertex on P. Moreover, if none of internal vertices on P belongs to any hole, then $j = i-1$ or $i + 1$.

Proof. Let Q be the shorter (v_i, v_j)-section of C. Firstly, consider the case where $|E(P)| = 1$. If $j \neq i-1$ or $i+1$, then the hypothesis of Lemma 2.3 is satisfied. However, none of (1), (2) holds, which is a contradiction. Thus, $j \in \{i-1, i+1\}$ and we are done.

Now suppose that $|E(P)| \geq 2$. Then v_iPQ^{-1} is a cycle of length at least 4. Since v is not on any hole on G, it cannot be a hole and has a chord. Take an internal vertex w on P. If w is adjacent to a vertex v_k for some k, $1 \leq |i-k| \leq |i-j|$, then v_i, the (v,w)-section of P, and v_k form a C-avoiding (v_i, v_k)-path, which contradicts the choice of v_j. Thus no internal vertex of P is adjacent to any vertex on the shorter (v_i, v_j)-section of C except v_i. Thus v_i is adjacent to an internal vertex of P. Let x be the first internal vertex on P and P' be the (v,x)-section of P. Then $v_iP'v_i$ is a hole or a triangle. However, the former cannot happen by the condition on v. Thus x immediately follows v on P. By repeating this argument, we can show that v_i is adjacent to every internal vertex on P.

Now assume that none of internal vertices on P does not belong to any hole. Let y be the vertex immediately preceding v_j on P. Then v_iyQ^{-1} is a hole or a triangle. By our assumption, the former does not hold. Thus Q is a path of length 1, that is, v_i and v_j are adjacent. Hence $j = i-1$ or $j = i+1$. □

3 Proof of Theorem 1.4

In this section, we shall show that the competition number of a graph with exactly two holes cannot exceed 3.

Let G be a graph with exactly two holes C_1 and C_2. We denote the holes of G by

$$C_1 : v_0v_1 \cdots v_{m-1}v_0, \quad C_2 : w_0w_1 \cdots w_{m'-1}w_0,$$

where m and m' are the length of the holes C_1 and C_2, respectively. In the following, we assume that all subscripts of vertices on a cycle are considered in modulo the length of the cycle.

Without loss of generality, we may assume that $m \geq m' \geq 4$. For $t \in \{1, 2\}$, let

$$X_t := X_{C_t} = \{x \in V(G) \mid xv \in E(G) \text{ for all } v \in V(C_t)\}.$$

In the following, we deal with the case that the two holes have a common edge since Theorem 1.3 covers the case that the two holes are edge disjoint.

Lemma 3.1. *If a graph G has exactly two holes C_1 and C_2, then both X_1 and X_2 are cliques.*

Proof. Suppose that two distinct vertices u_1 and u_2 in X_1 are not adjacent. Then $u_1v_0u_2v_2u_1$ and $u_1v_1u_2v_3u_1$ are two holes other than C_1. That is, G has at least three holes, which is a contradiction. □

Lemma 3.2. *Let G be a connected graph having exactly two holes C_1 and C_2. If C_1 and C_2 have a common edge, then $G[E(C_1) \cap E(C_2)]$ is a path.*

Proof. Suppose that $G[E(C_1) \cap E(C_2)]$ is not a path. Without loss of generality, we may assume that v_0v_1 is a common edge but v_1v_2 is not common. Let v_i is the first vertex on C after v_1 common to C_1 and C_2. Then $i \in \{2, \ldots, m-2\}$. Other than v_0, let w be the vertex on C_2 that is adjacent to v_1. In addition, let Z be the (w, v_i)-section of C_2 not containing v_0.

Now, consider a (w, v_{m-1})-walk $W = Zv_{i+1} \cdots v_{m-1}$. Then $G[W]$ contains a (w, v_{m-1})-path. Let P be a shortest path among such paths. We shall claim that $C = v_0v_1Pv_0$ is a hole. Since neither v_0 nor v_1 is on W, none of v_0, v_1 is on P. Thus C is a cycle. By the definition of P, there is no chord between any pair of vertices on P. Since C_1 is a hole, v_0 is not adjacent to any of v_{i+1}, …, v_{m-2}. On the other hand, $V(Z) \subset V(C_2)$ and $v_0 \in V(C_2)$. Thus v_0 is not adjacent to any vertex on Z. Thus v_0 is not adjacent to any vertex on P. By a similar argument, we can claim that v_1 is not adjacent to any vertex on P except u. Hence we have shown that C is a hole of G. Since C does not contain the edge v_1v_2, C is not C_1 and so $C = C_2$.

Suppose that v_j is adjacent to a vertex v on Z for some $j \in \{i+1, \ldots, m-1\}$. Then v_jv is shorter than any (v, v_j)-path containing v_i in $G[W]$ and so P does not contain v_i. Thus $v_i \notin V(C)$. Hence C does not contain v_i and so C is distinct from C_2, which is a contradiction. Therefore, v_j is not adjacent to any vertex on Z for any $j \in \{i+1, \ldots, m-1\}$. If v_j is on Z, then v_j is adjacent to a vertex on Z, which is impossible. This implies that no vertex on W repeats and that no two nonconsecutive vertices in W are adjacent. Thus $W = P$. Then $G[E(C_1) \cap E(C_2)] = v_iv_{i+1} \cdots v_{m-1}v_0v_1$ is a path and we reach a contradiction. □

Lemma 3.3. *Let G be a connected graph having exactly two holes C_1 and C_2. If $|E(C_1) \cap E(C_2)| \geq 2$, then $X_1 = X_2$.*

Proof. By Lemma 3.2, $G[E(C_1) \cap E(C_2)] = w_iw_{i+1} \cdots w_j$ where $|j - i| \geq 2$. We take a vertex $v \in X_1$. Note that $v \notin \{w_i, \ldots, w_j\}$. Then v must be contained in $V(C_2)$ or X_2 by the Lemma 2.3 since v is adjacent to non-adjacent vertices w_i and w_j in $V(C_2)$. If $v \in V(C_2)$, then C_2 has a chord vw_{i+1}, which is a contradiction. Therefore, $v \in X_2$. Thus, $X_1 \subseteq X_2$. Similarly, it can be shown that $X_2 \subseteq X_1$. □

Lemma 3.4. *Let G be a connected graph having exactly two holes C_1 and C_2. If there is no C_t-avoiding (u, v)-path for consecutive vertices u, v on C_t for $t \in \{1, 2\}$, then $G - uv$ has at most one hole.*

Proof. First consider the case where uv is not a common edge of C_1 and C_2. We may assume that uv is an edge of C_1. Suppose that $G - uv$ has at least two holes. Let C^* be a hole of $G - uv$ distinct from C_2. Now $C^* + uv$ are two cycles C_1 and C' sharing exactly

one edge that is uv. Since uv does not belong to C_2, C' cannot be C_2. If the length of C' is greater than 3, then C' is a hole, contradicting the hypothesis. Thus, $C' - uv$ is a path of length 2. Let x be the internal vertex of $C' - uv$. Since there is no C_1-avoiding (u, v)-path, it is true that $x \in X_1$. However, this implies that C^* has a chord joining x and every vertex in $V(C_1) \setminus \{u, v\}$, which is a contradiction.

Now we consider the case where uv is a common edge of C_1 and C_2. Then $G - uv$ contains neither C_1 nor C_2. Moreover, $X_1 = X_2$. For, if there exist a vertex $x \in X_1 \setminus X_2$ (resp. $x \in X_2 \setminus X_1$), uxv is a C_2-avoiding (resp. C_1-avoiding) path, which is a contradiction. We let $X = X_1 = X_2$.

Now we show that $G - uv$ does not contain a hole. Suppose that $G - uv$ contains a hole H. We will reach a contradiction. Since H is not a hole of G, uv is a chord of H in G. In fact, uv is the only chord of H in G and so $|H| = 4$. Let $H = uxvyu$. Then since there is no C_1-avoiding (u, v)-path in G, both x and y belong to X. Then x and y are adjacent by Lemma 3.1 and so xy is a chord of H in $G - uv$, which contradicts the assumption that H is a hole of $G - uv$ □

Lemma 3.5. *Let G be a graph with exactly two holes C_1 and C_2. Suppose that there exists $i \in \{0, 1, \ldots, m-1\}$ such that $v_i v v_{i+1}$ is a C_1-avoiding (v_i, v_{i+1})-path for some v not in $V(C_1) \cup V(C_2)$. In addition, suppose that $v_i v_{i+1}$ is an edge of C_2 if C_1 and C_2 share at least two edges. Then $X_1 \cup \{v_i, v_{i+1}\}$ is a vertex cut of G.*

Proof. Suppose that there is a vertex in $V(C_1) \setminus \{v_i, v_{i+1}\}$ that is reachable from v by a C_1-avoiding path. Let v_j be the vertex in $V(C_1) \setminus \{v_i, v_{i+1}\}$ of the smallest index that is reachable from v by a C_1-avoiding path. Let P be a shortest C_1-avoiding (v, v_j)-path. By Lemma 2.4, $j = i + 2$ and v_{i+1} is adjacent to all the vertices of P. If $|P| = 1$, then v is adjacent to two nonadjacent vertices on C_1. Since $v \notin X_1$, v is contained in a hole C by Lemma 2.3. Since $\{v_i, v, v_{i+1}\}$ forms a clique, C contains at most one of v_i, v_{i+1}. Thus, C is different from C_1. Since $v \notin V(C_2)$, C is different from C_2 and we reach a contradiction. Hence $|P| \geq 2$. Let $P = vw_1w_2 \cdots w_l v_{i+2}$ for $l \geq 2$. We first consider the cycle $C' = v_2 P v_2$. Then $|C'| \geq 4$. Now we consider the cycle $C'' = v_i P v_{i+3} v_{i+1} \cdots v_{i-1} v_i$. Then $|C''| \geq 4$. If C'' is a hole, then C'' is a hole distinct from C_1 and C_2 since it contains v, which is a contradiction. Thus C'' has a chord. Note that any two nonconsecutive vertices on P cannot be adjacent and that any two nonconsecutive vertices in $V(C'') \setminus V(P)$ cannot be adjacent. Thus a vertex $u \in V(P) \setminus \{v_i, v_{i+2}\}$ is adjacent to a vertex v_k on the (v_{i+3}, v_{i-1})-section of C'' not containing v. Therefore u is adjacent to two nonconsecutive vertex v_{i+1} and v_k on C_1. If C_1 and C_2 share at most one edge, then we reach a contradiction by Lemma 2.3. Consider the case where C_1 and C_2 share at least two edges. Then u is contained in a hole which has at least two common edges with C_1. Since C_2 is the only hole other than C_1, u is on C_2. Then $v_{i+1}u$ is a chord of C_2, which is a contradiction. □

Lemma 3.6. *Let G be a graph with exactly two holes C_1 and C_2 sharing exactly one edge v_iv_{i+1}. Suppose that there exists a C_1-avoiding (v_{i+2}, v_{i+3})-path. Then $X_1 \cup \{v_{i+2}, v_{i+3}\}$ is a vertex cut.*

Proof. Without loss of generality, we may assume that $v_i = w_0$ and $v_{i+1} = w_1$. Then $v_{i+2}v_{i+3}$ in $E(C_1) \setminus E(C_2)$. If a shortest C_1-avoiding (v_{i+2}, v_{i+3})-path has length greater than 2, then the path together with $v_{i+2}v_{i+3}$ form a hole sharing an edge. Since $v_{i+2}v_{i+3}$ is not an edge of C_2, this hole is distinct from C_2, which is a contradiction. Thus a shortest C_1-avoiding (v_{i+2}, v_{i+3})-path is $v_{i+2}vv_{i+3}$ for some $v \in V(G) \setminus V(C_1)$. If $v \in V(C_2)$, then $v = w_{n-1}$ by Theorem 2.3. Then w_{n-1} is adjacent to v_{i+1} and v_{i+3}, which contradicts Lemma 2.3 as $w_{n-1} \notin X_1$ by the definition of a C_1-avoiding path. Thus $v \notin V(C_2)$. By Lemma 3.5, $X_1 \cup \{v_{i+2}, v_{i+3}\}$ is a vertex cut. □

Lemma 3.7. *Let G be a graph with exactly two holes C_1 and C_2 sharing at least two edges. Suppose that for any $i \in \{0, 1 \ldots, m-1\}$, there exists a C_1-avoiding (v_i, v_{i+1})-path. Then there exists a vertex cut consisting of X_1 together with two consecutive vertices on both C_1 and C_2.*

Proof. By Lemma 3.2, $G[E(C_1) \cap E(C_2)]$ is a path. Without loss of generality, we may assume that $G[E(C_1) \cap E(C_2)] = v_1v_2...v_k$ for some integer $k \geq 2$. We shall claim that $X_1 \cup \{v_1, v_2\}$ is a vertex cut as follows. If a shortest C_1-avoiding (v_1, v_2)-path has length greater than 2, then the path together with v_1v_2 form a hole sharing exactly one edge, which is a contradiction. Thus a shortest C_1-avoiding (v_1, v_2)-path is v_1vv_2 for some $v \in V(G) \setminus V(C_1)$. From the fact that C_1 and C_2 are holes and $v_1v_2 \in E(C_1) \cap E(C_2)$, it follows that v is not on C_2. Thus, by Lemma 3.5, $X_1 \cup \{v_1, v_2\}$ is a vertex cut. □

Now, we prove our main theorem.

Proof of Theorem 1.4. Suppose that there is no C_1-avoiding (v_i, v_{i+1})-path for some $i \in \{0, \ldots, m-1\}$. Then $G_1 := G - v_iv_{i+1}$ has at most one hole by Lemma 3.4 and so $k(G_1) \leq 2$ by Theorem 1.1. Let $G_2 := v_iv_{i+1}$. Then G_2 is chordal, $E(G_1) \cup E(G_2) = E(G)$, and $V(G_1) \cap V(G_2) = \{v_i, v_{i+1}\}$ is a clique of G_2. By Theorem 2.2, we have $k(G) \leq 3$. Similarly, if there is no C_2-avoiding (w_i, w_{i+1})-path for some $i \in \{0, \ldots, m'-1\}$, then we can show that $k(G) \leq 3$.

Now we suppose that there are a C_1-avoiding (v_i, v_{i+1})-path for any $i \in \{0, \ldots, m-1\}$ and a C_2-avoiding (w_i, w_{j+1})-path for any $j \in \{0, \ldots, m'-1\}$. We note that we deal with the case where C_1 and C_2 share an edge.

If C_1 and C_2 share exactly one edge, say v_iv_{i+1} for some $i \in \{0, 1, \ldots, m-1\}$, then $X_1 \cup \{v_{i+2}, v_{i+3}\}$ is a vertex cut of G by Lemma 3.6. Let Q be the components of $G - X_1 - \{v_{i+2}, v_{i+3}\}$ that contains $V(C_1) \setminus \{v_{i+2}, v_{i+3}\}$. Let G_2 be the subgraph of G induced by the vertex set of remaining components and $X_1 \cup \{v_{i+2}, v_{i+3}\}$. Since v_iv_{i+1} is contained in Q, C_2 is not contained in G_2 and so G_2 is chordal. Note that the

subgraph induced by $V(Q) \cup X_1 \cup \{v_{i+2}, v_{i+3}\}$ contains no C_1-avoiding (v_{i+2}, v_{i+3})-path and so $G_1 := Q - v_{i+2}v_{i+3}$ has exactly one hole by Lemma 3.4. Thus $k(G_1) \leq 2$ by Theorem 1.1. Thus $k(G) \leq 3$ by Theorem 2.2.

Suppose that C_1 and C_2 share at least two edges. Then by Lemma 3.7, there exists $i \in \{0, 1, \ldots, m-1\}$ such that $X_1 \cup \{v_i, v_{i+1}\}$ is a vertex cut of G and v_iv_{i+1} is a common edge of C_1 and C_2. Let Q^* be the connected component of $G - X_1 - \{v_i, v_{i+1}\}$ that contains $V(C_1) \setminus \{v_i, v_{i+1}\}$. If C_1 and C_2 share at least two edges, C_2. Let G_2 be the subgraph of G induced by the vertices in $V(Q) \cup X_1 \cup \{v_i, v_{i+1}\}$. Then C_2 is not contained in G_1 since at least one edge other than v_iv_{i+1} is contained in Q^* by the assumption that C_1 and C_2 share at least two edges. Thus G_2 is chordal. Note that the subgraph induced by $V(Q) \cup X_1 \cup \{v_i, v_{i+1}\}$ contains no C_1-avoiding (v_i, v_{i+1})-path and so $G_1 := Q - v_iv_{i+1}$ has exactly one hole by Lemma 3.4. Thus $k(G_1) \leq 2$ by Theorem 1.1. Thus $k(G) \leq 3$ by Theorem 2.2. □

References

[1] J. A. Bondy and U. S. R. Murty: *Graph theory with applications*, (North Holland, New York, 1976).

[2] H. H. Cho and S. -R. Kim: The competition number of a graph having exactly one hole, *Discrete Math.* **303** (2005) 32–41.

[3] J. E. Cohen: Interval graphs and food webs: a finding and a problem, *Document 17696-PR*, RAND Corporation, Santa Monica, CA (1968).

[4] S. -R. Kim: The competition number and its variants, *Quo Vadis, Graph Theory*, (J. Gimbel, J. W. Kennedy, and L. V. Quintas, eds.), *Annals of Discrete Mathematics* **55**, North-Holland, Amsterdam (1993) 313–326.

[5] S. -R. Kim: Graphs with one hole and competition number one, *J. Korean Math. Soc.* **42** (2005) 1251–1264.

[6] S. -R. Kim and F. S. Roberts: Competition numbers of graphs with a small number of triangles, *Discrete Appl. Math.* **78** (1997) 153–162.

[7] S. -R. Kim, J. Y. Lee, and Y. Sano: The competition number of a graph whose holes do not overlap much, submitted.

[8] B. -J. Li and G. J. Chang: The competition number of a graph with exactly h holes, all of which are independent, *Discrete Appl. Math.* **157** (2009) 1337–1341.

[9] J. R. Lundgren: Food Webs, Competition Graphs, Competition-Common Enemy Graphs, and Niche Graphs, in Applications of Combinatorics and Graph Theory to the Biological and Social Sciences, *IMH Volumes in Mathematics and Its Application* **17** Springer-Verlag, New York, (1989) 221–243.

[10] R. J. Opsut: On the computation of the competition number of a graph, *SIAM J. Algebraic Discrete Methods* **3** (1982) 420–428.

[11] A. Raychaudhuri and F. S. Roberts: Generalized competition graphs and their applications, *Methods of Operations Research*, **49** Anton Hain, Königstein, West Germany, (1985) 295–311.

[12] F. S. Roberts: Food webs, competition graphs, and the boxicity of ecological phase space, *Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976)* (1978) 477–490.

[13] F. S. Roberts: *Graph theory and its applications to problems of society*, (SIAM, Pennsylvania, 1978).